Wigner's Theorem for Semi-finite von Neumann Algebras

Devin Daley Gent
University of New Hampshire, Durham

WIGNER'S THEOREM FOR SEMI-FINITE VON NEUMANN ALGEBRAS

By

Devin Daley Gent

BA, Rivier University, 2016

DISSERTATION

Submitted to the University of New Hampshire

in Partial Fulfillment of

the Requirements for the Degree of

Doctor of Philosophy

in

Mathematics

May, 2023

This dissertation has been examined and approved in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics by:

Dissertation Director, Junhao Shen, Professor of Mathematics

Maria Basterra, Professor of Mathematics

Donald Hadwin, Professor of Mathematics

Rita Hibschweiler, Professor of Mathematics

Edward Hinson, Associate Professor of Mathematics

On April 6th, 2023

Original approval signatures are on file with the University of New Hampshire Graduate School.

ACKNOWLEDGEMENTS

In pursuing this degree there have been many people who have helped me, supported me, and made this a wonderful experience.

I would like to thank my family for always encouraging my education, interests, and passions. I am lucky to have been surrounded by people willing to support me in whichever path I chose to pursue.

I would like to thank Crystal for believing in me the most and for giving me reasons to smile on cloudy days.

I would like to thank Olga Chuyan and Teresa Magnus of Rivier University, who nurtured and developed my love of mathematics. They laid the foundation that led me to this point.

I would like to thank the *entirety* of the UNH Mathematics department– faculty, staff, and fellow students– for their endless support and for making the third floor of Kingsbury Hall a home away from home. This experience has been everything I could have wanted. In particular, I would like to thank Emily and Sukitha for sharing this journey, their kindness, and their friendship with me.

I would like to thank my dissertation committee; Junhao Shen, Maria Basterra, Don Hadwin, Rita Hibschweiler, and Ed Hinson; for not only their help in preparing this dissertation but also for each playing a huge part in my development as a graduate student through classes, meetings, and discussions.

Finally, I would like to thank my advisor, Junhao Shen, for being an incredible mentor to me over these years. His kindness, enthusiasm, and encouragement have not only helped me reach this point but have also shaped who I want to be as a mathematician and scholar.

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ABSTRACT

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Wigner's theorem, an important result in quantum mechanics, shows that a transition probability preserving bijection on the set of rank-one projections on a Hilbert space $\mathcal{H}$ extends to a Jordan $*$-isomorphism ρ on the algebra of all bounded linear operators on $\mathcal{H}$. In this work we consider generalizations of Wigner's theorem within the context of von Neumann algebras.

Let $\mathcal{M}$ and $\mathcal{N}$ be two semi-finite von Neumann algebras with faithful normal semi-finite tracial weights $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively. Suppose that $\mathcal{M}$ has no direct summand of type I_2 and is either diffuse or atomic such that every minimal projection in $\mathcal{M}$ has trace 1. Let $c \in (0, \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2})$, $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) = c\}$, and $0 < p < \infty$. In this work we prove generalizations of Wigner's theorem along two main lines. First, we show that every ortho-isomorphism $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ can be extended to a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$. Second, we show that if $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ acts as an L^p-isometry on commuting pairs of projections, then φ extends to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ when $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) < \infty$ or $0 < p \leq 2$.

CHAPTER 1
INTRODUCTION

A major result in quantum mechanics, proved by Eugene Wigner [30] in 1931, is that every symmetry transformation on ray space is induced by a unitary or anti-unitary transformation. More specifically, let $\mathcal{H}$ be a complex Hilbert space, $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{P}_1$ be the set of all rank-one projections in $B(\mathcal{H})$. With τ the canonical trace on $B(\mathcal{H})$, we can restate Wigner's theorem in mathematical language as follows.

Theorem 1.1.1 (Wigner [30]). *If $\varphi : \mathcal{P}_1 \to \mathcal{P}_1$ is a bijection which satisfies*

$$\tau(EF) = \tau(\varphi(E)\varphi(F))$$

for all $E, F \in \mathcal{P}_1$, then there exists a unitary or anti-unitary operator $U \in B(\mathcal{H})$ such that $\varphi(E) = UEU^*$ for all $E \in \mathcal{P}_1$.

In the literature such a map φ satisfying $\tau(EF) = \tau(\varphi(E)\varphi(F))$ for all $E, F \in \mathcal{P}_1$ is said to be *transition probability preserving*. Since its initial formulation Wigner's theorem has been studied, extended, and generalized in different directions such as in [1, 4, 6–9, 15–19, 21, 22, 24–26, 29]. Notably, in 1962 Uhlhorn [28] showed that when $\dim \mathcal{H} \geq 3$ the condition that $\tau(EF) = \tau(\varphi(E)\varphi(F))$ for *all* $E, F \in \mathcal{P}_1$ could be replaced by a weaker condition:

$$\tau(EF) = 0 \text{ if and only if } \tau(\varphi(E)\varphi(F)) = 0.$$

As $B(\mathcal{H})$ is a prototypical example of a type I factor, much work has been done to generalize Wigner's theorem by replacing $B(\mathcal{H})$ with various different factors. In this work we consider extensions of Wigner's theorem in the context of von Neumann algebras more generally.

Suppose $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. We let $\mathcal{P}(\mathcal{M})$ denote the set of projections in $\mathcal{M}$. If $c \in [0, \tau_{\mathcal{M}}(I_{\mathcal{M}})]$, we let $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ denote the Grassmann space of projections $E \in \mathcal{M}$ with trace $\tau_{\mathcal{M}}(E) = c$. For each $0 < p < \infty$, we let $\|X\|_p = (\tau_{\mathcal{M}}(|X|^p))^{1/p}$ for each X in $\mathcal{M}$ and let $L^p(\mathcal{M}, \tau_{\mathcal{M}})$ be the non-commutative L^p-space associated with $(\mathcal{M}, \tau_{\mathcal{M}})$. It is easily verified that the requirement that φ be transition probability preserving in Wigner's theorem is equivalent to the requirement that φ be an L^2-isometry. Alternatively, if $c \in (0, \tau_{\mathcal{M}}(I_{\mathcal{M}}))$ and $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, we have that $E \perp F$ precisely when

$$\tau_{\mathcal{M}}(EF) = \tau_{\mathcal{M}}(FEF) = \tau_{\mathcal{M}}((EF)^*EF) = 0.$$

In particular a map $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ satisfies

$$\tau_{\mathcal{M}}(EF) = 0 \text{ if and only if } \tau_{\mathcal{M}}(\psi(E)\psi(F)) = 0$$

for every pair $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ if and only if φ preserves orthogonality in both directions. By a result of Kadison [13], a map $\rho : B(\mathcal{H}) \to B(\mathcal{H})$ is induced by a unitary operator or anti-unitary operator if and only if ρ is a Jordan $*$-isomorphism. With updated terminology Uhlhorn's result becomes the following.

Theorem 1.1.2 (Uhlhorn [28]). *Suppose $\mathcal{H}$ is a complex Hilbert space such that* $\dim \mathcal{H} \geq 3$. *If $\varphi : \mathcal{P}_1 \to \mathcal{P}_1$ is an ortho-isomorphism on the set of rank one projections in $B(\mathcal{H})$, then φ extends to a Jordan $*$-isomorphism $\rho : B(\mathcal{H}) \to B(\mathcal{H})$.*

Thus Wigner and Uhlhorn present two different lines of inquiry: L^p-isometries and ortho-isomorphisms.

Uhlhorn's result has been studied extensively. For example, it was shown in [24] that, for an infinite dimensional Hilbert space, an ortho-isomorphism φ from $\mathcal{P}_n$ onto $\mathcal{P}_n$ is induced by a unitary or anti-unitary operator. A combination of results in [25, 26] proves that, for a factor $\mathcal{M}$ of type II and $c \in (0, \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2})$, an ortho-isomorphism φ from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ onto $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ can be extended to a Jordan $*$-isomorphism on $\mathcal{M}$.

On the subject of L^p-isometries, it has been shown in [22] that, given a semi-finite factor $\mathcal{M}$ and $c \in (0, \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2})$, any, potentially non-surjective, L^2-isometry $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ can be extended to a Jordan $*$-homomorphism. In the surjective case, there have been a number of results including for L^2-isometries on the projection lattice [29], L^p-isometries on a type II factor [25], and L^p-isometries between different semi-finite factors [21].

In pursuing these subjects, most generalizations deal with surjective mappings on a semi-finite factor $\mathcal{M}$. In this work we will study, more generally, maps from a semi-finite *von Neumann algebra* $\mathcal{M}$, which is either diffuse or atomic such that all minimal projections have the same trace. We will pursue two lines of inquiry. First, we consider when an ortho-isomorphism on a subset $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ of a semi-finite von Neumann $\mathcal{M}$ can be extended to a Jordan $*$-isomorphism on $\mathcal{M}$. Adapting methods used in [25,26], we shall prove the following result in this project.

Theorem 4.3.1. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra without a direct summand of type I_2 and with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. Let $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$. Suppose the following are true.*

(a) $\mathcal{M}$ is either diffuse or atomic.

(b) For an atomic von Neumann algebra $\mathcal{M}$, we assume that

(b_1) $\tau_{\mathcal{M}}$ is the canonical tracial weight satisfying

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b_2) c is a positive integer.

Suppose $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is a bijection such that, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*,*

$$E \perp F \text{ if and only if } \varphi(E) \perp \varphi(F).$$

Then there exists a Jordan $*$*-isomorphism* $\rho : \mathcal{M} \to \mathcal{M}$ *such that*

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

Second, we will consider when a possibly *non-surjective* map between Grassmann spaces of semi-finite von Neumann algebras which acts as an L^p-isometry *on commuting pairs of projections* can be extended to a trace-preserving Jordan $*$-homomorphism. We prove the following two theorems in this work.

Theorem 5.5.2. *Let* $\mathcal{M}$ *be a semi-finite von Neumann algebra without a direct summand of type* I_2 *and with a faithful normal semi-finite tracial weight* $\tau_{\mathcal{M}}$*. Let* $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$*. Suppose that the following are true.*

(a) $\mathcal{M}$ *is either diffuse or atomic.*

(b) For an atomic von Neumann algebra $\mathcal{M}$*, we assume that*

(b$_1$*)* $\tau_{\mathcal{M}}$ *is the canonical tracial weight satisfying*

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b$_2$*)* c *is a positive integer.*

Let $\mathcal{N}$ *be a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight* $\tau_{\mathcal{N}}$*.*

Assume that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) < \infty$ *and* $0 < p < \infty$*. Suppose* $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to$

$\mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *is a map such that, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\text{if } EF = FE, \text{ then } \|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p \,.$$

Then there exists a trace-preserving Jordan $*$*-homomorphism* $\rho : \mathcal{M} \to \mathcal{N}$ *such that*

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

Theorem 5.5.5. *Let* $\mathcal{M}$ *be a semi-finite von Neumann algebra without a direct summand of type* I_2 *and with a faithful normal semi-finite tracial weight* $\tau_{\mathcal{M}}$*. Let* $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$*. Suppose that the following are true.*

(a) $\mathcal{M}$ *is either diffuse or atomic.*

(b) For an atomic von Neumann algebra $\mathcal{M}$*, we assume that*

(b_1) $\tau_{\mathcal{M}}$ *is the canonical tracial weight satisfying*

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b_2) c *is a positive integer with* $c \geq 2$.

Let $\mathcal{N}$ *be a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight* $\tau_{\mathcal{N}}$.

Assume that $0 < p \leq 2$*. Suppose* $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *is a map such that, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\text{if } EF = FE, \text{ then } \|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p \,.$$

Then there exists a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

This work is structured as follows.

In Chapter 2, we provide an outline, without proofs, of the essentials of von Neumann algebra theory which will be required for our investigation into the extension of Wigner's theorem.

In Chapter 3, we fix terminology and notation specific to the extension of maps on Grassmann spaces. We also develop necessary lemmas on tracial weights, orthogonality preserving maps, and diffuse and atomic von Neumann algebras.

In Chapter 4, we consider the extension of ortho-isomorphisms to Jordan $*$-isomorphisms. We consider the case when $\mathcal{M}$ is diffuse in Section 4.1. In Section 4.2, we assume $\mathcal{M}$ is atomic and that every minimal projection in $\mathcal{M}$ has the same trace. We complete our coverage of ortho-isomorphisms with the proof of our first main result and provide a few examples of its application.

In Chapter 5, we turn our attention to potentially non-surjective mappings of the form $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$, with $\mathcal{M}, \mathcal{N}$ semi-finite von Neumann algebras, where φ is transition probability preserving for commuting pairs of projections. We proceed by cases according to the values of $\tau_{\mathcal{M}}(I_{\mathcal{M}})$ and $\tau_{\mathcal{N}}(I_{\mathcal{N}})$. First, in Section 5.1, we consider the case where $\mathcal{M}$ and $\mathcal{N}$ are finite von Neumann algebras with tracial states such that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = 1$. In Section 5.2, we investigate the case where $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = \infty$. In Section 5.3, we give an extension when $\tau_{\mathcal{M}}(I_{\mathcal{M}}) \leq \tau_{\mathcal{N}}(I_{\mathcal{N}})$. We collect these results in Section 5.4. Finally, in Section 5.5, we observe that a map which acts as an L^p-isometry on commuting pairs of projections is also transition probability preserving for commuting pairs. We end Chapter 5 by combining this observation with our findings from Sections 5.1 through 5.3 to prove our remaining two main results.

CHAPTER 2

BACKGROUND

In this chapter we outline the fundamentals of von Neumann algebra theory. This chapter is not intended to be a comprehensive review, but rather a convenient overview of the core concepts forming the backbone of this work. Our focus will remain on how these concepts relate to the overarching goals of proving a Wigner's theorem type extension for semi-finite von Neumann algebras. For simplicity, we will only consider vector spaces over the complex numbers in our overview. Our development in this chapter is primarily based on Kadison's comprehensive coverage in [14]. For further reading see [11,23,27]. In particular, we use [27] as our reference for faithful normal semi-finite tracial weights. For additional coverage of Jordan $*$-homomorphisms and Jordan $*$-isomorphisms, see [11]. For more information on non-commutative L^p-spaces, see [20].

2.1 Linear spaces

We start by considering complex vector spaces.

Definition 2.1.1. A *norm* on a complex vector space $\mathcal{X}$ is a map $\|\cdot\| : \mathcal{X} \to [0, \infty)$ such that, for all $x, y \in \mathcal{X}$ and $\alpha \in \mathbb{C}$,

(i) $\|x\| > 0$ if $x \neq 0$;

(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

We say, in this case, that $(\mathcal{X}, \|\cdot\|)$ is a *normed space.*

Remark 2.1.2. If $(\mathcal{X}, \|\cdot\|)$ is a normed space, then $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ given by

$$d(x, y) = \|x - y\|, \qquad \text{for all } x, y \in \mathcal{X}$$

defines a metric d on $\mathcal{X}$.

Of particular interest to us are complete normed spaces whose norm has been induced from an inner product.

Definition 2.1.3. An *inner product* on a complex vector space $\mathcal{H}$ is a mapping $(x, y) \mapsto \langle x, y\rangle$ from $\mathcal{H} \times \mathcal{H}$ into $\mathbb{C}$ such that, for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$,

(i) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

(ii) $\langle y, x\rangle = \overline{\langle x, y\rangle}$, where $\overline{\langle x, y\rangle}$ is the complex conjugate of $\langle x, y\rangle$;

(iii) $\langle x, x\rangle \geq 0$.

If, further,

(iv) $\langle x, x\rangle \neq 0$ for all $x \neq 0$;

then $\|x\| = \langle x, x\rangle^{1/2}$ defines a norm on $\mathcal{H}$. If $\mathcal{H}$ is complete with respect to the metric induced by this norm (i.e. if every Cauchy sequence converges), then $\mathcal{H}$ is said to be a *Hilbert space.*

Example 2.1.4. For each n, $\mathbb{C}^n$ is a Hilbert space with an inner product given by $\langle z, w\rangle = z \cdot \overline{w}$ for all $z, w \in \mathbb{C}^n$.

In the following, we turn our attention to spaces which have a multiplicative structure.

Definition 2.1.5. A *Banach algebra* is an associative algebra $\mathcal{A}$ over $\mathbb{C}$ (i.e. with addition, multiplication, and scalar multiplication) which is simultaneously a complete normed space satisfying the following:

(i) $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in \mathcal{A}$;

(ii) $\|I\| = 1$ where I is the multiplicative identity of $\mathcal{A}$.

Definition 2.1.6. An *involution* on a Banach algebra $\mathcal{A}$ is a mapping $A \mapsto A^*$, from $\mathcal{A}$ into $\mathcal{A}$, such that, for all $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$,

(i) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$;

(ii) $(AB)^* = B^* A^*$;

(iii) $(A^*)^* = A$.

If, further,

(iv) $\|A^* A\| = \|A\|^2$;

then $\mathcal{A}$ is said to be a *C^*-algebra.*

2.2 The operator algebra $B(\mathcal{H})$

Let $\mathcal{H}$ be a Hilbert space.

Definition 2.2.1. A linear map $T : \mathcal{H} \to \mathcal{H}$ is said to be a *bounded linear operator* if

$$\sup_{x \in \mathcal{H},\ \|x\| \leq 1} \|Tx\| < \infty.$$

The space, $B(\mathcal{H})$, of all bounded linear operators of the form $T : \mathcal{H} \to \mathcal{H}$ is a C^*-algebra. Addition, multiplication, and scalar multiplication are defined for $S, T \in B(\mathcal{H})$ and $\alpha \in \mathbb{C}$ via

- $(S + T)(x) = S(x) + T(x)$;
- $(ST)(x) = (S \circ T)(x)$;
- $(\alpha T)(x) = \alpha \cdot T(x)$;

for all $x \in \mathcal{H}$.

The norm of a bounded operator $T \in B(\mathcal{H})$ is given by

$$\|T\| = \sup_{x \in \mathcal{H},\ \|x\| \leq 1} \|Tx\| .$$

For each $T \in B(\mathcal{H})$ there is a unique $T^* \in B(\mathcal{H})$, called the *adjoint* of T, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$. The mapping $T \mapsto T^*$ defines an involution on $B(\mathcal{H})$.

Definition 2.2.2. We say that

(a) $T \in B(\mathcal{H})$ is *self-adjoint* if $T = T^*$;

(b) $T \in B(\mathcal{H})$ is *positive*, denoted $0 \leq T$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$;

(c) $U \in B(\mathcal{H})$ is *unitary* if $T^*T = TT^* = I$, or equivalently if $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$;

(d) $U : \mathcal{H} \to \mathcal{H}$ is *anti-unitary* if $\langle Ux, Uy\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in \mathcal{H}$;

Remark 2.2.3. If $T \in B(\mathcal{H})$, then $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle \geq 0$. Thus $T^*T \geq 0$.

2.3 von Neumann algebras

Let $\mathcal{H}$ be a Hilbert space. We concern ourselves with self-adjoint unital subalgebras of $B(\mathcal{H})$ which are closed under the following topology.

Definition 2.3.1. A net $\{T_\lambda\} \subseteq B(\mathcal{H})$ converges to $T \in B(\mathcal{H})$ in the *weak-operator topology* if and only if $\langle T_\lambda x, y\rangle$ converges to $\langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$.

Definition 2.3.2. A *von Neumann algebra* is a C^*-algebra $\mathcal{M}$, contained in $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, which is closed in the weak-operator topology and contains I. If every pair of elements in $\mathcal{M}$ commute, then $\mathcal{M}$ is said to be an *abelian* von Neumann algebra. A *factor*, $\mathcal{R}$, is a von Neumann algebra whose center consists solely of scalar multiples of I.

Example 2.3.3. For any Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ is a factor.

Remark 2.3.4. We denote the set of positive elements in $\mathcal{M}$ by $\mathcal{M}^+$. This set induces an ordering on the elements of $\mathcal{M}$ by defining, for $S, T \in \mathcal{M}$, that $S \leq T$ if and only if $(T - S) \in \mathcal{M}^+$.

Definition 2.3.5. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two von Neumann algebras. A linear map $\rho : \mathcal{M} \to \mathcal{N}$ satisfying $\rho(A^*) = \rho(A)^*$ for all $A \in \mathcal{M}$ and $\rho(A^2) = \rho(A)^2$ for all self-adjoint $A \in \mathcal{M}$ is said to be a *Jordan $*$-homomorphism*. A bijective Jordan $*$-homomorphism is called a *Jordan $*$-isomorphism*.

2.4 Projections

Let $\mathcal{M}$ be a von Neumann algebra. For our work on an extension of Wigner's theorem, we focus primarily on the set of projections in $\mathcal{M}$.

Definition 2.4.1. A *projection* in $\mathcal{M}$ is a self-adjoint, idempotent, linear operator. We denote the set of projections in $\mathcal{M}$ by $\mathcal{P}(\mathcal{M}) = \{E \mid E = E^2 = E^*\}$.

Remark 2.4.2. If $E \in \mathcal{P}(\mathcal{M})$ is a projection, then $E\mathcal{M}E = \{ETE \mid T \in \mathcal{M}\}$ is a von Neumann algebra.

The following notation and terminology on projections in a von Neumann algebra will be essential in this work.

Definition 2.4.3. Let $E, F \in \mathcal{P}(\mathcal{M})$. We say that

(a) E is a *subprojection* of F, denoted $E \leq F$, if $EF = FE = E$;

(b) $E \neq 0$ is *minimal* if E has no non-zero subprojections;

(c) E and F are *orthogonal*, denoted $E \perp F$, if $EF = FE = 0$;

(d) E and F are *equivalent* relative to $\mathcal{M}$, denoted $E \sim F$, if $E = V^*V$ and $F = VV^*$ for some $V \in \mathcal{M}$;

(e) E is *weaker* than F, denoted $E \precsim F$, if E is equivalent to a subprojection of F;

(f) E is *abelian* if $E\mathcal{M}E$ is abelian;

(g) E is *infinite* relative to $\mathcal{M}$ if there exists some $P \in \mathcal{P}(\mathcal{M})$ such that $P \leq E$, $P \neq E$, and $P \sim E$;

(h) E is *properly infinite* relative to $\mathcal{M}$ if E is infinite and for each central $P \in \mathcal{P}(\mathcal{M})$ either $PE = 0$ or PE is infinite;

(i) E is *finite* relative to $\mathcal{M}$ if E is not infinite relative to $\mathcal{M}$.

Remark 2.4.4. Let $E, F \in \mathcal{P}(\mathcal{M})$. E is a subprojection of F if and only if $F - E$ is a positive operator. Thus the ordering of Definition 2.4.3 (a) agrees with that of Remark 2.3.4

Remark 2.4.5. The set of projections in $B(\mathcal{H})$ is in one-to-one correspondence with the set of closed subspaces of $\mathcal{H}$, where each projection E is associated to the subspace $E(\mathcal{H})$. In particular, we can define the *range projection* of an operator $T \in B(\mathcal{H})$, denoted $R(T)$, as the projection associated to the closure of $T(\mathcal{H})$.

Example 2.4.6. The set of minimal projections in $B(\mathcal{H})$ is precisely the set of rank-one projections, $\mathcal{P}_1$.

Remark 2.4.7. Every projection is a subprojection of I, and 0 is a subprojection of every projection. Two projections E and F are orthogonal if and only if $E \leq I - F$. We will denote $F^\perp = I - F$. If $E \leq F$, then $F - E$ is a projection and $E \precsim F$. If $E \perp F$, then $E + F$ is a projection.

Definition 2.4.8. Let $\{E_\lambda\}$ be a family of projections in $\mathcal{M}$. We define the *union* of the family $\{E_\lambda\}$, denoted $\bigvee_\lambda E_\lambda$, as the least upper bound of the family within the set of projections. Correspondingly we define the *intersection*, $\bigwedge_\lambda E_\lambda$, as the greatest lower bound of the family $\{E_\lambda\}$ within the set of projections.

