Winter 2023

Aspects of Localizing Quantum Information in Diffeomorphism-Invariant Theories

Adam Dukehart
University of New Hampshire

Aspects of Localizing Quantum Information in Diffeomorphism-Invariant Theories

BY

Adam M. Dukehart

Master of Science, University of New Hampshire, 2021

Bachelor of Science, Siena College, 2016

DISSERTATION

Submitted to the University of New Hampshire

in Partial Fulfillment of

the Requirements for the Degree of

Doctor of Philosophy

in

Physics

December 2023

This thesis/dissertation has been examined and approved in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics by:

Dissertation Advisor, Dr. David M. Mattingly, Associate Professor of Physics

Dr. Per Berglund, Professor of Physics

Dr. Chanda Prescod-Weinstein, Associate Professor of Physics

Dr. Benjamin D. G. Chandran, Professor of Physics

Dr. Fabian Kislat, Assistant Professor of Physics

On Sept. 14, 2023

Original approval signatures are on file with the University of New Hampshire Graduate School.

DEDICATION

To Joseph C. Dukehart and Elmer V. Lanci.

We began this chapter together, and I am delighted to conclude it in your honor.

I appreciate your kindness and encouragement. May you find eternal peace and rest.

ACKNOWLEDGEMENTS

I want to express my deepest gratitude to my research advisor and defense chair, Prof. David Mattingly, whose expertise, wisdom, and incredible patience helped me build the necessary skills and knowledge to begin a career in Quantum Physics. Furthermore, I am incredibly appreciative to my defense committee, Prof. Per Perglund, Prof. Chanda Prescod-Weinstein, Prof. Benjamin Chandran, and Prof. Fabian Kislat, who graciously provided their knowledge and expertise. A special thank you to Prof. Shawna Hollen for helping me hone my teaching skills, providing me with networking opportunities, and offering career advice.

I wish to extend a special thank you to my graduate school classmates, Abigail Andreoli, Steve Arias, Ben St. Laurent, and Jason Jaung, for your friendship and encouragement in getting me to where I am today. Thank you to the many friends and colleagues I've made inside and outside the Department of Physics for their help in completing coursework, moral support, and company when going on adventures around New England and beyond. I would like to sincerely thank Steven Arias, Moustafa Ismail, Luke Martin, and Nathan Rutheford, who graciously read and provided feedback on several drafts of this dissertation.

Finally, this endeavor would not have been possible without my mother, Lynette Dukehart, who taught me the importance of honesty, hard work, and perseverance. I am also extremely grateful to my father, Matthew Dukehart, stepmother Anne Dukehart, and sister Katryna Dukehart, who taught me the value of personal achievement, determination, and merit.

TABLE OF CONTENTS

ABSTRACT

Aspects of Localizing Quantum Information in Diffeomorphism-Invariant Theories

by

Adam M. Dukehart

University of New Hampshire, December 2023

Quantum Field Theory (QFT) is often formulated using local operators, i.e., operators defined at points on a fixed background spacetime. However, General Relativity (GR) requires that observables be fully diffeomorphism invariant. This includes invariance not only under passive diffeomorphisms, which change the coordinate system but also active diffeomorphisms, which imply background independence and a dynamical metric. Localizing such systems and observables in quantum gravity has been a long-standing problem. This dissertation makes progress on two facets of this problem. Part I shows that a typical quantum information paradox using local operators cannot be embedded in a self-consistent, low-energy prescription which makes it diffeomorphism invariant. Part II explores the energy cost of localizing quantum information using the relational frameworks of the G-twirl and the Z-model. Moreover, it is shown that these two frameworks are related since both can be obtained via limits of positive-operator valued measurements. Part I is based on work currently under review at the International Journal of Modern Physics D and Part II is based on work published in Physical Review D.

CHAPTER 1

Introduction

Quantum Field Theory (QFT) is one of the most well-tested frameworks in modern physics. With many of its predictions having experimental support [1], QFT is regarded as a fundamental framework that governs experimentally accessible physical reality. Fields in QFT implement and maintain the principle of locality for quantum dynamics and interactions [2]. The *principle of locality*[1] states that physical interactions occur at a point in spacetime[2], i.e., the interacting objects must be co-located or "touching." Thus, the introduction of locality removes the notion of action at a distance. While this may be obvious if one considers interactions such as collisions or hard scattering, it may be less evident if one considers long-range interactions such as electron-electron repulsion. In QFT, long-range interactions are described as a set of local interactions with a mediator field[3]. For example, in electron-electron repulsion, each electron locally interacts with the mediating electromagnetic field. This was first observed classically by Faraday, later on, formalized classically by Maxwell, and then formalized quantum mechanically by Dyson, Feynmann, Schwinger, and Tomonaga.

In QFT, physical interactions are generally described by polynomial terms in the Lagrangian, where the polynomials are formed from phase space variables at the same spacetime point, i.e., polynomials of local operators. Mathematically, local operators, written as $\hat{O}$, act on a set of L^2-integrable test functions that have compact support on an open region of spacetime, i.e., the set $f(x) \in L^2$, where $x \in \mathcal{U}$ and $\mathcal{U} \subset \mathbb{R}^4$. Since the test functions

[1] Referred to from now on as *locality*.

[2] This concept is also referred to as *cluster decomposition*, which states that the vacuum expectation values of operators localized to bounded regions become factorized when the regions are sufficiently separated [3].

[3] It is important to note that there are subtleties here that will be discussed later on in this chapter.

are of compact support, they vanish outside their region of support. When the region of support is shrunk to a point $p \in \mathcal{U}$, the test functions become Dirac delta functions. Thus, local operators $\hat{O}(x)$, defined at a point $x \in \mathbb{R}^4$, act on the states of a rigged Hilbert space, $\mathcal{H}$, defined at the same point. In Lorentz invariant QFT, locality is further implemented by imposing microcausality relations. Given two local operators defined at different spacetime points, $\hat{O}_1(x)$, $\hat{O}_2(y)$ where $x, y \in \mathbb{R}^4$, their commutator is

$$\left[\hat{O}_1(x), \hat{O}_2(y)\right] = 0 \text{ for } (x-y)^2 > 0, \tag{1.1}$$

i.e., microcausality requires that two spacelike separated local operators commute. This implies that the set of local operators defined at a point x acts trivially on the states in a Hilbert space defined at a point y, for spacelike separated x, y. If non-local operators were allowed, then the microcausality relations would break, i.e., the commutation in (1.1) would not vanish, and the theory would allow for unobserved effects such as paradoxes [4] and contain a necessary condition for superluminal communication [5]. It is important to note that there exists a subset of operators that represent physical quantities, e.g., position, momentum, spin, etc. These operators are Hermitian and are commonly referred to as *observables* [6, 7].

There is an important distinction that now needs to be made. The locality conditions above describe operator locality, not state locality. The microcausality relations in (1.1) are an example of commuting subalgebras for a quantum theory. The existence of commuting subalgebras and an appropriate UV regulator [8] allow for the factorization of the Hilbert space into a tensor product of subsystem Hilbert spaces, e.g., the factorization of the Hilbert space $\mathcal{H}_{AB}$ is defined as $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces describing subsystems A and B. This is the Hilbert space manifestation of locality. Such factorizable Hilbert spaces are required for the construction of entangled states [9]. States solely belonging to a subsystem Hilbert space are local, however entangled states, such as the

Bell state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |1\rangle_B)$, belong to both subsystem Hilbert spaces in a non-separable way [9]. This is an example of state non-locality since the information within the state cannot be assigned to solely one subsystem Hilbert space. Thus, quantum entanglement can be an example of state non-locality.

In contrast, operator non-locality is a feature of gauge theories—specifically theories that describe long-range gauge fields such as electromagnetism and gravity. Since observables are operators that represent physical quantities, and physical quantities are gauge invariant, observables must be gauge invariant, i.e., observables must commute with the gauge constraints of the theory [10]. Not all operators are gauge invariant, but operators can be made gauge invariant with the inclusion of a dressing function. However, particular dressings can make the resultant operator non-local. For example, consider quantum electromagnetism, a $U(1)$ gauge theory. The two constraints of the theory are the momentum constraint $\phi_1(x) = \Pi^0(x) = 0$ and the Gauss Law constraint $\phi_2(x) = \partial_i \Pi^i(x) = \psi^\dagger(x)\psi(x) = 0$, where $\Pi^\mu(x)$ is the conjugate momentum to the vector potential and $\psi(x)$ is some charged matter field, a local operator. The charged matter field operator has a non-vanishing commutator with both constraints, meaning it is not a gauge-invariant quantity. Thus, the field transforms non-trivially under the $U(1)$-gauge transformation $\psi(x) \to e^{iq\theta(x)}\psi(x)$. To construct a gauge-invariant operator, Dirac [11] introduced the dressing function

$$V(x) = e^{iq\int d^4x' f^\mu(x,x')A_\mu(x')} \tag{1.2}$$

where $f^\mu(x,x')$ satisfies $\partial_\mu f^\mu(x,x') = \delta^4(x-x')$ but is otherwise arbitrary. The dressed operator, $\Psi(x) = V(x)\psi(x)$, transforms trivially under the standard $U(1)$-gauge transformations,

$$A_\mu(x) \to A'_\mu = A_\mu(x) - \partial_\mu\theta(x), \text{ and} \tag{1.3a}$$

$$\psi(x) \to \psi'(x) = e^{-iq\theta(x)}\psi(x), \tag{1.3b}$$

where $\theta(x)$ is an arbitrary scalar function. To show the dressed operator $\Psi(x)$ transforms trivially under the $U(1)$ gauge transformations, apply (1.3) to $\Psi(x)$,

$$\begin{aligned}\Psi(x) \to \Psi'(x) &= e^{-iq\theta(x)}\Psi(x) = e^{-iq\theta(x)}\psi(x)e^{iq\int d^4x' f^\mu(x,x')(A_\mu(x')-\partial_\mu\theta(x'))} \\ &= e^{-iq\theta(x)}\Psi(x)e^{-iq\int d^4x'(\partial_\mu(f^\mu(x,x')\theta(x'))-\theta(x')\partial_\mu f^\mu(x,x'))} \\ &= e^{-iq\theta(x)}\Psi(x)e^{iq\theta(x)} = \Psi(x)\end{aligned} \tag{1.4}$$

Going between the second and third equality, the total derivative term vanishes since the boundary is taken to be at infinity and the second term computes to $\theta(x)$ due to the condition $f^\mu(x,x')$ must satisfy. When acting on the vacuum, the dressed operator $\Psi(x)$ creates a particle with charge q and electromagnetic field response with support over all space, i.e., the operator is non-local but the resultant state is gauge-invariant. While the particle field operator is not an observable, the propagator $\langle 0|\, \Psi^\dagger(x)\Psi(y)\, |0\rangle$ is an observable. Since the $U(1)$ gauge allows for opposing charges, the observable can be a local object—the non-local aspects of the field operators can be screened and inaccessible to an observer.

Similar to QFT, General Relativity (GR) is another well-tested physical framework [12]. While GR is considered to be a classical approximation of the theory of quantum gravity, many of its underlying principles are considered to be fundamental and are thought to be underlying principles in the theory of quantum gravity, whatever it may be (see [13, 14] for more details). Outside any theory of quantum gravity, quantum models that include gravity[4] must be able to handle principles from both QFT and GR. One such principle is the *principle of general covariance*[5], a key concept underlying gravitational physics. General covariance states that (A) everything is dynamic and (B) physics is the same in every reference frame. Mathematically, observables and states in any generally covariant framework are invariant under diffeomorphisms. Thus diffeomorphisms serve as the gauge group of gravity, meaning any physical observable in a "quantum + gravity" framework must be

[4]Such models will be referred to from now on as "quantum + gravity."
[5]Referred to from now on as *general covariance*.

diffeomorphism invariant. However, difficulties arise when constructing local operators in diffeomorphism-invariant theories. Diffeomorphisms move points around arbitrarily and, as a result, the notion of locality cannot be defined since there is no "here" or "there." This means that the operators in such theories cannot have compact support, not only violating locality by also microcausality. Therefore, there is tension[6] between locality and general covariance; operators that are local relative to a fixed background reference frame are not gauge invariant and gauge-invariant operators are non-local.

While the full theory of quantum gravity has yet to be discovered, there has been recent interest in utilizing quantum information theory to construct tests and determine properties the full theory of quantum gravity must satisfy [16, 17]. However, quantum information theory is not inherently diffeomorphism invariant and there are difficulties when coupling to gravity. An implication is that quantum information theory tests of quantum gravity may not be accurate since one must be careful when determining whether the results are physical or pure gauge. Furthermore, key aspects of quantum information theory can be incompatible with diffeomorphism invariance such as local observables, or entanglement. The question—and the main focus of this dissertation—is how do diffeomorphism invariance observables affect certain questions in quantum information theory?

1.1 Diffeomorphism-Invariant Operators in Quantum Paradoxes

As discussed above, one way to construct gauge-invariant operators in theories coupled to electromagnetism is to act a dressing function on an operator that has a non-trivial gauge transformation. A similar method, as demonstrated by Giddings and Donnelly [18], can be used to construct diffeomorphism-invariant operators in theories coupled to gravity. In a departure from Dirac's electromagnetic dressing, which is constructed non-perturbatively, Giddings and Donnelly use a first-order perturbation $h_{\mu\nu}$, defined by $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ to

[6]There are some theories that require observable non-locality. For example, in AdS/CFT the non-locality of bulk AdS observables allows them to be mapped to local CFT observables on the boundary [15].

construct the dressing. This dressing has the following definition,

$$V^{\mu}(x) = \kappa \int d^4x' f^{\mu\nu\lambda}(x, x') h_{\nu\lambda}(x') \tag{1.5}$$

where f is symmetric in the last two indices and satisfies, $2\partial_{\nu'} f^{\mu\nu\lambda}(x, x') = \delta^4(x - x')\eta^{\mu\nu}$. For a local operator $\hat{O}(x)$, the resultant observable, $\hat{\mathcal{O}}(x) = \hat{O}(x) + V^{\mu}(x)\partial_{\mu}\hat{O}(x)$, while diffeomorphism invariant up to first order in the gravitational coupling, does not have compact support due to the integral over all space. This behavior is similar to the behavior of the electromagnetic dressing. However, unlike electromagnetism, gravity does not have opposite gauge charges that can be used to construct local observables, i.e., there is no negative mass.

Theories coupled to electromagnetism are able to hide (or localize) quantum information using a screening effect provided by edge-mode charges. For example, consider an electron located within a spacetime region $\mathcal{U}$. Due to the non-local nature of the gauge dressing, the electron will generate an everywhere-permeating electromagnetic field response. This means that an observer, located outside the $\mathcal{U}$ can detect and determine the state of the electron by measuring the state of the electromagnetic field. Effectively, the quantum information of the electron is currently non-local. The quantum information can be localized to $\mathcal{U}$, hiding the presence of the electron, by introducing charged edge modes to the boundary of $\mathcal{U}$, whose total charge is equal to but opposite of the electron's charge. Since the edge modes are observables, they must be dressed as well, and due to their opposing charge, the dressing components perpendicular to the boundary will cancel with the dressing components of the electron. Therefore, the quantum information of the electron won't leak out of $\mathcal{U}$ since the electromagnetic field components perpendicular to the boundary vanish outside $\mathcal{U}$.

Since there are no opposite gauge charges in theories coupled to linearized gravity, such theories are incapable of perfectly hiding quantum information in the same way as theories coupled to electromagnetism. However, it is possible to choose a gravitational dressing that is "insensitive" to the quantum information of a state, located within a spacetime region

$\mathcal{U}$ [18]. Consider a matter distribution with the state $|\psi\rangle$ located within a spacetime region $\mathcal{U}$ and an ϵ-extended region $\mathcal{U}_\epsilon$ such that $\mathcal{U} \subset \mathcal{U}_\epsilon$. One can then choose the gravitational split dressing to dress the state. The split dressing is constructed from two additive components, a "standard" dressing $V_S^\mu(y)$, defined at $y \in \mathcal{U}$, and a dressing that connects the point y to another point $x \in \mathcal{R}_\epsilon$, $V_L^\mu(x, y)$, such that $V_L^\mu(x, y) = 0$ for all $x \notin \mathcal{U}_\epsilon$. As a result, the expectation value of the metric perturbation outside $\mathcal{U}_\epsilon$ depends only on the total Poincare charges of the state. In other words, the only information an observer, located outside $\mathcal{U}_\epsilon$ can ascertain from a measurement of the gravitational field is the total Poincare charge of the matter inside $\mathcal{U}$. Furthermore, two states $|\psi_1\rangle$ and $|\psi_2\rangle$ will be indistinguishable via measurements outside $\mathcal{U}_\epsilon$ since such measurements are dependent on the total Poincare charge. Therefore, the split dressing makes it difficult to fully determine the quantum information located within $\mathcal{U}$. Note that one can encode information within the Poincare charges of the system, but not all information can be encoded in this manner.

Part I of this dissertation is based on the work found in [19]. Here, the focus is on investigating the effects of gravity on quantum paradoxes. In particular, the effects diffeomorphism-invariant operators have on the construction of the Frauchiger-Renner (FR) paradox. While there is debate surrounding the validity of the FR paradox (cf. [20–23]), the paradox will be used as a way to examine the subtle role of (quantum) gravity, in particular, local observables in perturbative paradoxical outcomes. The authors of the FR paradox argue, using a set of reasonable assumptions in the framework of non-relativistic quantum mechanics, that one can construct an experiment where two observers measure conflicting results, an outcome contradictory to what is predicted by the theory. They then use these results to argue non-relativistic quantum mechanics is not self-consistent. However, gravity is part of reality, so from a fundamental perspective, the quantum theory that needs to be self-consistent is the full theory of quantum gravity. Structurally, the physical Hilbert space and dynamics of quantum gravity are heavily constrained by the requirement of diffeomorphism invariance, (cf. [24]), which is typically not imposed in non-relativistic quantum mechanics. How-

ever, when using paradoxes to make claims about underlying fundamental theories, such as quantum mechanics, it is necessary to at least ensure the paradoxical observables are gauge invariant. In the gravitational case, this would mean introducing the gravitational field as a dynamical object and imposing diffeomorphism invariance. While this may be difficult to do non-perturbatively, one approach would be to match the field content of general relativity and impose diffeomorphism invariance perturbatively to first-order in Newton's constant G. If such an approach can be made self-consistent and generate only small corrections to the non-relativistic results, then ignoring gravity is justified. This part examines whether one can build a "paradoxical" thought experiment that involves local observables while maintaining gauge invariance to first order in G in a self-consistent manner (up to the paradoxical result occurring at some time t). It is shown below that perturbative gauge invariance can't be self-consistently imposed if the experiment is to eventually generate a paradox—higher order corrections in G and fundamental aspects of full quantum gravity become important. This is surprising as it seems that a low-energy thought experiment is somehow sensitive to quantum gravity. However, this is actually already a known occurrence in the gravitational literature for precisely those thought experiments that incorporate paradoxes and quantum mechanics. Classically, spacetimes that allow closed timelike curves (CTCs) may have low curvature yet contain grandfather paradoxes. Quantum mechanically, these spacetimes lead to the breakdown of semi-classical gravity due to unstable energy-momentum tensor fluctuations [25] and the necessity of the chronology protection conjecture [26], a statement about quantum gravity. Similarly, the Almheiri-Marolf-Polchinski-Sully (AMPS) or infaller paradox in black hole thermodynamics, which is built in spacetime regions with low curvature, generates extreme gravitational responses (e.g. firewalls) [27] once quantum mechanics is taken into account. Firewalls have also been argued to be one manifestation of a more generic paradox, due to the violation of boundary unitary [15], that can occur in scattering experiments [28] with local bulk operators. In all these scenarios, the existence of the paradox leads to a strong gravitational and even quantum gravitational response—one cannot stay in the weak field

regime self-consistently and still assume there exists a quantum mechanical paradox.