Remark 2.4.9. Let $E, F \in \mathcal{P}(\mathcal{M})$. If $E \perp F$, then $E \vee F = E + F$. The projections E and F commute if and only if $EF = E \wedge F$. When this is the case, so that $EF = FE$, we have $E \vee F = E + F - EF$.

We can classify von Neumann algebras by the behavior of projections as follows.

Definition 2.4.10. For a von Neumann algebra $\mathcal{M}$ with idenity I, we say that

(a) $\mathcal{M}$ is a *finite* von Neumann algebra if I is finite relative to $\mathcal{M}$;

(b) $\mathcal{M}$ is a *properly infinite* von Neumann algebra if I is properly infinite relative to $\mathcal{M}$;

(c) $\mathcal{M}$ is a *diffuse* von Neumann algebra if M has no minimal projections;

(d) $\mathcal{M}$ is an *atomic* von Neumann algebra if for every non-zero projection $P \in \mathcal{P}(\mathcal{M})$ there exists a non-zero minimal projection $E \in \mathcal{P}(\mathcal{M})$ with $E \leq P$.

Remark 2.4.11. In an atomic von Neumann algebra every projection can be written as a sum of minimal projections.

2.5 Type decomposition of a von Neumann algebra

Let $\mathcal{M}$ be a von Neumann algebra.

Definition 2.5.1. The *central carrier*, C_A, of an operator $A \in \mathcal{M}$ is the projection $C_A = I - \bigvee_\lambda P_\lambda$ where $\{P_\lambda\}$ is the family of all projections $P_\lambda \in \mathcal{P}(\mathcal{M})$ satisfying $P_\lambda A = 0$.

The Murray-von Neumann type decomposition allows us to view all von Neumann algebras as direct sums of von Neumann subalgebras of classifiable types.

Definition 2.5.2. Let $n \in \{1, 2, \ldots, \infty\}$. We say that $\mathcal{M}$ is of

- type I if $\mathcal{M}$ has an abelian projection with central carrier I,
  - type I_n if I is the sum of n equivalent abelian projections;

- type II if $\mathcal{M}$ has no non-zero abelian projections but has a finite projection with central carrier I,
  - type II_1 if I is finite;
  - type II_∞ if I is properly infinite;
- type III if $\mathcal{M}$ has no non-zero finite projections.

Remark 2.5.3. A factor $\mathcal{R}$ must be of either type I_n for some n, or type II_1, or type II_∞, or type III.

Example 2.5.4. Let $M_n(\mathbb{C})$ be a complex matrix algebra. Then $M_n(\mathbb{C})$ is a factor of type I_n.

Example 2.5.5. If G is a group with at least two conjugacy classes where the conjugacy class of every non-identity element is infinite (an i.c.c. group), then $\ell^2(G)$, the space of functions $x : G \to \mathbb{C}$ with $\sum_{g\in G} |x(g)|^2 < \infty$, is a Hilbert space. With $x, y \in \ell^2(G)$, the convolution $x * y$ given by

$$(x * y)(g_0) = \sum_{g\in G} x(g_0 g^{-1})y(g)$$

is a function in $\ell^2(G)$. With $L_x(y) = x * y$, the space

$$\mathscr{L}_G = \{L_x \mid x \in \ell^2(G) \text{ and } L_x \in B(\ell^2(G))\}$$

is a factor of type II_1.

Example 2.5.6. If $\mathcal{R}$ is a type II_1 factor and $\mathcal{H}$ is a Hilbert space of infinite dimension, then $\mathcal{R} \otimes B(\mathcal{H})$ is a type II_∞ factor.

Remark 2.5.7. Every von Neumann algebra can be decomposed as the direct sum

$$\mathcal{M} \cong \mathcal{M}_{\text{I}} \oplus \mathcal{M}_{\text{II}} \oplus \mathcal{M}_{\text{III}}$$

of von Neumann subalgebras $M_{\mathrm{I}}, M_{\mathrm{II}}, M_{\mathrm{III}}$ such that $M_{\mathrm{I}} = \{0\}$ or is of type I, $M_{\mathrm{II}} = \{0\}$ or is of type II, and $M_{\mathrm{III}} = \{0\}$ or is of type III. Moreover every type I von Neumann algebra can be decomposed as the direct sum of subalgebras of type I_n with $n \in \{1, 2, \ldots, \infty\}$, while every type II von Neumann algebra can be decomposed as the direct sum of a type II_1 subalgebra and a type II_∞ subalgebra.

Definition 2.5.8. A von Neumann algebra $\mathcal{M}$ is said to be *semi-finite* if $\mathcal{M}$ has no type III direct summand.

2.6 Tracial weights

A key assumption in Wigner's theorem is that the map to be extended must be transition probability preserving. The concept of a faithful normal semi-finite tracial weight allows us to generalize the canonical trace from Wigner's theorem.

Definition 2.6.1. Let $\mathcal{M}^+$ be the set of positive operators in a von Neumann algebra $\mathcal{M}$. A map $\tau_{\mathcal{M}} : \mathcal{M}^+ \to [0, \infty]$ is called a *tracial weight* if it satisfies the following properties:

(i) $\tau_{\mathcal{M}}(H + K) = \tau_{\mathcal{M}}(H) + \tau_{\mathcal{M}}(K)$ for all $H, K \in \mathcal{M}^+$;

(ii) $\tau_{\mathcal{M}}(aH) = a\tau_{\mathcal{M}}(H)$ for each $H \in \mathcal{M}^+$ and $a \geq 0$;

(iii) $\tau_{\mathcal{M}}(AA^*) = \tau_{\mathcal{M}}(A^*A)$ for each $A \in \mathcal{M}$.

A tracial weight $\tau_{\mathcal{M}}$ is said to be *faithful* if $\tau_{\mathcal{M}}(H) > 0$ for every non-zero $H \in \mathcal{M}^+$, *normal* if $\tau_{\mathcal{M}}(\sup H_\lambda) = \sup \tau_{\mathcal{M}}(H_\lambda)$ for every bounded increasing net $\{H_\lambda\}$ in $\mathcal{M}^+$, and *semi-finite* if for every non-zero $H \in \mathcal{M}^+$ there exists a non-zero $K \in \mathcal{M}^+$ satisfying $K \leq H$ and $\tau_{\mathcal{M}}(K) < \infty$. $\tau_{\mathcal{M}}$ is said to be a *tracial state* if $\tau_{\mathcal{M}}(I) = 1$.

Remark 2.6.2. Let $\mathcal{M}$ be a von Neumann algebra with tracial weight $\tau_{\mathcal{M}}$. If E and F are two projections in $\mathcal{M}$, then $E \sim F$ implies $\tau_{\mathcal{M}}(E) = \tau_{\mathcal{M}}(F)$ and $E \precsim F$ implies $\tau_{\mathcal{M}}(E) \leq \tau_{\mathcal{M}}(F)$. If $H \in \mathcal{M}^+$ satisfies $\tau_{\mathcal{M}}(H) < \infty$, then for every (not necessarily positive) operator T in $\mathcal{M}$ we have $\tau_{\mathcal{M}}(HT) = \tau_{\mathcal{M}}(TH)$.

Remark 2.6.3. A von Neumann algebra is semi-finite, in the sense of Definition 2.5.8, if and only if it admits a faithful normal semi-finite tracial weight.

Remark 2.6.4. If E is a non-zero projection in a diffuse von Neumann algebra $\mathcal{M}$, then there exists a non-zero projection $F \neq E$ such that $F \leq E$. This implies, when $\tau_{\mathcal{M}}$ is a faithful normal semi-finite tracial weight of $\mathcal{M}$, that E has subprojections of arbitrarily small trace.

Remark 2.6.5. If $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$ and $E \in \mathcal{P}(\mathcal{M})$, then

$$\tau_{\mathcal{M}}(E) = \sup\{\tau_{\mathcal{M}}(F) \mid F \in \mathcal{P}(\mathcal{M}),\ F \leq E,\ \text{and}\ \tau_{\mathcal{M}}(F) < \infty\}.$$

Definition 2.6.6. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two von Neumann algebras with tracial weights $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively. A Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ is said to be *trace-preserving* if $\tau_{\mathcal{N}}(\rho(H)) = \tau_{\mathcal{M}}(H)$ for all $H \in \mathcal{M}^+$.

2.7 Non-commutative L^p-spaces

We conclude this chapter by examining the construction of a non-commutative L^p-space for $0 < p < \infty$.

Definition 2.7.1. Suppose $\mathcal{M}$ is a von Neumann algebra. If $A \in \mathcal{M}^+$ is a positive operator in $\mathcal{M}$, then there exists a unique positive operator $H \in \mathcal{M}^+$ such that $H^2 = A$. H is called the *positive square root* of A, and is denoted $A^{1/2} = H$.

Assume that $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. If $A \in \mathcal{M}$, we define the modulus of A to be $|A| = (A^*A)^{1/2}$. Note that A^*A is positive for any operator A, whence A^*A must indeed have a positive square root.

Let $\mathcal{F}$ denote the linear span of the set of self-adjoint elements A in $\mathcal{M}$ such that $\tau_{\mathcal{M}}(R(A)) < \infty$, where $R(A)$ is the range projection of A. For $0 < p < \infty$ we define, for each $X \in \mathcal{F}$,

$$\|X\|_p = \tau_{\mathcal{M}}(|X|^p)^{1/p}.$$

When $1 \leq p < \infty$, it can be shown that $\|\cdot\|_p$ defines a norm on $\mathcal{F}$. When $0 < p < 1$, it can be shown that $\|\cdot\|_p$ defines a quasi-norm on $\mathcal{F}$. That is, for $0 < p < 1$, $\|\cdot\|_p$ satisfies conditions (i) and (ii) of Definition 2.1.1 and the following weaker alternative to condition (iii): there exists a real number $c > 0$ such that, for all $X, Y \in \mathcal{F}$, $\|X + Y\|_p \leq c(\|X\|_p + \|Y\|_p)$. We define the non-commutative L^p-space $L^p(\mathcal{M}, \tau_{\mathcal{M}})$ as the completion of $\mathcal{F}$ with respect to $\|\cdot\|_p$.

CHAPTER 3

PRELIMINARIES

Throughout this work $\mathcal{M}$ and $\mathcal{N}$ will be two semi-finite von Neumann algebras with faithful normal semi-finite tracial weights $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively. In this chapter we develop the tools necessary for our investigation into Wigner's theorem for semi-finite von Neumann algebras.

3.1 Grassmann spaces of a von Neumann algebra

We introduce the following notation for Grassmann spaces of von Neumann algebras.

Definition 3.1.1. For $S, \{c\} \subseteq [0, \tau_{\mathcal{M}}(I)]$, we let

$$\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) = c\};$$

$$\mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \geq c \text{ and } \tau_{\mathcal{M}}(I - P) \geq c\};$$

$$\mathcal{P}_S(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \in S\}.$$

3.2 Traces

The following elementary lemma relates properties of a pair of projections to properties of their tracial weights.

Lemma 3.2.1. *Suppose $E, F \in \mathcal{P}(\mathcal{M})$ are two projections such that $\tau_{\mathcal{M}}(E) < \infty$.*

(i) If $\tau_{\mathcal{M}}(EF) = 0$, then $EF = 0$.

(ii) If $\tau_{\mathcal{M}}(EF) = \tau_{\mathcal{M}}(E)$, then $E \leq F$.

Proof. (i) If $\tau_{\mathcal{M}}(EF) = 0$, then

$$0 = \tau_{\mathcal{M}}(EF) = \tau_{\mathcal{M}}(EFE) = \tau_{\mathcal{M}}([FE]^*[FE]),$$

which implies, since $\tau_{\mathcal{M}}$ is faithful, that $EF = 0$.

(ii) If $\tau_{\mathcal{M}}(EF) = \tau_{\mathcal{M}}(E)$, then $0 = \tau_{\mathcal{M}}(E - EF) = \tau_{\mathcal{M}}(E(I_{\mathcal{M}} - F)) = 0$. By (i) we have $E(I_{\mathcal{M}} - F) = 0$, whence $E \leq F$. □

A useful tool in computing the trace of a union or intersection of projections is a direct consequence of Kaplansky's formula (Theorem 6.1.17 in [14]).

Lemma 3.2.2. *If $E, F \in \mathcal{P}(\mathcal{M})$ are two projections with finite trace, then*

$$\tau_{\mathcal{M}}(E \vee F) + \tau_{\mathcal{M}}(E \wedge F) = \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F).$$

Proof. The proof follows directly from Theorem 6.1.17 in [14] and the fact that any two equivalent projections have the same trace. □

The following lemma asserts that a trace-preserving Jordan $*$-homomorphism preserves certain properties of a pair of projections with finite trace.

Lemma 3.2.3. *Suppose $\rho : \mathcal{M} \to \mathcal{N}$ is a trace-preserving Jordan $*$-homomorphism. If $E, F \in \mathcal{P}(\mathcal{M})$ are two projections with finite trace, then*

(i) $\tau_{\mathcal{N}}(\rho(E)\rho(F)) = \tau_{\mathcal{M}}(EF)$.

Further, if $EF = FE$, then

(ii) $\rho(EF) = \rho(E)\rho(F) = \rho(F)\rho(E)$.

(iii) $\rho(E \vee F) = \rho(E) \vee \rho(F)$.

Proof. (i). By Lemma 2 in [12] we have

$$\rho(E)\rho(F)\rho(E) = \rho(EFE),$$

thus

$$\tau_{\mathcal{N}}(\rho(E)\rho(F)) = \tau_{\mathcal{N}}(\rho(E)\rho(F)\rho(E)) = \tau_{\mathcal{N}}(\rho(EFE)) = \tau_{\mathcal{M}}(EFE) = \tau_{\mathcal{M}}(EF).$$

(ii). Suppose $EF = FE$, so that EF is a projection. Observe that

$$\begin{aligned}\tau_{\mathcal{N}}([\rho(E)\rho(F) - \rho(EF)]^*[\rho(E)\rho(F) - \rho(EF)]) &= \tau_{\mathcal{N}}(\rho(F)\rho(E)\rho(F)) - \tau_{\mathcal{N}}(\rho(F)\rho(E)\rho(EF)) \\ &\quad - \tau_{\mathcal{N}}(\rho(EF)\rho(E)\rho(F)) + \tau_{\mathcal{N}}(\rho(EF)).\end{aligned}$$

We use the fact that $\rho(EF) = \rho(EFE) = \rho(E)\rho(F)\rho(E)$, the above equation, and (i) to obtain

$$\begin{aligned}\tau_{\mathcal{N}}([\rho(E)\rho(F) - \rho(EF)]^*[\rho(E)\rho(F) - \rho(EF)]) &= \tau_{\mathcal{N}}(\rho(F)\rho(E)\rho(F)) - \tau_{\mathcal{N}}(\rho(F)\rho(E)\rho(EF)) \\ &\quad - \tau_{\mathcal{N}}(\rho(EF)\rho(E)\rho(F)) + \tau_{\mathcal{N}}(\rho(EF)) \\ &= 2(\tau_{\mathcal{M}}(EF) - \tau_{\mathcal{N}}(\rho(E)\rho(F)\rho(E)\rho(F))) \\ &= 2(\tau_{\mathcal{M}}(EF) - \tau_{\mathcal{N}}([\rho(E)\rho(F)\rho(E)]^2)) \\ &= 2(\tau_{\mathcal{M}}(EF) - \tau_{\mathcal{N}}([\rho(EF)]^2)) \\ &= 0.\end{aligned}$$

As $\tau_{\mathcal{N}}$ is faithful, we conclude $\rho(EF) = \rho(E)\rho(F)$. It follows that $\rho(EF) = \rho(FE) = \rho(F)\rho(E)$, so $\rho(E)$ and $\rho(F)$ commute.

(iii). Using (ii) we find that

$$\rho(E \vee F) = \rho(E + F - EF) = \rho(E) + \rho(F) - \rho(E)\rho(F) = \rho(E) \vee \rho(F).$$

□

3.3 Mappings

We introduce the following terminology for projection mappings.

Definition 3.3.1. Let $\mathcal{A}$ be a subset of $\mathcal{P}(\mathcal{M})$, $\mathcal{B}$ a subset of $\mathcal{P}(\mathcal{N})$, and $\phi : \mathcal{A} \to \mathcal{B}$ be a map.

(a) ϕ is trace-preserving if $\tau_{\mathcal{M}}(E) = \tau_{\mathcal{N}}(\phi(E))$ for all $E \in \mathcal{A}$.

(b) ϕ is order-preserving if, for all $E, F \in \mathcal{A}$, $E \leq F$ implies $\phi(E) \leq \phi(F)$.

(c) ϕ is ortho-preserving if, for all $E, F \in \mathcal{A}$, $E \perp F$ implies $\phi(E) \perp \phi(F)$.

(d) ϕ is an orthomorphism if ϕ satisfies the following two conditions:

   (i) ϕ is ortho-preserving.

   (ii) $\phi(E+F) = \phi(E)+\phi(F)$ for all orthogonal projections E, F in $\mathcal{A}$ with $E+F \in \mathcal{A}$.

When ϕ is a bijection we say that

(e) ϕ is an order-isomorphism if ϕ and ϕ^{-1} are both order-preserving.

(f) ϕ is an ortho-isomorphism if ϕ and ϕ^{-1} are both ortho-preserving.

(g) ϕ is an order-ortho-isomorphism if ϕ is both an order-isomorphism and an ortho-isomorphism.

Remark 3.3.2. We say, for instance, that ϕ is order-ortho-preserving if ϕ is both order-preserving and ortho-preserving. Likewise, we say that ϕ is trace-ortho-preserving if ϕ is both trace-preserving and ortho-preserving.

Lemma 3.3.3. *Let $\mathcal{A}$ be a subset of $\mathcal{P}(\mathcal{M})$ and $\mathcal{B}$ a subset of $\mathcal{P}(\mathcal{N})$. If $\phi : \mathcal{A} \to \mathcal{B}$ is trace-order-ortho-preserving, then ϕ is an orthomorphism.*

Proof. Suppose E and F are two projections in $\mathcal{A}$ such that $E \perp F$ and $E+F \in \mathcal{A}$. Since ϕ is order-ortho-preserving, we have $\phi(E), \phi(F) \leq \phi(E+F)$ and $\phi(E) \perp \phi(F)$. Therefore $\phi(E)+\phi(F) \leq \phi(E+F)$. Further, since ϕ is trace-preserving, we observe

$$\tau_{\mathcal{N}}(\phi(E)+\phi(F)) = \tau_{\mathcal{M}}(E)+\tau_{\mathcal{M}}(F) = \tau_{\mathcal{M}}(E+F) = \tau_{\mathcal{N}}(\phi(E+F)).$$

Since $\tau_{\mathcal{N}}$ is faithful, we conclude $\phi(E)+\phi(F) = \phi(E+F)$. □

The following result asserts that a Jordan $*$-isomorphism of von Neumann algebras restricts to an order-ortho-isomorphism on the set of projections.

Lemma 3.3.4. *Suppose $\rho : \mathcal{M} \to \mathcal{N}$ is a Jordan $*$-isomorphism of von Neumann algebras.*

(i) If $A \in \mathcal{M}$, then A is a projection if and only if $\rho(A)$ is a projection.

(ii) If E, F are projections in $\mathcal{M}$, then $E \leq F$ if and only if $\rho(E) \leq \rho(F)$.

(iii) If E, F are projections in $\mathcal{M}$, then $E \perp F$ if and only if $\rho(E) \perp \rho(F)$.

(iv) If $\{E_\lambda\}$ is a family of projections in $\mathcal{M}$, then $\rho(\bigvee_\lambda E_\lambda) = \bigvee_\lambda \rho(E_\lambda)$.

Proof. (i). With A a projection in the domain, we have

$$\rho(A)^2 = \rho(A^2) = \rho(A) \quad \text{and} \quad \rho(A)^* = \rho(A^*) = \rho(A)$$

so that $\rho(A)$ is a projection. Conversely, if $\rho(A)$ is a projection we have $\rho(A^*) = \rho(A)^* = \rho(A)$. As ρ is an injection, we find that $A = A^*$ and A is self-adjoint. Therefore $\rho(A^2) = \rho(A)^2 = \rho(A)$. Since ρ is an injection, we conclude $A = A^2 = A^*$ and A is a projection.

(ii). If $E \leq F$ then $F - E$ is a projection and by (i) so is $\rho(F-E) = \rho(F) - \rho(E)$. Thus $\rho(E) \leq \rho(F)$. If $\rho(E) \leq \rho(F)$, then we have that $\rho(F-E) = \rho(F) - \rho(E)$ is a projection. By (i) we find $F - E$ is a projection, thus $E \leq F$.

(iii). First note, as a consequence of (i) and (ii), that $\rho(I_{\mathcal{M}}) = I_{\mathcal{N}}$. For if C is the pullback of $I_{\mathcal{N}}$ under the surjection ρ, so that $\rho(C) = I_{\mathcal{N}}$, we would have that C is a projection by (i). By (ii), we observe $\rho(I_{\mathcal{M}}) \leq I_{\mathcal{N}} = \rho(C)$ implies $I_{\mathcal{M}} \leq C$, whence $I_{\mathcal{M}} = C$. Likewise $\rho(0_{\mathcal{M}}) = 0_{\mathcal{N}}$, for $\rho(C) = 0_{\mathcal{N}} \leq \rho(0_{\mathcal{M}})$ implies $C \leq 0_{\mathcal{M}}$ so that $C = 0_{\mathcal{M}}$.

Now suppose $E \perp F$. Since $E \leq I_{\mathcal{M}} - F$, by (ii) and the above argument we have

$$\rho(E) \leq \rho(I_{\mathcal{M}} - F) = \rho(I_{\mathcal{M}}) - \rho(F) = I_{\mathcal{N}} - \rho(F).$$

Thus $\rho(E) \perp \rho(F)$. Likewise $\rho(E) \leq I_{\mathcal{N}} - \rho(F) = \rho(I_{\mathcal{M}} - F)$ implies $E \leq I_{\mathcal{M}} - F$, and $E \perp F$.

(iv). Since ρ is surjective there exists a projection F with $\rho(F) = I_{\mathcal{N}} - \bigvee_{\lambda} \rho(E_{\lambda})$. In particular, $\rho(F) \perp \rho(E_{\lambda})$ for all λ, which implies $F \perp E_{\lambda}$ for all λ. Thus $\bigvee_{\lambda} E_{\lambda} \leq I_{\mathcal{M}} - F$. This implies

$$\rho(\bigvee_{\lambda} E_{\lambda}) \leq \rho(I_{\mathcal{M}} - F) = I_{\mathcal{N}} - \rho(F) = I_{\mathcal{N}} - (I_{\mathcal{N}} - \bigvee_{\lambda} \rho(E_{\lambda})) = \bigvee_{\lambda} \rho(E_{\lambda}).$$

On the other hand, $E_{\lambda} \leq \bigvee_{\lambda} E_{\lambda}$ implies $\rho(E_{\lambda}) \leq \rho(\bigvee_{\lambda} E_{\lambda})$ for each λ, hence

$$\rho(\bigvee_{\lambda} E_{\lambda}) = \bigvee_{\lambda} \rho(E_{\lambda}).$$

□

For each T in $\mathcal{M}$, we let $R(T)$ denote the range projection of T in $\mathcal{M}$. That is, $R(T)$ is the projection associated to the closure of the range of T. The next result from [3] plays a critical role in this work.

Lemma 3.3.5. *Suppose $\phi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ is a trace-preserving orthomorphism such that $\phi(0) = 0$ and $\phi(I_{\mathcal{M}}) = I_{\mathcal{N}}$. Then ϕ can be extended to a Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that, for each positive operator T in $\mathcal{M}$, $\tau_{\mathcal{M}}(R(T)) < \infty$ implies $\tau_{\mathcal{M}}(T) = \tau_{\mathcal{N}}(\rho(T))$. In particular, if $\tau_{\mathcal{M}}(I_{\mathcal{M}}) < \infty$, then ϕ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$.*

Proof. It follows from Corollary 1 in [3] that ϕ can be extended to a Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that, for all $E \in \mathcal{P}(\mathcal{M})$, $\tau_{\mathcal{N}}(\rho(E)) = \tau_{\mathcal{M}}(E)$. Suppose $T \geq 0$ is a positive operator in $\mathcal{M}$ with $\tau_{\mathcal{M}}(R(T)) < \infty$. Note that $T = R(T)T = TR(T)$. For each $\epsilon > 0$ there exists a family $\{E_1, \ldots, E_n\}$ of mutually orthogonal subprojections of $R(T)$ and a family of real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\|T - (\alpha_1 E_1 + \ldots + \alpha_n E_n)\| < \epsilon.$$

Since ρ is a Jordan $*$-homomorphism, $\rho(T) = \rho(R(T)T) = \rho(R(T))\rho(T)$, each $\rho(E_i)$ is a subprojection of $\rho(R(T))$ for $1 \leq i \leq n$, and

$$\|\rho(T) - (\alpha_1 \rho(E_1) + \ldots + \alpha_n \rho(E_n))\| < \epsilon.$$

As ϕ is trace-preserving, we conclude that

$$\begin{aligned}
|\tau_{\mathcal{M}}(T) - \tau_{\mathcal{N}}(\rho(T))| &\leq |\tau_{\mathcal{M}}(R(T)T - (\alpha_1 E_1 + \ldots + \alpha_n E_n))| \\
&\quad + |\tau_{\mathcal{M}}(\alpha_1 E_1 + \ldots + \alpha_n E_n) - \tau_{\mathcal{N}}(\alpha_1 \rho(E_1) + \ldots + \alpha_n \rho(E_n))| \\
&\quad + |\tau_{\mathcal{N}}(\alpha_1 \rho(E_1) + \ldots + \alpha_n \rho(E_n) - \rho(R(T))\rho(T))| \\
&\leq \|R(T)\|_1 \|T - (\alpha_1 E_1 + \ldots + \alpha_n E_n)\| \\
&\quad + \|\rho(R(T))\|_1 \|\alpha_1 \rho(E_1) + \ldots + \alpha_n \rho(E_n) - \rho(T)\| \\
&\leq 2\epsilon \cdot \tau_{\mathcal{M}}(R(T)).
\end{aligned}$$

This implies $\tau_{\mathcal{M}}(T) = \tau_{\mathcal{N}}(\rho(T))$ and thus completes the proof. □

The following important lemma, a combination of Theorem 8.1.1 and Theorem 8.1.2 in [11], is a generalization of a result of Dye [5].

Lemma 3.3.6. *Suppose $\mathcal{M}$ has no direct summand of type I_2. If $\varphi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ is an orthomorphism, then φ can be extended to a Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{M}$. If φ is* *an ortho-isomorphism on $\mathcal{P}(\mathcal{M})$, then ρ is a Jordan $*$-isomorphism.*

3.4 A corner of a von Neumann algebra

We require the following lemma about a corner of a von Neumann algebra which states that, in a semi-finite von Neumann algebra with no direct summand of type I_2, a projection of finite tracial weight is contained in a finite von Neumann subalgebra with no direct summand of type I_2.

Lemma 3.4.1. *Assume $\mathcal{M}$ has no direct summand of type I_2. For each projection E in $\mathcal{M}$ with $\tau_{\mathcal{M}}(E) < \infty$, there exists a projection F in $\mathcal{M}$ such that*

(i) $E \le F$ and $\tau_{\mathcal{M}}(F) < \infty$.

(ii) $F\mathcal{M}F$ has no direct summand of type I_2.

Proof. From the type decomposition of a von Neumann algebra (see Theorem 6.5.2 in [14]), there are mutually orthogonal central projections $\{P_n\}_{n=0}^{\infty}$ and R_0 of $\mathcal{M}$, with $\sum_{n=0}^{\infty} P_n + R_0 = I$, such that $P_0\mathcal{M}$ is properly infinite or $P_0 = 0$, $R_0\mathcal{M}$ is of type II_1 or $R_0 = 0$, and, for each $n \ge 1$, $P_n\mathcal{M}$ is of type I_n or $P_n = 0$. Notice $P_2 = 0$, since $\mathcal{M}$ has no direct summand of type I_2.

If $P_0 = 0$, we let $F_{0,1} = F_{0,2} = 0$. Now assume $P_0 \ne 0$, so that P_0 is a properly infinite central projection in $\mathcal{M}$. By Lemma 6.3.3 in [14] there exist mutually orthogonal projections H_1, H_2, H_3 in $P_0\mathcal{M}$ such that $P_0 = H_1 + H_2 + H_3$ and $P_0 \sim H_1 \sim H_2 \sim H_3$. Since $P_0E \le P_0 \sim H_i$, it follows that $P_0E \precsim H_i$ for $1 \le i \le 3$. By Exercise 6.9.7 in [14], we may assume that $P_0E \le H_1$. Let $F_{0,1}$ and $F_{0,2}$ be subprojections of H_2 and H_3, respectively, such that $P_0E, F_{0,1}, F_{0,2}$ are mutually orthogonal and mutually equivalent in $P_0\mathcal{M}$. Then

$$\tau_{\mathcal{M}}(P_0E + F_{0,1} + F_{0,2}) = 3\tau_{\mathcal{M}}(P_0E)$$

and $(P_0E + F_{0,1} + F_{0,2})\mathcal{M}(P_0E + F_{0,1} + F_{0,2})$ is a finite von Neumann algebra with no direct summand of type I_2.

For each positive integer $n \geq 3$, recall that $P_n\mathcal{M}$ is a von Neumann algebra of type I_n if $P_n \neq 0$. By the proof of Theorem 8.4.4 in [14], for a projection P_nE in $P_n\mathcal{M}$ there exists a family $\{Q_0^{(n)}, Q_1^{(n)}, \ldots, Q_n^{(n)}\}$ of mutually orthogonal projections in the center of $P_n\mathcal{M}$, with $P_n = Q_0^{(n)} + \ldots + Q_n^{(n)}$, such that $Q_j^{(n)}E$ is the sum of j abelian projections each having central carrier $Q_j^{(n)}$ (where $Q_0^{(n)} = P_n - C_{P_nE}$). Hence

$$\begin{aligned}(P_nE)\mathcal{M}(P_nE) &= ((Q_0^{(n)} + \ldots + Q_n^{(n)})E)\mathcal{M}((Q_0^{(n)} + \ldots + Q_n^{(n)})E) \\ &= (Q_1^{(n)}E)\mathcal{M}(Q_1^{(n)}E) + \ldots + (Q_n^{(n)}E)\mathcal{M}(Q_n^{(n)}E)\end{aligned}$$

where, if $Q_j^{(n)} \neq 0$, $(Q_j^{(n)}E)\mathcal{M}(Q_j^{(n)}E) \subseteq P_n\mathcal{M}$ is a von Neumann subalgebra of type I_j. If $Q_2^{(n)} = 0$, let $F_n = 0$. If $Q_2^{(n)} \neq 0$, then $Q_2^{(n)}E$ is a sum of abelian projections H_1, H_2 each having central carrier $Q_2^{(n)}$. Since $n \geq 3$ and $Q_2^{(n)}$ is a sum of n abelian projections each having central carrier $Q_2^{(n)}$, there exists a subprojection F_n of $Q_2^{(n)}$ such that F_n, H_1, H_2 are mutually orthogonal and mutually equivalent. Now $(H_1 + H_2 + F_n)\mathcal{M}(H_1 + H_2 + F_n)$ is of type I_3. Furthermore, $F_n \leq Q_2^{(n)}$, $\tau_{\mathcal{M}}(P_nE + F_n) \leq 2\tau_{\mathcal{M}}(P_nE)$ and

$$\begin{aligned}(P_nE + F_n)\mathcal{M}(P_nE + F_n) &= (Q_1^{(n)}E)\mathcal{M}(Q_1^{(n)}E) + (Q_2^{(n)}E + F_n)\mathcal{M}(Q_2^{(n)}E + F_n) \\ &\quad + (Q_3^{(n)}E)\mathcal{M}(Q_3^{(n)}E) + \ldots + (Q_n^{(n)}E)\mathcal{M}(Q_n^{(n)}E) \\ &= (Q_1^{(n)}E)\mathcal{M}(Q_1^{(n)}E) + (H_1 + H_2 + F_n)\mathcal{M}(H_1 + H_2 + F_n) \\ &\quad + (Q_3^{(n)}E)\mathcal{M}(Q_3^{(n)}E) + \ldots + (Q_n^{(n)}E)\mathcal{M}(Q_n^{(n)}E)\end{aligned}$$

has no direct summand of type I_2.

Let

$$F = E + F_{0,1} + F_{0,2} + \sum_{n \geq 3} F_n.$$

Then, by the choice of $F_{0,1}, F_{0,2}, F_3, \ldots$, we conclude

$$\begin{aligned}\tau_{\mathcal{M}}(F) &= \tau_{\mathcal{M}}(P_0E + F_{0,1} + F_{0,2} + P_1E + R_0E + \sum_{n\geq 3}(P_nE + F_n)) \\ &\leq 3\tau_{\mathcal{M}}(P_0E) + \tau_{\mathcal{M}}(P_1E) + \tau_{\mathcal{M}}(R_0E) + \sum_{n\geq 3} 2\tau_{\mathcal{M}}(P_nE) \\ &\leq 3\tau_{\mathcal{M}}(E) < \infty\end{aligned}$$

and $F\mathcal{M}F$ has no direct summand of type I_2. □

3.5 Diffuse von Neumann algebras

In this section, we assume that $\mathcal{M}$ is a diffuse semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. The following lemma outlines a few important properties of the lattice of projections in $\mathcal{M}$. This result will be used as a foundation, without explicit reference, in much of the remainder of this work. A sketch of the proof has been included for the reader's convenience.

Lemma 3.5.1. *Suppose $\mathcal{M}$ is diffuse. Let E, P be two projections in $\mathcal{M}$ with $E \leq P$, and let $c, d \in [0, \tau_{\mathcal{M}}(I)]$ be real numbers satisfying $c \leq \tau_{\mathcal{M}}(E) \leq d \leq \tau_{\mathcal{M}}(P)$. The following statements are true.*

(i) There exist two projections, $F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and $G \in \mathcal{P}_d(\mathcal{M}, \tau_{\mathcal{M}})$, such that

$$F \leq E \leq G \leq P.$$

(ii) If $d < \frac{\tau_{\mathcal{M}}(I)}{2}$, then there exist two commuting projections $G_1, G_2 \in \mathcal{P}_d(\mathcal{M}, \tau_{\mathcal{M}})$ with $E = G_1 \wedge G_2$.

(iii) If $c > 0$, then $E = \bigvee\{F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \mid F \leq E\}$.

Proof. (i). We may assume $0 < c < \tau_{\mathcal{M}}(E) < \infty$. By Zorn's lemma there is a maximal directed set $\{F_\lambda\}_{\lambda\in\Lambda}$ of subprojections of E such that $\tau_{\mathcal{M}}(F_\lambda) \leq c$ for each $\lambda \in \Lambda$. Since $\tau_{\mathcal{M}}$

is normal, $\{F_\lambda\}_{\lambda\in\Lambda}$ is an increasing net with limit $\bigvee_{\lambda\in\Lambda} F_\lambda$ such that

$$\tau_{\mathcal{M}}(\bigvee_{\lambda\in\Lambda} F_\lambda) = \sup\{\tau_{\mathcal{M}}(F_\lambda) \mid \lambda \in \Lambda\} \leq c.$$

Assume, by way of contradiction, that $\tau_{\mathcal{M}}(\bigvee_{\lambda\in\Lambda} F_\lambda) < c$. Then there is a subprojection G of the non-zero projection $E - \bigvee_{\lambda\in\Lambda} F_\lambda$ such that $\tau_{\mathcal{M}}(\bigvee_{\lambda\in\Lambda} F_\lambda + G) < c$. By maximality $\bigvee_{\lambda\in\Lambda} F_\lambda + G$ is an element of the family $\{F_\lambda\}_{\lambda\in\Lambda}$. This implies $G \leq \bigvee_{\lambda\in\Lambda} F_\lambda$, which is impossible. Thus we must have $\tau_{\mathcal{M}}(\bigvee_{\lambda\in\Lambda} F_\lambda) = c$. With $F = \bigvee_{\lambda\in\Lambda} F_\lambda$ the first part of the claim is shown. The second is trivial, as $P - E$ has a subprojection G' such that $\tau_{\mathcal{M}}(E + G') = d$. With $G = E + G'$ we clearly have $E \leq G \leq P$.

(ii). We may choose $G_1' \leq I - E$ and $G_2' \leq I - (E + G_1')$ satisfying

$$\tau_{\mathcal{M}}(E + G_1') = \tau_{\mathcal{M}}(E + G_2') = d.$$

Set $G_i = E + G_i'$ for $i = 1, 2$.

(iii). We may assume $0 < c < \tau_{\mathcal{M}}(E)$. Let $\{F_\lambda\}$ denote the family of all subprojections of E with trace c. Let $F_1 = E - \bigvee_\lambda F_\lambda$. By the definition of the family $\{F_\lambda\}$ and part (i) we have $\tau_{\mathcal{M}}(F_1) < c$. We can choose $F_2 \leq \bigvee_\lambda F_\lambda$ such that $(F_1 + F_2) \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Again, by definition, $F_1 + F_2$ belongs to the family $\{F_\lambda\}$. Thus $F_1 \leq \bigvee_\lambda F_\lambda$. We conclude $F_1 = 0$ as desired. $\square$

The next two results relate the study of subprojections in a diffuse von Neumann algebra to certain properties of commutative projections.

Lemma 3.5.2. *Suppose that $\mathcal{M}$ is diffuse. Assume that E_1, E_2 and F are projections in $\mathcal{M}$ such that $\tau_{\mathcal{M}}(E_1) = \tau_{\mathcal{M}}(E_2) < \infty$, $E_1E_2 = 0$ and $F \leq E_1 + E_2$. Then there exist projections P_1, P_2 in $\mathcal{M}$ such that*

(i) $\tau_{\mathcal{M}}(P_i) = \tau_{\mathcal{M}}(E_i)$ for $i = 1, 2$.

(ii) $P_1P_2 = 0$ and $P_1 + P_2 = E_1 + E_2$.

(iii) $FP_i = P_iF$ *and* $E_jP_i = P_iE_j$ *for* $i, j = 1, 2$.

Proof. Let $\mathcal{R} = (E_1 + E_2)\mathcal{M}(E_1 + E_2)$ be a von Neumann subalgebra of $\mathcal{M}$ and $\mathcal{W}$ be the von Neumann subalgebra generated by F, E_1, E_2 in $\mathcal{R}$. Note that $\mathcal{R}$ is also diffuse. By Theorem V.1.41 in [27], $\mathcal{W}$ is a direct sum of an abelian von Neumann subalgebra and a von Neumann subalgebra of type I_2. Let $\mathcal{A}$ be a MASA (maximal abelian self-adjoint subalgebra) of $\mathcal{W}' \cap \mathcal{R}$. Thus $\mathcal{A}$ is also diffuse with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. Then $\tau_{\mathcal{M}}(I_{\mathcal{A}}) = \tau_{\mathcal{M}}(E_1) + \tau_{\mathcal{M}}(E_2)$, where $I_{\mathcal{A}}$ is the identity of $\mathcal{A}$. Since $\mathcal{A}$ is diffuse, there exist two mutually orthogonal projections P_1, P_2 in $\mathcal{R}$ such that

(i) $\tau_{\mathcal{M}}(P_i) = \tau_{\mathcal{M}}(E_i)$ for $i = 1, 2$.

(ii) $P_1P_2 = 0$ and $P_1 + P_2 = E_1 + E_2$.

(iii) $FP_i = P_iF$ and $E_jP_i = P_iE_j$ for $i, j = 1, 2$.

□

Proposition 3.5.3. *Suppose* $\mathcal{M}$ *is diffuse. Let* $0 < c < \infty$ *and* $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$ *be a trace-ortho-preserving map. The following statements are equivalent.*

(a) If E_1, E_2, F *are mutually commutative projections in* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *such that* $E_1E_2 = 0$ *and* $F \le E_1 + E_2$, *then* $\varphi(F) \le \varphi(E_1) + \varphi(E_2)$.

(b) If E_1, E_2, F *are projections in* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *such that* $E_1E_2 = 0$ *and* $F \le E_1 + E_2$, *then* $\varphi(F) \le \varphi(E_1) + \varphi(E_2)$.

Proof. Obviously (b) implies (a). Now assume (a) is true. Suppose that E_1, E_2, F are projections in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E_1E_2 = 0$ and $F \le E_1 + E_2$. By Lemma 3.5.2, there exist projections P_1, P_2 in $\mathcal{M}$ such that

(i) $\tau_{\mathcal{M}}(P_i) = \tau_{\mathcal{M}}(E_i)$ for $i = 1, 2$.

(ii) $P_1P_2 = 0$ and $P_1 + P_2 = E_1 + E_2$.

(iii) $FP_i = P_iF$ and $E_jP_i = P_iE_j$ for $i, j = 1, 2$.

By assumption (a) and conditions (ii) and (iii),

$$\varphi(P_1) + \varphi(P_2) \leq \varphi(E_1) + \varphi(E_2) \qquad \text{and} \qquad \varphi(F) \leq \varphi(P_1) + \varphi(P_2).$$

Thus

$$\varphi(F) \leq \varphi(E_1) + \varphi(E_2).$$

□

The following proposition gives a slight generalization of Lemma 3.4.1 by adding a restriction on the tracial weight of a corner of a diffuse von Neumann algebra.

Proposition 3.5.4. *Suppose $\mathcal{M}$ is diffuse and has no direct summand of type I_2. Let c be a positive number. Assume that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \infty$. For each projection E in $\mathcal{M}$ with $\tau_{\mathcal{M}}(E) < \infty$, there exists a diffuse von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ such that*

(i) $\tau_{\mathcal{M}}(I_{\mathcal{M}_1}) = nc$ for some positive integer $n \geq 4$.

(ii) $E\mathcal{M}E \subseteq \mathcal{M}_1$.

(iii) $\mathcal{M}_1$ has no direct summand of type I_2.

Proof. By Lemma 3.4.1, there exists a projection F in $\mathcal{M}$ such that $E \leq F$, $\tau_{\mathcal{M}}(F) < \infty$ and $F\mathcal{M}F$ has no direct summand of type I_2. Note that $\mathcal{M}$ is diffuse. Let P be a projection in $\mathcal{M}$ such that $PF = 0$ and $\tau_{\mathcal{M}}(P + F) = nc$ for some positive integer $n \geq 4$. Let $\mathcal{A}$ be a MASA of $P\mathcal{M}P$ and $\mathcal{M}_1 = \mathcal{A} + F\mathcal{M}F$. Then $\mathcal{M}_1$ is a diffuse von Neumann algebra with (i) $\tau_{\mathcal{M}}(I_{\mathcal{M}_1}) = \tau_{\mathcal{M}}(F + P) = nc$, (ii) $E\mathcal{M}E \subseteq F\mathcal{M}F \subseteq \mathcal{M}_1$, and (iii) $\mathcal{M}_1$ has no direct summand of type I_2. □

3.6 Atomic von Neumann algebras

In this section, we assume that $\mathcal{M}$ is atomic and $\tau_{\mathcal{M}}$ is the canonical, faithful, normal, semi-finite, tracial weight of $\mathcal{M}$ such that

$$\tau_{\mathcal{M}}(H) = 1 \text{ for each minimal projection } H \text{ in } \mathcal{M}.$$

Given such an atomic von Neumann algebra $\mathcal{M}$, we can develop analogues to the results obtained in the case of a diffuse semi-finite von Neumann algebra.

Lemma 3.6.1. *Suppose $\mathcal{M}$ is atomic and $\tau_{\mathcal{M}}$ is the canonical, faithful, normal, semi-finite, tracial weight of $\mathcal{M}$ such that*

$$\tau_{\mathcal{M}}(H) = 1 \textit{ for each minimal projection } H \textit{ in } \mathcal{M}.$$

Let E, P be two projections in $\mathcal{M}$ such that $E \leq P$, and let $c, d \in [0, \tau_{\mathcal{M}}(I)]$ be two natural numbers which satisfy $c \leq \tau_{\mathcal{M}}(E) \leq d \leq \tau_{\mathcal{M}}(P)$. Then the following statements are true.

(i) There exist two projections, $F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and $G \in \mathcal{P}_d(\mathcal{M}, \tau_{\mathcal{M}})$, such that

$$F \leq E \leq G \leq P.$$

(ii) If $d < \frac{\tau_{\mathcal{M}}(I)}{2}$, then there exist two commuting projections $G_1, G_2 \in \mathcal{P}_d(\mathcal{M}, \tau_{\mathcal{M}})$ with $E = G_1 \wedge G_2$.

(iii) If $c > 0$, then $E = \bigvee\{F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \mid F \leq E\}$.

Proof. The proof is elementary and has been omitted. □

Lemma 3.6.2. *Suppose $\mathcal{M}$ is atomic and $\tau_{\mathcal{M}}$ is the canonical, faithful, normal, semi-finite,* *tracial weight of* $\mathcal{M}$ *such that*

$$\tau_{\mathcal{M}}(H) = 1 \text{ for each minimal projection } H \text{ in } \mathcal{M}.$$

Let k *be a positive integer such that* $k \geq 2$. *Assume that* E_1, E_2 *and* F *are projections in* $\mathcal{M}$ *such that*

$$\tau_{\mathcal{M}}(E_1) = \tau_{\mathcal{M}}(E_2) = k, \ E_1E_2 = 0 \text{ and } F \leq E_1 + E_2.$$

Then there exist projections $P_1, P_2, Q_1,$ *and* Q_2 *in* $\mathcal{M}$ *such that*

(i) $\tau_{\mathcal{M}}(P_1) = \tau_{\mathcal{M}}(P_2) = \tau_{\mathcal{M}}(Q_1) = \tau_{\mathcal{M}}(Q_2) = k.$

(ii) $P_1P_2 = 0,\ Q_1Q_2 = 0,$ *and* $P_1 + P_2 = Q_1 + Q_2 = E_1 + E_2.$

(iii) $FP_i = P_iF,\ P_iQ_j = Q_jP_i,$ *and* $E_jQ_i = Q_iE_j$ *for* $i, j = 1, 2.$

Proof. Let $\mathcal{R} = (E_1+E_2)\mathcal{M}(E_1+E_2)$. Then $\mathcal{R}$ is a finite dimensional von Neumann subalgebra of $\mathcal{M}$ and $\tau_{\mathcal{M}}$ is a trace on $\mathcal{R}$ such that $\tau_{\mathcal{M}}(H) = 1$ for each minimal projection H in $\mathcal{R}$. Note that $\tau_{\mathcal{M}}(I_{\mathcal{R}}) = \tau_{\mathcal{M}}(E_1 + E_2) = 2k$, where $I_{\mathcal{R}} = E_1 + E_2$ is the identity of $\mathcal{R}$. Let $\mathcal{W}$ be the von Neumann subalgebra generated by E_1, E_2, F in $\mathcal{R}$. By Theorem V.1.41 in [27], there exists a projection Z in the center of $\mathcal{W}$ such that $Z\mathcal{W}$ is of type I_2 and $(I_{\mathcal{R}} - Z)\mathcal{W}$ is abelian. Thus we may assume that there exist abelian von Neumann subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{W}$ such that

$$\mathcal{A}_1 \otimes M_2(\mathbb{C}) = Z\mathcal{W} \subseteq Z\mathcal{R}Z \qquad \text{and} \qquad \mathcal{A}_2 = (I_{\mathcal{R}} - Z)\mathcal{W} \subseteq (I_{\mathcal{R}} - Z)\mathcal{R}(I_{\mathcal{R}} - Z).$$

We may further assume that $Z\mathcal{R}Z = \mathcal{R}_1 \otimes M_2(\mathbb{C})$, where $\mathcal{R}_1$ is a finite dimensional von Neumann subalgebra such that $\mathcal{A}_1$ is an abelian subalgebra of $\mathcal{R}_1$.