1.2 Localizing Relational Quantum Information

Since diffeomorphisms move points around arbitrarily, any labeling of spacetime points has no physical meaning in diffeomorphism-invariant theories. Moreover, as was mentioned earlier, one way to work within the framework of diffeomorphism-invariant theories is to dress operators to make them gauge invariant. While the dressed operators are gauge-invariant, they are no longer local. However, local operators can be defined by working in a relational framework. In a relational framework, local operators are defined relative to some reference observable in such a way that the new operator is diffeomorphism invariant. An example of a relational framework is the Z-model, developed by Giddings, Marolf, and Hartle [29]. In the Z-model, along with the preexisting operators $\hat{O}(x)$, four additional massless, free, scalar auxiliary fields Z^i, $i = 0, 1, 2, 3$ are also defined. If the state of the auxiliary field is initially $|\psi_Z\rangle$ in some spacetime region, then the spacetime coordinates can be defined as,

$$\langle\psi_Z|\,\hat{Z}^i\,|\psi_Z\rangle = \lambda\delta^i_\mu x^\mu. \tag{1.6}$$

In other words, the spacetime coordinates of a region are proportional to the expectation value of the auxiliary field in that region. While the states $|\psi_Z\rangle$ spontaneously break diffeomorphism invariance, local operators may be defined relative to the expectation value in (1.6). A local operator $\hat{O}(x)$, may be made relational by defining,

$$\hat{\mathcal{O}}_{0,\xi} = \int d^4x \hat{O}(x)\delta\left(\hat{Z}^i(x) - \xi^i\right)\left|\frac{\partial\hat{Z}^i(x)}{\partial x^\mu}\right|, \tag{1.7}$$

where x represents some background coordinate system, not to be confused with the coordinate system constructed from the expectation values of the auxiliary fields. The new operator $\hat{\mathcal{O}}_{0,\xi}$ is diffeomorphism invariant under the coordinate change $x^\mu \to x^{\mu'}(x^\nu)$. Furthermore, this operator is local since the delta function will pick out a definite point, in the

classical limit. It is important to note that the construction in (1.7) cannot work in QFT due to the delta function. However, one can use a coarse-graining of the delta function to write a similar operator compatible with QFT (see [29] for more details).

Relational observables allow one to properly define quantum entanglement and thus non-local quantum information. In the past, researchers have considered localizing pure states in a "quantum + gravity" framework [30–33]. However, to the author's knowledge, no one has examined localizing the non-local quantum information in an entangled state in a gravitational setting. To localize pure states, consider a massless particle trapped in a box of length L. The energy of the trapped particle, in the rest frame of the box, is given by $E(L) = hc/L$, neglecting dimensionless constants. As the box shrinks, $L \to 0$, the energy of the particle increases and eventually passes the energy necessary to form a black hole. In other words, if $R(E) = 2GE/c^2$ is the Schwarzschild radius associated with energy E, then when $L \approx R(E(L))$, a black hole will begin to form, rapid emission of Hawking radiation occurs, and one is quickly pushed into the quantum gravity regime. This occurs around the Planck length. The above argument demonstrates the breakdown of state locality in a gravitational setting: distances below the Planck scale cannot be measured because the act of doing so would disturb spacetime to the degree that would form black holes. In quantum information theory, one can construct non-local quantum information using qubits. The non-local information contained in a set of entangled qubits can be localized via an entanglement extraction procedure [34, 35].

Part II of this dissertation is based on the work in [36]. It examines the energy cost required to fully localize non-local quantum information contained in bipartite entangled Gaussian states. In this approach, the question of localization can be recast as one of entanglement extraction, where extracting the entanglement corresponds to localizing the system. Hackl and Jonsson have recently made progress on calculating the minimum energy necessary to extract such bipartite entanglement from Gaussian systems, which will provide the tools necessary to derive the energetics for the quantum information localization process [35].

There is a complication, however. The entanglement extraction process relies on the existence of an external observer to set the energy scale and reference frame. In gravity, there are no external observers and observables are expected to be diffeomorphism invariant. In order to apply Hackl and Jonsson's approach in a "quantum + gravity " scenario, one must also construct a system where the degrees of freedom are diffeomorphism-invariant and Gaussian. If not, then one would not be sure whether the answers are gauge invariant and hence physical. However, constructing fully diffeomorphism-invariant degrees of freedom can be difficult and so this work focuses on translationally invariant degrees of freedom, as a toy model. Translational invariance is established in two ways, utilizing a relational framework. First, one can apply a procedure known as G-twirling to a set of Gaussian states in the presence of an external frame. By G-twirling over translations, the notion of absolute position in a system is removed, leaving only relational and entangled positional degrees of freedom [37] (as well as an irrelevant center-of-mass momentum degree of freedom). As demonstrated below, the simplest version of G-twirling implies that the energy cost from entanglement extraction vanishes. This is consistent with implementing the G-twirl as a simple transformation on the Hilbert space. The underlying translation invariance and effect on the energy cost is the toy model equivalent of the effect of diffeomorphism invariance and the vanishing of the Hamiltonian on physical states in quantum gravity. In contrast, one can also implement an external frame as a dynamical system, thereby also relationalizing the quantum states. In this case, one can naturally recover a non-zero energy cost. To implement this frame, a $U(1)$ symmetry and corresponding gauge field are introduced and a simple Z-model is built for relational observables in the language of Giddings, Marolf, and Hartle [29].

Part I

Diffeomorphism Invariance and Quantum Mechanical Paradoxes

CHAPTER 2

The Frauchiger-Renner Thought Experiment

Thought experiments play a large role in quantum mechanics, both in popular treatments, such as the famous Schrödinger's cat, and in professional discourse. Quantum mechanics is often counterintuitive to a classically trained intuition and thought experiments have a long history of illuminating non-classical logical results and fallacies. One such line of thought experiments is that of "Wigner's Friend" [38] and various extensions, such as the Frauchiger-Renner (FR) [39] experiment, which has recently generated interest [20,39,40]. In the FR thought experiment, used as a prototypical example in this work, the authors argue that under some fairly reasonable assumptions in the framework of non-relativistic quantum mechanics, one can construct an experiment where two observers measure conflicting results. They then use this result to argue that non-relativistic quantum mechanics is not self-consistent.

While the FR thought experiment is a matter of debate among the quantum information community, it is a useful tool to examine the subtle role of (quantum) gravity in such constructions. This chapter discusses the details of the FR thought experiment and its possible connections to (quantum) gravity.

2.1 The Assumptions of the FR Thought Experiment

The FR thought experiment [39] works in the framework of non-relativistic quantum mechanics and makes the following assumptions, labeled (Q), (C), and (S) in its original construction:

(Q) Every observer evolves under the rules of standard quantum mechanics on their local Hilbert spaces. Within that framework, there is a process that allows observers to make conclusive, classically stable measurements. Specifically, let a system S be in state $|\psi\rangle_S$ at time t_0, and let there exist a family of Heisenberg measurement operators, $\{\hat{\pi}_x\}_{x\in\mathbb{R}}$ that allows one to measure an observable x by time $t > t_0$. If $\langle\psi|\,\hat{\pi}_\xi\,|\psi\rangle = 1$ for ξ as some possible value of x, then one can make the definitive statement at time t that $x = \xi$. The ability to make such definitive statements about the outcomes of measurements will be referred to as *with certainty*. Furthermore, since the measurement is stable, $x = \xi$ for all time after t.

(C) Inferred knowledge is transferable between two observers. If observer A knows that observer A' has concluded that $x = \xi$ at some time t and that A' is reasoning using the exact same theory as A, then A can conclusively state that $x = \xi$ at time t.

(S) Contradictory logical statements cannot be true simultaneously. If A believes that $x = \xi$ at time t is true, then they cannot also believe that $x \neq \xi$ at time t is true.

With the above assumptions in hand, the thought experiment itself can now be explained in detail.

2.2 The Experimental Setup of the FR Thought Experiment

The setup for the FR thought experiment is similar to constructions employed in Hardy's and Wigner's Friend thought experiments. Consider four quantum observers, Alice, Bob, Charlie, and Diane. Charlie and Diane are each locked away in different local, isolated laboratories, while Alice and Bob are outside observing Charlie's and Diane's laboratories, respectively, in separate isolated regions. The experiment proceeds as follows, in rounds $n = 0, 1, 2, \ldots$. The number on the right indicates the timing of the steps and it is assumed that each step takes one unit of time:

$n:00$ Charlie measures the state of a quantum coin toss, $|coin\rangle \equiv (|heads\rangle + \sqrt{2}\,|tails\rangle)/\sqrt{3}$, in the $\{|heads\rangle\,, |tails\rangle\}$ basis, and records the result as $r \in \{heads,\ tails\}$. He then sets the spin state S of a particle to $|\downarrow\rangle_S$ if the result is $r = heads$ and to $|{\rightarrow}\rangle_S \equiv (|\uparrow\rangle_S + |\downarrow\rangle_S)/\sqrt{2}$ if the result is $r = tails$. Charlie then sends the particle to Diane.

$n:10$ Diane measures S with respect to the $\{|\uparrow\rangle_S\,, |\downarrow\rangle_S\}$ basis and records the result as $z \in \{-\frac{1}{2}, +\frac{1}{2}\}$.

$n:20$ Alice measures Charlie's laboratory with respect to a basis containing the state $|\overline{ok}\rangle \equiv (|\overline{h}\rangle - |\overline{t}\rangle)/\sqrt{2}$, where the states $\{|\overline{h}\rangle\,, |\overline{t}\rangle\}$ are the possible states of Charlie's laboratory after he has obtained the result of the coin toss. If the result associated with the $|\overline{ok}\rangle$ state occurs Alice records $\overline{w} = \overline{ok}$ otherwise she records $\overline{w} = \overline{fail}$.

$n:30$ Bob measures Diane's laboratory with respect to a basis containing the state $|ok\rangle \equiv (|-\frac{1}{2}\rangle - |\frac{1}{2}\rangle)/\sqrt{2}$, where the states $\{|-\frac{1}{2}\rangle\,, |\frac{1}{2}\rangle\}$ are the possible states of Diane's laboratory after she has obtained the result of the spin measurement. If the outcome associated with the $|ok\rangle$ state occurs, Bob records $w = ok$, otherwise, he records $w = fail$.

$n:40$ If both Alice and Bob measure $\overline{w} = \overline{ok}$ and $w = ok$, respectively, the experiment is stopped, otherwise it continues to the next round.

2.3 The Paradox

Alice and Bob can construct an entangled qubit from the correlations between the measurements of Charlie and Diane,

$$|\psi\rangle_{AB} = \frac{|\overline{h}\rangle_A \otimes |-\frac{1}{2}\rangle_B + |\overline{t}\rangle_A \otimes |-\frac{1}{2}\rangle_B + |\overline{t}\rangle_A \otimes |+\frac{1}{2}\rangle_B}{\sqrt{3}}, \tag{2.1}$$

where the first qubit is associated with Alice and the second qubit is associated with Bob. Alice and Bob can calculate the probability they will measure $\overline{w} = \overline{ok}$ and $w = ok$, respectively,

by acting the states $|\overline{ok}\rangle$ and $|ok\rangle$ on $|\psi\rangle$,

$$\left|\left(\langle ok|_B \otimes \langle \overline{ok}|_A\right)|\psi\rangle_{AB}\right|^2 = \frac{1}{12}. \tag{2.2}$$

Another analysis can be performed by examining the inferences that each observer—Alice, Bob, Charlie, and Diane—can construct via assumptions (Q), (C), and (S). Examination of (2.1) shows that if Diane measures $z = -1/2$, then Alice can only measure the state $(|\overline{h}\rangle + |\overline{t}\rangle)/\sqrt{2}$. Hence, from Alice's perspective, if she measures $\overline{w} = \overline{ok}$, then she can infer Diane has measured $z = +1/2$. From Diane's perspective, if she measures $z = +1/2$, then she can infer that Charlie has measured $r = tails$ since there is no $|\overline{h}\rangle_A \otimes |+\frac{1}{2}\rangle_B$ component in (2.1). From Charlie's perspective, if he measures $r = tails$, then he can infer that Diane will be in the $(|-\frac{1}{2}\rangle + |+\frac{1}{2}\rangle)/\sqrt{2}$ state, implying that Bob will measure $w = fail$. Thus, if Alice measures $\overline{w} = \overline{ok}$, then she knows with certainty that Diane knows that Charlie knows that Bob measured $w = fail$, even though Bob and Alice will have a non-zero probability of "agreement". From this analysis Frauchiger and Renner have concluded that a contradiction arises, yielding a paradox.

There are counterarguments to the logic of the FR thought experiment [41–43]. One counterargument, proposed by [44], highlights the issues of using inferred information before and after a measurement is performed. The inference—if Charlie is in the $|\overline{t}\rangle$ state then Bob measures $w = fail$—only holds for the original qubit state in (2.1). However, Alice *first* measures (2.1) in the $|\overline{ok}\rangle$ bases, thus transforming (2.1), implying Charlie no longer has access to the information required to make his inference. Therefore Charlie could never conclude that Bob has measured $w = fail$, since the non-measured version of (2.1) does not exist after Alice's measurement. The rejoinder of Renner [45] is that no information dependent on Alice's measurement is used in Charlie's logic and hence the paradox is logically sound. This argument is of no concern, rather the focus is on the consequences of gauge invariance, in particular diffeomorphism invariance, in the construction of the thought experiment.

The FR thought experiment, as constructed, requires classically certain measurements, local observers, and logical inferences to generate a paradox, as codified in assumptions (Q), (C), and (S). Under assumption (Q) Alice and Bob are defined to be local observers with separable Hilbert spaces. This is, however, in tension with general covariance (which should be preserved in low energy quantum gravity as it is a symmetry of general relativity), as constructing separable Hilbert spaces in generally covariant theories is non-trivial [29, 46]. In other words, the formulation is not a priori gauge invariant.

Any claim that asserts quantum mechanics is not self-consistent, based on non-gauge-invariant observables, must be eyed with caution. Such a claim is equivalent to observers who use different $U(1)$ gauges to claim that electromagnetism is not self-consistent because they each record different values for the vector potential. Certainly, it is possible that there is a physical difference but the difference could be pure gauge. Remedying this in the gravitational context means dressing any local observables so that they are gauge invariant under diffeomorphisms. Only then can one explore the ramifications for quantum mechanics. The key question is whether or not making the construction gauge invariant can be done while remaining in a weak gravity regime. Before this is examined in more detail, the case in electromagnetism is analyzed, which should be more familiar to readers.

CHAPTER 3

QED Gauge-Invariant Observables and the FR Thought Experiment

Not all degrees of freedom within a theory are physical. Sometimes there exist degrees of freedom that are defined with respect to a "reference frame" whose choice is arbitrary at every instance in time [10]. Such degrees of freedom are known as gauge degrees of freedom and are not physical; theories that contain these degrees of freedom are known as gauge theories. Physical degrees of freedom, e.g. observables, are independent of reference frame choice [10]. In other words, observables are invariant under transformations of the reference frame, i.e., gauge transformations, whereas gauge degrees of freedom are susceptible to such transformations.

In gauge theories, the dynamics of physical observables cannot be completely determined by the equations of motion for all times t, since one could always make a gauge transformation in the future given some set of initial conditions. This means that one set of initial conditions is not enough to specify the trajectory of physical observables through time, since a gauge transformation allows for two or more solutions to the equations of motion, given some set of initial conditions, to exist simultaneously. Thus gauge theories contain arbitrary functions of time within the general solution to the equations of motion [10].

The existence of such arbitrary functions of time in the equations of motion implies that not all canonical variables are linearly independent [10]. There exist relationships between the linearly dependent canonical variables that are referred to as constraints. Thus, gauge systems contain a set of constraints that relate non-physical canonical variables[1]. Such

[1]The converse of this statement is not always true. Systems with constraints are not necessarily gauge systems.

constraints can be used to construct gauge transformations and to define (even construct) physical observables.

3.1 Gauge Constraints and Transformations

Following [10], in N-dimensional systems with gauge symmetries there exist a set of *primary constraints*, functions $\phi_m(p,q)$ such that,

$$\phi_m(p,q) = 0, \tag{3.1}$$

where $m = 1, \ldots, M$, which do not arise from the equations of motion. The set of constraints in (3.1) define a constraint hypersurface in phase space, on which the solutions to the equations of motion are unique and the canonical Hamiltonian is well defined. Since the constraints vanish on the hypersurface the canonical Hamiltonian remains unchanged under the replacement,

$$H \to H' = H + u^m \phi_m, \tag{3.2}$$

where the u^m are arbitrary functions of the configuration space coordinates and velocities and ensure that the Legendre transformation is invertible.

A basic consistency requirement is that the primary constraints are constants in time and thus have vanishing Poisson brackets with (3.2). The resulting equations of motion are either independent of u^m, and thus only involve the canonical coordinates, or they may impose restrictions on u^m. If the equations of motion, for $\phi_k \notin \{\phi_m\}_{m=1}^{M}$, where $k = M+1, \ldots, M+K$, are independent of u^m and the canonical coordinates are independent of the other primary constraints, then the constraints, ϕ_k are considered *secondary constraints*. Note that secondary constraints also vanish on the constraint hypersurface, however, their defining feature is that they only hold when the equations of motion are satisfied.

Again consider the consistency requirement, that both the primary and secondary constraints are constants in time and thus have weakly vanishing Poisson brackets with (3.2).

The resulting equations of motion impose restrictions on u^m. Using the equations of motion for the constraints to find solutions for u^m, it follows that,

$$u^m \approx U^m + v^a V_a^m, \tag{3.3}$$

where U^m denotes particular solutions, V_a^m denotes general solutions, and the coefficient functions, v^a, are totally arbitrary. One can now construct a total Hamiltonian

$$H_T = H + U^m \phi_m + v^a V_a^m \phi_m, \tag{3.4}$$

defined everywhere in phase space.

A *first-class constraint* is any constraint that has a weakly vanishing Poisson bracket with all other constraints, i.e., for some constraint ϕ

$$\{\phi, \phi_j\} \approx 0, \tag{3.5}$$

where $j = 1, \ldots, M + K$. The importance of the first-class constraints is fully realized in the following statement, *all first-class, primary constraints generate gauge transformations*, i.e.,

$$\delta F = v^a \{F, \phi_a\}, \tag{3.6}$$

where δF is the gauge transformation of a canonical function F, and v^a is the difference between two choices of arbitrary functions at different times, related to the first-class, primary constraints ϕ. It should be noted, as counterexamples to the Dirac conjecture have shown, that only some first-class, secondary constraints generate gauge transformations. Any gauge transformation allows one to determine the value of canonical functions F on different hypersurfaces in phase space, allowing for the construction of dressings that, when applied to observables make them gauge-invariant.

As an example, consider the Lagrangian density for QED,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - q\bar{\psi}A\!\!\!/\psi. \tag{3.7}$$

One can show that the temporal component of the canonical momentum for the vector potential, $\hat{A}_\mu(x)$, vanishes, thus defining the primary constraint hypersurface, i.e. $\phi_1(p,q) = \pi^0_A = 0$, for the theory. The replacement Hamiltonian density for this theory is,

$$\begin{aligned} H' = &-\frac{1}{2}\pi^i_A\pi_{i,A} + \frac{1}{4}F_{ij}F^{ij} - \pi^i_\psi(\partial_i\psi) + m\bar{\psi}\psi \\ &- iq\pi^\mu_\psi A_\mu\psi + \pi^0_A(\partial_0 A_0) - A_0(\partial_i\pi^i_\psi), \end{aligned} \tag{3.8}$$

where Roman indices sum over spatial quantities and Greek indices sum over spacetime quantities. Note that the last three terms in (3.8) vanish on the constraint hypersurface. The equations of motion for π^0_A yield the secondary constraint, $\phi_2(p,q) = \partial_i\pi^i_A - q\psi^\dagger\psi = 0$, which is Gauss' Law, written in differential form. Since the Poisson bracket between constraints ϕ_1 and ϕ_2 weakly vanishes, both constraints are first-class.

The Poisson bracket between $A_\mu(x)$ and the first-class constraints generates the typical gauge transformation of electrodynamics, i.e., $A_\mu(x) \to A_\mu(x) - \partial_\mu\theta(x)$, where $\theta(x)$ is arbitrary. Similarly the Poisson bracket between the local, electron field $\psi(x)$ and the first-class constraints generates the gauge transformations of $\psi(x)$,

$$\delta\psi(x) = iq\theta(x)\psi(x). \tag{3.9}$$

.