Let $\mathcal{B}$ be a MASA in $\mathcal{W}' \cap \mathcal{R}$. Since Z is in the center of $\mathcal{W}$, we find that $Z \in \mathcal{B}$ and $\mathcal{B} = Z\mathcal{B} + (I_{\mathcal{R}} - Z)\mathcal{B}$. Since the center of $\mathcal{W}$ is contained in $\mathcal{B}$, it follows that

$$\mathcal{A}_1 \otimes I_2 \subseteq Z\mathcal{B} \subseteq \mathcal{R}_1 \otimes I_2 \quad \text{and} \quad \mathcal{A}_2 \subseteq (I_{\mathcal{R}} - Z)\mathcal{B} \subseteq (I_{\mathcal{R}} - Z)\mathcal{R}(I_{\mathcal{R}} - Z),$$

where I_2 is the identity of $M_2(\mathbb{C})$. From the assumption that $\mathcal{B}$ is a MASA in $\mathcal{W}' \cap \mathcal{R}$, we find that

$$Z\mathcal{B} \text{ is a MASA of } \mathcal{R}_1 \otimes I_2, \text{ and } (I_{\mathcal{R}} - Z)\mathcal{B} \text{ is a MASA of } (I_{\mathcal{R}} - Z)\mathcal{R}(I_{\mathcal{R}} - Z).$$

Therefore,

(a) $\tau_{\mathcal{M}}(K) = 2$ for each minimal projection K in $Z\mathcal{B}$ and $\dim(Z\mathcal{B}) = \tau_{\mathcal{M}}(Z)/2$.

(b) $\tau_{\mathcal{M}}(H) = 1$ for each minimal projection H in $(I_{\mathcal{R}} - Z)\mathcal{B}$ and $\dim((I_{\mathcal{R}} - Z)\mathcal{B}) = \tau_{\mathcal{M}}(I_{\mathcal{R}} - Z)$.

Claim 3.6.2.1. *If $\tau_{\mathcal{M}}(I_{\mathcal{R}} - Z) \neq 0$, then there exist projections P_1 and P_2 in $\mathcal{R}$ such that*

(a_1) $\tau_{\mathcal{M}}(P_1) = \tau_{\mathcal{M}}(P_2) = k$.

(b_1) $P_1P_2 = 0$ and $P_1 + P_2 = E_1 + E_2$.

(c_1) $FP_i = P_iF$ and $E_jP_i = P_iE_j$ for $i, j = 1, 2$.

Proof of Claim 3.6.2.1. Recall $\tau_{\mathcal{M}}(I_{\mathcal{R}}) = 2k$ and let $\tau_{\mathcal{M}}(Z) = 2x$ for some nonnegative integer x. Then $2k - 2x = \tau_{\mathcal{M}}(I_{\mathcal{R}} - Z) > 0$. That is, $k > x$. We will consider the cases where x is even and x is odd separately.

First, we assume that $x = 2s$ is an even integer with $s \geq 0$. By condition (a) on $Z\mathcal{B}$, we can choose a projection G_1, a sum of s minimal projections, in $Z\mathcal{B}$ such that $\tau_{\mathcal{M}}(G_1) = 2s = x$. By condition (b) on $(I_{\mathcal{R}} - Z)\mathcal{B}$, we know that $\dim((I_{\mathcal{R}} - Z)\mathcal{B}) = 2k - 2x > 0$, and we can choose a projection G_2, a sum of $k - x$ minimal projections, in $(I_{\mathcal{R}} - Z)\mathcal{B}$ such that $\tau_{\mathcal{M}}(G_2) = k - x$. Let $P_1 = G_1 + G_2$ and $P_2 = I_{\mathcal{R}} - P_1$. Then P_1 and P_2 are desired projections satisfying conditions (a_1), (b_1), (c_1).

Second, we assume that $x = 2s + 1$ is an odd integer with $s \geq 0$. By condition (a) on $Z\mathcal{B}$, we can choose a projection G_1, a sum of s minimal projections, in $Z\mathcal{B}$ such that $\tau_{\mathcal{M}}(G_1) = 2s = x - 1$. By condition (b) on $(I_{\mathcal{R}} - Z)\mathcal{B}$, we know that $\dim((I_{\mathcal{R}} - Z)\mathcal{B}) =$

$2k-2x \geq k-x+1 \geq 0$, and we can choose a projection G_2, a sum of $k-x+1$ minimal projections, in $(I_{\mathcal{R}}-Z)\mathcal{B}$ such that $\tau_{\mathcal{M}}(G_2) = k-x+1$. Let $P_1 = G_1+G_2$ and $P_2 = I_{\mathcal{R}}-P_1$. Then P_1 and P_2 are desired projections satisfying conditions (a_1), (b_1), (c_1).

This ends the proof of Claim 3.6.2.1.

Claim 3.6.2.2. *If $\tau_{\mathcal{M}}(I_{\mathcal{R}}-Z)=0$, then there exist projections $P_1, P_2, Q_1,$ and Q_2 in $\mathcal{R}$ such that*

(a_2) $\tau_{\mathcal{M}}(P_1) = \tau_{\mathcal{M}}(P_2) = \tau_{\mathcal{M}}(Q_1) = \tau_{\mathcal{M}}(Q_2) = k.$

(b_2) $P_1P_2 = 0$, $Q_1Q_2 = 0$, *and* $P_1+P_2 = Q_1+Q_2 = E_1+E_2.$

(c_2) $FP_i = P_iF$, $P_iQ_j = Q_jP_i$, *and* $E_jQ_i = Q_iE_j$ *for* $i, j = 1, 2.$

Proof of Claim 3.6.2.2. Assume that $\tau_{\mathcal{M}}(I_{\mathcal{R}}-Z)=0$. Then $Z = I_{\mathcal{R}}$ and $\tau_{\mathcal{M}}(Z) = \tau_{\mathcal{M}}(I_{\mathcal{R}}) = 2k$. Furthermore $E_1, E_2, F, I_{\mathcal{R}}-F$ are mutually equivalent projections in $Z\mathcal{W} = \mathcal{W}$.

By condition (a) on $\mathcal{B}$, we know that $\dim(\mathcal{B}) = \dim(Z\mathcal{B}) = k \geq 2$ and we let K_1, K_2 be two orthogonal minimal projections in $\mathcal{B}$ such that

$$K_1E_1 \neq 0.$$

Let

$$Q_1 = K_1 + (I_{\mathcal{R}} - K_1 - K_2)F \qquad \text{and} \qquad Q_2 = I_{\mathcal{R}} - Q_1.$$

Then, since $F, I_{\mathcal{R}}-F$ are mutually equivalent in $\mathcal{W}$ and K_1, K_2 are in $\mathcal{W}' \cap \mathcal{R}$, we have

$$\begin{aligned}
\tau_{\mathcal{M}}(Q_1) &= \tau_{\mathcal{M}}(K_1) + \tau_{\mathcal{M}}((I_{\mathcal{R}} - K_1 - K_2)F) \\
&= 2 + k - \tau_{\mathcal{M}}(K_1F) - \tau_{\mathcal{M}}(K_2F) \\
&= 2 + k - \frac{\tau_{\mathcal{M}}(K_1(F + (I_{\mathcal{R}} - F)))}{2} - \frac{\tau_{\mathcal{M}}(K_2(F + (I_{\mathcal{R}} - F)))}{2} \\
&= k,
\end{aligned}$$

and

$$\tau_{\mathcal{M}}(Q_2) = \tau_{\mathcal{M}}(I_{\mathcal{R}} - Q_1) = k.$$

Moreover,

$$FQ_i = Q_iF \text{ for } i = 1, 2.$$

Since K_1E_1 is assumed to be a non-zero projection in $\mathcal{R}$, it follows that $Q_1 \wedge E_1 \geq K_1E_1 \neq 0$. Let $\mathcal{W}_1$ be the von Neumann subalgebra generated by Q_1, E_1, E_2 in $\mathcal{R}$ and Z_1 a projection in the center of $\mathcal{W}_1$ such that $Z_1\mathcal{W}_1$ is of type I_2 and $(I_{\mathcal{R}} - Z_1)\mathcal{W}_1$ is abelian. Note that $I_{\mathcal{R}} - Z_1 \geq Q_1 \wedge E_1 \geq K_1E_1 \neq 0$. Applying Claim 3.6.2.1 to Q_1, E_1, E_2, there exist projections P_1 and P_2 in $\mathcal{R}$ such that

(a$_3$) $\tau_{\mathcal{M}}(P_1) = \tau_{\mathcal{M}}(P_2) = k$.

(b$_3$) $P_1P_2 = 0$ and $P_1 + P_2 = E_1 + E_2$.

(c$_3$) $Q_1P_i = P_iQ_1$ and $E_jP_i = P_iE_j$ for $i, j = 1, 2$.

It is easy to verify that

$$Q_2P_i = P_iQ_2 \text{ for } i = 1, 2.$$

Therefore, P_1, P_2, Q_1, Q_2 are desired projections in $\mathcal{R}$ satisfying conditions (a$_2$), (b$_2$), (c$_2$). This ends the proof of Claim 3.6.2.2.

The proof of Lemma 3.6.2 is now completed by Claim 3.6.2.1 and Claim 3.6.2.2. □

We are now positioned to provide analogues of Proposition 3.5.3 and Proposition 3.5.4 in the case of atomic von Neumann algebras.

Proposition 3.6.3. *Suppose $\mathcal{M}$ is atomic and $\tau_{\mathcal{M}}$ is the canonical, faithful, normal, semi-finite, tracial weight of $\mathcal{M}$ such that*

$$\tau_{\mathcal{M}}(H) = 1 \textit{ for each minimal projection } H \textit{ in } \mathcal{M}.$$

Let k be a positive integer such that $k \geq 2$ and $\varphi : \mathcal{P}_k(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$ be a trace-ortho-preserving map. Then the following statements are equivalent.

(i) If E_1, E_2, F are mutually commutative projections in $\mathcal{P}_k(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E_1E_2 = 0$ and $F \leq E_1 + E_2$, then $\varphi(F) \leq \varphi(E_1) + \varphi(E_2)$.

(ii) If E_1, E_2, F are projections in $\mathcal{P}_k(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E_1E_2 = 0$ and $F \leq E_1 + E_2$, then $\varphi(F) \leq \varphi(E_1) + \varphi(E_2)$.

Proof. The proof is similar to that of Proposition 3.5.3, where Lemma 3.5.2 should be replaced by Lemma 3.6.2. □

Proposition 3.6.4. *Suppose $\mathcal{M}$ is an atomic von Neumann algebra with no type I_2 direct summand and $\tau_{\mathcal{M}}$ is the canonical, faithful, normal, semi-finite, tracial weight of $\mathcal{M}$ such that*

$$\tau_{\mathcal{M}}(H) = 1 \text{ for each minimal projection } H \text{ in } \mathcal{M}.$$

Let k be a positive integer. Assume that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \infty$. For each projection E in $\mathcal{M}$ with $\tau_{\mathcal{M}}(E) < \infty$, there exists an atomic von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ such that

(i) $\tau_{\mathcal{M}}(I_{\mathcal{M}_1}) = nk$ for some positive integer $n \geq 4$.

(ii) $E\mathcal{M}E \subseteq \mathcal{M}_1$.

(iii) $\mathcal{M}_1$ has no direct summand of type I_2.

Proof. The proof is similar to that of Proposition 3.5.4 and has been omitted. □

3.7 Two projections theory

In this section, $\mathcal{M}$ is assumed to be a semi-finite von Neumann algebra which is not necessarily diffuse nor atomic. A useful tool when computing L^p-norms of projections is Halmos' two projections theorem. We state a version of this theorem below.

Theorem 3.7.1. *(Halmos's Two Projections Theorem) Let E, F be two projections in $\mathcal{M}$. Then E and F have matrix representations*

$$E = I_1 \oplus I_2 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} I_5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_5 & 0 \\ 0 & R \end{pmatrix}$$

and

$$F = I_1 \oplus 0 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} I_5 & 0 \\ 0 & R \end{pmatrix}$$

where the I_j are projections in $\mathcal{M}$ with $I_1 = E \wedge F$, $I_2 = E \wedge F^\perp$, $I_3 = E^\perp \wedge F$, $I_4 = E^\perp \wedge F^\perp$, $I_5 = E - (I_1 + I_2)$, and $I_6 = E^\perp - (I_3 + I_4)$; C and S are self-adjoint operators in $\mathcal{M}$ vanishing on $I_5^\perp(H)$ which satisfy $0 \leq C, S \leq I_5$, $S^2 + C^2 = I_5$, and $CS = SC$; and R is a partial isometry in $\mathcal{M}$ such that $R^*R = I_6$ and $RR^* = I_5$.

For further reading on the two projections theorem, see [10], Theorem 1.1 in [2], and the argument preceding Theorem 1.41 in [27]. One application of this useful theorem is the following.

Lemma 3.7.2. *Suppose E, F are projections in $\mathcal{M}$ such that $\tau_{\mathcal{M}}(E \vee F) < \infty$. The following statements are true.*

(i) $\tau_{\mathcal{M}}(E \wedge F) \leq \tau_{\mathcal{M}}(EF)$ *with equality if and only if $EF = FE$.*

(ii) Assume that $0 < p < \infty$. Then

$$\|E - F\|_p^p \leq \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(E \wedge F),$$

with equality if and only if $EF = FE$.

(iii) Assume that $0 < p \leq 2$. Then

$$\|E - F\|_p^p \geq \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(EF).$$

Proof. By Theorem 3.7.1, E and F can be represented as

$$E = I_1 \oplus I_2 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} I_5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_5 & 0 \\ 0 & R \end{pmatrix}$$

and

$$F = I_1 \oplus 0 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} I_5 & 0 \\ 0 & R \end{pmatrix}.$$

(i). Observe that

$$EF = I_1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} I_5 & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} C^2 & CS \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_5 & 0 \\ 0 & R \end{pmatrix}, \tag{3.1}$$

whence

$$\tau_{\mathcal{M}}(EF) = \tau_{\mathcal{M}}(I_1) + \tau_{\mathcal{M}}(C^2) = \tau_{\mathcal{M}}(E \wedge F) + \tau_{\mathcal{M}}(C^2).$$

We conclude that $\tau_{\mathcal{M}}(EF) \geq \tau_{\mathcal{M}}(E \wedge F)$, with equality if and only if $C = 0$ if and only if $EF = I_1 = E \wedge F$.

(ii). We will compute $\|E - F\|_p^p$ in steps. First observe

$$E - F = 0 \oplus I_2 \oplus -I_3 \oplus 0 \oplus \begin{pmatrix} S^2 & -CSR \\ -R^*CS & -R^*S^2R \end{pmatrix}.$$

By a calculation

$$
\begin{aligned}
(E-F)^2 &= 0 \oplus I_2 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} S^4 + C^2S^2 & -S^3CR + S^3CR \\ -R^*CS^3 + R^*CS^3 & R^*C^2S^2R + R^*S^4R \end{pmatrix} \\
&= 0 \oplus I_2 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} S^2(S^2+C^2) & 0 \\ 0 & R^*(C^2+S^2)S^2R \end{pmatrix} \\
&= 0 \oplus I_2 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} S^2 & 0 \\ 0 & R^*S^2R \end{pmatrix}.
\end{aligned}
$$

Since E, F are both self-adjoint we have $(E-F)^*(E-F) = (E-F)^2$. Therefore

$$
|E-F| = ((E-F)^*(E-F))^{1/2} = 0 \oplus I_2 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} S & 0 \\ 0 & R^*SR \end{pmatrix},
$$

from which it follows that

$$
|E-F|^p = 0 \oplus I_2 \oplus I_3 \oplus 0 \oplus \begin{pmatrix} S^p & 0 \\ 0 & R^*S^pR \end{pmatrix}.
$$

We conclude that

$$
\|E-F\|_p^p = \tau_{\mathcal{M}}(|E-F|^p) = \tau_{\mathcal{M}}(I_2) + \tau_{\mathcal{M}}(I_3) + 2\tau_{\mathcal{M}}(S^p). \tag{3.2}
$$

From their representations we find the traces of E and F are

$$
\tau_{\mathcal{M}}(E) = \tau_{\mathcal{M}}(I_1) + \tau_{\mathcal{M}}(I_2) + \tau_{\mathcal{M}}(I_5)
$$

and

$$\begin{aligned}\tau_{\mathcal{M}}(F) &= \tau_{\mathcal{M}}(I_1) + \tau_{\mathcal{M}}(I_3) + \tau_{\mathcal{M}}(C^2) + \tau_{\mathcal{M}}(R^* S^2 R) \\ &= \tau_{\mathcal{M}}(I_1) + \tau_{\mathcal{M}}(I_3) + \tau_{\mathcal{M}}(C^2) + \tau_{\mathcal{M}}(S^2) \\ &= \tau_{\mathcal{M}}(I_1) + \tau_{\mathcal{M}}(I_3) + \tau_{\mathcal{M}}(I_5).\end{aligned}$$

Recall that $S^p \leq I_5$ and $I_1 = E \wedge F$. Therefore

$$\begin{aligned}\|E - F\|_p^p &= \tau_{\mathcal{M}}(I_2) + \tau_{\mathcal{M}}(I_3) + 2\tau_{\mathcal{M}}(S^p) \\ &= \tau_{\mathcal{M}}(E) - \tau_{\mathcal{M}}(I_1) - \tau_{\mathcal{M}}(I_5) + \tau_{\mathcal{M}}(F) - \tau_{\mathcal{M}}(I_1) - \tau_{\mathcal{M}}(I_5) + 2\tau_{\mathcal{M}}(S^p) \\ &= \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(E \wedge F) - 2\tau_{\mathcal{M}}(I_5 - S^p) \\ &\leq \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(E \wedge F)\end{aligned}$$

with equality if and only if $S^p = I_5$. Clearly this is equivalent to the condition that $C = 0$. It follows from (3.1) that $\|E - F\|_p^p = \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(E \wedge F)$ if and only if $C = 0$ if and only if $EF = FE$.

(iii). Assume that $0 < p \leq 2$. Recall that $S^2 + C^2 = I_5$. We have

$$\begin{aligned}\|E - F\|_p^p &= \tau_{\mathcal{M}}(I_2) + \tau_{\mathcal{M}}(I_3) + 2\tau_{\mathcal{M}}(S^p) && \text{(by (3.2))} \\ &\geq \tau_{\mathcal{M}}(I_2) + \tau_{\mathcal{M}}(I_3) + 2\tau_{\mathcal{M}}(S^2) \\ &= \tau_{\mathcal{M}}(E) - \tau_{\mathcal{M}}(I_1) - \tau_{\mathcal{M}}(I_5) + \tau_{\mathcal{M}}(F) - \tau_{\mathcal{M}}(I_1) - \tau_{\mathcal{M}}(I_5) + 2\tau_{\mathcal{M}}(S^2) \\ &= \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(I_1) - 2\tau_{\mathcal{M}}(I_5 - S^2) \\ &= \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(I_1) - 2\tau_{\mathcal{M}}(C^2) \\ &= \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - 2\tau_{\mathcal{M}}(EF). && \text{(by (3.1))}\end{aligned}$$

This completes the last part of the proof. □

CHAPTER 4
WIGNER'S THEOREM FOR ORTHO-ISOMORPHISMS ON SEMI-FINITE VON NEUMANN ALGEBRAS

In this chapter, we assume $\mathcal{M}$ is a semi-finite von Neumann algebra with faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$ and identity I. For a positive number c, let φ be an ortho-isomorphism from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ onto $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Motivated by [25, 26], we consider the question of when φ can be extended to a Jordan $*$-isomorphism on $\mathcal{M}$. We will consider two cases: first, when $\mathcal{M}$ is diffuse and second, when $\mathcal{M}$ is atomic and every minimal projection has the same trace.

4.1 Extension for ortho-isomorphisms on diffuse von Neumann algebras

4.1.1 Assumptions and statement of main result

Assumption 4.1.1. *We assume the following are true in this section.*

(i) $0 < c < \tau_{\mathcal{M}}(I)$.

(ii) $\mathcal{M}$ *is diffuse and has no type* I_2 *direct summand.*

(iii) $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is an ortho-isomorphism.*

Remark 4.1.2. By Assumption 4.1.1, we may make full use of Lemma 3.5.1 throughout this section.

This section is devoted to the following result, whose proof will be postponed until subsection 4.1.4.

Theorem 4.1.3. *Under Assumption* 4.1.1*, if* $0 < c < \frac{\tau_{\mathcal{M}}(I)}{2}$*, then* φ *can be extended to a Jordan* $*$*-isomorphism* $\rho : \mathcal{M} \to \mathcal{M}$*.*

We aim to employ Lemma 3.3.6, therefore we first need to extend the map φ on $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ to a map on $\mathcal{P}(\mathcal{M})$ by considering two cases: $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{4})$ and $c \in [\frac{\tau_{\mathcal{M}}(I)}{4}, \frac{\tau_{\mathcal{M}}(I)}{2})$.

4.1.2 Extension when $0 < c < \frac{\tau_{\mathcal{M}}(I)}{4}$

We will first demonstrate that φ can be extended when $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{4})$. Later in this section we will use a more advanced method for the case where $c \in [\frac{\tau_{\mathcal{M}}(I)}{4}, \frac{\tau_{\mathcal{M}}(I)}{2})$. Before continuing, we recall notation and terminology which is relevant to this chapter.

For $S, \{c\} \subseteq [0, \tau_{\mathcal{M}}(I)]$, we let

$$\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) = c\};$$

$$\mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \geq c \text{ and } \tau_{\mathcal{M}}(I - P) \geq c\};$$

$$\mathcal{P}_S(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \in S\}.$$

Given a bijection $\phi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ we say that

(i) ϕ is an ortho-isomorphism if, for all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, $E \perp F$ if and only if $\phi(E) \perp \phi(F)$.

(ii) ϕ is an order-isomorphism if, for all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, $E \leq F$ if and only if $\phi(E) \leq \phi(F)$.

(iii) ϕ is an order-ortho-isomorphism if ϕ is both an ortho-isomorphism and an order-isomorphism.

Lemma 4.1.4. *Let* $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{2})$*. If* $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is a projection and* $\{E_\lambda\} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is a family of projections with* $\bigvee_\lambda E_\lambda \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$*, then*

(i) $\bigvee_\lambda \varphi(E_\lambda) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$*;*

(ii) $E \leq \bigvee_\lambda E_\lambda$ *if and only if* $\varphi(E) \leq \bigvee_\lambda \varphi(E_\lambda)$.

Proof. (i). Since $\tau_{\mathcal{M}}(I - \bigvee_\lambda E_\lambda) \geq c$, there exists a projection $F \leq I - \bigvee_\lambda E_\lambda$ with trace c. For each λ we have $F \perp E_\lambda$, hence $\varphi(F) \perp \varphi(E_\lambda)$. This implies

$$\tau_{\mathcal{M}}(I - \bigvee_\lambda \varphi(E_\lambda)) \geq \tau_{\mathcal{M}}(\varphi(F)) = c.$$

Also $\tau_{\mathcal{M}}(\bigvee_\lambda \varphi(E_\lambda)) \geq \tau_{\mathcal{M}}(\varphi(E_\lambda)) = c$ for each λ. Hence $\varphi(E_\lambda) \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, and $\{\varphi(E_\lambda)\} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $\bigvee_\lambda \varphi(E_\lambda) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

(ii). First assume $E \leq \bigvee_\lambda E_\lambda$. Since $\bigvee_\lambda \varphi(E_\lambda) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$, there is a family $\{F_\mu\}$ of projections, each with trace c, such that $\bigvee_\mu F_\mu = I - \bigvee_\lambda \varphi(E_\lambda)$. Therefore $F_\mu \perp \varphi(E_\lambda)$ for any choice of μ, λ. As φ is a surjection, to each F_μ there corresponds some $G_\mu \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $\varphi(G_\mu) = F_\mu$. Hence $\varphi(G_\mu) \perp \varphi(E_\lambda)$ for any choice of μ, λ. Since φ is an ortho-isomorphism, $G_\mu \perp E_\lambda$ for all μ, λ. We conclude that, for all μ, $E \leq \bigvee_\lambda E_\lambda \leq I - G_\mu$. As φ preserves orthogonality, we must have $\varphi(E) \perp \varphi(G_\mu)$ for all μ. Thus

$$\varphi(E) \leq I - \bigvee_\mu \varphi(G_\mu) = I - \bigvee_\mu F_\mu = \bigvee_\lambda \varphi(E_\lambda).$$

To prove the converse we apply the preceding proof to the ortho-isomorphism φ^{-1} given the projection $\varphi(E_\lambda) \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and family $\{\varphi(E_\lambda)\} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $\bigvee_\lambda \varphi(E_\lambda) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. □

Proposition 4.1.5. *Let* $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{2})$. *The ortho-isomorphism* φ *can be extended to an order-ortho-isomorphism* $\varphi^+ : \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *satisfying the following properties:*

(i) $\varphi^+(E) = \varphi(E)$ *for each* $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*;*

(ii) $\varphi^+(I - P) = I - \varphi^+(P)$ *for each* $P \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$*;*

(iii) $\varphi^+(P_1 \vee P_2) = \varphi^+(P_1) \vee \varphi^+(P_2)$ *for all* $P_1, P_2 \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *with* $P_1 \vee P_2 \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

Proof. Define a map $\varphi^+ : \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{M})$ by

$$\varphi^+(P) = \bigvee\{\varphi(E) \mid E \leq P \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

For each $P \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$, observe that $\bigvee\{E \mid E \leq P \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\} = P \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. By Lemma 4.1.4 we have $\varphi^+(P) = \bigvee\{\varphi(E) \mid E \leq P \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\} \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$, so that φ^+ is a map from $\mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ to $\mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

Claim 4.1.5.1. *If $P \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and $\{E_\lambda\}$ is a family of projections in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P = \bigvee_\lambda E_\lambda$, then $\varphi^+(P) = \bigvee_\lambda \varphi(E_\lambda)$.*

Proof of Claim 4.1.5.1. Note, first, that by the definition of φ^+ and the fact that $E_\lambda \leq P$ for each λ, we have $\bigvee_\lambda \varphi(E_\lambda) \leq \varphi^+(P)$. For the other inequality, observe that if $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and $E \leq P = \bigvee_\lambda E_\lambda$, then $\varphi(E) \leq \bigvee_\lambda \varphi(E_\lambda)$ by Lemma 4.1.4. Thus $\varphi^+(P) \leq \bigvee_\lambda \varphi(E_\lambda)$. This finishes the proof of the claim.

For the remainder of the proof, let $P_1, P_2 \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Then there are families $\{E_\lambda\}, \{F_\mu\} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P_1 = \bigvee_\lambda E_\lambda$ and $P_2 = \bigvee_\mu F_\mu$.

We first show φ^+ is a bijection. Suppose $\varphi^+(P_1) = \varphi^+(P_2)$. By Claim 4.1.5.1, observe that for each λ we have

$$\varphi(E_\lambda) \leq \bigvee_\lambda \varphi(E_\lambda) = \varphi^+(P_1) = \varphi^+(P_2) = \bigvee_\mu \varphi(F_\mu).$$

Hence, by Lemma 4.1.4, we have $E_\lambda \leq \bigvee_\mu F_\mu = P_2$ for every λ. We conclude $P_1 \leq P_2$. Using the same process for each μ we obtain $P_2 \leq P_1$, whence $P_1 = P_2$ and φ^+ is injective.

For arbitrary $P_1 \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$, since $P_1 = \bigvee_\lambda E_\lambda$ and φ is a bijection there is a family $\{G_\lambda\} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $\varphi(G_\lambda) = E_\lambda$ for each λ. Applying Lemma 4.1.4 to the ortho-isomorphism φ^{-1} we find that $\bigvee_\lambda G_\lambda \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Therefore. by Claim 4.1.5.1,

$$\varphi^+(\bigvee_\lambda G_\lambda) = \bigvee_\lambda \varphi(G_\lambda) = \bigvee_\lambda E_\lambda = P_1.$$

Thus φ^+ is surjective.

Next we show φ^+ is ortho-preserving in both directions. Indeed, $P_1 \perp P_2$ if and only if $E_\lambda \perp F_\mu$ for all λ, μ. As φ is an ortho-isomorphism, this latter statement is equivalent to the requirement that $\varphi(E_\lambda) \perp \varphi(F_\mu)$ for all λ, μ, or, stated differently, that $\bigvee_\lambda \varphi(E_\lambda) \perp \bigvee_\mu \varphi(F_\mu)$. In light of Claim 4.1.5.1, this last condition is precisely $\varphi^+(P_1) \perp \varphi^+(P_2)$. We have therefore shown that $P_1 \perp P_2$ if and only if $\varphi^+(P_1) \perp \varphi^+(P_2)$.

To show φ^+ is order-preserving in both directions, note that $P_1 \leq P_2$ if and only if $E_\lambda \leq \bigvee_\mu F_\mu$ for each λ. Combining Lemma 4.1.4 and Claim 4.1.5.1, this is equivalent to the condition $\varphi(E_\lambda) \leq \bigvee_\mu \varphi(F_\mu) = \varphi^+(P_2)$, which is to say that $\varphi^+(P_1) = \bigvee_\lambda \varphi(E_\lambda) \leq \varphi^+(P_2)$. This completes the proof that $\varphi^+ : \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ is an order-ortho-isomorphism. We now verify the three additional properties listed in the proposition.

(i). This is immediate from the definition of φ^+.

(ii). Suppose $P \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Since $\varphi^+(P) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$, we must have $I - \varphi^+(P) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. As φ^+ is a surjection, there must be some $Q \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $\varphi^+(Q) = I - \varphi^+(P)$. We need only show $Q = I - P$. Since $\varphi^+(Q) \perp \varphi^+(P)$ and φ^+ is ortho-preserving in both directions, we find $Q \perp P$. Hence $Q \leq I - P$. Further, since $(I - P) \perp P$ we must also have $\varphi^+(I - P) \perp \varphi^+(P)$. That is, $\varphi^+(I - P) \leq I - \varphi^+(P) = \varphi^+(Q)$. As φ^+ preserves order in both directions, this implies $I - P \leq Q$. Thus $Q = I - P$, so that $\varphi^+(I - P) = \varphi^+(Q) = I - \varphi^+(P)$.

(iii). Suppose $P_1 \vee P_2 \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. We consider the image of $P_1 \vee P_2$ under the map φ^+. Since $P_1 \vee P_2$ is simply the union of all the E_λ and F_μ, by Claim 4.1.5.1 we see that

$$\varphi^+(P_1 \vee P_2) = \varphi^+(\bigvee_\lambda E_\lambda \vee \bigvee_\mu F_\mu) = \bigvee_\lambda \varphi(E_\lambda) \vee \bigvee_\mu \varphi(F_\mu) = \varphi^+(P_1) \vee \varphi^+(P_2).$$

This finishes the proof. $\square$

Lemma 4.1.6. *Let $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{4})$. For all $E_1, E_2, E_3 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ we have*

$$E_1 \wedge E_2 \leq E_3 \text{ if and only if } \varphi(E_1) \wedge \varphi(E_2) \leq \varphi(E_3).$$

Proof. Let $E_1, E_2, E_3 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. As a consequence of Kaplansky's formula on traces, Lemma 3.2.2,

$$\tau_{\mathcal{M}}(E_1 \vee E_2 \vee E_3) \leq \tau_{\mathcal{M}}(E_1) + \tau_{\mathcal{M}}(E_2) + \tau_{\mathcal{M}}(E_3) = 3c < \tau_{\mathcal{M}}(I).$$

Hence there is some projection $G \in \mathcal{P}_{3c}(\mathcal{M}, \tau_{\mathcal{M}})$ with $E_1 \vee E_2 \vee E_3 \leq G$. For each $1 \leq i \leq 3$ we have $\tau_{\mathcal{M}}(G - E_i) = 2c > c$. Further,

$$\tau_{\mathcal{M}}(I - (G - E_i)) = \tau_{\mathcal{M}}(I) - \tau_{\mathcal{M}}(G - E_i) = \tau_{\mathcal{M}}(I) - 2c > 4c - 2c = 2c > c,$$

so that $G - E_i \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ for all $1 \leq i \leq 3$. Likewise,

$$\tau_{\mathcal{M}}((G - E_1) \vee (G - E_2)) \geq \tau_{\mathcal{M}}(G - E_1) = 2c > c,$$

and since $(G - E_1) \vee (G - E_2) = G - (E_1 \wedge E_2)$ we observe that

$$\begin{aligned}
\tau_{\mathcal{M}}(I - (G - E_1) \vee (G - E_2)) &= \tau_{\mathcal{M}}(I - (G - (E_1 \wedge E_2))) \\
&= \tau_{\mathcal{M}}(I) - \tau_{\mathcal{M}}(G - (E_1 \wedge E_2)) \\
&\geq \tau_{\mathcal{M}}(I) - 3c \\
&> c.
\end{aligned}$$

Thus $(G - E_1) \vee (G - E_2) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Lastly, since $\tau_{\mathcal{M}}(I - G) > 4c - 3c = c$, we must have $G \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

Let $\varphi^+ : \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be the extension of φ guaranteed by Proposition 4.1.5.

By part (iii) of the same proposition,

$$\varphi^+(G) = \varphi^+(E_i \vee (G - E_i)) = \varphi^+(E_i) \vee \varphi^+(G - E_i)$$

for each $1 \le i \le 3$. Since φ^+ preserves orthogonality we have $\varphi^+(E_i) \perp \varphi^+(G - E_i)$ and, as φ^+ extends φ, we observe $\varphi^+(E_i) = \varphi(E_i)$. Therefore

$$\varphi^+(G) = \varphi^+(E_i) \vee \varphi^+(G - E_i) = \varphi(E_i) + \varphi^+(G - E_i).$$

We have shown

$$\varphi^+(G - E_i) = \varphi^+(G) - \varphi(E_i), \quad \text{for all } 1 \le i \le 3. \tag{4.1}$$

Thus

$$\begin{aligned}
E_1 \wedge E_2 \le E_3 &\iff G - E_3 \le G - (E_1 \wedge E_2) \\
&\iff G - E_3 \le (G - E_1) \vee (G - E_2) \\
&\iff \varphi^+(G - E_3) \le \varphi^+((G - E_1) \vee (G - E_2)) \quad (\varphi^+ \text{ an order-isomorphism}) \\
&\iff \varphi^+(G - E_3) \le \varphi^+(G - E_1) \vee \varphi^+(G - E_2) \quad (\text{by Proposition 4.1.5(iii)}) \\
&\iff \varphi^+(G) - \varphi(E_3) \le [\varphi^+(G) - \varphi(E_1)] \vee [\varphi^+(G) - \varphi(E_2)] \qquad (\text{by (4.1)}) \\
&\iff \varphi^+(G) - \varphi(E_3) \le \varphi^+(G) - (\varphi(E_1) \wedge \varphi(E_2)) \\
&\iff \varphi(E_1) \wedge \varphi(E_2) \le \varphi(E_3).
\end{aligned}$$

□

In Proposition 4.1.5 we found an order-ortho-isomorphism,

$$\varphi^+ : \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}),$$

extending φ on $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Analogously we now develop an order-ortho-isomorphism $\varphi^- :$

$\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ which extends φ for $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{4})$.

Proposition 4.1.7. *Let $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{4})$. The ortho-isomorphism φ can be extended to an order-ortho-isomorphism $\varphi^- : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$.*

Proof. Define a map $\varphi^- : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{M})$ by

$$\varphi^-(P) = \bigwedge\{\varphi(E) \mid P \leq E \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

It is easily verified that $\varphi^-(P) \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ and, for any $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, we have $\varphi^-(E) = \varphi(E)$. Thus $\varphi^- : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ is an extension of φ.

Claim 4.1.7.1. *If $P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $E_1, E_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ satisfy $P = E_1 \wedge E_2$, then $\varphi^-(P) = \varphi(E_1) \wedge \varphi(E_2)$.*

Proof of Claim 4.1.7.1. Clearly $\varphi^-(P) \leq \varphi(E_1) \wedge \varphi(E_2)$ by the definition of φ^-. Conversely, if $P \leq E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, then by Lemma 4.1.6 we must have $\varphi(E_1) \wedge \varphi(E_2) \leq \varphi(E)$. Since this happens for every such E, by the definition of φ^- we obtain $\varphi(E_1) \wedge \varphi(E_2) \leq \varphi^-(P)$. Thus $\varphi^-(P) = \varphi(E_1) \wedge \varphi(E_2)$ as desired.

For the remainder of the proof suppose $P_1, P_2 \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$, so that there must exist $E_1, E_2, F_1, F_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ which satisfy $P_1 = E_1 \wedge E_2$ and $P_2 = F_1 \wedge F_2$.

We will show φ^- is bijective. Indeed, if $\varphi^-(P_1) = \varphi^-(P_2)$ we have

$$\varphi(E_1) \wedge \varphi(E_2) = \varphi(F_1) \wedge \varphi(F_2) \leq \varphi(F_1), \varphi(F_2)$$

by Claim 4.1.7.1. Hence, by Lemma 4.1.6, we find $E_1 \wedge E_2 \leq F_1, F_2$, so that $P_1 = E_1 \wedge E_2 \leq F_1 \wedge F_2 = P_2$. Reversing the roles of P_1 and P_2 we immediately obtain the reverse inequality. Thus $P_1 = P_2$ and φ^- is injective.

For arbitrary $P_1 \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ we observe that since $P_1 = E_1 \wedge E_2$ and φ is surjective, there are $E_1', E_2' \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $\varphi(E_1') = E_1$ and $\varphi(E_2') = E_2$. Clearly $E_1' \wedge E_2'$ is an element of $\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ and, by Claim 4.1.7.1, $\varphi^-(E_1' \wedge E_2') = P_1$. Thus φ^- is surjective.

Next we show φ^- preserves order in both directions. If $P_1 \le P_2$, then by definition $\varphi^-(P_1) \le \varphi^-(P_2)$, as $\varphi^-(P_2)$ is the intersection of only some of the projections $E \ge P_1$ with $\tau_{\mathcal{M}}(E) = c$. If, conversely, we assume $\varphi^-(P_1) \le \varphi^-(P_2)$, then as in the proof that φ^- is injective we find that

$$\varphi(E_1) \wedge \varphi(E_2) \le \varphi(F_1) \wedge \varphi(F_2) \le \varphi(F_1), \varphi(F_2)$$

implies $P_1 \le P_2$. Thus φ^- preserves order in both directions.

Finally, we show φ^- preserves orthogonality in both directions. Indeed, if $P_1 \perp P_2$, then, since $\tau_{\mathcal{M}}(I - P_1) = \tau_{\mathcal{M}}(I) - \tau_{\mathcal{M}}(P_1) > 4c - c = 3c$ and $P_2 \le I - P_1$, there is some $F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P_2 \le F \le I - P_1$. Now $P_1 \le I - F$ and $\tau_{\mathcal{M}}(I - F) > 4c - c > c$, so there exists some projection $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P_1 \le E \le I - F$. Note that $E \perp F$ implies $\varphi(E) \perp \varphi(F)$ since φ is an ortho-isomorphism. As φ^- preserves order and extends φ we find

$$\varphi^-(P_1) \le \varphi(E) \le I - \varphi(F) \le I - \varphi^-(P_2).$$

Thus $\varphi^-(P_1) \perp \varphi^-(P_2)$. A similar argument shows that φ^- preserves orthogonality in the opposite direction. This completes the proof that φ^- is an order-ortho-isomorphism. □

In the following, we show that the map $\varphi^- : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ obtained in Proposition 4.1.7 can in turn be extended to an order-ortho-isomorphism $\psi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$.

Proposition 4.1.8. *If $\phi : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ is an order-ortho-isomorphism, then ϕ can be extended to an order-ortho-isomorphism $\psi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$.*

Proof. By Assumption 4.1.1 and Lemma 3.5.1, for each $P \in \mathcal{P}(\mathcal{M})$ we can find a family $\{E_\lambda\} \subseteq \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ with $P = \bigvee_\lambda E_\lambda$. We define the map $\psi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ by

$$\psi(P) = \bigvee \{\phi(E) \mid E \le P \text{ and } E \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}(\mathcal{M}).$$

Clearly ψ is well-defined and extends ϕ. We will prove a few claims before showing that ψ is an order-ortho-isomorphism.

Claim 4.1.8.1. *Suppose $E \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $\{E_\lambda\} \subseteq \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$. Then*

$$E \leq \bigvee_\lambda E_\lambda \text{ if and only if } \phi(E) \leq \bigvee_\lambda \phi(E_\lambda).$$

Proof of Claim 4.1.8.1. First, suppose $E \leq \bigvee_\lambda E_\lambda$. There is a family $\{G_\mu\} \subseteq \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ with $I - \bigvee_\lambda \phi(E_\lambda) = \bigvee_\mu G_\mu$. As ϕ is surjective, there is some family $\{F_\mu\} \subseteq \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ with $\phi(F_\mu) = G_\mu$. We now have $\bigvee_\lambda \phi(E_\lambda) \perp \bigvee_\mu \phi(F_\mu)$, which implies $\phi(E_\lambda) \perp \phi(F_\mu)$ for all μ, λ. Since ϕ is ortho-preserving in both directions, we have $E_\lambda \perp F_\mu$ for all μ, λ. Thus, for all μ, we observe $E \leq \bigvee_\lambda E_\lambda \leq I - F_\mu$. Now $E \perp F_\mu$ implies $\phi(E) \perp \phi(F_\mu)$ for every μ, so we conclude

$$\phi(E) \leq I - \bigvee_\mu \phi(F_\mu) = I - \bigvee_\mu G_\mu = \bigvee_\lambda \phi(E_\lambda).$$

Conversely, if we assume $\phi(E) \leq \bigvee_\lambda \phi(E_\lambda)$, then we find that $E \leq \bigvee_\lambda E_\lambda$ by applying the forward direction to the order-ortho-isomorphism $\phi^{-1} : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$. This completes the proof of the claim.

Claim 4.1.8.2. *If $\{E_\lambda\}$ is a family of projections in $\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$, then*

$$\psi(\bigvee_\lambda E_\lambda) = \bigvee_\lambda \phi(E_\lambda).$$

Proof of Claim 4.1.8.2. By the definition of ψ we have $\bigvee_\lambda \phi(E_\lambda) \leq \psi(\bigvee_\lambda E_\lambda)$. Conversely, if $E \leq \bigvee_\lambda E_\lambda$ is a projection in $\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$, then Claim 4.1.8.1 shows $\phi(E) \leq \bigvee_\lambda \phi(E_\lambda)$. Since E was arbitrary, we conclude $\psi(\bigvee_\lambda E_\lambda) \leq \bigvee_\lambda \phi(E_\lambda)$. Thus $\psi(\bigvee_\lambda E_\lambda) = \bigvee_\lambda \phi(E_\lambda)$.

For the remainder of the proof, let $P_1, P_2 \in \mathcal{P}(\mathcal{M})$ be two projections. Then there exist two families, $\{E_\lambda\}, \{F_\mu\} \subseteq \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$, such that $P_1 = \bigvee_\lambda E_\lambda$ and $P_2 = \bigvee_\mu F_\mu$.

We will prove ψ is a bijection. If $\psi(P_1) = \psi(P_2)$, then $\bigvee_\lambda \phi(E_\lambda) = \bigvee_\mu \phi(F_\mu)$ by

Claim 4.1.8.2. Since $\phi(E_\lambda) \leq \bigvee_\mu \phi(F_\mu)$ for each λ, using Claim 4.1.8.1 we find that $E_\lambda \leq \bigvee_\mu F_\mu = P_2$ for each λ. Thus $P_1 = \bigvee_\lambda E_\lambda \leq P_2$. Reversing the roles of λ and μ we obtain $P_1 = P_2$, so ψ is injective.

For any P_1, we have that $P_1 = \bigvee_\lambda E_\lambda$ for some $\{E_\lambda\} \subseteq \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_\mathcal{M})$. As ϕ is a surjection, there is a family $\{G_\lambda\} \subseteq \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_\mathcal{M})$ such that $\phi(G_\lambda) = E_\lambda$ for each λ. By Claim 4.1.8.2 $\bigvee_\lambda G_\lambda$ is a projection whose image under ψ is

$$\psi(\bigvee_\lambda G_\lambda) = \bigvee_\lambda \phi(G_\lambda) = \bigvee_\lambda E_\lambda = P_1.$$

Thus ψ is surjective.

We show ψ preserves orthogonality in both directions by the following equivalencies:

$$\begin{aligned} P_1 \perp P_2 &\iff E_\lambda \perp F_\mu, \quad \text{for all } \mu, \lambda \\ &\iff \phi(E_\lambda) \perp \phi(F_\mu), \quad \text{for all } \mu, \lambda && (\phi \text{ an ortho-isomorphism}) \\ &\iff \bigvee_\lambda \phi(E_\lambda) \perp \bigvee_\mu \phi(F_\mu) \\ &\iff \psi(P_1) \perp \psi(P_2). && (\text{by Claim 4.1.8.2}) \end{aligned}$$

Lastly, ψ preserves order in both directions by the following:

$$\begin{aligned} P_1 \leq P_2 &\iff \bigvee_\lambda E_\lambda \leq \bigvee_\mu F_\mu \\ &\iff E_\lambda \leq \bigvee_\mu F_\mu, \quad \text{for all } \lambda \\ &\iff \phi(E_\lambda) \leq \bigvee_\mu \phi(F_\mu), \quad \text{for all } \lambda && (\text{by Claim 4.1.8.1}) \\ &\iff \bigvee_\lambda \phi(E_\lambda) \leq \bigvee_\mu \phi(F_\mu) \\ &\iff \psi(P_1) \leq \psi(P_2). && (\text{by Claim 4.1.8.2}) \end{aligned}$$

This completes the proof that $\psi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ is an order-ortho-isomorphism which extends ϕ.

$\square$

Theorem 4.1.9. *Under Assumption* 4.1.1*, if* $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{4})$*, then* φ *can be extended to a Jordan* $*$*-isomorphism* $\rho : \mathcal{M} \to \mathcal{M}$*.*

Proof. By Proposition 4.1.7, φ extends to an order-ortho-isomorphism $\varphi^- : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ which, in turn, extends to an order-ortho-isomorphism $\psi : \mathcal{P} \to \mathcal{P}$ by Proposition 4.1.8. By Dye's Theorem, Lemma 3.3.6, ψ extends to a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$. Therefore ρ must also extend φ and the proof is complete. $\square$

4.1.3 Extension when $\frac{\tau_{\mathcal{M}}(I)}{4} \leq c < \frac{\tau_{\mathcal{M}}(I)}{2}$

We continue under Assumption 4.1.1 and aim to extend Theorem 4.1.9 to the case when $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{2})$. Observe that if $\tau_{\mathcal{M}}(I) = \infty$, then $(0, \frac{\tau_{\mathcal{M}}(I)}{4}) = (0, \infty)$ and Theorem 4.1.9 would apply for *any* non-zero finite c. We therefore assume that $\tau_{\mathcal{M}}(I) < \infty$ for the remainder of this subsection.