3.2 Gauge-Invariant Fields and Asymptotic Dependence

A gauge-invariant field can be constructed by promoting $\psi(x) \to \Psi(x) = e^{iq\zeta(x)}\psi(x)$. A key aspect is that $\zeta(x)$ does not have to have compact support. Following Dirac's original analysis

in [11], one could choose $\zeta(x) = \int d^4x' f^\mu(x,x') A_\mu(x')$. The new field, $\Psi(x)$, is then gauge invariant as long as $\partial_\mu f^\mu(x,x') = \delta^{(4)}(x-x')$ is satisfied. The choice of $f^\mu(x,x')$, or dressing of the local electron field, is arbitrary up to the condition on $f^\mu(x,x')$. Physically, this corresponds to the fact that there are infinitely many gauge-invariant dressings of the local field, all of which are equivalent up to solutions of the source-free Maxwell's Equations [47].

Not only can $\zeta(x)$ be non-compact, but in certain cases it must be. Specifically, if O is a local field with compact support that transforms non-trivially under $U(1)$-gauge transformations, then the corresponding gauge invariant field $\mathcal{O}$ depends on the asymptotic field configuration. To see this, expand $\mathcal{O}$ in a power series in q,

$$\mathcal{O} = O + qO_1 + q^2 O_2 + ... \tag{3.10}$$

Since $\mathcal{O}$ is gauge invariant, its Poisson bracket with the constraints vanishes. In particular, consider the integrated Gauss' Law constraint, ϕ_2, on a hypersurface in Minkowski space

$$C_G = \int d^3x (\partial^\mu F_{\mu\nu} - J_\nu)\xi^\nu = 0 \tag{3.11}$$

where J_μ is the current for some matter distribution and ξ^μ is the timelike normal to the three-dimensional hypersurface. Without loss of generality, we can take $\xi^\mu = (1,0,0,0)$. Integrating by parts yields

$$C_G = \int d^3x (\partial^\mu (F_{\mu\nu}\xi^\nu) - J_\nu \xi^\nu) = 0. \tag{3.12}$$

The first term is just a boundary term while the second is the total charge $Q = q$. Therefore, the constraint is $C_G = C_S - Q$ where C_S is a pure boundary term. Now consider the Poisson bracket of C_G with $\mathcal{O}$. Since $\mathcal{O}$ is gauge-invariant, its Poisson bracket with C_G must vanish, i.e.,

$$\{C_G, \mathcal{O}\} = \{C_S, O\} + q\{C_S, O_1\} - \{Q, O\} - q\{Q, O_1\} + ... \tag{3.13}$$

is equal to zero.

Since O is compact, $\{C_S, O\} = 0$. Moreover, Q is the generator of gauge transformations and hence $\{Q, O\} \neq 0$. Therefore the remaining term must be canceled by the $q\{C_S, O_1\}$ term, which implies that O_1 must depend on the asymptotic electromagnetic field, i.e., the dressed gauge-invariant field $\mathcal{O}$ always has non-compact support.

Gravitationally, a physical observer will transform non-trivially under at least one of the transformations of the Poincare group—there is no ability, for example, to set the total energy of a system to zero. The electromagnetic parallel is the total electric charge of the observing system that is non-vanishing. The consequence of this assumption will now be examined for the construction of the FR thought experiment, as a pedagogical introduction to the gravitational case.

3.3 Consequences for the FR Thought Experiment

Consider the following as a possible implementation of a measurement system for Alice and Bob that parallels the gravitational case and illuminates the difficulty. Assume messages and information between observers, such as the particle of defined spin in $n : 00$ of the FR thought experiment, are sent using polarized photons. Alice and Bob measure the polarization via a QED system that correlates the polarization with an electron, i.e. a net charge being present in their local detectors. This can be done as the scattering amplitude in QED is dependent on both photon and electron polarizations [48]. The system is designed so that the measurement results, $ok, \overline{ok}$, correlate with the electron state $|e\rangle$ while $fail, \overline{fail}$, correlate with the vacuum $|0\rangle$.

The local field operator associated with the electron, $\hat{\psi}_e(x)$, transforms non-trivially under $U(1)$-gauge transformations, and hence a local excitation in Alice or Bob's detector is not gauge-invariant. To maintain gauge invariance, there can be a local excitation and an electromagnetic field extending to infinity. This is just Gauss' Law applied to the measurement system above. For a related derivation see [49].

Now, consider Alice's role in the original FR thought experiment. With certainty implies that all measurements accessible to Alice are consistent with her inference that Bob measured $w = fail$ (while Bob actually measured $w = ok$) and the existence of a vacuum between them. Her measurements of the electromagnetic field, at her position, must therefore be consistent with Bob measuring $w = fail$ and the rest of the assumptions of the experiment. By superposition, the electric field can be split and measured by Alice as $\vec{E}_{AA} + \vec{E}_{AB}$, where $\vec{E}_{AA}$ is the self-consistent electric field generated by Alice's apparatus measuring $\overline{w} = \overline{ok}$ and $\vec{E}_{AB}$ is the electric field at Alice generated by Bob's measurement. Since Alice's measurements must be consistent with her inference that Bob measured $|fail\rangle$ and vacuum in between them, $\vec{E}_{AB} = 0$.

It is clear that there is an immediate problem when implementing the Gauss' Law constraint on the electric fields. Since the measurements are stable, the measured electric fields must be time-independent. Therefore, via Maxwell's Equations, the electric field is curl-free and the derivative of a potential Φ. Therefore the Gauss' Law constraint reduces to Poisson's equation, $\nabla^2\Phi = \rho$, where ρ is the charge density. By superposition, Alice can definitively state that the potential due to Bob and its derivative vanish in her region, i.e., $\Phi_{BA} = 0, \vec{\nabla}\Phi_{BA} = 0$. However, the unique solution to this is that the potential due to Bob everywhere is, $\Phi_B(r) = 0$. Therefore if Bob measures $\Phi_{BB} \neq 0$, the Gauss' Law constraint is violated. There is no way to even define the paradox in a gauge-invariant sense.

The escape from this conclusion, in the electromagnetic case, is simple: use detectors/measurement devices that don't carry a net $U(1)$ charge. This is easy to do since electromagnetism permits screening [11] due to the existence of both positive and negative charges. In this case, either outcome of Bob's measurement would leave no electric field imprint for Alice to measure, allowing the FR thought experiment to go through as it was originally proposed. This leaves higher moments (e.g. electric or magnetic dipoles or quadrupoles) that could still be detected by Alice, but one could in principle cancel those as well, out to some finite order. At the monopole level, however, since there are no neg-

ative masses available in general relativity to screen the gravitational field, there are no measurement systems that are uncharged under the diffeomorphism group. Therefore the key question is whether such a screening procedure can work for gravitationally charged measurement devices. The next chapter examines this question in more detail.

CHAPTER 4

Diffeomorphism-Invariant Observables and the FR Thought Experiment

Similar to the electromagnetic case, gravity also possesses a gauge symmetry: diffeomorphism invariance. Local fields are not diffeomorphism-invariant. For example, let $\phi(x)$ be a local, massive, scalar field. Under a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, $\phi(x)$ transforms as,

$$\phi(x) \to (f_*\phi)(x) = \phi(f^{-1}(x)). \tag{4.1}$$

Under an infinitesimal transformation generated by a vector field $\xi^\mu(x)$, the diffeomorphism takes the form, $f^\mu(x) = x^\mu + \kappa\xi^\mu$ and the variation of the transformed field is,

$$\begin{aligned}\delta\phi(x) &= \phi(f^{-1}(x)) - \phi(x) \\ &= \kappa\xi^\mu\partial_\mu\phi(x) + \frac{1}{2!}\kappa^2\xi^\mu\xi^\nu\partial_\mu\partial_\nu\phi(x) + \frac{1}{3!}\kappa^3\xi^\mu\xi^\nu\xi^\sigma\partial_\mu\partial_\nu\partial_\sigma\phi(x) + \mathcal{O}(\kappa^4)\end{aligned} \tag{4.2}$$

up to second order in $\kappa = \sqrt{32\pi G}$. Thus, local fields are not inherently diffeomorphism invariant since the variation of the transformed field is non-vanishing. And, similar to the electromagnetic case, one must construct diffeomorphism-invariant measurements to claim that the theory can make physical predictions.

One can examine the full diffeomorphism invariance of general relativity but, since the fields are weak in the FR thought experiment, it will suffice to consider diffeomorphism invariance in linearized gravity, up to first order in Newton's constant G. It is therefore advantageous to work with perturbations around Minkowski space such that the metric is

given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{4.3}$$

where $\eta_{\mu\nu}$ is the background Minkowksi metric and $h_{\mu\nu}$ is the metric perturbation.

Any field excitation that is diffeomorphism-invariant must have a vanishing variation under diffeomorphisms, i.e., for a scalar field

$$\delta\Phi(x) = 0 \tag{4.4}$$

under a diffeomorphism. One may construct such an excitation from a dressing $V^\mu(x)$ and the local field excitation $\phi(x)$. Generally, the dressing V expanded to n^{th} order in κ will contain higher order derivatives, however, this work follows [50] and considers the following construction, up to first-order in κ

$$\Phi(x) = \phi(x + V(x)) = \phi(x) + V^\mu(x)\partial_\mu\phi(x) + \mathcal{O}(\kappa^2). \tag{4.5}$$

Combining the transformation laws from (4.2) and (4.4) implies that, at first order in κ, the dressing must transform as

$$\delta V^\mu(x) = \kappa\xi^\mu(x), \tag{4.6}$$

under diffeomorphisms.

One can now determine the form of $V^\mu(x)$ that allows local field excitations to be diffeomorphism-invariant. Applying the logic from [11], let $V^\mu(x)$ be a linear functional of the metric perturbation over the spacetime volume. The dressing takes the form,

$$V^\mu(x) = \kappa \int d^4x' f^{\mu\nu\sigma}(x, x')h_{\nu\sigma}(x), \tag{4.7}$$

where the tensor $f^{\mu\nu\sigma}(x, x')$ is symmetric in the second and third indices. Enforcement of the transformation law in (4.6) and the transformation laws for the metric perturbation imply

that $f^{\mu\nu\sigma}(x,x')$ must satisfy,

$$2\partial_{\nu'} f^{\mu\nu\sigma}(x,x') = \delta^4(x-x')\eta^{\mu\sigma}. \tag{4.8}$$

The function $f^{\mu\nu\sigma}(x,x')$ determines the structure of the gravitational field that is created by excitations of the ϕ field. There is freedom in the choice of $f^{\mu\nu\sigma}(x,x')$, however, this freedom does not allow for physically inequivalent dressings. For examples of the different choices for $f^{\mu\nu\sigma}(x,x')$ see [50].

4.1 Consequences for the FR Thought Experiment

As in the electromagnetic case, the question for the construction of the FR thought experiment is whether there exists an f that has local support given some measurement apparatus. In GR, there is also a dressing theorem [50], which states that if a matter distribution in a compact region has non-vanishing Poincare charge, then the gravitational field depends on the asymptotic metric, i.e., there are no local functions f such that $V^\mu(x)$ and hence $\hat{\Phi}(x)$, the gauge-invariant operator, is of compact support. Therefore any measurement device that stores information in, e.g., the energy eigenstate of a detector or the spin of a particle, would generate a gravitational field that extends to infinity (with some falloff). In this situation, the problem is the same as the charged electromagnetic case—there is no gauge-invariant construction of the paradox since the field measured by Alice must be compatible with her belief that Bob measures $w = fail$ and this would violate at least one of the constraint equations of GR.

The key question is then: can Bob make a measurement and store information in such a way that, outside his local region, the gravitational field is indifferent to the results of the measurement? In other words, is it possible to screen the gravitational field? Gravitational screening has been investigated by a number of authors. Carlotto and Schoen proved that one can construct classical solutions to the vacuum Einstein Equations that satisfy the constraint

equations and have support only in conical regions [51]. This allowed them to build global solutions by gluing; where two regions of non-trivial gravitational field are separated by a region of flat space for an arbitrarily long time T. Similar vacuum solutions have been found that are identical to the Schwarzschild-AdS spacetime outside a bounded region on a spacelike slice but differ inside [52]. However, such solutions share a generic feature: they are non-analytic somewhere in the spacetime and, as pointed out in [28], quantum gravity theories that have unitarity can forbid non-analytic solutions. For the purposes of this work, the relevant arena to examine this line of reasoning is in the perturbative regime, which has been explored in [18] and which is paraphrased in Sec. 1.1 and summarized below.

Consider a neighborhood $\mathcal{U}$ containing a local state $|\psi\rangle$ and an ϵ-extended neighborhood $\mathcal{U}_\epsilon$, such that $\mathcal{U} \subset \mathcal{U}_\epsilon$. One can construct, up to first order in κ, a split gravitational dressing that shields information from observers outside of $\mathcal{U}_\epsilon$. Specifically, outside $\mathcal{U}_\epsilon$, the gravitational field is only a function of the total Poincare charges of the fields inside $\mathcal{U}$. Therefore if two states $|\psi_1\rangle$ and $|\psi_2\rangle$ share the same Poincare charge, they are indistinguishable to an outside observer, i.e.,

$$\langle\psi_1|\,\hat{h}_{\mu\nu}\,|\psi_1\rangle = \langle\psi_2|\,\hat{h}_{\mu\nu}\,|\psi_2\rangle \tag{4.9}$$

for all observers outside $\mathcal{U}_\epsilon$.

Now ask if such a pair of states can contain information. The issue with charged fields has already been shown in Ch. 3, so consider uncharged fields that contain information strictly in the gravitational sector. Let the region $\mathcal{U}$ be Bob's local measuring device. Assign the states $|\psi_1\rangle, |\psi_2\rangle$ to $|ok\rangle, |fail\rangle$, and let the information stored in the classically stable measuring device be stored in two differing energy-momentum tensors inside Bob's region, call them $T^n_{\mu\nu} = \langle\psi_n|\,\hat{T}_{\mu\nu}\,|\psi_n\rangle$ for $n = 1, 2$. This then also implies that inside Bob's region, the gravitational fields must also differ, i.e., inside $\mathcal{U}_\epsilon$,

$$\langle ok|\,\hat{h}_{\mu\nu}\,|ok\rangle \neq \langle fail|\,\hat{h}_{\mu\nu}\,|fail\rangle\,. \tag{4.10}$$

Furthermore, since the information must be stable according to the assumptions in the construction of the FR thought experiment, the time derivative $\dot{h}_{\mu\nu}$ must vanish. As long as such a situation is possible then there are no consequences for the FR thought experiment and gravity, quantum or otherwise, can indeed be neglected.

4.2 Inconsistency of the Weak Field General Relativistic Approximation

The argument above is not self-consistent, however. In order for the linearized, weak-field approximation to properly reflect gauge invariance, via imposition of the general relativistic constraint equation, two things need to hold. First, the perturbations $h_{\mu\nu}$ that are solutions to the quadratic action need to be small so that the full action is well approximated by the quadratic action. Second, the derivatives of the perturbations also must remain small. If they do not, then one cannot use the quadratic approximation to the general relativistic action, and corresponding constraint equations, but instead must use a quantum-corrected effective action that includes higher curvature terms. Note, that there are higher curvature corrections that are still quadratic in $h_{\mu\nu}$. As an example, consider the addition of a putative, quantum gravity-induced, subdominant Weyl squared term to the Einstein-Hilbert action

$$S_{grav} = -\int \sqrt{-g} d^4x \ \frac{M_{Pl}^2}{2} R + \frac{\alpha}{2} W^2 \tag{4.11}$$

where $W^2 = W_{\mu\nu\sigma\rho} W^{\mu\nu\sigma\rho}$ and $W_{\mu\nu\sigma\rho}$ is the Weyl tensor.

The quadratic action in the Newtonian gauge [53] is then

$$\begin{aligned} S_{quad} = \frac{\alpha}{4} \int d^4x \Bigg[& - \ddot{h}^{ij} \ddot{h}_{ij} + \dot{h}^{ij} \vec{\nabla}^2 \dot{h}_{ij} - h^{ij} \vec{\nabla}^4 h_{ij} \\ & + \frac{M_{Pl}^2}{8} \left(\dot{h}^{ij} \dot{h}_{ij} + h^{ij} \vec{\nabla}^2 h_{ij} \right) \Bigg] \end{aligned} \tag{4.12}$$

where h_{ij} are the usual spatial excitations of the metric. Similar expressions hold for the scalar and vector sectors of $h_{\mu\nu}$ as well. Note the presence of terms that are quadratic in the

field, yet have higher derivatives in both space and time. If derivatives of a putative quadratic solution in GR ($\alpha \to 0$) become large compared to the Planck scale, then these additional terms will contribute to the action and the GR approximation is then not self-consistent for these solutions. Quadratic in field terms containing arbitrary higher derivatives exist as well, e.g., via the insertion of a term in the Lagrangian with a d'Alembertian between the Weyl tensors.

Now examine the localization procedure on gravitational states relevant for the FR thought experiment, i.e., those defined in (4.10), in this context. First, since it is assumed $h_{\mu\nu}$ is time-independent, any time derivatives can be ignored. Now consider the difference $\Delta h_{\mu\nu} = \langle ok| \hat{h}_{\mu\nu} |ok\rangle - \langle fail| \hat{h}_{\mu\nu} |fail\rangle$ over the whole spacetime. The function $\Delta h_{\mu\nu}$ is of compact support, as it is non-zero inside $\mathcal{U}_\epsilon$ and identically zero outside, by construction. Therefore, $\Delta h_{\mu\nu}$ can be considered to be a non-analytic bump function. In the linearized approximation, superposition holds and thus $\Delta h_{\mu\nu}$ is also a solution to the linearized constraint equations. If one is to be able to neglect higher curvature terms then the n^{th} derivative of $\Delta h_{\mu\nu}$ must grow no faster than a polynomial in n. In other words, there exists some mass scale $M < M_{Pl}$, such that $\partial^n h_{\mu\nu}/\partial x^n < M^n$. If this is true, the higher curvature terms (with generic $\mathcal{O}(1)$ coefficients) evaluated on the lowest order solutions remain subdominant to the Einstein-Hilbert term. Only then can the general relativistic approximation and use of the corresponding constraint equations remain valid.

However, since $\Delta h_{\mu\nu}$ is a function of compact support and non-analytic, its derivatives must violate precisely this condition! Real analytic functions, $f(x)$, satisfy the following property [54]: for each point x_0 within the domain of $f(x)$, there exists a neighborhood $\mathcal{V}$ of x_0 such that

$$|\partial^m f(x)| \leq m! C^m, \tag{4.13}$$

for all $x \in \mathcal{V}$, where $C \in \mathbb{R}_+$ and $m \in \mathbb{N}$. Non-analytic functions must have at least one derivative, of order m, that violates (4.13). In fact, an infinite number of derivatives violate this bound for $\Delta h_{\mu\nu}$, a bump function solution to the constraint equations. To see

this, work by contradiction and assume that only a finite number of derivatives violate this bound. Thus, there exists some number q such that all q^{th} and higher derivatives of $\Delta h_{\mu\nu}$ are analytic. However, the derivative of a bump function is another bump function. Hence, the q^{th} and higher derivatives cannot be analytic. Therefore, there are an infinite number of derivatives of $\Delta h_{\mu\nu}$ that violate the bound (4.13).

The consequence of this argument is that one cannot state that a non-analytic solution to linearized GR of compact support is an actual solution of quantum gravity. In other words, there is no way to consistently implement a weak gravity, gauge invariant construction of a paradox within the framework of general relativity, neglecting higher curvature quantum gravity corrections.

4.3 Boundary Unitarity vs. Non-Analytic, Localized Solutions

The above argument is analogous to the path leading to the chronology protection conjecture where a paradox even in low curvature regions leads unavoidably to quantum gravity. Of course, this does not, by itself, lead to any conclusion on whether or not the construction is possible in quantum gravity. One has to invoke additional assumptions about quantum gravity to make progress. The work below examines how the principle of boundary unitarity, a proposed property of holographic quantum gravity, sheds light on this latter question.

The FR thought experiment generates a paradox by isolating two observers, allowing each to make measurements with certainty, stably record the results, and then meet. It has been shown that whether one can formulate this procedure in a gauge-invariant manner is sensitive to the behavior of higher curvature terms and is hence a question for quantum gravity. It is illuminating to ask whether there are any other resolutions to paradoxes in gravitational physics that involve diffeomorphism invariance, non-analytic behavior, and unitary evolution. Marolf [15] has examined the consequences of diffeomorphism invariance, which yields a property of quantum gravity known as *boundary unitarity*. This property has been proposed as a solution to the infaller paradox of black hole physics [28].

Consider the classical, diffeomorphism-invariant, general relativistic Hamiltonian

$$H = \int_{\Sigma} N_\mu C^\mu + H_\partial \tag{4.14}$$

where N_μ are the lapse and shift, C^μ are the constraints, and H_∂ is the boundary Hamiltonian. At asymptotic times (4.14) generates time translations via Hamilton's equations. Let $\mathcal{O}$ be an element of the boundary algebra $\mathcal{A}_\partial$—the algebra of diffeomorphism-invariant observables that exist on the past/future null infinities. Since $\mathcal{A}_\partial$ is closed under the Poisson bracket, the Poisson bracket of $\mathcal{O}$ with the constraints vanishes. Hence, the evolution of diffeomorphism-invariant observables on the past/future null boundary is governed by the boundary Hamiltonian

$$\frac{\partial \mathcal{O}}{\partial t} = \{H, \mathcal{O}\} = \{H_\partial, \mathcal{O}\}. \tag{4.15}$$

As pointed out in [28], (4.15) admits solutions that are non-analytic in time, which through simple propagation would also lead to non-analytic in space solutions. Therefore, one cannot classically forbid non-analytic solutions.

Can the same be said for quantum gravity? Assuming the quantum gravitational Hamiltonian—whatever it may be—is diffeomorphism-invariant, mirroring the structure of (4.14), and the constraint algebra is closed under the commutator, then the evolution of boundary observables is given by

$$\frac{\partial \hat{\mathcal{O}}}{\partial t} = \frac{1}{i\hbar}[\hat{H}_\partial, \hat{\mathcal{O}}]. \tag{4.16}$$

Thus, given the above requirements, the evolution of boundary observables is governed by the boundary Hamiltonian of the theory.

The two evolution equations, (4.15) and (4.16), are similar. However there is a key difference—while the classical evolution equation admits solutions that are non-analytic in time, the quantum evolution equation does not. Indeed, the Hamiltonian would be an element of $\mathcal{A}_\partial$, as would be the commutator of the Hamiltonian with any other observable. Thus, (4.16) would be an element of $\mathcal{A}_\partial$ as well. This implies that the boundary algebra

associated with quantum gravity is closed under unitary evolution. In particular, quantum mechanically, the solution to (4.16) for an observable $\hat{\mathcal{O}}$ is

$$\hat{\mathcal{O}}(t) = e^{i\hat{H}_\partial t}\hat{\mathcal{O}}(0)e^{-i\hat{H}_\partial t}. \tag{4.17}$$

Assuming that $\hat{H}_\partial$ is self-adjoint and $\hat{\mathcal{O}}$ is bounded, (4.17) is an analytic function in time. The inner product of (4.17) with eigenstates of $\hat{H}_\partial$ has the capacity to be analytic in time, however, there are infinite terms in the inner product, which could hide non-analytic behavior. A remedy is to assume $\hat{H}_\partial$ is bounded from below and then apply an energy cut-off as an upper bound.

Boundary unitarity forbids non-analytic behavior in time, while the FR thought experiment yields non-analytic behavior on a spatial slice. One can, of course, translate the non-analytic, spacelike solutions of compact support to the boundary by considering one of the localized, weak field solutions necessary for the FR thought experiment and allowing it to propagate outwards. Since the excitation is localized, upon arrival at the boundary, or in some suitable asymptotic region, it would constitute new information and generate non-analytic behavior in time. The gravitational degrees of freedom would be static and consistent with the background spacetime up until some time t, then, upon arrival of the excitation, suddenly change, i.e., behave non-analytically, which would be forbidden by boundary unitarity. Hence, if boundary unitarity holds for quantum gravity, the local excitations necessary for the FR paradox to even be stated would be forbidden.

Part II

Energy Cost of Localizing Relational Quantum Information

CHAPTER 5

Gaussian States with External Partitions

Gaussian states provide a versatile analytical tool in many areas of physics (cf. [55]). They get their name from the Gauss function $f(x) = e^{-x^2}$, most notably used in many areas of probability and statistics. In quantum physics, Gaussian states are often associated with the ground state of the Harmonic oscillator, making them an excellent tool for analyzing quantum systems.

When constructing Gaussian states, the standard approach assumes the ability to establish the usual p_i, q_i basis of an N particle phase space, where i runs from 1 to N. As such, it implicitly uses an external partition—the division of the overall phase space into subspaces associated with each particle and the labeling of position and momentum as measured by some external measurement system. In the language of quantum reference frames, such an external partition is called a "perfect" quantum reference frame [56]. In such a frame each single particle Hilbert space is spanned by some continuous set of kets $|g\rangle$ that are completely distinguishable. In this case, the typical choice is the position basis kets $|x\rangle$, with distinguishability implemented as $\langle x|x'\rangle = \delta(x' - x)$ (or equivalently the momentum basis kets). This distinguishability is then reflected in the classical p, q phase space in some associated classical reference frame.

With such a frame, for a system with N particle degrees of freedom, there exists the classical phase space $V \simeq \mathbb{R}^{2N}$, consisting of the p's and q's from above, and its dual $V^* \simeq \mathbb{R}^{2N}$. The system can be quantized by promoting the $2N$ phase space coordinates to operators representing observables. These operators can be put in an operator-valued vector

$\hat{\xi}^a = (\hat{q}_1, \hat{q}_2, ..., \hat{q}_N, \hat{p}_1, \hat{p}_2, ..., \hat{p}_N)$. A Gaussian state $|\psi\rangle$, whether it is bosonic or fermionic, is completely described by its one-point function[1],

$$z^a = \langle\psi| \hat{\xi}^a |\psi\rangle \tag{5.1}$$

and its two-point function

$$C_2^{ab} = \langle\psi| (\hat{\xi} - z)^a (\hat{\xi} - z)^b |\psi\rangle . \tag{5.2}$$

The two-point correlation function can be decomposed into a symmetric piece G^{ab} and an antisymmetric piece Ω^{ab} via

$$C_2^{ab} = \frac{1}{2} \left(G^{ab} + i\Omega^{ab} \right) . \tag{5.3}$$

All other n-point functions vanish for $n > 2$. In the case where the one-point function vanishes—as is when dealing with fermions—all n-point functions vanish for odd n while the even n-point functions can be determined via Wick's Theorem [57].

5.1 Bosonic vs. Fermionic Degrees of Freedom

For bosonic and fermionic degrees of freedom, the roles and behavior of G^{ab} and Ω^{ab} differ. Bosonic degrees of freedom are characterized by the commutation relations

$$\left[\hat{x}^i, \hat{p}^j\right] = i\hbar\delta^{ij}\hat{\mathbf{I}}, \tag{5.4}$$

$$\left[\hat{x}^i, \hat{x}^j\right] = [\hat{p}^i, \hat{p}^j] = 0. \tag{5.5}$$

The antisymmetric piece of the two-point function, Ω^{ab}, can be isolated via

$$\langle\psi| \hat{\xi}^a \hat{\xi}^b - \hat{\xi}^b \hat{\xi}^a |\psi\rangle = i\Omega^{ab} \tag{5.6}$$

[1]For fermions $z^a = 0$.

and by expressing $\hat{\xi}^a$ in terms of the phase space operators it is clear that Ω^{ab} is simply the symplectic form inherited from the classical Poisson brackets after quantization. In other words, for bosons

$$\Omega^{ab} = \begin{pmatrix} 0 & -i\mathbb{I} \\ i\mathbb{I} & 0 \end{pmatrix} \tag{5.7}$$

and is not state-dependent. On the other hand,

$$\langle\psi|\, \hat{\xi}^a\hat{\xi}^b + \hat{\xi}^b\hat{\xi}^a \,|\psi\rangle = G^{ab} \tag{5.8}$$

shows that G^{ab} is a state-dependent quantity for bosons. This dependence is one-to-one, i.e., any bosonic Gaussian state can be uniquely specified (up to a phase) by G^{ab}.

Conversely, for fermionic degrees of freedom the (anti)-commutation relations are

$$\{\hat{x}^i, \hat{x}^j\} = \{\hat{p}^i, \hat{p}^j\} = \hbar\delta^{ij}\hat{\mathbf{I}}, \tag{5.9}$$

$$\{\hat{x}^i, \hat{p}^j\} = 0. \tag{5.10}$$

For fermionic degrees of freedom, the roles and behavior of G^{ab} and Ω^{ab} are reversed, but determined in the same manner as their bosonic counterparts. The symmetric G^{ab} can be isolated via (5.8). Given the (anti)-commuting nature of fermions, it is clear that G^{ab} is the symmetric, positive-definite, bilinear form inherited from the classical (anti)-commuting Poisson brackets after quantization. This implies that G^{ab} takes the form,

$$G^{ab} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{5.11}$$

and is state-independent. The antisymmetric form Ω^{ab} is determined via (5.6), however for fermions Ω^{ab} is now a state-dependent quantity and is in one-to-one correspondence with each Gaussian state up to a phase.

5.2 Combined Kähler Structure

While this work will concentrate primarily on fermions, it is useful to do so in a notation that allows for both bosonic and fermionic analysis. This can be accomplished by unifying the mathematical description of bosonic and fermionic Gaussian states via Kähler structures. A *Kähler space* is a real vector space that is equipped with the following linear operators:

- **Metric**, a symmetric, positive-definite, bilinear form G^{ab}, with inverse G^{-1}_{ab} such that $G^{ac}G^{-1}_{cb} = \delta^a_b$,
- **Symplectic form**, an antisymmetric, non-degenerate form Ω^{ab}, with inverse Ω^{-1}_{ab} such that $\Omega^{ac}\Omega^{-1}_{cb} = \delta^a_b$,
- **Complex structure**, denoted J^a_b, satisfies the property $J^a_c J^c_b = -\delta^a_b$.

The triple of these three operators (G, Ω, J) is referred to as a *Kähler structure*. The three operators are related via,

$$J^a_b = -G^{ac}\Omega^{-1}_{cb} = \Omega^{ac}G^{-1}_{cb}. \tag{5.12}$$

It is apparent that bosonic and fermionic Gaussian states have two of the three required linear operators for a Kähler structure, particularly a metric G^{ab} and a symplectic form Ω^{ab}. However, this is not enough to imply that G^{ab} and Ω^{ab} are compatible Kähler structures. It is required that J^a_b, defined by G^{ab} and Ω^{ab} via (5.12), satisfies the condition $J^2 = -\mathbb{I}$ for G^{ab} and Ω^{ab} to be compatible Kähler structures.

Bosonic Gaussian states have an associated metric G^{ab} that is state-dependent and a symplectic form Ω^{ab} that is state-independent. Assuming both G^{ab} and Ω^{ab} are Kähler compatible, the complex structure J^a_b, that relates G^{ab} and Ω^{ab}, is therefore a unique state-dependent quantity. Similarly, fermionic Gaussian states have a metric G^{ab} that is state-independent and a symplectic form Ω^{ab} that is state-dependent. Again, assuming G^{ab} and Ω^{ab} are Kähler compatible, the complex structure that relates the two is also state-dependent in a similar way. Thus, for both bosonic and fermionic Gaussian states, the complex structure

J_b^a is uniquely determined by the state, up to a phase. Additionally, J_b^a can be used as an ideal label for either bosonic or fermionic Gaussian states.

Moreover, J_b^a can be used to explicitly define an operator that annihilates a specified Gaussian state. Given the Kähler structure, every Gaussian state $|\psi\rangle$ solves the equation

$$\frac{1}{2}\left(\delta_b^a + iJ_b^a\right)\left(\xi - z\right)^b |\psi\rangle = 0, \tag{5.13}$$

such that $|\psi\rangle$ generates the unique J_b^a used to define the operator.

5.3 Annihilation and Creation Operators

While Gaussian states can be described using sets of phase space operators $\hat{q}_i, \hat{p}_i$ it is also convenient to describe Gaussian states using the Fock basis construction and annihilation/creation operators.

As is familiar from introductory quantum mechanics, a Hilbert space representation of the algebra of observables in the Fock basis can be defined by a set of annihilation and creation operators $\hat{a}_i, \hat{a}_i^\dagger$, where $i = 1 \ldots N$ for a system with N particles, respectively. For a system of bosonic particles, the canonical commutation relations are imposed on the annihilation and creation operators,

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}\hat{\mathbf{I}}, \tag{5.14}$$

$$[\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0. \tag{5.15}$$

For a system of fermionic particles, (anti)-commutation relations are imposed on the annihilation and creation operators,

$$\left\{\hat{a}_i, \hat{a}_j^\dagger\right\} = \delta_{ij}\hat{\mathbf{I}}, \tag{5.16}$$

$$\{\hat{a}_i, \hat{a}_j\} = \left\{\hat{a}_i^\dagger, \hat{a}_j^\dagger\right\} = 0. \tag{5.17}$$

For both bosons and fermions the vacuum state $|0,\ldots,0\rangle$ is the state annihilated by for all $\hat{a}_i$ where $i = 1,\ldots,N$,

$$\hat{a}_i\,|0,\ldots,0\rangle = 0. \tag{5.18}$$

Orthonormal basis states are given by $|n_1,\ldots,n_N\rangle$ where $n_i \in \mathbb{N}$ for bosonic systems and $n_i \in \{0,1\}$ for fermionic systems. The action of the annihilation and creation operators on these states satisfies,

$$\hat{a}_i\,|n_1,\ldots,n_N\rangle = \sqrt{n_i}\,|n_1,\ldots,n_i-1,\ldots,n_N\rangle\,, \tag{5.19}$$

$$\hat{a}_i^\dagger\,|n_1,\ldots,n_N\rangle = \sqrt{n_i+1}\,|n_1,\ldots,n_i+1,\ldots,n_N\rangle \tag{5.20}$$

and they can be obtained from the vacuum state via

$$|n_1,\ldots,n_N\rangle = \prod_{i=1}^{N}\left(\frac{(\hat{a}_i^\dagger)^{n_i}}{\sqrt{n_i!}}\right)|0,\ldots,0\rangle\,. \tag{5.21}$$

To relate the annihilation and creation operators to the operator-valued vector $\hat{\xi}^a$ one needs to define vectors $v_{ia} \in V_{\mathbb{C}}^*$ in a complex vector space $V_{\mathbb{C}}^*$ such that, for a $\hat{\xi}^a$ in some basis,

$$\hat{a}_i = v_{ia}\hat{\xi}^a, \tag{5.22}$$

$$\hat{a}_i^\dagger = v_{ia}^*\hat{\xi}^a. \tag{5.23}$$

Given the relation between G^{ab}, Ω^{ab} and $\hat{\xi}^a$ from (5.8) and (5.6), respectively, it is easy to see that the $\hat{\xi}^a$ associated with bosonic systems inherits the commutation relations and the (anti)-commutation relations for fermionic systems. Along with the algebras for bosonic and fermionic annihilation and creation operators, this implies that there are conditions that the

vectors v_{ia} must satisfy. For bosons, the v_{ia} must satisfy,

$$\Omega^{ab} v_{ia} v_{jb} = 0, \tag{5.24}$$

$$\Omega^{ab} v^*_{ia} v_{jb} = i\delta_{ij}. \tag{5.25}$$

And similarly for fermions the v_{ia} must satisfy,

$$G^{ab} v_{ia} v_{jb} = 0, \tag{5.26}$$

$$G^{ab} v^*_{ia} v_{jb} = \delta_{ij} \tag{5.27}$$

Using these conditions a set of vectors u^a_j, dual to v_{ia}, can be constructed to define a basis transformation between the Fock basis $(\hat{a}_1, \hat{a}^\dagger_1, \ldots, \hat{a}_N, \hat{a}^\dagger_N)$ and $\hat{\xi}^a$. The vectors u^a_i are defined by

$$u^a_i = i\Omega^{ab} v^*_{ib} \tag{5.28}$$

for bosons and

$$u^a_i = G^{ab} v^*_{ib} \tag{5.29}$$

for fermions. Given a set of u^a_i the transformation between the Fock basis and $\hat{\xi}^a$ is

$$\hat{\xi}^a = z^a + \sum_{i=1}^{N} \left(u^a_i \hat{a}_i + u^a_i \hat{a}^\dagger_i \right), \tag{5.30}$$

with $z^a = 0$ for fermions. This work will move back and forth between the $\hat{x}, \hat{p}$ and $\hat{a}, \hat{a}^\dagger$ bases via

$$\hat{a} = \sqrt{\frac{\omega}{2}} \left(\hat{x} + \frac{i}{\omega} \hat{p} \right). \tag{5.31}$$

In the following discussions—results are, of course, basis independent.

5.4 Example Construction for a Fermionic Gaussian State

As a simple example, consider the ground state, $|0\rangle$, of the fermionic harmonic oscillator $\hat{H} = i\omega\hat{x}\hat{p} = \omega(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)/2$. For ease of calculation, work in the Fock basis so that the operator-valued vector is $\hat{\xi}^a = (\hat{a}, \hat{a}^\dagger)$. The Hamiltonian can be written using $\hat{\xi}^a$ in the following manner, $\hat{H} = \frac{1}{2}h_{ab}\hat{\xi}^a\hat{\xi}^b$. It follows that the Hamiltonian matrix h_{ab} and the metric G^{ab} for the system take the form,

$$h_{ab} = \begin{pmatrix} 0 & i\omega \\ -i\omega & 0 \end{pmatrix}, \text{ and } G^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.32}$$

respectively. The state-dependent symplectic form associated with the ground state of the fermionic harmonic oscillator is

$$\Omega^{ab} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{5.33}$$

Thus the linear complex structure is,

$$J^a_b = \Omega^{ac}G^{-1}_{cb} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{5.34}$$

It is easy to see that $J^2 = -\mathbb{I}$, implying that G^{ab} and Ω^{ab} are Kähler compatible. Given the form of the complex structure J^a_b the operator from (5.13) takes on the form

$$\frac{1}{2}(\delta^a_b + iJ^a_b)\hat{\xi}^b\,|0\rangle = \begin{pmatrix} \hat{a} \\ 0 \end{pmatrix}|0\rangle = 0. \tag{5.35}$$

Since (5.35) annihilates the ground state of the fermionic harmonic oscillator, the ground state is a Gaussian state. By similar means, one can also show that the first excited state of

the fermionic harmonic oscillator, $|1\rangle$, with associated linear complex structure,

$$J_b^a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{5.36}$$

is also a Gaussian state.

5.5 Squeezing, Displacement, and General Transformations of Gaussian States

A general one-particle Gaussian state[2] can be constructed by acting the squeezing operator

$$\hat{S}(r) = \exp\left[r\left(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\right)\right] \tag{5.37}$$

and the displacement operator

$$\hat{D}(\alpha) = \exp\left[\hat{a}^\dagger\gamma - \gamma^*\hat{a}\right]. \tag{5.38}$$

on either the fermonic ground state $|0\rangle$ or first excited state $|1\rangle$. The squeezing operator $\hat{S}$ is parameterized by a squeezing parameter $r \in \mathbb{R}$, where $r \to \infty$ indicates a highly localized state[3]. Due to the Grassmann properties of fermionic states, the squeezing operator preserves both the ground and first excited state, i.e., $\hat{S}(r)|0\rangle = |0\rangle$ and $\hat{S}(r)|1\rangle = |1\rangle$, respectively, for all values of the squeezing parameters r. The squeezing operator is unitary, satisfying $\hat{S}(r)\hat{S}^\dagger(r) = \hat{S}^\dagger(r)\hat{S}(r) = \hat{\mathbf{I}}$. The displacement operator $\hat{D}$ is parameterized by the variable γ, the amount of phase space displacement. For bosons γ is a complex number and for fermions γ is Grassmannian. The action of the displacement operator on, specifically, the fermionic ground state or the first excited state produces a coherent state[4], i.e., $\hat{D}(\gamma)|0\rangle = |\gamma\rangle$ or

[2]The most general Gaussian state can be constructed via the action of the squeezing and displacement operators on a thermal state, that is inherently Gaussian.