Remark 4.1.10. Suppose $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{2})$ and $\tau_{\mathcal{M}}(I) < \infty$. Consider $\hat{\tau}_{\mathcal{M}}$ defined by $\hat{\tau}_{\mathcal{M}}(H) = \frac{\tau_{\mathcal{M}}(H)}{\tau_{\mathcal{M}}(I)}$ for each $H \in \mathcal{M}^+$. It is easily verified that $\hat{\tau}_{\mathcal{M}}$ is a faithful normal semi-finite tracial weight on $\mathcal{M}$ such that $\hat{\tau}_{\mathcal{M}}(I) = 1$. That is, $\hat{\tau}_{\mathcal{M}}$ is a tracial state. Let $d = \frac{c}{\tau_{\mathcal{M}}(I)} < \frac{1}{2}$. We have that φ is an ortho-isomorphism on the set $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \mathcal{P}_d(\mathcal{M}, \hat{\tau}_{\mathcal{M}})$. In particular, to show φ extends to a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$, it suffices to prove the claim in the case where $\tau_{\mathcal{M}}(I) = 1$.

By Remark 4.1.10, for the remainder of this subsection we will proceed under Assumption 4.1.1 and further assume that

(iv) $\tau_{\mathcal{M}}(I) = 1$.

The following lemma outlines a few helpful properties of an order-isomorphism on a space of the form $\mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$.

Lemma 4.1.11. *Suppose* $0 < a \leq b < 1$. *If* $\phi : \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$ *is an order-isomorphism, then the following are true.*

(i) $\phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})$ *and* $\phi(\mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})$.

(ii) If $P, Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$ *and* $P \vee Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$, *then* $\phi(P \vee Q) = \phi(P) \vee \phi(Q)$.

(iii) If $P, Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$ *and* $P \wedge Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$, *then* $\phi(P \wedge Q) = \phi(P) \wedge \phi(Q)$.

Proof. (i). Observe that $\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})$ is the set of minimal elements in $\mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$. Suppose, by way of contradiction, that $\phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})) \neq \mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})$. If $\tau_{\mathcal{M}}(E) = a$ and $\tau_{\mathcal{M}}(\phi(E)) > a$, then there exists some proper subprojection of $\phi(E)$, necessarily of the form $\phi(F)$ with $\tau_{\mathcal{M}}(F) \in [a, b]$ since ϕ is surjective. Since ϕ is order-preserving in both directions, we find that $F < E$ which contradicts the minimality of E. In particular, we have shown $\phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})) \subseteq \mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})$. The same argument, this time using the order-isomorphism ϕ^{-1}, produces $\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}}) \subseteq \phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}}))$. Thus $\phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})$. As $\mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})$ is the set of maximal elements in $\mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$, a similar proof yields $\phi(\mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})$.

(ii). Suppose $P, Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \vee Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$. Since ϕ is order-preserving, we find $\phi(P), \phi(Q) \leq \phi(P \vee Q)$. In particular $\phi(P) \vee \phi(Q) \leq \phi(P \vee Q)$. Moreover,

$$a \leq \tau_{\mathcal{M}}(\phi(P)) \leq \tau_{\mathcal{M}}(\phi(P) \vee \phi(Q)) \leq \tau_{\mathcal{M}}(\phi(P \vee Q)) \leq b$$

implies $\phi(P) \vee \phi(Q)$ is an element of $\mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$, the domain of the order-isomorphism ϕ^{-1}. Now $\phi(P), \phi(Q) \leq \phi(P) \vee \phi(Q)$ implies $P, Q \leq \phi^{-1}(\phi(P) \vee \phi(Q))$. Hence

$$P \vee Q \leq \phi^{-1}(\phi(P) \vee \phi(Q)).$$

Since ϕ preserves order this implies, in turn, that

$$\phi(P \vee Q) \leq \phi(P) \vee \phi(Q).$$

Thus $\phi(P \vee Q) = \phi(P) \vee \phi(Q)$ as desired.

(iii). Suppose $P, Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \wedge Q \in \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$. Since $P \wedge Q \leq P, Q$ and ϕ preserves order, we have $\phi(P \wedge Q) \leq \phi(P), \phi(Q)$. Thus $\phi(P \wedge Q) \leq \phi(P) \wedge \phi(Q)$. To prove the other inequality, note that we now have

$$a \leq \tau_{\mathcal{M}}(\phi(P \wedge Q)) \leq \tau_{\mathcal{M}}(\phi(P) \wedge \phi(Q)) \leq \tau_{\mathcal{M}}(\phi(P)) \leq b,$$

so that $\phi(P) \wedge \phi(Q)$ is in $\mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}})$, the domain of ϕ^{-1}. Since ϕ^{-1} is an order-isomorphism and $\phi(P) \wedge \phi(Q) \leq \phi(P), \phi(Q)$, we observe

$$\phi^{-1}(\phi(P) \wedge \phi(Q)) \leq P, Q,$$

whence $\phi^{-1}(\phi(P) \wedge \phi(Q)) \leq P \wedge Q$. Since ϕ is an order-isomorphism, this implies

$$\phi(P) \wedge \phi(Q) \leq \phi(P \wedge Q).$$

As $\phi(P \wedge Q) \leq \phi(P) \wedge \phi(Q)$ was already shown, this completes the proof. $\square$

Lemma 4.1.12. *Let $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$ be a real number and $\phi : \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$ be an order-isomorphism. Suppose $a, b \in [y, 1-y]$ are real numbers, with $a \leq b$, such that*

$$\phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}}) \quad \textit{and} \quad \phi(\mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}}).$$

Then

$$\phi(\mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}}).$$

Proof. Clearly $1 > 1-y > \frac{1}{2} > y > 0$. We first show $\phi(\mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}})) \subseteq \mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}})$. If $\tau_{\mathcal{M}}(P) = \frac{a+b}{2} \geq a$, there exists a subprojection $Q \leq P$ with trace a. Define $E_1 = P - Q$, so that E_1 has trace $\frac{a+b}{2} - a = \frac{b-a}{2}$. By a calculation

$$\frac{a+b}{2} + 2\left(\frac{b-a}{2}\right) = \frac{3b-a}{2} \leq \frac{3(1-y)-y}{2} = \frac{3-4y}{2} \leq \frac{3-4(\frac{1}{4})}{2} = 1,$$

which implies

$$2\left(\frac{b-a}{2}\right) \leq 1 - \frac{a+b}{2} = \tau_{\mathcal{M}}(I-P).$$

Hence there exist two mutually orthogonal subprojections $E_2, E_3 \leq I - P$ which satisfy $\tau_{\mathcal{M}}(E_2) = \tau_{\mathcal{M}}(E_3) = \frac{b-a}{2}$. Moreover, since $E_1, Q \leq P \leq I - E_2, I - E_3$, and $E_1 \leq I - Q$, we observe that E_1, E_2, E_3 and Q are all pairwise orthogonal. We now define projections $P_i = Q + E_i$ for $1 \leq i \leq 3$. For all $i \neq j$, we have

$$P_i \vee P_j = Q + E_i + E_j \quad \text{and} \quad P_i \wedge P_j = Q.$$

Thus

$$\tau_{\mathcal{M}}(P_i \vee P_j) = \tau_{\mathcal{M}}(Q + E_i + E_j) = a + 2\left(\frac{b-a}{2}\right) = b$$

and

$$\tau_{\mathcal{M}}(P_i \wedge P_j) = \tau_{\mathcal{M}}(Q) = a.$$

Since $\phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})$ and $\phi(\mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_b(\mathcal{M}, \tau_{\mathcal{M}})$ by hypothesis, we find

$$\tau_{\mathcal{M}}(\phi(P_i \vee P_j)) = b \quad \text{and} \quad \tau_{\mathcal{M}}(\phi(P_i \wedge P_j)) = a.$$

Since the intersections and unions of the P_i, P_j have trace in the interval $[y, 1-y]$, we may apply Lemma 4.1.11 to find

$$\phi(P_i \vee P_j) = \phi(P_i) \vee \phi(P_j) \quad \text{and} \quad \phi(P_i \wedge P_j) = \phi(P_i) \wedge \phi(P_j).$$

By Lemma 3.2.2 we have

$$\begin{aligned}\tau_{\mathcal{M}}(\phi(P_i)) + \tau_{\mathcal{M}}(\phi(P_j)) &= \tau_{\mathcal{M}}(\phi(P_i) \vee \phi(P_j)) + \tau_{\mathcal{M}}(\phi(P_i) \wedge \phi(P_j)) \\ &= \tau_{\mathcal{M}}(\phi(P_i \vee P_j)) + \tau_{\mathcal{M}}(\phi(P_i \wedge P_j)) = b + a\end{aligned}$$

whenever the indices i and j are distinct. Therefore

$$2(a+b) = \tau_{\mathcal{M}}(\phi(P_1)) + \tau_{\mathcal{M}}(\phi(P_2)) + \tau_{\mathcal{M}}(\phi(P_1)) + \tau_{\mathcal{M}}(\phi(P_3)) = 2\tau_{\mathcal{M}}(\phi(P_1)) + a + b,$$

so that $\tau_{\mathcal{M}}(\phi(P_1)) = \frac{a+b}{2}$. We recall that $P_1 = Q + E_1 = P$, and so conclude $\tau_{\mathcal{M}}(\phi(P)) = \frac{a+b}{2}$. This shows $\phi(\mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}})) \subseteq \mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}})$.

For the opposite containment we use the fact that $\phi(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}})$ implies $\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}}) = \phi^{-1}(\mathcal{P}_a(\mathcal{M}, \tau_{\mathcal{M}}))$, and likewise for b, and so we can apply the above argument to the order-isomorphism $\phi^{-1} : \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$ to obtain $\phi^{-1}(\mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}})) \subseteq \mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}})$. Taking the image under ϕ of each set we have

$$\mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}}) \subseteq \phi(\mathcal{P}_{\frac{a+b}{2}}(\mathcal{M}, \tau_{\mathcal{M}}))$$

which completes the proof. $\square$

Using Lemmas 4.1.11 and 4.1.12 we can show that, when $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$, every order-isomorphism on $\mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$ is trace-preserving.

Proposition 4.1.13. *Let $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$ be a real number and*

$$\phi : \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$$

be an order-isomorphism. If $P \in \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$, then $\tau_{\mathcal{M}}(P) = \tau_{\mathcal{M}}(\phi(P))$.

Proof. Let $S = \{x \in [y, 1-y] \mid \phi(\mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})\}$. We want to show $S = [y, 1-y]$. By Lemma 4.1.11 and Lemma 4.1.12, we have that S contains y, $1-y$, and the midpoint of any two elements in S. Thus S is dense in $[y, 1-y]$.

Suppose P is a projection with trace $x \in [y, 1-y]$. We will show $\tau_{\mathcal{M}}(\phi(P)) = x$. Let $\epsilon > 0$. Since S is dense in $[y, 1-y]$, S intersects the open sets $(x-\epsilon, x)$ and $(x, x+\epsilon)$ at some real numbers z_1 and z_2, respectively. Now $z_1 < x = \tau_{\mathcal{M}}(P) < z_2$ implies that there are projections E, F that satisfy

(i) $F \leq P \leq E$;

(ii) $\tau_{\mathcal{M}}(F) = z_1 \in S \quad$ and $\quad \tau_{\mathcal{M}}(E) = z_2 \in S$.

Since ϕ is an order-isomorphism, by (i) we obtain $\phi(F) \leq \phi(P) \leq \phi(E)$. By (ii) and the definition of the set S we have

$$\tau_{\mathcal{M}}(\phi(F)) = \tau_{\mathcal{M}}(F) \quad \text{and} \quad \tau_{\mathcal{M}}(\phi(E)) = \tau_{\mathcal{M}}(E).$$

Thus

$$x - \epsilon < \tau_{\mathcal{M}}(F) = \tau_{\mathcal{M}}(\phi(F)) \leq \tau_{\mathcal{M}}(\phi(P)) \leq \tau_{\mathcal{M}}(\phi(E)) = \tau_{\mathcal{M}}(E) < x + \epsilon.$$

For every $\epsilon > 0$ we now have

$$|\tau_{\mathcal{M}}(\phi(P)) - x| < \epsilon,$$

hence $\tau_{\mathcal{M}}(\phi(P)) = x = \tau_{\mathcal{M}}(P)$. □

Lemma 4.1.14. *Let $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$ be a real number and $\phi : \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$ be an order-isomorphism. Let $\{Q, E_1, E_2, E_3\}$ be a family of projections such that*

(1) $Q \perp (E_1 \vee E_2 \vee E_3)$;

(2) $y = \tau_{\mathcal{M}}(Q) + \tau_{\mathcal{M}}(E_i)$ for all $1 \leq i \leq 3$;

(3) $(E_i \vee E_j) \wedge E_k = 0$ for all distinct indices i, j, k;

and set $P_i = Q + E_i$ for each $1 \leq i \leq 3$. If $\tau_{\mathcal{M}}(Q) \geq \frac{6y-1}{4}$, then $P_1, P_2, P_3 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ are projections that satisfy, for distinct $i, j, k \in \{1, 2, 3\}$, *(i)* $\tau_{\mathcal{M}}(\phi(P_i) \wedge \phi(P_k)) = \tau_{\mathcal{M}}(Q)$;

(ii) $\phi(P_i) \wedge \phi(P_k) = \phi(P_j) \wedge \phi(P_k)$.

Proof. Let $x = \tau_{\mathcal{M}}(Q)$ and assume $x \geq \frac{6y-1}{4}$. Since $E_i \perp Q$ for each index i, we have that each P_i is a projection, and each has trace y by condition (2).

(i). By condition (2) we see that $\tau_{\mathcal{M}}(E_i) = y - x$ for all $1 \leq i \leq 3$, and clearly $E_i \wedge E_j \leq (E_i \vee E_k) \wedge E_j = 0$ for distinct i, j, k by (3). By Lemma 3.2.2 we find that

$$\tau_{\mathcal{M}}(E_i \vee E_j) = \tau_{\mathcal{M}}(E_i) + \tau_{\mathcal{M}}(E_j) - \tau_{\mathcal{M}}(E_i \wedge E_j) = 2(y - x) + \tau_{\mathcal{M}}(0) = 2(y - x)$$

for distinct $i, j \in \{1, 2, 3\}$. We use Lemma 3.2.2 again on the projections $E_i \vee E_j$ and E_k, along with the above result and condition (3), to find

$$\begin{aligned}\tau_{\mathcal{M}}(E_i \vee E_j \vee E_k) &= \tau_{\mathcal{M}}(E_i \vee E_j) + \tau_{\mathcal{M}}(E_k) - \tau_{\mathcal{M}}((E_i \vee E_j) \wedge E_k) \\ &= 2(y - x) + (y - x) + \tau_{\mathcal{M}}(0) \\ &= 3(y - x).\end{aligned}$$

Suppose $i, j, k \in \{1, 2, 3\}$ are distinct. We have

$$P_i \vee P_j = Q + (E_i \vee E_j) \quad \text{and} \quad P_i \vee P_j \vee P_k = Q + (E_i \vee E_j \vee E_k).$$

Considering traces, the results of the preceding paragraph, and the fact that $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$, we obtain

$$\begin{aligned}\tau_{\mathcal{M}}(P_i \vee P_j) &= \tau_{\mathcal{M}}(Q) + \tau_{\mathcal{M}}(E_i \vee E_j) \\ &= x + 2(y - x) \\ &= 2y - x \\ &\leq 2y - \frac{6y - 1}{4} \\ &= \frac{2y + 1}{4} < \frac{1}{2} < 1 - y,\end{aligned}$$

and

$$\begin{aligned}\tau_{\mathcal{M}}(P_i \vee P_j \vee P_k) &= \tau_{\mathcal{M}}(Q) + \tau_{\mathcal{M}}(E_i \vee E_j \vee E_k)\\ &= x + 3(y-x)\\ &= 3y - 2x\\ &\leq 3y - \frac{6y-1}{2}\\ &= \frac{1}{2} < 1 - y.\end{aligned}$$

Since $\tau_{\mathcal{M}}(P_i) = y$ for $1 \leq i \leq 3$, we must have that $P_i \vee P_j$ and $P_i \vee P_j \vee P_k$ belong to $\mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$ for all $i, j, k \in \{1, 2, 3\}$. As elements of the domain of the order-isomorphism ϕ, we may apply Lemma 4.1.11 and Proposition 4.1.13 to obtain $\tau_{\mathcal{M}}(\phi(P_i)) = y$,

$$\tau_{\mathcal{M}}(\phi(P_i) \vee \phi(P_j)) = \tau_{\mathcal{M}}(\phi(P_i \vee P_j)) = \tau_{\mathcal{M}}(P_i \vee P_j) = 2y - x,$$

and

$$\tau_{\mathcal{M}}(\phi(P_i) \vee \phi(P_j) \vee \phi(P_k)) = \tau_{\mathcal{M}}(\phi(P_i \vee P_j \vee P_k)) = \tau_{\mathcal{M}}(P_i \vee P_j \vee P_k) = 3y - 2x$$

for all distinct indices. Applying Lemma 3.2.2 to $\phi(P_i)$ and $\phi(P_k)$ we have, therefore,

$$\tau_{\mathcal{M}}(\phi(P_i) \wedge \phi(P_k)) = \tau_{\mathcal{M}}(\phi(P_i)) + \tau_{\mathcal{M}}(\phi(P_k)) - \tau_{\mathcal{M}}(\phi(P_i) \vee \phi(P_k)) = 2y - (2y - x) = x = \tau_{\mathcal{M}}(Q)$$

as desired.

(ii). Suppose $i, j, k \in \{1, 2, 3\}$ are distinct indices. Continuing as above, we apply

Lemma 3.2.2 to $\phi(P_i) \vee \phi(P_j)$ and $\phi(P_k)$ to find

$$\begin{aligned}\tau_{\mathcal{M}}((\phi(P_i) \vee \phi(P_j)) \wedge \phi(P_k)) &= \tau_{\mathcal{M}}(\phi(P_i) \vee \phi(P_j)) + \tau_{\mathcal{M}}(\phi(P_k)) - \tau_{\mathcal{M}}(\phi(P_i) \vee \phi(P_j) \vee \phi(P_k)) \\ &= 2y - x + y - (3y - 2x) \\ &= x.\end{aligned}$$

Since $\phi(P_i) \wedge \phi(P_k)$ is a subprojection of $(\phi(P_i) \vee \phi(P_j)) \wedge \phi(P_k)$ which also has trace x, we must have that

$$\phi(P_i) \wedge \phi(P_k) = (\phi(P_i) \vee \phi(P_j)) \wedge \phi(P_k).$$

By a similar argument we likewise have

$$\phi(P_j) \wedge \phi(P_k) = (\phi(P_i) \vee \phi(P_j)) \wedge \phi(P_k).$$

Thus

$$\phi(P_i) \wedge \phi(P_k) = (\phi(P_i) \vee \phi(P_j)) \wedge \phi(P_k) = \phi(P_j) \wedge \phi(P_k)$$

when the indices $i, j, k \in \{1, 2, 3\}$ are distinct. □

Lemma 4.1.15. *Let $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$ be a real number and $\phi : \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$ be an order-isomorphism. If $\{P_1, P_2, P_3\}$ are a family of projections in $\mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ such that*

$$\tau_{\mathcal{M}}(P_1 \wedge P_2) = \frac{6y-1}{4} \quad \textit{and} \quad P_1 \wedge P_2 \leq P_3,$$

then

$$\tau_{\mathcal{M}}(\phi(P_1) \wedge \phi(P_2)) = \frac{6y-1}{4} \quad \textit{and} \quad \phi(P_1) \wedge \phi(P_2) \leq \phi(P_3).$$

Proof. Set $Q = P_1 \wedge P_2$ and let $x = \tau_{\mathcal{M}}(Q) = \frac{6y-1}{4}$. Since $Q \leq P_i$ for all $1 \leq i \leq 3$ we can define, for each i, projections $F_i = P_i - Q$ with $\tau_{\mathcal{M}}(F_i) = y - x$. We have

$$F_1 \wedge F_2 = (P_1 \wedge P_2) - Q = 0$$

and

$$\tau_{\mathcal{M}}(P_1 \vee P_2 \vee P_3) = \tau_{\mathcal{M}}(Q + F_1 \vee F_2 \vee F_3) \leq x + 3(y - x) = 3y - 2x = 3y - \frac{6y-1}{2} = \frac{1}{2}.$$

Since

$$2(y-x) = \frac{1-2y}{2} < \frac{1}{2},$$

we find

$$\tau_{\mathcal{M}}(I - (P_1 \vee P_2 \vee P_3)) \geq 1 - \frac{1}{2} > 2(y-x).$$

Hence there exist mutually orthogonal subprojections $F_4, F_5 \leq I - (P_1 \vee P_2 \vee P_3)$ such that $\tau_{\mathcal{M}}(F_4) = \tau_{\mathcal{M}}(F_5) = y - x$. Note that $Q \leq P_1 \leq I - F_4, I - F_5$, hence we can define projections $P_4 = Q + F_4$ and $P_5 = Q + F_5$ with $\tau_{\mathcal{M}}(P_4) = \tau_{\mathcal{M}}(P_5) = y$. By the choice of the F_i, we have $\tau_{\mathcal{M}}(F_i) = y - x$ and $Q \perp F_i$ for all $1 \leq i \leq 5$. Further, for each $1 \leq j \leq 3$ we have $F_j \leq P_j \perp (F_4 \vee F_5)$, hence

$$(F_1 \vee F_2 \vee F_3) \perp (F_4 \vee F_5).$$

Recall that $F_1 \wedge F_2 = 0$ so that

(1) $Q \perp (F_1 \vee F_2 \vee F_4)$;

(2) $y = \tau_{\mathcal{M}}(Q) + \tau_{\mathcal{M}}(F_i)$ for all $i \in \{1, 2, 4\}$;

(3) $(F_i \vee F_j) \wedge F_k = 0$ for all distinct indices $i, j, k \in \{1, 2, 4\}$.

Thus we may apply Lemma 4.1.14 to the family $\{Q, F_1, F_2, F_4\}$ to obtain

$$\tau_{\mathcal{M}}(\phi(P_1) \wedge \phi(P_2)) = \tau_{\mathcal{M}}(Q) = \frac{6y-1}{4} \quad \text{and} \quad \phi(P_1) \wedge \phi(P_2) = \phi(P_1) \wedge \phi(P_4).$$

This completes the first part of the proof. Further, it is easy to verify, by the choice of Q and the F_i, that both $\{Q, F_1, F_4, F_5\}$ and $\{Q, F_3, F_4, F_5\}$ also satisfy the conditions of

Lemma 4.1.14. Hence

$$\phi(P_1) \wedge \phi(P_4) = \phi(P_5) \wedge \phi(P_4) \quad \text{and} \quad \phi(P_5) \wedge \phi(P_4) = \phi(P_5) \wedge \phi(P_3).$$

We conclude

$$\phi(P_1) \wedge \phi(P_2) = \phi(P_1) \wedge \phi(P_4) = \phi(P_5) \wedge \phi(P_4) = \phi(P_5) \wedge \phi(P_3) \leq \phi(P_3),$$

which finishes the proof. □

Proposition 4.1.16. *Let $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$ be a real number and $\phi_y : \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ be an ortho-isomorphism. Let $x = \frac{6y-1}{4}$. The following are true.*

(i) $0 < x < y$.

(ii) There exists an ortho-isomorphism $\phi_x : \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ such that, for all $E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$,

$$E \leq P \quad \textit{if and only if} \quad \phi_x(E) \leq \phi_y(P).$$

Proof. (i). Since $y \in \left[\frac{1}{4}, \frac{1}{2}\right)$, we observe

$$x = \frac{6y-1}{4} \geq \frac{\frac{6}{4}-1}{4} > 0$$

and

$$y - x = \frac{1-2y}{4} > \frac{1-1}{4} = 0,$$

hence $0 < x < y$.

(ii). By Proposition 4.1.5, ϕ_y extends to an order-ortho-isomorphism, which we will also denote by ϕ_y, from $\mathcal{Q}_y(\mathcal{M}, \tau_{\mathcal{M}})$ onto $\mathcal{Q}_y(\mathcal{M}, \tau_{\mathcal{M}})$. By definition

$$\mathcal{Q}_y(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \geq y \text{ and } \tau_{\mathcal{M}}(I - P) \geq y\} = \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

Therefore ϕ_y extends to an order-ortho-isomorphism $\phi_y : \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{[y,1-y]}(\mathcal{M}, \tau_{\mathcal{M}})$.

Claim 4.1.16.1. *For all $P_1, P_2, P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$, the following are equivalent.*

(a) $\tau_{\mathcal{M}}(P_1 \wedge P_2) = x$ and $P_1 \wedge P_2 \leq P$.

(b) $\tau_{\mathcal{M}}(\phi_y(P_1) \wedge \phi_y(P_2)) = x$ and $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(P)$.

Proof of Claim 4.1.16.1. The claim follows directly from Lemma 4.1.15. When we apply Lemma 4.1.15 to the order-isomorphism ϕ_y we find that (a) implies (b). The converse is given by applying Lemma 4.1.15 to the order-isomorphism ϕ_y^{-1}.

We are now prepared to define a map $\phi_x : \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{M})$ by

$$\phi_x(E) = \bigwedge\{\phi_y(P) \mid E \leq P \text{ and } P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}}).$$

Claim 4.1.16.2. *If $P_1, P_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ satisfy $\tau_{\mathcal{M}}(P_1 \wedge P_2) = x$, then $\phi_x(P_1 \wedge P_2) = \phi_y(P_1) \wedge \phi_y(P_2)$.*

Proof of Claim 4.1.16.2. By the definition of ϕ_x, we have $\phi_x(P_1 \wedge P_2) \leq \phi_y(P_1) \wedge \phi_y(P_2)$. Conversely, if $P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ satisfies $P_1 \wedge P_2 \leq P$, then Claim 4.1.16.1 implies that $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(P)$. Taking the intersection over all such P, we find $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_x(P_1 \wedge P_2)$. Thus $\phi_x(P_1 \wedge P_2) = \phi_y(P_1) \wedge \phi_y(P_2)$.

Claim 4.1.16.1 and Claim 4.1.16.2 show that the codomain of the map ϕ_x is $\mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$. For, given any $E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$, there exist projections $P_1, P_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ with $E = P_1 \wedge P_2$. By Claim 4.1.16.2 we find $\phi_x(E) = \phi_y(P_1) \wedge \phi_y(P_2)$. Therefore $\tau_{\mathcal{M}}(\phi_x(E)) = x$ by Claim 4.1.16.1.

For the remainder of the proof, suppose $E, F \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$. Then there exist projections $P_1, P_2, Q_1, Q_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ with $E = P_1 \wedge P_2$ and $F = Q_1 \wedge Q_2$.

Suppose $P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$. If $E \leq P$, then $\phi_x(E) \leq \phi_y(P)$ by the definition of ϕ_x. If, conversely, $\phi_x(E) \leq \phi_y(P)$, then by Claim 4.1.16.2 we have $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(P)$, whence $E = P_1 \wedge P_2 \leq P$ by Claim 4.1.16.1. It remains to prove that ϕ_x is an ortho-isomorphism.

If $\phi_x(E) = \phi_x(F)$, then, by Claim 4.1.16.2,

$$\phi_y(P_1) \wedge \phi_y(P_2) = \phi_y(Q_1) \wedge \phi_y(Q_2) \leq \phi_y(Q_1), \phi_y(Q_2).$$

By Claim 4.1.16.1 we have $P_1 \wedge P_2 \leq Q_1, Q_2$, therefore $E = P_1 \wedge P_2 \leq Q_1 \wedge Q_2 = F$. We obtain the reverse inequality by reversing the roles of E and F, ergo $E = F$ and ϕ_x is injective.

To prove ϕ_x is surjective, observe that for $E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ we have $E = P_1 \wedge P_2$ with $P_1, P_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$. Since ϕ_y is a surjection, we have $\phi_y(P_1') = P_1$ and $\phi_y(P_2') = P_2$ for some $P_1', P_2' \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$. By Claim 4.1.16.1, $P_1' \wedge P_2' \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$, so that we may apply Claim 4.1.16.2 to obtain

$$\phi_x(P_1' \wedge P_2') = \phi_y(P_1') \wedge \phi_y(P_2') = P_1 \wedge P_2 = E.$$

Thus ϕ_x is a surjection.

We now show ϕ_x preserves orthogonality in both directions. Suppose $E \perp F$. By a calculation,

$$\tau_{\mathcal{M}}(I - F) = 1 - x > 1 - y > y,$$

so there exists a projection $P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E \leq P \leq I - F$. Since $F \leq I - P$ and $\tau_{\mathcal{M}}(I - P) = 1 - y > y$, there must also exist a projection $Q \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ with $F \leq Q \leq I - P$. As ϕ_y is an ortho-isomorphism, we have that $P \perp Q$ implies $\phi_y(P) \perp \phi_y(Q)$. We have shown that $E \leq P$ implies $\phi_x(E) \leq \phi_y(P)$. Therefore $\phi_x(E) \perp \phi_y(Q)$. Likewise we find $\phi_x(F) \le \phi_y(Q) \le I - \phi_x(E)$ so that $\phi_x(E) \perp \phi_x(F)$. A similar approach yields the converse. $\square$

4.1.4 Proof of Theorem 4.1.3

We are now ready to present the proof of the main result of this section.

Proof of Theorem 4.1.3. If $c < \frac{\tau_{\mathcal{M}}(I)}{4}$ we apply Theorem 4.1.9. If, instead, $\frac{\tau_{\mathcal{M}}(I)}{4} \le c < \frac{\tau_{\mathcal{M}}(I)}{2}$, then we must have $\tau_{\mathcal{M}}(I) < \infty$. As observed in Remark 4.1.10, it suffices to show in the case where $\tau_{\mathcal{M}}(I) = 1$ and $c \in \left[\frac{1}{4}, \frac{1}{2}\right)$.

We define a recursive sequence by setting $y_1 = c$ and, for each integer $n \ge 1$, setting

$$y_{n+1} = \frac{6y_n - 1}{4}.$$

Note that $y_1 = c < \frac{1}{2}$, and if $y_n < \frac{1}{2}$ for some n we have

$$y_{n+1} = \frac{6y_n - 1}{4} < \frac{3 - 1}{4} = \frac{1}{2}.$$

Thus $y_n < \frac{1}{2}$ for all n by simple induction. Therefore, for all $n \ge 1$, we have that

$$y_{n+1} - y_n = \frac{6y_n - 1}{4} - y_n = \frac{2y_n - 1}{4} < 0,$$

hence $y_{n+1} < y_n$ and $\{y_n\}$ is a strictly decreasing sequence of real numbers converging to its infimum. Suppose, by way of contradiction, that this infimum is a finite real number L. Thus, if $\epsilon > 0$, there must be some y_k with $y_k < L + \epsilon$. As $L \le y_n$ for all n, we must therefore have

$$L \le y_{k+1} = \frac{6y_k - 1}{4} < \frac{6L + 6\epsilon - 1}{4},$$

hence

$$1 < 2L + 6\epsilon$$

for every $\epsilon > 0$. We arrive at a contradiction: $L \geq \frac{1}{2} > y_1$. Thus $\lim_{n\to\infty} y_n = -\infty$.

We have shown $\{y_n\}$ is a decreasing sequence with limit $-\infty$ and first term $y_1 \geq \frac{1}{4}$. Therefore there is some $k \geq 2$ such that $y_k < \frac{1}{4} \leq y_{k-1}$. Note that $y_i \in \left[\frac{1}{4}, \frac{1}{2}\right)$ for all $1 \leq i < k$. We apply Proposition 4.1.16 inductively to the y_i with $1 \leq i < k$ to obtain a family of ortho-isomorphisms $\varphi_{y_i} : \mathcal{P}_{y_i}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{y_i}(\mathcal{M}, \tau_{\mathcal{M}})$ for $2 \leq i \leq k$, such that, for all $E \in \mathcal{P}_{y_i}(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_{y_{i-1}}(\mathcal{M}, \tau_{\mathcal{M}})$,

$$E \leq P \quad \text{if and only if} \quad \varphi_{y_i}(E) \leq \varphi_{y_{i-1}}(P)$$

where $\varphi_{y_1} = \varphi$. Suppose $E \in \mathcal{P}_{y_k}(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_{y_1}(\mathcal{M}, \tau_{\mathcal{M}})$. If $E \leq P$, there is a family of projections $\{P_{y_i}\}$ with $\tau_{\mathcal{M}}(P_{y_i}) = y_i$ for $2 \leq i < k$ and

$$E \leq P_{y_{k-1}} \leq ... \leq P_{y_2} \leq P.$$

This sequence of inequalities implies, inductively, that

$$\varphi_{y_k}(E) \leq \varphi_{y_{k-1}}(P_{y_{k-1}}) \leq ... \leq \varphi_{y_2}(P_{y_2}) \leq \varphi_{y_1}(P) = \varphi(P).$$

Similarly, if $\varphi_{y_k}(E) \leq \varphi(P)$, we use the surjectivity of each φ_{y_i} to find a family $\{\varphi_{y_i}(P_{y_i})\}$ with $\tau_{\mathcal{M}}(P_{y_i}) = y_i$ for $2 \leq i < k$ and

$$\varphi_{y_k}(E) \leq \varphi_{y_{k-1}}(P_{y_{k-1}}) \leq ... \leq \varphi_{y_2}(P_{y_2}) \leq \varphi_{y_1}(P) = \varphi(P).$$

This implies, in turn, that

$$E \leq P_{y_{k-1}} \leq ... \leq P_{y_2} \leq P.$$

Thus for all $E \in \mathcal{P}_{y_k}(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_{y_1}(\mathcal{M}, \tau_{\mathcal{M}}) = \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ we have

$$E \leq P \quad \text{if and only if} \quad \varphi_{y_k}(E) \leq \varphi(P). \tag{4.2}$$

We observe that since $y_{k-1} \in \left[\frac{1}{4}, \frac{1}{2}\right)$ we have $0 < y_k$ by part (i) of Proposition 4.1.16. As $0 < y_k < \frac{1}{4}$, we have, by Theorem 4.1.9, that φ_k extends to a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$. For any $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ we have

$$\begin{aligned}
\varphi(P) &= \bigvee\{F \mid F \in \mathcal{P}_{y_k}(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } F \leq \varphi(P)\} \\
&= \bigvee\{\varphi_{y_k}(E) \mid E \in \mathcal{P}_{y_k}(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } \varphi_{y_k}(E) \leq \varphi(P)\} && (\varphi_{y_k} \text{ a bijection}) \\
&= \bigvee\{\varphi_{y_k}(E) \mid E \in \mathcal{P}_{y_k}(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } E \leq P\} && (\text{by } (4.2)) \\
&= \bigvee\{\rho(E) \mid E \in \mathcal{P}_{y_k}(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } E \leq P\} && (\rho \text{ extends } \varphi_{y_k}) \\
&= \rho(P). && (\text{by Lemma 3.3.4 (iv)})
\end{aligned}$$

Thus the Jordan $*$-isomorphism ρ is also an extension of φ and the proof is complete. □

4.2 Extension for ortho-isomorphisms on atomic von Neumann algebras

4.2.1 Assumptions and statement of main result

In the previous section we considered von Neumann algebras without any minimal projections. We now concern ourselves with atomic algebras, in which every non-zero projection has a minimal non-zero subprojection.

Assumption 4.2.1. *We assume the following are true in this section.*

(i) $0 < c < \tau_{\mathcal{M}}(I)$.

(ii) $\mathcal{M}$ *is atomic and has no type* I_2 *direct summand.*

(iii) $\tau_{\mathcal{M}}$ *is the canonical tracial weight satisfying*

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M}.$$

(iv) $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is an ortho-isomorphism.*

Remark 4.2.2. Suppose $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \emptyset$. Then φ can be trivially extended to a Jordan $*$-isomorphism, namely the identity mapping.

Remark 4.2.3. By Assumption 4.2.1, we may make full use of Lemma 3.6.1 throughout this section.

We aim to provide the following analogue to Theorem 4.1.3 in the atomic case.

Theorem 4.2.4. *Under Assumption 4.2.1, if $0 < c < \frac{\tau_{\mathcal{M}}(I)}{2}$, then φ can be extended to a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$.*

Remark 4.2.5. If we can show the above result holds, then it must also hold when $\tau_{\mathcal{M}}$ is a faithful normal semi-finite tracial weight such that all minimal projections have the same trace d, with d a fixed real number not necessarily equal to 1. Clearly $0 < d < \infty$. Note that $\hat{\tau}_{\mathcal{M}}$, given by $\hat{\tau}_{\mathcal{M}}(A) = \frac{\tau_{\mathcal{M}}(A)}{d}$, defines another faithful normal semi-finite tracial weight such that each minimal projection has trace 1. Clearly $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) = \mathcal{P}_{c/d}(\mathcal{M}, \hat{\tau}_{\mathcal{M}})$ for any $c \in (0, \tau_{\mathcal{M}}(I))$. Thus to show that an ortho-isomorphism $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ extends to a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$, it suffices to consider the case when $d = 1$.

The key tool in the proofs of Lemma 4.1.4 and Proposition 4.1.5 for a diffuse von Neumann algebra was Lemma 3.5.1. By utilizing Lemma 3.6.1, instead, we obtain the following analogues under Assumption 4.2.1.

Lemma 4.2.6. *Let $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{2})$. If $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ is a projection and $\{E_\lambda\} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ is a family of projections with $\bigvee_\lambda E_\lambda \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$, then*

(i) $\bigvee_\lambda \varphi(E_\lambda) \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$*;*

(ii) $E \leq \bigvee_\lambda E_\lambda$ *if and only if* $\varphi(E) \leq \bigvee_\lambda \varphi(E_\lambda)$*.*

Proof. The proof is similar to that of Lemma 4.1.4 and has been omitted. □

Proposition 4.2.7. *Let $c \in (0, \frac{\tau_{\mathcal{M}}(I)}{2})$. The ortho-isomorphism φ can be extended to an order-ortho-isomorphism $\varphi^+ : \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ satisfying the following properties:*

(i) $\varphi^+(E) = \varphi(E)$ *for each* $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*;*

(ii) $\varphi^+(I - P) = I - \varphi^+(P)$ *for each* $P \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$*;*

(iii) $\varphi^+(P_1 \vee P_2) = \varphi^+(P_1) \vee \varphi^+(P_2)$ *for all* $P_1, P_2 \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *with* $P_1 \vee P_2 \in \mathcal{Q}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

Proof. The proof is similar to that of Proposition 4.1.5 and has been omitted. □

4.2.2 Technical lemmas

Our approach in the remainder of this section will be to show that φ induces an ortho-isomorphism on the set of minimal projections.

Lemma 4.2.8. *Suppose* S *is a set of projections, containing* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*, such that* $\tau_{\mathcal{M}}(E) \geq c$ *for each* $E \in S$*. If* $\phi : S \to S$ *is an order-isomorphism, then* $\phi(\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

Proof. Suppose, by way of contradiction, that $\phi(\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})) \neq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. If $\tau_{\mathcal{M}}(E) = c$ and $\tau_{\mathcal{M}}(\phi(E)) > c$, then there exists some proper subprojection of $\phi(E)$ with trace c, necessarily of the form $\phi(F)$ with $F \in S$ since ϕ is surjective. Since the bijection ϕ is order-preserving in both directions, we find that $F < E$. Therefore $\tau_{\mathcal{M}}(E) - \tau_{\mathcal{M}}(F) = \tau_{\mathcal{M}}(E - F) > 0$, so that $\tau_{\mathcal{M}}(F) < c$ and $F \notin S$. Hence $F \in S$ and $F \notin S$, a clear contradiction. In particular, we have shown $\phi(\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})) \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. The same argument, this time using the order-isomorphism ϕ^{-1}, produces $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \subseteq \phi(\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}))$. Thus $\phi(\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. □

Lemma 4.2.9. *Let* $\tau_{\mathcal{M}}(I) \geq 5$*,* $y \in \left[2, \frac{\tau_{\mathcal{M}}(I)}{2}\right)$ *be a natural number, and* $\phi_y : \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ *be an ortho-isomorphism. Let* $x = y - 1$*. For all* $P_1, P_2, P_3 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$*, the following are equivalent:*

(i) $\tau_{\mathcal{M}}(P_1 \wedge P_2) = x$ *and* $P_1 \wedge P_2 \leq P_3$.

(ii) $\tau_{\mathcal{M}}(\phi_y(P_1) \wedge \phi_y(P_2)) = x$ *and* $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(P_3)$.

Proof. By Proposition 4.2.7, ϕ_y extends to an order-ortho-isomorphism $\phi : \mathcal{Q}_y(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{Q}_y(\mathcal{M}, \tau_{\mathcal{M}})$. We first prove two preliminary claims.

Claim 4.2.9.1. $\phi(\mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}})$.

Proof of Claim 4.2.9.1. Note that $\phi(\mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ by Lemma 4.2.8. Hence ϕ acts as an order-isomorphism $\phi : S \to S$ on the complement $S = \mathcal{Q}_y(\mathcal{M}, \tau_{\mathcal{M}}) \setminus \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$. Clearly $\tau_{\mathcal{M}}(E) \geq y+1$ for each $E \in S$. Further, if $F \in \mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}})$, then $\tau_{\mathcal{M}}(I - F) = \tau_{\mathcal{M}}(I) - (y+1) \geq y$ since $2y+1 \leq \tau_{\mathcal{M}}(I)$. Hence $\mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}}) \subseteq S$. Applying Lemma 4.2.8 to $\phi : S \to S$ we find $\phi(\mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}})) = \mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}})$ as claimed.

Claim 4.2.9.2. *If $P, Q \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ satisfy $\tau_{\mathcal{M}}(P \wedge Q) = x$, then $\tau_{\mathcal{M}}(\phi_y(P) \wedge \phi_y(Q)) = x$.*

Proof of Claim 4.2.9.2. By Lemma 3.2.2 we have

$$\tau_{\mathcal{M}}(P \vee Q) = \tau_{\mathcal{M}}(P) + \tau_{\mathcal{M}}(Q) - \tau_{\mathcal{M}}(P \wedge Q) = 2y - (y-1) = y+1.$$

As observed in the proof of Claim 4.2.9.1, $\mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}}) \subseteq \mathcal{Q}_y$. Hence $P \vee Q \in \mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}})$ is in the domain of ϕ. We apply Lemma 3.2.2, Proposition 4.2.7 (iii), and Claim 4.2.9.1 to obtain

$$\begin{aligned}
\tau_{\mathcal{M}}(\phi_y(P) \wedge \phi_y(Q)) &= \tau_{\mathcal{M}}(\phi(P) \wedge \phi(Q)) && (\phi \text{ extends } \phi_y) \\
&= \tau_{\mathcal{M}}(\phi(P)) + \tau_{\mathcal{M}}(\phi(Q)) - \tau_{\mathcal{M}}(\phi(P) \vee \phi(Q)) && (\text{by Lemma 3.2.2}) \\
&= 2y - \tau_{\mathcal{M}}(\phi(P \vee Q)) && (\text{by Proposition 4.2.7}) \\
&= 2y - (y+1) && (\text{by Claim 4.2.9.1}) \\
&= y - 1 = x.
\end{aligned}$$

This ends the proof of the claim.

We now proceed to the proof of the lemma. To show (i) implies (ii), let $P_1, P_2, P_3 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ be given such that $\tau_{\mathcal{M}}(P_1 \wedge P_2) = x$ and $P_1 \wedge P_2 \leq P_3$. Let $E = P_1 \wedge P_2$. By

Claim 4.2.9.2 we immediately obtain $\phi_y(P_1) \wedge \phi_y(P_2) \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$. Note $E \leq P_i$ for each $1 \leq i \leq 3$, so there exist minimal projections $F_i = P_i - E \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}})$ for each i. For each pair of distinct indices $i, j \in \{1, 2, 3\}$ we have $P_i \wedge P_j = E + F_i \wedge F_j$. In particular, this implies $F_1 \wedge F_2 = 0$. If $F_3 = F_i$ for some $i \in \{1, 2\}$, then $P_3 = P_i$ and $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(P_3)$. Assume, then, that $F_3 \neq F_1, F_2$. As each F_i is a minimal projection, we must now have $F_3 \wedge F_1 = F_3 \wedge F_2 = 0$. Observe

$$\tau_{\mathcal{M}}(E \vee F_1 \vee F_2 \vee F_3) \leq x + 1 + 1 + 1 = y + 2,$$

hence

$$\tau_{\mathcal{M}}(I - E \vee F_1 \vee F_2 \vee F_3) \geq \tau_{\mathcal{M}}(I) - y - 2 > \frac{\tau_{\mathcal{M}}(I)}{2} - 2 > 0.$$

Thus there exists some $F_4 \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}})$ with $F_4 \perp E \vee F_1 \vee F_2 \vee F_3$. Let $P_4 = E + F_4$.

Claim 4.2.9.3. *$\phi_y(P_i) \wedge \phi_y(P_j) = \phi_y(P_i) \wedge \phi_y(P_4)$ for all distinct $i, j \in \{1, 2, 3\}$.*

Proof of Claim 4.2.9.3. Assume, by way of contradiction, that $G_1 = \phi_y(P_i) \wedge \phi_y(P_j)$ and $G_2 = \phi_y(P_i) \wedge \phi_y(P_4)$ are distinct for some pair of fixed, distinct $i, j \in \{1, 2, 3\}$. Then $G_1 \vee G_2$ is a projection with trace strictly greater than $\tau_{\mathcal{M}}(G_1) = x = y - 1$. Hence $G_1 \vee G_2$ has some subprojection of trace y, and we may assume it is of the form $\phi_y(Q)$ for some $Q \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ by the surjectivity of ϕ_y. Note $G_1, G_2 \leq \phi_y(P_i)$ implies $\phi_y(Q) \leq \phi_y(P_i)$. In particular, as both images have the same trace, $\phi_y(Q) = \phi_y(P_i)$. By injectivity $Q = P_i$. On the other hand,

$$G_1 \leq \phi_y(P_j) \leq \phi_y(P_j) \vee \phi_y(P_4)$$

and

$$G_2 \leq \phi_y(P_4) \leq \phi_y(P_j) \vee \phi_y(P_4)$$

imply $\phi_y(Q) \leq \phi_y(P_j) \vee \phi_y(P_4)$. As $P_j \vee P_4 = E + F_j \vee F_4 = E + F_j + F_4 \in \mathcal{P}_{y+1}(\mathcal{M}, \tau_{\mathcal{M}}) \subseteq \mathcal{Q}_y$, we apply Proposition 4.2.7 to obtain

$$\phi(Q) = \phi_y(Q) \leq \phi_y(P_j) \vee \phi_y(P_4) = \phi(P_j \vee P_4).$$

As ϕ preserves order in both directions, we must have $Q \leq P_j \vee P_4 = E + F_j + F_4$. But now $E + F_i = P_i = Q \leq E + F_j + F_4$. As $F_i \leq F_j + F_4$ we have $F_i = F_i(F_j + F_4) = F_iF_j + 0$, whence $F_i \leq F_j$. This is impossible, however, as F_i, F_j were chosen to be non-zero projections with $F_i \wedge F_j = 0$. We must have, therefore, that $\phi_y(P_i) \wedge \phi_y(P_j) = \phi_y(P_i) \wedge \phi_y(P_4)$.