[3]Generally, the squeezing operator is a function of a complex parameter $\zeta = re^{i\phi}$, where ϕ is an arbitrary phase. For simplicity, $\phi = 0$ has been assumed.

[4]A coherent state can be constructed by acting the displacement operator on the bosonic ground state as well.

$\hat{D}(\gamma)\,|1\rangle = |\gamma'\rangle$ for all values of the displacement parameter γ [58]. Here, the unprimed coherent state is an eigenstate of the annihilation operator while the primed coherent state is an eigenstate of the creation operator. Like the squeezing operator, the displacement operator is also unitary, satisfying $\hat{D}(\gamma)\hat{D}^\dagger(\gamma) = \hat{D}^\dagger(\gamma)\hat{D}(\gamma) = \hat{\mathbf{I}}$.

Given a Gaussian state, parameterized by its complex structure $|J\rangle$, the action of the squeezing and displacement operators on the state, i.e., $\hat{D}(\gamma)\hat{S}(r)\,|J\rangle = |J'\rangle$, produces a new Gaussian state $|J'\rangle$[5]. To see this, begin by defining the unitary operator,

$$\hat{U}(r,\gamma) = \hat{D}(\gamma)\hat{S}(r), \tag{5.39}$$

The action of (5.39)) on $\hat{\xi}^a$ results in,

$$\hat{U}^\dagger(r,\gamma)\hat{\xi}^a\hat{U}(r,\gamma) = R^a_b\hat{\xi}^b + z^a, \tag{5.40}$$

where R^a_b is the matrix representation of the squeezing transformation $\hat{S}(r)$ and z^a is the matrix representation of the displacement transformation $\hat{D}(\gamma)$ [59]. Let $|J'\rangle = \hat{U}(r,\gamma)\,|J\rangle$ be a state generated by the action of the unitary operator (5.39) on a Gaussian state $|J\rangle$ with associated complex structure $J = \Omega G^{-1}$. From (5.40) it follows that the metric and symplectic form associated with the state $|J'\rangle$ are given by, $G' = RGR^\top$ and $\Omega' = R\Omega R^\top$, respectively. It follows that the complex structure associated with $|J'\rangle$ is $J' = \Omega' G'^{-1} = R\Omega R^\top (R^{-1})^\top G^{-1}R^{-1} = RJR^{-1}$, where R^{-1} is the inverse transformation associated with R. For $|J'\rangle$ to be a Gaussian state, the associated complex structure must satisfy $J'^2 = -\mathbb{I}$. Since $J' = RJR^{-1}$ it is easy to see that $J'^2 = RJR^{-1}RJR^{-1} = RJ^2R^{-1}$. Thus, since $|J\rangle$ is Gaussian, $J^2 = -\mathbb{I}$, and since $RR^{-1} = \mathbb{I}$ by the definition of unitarity, it follows that $J'^2 = -\mathbb{I}$, meaning $|J'\rangle$ is a Gaussian state. Generally, unitary transformations composed of the squeezing and displacement operators will map any bosonic or fermionic Gaussian state to another bosonic or fermionic Gaussian state, respectively [59].

[5]Such states can be either bosonic or fermionic.

5.6 Localized States as Squeezed Gaussian States

In the position basis of a rigged Hilbert space, eigenkets of the position operator, or states $|x_0\rangle$ such that $\hat{x}\,|x_0\rangle = x_0\,|x_0\rangle$, can be represented as δ-functions: $\langle x|x_0\rangle = \delta(x - x_0)$. A δ function is a limit of a sequence of normalized, narrowing Gaussian functions. Reducing the width of a Gaussian function is, however, simply squeezing the state. Hence completely localized δ-function states are equivalently Gaussian squeezed states in the infinitely squeezed limit. As discussed previously, Gaussian states can be constructed via the action of the squeezing operator (5.37) and displacement operator (5.38) on the Fock vacuum, i.e., $|J\rangle = \hat{D}(\gamma)\hat{S}(r)\,|0\rangle$. The position basis equivalence of such states to Gaussian functions centered around some x_0 and the equivalency between the position eigenkets $|x'\rangle$ and $|J\rangle$ in the minimal uncertainty, infinitely squeezed limit is given in detail in equations 18-47 of [60]. The necessary result presented here is,

$$|\chi_{x'}\rangle = \left(\frac{\omega}{\pi}\right)^{1/4} \exp\left[-ix'\hat{p}\right] \exp\left[-\frac{1}{2}\left(\hat{a}^\dagger\right)^2\right] |0\rangle\,, \tag{5.41}$$

where $\hbar = m = 1$. The state $|\chi_{x'}\rangle$ is the Gaussian state that, in the infinitely squeezed limit, becomes a position eigenstate. Since the momentum basis of a rigged Hilbert space shares the same properties as its position basis counterpart, the eigenkets of the momentum basis can be written as shifted, squeezed Gaussian states, as shown in equation 48 of [60], presented below.

$$|\rho_{p'}\rangle = \left(\frac{1}{\pi\omega}\right)^{1/4} \exp\left[ip'\hat{x}\right] \exp\left[\frac{1}{2}\left(\hat{a}^\dagger\right)^2\right] |0\rangle\,. \tag{5.42}$$

An important consequence of the Gaussian description of the position basis is the structure of the inner product between eigenkets $|\chi_x\rangle$ and $|\chi_{x'}\rangle$. Given the Gaussian description of these eigenkets in (5.41), the inner product may be written as,

$$\langle\chi_x|\chi_{x'}\rangle = \left(\frac{\omega}{\pi}\right)^{1/2} \langle\psi| \exp\left[-i(x' - x)\hat{p}\right] |\psi\rangle\,, \tag{5.43}$$

where the state $|\psi\rangle = \exp\left[-\left(\hat{a}^\dagger\right)^2/2\right]|0\rangle$. It is clear the inner product can be simplified by inserting a resolution of identity,

$$\langle\chi_x|\chi_{x'}\rangle = \left(\frac{\omega}{\pi}\right)^{1/2}\int dp\; e^{-i(x'-x)p}\,\langle\psi|p\rangle\,\langle p|\psi\rangle\,. \tag{5.44}$$

To determine the projection of $|\psi\rangle$ onto the momentum basis states, it is easiest to write the momentum eigenkets as functions of the annihilation/creation operators acting on the Fock vacuum. The resulting state has the same form as (5.42), i.e., $|p\rangle \to |\rho_p\rangle$. It is a straightforward calculation to show that $\langle\rho_p|\psi\rangle = (1/\pi\omega)^{1/2}\exp[-p^2/4\omega]$. Substituting this into (5.44) the inner product has the expected structure,

$$\langle\chi_x|\chi_{x'}\rangle = \frac{1}{b\sqrt{\pi}}e^{-(x-x')^2/(2b)^2}, \tag{5.45}$$

where $b = (1/2\omega)^{1/2}$ is the width of the distribution. The inner product is a Gaussian in the difference of positions. In the limit where the width of the distribution becomes infinitesimally small, i.e., $b \to 0$, the completely localized structure of the position eigenkets is recovered.

CHAPTER 6

Entanglement Extraction

Entanglement is considered to be a key, physical resource in the realm of quantum information. It can be manipulated and transformed in various ways for a multitude of applications [61, 62]. Using local operations and classical communication (LOCC), two parties, who share an entangled state, can enact a protocol that changes the state while keeping the overall amount of entanglementi.e., the entanglement entropy, or any other entanglement measure—the same over the course of the protocol. An example of entanglement manipulation can be seen in its exploitation to communicate quantum information over large distances—as is the case in quantum teleportation [63]. Another example is entanglement distillation, wherein two parties that share a large number m of copies of an entangled state, use LOCC to create as many n copies of a maximally entangled state, such as a Bell state. While these cases are classic examples of entanglement transformations a case that is of particular interest is when entanglement is transferred between different subsystems. The transfer of entanglement between subsystems is known as *entanglement extraction* and will be useful in the context of localizing quantum information.

The goal of entanglement extraction is to transfer entanglement between two systems. Particularly, entanglement between states in one system is transferred onto initially, unentangled states in another. However, the extraction protocol must be chosen with care since no new entanglement should be introduced into the total system. If new entanglement was introduced, then the entanglement contained in one system would not necessarily be transferred to the other, rather, both systems could become entangled and the total amount

of entanglement within the total system would increase. Thus, extraction protocols must utilize LOCC since it does not generate new entanglement within the system, a necessary requirement to show that entanglement is transferred between the subsystems.

Suppose Alice and Bob, denoted by A and B respectively, have access to a shared system with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. This shared system, considered to be the *source* system, is physical, contains at least one copy of some entangled state $|\psi\rangle_{AB}$, and cannot be modified in any way. That is, Alice and Bob are not allowed to perform operations that solely act on any state in the source system. Alice and Bob also have access to local ancillary subsystems with Hilbert spaces $\mathcal{H}_{A'}$ and $\mathcal{H}_{B'}$, where the subscripts denote the association with Alice's and Bob's subsystems, respectively. The ancilla subsystems are considered to be the *target* subsystems where the entanglement extracted from the source system will be stored. Initially, the target systems are not entangled. The composite Hilbert space that describes the total target system is $\mathcal{H}_{A'B'} = \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$ and the total Hilbert space for the source + target systems is given by

$$\mathcal{H}_{AA'BB'} = \mathcal{H}_{AB} \otimes \mathcal{H}_{A'B'} \,. \tag{6.1}$$

Since no new entanglement should be introduced into the source or target system, Alice and Bob can only use LOCC to manipulate the total system. Note that, by enacting LOCC on the total system, Alice and Bob are still following the previous rule stating they are not allowed to directly manipulate the source system. By using a LOCC protocol, described in more detail below, Alice and Bob attempt to distill as many m high-fidelity copies of some Bell state as possible. The states created in the target Hilbert space are entangled states shared between the two target subsystems. Concurrently, the LOCC process indirectly transforms the entanglement found in the source system, leaving the source system in some state $|\phi\rangle_{AB}$. Since no new entanglement has been generated during this process, the entanglement needed to generate high-fidelity copies of the Bell state must come from the

source system. This implies that the final state of the source system has a lower entanglement entropy compared to the initial state. In the work below, it is the case that the source system becomes completely unentangled.

An interesting and useful consequence of the entanglement extraction procedure is its energy cost. Let $\hat{H}_{AB}$ represent the Hamiltonian of the source system and assume that the local operations carried out on the target subsystems are done with no energy cost. It follows that the source + target system Hamiltonian is thus, $\hat{H} = \hat{H}_{AB} \otimes \hat{\mathbf{I}}_{A'B'}$. Since the source system, and by extension its Hamiltonian, cannot be modified, the Hamiltonian $\hat{H}$ is the Hamiltonian before and after the extraction procedure. The difference in energy expectation values before and after extraction is,

$$\Delta E = \langle\psi|_{AB} \hat{H} |\psi\rangle_{AB} - \langle\phi|_{AB} \hat{H} |\phi\rangle_{AB} . \tag{6.2}$$

Thus, as long as the states before and after extraction produce different energy expectation values, there will be a change in the energy of the system as a result of the extraction protocol. This change is discussed in more detail below.

6.1 The Entanglement Extraction Protocol

As mentioned above, the entanglement extraction process requires a source system S that contains entangled states and two target subsystems that are initially unentangled. While the location of these systems is generally unimportant, target subsystems are chosen to be located at spatial infinity. The source system is composed of Alice's and Bob's localized subsystems A and B, with associated Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Similarly, the two target subsystems 1 and 2 have associated Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Since the source system is composed of local subsystem Hilbert spaces, the composite Hilbert space is factorizable, i.e., $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$, and similarly for the composite Hilbert space of the target subsystems after the extraction of entanglement from the source system to the

target subsystems has occurred, i.e., $\mathcal{H}_F = \mathcal{H}_1 \otimes \mathcal{H}_2$. Factorizability allows one to define entanglement on the composite Hilbert spaces.

Suppose the source system contains a set of entangled states and the target subsystems each contain one state[1]. Initially, the total system is in the product state,

$$\hat{\rho}_I = \hat{\rho}_1 \otimes \hat{\rho}_2 \otimes |\Psi\rangle_S \langle\Psi|_S \,, \tag{6.3}$$

where $|\Psi\rangle_S$ is the initial, entangled state of the source system and $\hat{\rho}_1, \hat{\rho}_2$ are the initial density matrices of the two target subsystems. Entanglement is extracted by "swapping" the unentangled modes of the target subsystem with the entangled modes of the source system. Once the swapping procedure is complete the source subsystems A and B are unentangled and the target subsystems 1 and 2 are now entangled.

Entanglement is extracted from the source system by a set of "swap" operations—two unitary operations $\hat{U}_1$ and $\hat{U}_2$ that map one target mode to one mode inside the source system [35]. For example, the target modes, $(\hat{a}_1, \hat{a}_1^\dagger)$ and $(\hat{a}_2, \hat{a}_2^\dagger)$, are swapped with one mode inside the source system, $(\hat{a}_A, \hat{a}_A^\dagger)$ and $(\hat{a}_B, \hat{a}_B^\dagger)$, using the following relations,

$$\begin{aligned} \hat{U}_1^\dagger \hat{a}_1 \hat{U}_1 &= \hat{a}_A, \qquad & \hat{U}_1^\dagger \hat{a}_A \hat{U}_1 &= \hat{a}_1, \\ \hat{U}_1^\dagger \hat{a}_1^\dagger \hat{U}_1 &= \hat{a}_A^\dagger, \qquad & \hat{U}_1^\dagger \hat{a}_A^\dagger \hat{U}_1 &= \hat{a}_1^\dagger, \end{aligned} \tag{6.4}$$

while $\hat{U}_2$ swaps $(\hat{a}_2, \hat{a}_2^\dagger)$ and $(\hat{a}_B, \hat{a}_B^\dagger)$. Note that these types of unitary operations always exist due to bit symmetry [64]. After the swap operations, the two target subsystems become entangled. The entanglement content of the state depends on the type of source and target subsystems, their couplings, and resources such as energy available to implement the extraction. To ensure the entanglement of the two target modes was preexisting in the system and not created by the swap operations, the source modes are restricted so they either commute or anti-commute depending on whether the source system is a set of bosons or fermions, re-

[1]Such states can be either bosonic or fermionic.

spectively. For example, suppose the source system is a set of entangled fermions. It follows that the restrictions on the fermionic modes are,

$$[\hat{a}_A, \hat{a}_B]_+ = [\hat{a}_A, \hat{a}_B^\dagger]_+ = [\hat{a}_A^\dagger, \hat{a}_B]_+ = [\hat{a}_A^\dagger, \hat{a}_B^\dagger]_+ = 0. \tag{6.5}$$

The conditions like those in (6.5) ensure that the entanglement of the two target modes was not created by the swap operations between the target and source modes [65].

An initial configuration can be chosen such that the source system is a product state between the two modes and the rest of the system

$$|\Psi\rangle_S = |\psi\rangle_{AB} \otimes |\phi\rangle_R \,, \tag{6.6}$$

where $|\psi\rangle_{AB}$ is the initial entangled state comprised of states localized to Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$, and $|\phi\rangle_R$ is the ground state for the rest of the system. When the target modes and source modes are swapped the total system is placed in the state

$$\hat{\rho}_F = |\psi\rangle \langle\psi|_{12} \otimes \hat{\rho}_A \otimes \hat{\rho}_B \otimes |\phi\rangle \langle\phi|_R \,, \tag{6.7}$$

where $|\psi\rangle_{12}$ is the final, entangled state of the two target systems and $\hat{\rho}_A$ and $\hat{\rho}_B$ are the final states of Alice's and Bob's subsystems which compose the source system. Notice that the rest of the source system is unaffected by the extraction process and remains in the ground state $|\phi\rangle_R$.

Hackl and Jonsson assume that the initial source system S possesses entanglement in its ground state. Hence after any entanglement extraction the states $\hat{\rho}_A$ and $\hat{\rho}_B$ must be in a higher energy state. It follows that the energy expectation value has increased due to the extraction process—this is the cost [35].

6.2 The Energy Cost of Entanglement Extraction

The energy cost of entanglement extraction is given by the difference between the expectation value of the source system's Hamiltonian before and after the extraction, i.e.,

$$\Delta E = \mathrm{Tr}\left(\hat{\rho}_A \otimes \hat{\rho}_B \otimes |\phi\rangle\langle\phi|_R \hat{H} - |\Psi\rangle\langle\Psi|_S \hat{H}\right). \tag{6.8}$$

The Hamiltonian $\hat{H}$ of the source system may be coupling different modes of the system, however for the calculation of the energy cost, only the parts acting on the modes in A, B are relevant. Therefore, the energy cost of entanglement extraction is solely determined by $\hat{H}_{AB}$ acting on only the A, B modes[2] i.e.,

$$\Delta E = \mathrm{Tr}\left(\hat{\rho}_A \otimes \hat{\rho}_B \hat{H}_{AB}\right) - \mathrm{Tr}\left(|\psi\rangle\langle\psi|_{AB} \hat{H}_{AB}\right). \tag{6.9}$$

If one wishes to minimize the extraction energy cost then two requirements need to be fulfilled. First, one must choose the source modes to be partner modes, i.e., $\hat{a}_B = \hat{a}_{\bar{A}}$ and $\hat{a}_B^\dagger = \hat{a}_{\bar{A}}^\dagger$. Such a choice maximizes the extracted entanglement. If non-partner modes are chosen, then the mixed-state entanglement between the target modes is never larger than between the mode and its partner. Second, the final partner modes of the source system, due to the swapping process, will be in the ground states of the single-mode restrictions $\hat{H}_A$ and $\hat{H}_B$ of $\hat{H}$ onto the individual partner modes. For bosonic and fermionic modes, these states are Gaussian. For more details see [35], particularly the discussion in Section 3.2.

For every mode in a given multi-particle Gaussian state system, there exists a mode that

[2]The values of $\hat{H}$ and $\hat{H}_{AB}$ acting on various states may be related of course. For example, if one started from a fully diffeomorphism invariant theory then all physical states would satisfy the Hamiltonian constraint and the energy cost would vanish. In this case, it is unclear how to define the entanglement as there is no natural, local partition into subsystems. In contrast, this work starts with a non-relational system and rewrites it relationally. There is still a reflection of the diffeomorphism invariant behavior when analyzing entanglement extraction under the G-twirl (for details see Chapter 7), as twirling over translations in a situation where translation invariance enforces the equality of the before and after Hamiltonians also destroys the possibility of entanglement extraction. For related work that does treat translation invariance in the language of quantum reference frames and constrained systems, see [66]

shares all of the first mode's entanglement [35]. This implies that one can always find partner modes within multi-particle Gaussian systems. Thus, as long as the entanglement extraction protocol can be performed, one can always choose partner modes to minimize the energy cost of entanglement extraction. If one does not choose partner modes, the energy cost of entanglement extraction is not necessarily minimized. In the work below, the partner modes are not explicitly chosen for the entanglement extraction protocol. Thus, the calculated energy cost is not necessarily minimized. Since this work merely seeks to demonstrate when there is an energy cost to this process for various relational approaches, minimizing the energy cost is beyond the scope of this work. Whether or not that cost is minimized is left for future research.

CHAPTER 7

Quantum Reference Frames and the G-Twirl

Reference frames are one of the most fundamental concepts in physics. Relativity defines reference frames as abstract mathematical coordinate systems, whose properties such as origin, orientation, scale, etc. are set by physical measurements made by an observer, relative to the observer's dynamics. This implies that reference frames are physical objects and can thus be quantized. The quantum mechanical counterpart to the classical reference frame is often referred to as a *quantum reference frame* [67, 68]. Just like classical reference frames, quantum reference frames are implicit in the definition of quantum states. For example, spin states are defined with respect to a frame with a specified orientation, often referred to as the lab frame. Thus, two labs with different orientations will define physically different spin-up states.

Quantum states, defined with respect to one quantum reference frame, may be written in terms of quantum states defined with respect to a different quantum reference frame, acted on by a reference frame transformation. Similar to their classical counterparts, quantum reference frame transformations are operator representations of group elements that map points between different coordinate systems. However, a unique property of quantum reference frame transformations is the ability to change a quantum superposition into quantum entanglement and visa versa. That is to say, a quantum superposition state in one frame can appear as an entangled state in another [68]. This implies the existence of an absolute quantum reference frame, antithetical to principles of relativity, particularly general covariance.

To be compatible with general covariance and, by extension, diffeomorphism invariance, quantum states must be defined in a relational framework. However, as is demonstrated below, if the Hilbert space of a quantum state, defined with respect to some external frame is repartitioned into a center-of-mass/relational partition, the dependence on the external frame, gets placed into the center-of-mass degrees of freedom. Thus the state is still frame-dependent. For a quantum state to be frame-independent, as is required by diffeomorphism invariance, frame-dependent degrees of freedom need to be removed. This can be done by group averaging the quantum state over all possible diffeomorphism, otherwise known as a G-twirl. Since the diffeomorphism group is infinite, this work has chosen to use translations as a toy model.

7.1 Entanglement Generation in a Relational Basis

Given that $|x\rangle$ is the limit of a highly squeezed Gaussian state, one can use approximate position basis states without leaving the Gaussian framework. More importantly, this work investigates how switching to a center-of-mass/relational partition affects the entanglement of Gaussian states. First, consider two particles 1 and 2, localized at some points x_1 and x_2 respectively, with respect to some external reference frame. This state can be written in an external partition as $|\psi\rangle = |x_1\rangle \otimes |x_2\rangle$ [1]. It is clear this state is a product state between the two particles without any entanglement.

In diffeomorphism invariant theories there is, of course, no external frame dependence. One common approach to preserving diffeomorphism invariance in quantum mechanics is to move to a relational framework, where outcomes are defined in relation to others and probabilities become conditional (for a review, see [69]). In this framework, a simple toy model that captures the relational aspect of a fully diffeomorphism invariant system by requiring that the quantum mechanical system be translationally invariant is constructed. Intuitively, since only position states are considered, forcing a system to be translationally

[1] For the rest of this work, $|x\rangle$ will be used as a shorthand to refer to a highly localized Gaussian state around x, rather than an exact position eigenstate.

invariant will erase any absolute position information, leaving only translationally invariant relational degrees of freedom, such as $x_1 - x_2$, in the reduced Hilbert space. The first step on this path is to define new operators and a corresponding basis that captures the center-of-mass and relational degrees of freedom. This basis will be referred to as the relational basis, in contrast to the external or absolute basis defined by the position states $|x\rangle$.

The states and operators with respect to the relational basis can be constructed via transformation from the external frame and partition. The explanation below follows the presentation in [70]. For a system with N degrees of freedom, the position and momentum operators of the external partition, $\{\hat{x}_k, \hat{p}_k\}_{k=1}^N$ are fully specified by the center-of-mass position and momentum operators $\{\hat{x}_{cm}, \hat{p}_{cm}\}$ and the relational position and momentum operators $\{\hat{x}_{i|1}, \hat{p}_{i|1}\}_{i=2}^N$. The transformation between the positions and momenta of the external partition and the positions and momenta in the center-of-mass/relational partition are given by,

$$\hat{x}_{cm} = \frac{1}{M} \sum_{k=1}^{n} m_k \hat{x}_k, \tag{7.1a}$$

$$\hat{p}_{cm} = \sum_{k=1}^{n} \hat{p}_k, \tag{7.1b}$$

$$\hat{x}_{i|1} = \hat{x}_i - \hat{x}_1 \text{ for } i \in \{2, n\}, \tag{7.1c}$$

$$\hat{p}_{i|1} = \hat{p}_i - \tilde{m}_i \hat{p}_{cm} \text{ for } i \in \{2, n\}, \tag{7.1d}$$

where $M = \sum_{k=1}^{n} m_k$ is the total mass of the system and $\tilde{m}_i = m_i/M$ is the mass fraction of the i^{th} particle. The canonical commutation relations for the operators in the center-of-mass/relational partition follow from the commutation relations between operators of the external partition, i.e., $[\hat{x}_{cm}, \hat{p}_{cm}] = [\hat{x}_{i|1}, \hat{p}_{i|1}] = i$ with all other combinations vanishing. Notice that the relational position and momentum operators are defined with respect to particle 1, however, any other particle may be chosen with the same effect.

Transformations between the external partition and the center-of-mass/relational par-

tition may engender entanglement if at least one of the particles is in a superposition in the external partition. As an example, consider a two-particle composite state where, for simplicity, only particle one is in superposition with respect to the external partition[2],

$$\begin{aligned} |\psi_{12}\rangle &= \frac{1}{\sqrt{2}} \left(|x_1\rangle + |x_1'\rangle \right) \otimes |x_2\rangle \\ &= \frac{1}{\sqrt{2}} \left(|x_1\rangle \otimes |x_2\rangle + |x_1'\rangle \otimes |x_2\rangle \right). \end{aligned} \tag{7.2}$$

Here $|x_1\rangle, |x_1'\rangle \in \mathcal{H}_1$ are the possible position states of particle 1 and $|x_2\rangle \in \mathcal{H}_2$ is the position state of particle 2. It is clear no entanglement exists between degrees of freedom in the external partition. Now transform into the center-of-mass/relational partition, with particle 1 considered to be the "reference" particle from which the position of particle 2 will be defined. The center-of-mass and relational position states for the two components of the superposition differ and $|\psi\rangle$ becomes

$$|\psi_{12}\rangle \to |\psi_{12,cm}\rangle = \frac{1}{\sqrt{2}} \left(|x_{cm}\rangle \otimes |x_{2|1}\rangle + |x_{cm}'\rangle \otimes |x_{2|1}'\rangle \right), \tag{7.3}$$

where the prime denotes the center-of-mass and relational position states between $|x_1'\rangle$ and $|x_2\rangle$. Now there is bipartite entanglement between the center-of-mass and the relational position states [68].

There is a subtle difference in the n-particle case when compared to the two-particle case, which can be demonstrated with the three-particle case. Consider a three-particle composite state in the external frame where, for simplicity, only the reference particle is placed in

[2]Gaussian states in the highly squeezed limit have been chosen, to allow for a trivial normalization. Generally, the Gaussian nature of the states does not allow for such a non-trivial normalization since $\langle x|x'\rangle \neq \delta(x - x')$.

superposition,

$$\begin{aligned} |\psi_{123}\rangle &= \frac{1}{\sqrt{2}}\left(|x_1\rangle + |x_1'\rangle\right) \otimes |x_2\rangle \otimes |x_3\rangle \\ &= \frac{1}{\sqrt{2}}\left(|x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle + |x_1'\rangle \otimes |x_2\rangle \otimes |x_3\rangle\right). \end{aligned} \tag{7.4}$$

As in the previous example $|x_1\rangle, |x_1'\rangle \in \mathcal{H}_1$ are the possible position states of particle 1, the reference particle, $|x_2\rangle \in \mathcal{H}_2$ is the position state of particle 2, and $|x_3\rangle \in \mathcal{H}_3$ is the position state of particle 3. Currently, there is no entanglement between any of the degrees of freedom. Different combinations of the states will generally have different centers-of-mass positions and different relational positions, however, we would like to note that there exist cases where the center-of-mass and/or relational positions between two or more combinations are equal. After the transformation into the center-of-mass/relational partition, the state $|\psi_{123}\rangle$ is described as,

$$|\psi_{123}\rangle \to |\psi_{123,cm}\rangle = \frac{1}{\sqrt{2}}\left(|x_{cm}\rangle \otimes |x_{2|1}\rangle \otimes |x_{3|1}\rangle + |x_{cm}'\rangle \otimes |x_{2|1}'\rangle \otimes |x_{3|1}'\rangle\right), \tag{7.5}$$

where the primes denote center-of-mass and relational position states that utilized $|x_1'\rangle$ in their definitions. Here there is tripartite entanglement between both the center-of-mass and relational degrees of freedom, which differs from the two-particle case. In general, with different transformations and n-particles entanglement can be generated between various parts of the partitioned Hilbert space—one is not restricted to center-of-mass/relational bipartite entanglement.

Below the generalization of (7.2) and (7.4) will be used extensively for calculational simplicity. For a system of N particles, where one particle is placed in a location superposition

with respect to the external frame, the state is given by

$$\begin{aligned} |\psi_{1,\ldots,N}\rangle &= \frac{1}{\sqrt{2}}\left(|x_1\rangle + |x_1'\rangle\right)|x_2\rangle \otimes \ldots \otimes |x_N\rangle \\ &= \frac{1}{\sqrt{2}}\left(|x_1\rangle \otimes |x_2\rangle \otimes \ldots \otimes |x_N\rangle + |x_1'\rangle \otimes |x_2\rangle \otimes \ldots \otimes |x_N\rangle\right) \end{aligned} \tag{7.6}$$

where $|x_1\rangle, |x_1'\rangle \in \mathcal{H}_1$ are the possible position states of particle 1 and $|x_i\rangle \in \mathcal{H}_i$ is the position state of the ith particle. The state in (7.6) written in the center-of-mass/relational partition is given by

$$\begin{aligned} |\psi_{1,\ldots,N}\rangle &\to |\psi_{1,\ldots,N,cm}\rangle \\ &= \frac{1}{\sqrt{2}}\left(|x_{cm}\rangle \otimes |x_{rel}\rangle + |x_{cm}'\rangle \otimes |x_{rel}'\rangle\right) \end{aligned} \tag{7.7}$$

where $|x_{rel}\rangle = |x_{2|1}\rangle \otimes \ldots \otimes |x_{N|1}\rangle$ represents the $N-1$ remaining relational degrees of freedom. Notice that the state in (7.7) is an n-partite entangled state and our discussion regarding the nature of entanglement within the two partitions still holds.

7.2 The G-Twirl

The entangled states in the center-of-mass/relational partition are not yet fully relational, in that the center-of-mass coordinate contains degrees of freedom relative to the external partition. To remove the center-of-mass degree of freedom one can group average over translations, which will reduce the state to one containing only relational degrees of freedom. The procedure from quantum information theory for group averaging over quantum reference frames is known as the G-twirl [67].

Consider a quantum state represented by the density matrix $\hat{\rho} \in \mathcal{H}$ in some Hilbert space $\mathcal{H}$, described with respect to some external reference frame. Changes to the orientation of the quantum state $\hat{\rho}$ with respect to the external reference frame are performed via the action of some unitary operation $\hat{U}(g) \in \mathcal{H}$ on the state $\hat{\rho}$. Here, $\hat{U}(g)$ is the unitary representation

of a group element $g \in G$, where G is the group of all possible changes of the external reference frame. It is important to note that in the definition of the G-twirl, G is a compact group [67], however, as is shown below, G can be non-compact albeit yielding slightly more complex results [37]. The result of the G-twirl is the description of the quantum state that does not contain any information about the external frame. This description is achieved by averaging over all possible orientations of $\hat{\rho}$ with respect to the external frame, where every possible orientation is weighted equally,

$$\hat{\rho}_R = \mathcal{G}[\hat{\rho}] \equiv \int dg \hat{U}(g) \hat{\rho} \hat{U}^\dagger(g), \tag{7.8}$$

where dg is the Haar measure of the group G and $\hat{\rho}_R \in \mathcal{H}$ is the relational description of $\hat{\rho}$.

Since all elements of the group that transforms the external frame are averaged over, any relation to the external reference frame that was used to describe the state $\hat{\rho}$ is removed. Only the relational degrees of freedom within the system remain, i.e., the information unaffected by changes in the external reference frame. For example, suppose $\hat{\rho}$ describes a composite state of two particles such that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. After a G-twirl is performed, the remaining information contains the relational degrees of freedom between the two particles. Note that the G-twirl is done via the product representation $\hat{U}(g) = \hat{U}_1(g) \otimes \hat{U}_2(g)$, where $\hat{U}_1(g) \in \mathcal{H}_1$ and $\hat{U}_2(g) \in \mathcal{H}_2$ are the unitary representations of the group G in each Hilbert space.

Since translation invariant states are of interest here, the focus will be on the G-twirl operation as it relates to translations. The presentation below follows from the work of [70]. For more details, the reader is encouraged to review the original work. Additionally, the reader is encouraged to see the related approach, where G-twirls on physical states can be understood as group averaging [71–73]. The action of the three-dimensional translation group $g = x \in \mathbb{R}^3$ on the external frame in the external partition, $\mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n$ is given by

$$\hat{U}(x) = \bigotimes_{n=1}^{N} e^{-ix\hat{p}_n}. \tag{7.9}$$

In the center-of-mass/relational partition, $\mathcal{H}_{cm} \otimes \mathcal{H}_R$, the action of the translation group is given by

$$\hat{U}(x) = e^{-ix\hat{p}_{cm}} \otimes \hat{\mathbf{I}}_R. \tag{7.10}$$

To carry out the G-twirl over translations, first express the state $\hat{\rho}$ in the center-of-mass/relational partition, in the momentum basis,

$$\hat{\rho} = \int dp_{cm} dp'_{cm} dp_R dp'_R \; \rho(p_{cm}, p'_{cm}, p_R, p'_R) \, |p_{cm}\rangle \langle p'_{cm}| \otimes |p_R\rangle \langle p'_R| \,, \tag{7.11}$$

where $|p_{cm}\rangle$ and $|p'_{cm}\rangle$ are the possible basis states of the center-of-mass momentum and similarly, $|p_R\rangle$ and $|p'_R\rangle$ are the basis states of the $N-1$ relational momentum vectors. It follows that the G-twirl over the possible translations of the external frame is given by

$$\begin{aligned}
\mathcal{G}_T[\hat{\rho}] &= \int dx \hat{U}(x) \left[\int dp_{cm} dp'_{cm} dp_R dp'_R \rho(p_{cm}, p'_{cm}, p_R, p'_R) \, |p_{cm}\rangle \langle p'_{cm}| \otimes |p_R\rangle \langle p'_R| \right] \hat{U}^\dagger(x) \\
&= \int dx dp_{cm} dp'_{cm} dp_R dp'_R \rho(p_{cm}, p'_{cm}, p_R, p'_R) e^{-ix\hat{p}_{cm}} \, |p_{cm}\rangle \langle p'_{cm}| \, e^{ix\hat{p}'_{cm}} \otimes |p_R\rangle \langle p'_R| \\
&= (2\pi)^3 \int dp_{cm} dp_R dp'_R \; \rho(p_{cm}, p_{cm}, p_R, p'_R) \, |p_{cm}\rangle \langle p_{cm}| \otimes |p_R\rangle \langle p'_R| \,.
\end{aligned} \tag{7.12}$$

The definition of the Dirac delta function $(2\pi)^3 \delta(p - p') \equiv \int dx e^{ix(p-p')}$ has been used to go from the second to the third equality. It is clear that the G-twirl operation is effectively the trace over the center-of-mass degrees of freedom in the center-of-mass/relational partition, i.e., $\mathcal{G}_T[\hat{\rho}] = \mathcal{I} \operatorname{Tr}_{cm} \hat{\rho}$, where $\mathcal{I}$ denotes a divergent constant originating from integral over the probability function $\rho(p_{cm}, p_{cm}, p_R, p'_R)$ in (7.12). The new state $\mathcal{G}_T[\hat{\rho}]$ is not normalized, since there are infinitely many states to trace over, due in part to the noncompact nature of the group of translations. However, the divergent nature of $\mathcal{I}$ will be inconsequential to the physics of interest for reasons discussed below.

7.3 Removal of the external partition via a G-twirl over translations

As an example, perform a G-twirl on the state in (7.3) over the set of one-dimensional translations. Since the relational state has been shown to be $\mathcal{G}_T[\hat{\rho}] = \mathcal{I}\,\mathrm{Tr}_{cm}\,\hat{\rho}$ it follows that

$$\mathcal{G}_T[|\psi_{12,cm}\rangle\langle\psi_{12,cm}|] = \mathcal{I}\,\mathrm{Tr}_{cm}[|\psi_{12,cm}\rangle\langle\psi_{12,cm}|] = \mathcal{I}\left(|x_{2|1}\rangle\langle x_{2|1}| + |x'_{2|1}\rangle\langle x'_{2|1}|\right). \tag{7.13}$$

Here the primes refer to center-of-mass and relational states defined with $|x'_1\rangle$ in (7.2). Note that any normalization factors are absorbed into $\mathcal{I}$. The state is now completely relational, one that only contains relational information and no information regarding the external frame. Furthermore, the entanglement between the center of mass and the relational degrees of freedom has been removed and the relational degrees of freedom are now in a mixed state. Notice that the G-twirl destroys the entanglement that existed between the center-of-mass and relational degrees of freedom.

A slightly more interesting example is the G-twirl of the three particle state in (7.5), over the set of one-dimensional translations. It follows that,

$$\begin{aligned}\mathcal{G}_T[|\psi_{123,cm}\rangle\langle\psi_{123,cm}|] &= \mathcal{I}\,\mathrm{Tr}_{cm}[|\psi_{123,cm}\rangle\langle\psi_{123,cm}|] \\ &= \mathcal{I}\left(|x_{2|1}\rangle\langle x_{2|1}| \otimes |x_{3|1}\rangle\langle x_{3|1}|\right. \\ &\qquad \left. + |x'_{2|1}\rangle\langle x'_{2|1}| \otimes |x'_{3|1}\rangle\langle x'_{3|1}|\right).\end{aligned} \tag{7.14}$$

Even though the original pure state was tripartite entangled, the G-twirl has removed all the entanglement from the system. This is clear since the post G-twirl state is a separable, mixed state over the center-of-mass/relational partition. Moreover, this implies that constructing a purely relational state by discarding information about the external frame removes any entanglement as well. As is demonstrated below, this phenomenon makes for an interesting, case study of the entanglement extraction protocol as applied to relational degrees of freedom.

CHAPTER 8

Gauge Induced External Frames via the Z-Model

The labeling of spacetime points provided by an external frame has no physical meaning. This implies that states in the external partition don't come from any specific measurement of an explicitly defined, physical observable, rather they can be thought of as a choice of a particular gauge. As was demonstrated in Section 7.3, the G-twirl effectively removes this gauge choice by integrating out a state's dependence on the external frame, leaving only relational information behind.

Alternatively, one can put the external partition into a relational framework by correlating each position state $|x\rangle$ with some dynamical observable. This will give the arbitrary labeling meaning since the position states are now linked to the value of said observable. However, such observables must be chosen carefully. For example, let the Hamiltonian $\hat{H}$ be the "reference" observable. Since $\hat{H}$ is fundamentally a generator of time translations, it has no intrinsic dependence on position, however. This allows $\hat{H}$ to be translationally invariant, which goes back to the G-twirl scenario described in the previous chapter—in such a case there would be no way to correlate $|x\rangle$ with a value of H. Instead, the values of the Hamiltonian for some field configurations need to be in one-to-one correspondence with the position states $|x\rangle$. To give $\hat{H}$ this position dependence the Z-model construction will be employed. This chapter will provide a brief overview of the construction and for more details, the reader is encouraged to review the original work [29].

The original construction of the Z-model was meant to localize non-local, diffeomorphism-invariant observables in low-energy, effective quantum gravity. For observables to be consid-

ered diffeomorphism-invariant they must form a commuting subalgebra with the constraints of the theory brought on by diffeomorphism-invariance. In the context of quantum gravity, these constraints are the Hamiltonian and momentum constraints [29]. As a result, any naturally diffeomorphism-invariant observable, or constructed via application of a dressing [47] necessarily extends throughout the entirety of the spacetime, i.e., observables at different locations do not commute with themselves.

Given a particular quantum state, the Z-model allows one to define the location of a local operator by specifying it relative to a structure determined by the expectation value of a pseudo-local operator. This gives physical meaning to the labeling provided by the external partition. The first step is to introduce a dynamical, massless, auxiliary Z-field with a local field operator relative to the external partition $\hat{Z}(x)$. The field will be assumed to be in some state $|\psi_Z\rangle$ and the expectation value of $\hat{Z}(x)$ with $|\psi_Z\rangle$ is given as

$$Z(x) = \langle\psi_Z|\, \hat{Z}(x)\, |\psi_Z\rangle\,. \tag{8.1}$$

Since any coordinate system is monotonically increasing, $|\psi_Z\rangle$ must be chosen such that the expectation value is also monotonically increasing, implying that the gradient of the expectation value $\langle\hat{Z}\rangle$ over the desired portion of spacetime is everywhere nonvanishing. Given such a monotonic map, one can always make a coordinate transformation on the underlying external frame to *define* x such that $Z(x) = x$ if one wishes. In other words, the coordinate system can now be completely defined by observations on some dynamical field in a particular state.

Note here that the position labeling, even though it formally involves the quantum state, is an expectation value that satisfies classical equations of motion. Expectation values do not automatically follow the classical equations of motion. In single-particle quantum mechanics, this can be seen when the potential V is a polynomial function of $\hat{x}$. The expectation value $\langle\hat{V}\rangle = \langle\hat{x}^n\rangle$, while what would appear in the classical equations of motion is $\langle\hat{x}\rangle^n$ and this

is only equal for certain states (cf. [74]). In field theory, a similar discrepancy between expectation values and classical equations of motion unfolds if the Z field is, for example, self-interacting. In this implementation of the Z-model, however, the background Z-field is in vacuum and free, e.g., the electromagnetic potential between two charged plates. Hence the equations of motion are linear in Z and the expectation value will follow the classical equations. Hence, while there is some quantum state of the Z-field, the quantum nature of $\hat{Z}$ can be essentially ignored in the following. For the moment, assume that such a field and operator exists.

In the above, the Z-field itself was used as an observable that can be correlated with position. There are other operators one could use that are linear in $\hat{Z}$. In this case, the most straightforward way to construct a useful operator is to modify the Hamiltonian to introduce a coupling between the Gaussian states and the Z-field, e.g., $\hat{H}_{int} = \hat{\psi}\hat{Z}$ or $\hat{H}_{int} = \hat{\overline{\psi}}\hat{\psi}\hat{Z}$ term, where ψ is a field of the original model, and Z is the auxiliary field used to define the external frame. Since the auxiliary field is dynamical and spontaneously breaks the translation invariance of the Hamiltonian, the fundamental underlying translational invariance of the theory remains intact. With such a construction, the expectation value of the Hamiltonian for a Gaussian state $|\chi_x\rangle$ localized around x, i.e., the position eigenstates in the previously mentioned limit gets correlated with the position. In other words

$$\langle \chi_x, \psi_Z | \, \hat{H} \, | \chi_x, \psi_Z \rangle = H_0 + \alpha Z(x), \tag{8.2}$$

where H_0 is a constant piece from the free part of the Hamiltonian and the x in $Z(x)$ is the position the localized Gaussian state $|\chi_x\rangle$ is peaked around. The parameter α contains expectation values of Gaussian operators with respect to the state $|\chi_x\rangle$. If one assumes these expectation values are all equal neglecting backreaction (as there is no underlying violation of translation invariance in the Gaussian state sector), α is just a constant. Since $Z(x)$ is monotonically increasing by construction and correlated with position by (8.1), the

expectation value of the Hamiltonian, the observable of interest, therefore also gets correlated with the external position. This allows one to define the external position variable in terms of the expectation value of the Hamiltonian, thereby making the external position coordinate relative to a dynamical field, a well-defined observable.

To concretely implement this idea within the framework, a gradient for the expectation value of $\hat{Z}$ needs to be generated. Consider a system within a parallel plate capacitor. A system of charged particles, with individual charge q located within the capacitor, will be represented by a family of Gaussian states. The plates, separated by distance L and with charge density $\pm\sigma$ on the left/right plates respectively, will produce an everywhere (inside the plates) non-vanishing electric field E. The potential field A_μ, therefore, has a non-vanishing gradient everywhere within the capacitor (in an appropriate gauge). Therefore the Z-model can be incorporated into the framework via the usual electromagnetic coupling, which does not affect the Gaussian nature of the allowed states of the theory due to the form of the interaction term. Since the potential field has a non-vanishing gradient between the plates, it will be monotonic within the capacitor, allowing one to map the position x to the value of the electric potential of a particle at point x, i.e., the expectation value of the interaction term in the Hamiltonian. Particularly, the energy of a Gaussian state, centered around x is given by,

$$\langle \hat{H}_{int}(x) \rangle = q\sigma \langle x \rangle , \tag{8.3}$$

the expectation value of the electromagnetic coupling term.

Consider the three-particle state in the center-of-mass/relational partition shown in (7.5). When determining the energy expectation value of this state in the Z-model, one finds that, while the relational position eigenstate is independent of the electric potential, the center-of-mass eigenstate is not. Hence, there is a 1-to-1 correspondence between x_{cm} and the observable $\hat{H}_{int}(x_{cm})$. Therefore, information about the external frame is unable to be removed. Moreover, since the coupling is with the expectation value, then as long as it is assumed there is never entanglement between the auxiliary field and the Gaussian states one

can think about entanglement extraction between the Gaussian states themselves. In other words, as long as the auxiliary field is coupled semi-classically, the entanglement extraction analysis for relational systems is still possible without worrying about Z-field entanglement. However, note that this assumption can be relaxed and investigated further in any future work.

CHAPTER 9

The Localization of Relational Quantum Information

It is now time to turn to the process of entanglement extraction for relational systems, using the pieces developed in previous chapters. In particular, the work below demonstrates how the entanglement extraction protocol from Chapter 6 can be implemented alongside the G-twirl from Chapter 7 and the Z-model from Chapter 8. This chapter also demonstrates how to relate these concepts.

9.1 Lack of Entanglement Extraction in G-Twirled Relational Partitions

The work in Section 7.1 demonstrated that a pure, non-entangled state in some external partition can become entangled by writing the state in the center-of-mass/relational partition [37,70]. By writing the state in the center-of-mass/relational partition, there is entanglement between the center-of-mass degrees of freedom and the relational degrees of freedom. For examples see (7.3), (7.5), or (7.7). However, the transformation into the center-of-mass/relational partition does not produce a purely relational state since the center-of-mass degrees of freedom, which couple to the external frame, still exist. For the purposes of this work, the energy cost of entanglement extraction for purely relational degrees of freedom is of particular interest. The absolute position of the center-of-mass degrees of freedom, in this case, x_{cm}, is unknown without specifying some measurement system (reference frame) that is capable of differentiating different x_{cm}'s. However, if x_{cm} is translationally invariant, then such a measurement system cannot exist. Therefore the center-of-mass degrees of freedom are not measurable degrees of freedom, rather they are gauge degrees of freedom. These

gauge degrees of freedom can be removed by G-twirling over the center-of-mass degrees of freedom. As was shown in Section 7.3 the G-twirl will produce a purely relational state but at the cost of destroying the entanglement within the center-of-mass and relational degrees of freedom. After the G-twirl is done, the relational degrees of freedom are left in a mixed state that is not entangled. Therefore there is no possibility of an energy cost of entanglement extraction—the entanglement extraction protocol cannot even be performed.

9.2 Entanglement Extraction via the Z-Model

The work in Chapter 8 demonstrated that the Z-model tied the absolute position of a state with the configuration of an auxiliary field. This implies that the coordinate system used to define the absolute position is no longer a gauge degree of freedom. Furthermore, since the coordinate system now depends on the configuration of the auxiliary degree of freedom, the external partition has been "relationalized". The entanglement extraction protocol from Chapter 6 can now be examined for purely relational states. This is in contrast to the use of the G-twirl, where the G-twirl process creates a relational state by destroying the entanglement between the relational and non-relational parts of the state.

In the framework with the Z-model, the absolute position of charged particles is given meaning by the gradient electromagnetic vector potential inside a parallel plate capacitor. This means that the two location configurations in the superposition found in (7.2), (7.4), or (7.6) will have different energies. As a simple example, consider the superposition state found in (7.2). Since $|x_1\rangle$ and $|x_1'\rangle$ will interact differently with the Z-model coupling in the Hamiltonian, the energy for the components of the superposition state would be,

$$E = \langle x_2| \otimes \langle x_1| \hat{H}_{12} |x_1\rangle \otimes |x_2\rangle , \tag{9.1}$$

$$E' = \langle x_2| \otimes \langle x_1'| \hat{H}_{12} |x_1'\rangle \otimes |x_2\rangle . \tag{9.2}$$

Since $|x_1\rangle$ and $|x_1'\rangle$ are now physically different states, $E \neq E'$. The same principle is appli-

cable to n particles, where at least one particle is in a location superposition. Furthermore, these energy differences are unchanged by the choice of partition of the Hilbert space. Therefore, one will see the same effect in the center-of-mass/relational partition as well. Since there is entanglement and an energy difference the entanglement extraction protocol is able to be performed as prescribed and results in a non-vanishing energy cost.

Let the n-particle initial state in the center-of-mass/relational partition from (7.7) be the initial entangled state of the source system,

$$|\psi_I\rangle = |x_{cm}\rangle \otimes |x_{rel}\rangle + |x'_{cm}\rangle \otimes |x'_{rel}\rangle \tag{9.3}$$

is relational. This state written as a density matrix is $\hat{\rho}_I = |\psi_I\rangle \langle\psi_I|$. After the extraction process, the system will be left in a mixed state

$$\hat{\rho}_F = |x_{cm}\rangle \langle x_{cm}| \otimes |x_{rel}\rangle \langle x_{rel}| + |x'_{cm}\rangle \langle x'_{cm}| \otimes |x'_{rel}\rangle \langle x'_{rel}| \tag{9.4}$$

In Hackl and Jonson's work, the final state of the source system is a statistical mixture of states. As such, the final energy will be a statistical average of the energy for the state $|x_{cm}\rangle \otimes |x_{rel}\rangle$ and the state $|x'_{cm}\rangle \otimes |x'_{rel}\rangle$. Since only one state is considered, the final state will not be mixed but rather be either the primed or the unprimed state.

Unlike the scenario of Hackl and Jonsson, the initial state is not in the ground state, by definition. In order to even run the Z-model, the position of the particles needs to be correlated with different energies via the Hamiltonian interaction. This means that any state, other than the single "ground state" position state, at the location of one of the plates, has a higher energy. Hence the extraction process won't yield a minimum energy, but instead an energy difference, as was the resource used in the Z-model to establish the non-gauge nature of the external partition. The energy difference is given by

$$\Delta E = \mathrm{Tr}\left[\hat{\rho}_F \hat{H}\right] - \mathrm{Tr}\left[\hat{\rho}_I \hat{H}\right]. \tag{9.5}$$

No matter the final state, the energy difference is non-vanishing so there will be a net energy transfer to/from the target modes for extracting entanglement from this relational system. Since the final state will be measured in either the primed or unprimed location, there is an energy change for localizing the relational quantum information contained in the state. Note the connection between entanglement extraction and how the system is made relational. In the first approach, when any local information is thrown out, there is no way to perform any quantum information process related to localization or entanglement extraction at the end. And indeed, the protocol fails. In the second, when the location is made relational via the Z-model, the necessary change to the Hamiltonian also automatically enables the extraction protocol to occur. The two processes were locked together.

Of course, these results are only possible inside the capacitor. Outside the capacitor, where the electric field used for the Z-model does not exist the entanglement extraction protocol does not work, since there is no way to "relationalize" the states. This implies that there is a limited domain where the entanglement extraction protocol works. This is appropriate for a relational set-up with finite experimental configurations. Outside the capacitor, the G-twirl is required to construct relational states, since there is no auxiliary field with a monotonic gradient to use for the Z-model. The question then becomes, how can one relate the Z-model with the entanglement extraction protocol to the G-twirl without the entanglement extraction protocol? Below, it is shown that the G-twirl can be obtained as a limit of the Z-model via the framework of positive operator valued measurements (POVM).

9.3 From G to Z via POVM

Heuristically, from the time-energy uncertainty relation, the time it takes to make a measurement scales as $\Delta t \sim 1/\Delta E$, where ΔE is the uncertainty in the energy of the system [75]. Thus, any physically realizable system has some innate uncertainty in its energy. Therefore,

resolving the position in a finite time has some inherent inaccuracy since,

$$\langle \Delta x \rangle \sim \frac{\langle \Delta \hat{H}_{int}(x) \rangle}{q\sigma}. \tag{9.6}$$

The expression in (9.6) is a result of considering the uncertainty in the position defined in the Z-model, i.e., the uncertainty of (8.3). It is clear from the above relationship that when the charge on the plates of the capacitor decreases, the uncertainty in x increases. In other words, there must be a limit where the Z-model produces complete uncertainty about the position. If one were to keep only relational degrees of freedom, this limit should also reproduce the G-twirl. Such a transition can be constructed via the framework of POVMs.

As a reminder, POVMs are a set of positive semi-definite Hermitian matrices $\{\hat{P}_m\}$, where m is the value of each measurement, on a Hilbert space $\mathcal{H}$, that sums to the identity, i.e., for every measurement m,

$$\sum_m \hat{P}_m = \hat{\mathbf{I}}. \tag{9.7}$$

Generally the exact form of each $\hat{P}_m$ is unknown, however for the purposes of this work, it is convenient to assume the operators describe perfect measurements, i.e., $\hat{P}_m = |m\rangle \langle m|$, where $|m\rangle$ is a measurement eigenstate with eigenvalue m. Given a pure state, $|\psi\rangle$ and a set of POVMs $\hat{P}_m$, the probability $|\psi\rangle$ is in the $|m\rangle$ state when measured is,

$$p(m) = \langle \psi | \hat{P}_m | \psi \rangle = \mathrm{Tr}(\hat{\rho}\hat{P}_m), \tag{9.8}$$

where $\hat{\rho} = |\psi\rangle \langle \psi|$ is the density matrix associated with the state $|\psi\rangle$ and the trace is over the eigenstates of the POVM operator. For mixed states, the probability is solely given by the trace over the product of the density matrix and the POVM operators. In the case where the measurement operator is constructed from continuous variables, the trace is replaced with an integral over the possible eigenstates of the POVM operator.

Through the lens of POVMs, the energy expectation value of the detector, given a par-

ticular center-of-mass of the particle system is given by,

$$\langle \hat{H}_{int}(x) \rangle = \sum_j \mathcal{E}_j(x_{cm}) p(x_{cm}) \tag{9.9}$$

where $\mathcal{E}_j(x_{cm})$ is a map between the center of mass of the system and the energy read by the detector. Assume that the detector has a minimum uncertainty. That is, assume that when the detector makes an energy measurement, the true energy of the system is placed into bins $\mathcal{E}_j(x_{cm})$. The true energy of the system is then within the bin energy and the energy uncertainty, i.e., $\mathcal{E}_j(x_{cm}) + \Delta E$. In the continuum limit of, the probability of measuring the state with a center of mass of x_{cm} within one particular bin is given by,

$$p(x_{cm}) = \int_{x_{cm,i}}^{x_{cm,i}+\Delta x} dx''_{cm} \, \langle x''_{cm}|\psi\rangle \, \langle \psi | x''_{cm}\rangle \, . \tag{9.10}$$

The notion of a perfect measurement is given by $\hat{P}_{x''_{cm}} = |x''_{cm}\rangle \langle x''_{cm}|$. The state $|\psi\rangle$ will be taken to be the n-particle entangled state in the center-of-mass/relational partition from (7.7). It follows that the reduced density matrix containing only relational degrees of freedom, constructed by tracing over the center-of-mass degrees of freedom, is represented by,

$$\begin{aligned} \hat{\rho}_{rel} \;\; = \;\; \int_{x_{cm,i}}^{x_{cm,i}+\Delta x_{cm}} dx''_{cm} \, [& \langle x''_{cm}|x_{cm}\rangle \, \langle x_{cm}|x''_{cm}\rangle \, |x_{rel}\rangle \, \langle x_{rel}| \\ & + \langle x''_{cm}|x_{cm}\rangle \, \langle x'_{cm}|x''_{cm}\rangle \, |x_{rel}\rangle \, \langle x'_{rel}| \\ & + \langle x''_{cm}|x'_{cm}\rangle \, \langle x_{cm}|x''_{cm}\rangle \, |x'_{rel}\rangle \, \langle x_{rel}| \\ & + \langle x''_{cm}|x'_{cm}\rangle \, \langle x'_{cm}|x''_{cm}\rangle \, |x'_{rel}\rangle \, \langle x'_{rel}|] \, . \end{aligned} \tag{9.11}$$

In the limit where the charge on the plates goes to zero, $q\sigma \to 0$, the uncertainty of the position measurement becomes infinite, $\Delta x \to \infty$. This implies that the integral in (9.11) is now over all space instead of two arbitrary bins of the detector. Since the integral is over

all space, the reduced density matrix simplifies to

$$\begin{aligned}\hat{\rho}_{rel} &= \langle x_{cm}|x_{cm}\rangle \, |x_{rel}\rangle \, \langle x_{rel}| + \langle x'_{cm}|x_{cm}\rangle \, |x_{rel}\rangle \, \langle x'_{rel}| \\ &\quad + \langle x_{cm}|x'_{cm}\rangle \, |x'_{rel}\rangle \, \langle x_{rel}| + \langle x'_{cm}|x'_{cm}\rangle \, |x'_{rel}\rangle \, \langle x'_{rel}| \, . \end{aligned} \tag{9.12}$$

The results of the inner products are Gaussians given by (5.45). In the highly localized limit, i.e., $b \to 0$ is considered, then the inner products become delta functions. It is easy to see that since $x_{cm} \neq x'_{cm}$ the second and third terms will vanish and the first and fourth terms will remain, albeit with a divergent coefficient. It follows that,

$$\hat{\rho}_{rel} = \mathcal{N} \left(|x_{rel}\rangle \, \langle x_{rel}| + |x'_{rel}\rangle \, \langle x'_{rel}| \right) , \tag{9.13}$$

where $\mathcal{N}$ is divergent. Notice that this result matches the G-twirl result from 7.3.

In the limit where the charge on the plates becomes strong, $q\sigma \to \infty$, the uncertainty of the position measurement becomes zero $\Delta x \to 0$. However, for these purposes is sufficient to consider Δx to be finite. As a consequence, one cannot utilize the resolution of identity as was done previously since the integral is over a finite subset of x_{cm}. Using the inner product from (5.45) the reduced density matrix becomes,

$$\begin{aligned}\hat{\rho}_{rel} &= \frac{1}{b^2\pi} \int_{x_{cm,i}}^{x_{cm,i}+\Delta x_{cm}} dx''_{cm} \Big[e^{-(x''_{cm}-x_{cm})^2/2b^2} \, |x_{rel}\rangle \, \langle x_{rel}| \\ &\qquad + e^{-(x_{cm}-x''_{cm})^2/4b^2} e^{-(x''_{cm}-x'_{cm})^2/4b^2} \, |x_{rel}\rangle \, \langle x'_{rel}| \\ &\qquad + e^{-(x'_{cm}-x''_{cm})^2/4b^2} e^{-(x''_{cm}-x_{cm})^2/4b^2} \, |x'_{rel}\rangle \, \langle x_{rel}| \\ &\qquad + e^{-(x''_{cm}-x'_{cm})^2/2b^2} \, |x'_{rel}\rangle \, \langle x'_{rel}| \Big] \, . \end{aligned} \tag{9.14}$$

After integrating the coefficients are written as,

$$\begin{aligned}\hat{\rho}_{rel} &= \frac{1}{b\sqrt{2\pi}}\left[\left(\operatorname{erf}\left(\frac{x_{cm}-x_{cm,i}}{b\sqrt{2}}\right)-\operatorname{erf}\left(\frac{x_{cm}-x_{cm,i}-\Delta x_{cm}}{b\sqrt{2}}\right)\right)|x_{rel}\rangle\langle x_{rel}|\right.\\ &\quad +e^{-(x_{cm}-x'_{cm})^2/8b^2}\left(\operatorname{erf}\left(\frac{x_{cm}+x'_{cm}-2x_{cm,i}}{b\sqrt{8}}\right)\right.\\ &\quad \left.\left.-\operatorname{erf}\left(\frac{x_{cm}+x'_{cm}-2(x_{cm,i}+\Delta x_{cm})}{b\sqrt{8}}\right)\right)\times\left(|x_{rel}\rangle\langle x'_{rel}|+|x'_{rel}\rangle\langle x_{rel}|\right)\right.\\ &\quad \left.+\left(\operatorname{erf}\left(\frac{x'_{cm}-x_{cm,i}}{b\sqrt{2}}\right)-\operatorname{erf}\left(\frac{x'_{cm}-x_{cm,i}-\Delta x_{cm}}{b\sqrt{2}}\right)\right)|x'_{rel}\rangle\langle x'_{rel}|\right] \quad (9.15)\end{aligned}$$

where erf denotes the Error function. When the position states become highly localized, i.e., $b \to 0$, the uncertainty of the detector becomes small,i.e., $\Delta x \to 0$. In this limit (9.15) becomes,

$$\begin{aligned}\hat{\rho}_{rel} &= \frac{1}{b}\left[\Theta(x_{cm}-x_{cm,i})-\Theta(x_{cm}-x_{cm,i}-\Delta x)\right]|x_{rel}\rangle\langle x_{rel}|\\ &\quad +\frac{1}{b}\left[\Theta(x'_{cm}-x_{cm,i})-\Theta(x'_{cm}-x_{cm,i}-\Delta x)\right]|x'_{rel}\rangle\langle x'_{rel}|\,, \quad (9.16)\end{aligned}$$

where $\Theta(x)$ denotes the Heaviside step-function. The cross-terms have vanished since the exponential becomes a delta function in the highly localized limit and $x_{cm} \neq x'_{cm}$. When the measured center-of-mass position is within a bin of the detector one of the coefficients of (9.16) are nonzero. As the energy of the system is increased the measured result will be close to $x_{cm,i}$, i.e., $\Delta x \to 0$. This means that the Θ-functions will become closer making the entire coefficient infinitesimally narrow and infinitely tall. The end result will be a mixed state with a density matrix

$$\hat{\rho}_{rel} = \delta(x_{cm}-x_{cm,i})\,|x_{rel}\rangle\langle x_{rel}| + \delta(x'_{cm}-x_{cm,i})\,|x'_{rel}\rangle\langle x'_{rel}|\,, \quad (9.17)$$

matching the results from the Z-model in the sense of (9.4) after a center of mass measurement.

CHAPTER 10

Summary and Outlook

A full theory of quantum gravity requires compatibility between quantum theory and general covariance. However, diffeomorphism-invariant observables, a consequence of general covariance, can be inherently non-local while observables in quantum theory are local, leading to tension. The work presented in this dissertation explored two aspects of this tension. Particularly, how a typical quantum information paradox using local operators cannot be made diffeomorphism invariant in a self-consistent, low-energy manner, and how the (related) relational frameworks of the G-twirl and Z-model affect the energy cost of localizing non-local quantum information.

The FR thought experiment generates a paradox between two local observers in the absence of gravity. However, any thought experiment, especially those that draw deep conclusions about quantum mechanics, should employ gauge-invariant observables to ensure that the conclusions are gauge-invariant and thus physical. Part I discussed the implementation of diffeomorphism invariance, the gauge symmetry of general relativity, on the observables in the FR thought experiment under the assumptions of the experiment, at first order in Newton's constant. Contrary to expectations, the necessary solutions of a perturbative construction were sensitive to quantum gravity due to their non-analytic nature. This result is similar to the results of other paradoxes in weak-field gravity. The paradox in the FR thought experiment can be argued to be ill-posed if quantum gravity forbids non-analytic, bulk solutions, similar to how the principle of boundary unitarity forbids non-analytic solutions in time [28]. The paradox also requires a quantum gravitational analysis, even starting

from weak field gravity, analogous to how semi-classical gravity breaks down even in low curvature spacetimes with closed timelike curves and grandfather paradoxes [26].

Note that no individual piece of physics used was inherently not self-consistent. Without gravity, the local excitations necessary for the FR experiment are perfectly valid. Similarly, the construction of diffeomorphism-invariant observables, at first order in Newton's constant, is perfectly well defined. It is only the combination of the two that leads one to difficulty. Specifically, when one combines *certainty*, which implies exact knowledge of the gravitational field on a hypersurface, *stability*, which implies that the knowledge can be kept immutably in a time-independent manner, and a discrepancy in gravitational observations between two separated observers does one find that the resulting construction cannot be made diffeomorphism invariant in a way that is not sensitive to quantum gravity. Furthermore, if quantum gravity satisfies boundary unitarity, then there would be no possible construction of the FR thought experiment that is gauge invariant.

There is one more similarity with the infaller paradox. An additional way to "resolve" the paradox in the FR thought experiment is with local excitations. Using the same assumptions about certainty and stability as before, let Alice and Bob classically record two different measurements from the same experiment, resulting in their respective, local stress-energy tensors to differ. It follows that Alice's and Bob's spacetime regions are described by differing local metrics, $g^A_{\mu\nu}$ for Alice and $g^B_{\mu\nu}$ for Bob. Assuming a vacuum between Alice and Bob, one could solve the respective constraint equations to extend their metric solutions uniquely onto the three-dimensional spacelike hypersurface at some time t. Of course, one can't do this globally, and therefore along some (otherwise arbitrary) $2D$ surface γ that intersects every curve between Alice and Bob one must have a discontinuity in the metric. The metric that describes the total spacetime is then $g_{\mu\nu} = \Theta(x)g^A_{\mu\nu} + \Theta(-x)g^B_{\mu\nu}$, where, for convenience, the boundary surface γ has been chosen to exist at $x = 0$. Since the metric is discontinuous, substituting it into the Einstein equations would generate a divergent stress-energy tensor along γ, similar to a firewall. This would be the necessary energy momentum configuration

to keep local excitations while maintaining gauge invariance. However, this is not an endorsement of the "solution", but rather an argument that the construction itself cannot be made gauge invariant and that such a solution would amount to a misunderstanding of the quantum gravitational Hilbert space.

Part II implemented the localization of quantum information as an entanglement extraction protocol in relational systems with Gaussian states. One might expect that when full translation invariance is implemented via the G-twirl, leaving only relational states, localization can't matter and there should be no notion of an entanglement extraction process. This is indeed the case, as the G-twirl not only wipes out information on the external frame, it also naturally wipes out any entanglement amongst center-of-mass and relational degrees of freedom that were present in the external partition. The resultant mixed state has no entanglement to extract.

In contrast, if one keeps the external partition information but implements it relationally via a Z-model, then there is entanglement that can be extracted and states can be localized. In this scenario, however, the implementation of the Z-model itself creates a Hamiltonian in which there is an energy difference between the initial (entangled) and final (localized, unentangled) states. Hence the entanglement extraction procedure as outlined in Hackl can be run as expected. It is expected that a similar outcome would be present if one used different otherwise conserved quantities than the energy to label states—entanglement extraction would always require some net gain or loss of some resource (for example charge or angular momentum). Furthermore, it was found there exists a map between the two relational constructions, which can be implemented in the language of POVMs.

The work described in Part II broke the translation invariance and hence the degeneracy of the Hamiltonian via a simple dynamical method imposed by hand, that of a Z-model, or external field. However, gravitational self-interactions would also in principle lead to a non-degenerate Hamiltonian for the initial and final states. In this type of scenario, the extraction and localization process would result in the following: (A) the amount of entanglement

extracted being proportional to the gravitational self-energy difference of the two states, and simultaneously, (B) the system becoming more localized in space. This qualitatively reflects black hole thermodynamics and other holographic approaches connecting entanglement and gravitational dynamics (see [28] for an example). Whether or not implementing such a localization/extraction protocol in this framework quantitatively matches black hole physics is left for future work.

LIST OF REFERENCES

[1] R. L. Workman and Others. Review of Particle Physics. *PTEP*, 2022:083C01, 2022.

[2] R. Haag. *Local Quantum Physics: Fields, Particles, Algebras*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 1996.

[3] Eyvind H. Wichmann and James H. Crichton. Cluster decomposition properties of the s matrix. *Phys. Rev.*, 132:2788–2799, Dec 1963.

[4] Steven Weinberg. What is quantum field theory, and what did we think it is?, 1997.

[5] Jan Walleczek and Gerhard Grössing. Nonlocal quantum information transfer without superluminal signalling and communication. *Foundations of Physics*, 46(9):1208–1228, jan 2016.

[6] J. von Neumann. *Mathematical Foundations of Quantum Mechanics*. Goldstine Printed Materials. Princeton University Press, 1955.

[7] P.A.M. Dirac. *The Principles of Quantum Mechanics*. Comparative Pathobiology - Studies in the Postmodern Theory of Education. Clarendon Press, 1981.

[8] Edward Witten. APS medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory. *Reviews of Modern Physics*, 90(4), oct 2018.

[9] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Reviews of Modern Physics*, 81(2):865–942, jun 2009.

[10] M. Henneaux and C. Teitelboim. *Quantization of gauge systems*. Princeton University Press, Princeton, NJ, USA, 1992.

[11] P. A. M. Dirac. Gauge-invariant formulation of quantum electrodynamics. *Canadian Journal of Physics*, 33(11):650–660, 1955.

[12] Clifford M. Will. The confrontation between general relativity and experiment. *Living Reviews in Relativity*, 17(1), jun 2014.

[13] C. J. Isham. Conceptual and geometrical problems in quantum gravity. In H. Mitter and H. Gausterer, editors, *Recent Aspects of Quantum Fields*, pages 123–229, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.

[14] Claus Kiefer. *Quantum Gravity*. Oxford University Press, 04 2012.

[15] Donald Marolf. Unitarity and holography in gravitational physics. *Physical Review D*, 79(4), Feb 2009.

[16] Chiara Marletto and Vlatko Vedral. Gravitationally-induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. *Phys. Rev. Lett.*, 119(24):240402, 2017.

[17] Sougato Bose, Anupam Mazumdar, Gavin W. Morley, Hendrik Ulbricht, Marko Toroš, Mauro Paternostro, Andrew A. Geraci, Peter F. Barker, M. S. Kim, and Gerard Milburn. Spin entanglement witness for quantum gravity. *Phys. Rev. Lett.*, 119:240401, Dec 2017.

[18] William Donnelly and Steven B. Giddings. Gravitational splitting at first order: Quantum information localization in gravity. *Physical Review D*, 98(8), oct 2018.

[19] Adam Dukehart and David Mattingly. Diffeomorphism invariance and quantum mechanical paradoxes. Under review at International Journal of Modern Physics D.

[20] Časlav Brukner. A no-go theorem for observer-independent facts. *Entropy*, 20(5):350, May 2018.

[21] Dustin Lazarovici and Mario Hubert. How quantum mechanics can consistently describe the use of itself, 2018.

[22] Marijn Waaijer and Jan van Neerven. Relational analysis of the frauchiger–renner paradox and interaction-free detection of records from the past, 2019.

[23] Florian J. Boge. Quantum information versus epistemic logic: An analysis of the frauchiger–renner theorem. *Foundations of Physics*, 49(10):1143–1165, Sep 2019.

[24] Marcus Gaul and Carlo Rovelli. Loop quantum gravity and the meaning of diffeomorphism invariance, 1999.

[25] J. Friedman, M. S. Morris, I. D. Novikov, F. Echeverria, G. Klinkhammer, K. S. Thorne, and U. Yurtsever. Cauchy problem in space-times with closed timelike curves. *Phys. Rev.*, D42:1915–1930, 1990.

[26] S. W. Hawking. The Chronology protection conjecture. *Phys. Rev.*, D46:603–611, 1992.

[27] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, and James Sully. Black Holes: Complementarity or Firewalls? *JHEP*, 02:062, 2013.

[28] Ted Jacobson and Phuc Nguyen. Diffeomorphism invariance and the black hole information paradox. *Phys. Rev. D*, 100:046002, Aug 2019.

[29] Steven B. Giddings, Donald Marolf, and James B. Hartle. Observables in effective gravity. *Physical Review D*, 74(6), sep 2006.

[30] Sabine Hossenfelder. Minimal length scale scenarios for quantum gravity. *Living Reviews in Relativity*, 16(1), jan 2013.

[31] Kenichi Konishi, Giampiero Paffuti, and Paolo Provero. Minimum physical length and the generalized uncertainty principle in string theory. *Physics Letters B*, 234(3):276–284, 1990.

[32] Tao Han and Scott Willenbrock. Scale of quantum gravity. *Physics Letters B*, 616(3-4):215–220, jun 2005.

[33] I. Hinchliffe, N. Kersting, and Y. L. Ma. Review of the phenomenology of noncommutative geometry. *International Journal of Modern Physics A*, 19(02):179–204, jan 2004.

[34] Cédric Bény, Christopher T. Chubb, Terry Farrelly, and Tobias J. Osborne. Energy cost of entanglement extraction in complex quantum systems. *Nature Communications*, 9(1), sep 2018.

[35] Lucas Hackl and Robert H. Jonsson. Minimal energy cost of entanglement extraction. *Quantum*, 3:165, jul 2019.

[36] Adam Dukehart and David Mattingly. Energy cost of localization of relational quantum information. *Phys. Rev. D*, 107:126020, 2023.

[37] Alexander R. H. Smith, Marco Piani, and Robert B. Mann. Quantum reference frames associated with noncompact groups: The case of translations and boosts and the role of mass. *Physical Review A*, 94(1), jul 2016.

[38] E. P. Wigner. *Remarks on the Mind-Body Question*, pages 247–260. Springer Berlin Heidelberg, Berlin, Heidelberg, 1995.

[39] Daniela Frauchiger and Renato Renner. Quantum theory cannot consistently describe the use of itself. *Nature Communications*, 9(1):3711, 2018.

[40] Kok-Wei Bong, Aníbal Utreras-Alarcón, Farzad Ghafari, Yeong-Cherng Liang, Nora Tischler, Eric G Cavalcanti, Geoff J Pryde, and Howard M Wiseman. A strong no-go theorem on the Wigner's friend paradox. *Nature Physics*, pages 1–7, 2020.

[41] Massimiliano Proietti, Alexander Pickston, Francesco Graffitti, Peter Barrow, Dmytro Kundys, Cyril Branciard, Martin Ringbauer, and Alessandro Fedrizzi. Experimental rejection of observer-independence in the quantum world. *arXiv e-prints*, page arXiv:1902.05080, Feb 2019.

[42] Sebastian Fortin and Olimpia Lombardi. The frauchiger-renner argument: A new no-go result? *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 70:1 – 7, 2020.

[43] Clive Aw, Michele Dall'Arno, and Valerio Scarani. Knowledge by direct measurement versus inference from steering. *Quantum Studies: Mathematics and Foundations*, 7(2):247–254, Jun 2020.

[44] Dustin Lazarovici and Mario Hubert. How quantum mechanics can consistently describe the use of itself. *Scientific Reports*, 9(1):470, Jan 2019.

[45] Renato Renner. Quantum Information and the Foundations of Quantum Physics. Quantum Information and the Foundations of Quantum Physics. Nov 2019.

[46] Winston Fairbairn and Carlo Rovelli. Separable hilbert space in loop quantum gravity. *Journal of Mathematical Physics*, 45(7):2802–2814, Jul 2004.

[47] William Donnelly and Steven B. Giddings. How is quantum information localized in gravity? *Physical Review D*, 96(8), Oct 2017.

[48] F.W. Lipps and H.A. Tolhoek. Polarization phenomena of electrons and photons. ii: Results for compton scattering. *Physica*, 20(1):395 – 405, 1954.

[49] R. Ferrari, L. E. Picasso, and F. Strocchi. Some remarks on local operators in quantum electrodynamics. *Comm. Math. Phys.*, 35(1):25–38, 1974.

[50] William Donnelly and Steven B. Giddings. Observables, gravitational dressing, and obstructions to locality and subsystems. *Phys. Rev.*, D94(10):104038, 2016.

[51] Alessandro Carlotto and Richard Schoen. Localizing solutions of the einstein constraint equations, 2014.

[52] James Isenberg, David Maxwell, and Daniel Pollack. A gluing construction for non-vacuum solutions of the einstein constraint equations, 2005.

[53] Alberto Salvio. Quadratic gravity. *Frontiers in Physics*, 6, Aug 2018.

[54] Steven Krantz and Harold Parks. *A Primer of Real Analytic Functions*. 01 2002.

[55] X WANG, T HIROSHIMA, A TOMITA, and M HAYASHI. Quantum information with gaussian states. *Physics Reports*, 448(1-4):1–111, aug 2007.

[56] Esteban Castro-Ruiz and Ognyan Oreshkov. Relative subsystems and quantum reference frame transformations, 2021.

[57] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory*. Advanced book classics. Avalon Publishing, 1995.

[58] Kevin E. Cahill and Roy J. Glauber. Density operators for fermions. *Physical Review A*, 59(2):1538–1555, feb 1999.

[59] Lucas Hackl and Eugenio Bianchi. Bosonic and fermionic gaussian states from kähler structures. *SciPost Physics Core*, 4(3), sep 2021.

[60] E. Munguía-González, S. Rego, and J. K. Freericks. Making squeezed-coherent states concrete by determining their wavefunction. *American Journal of Physics*, 89(9):885–896, sep 2021.

[61] Richard Jozsa and Noah Linden. On the role of entanglement in quantum-computational speed-up. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 459(2036):2011–2032, aug 2003.

[62] Soumik Ghosh, Abhinav Deshpande, Dominik Hangleiter, Alexey V. Gorshkov, and Bill Fefferman. Sharp complexity phase transitions generated by entanglement. *Physical Review Letters*, 131(3), jul 2023.

[63] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, Mar 1993.

[64] Markus P. Müller and Cozmin Ududec. Structure of reversible computation determines the self-duality of quantum theory. *Physical Review Letters*, 108(13), mar 2012.

[65] Petar Simidzija, Robert H. Jonsson, and Eduardo Martín-Martínez. General no-go theorem for entanglement extraction. *Physical Review D*, 97(12), jun 2018.

[66] Augustin Vanrietvelde, Philipp A. Hoehn, Flaminia Giacomini, and Esteban Castro-Ruiz. A change of perspective: switching quantum reference frames via a perspective-neutral framework. *Quantum*, 4:225, jan 2020.

[67] Stephen D. Bartlett, Terry Rudolph, and Robert W. Spekkens. Reference frames, superselection rules, and quantum information. *Reviews of Modern Physics*, 79(2):555–609, apr 2007.

[68] Renato M Angelo, Nicolas Brunner, Sandu Popescu, Anthony J Short, and Paul Skrzypczyk. Physics within a quantum reference frame. *Journal of Physics A: Mathematical and Theoretical*, 44(14):145304, mar 2011.

[69] Philipp A. Höhn, Alexander R. H. Smith, and Maximilian P. E. Lock. Trinity of relational quantum dynamics. *Physical Review D*, 104(6), sep 2021.

[70] Smith, Alexander R. H. *Detectors, Reference Frames, and Time*. PhD thesis, 2017.

[71] Marius Krumm, Philipp A. Höhn, and Markus P. Müller. Quantum reference frame transformations as symmetries and the paradox of the third particle. *Quantum*, 5:530, aug 2021.

[72] Philipp A. Höhn, Marius Krumm, and Markus P. Müller. Internal quantum reference frames for finite abelian groups. *Journal of Mathematical Physics*, 63(11):112207, nov 2022.

[73] Anne-Catherine de la Hamette, Thomas D. Galley, Philipp A. Hoehn, Leon Loveridge, and Markus P. Mueller. Perspective-neutral approach to quantum frame covariance for general symmetry groups, 2021.

[74] Nicholas Wheeler. Remarks concerning the status & some ramifications of ehrenfest's theorem, 1998.

[75] Sebastian Deffner and Steve Campbell. Quantum speed limits: from heisenberg's uncertainty principle to optimal quantum control. *Journal of Physics A: Mathematical and Theoretical*, 50(45):453001, oct 2017.