To complete the proof that (i) implies (ii) we simply apply Claim 4.2.9.3 to the triples $\{P_1, P_2, P_4\}$ and $\{P_1, P_3, P_4\}$ to obtain

$$\phi_y(P_1) \wedge \phi_y(P_2) = \phi_y(P_1) \wedge \phi_y(P_4) = \phi_y(P_1) \wedge \phi_y(P_3) \leq \phi_y(P_3)$$

as desired.

To show (ii) implies (i), we apply the preceding result to the ortho-isomorphism ϕ_y^{-1}. $\square$

Proposition 4.2.10. *Let $\tau_{\mathcal{M}}(I) \geq 5$ and let $y \in \left[2, \frac{\tau_{\mathcal{M}}(I)}{2}\right)$ be a natural number. Set $x = y - 1$. If $\phi_y : \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ is an ortho-isomorphism, then there is an ortho-isomorphism $\phi_x : \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ such that, for all $E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$,*

$$E \leq P \quad \textit{if and only if} \quad \phi_x(E) \leq \phi_y(P).$$

Proof. Define ϕ_x by

$$\phi_x(E) = \bigwedge\{\phi_y(P) \mid E \leq P \text{ and } P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}}).$$

Claim 4.2.10.1. *If $P_1, P_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ satisfy $\tau_{\mathcal{M}}(P_1 \wedge P_2) = x$, then $\phi_x(P_1 \wedge P_2) = \phi_y(P_1) \wedge \phi_y(P_2)$ and $\tau_{\mathcal{M}}(\phi_y(P_1) \wedge \phi_y(P_2)) = x$.*

Proof of Claim 4.2.10.1. Clearly $\phi_x(P_1 \wedge P_2) \leq \phi_y(P_1) \wedge \phi_y(P_2)$. Suppose $P_1 \wedge P_2 \leq P_3 \in$

$\mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$. By Lemma 4.2.9, we have $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(P_3)$. Thus $\phi_y(P_1) \wedge \phi_y(P_2)$ is a subprojection of the intersection of all such $\phi_y(P_3)$. That is, $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_x(P_1 \wedge P_2)$. Therefore $\phi_x(P_1 \wedge P_2) = \phi_y(P_1) \wedge \phi_y(P_2)$. Again by Lemma 4.2.9, we have $\phi_y(P_1) \wedge \phi_y(P_2) \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$. □

Note that for each $E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ there exist projections $P_1, P_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E = P_1 \wedge P_2$. It follows from Claim 4.2.10.1 that ϕ_x maps $\mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ into $\mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$.

For the remainder of the proof let $E, F \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$. Then there exist $P_1, P_2, Q_1, Q_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ with $E = P_1 \wedge P_2$ and $F = Q_1 \wedge Q_2$. To show ϕ_x is injective note that, by Claim 4.2.10.1, $\phi_x(E) = \phi_x(F)$ implies

$$\phi_y(P_1) \wedge \phi_y(P_2) = \phi_x(E) = \phi_x(F) = \phi_y(Q_1) \wedge \phi_y(Q_2).$$

Therefore $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(Q_1), \phi_y(Q_2)$. By Lemma 4.2.9, we must have $P_1 \wedge P_2 \leq Q_1, Q_2$, hence $E \leq Q_1 \wedge Q_2 = F$. Reversing the roles of the P_i and Q_i we obtain $F \leq E$, therefore $E = F$.

To show ϕ_x is surjective, observe that any $F \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ can be written as $F = Q_1 \wedge Q_2$ with $Q_1, Q_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$. By surjectivity of the map ϕ_y we have $Q_1 = \phi_y(P_1)$ and $Q_2 = \phi_y(P_2)$ with $P_1, P_2 \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$. By Lemma 4.2.9, $P_1 \wedge P_2 \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ and by Claim 4.2.10.1, $\phi_x(P_1 \wedge P_2) = \phi_y(P_1) \wedge \phi_y(P_2) = F$ as desired.

Next we will prove, for all $E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$,

$$E \leq P \quad \text{if and only if} \quad \phi_x(E) \leq \phi_y(P). \tag{4.3}$$

In fact, this follows directly from Lemma 4.2.9 and Claim 4.2.10.1 since $E = P_1 \wedge P_2$ implies that $\phi_x(E) = \phi_y(P_1) \wedge \phi_y(P_2)$, and $P_1 \wedge P_2 \leq P$ if and only if $\phi_y(P_1) \wedge \phi_y(P_2) \leq \phi_y(P)$.

Finally we show ϕ_x preserves orthogonality in both directions. If $E \perp F$, then $E \leq I - F$. Calculating we see that $\tau_{\mathcal{M}}(I - F) > 2y - (y - 1) = y + 1 > y$, hence there exists a projection $P \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ with $E \leq P \leq I - F$. Likewise since $F \leq I - P$ and $\tau_{\mathcal{M}}(I - P) > 2y - y = y$, there exists some $Q \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ with $F \leq Q \leq I - P$. Since ϕ_y preserves orthogonality we have $\phi_y(Q) \leq I - \phi_y(P)$. Applying (4.3) twice we find $\phi_x(F) \leq I - \phi_y(P)$ and $\phi_x(E) \leq \phi_y(P) \leq I - \phi_x(F)$, hence $\phi_x(E) \perp \phi_x(F)$. To prove the converse we use the same technique. If $\phi_x(E) \perp \phi_x(F)$, we can find projections, necessarily of the form $\phi_y(P), \phi_y(Q)$ by the surjectivity of ϕ_y, such that $\phi_x(E) \leq \phi_y(P) \leq I - \phi_x(F)$ and $\phi_x(F) \leq \phi_y(Q) \leq I - \phi_y(P)$. This implies $P \perp Q$. Hence, by (4.3), $E \perp F$. This completes the proof that ϕ_x is an ortho-isomorphism satisfying (4.3). $\square$

4.2.3 Proof of Theorem 4.2.4

We are now prepared to prove our main result on ortho-isomorphisms on an atomic semi-finite von Neumann algebra.

Proof of Theorem 4.2.4. If $c = 1$ then φ is an ortho-isomorphism on the set of minimal projections. By Proposition 4.2.7 we observe, in this case, that φ extends to an order-ortho-isomorphism $\varphi^+ : \mathcal{Q}_1(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{Q}_1(\mathcal{M}, \tau_{\mathcal{M}})$ where

$$\mathcal{Q}_1(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \geq 1 \text{ and } \tau_{\mathcal{M}}(I - P) \geq 1\} = \mathcal{P}(\mathcal{M}) \setminus \{0, I\}.$$

It is easy to verify that φ^+ becomes an order-ortho-isomorphism on $\mathcal{P}(\mathcal{M})$ by additionally defining $\varphi^+(0) = 0$ and $\varphi^+(I) = I$. Thus φ^+, and therefore φ, can be extended to a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$ by Lemma 3.3.6.

Next we consider the case where $c \geq 2$. Then $\tau_{\mathcal{M}}(I) \geq 5$. We apply Proposition 4.2.10 inductively for each n with $1 < n \leq c$, starting with $n = c$, to obtain a family of ortho-isomorphisms $\varphi_n : \mathcal{P}_n(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_n(\mathcal{M}, \tau_{\mathcal{M}})$ for $1 \leq n < c$, such that, for each n and all $E \in \mathcal{P}_n(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_{n+1}(\mathcal{M}, \tau_{\mathcal{M}})$,

$$E \leq P \quad \text{if and only if} \quad \varphi_n(E) \leq \varphi_{n+1}(P),$$

where $\varphi_c = \varphi$.

Suppose $E \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. If $E \leq P$, there is a family of projections $\{P_n\}$ with $\tau_{\mathcal{M}}(P_n) = n$ for $2 \leq n < c$ and

$$E \leq P_2 \leq ... \leq P_{c-1} \leq P.$$

This sequence of inequalities implies that

$$\varphi_1(E) \leq \varphi_2(P_2) \leq ... \leq \varphi_{c-1}(P_{c-1}) \leq \varphi_c(P) = \varphi(P).$$

Similarly, if $\varphi_1(E) \leq \varphi(P)$, we use the surjectivity of each φ_n to find a family $\{\varphi_n(P_n)\}$ with $\tau_{\mathcal{M}}(P_n) = n$ for $2 \leq n < c$ and

$$\varphi_1(E) \leq \varphi_2(P_2) \leq ... \leq \varphi_{c-1}(P_{c-1}) \leq \varphi_c(P) = \varphi(P).$$

This implies, in turn, that

$$E \leq P_2 \leq ... \leq P_{c-1} \leq P.$$

Thus for all $E \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}})$ and all $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ we have

$$E \leq P \quad \text{if and only if} \quad \varphi_1(E) \leq \varphi(P). \tag{4.4}$$

Since φ_1 is an ortho-isomorphism on the set of minimal projections, by the $c = 1$ case there exists a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$ which extends φ_1. To complete the proof, we show ρ also extends φ. For any $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ we have

$$\begin{aligned}
\varphi(P) &= \bigvee\{F \mid F \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } F \leq \varphi(P)\} \\
&= \bigvee\{\varphi_1(E) \mid E \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } \varphi_1(E) \leq \varphi(P)\} && (\varphi_1 \text{ a bijection}) \\
&= \bigvee\{\varphi_1(E) \mid E \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } E \leq P\} && (\text{by } (4.4)) \\
&= \bigvee\{\rho(E) \mid E \in \mathcal{P}_1(\mathcal{M}, \tau_{\mathcal{M}}) \text{ and } E \leq P\} && (\rho \text{ extends } \varphi_1) \\
&= \rho(P). && (\text{by Lemma 3.3.4})
\end{aligned}$$

Thus the Jordan $*$-isomorphism ρ is an extension of φ and the proof is complete. □

4.3 Wigner's theorem for ortho-isomorphisms on von Neumann algebras

Our main result on ortho-isomorphisms is now a direct consequence of Theorem 4.1.3 and Theorem 4.2.4.

Theorem 4.3.1. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra without a direct summand of type I_2 and with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. Let $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$. Suppose the following are true.*

(a) $\mathcal{M}$ is either diffuse or atomic.

(b) For an atomic von Neumann algebra $\mathcal{M}$, we assume that

(b_1) $\tau_{\mathcal{M}}$ is the canonical tracial weight satisfying

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b_2) c is a positive integer.

Suppose $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ is a bijection such that, for all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

$$E \perp F \text{ if and only if } \varphi(E) \perp \varphi(F).$$

Then there exists a Jordan $*$-isomorphism $\rho : \mathcal{M} \to \mathcal{M}$ such that

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

Proof. If $\mathcal{M}$ is diffuse, then Theorem 4.3.1 is a consequence of Theorem 4.1.3. If $\mathcal{M}$ is atomic, then $\tau_{\mathcal{M}}(H) = 1$ for every minimal projection H in $\mathcal{M}$. In this case Theorem 4.3.1 is a consequence of Theorem 4.2.4. □

Example 4.3.2. Let $\mathcal{R}$ be a factor of type I_∞ and τ the canonical tracial weight of $\mathcal{R}$. Assume that c is a positive integer. By Theorem 4.3.1, a map $\varphi : \mathcal{P}_c(\mathcal{R}, \tau) \to \mathcal{P}_c(\mathcal{R}, \tau)$ satisfying

$$E \perp F \text{ if and only if } \varphi(E) \perp \varphi(F)$$

for all $E, F \in \mathcal{P}_c(\mathcal{R}, \tau)$ can be extended to a Jordan $*$-isomorphism $\rho : \mathcal{R} \to \mathcal{R}$.

Example 4.3.3. Let $\mathcal{R}$ be a factor of type II_1 with tracial state τ. Let $0 < c < \frac{1}{2}$. By Theorem 4.3.1, a map $\varphi : \mathcal{P}_c(\mathcal{R}, \tau) \to \mathcal{P}_c(\mathcal{R}, \tau)$ satisfying

$$E \perp F \text{ if and only if } \varphi(E) \perp \varphi(F)$$

for all $E, F \in \mathcal{P}_c(\mathcal{R}, \tau)$ can be extended to a Jordan $*$-isomorphism $\rho : \mathcal{R} \to \mathcal{R}$.

4.4 L^p-isometries

We conclude this chapter with an observation on surjective mappings which are L^p-distance-preserving on orthogonal pairs of projections. As a consequence of Lemma 3.7.2 and Theorem 4.3.1 we have the following.

Theorem 4.4.1. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra without a direct summand of type I_2 and with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. Let $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$. Suppose the following are true.*

(a) $\mathcal{M}$ *is either diffuse or atomic.*

(b) For an atomic von Neumann algebra $\mathcal{M}$*, we assume that*

(b₁) $\tau_{\mathcal{M}}$ *is the canonical tracial weight satisfying*

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b₂) c *is a positive integer.*

Assume $0 < p < \infty$*. Suppose* $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is a bijection such that, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*,*

$$\|E - F\|_p^p = 2c \text{ if and only if } \|\varphi(E) - \varphi(F)\|_p^p = 2c.$$

Then there exists a Jordan $*$*-isomorphism* $\rho : \mathcal{M} \to \mathcal{M}$ *such that*

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

Proof. Let $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. We will show that $E \perp F$ if and only if $\|E - F\|_p^p = 2c$. By Lemma 3.7.2,

$$\|E - F\|_p^p \leq 2c - \tau_{\mathcal{M}}(E \wedge F)$$

with equality if and only if $EF = FE$. Hence, if we suppose $E \perp F$, then $EF = FE = 0$ so that $\|E - F\|_p^p = 2c - 0$. Conversely, suppose $\|E - F\|_p^p = 2c$. Then we must have $E \wedge F = 0$ and $EF = FE$ by Lemma 3.7.2. Now $EF = E \wedge F = 0$. Thus $E \perp F$ if and only if $\|E - F\|_p^p = 2c$. Likewise $\varphi(E) \perp \varphi(F)$ if and only if $\|\varphi(E) - \varphi(F)\|_p^p = 2c$. Therefore, by the condition on φ, $E \perp F$ if and only if $\varphi(E) \perp \varphi(F)$. The result now follows from Theorem 4.3.1. □

CHAPTER 5

WIGNER'S THEOREM FOR L^p-ISOMETRIES ON SEMI-FINITE VON NEUMANN ALGEBRAS

In this chapter, we assume $\mathcal{M}$ and $\mathcal{N}$ are semi-finite von Neumann algebras with faithful normal tracial weights $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively.

For a positive number c, let φ be a map from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ into $\mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$. We begin our investigation into the extension of L^p-isometries by considering the possible values of $\tau_{\mathcal{M}}(I_{\mathcal{M}})$ and $\tau_{\mathcal{N}}(I_{\mathcal{N}})$.

5.1 Extension for maps on von Neumann algebras with tracial states

5.1.1 Assumptions and statement of main result

In this section, we assume $\mathcal{M}$ and $\mathcal{N}$ are finite von Neumann algebras with faithful normal tracial states $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively.

For a positive number c, let φ be a map from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ into $\mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$.

Assumption 5.1.1. *We assume the following are true in this section.*

(i) $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = 1$.

(ii) $\mathcal{M}$ *is either diffuse or atomic, and* $\mathcal{M}$ *has no type* I_2 *direct summand.*

(iii) For an atomic von Neumann algebra $\mathcal{M}$*, we assume that* $\tau_{\mathcal{M}}$ *is a tracial state of* $\mathcal{M}$ *such that every minimal projection in* $\mathcal{M}$ *has the same trace.*

(iv) The map $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *satisfies, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

- *Condition (A): If $EF = FE$ and $\tau_{\mathcal{M}}(E \vee F) \leq \tau_{\mathcal{M}}(I_{\mathcal{M}}) - c$, then*

$$\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF).$$

- *Condition (B): If $E \perp F$, then $\varphi(E) \perp \varphi(F)$.*

Remark 5.1.2. Observe that if $c \leq \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{3}$, then $\tau_{\mathcal{M}}(E \vee F) \leq \tau_{\mathcal{M}}(I_{\mathcal{M}}) - c$ for any $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and thus Condition (A) is equivalent to the condition that $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF)$ for any $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $EF = FE$. It is easy to verify in this case that Condition (A) implies Condition (B).

Remark 5.1.3. Assume that $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ is a non-empty set. In light of Assumption 5.1.1, we may apply Lemma 3.5.1 and Lemma 3.6.1 to obtain the following.

(i) For any projections P, Q in $\mathcal{M}$ with $P \leq Q$ and $\tau_{\mathcal{M}}(P) \leq c \leq \tau_{\mathcal{M}}(Q)$, there exists a projection E in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, such that $P \leq E \leq Q$.

(ii) If $c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$, then, for any projection P in $\mathcal{M}$ with $\tau_{\mathcal{M}}(P) \leq c$, there exist two commuting projections E_1, E_2 in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P = E_1 \wedge E_2$.

It is also easily verified that

(iii) For any projection P in $\mathcal{M}$ with $\tau_{\mathcal{M}}(P) \geq c$, there exist a family $\{E_\lambda\}$ of mutually orthogonal projections and two commuting projections F_1, F_2 in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $(\sum_\lambda E_\lambda) \perp (F_1 \vee F_2)$ and $P = (\sum_\lambda E_\lambda) + F_1 \vee F_2$.

We devote this section to show the following result, whose proof will be postponed until the end of this section.

Theorem 5.1.4. *Under Assumption* 5.1.1*, if* $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$ *and* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is not an empty set, then φ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$.*

5.1.2 Properties of maps of Grassmann spaces of von Neumann algebras

The following properties on maps of Grassmann spaces, made possible by Assumption 5.1.1, will be useful. Recall Definition 3.3.1 for terminology.

Lemma 5.1.5. *Let $0 \le x < y < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$ be real numbers. Suppose $\phi : \mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$ is an order-preserving map such that, for all $E, F \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$, $E \perp F$ implies $\phi(E) \perp \phi(F)$. Then ϕ is ortho-preserving on $\mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}})$.*

Proof. Assume that P, Q are in $\mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \perp Q$. Then there exist E, F in $\mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P \le E$, $Q \le F$, and $E \perp F$. By assumption, $\phi(E) \perp \phi(F)$. Since ϕ is order-preserving, we have $\phi(P) \perp \phi(Q)$. That is, ϕ is ortho-preserving on $\mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}})$. □

Lemma 5.1.6. *Let $0 < y < z < \infty$ be real numbers. Suppose $\phi : \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$ is a map satisfying*

(i) $\phi(P) = \bigvee\{\phi(E) \mid E \le P \text{ and } E \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})\}$, *for all* $P \in \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}})$.

(ii) For all $E, F \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$, $E \perp F$ implies $\phi(E) \perp \phi(F)$.

Then ϕ is order-ortho-preserving on $\mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}})$.

Proof. It is clear that ϕ is order-preserving. Assume that $P, Q \in \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $P \perp Q$. Then for each $E, F \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ with $E \le P$ and $F \le Q$, we have $E \perp F$. Thus it follows from assumption (ii) of this lemma that $\phi(E) \perp \phi(F)$. By assumption (i), $\phi(P) \perp \phi(Q)$. Thus ϕ is also ortho-preserving. □

Lemma 5.1.7. *Let $0 \le x < y < z < \infty$ be real numbers. Suppose*

$$\phi_1 : \mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N}) \quad \text{and} \quad \phi_2 : \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$$

are trace-order-ortho-preserving maps satisfying

(i) $\phi_2(P) = \bigvee\{\phi_2(E) \mid E \le P \text{ and } E \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})\}$, *for all* $P \in \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}})$.

(ii) $\phi_1(E) = \phi_2(E), \qquad$ *for all* $E \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$*.*

Then there exists a unique trace-order-ortho-preserving map

$$\phi : \mathcal{P}_{[x,z]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$$

extending both ϕ_1 *and* ϕ_2*. Furthermore, if*

(iii) $\phi_1(P) = \bigvee\{\phi_1(F) \mid F \le P \text{ and } F \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad$ *for all* $P \in \mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}})$*,*

then

$$\phi(Q) = \bigvee\{\phi(E) \mid E \le Q \text{ and } E \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } Q \in \mathcal{P}_{[x,z]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

Proof. We need only show the existence of the map. Define $\phi : \mathcal{P}_{[x,z]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$ by

$$\phi(P) = \begin{cases} \phi_1(P) & \text{if } P \in \mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}}) \\ \phi_2(P) & \text{if } P \in \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}}) \end{cases}.$$

Clearly ϕ is well-defined by assumption (ii) of this lemma. We will verify that ϕ is trace-order-ortho-preserving.

It is clear that ϕ is trace-preserving.

To show that ϕ is order-preserving, we need only consider the case when P and Q are projections with $P \in \mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}})$, $Q \in \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}})$, and $P \le Q$. Then there exists an $E \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P \le E \le Q$. It follows that $\phi(P) = \phi_1(P) \le \phi_1(E) = \phi_2(E) \le \phi_2(Q) = \phi(Q)$. Hence ϕ is order-preserving.

To show that ϕ is ortho-preserving, we need only consider the case when P and Q are projections with $P \in \mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}})$, $Q \in \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}})$, and $P \perp Q$. By assumption (i) on ϕ_2, we have

$$\phi(Q) = \phi_2(Q) = \bigvee\{\phi_2(E) \mid E \le Q \text{ and } E \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})\}.$$

If $E \leq Q$ and $E \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})$, then $E \perp P$ and $\phi(E) = \phi_1(E) \perp \phi_1(P) = \phi(P)$. Hence $\phi(Q) \perp \phi(P)$.

Furthermore, if

$$\phi_1(P) = \bigvee\{\phi_1(F) \mid F \leq P \text{ and } F \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{[x,y]}(\mathcal{M}, \tau_{\mathcal{M}}),$$

then, for each $Q \in \mathcal{P}_{[y,z]}(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\phi(Q) = \bigvee\{\phi(E) \mid E \leq Q \text{ and } E \in \mathcal{P}_y(\mathcal{M}, \tau_{\mathcal{M}})\} = \bigvee\{\phi(F) \mid F \leq Q \text{ and } F \in \mathcal{P}_x(\mathcal{M}, \tau_{\mathcal{M}})\}.$$

This finishes the proof of the lemma. □

Lemma 5.1.8. *Let $\mathcal{M}, \mathcal{N}$ be finite von Neumann algebras with faithful normal tracial states $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively. If $\phi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ is a trace-ortho-preserving map, then*

(i) $\phi(0) = 0$ and $\phi(I_{\mathcal{M}}) = I_{\mathcal{N}}$;

(ii) ϕ is order-preserving. For all $P, Q \in \mathcal{P}(\mathcal{M})$, if $P \leq Q$, then $\phi(P) \leq \phi(Q)$;

(iii) ϕ is an orthomorphism. For all $P, Q \in \mathcal{P}(\mathcal{M})$, if $P \perp Q$, then $\phi(P+Q) = \phi(P)+\phi(Q)$;

(iv) For all $P, Q \in \mathcal{P}(\mathcal{M})$, if $PQ = QP$, then $\phi(P)\phi(Q) = \phi(Q)\phi(P) = \phi(PQ)$;

(v) For all $P, Q \in \mathcal{P}(\mathcal{M})$, if $PQ = QP$, then $\phi(P \vee Q) = \phi(P) \vee \phi(Q)$.

Proof. (i). Since $\tau_{\mathcal{N}}(\phi(0)) = \tau_{\mathcal{M}}(0) = 0$ and $\tau_{\mathcal{N}}(\phi(I_{\mathcal{M}})) = \tau_{\mathcal{M}}(I_{\mathcal{M}}) = 1$, we have $\phi(0) = 0$ and $\phi(I_{\mathcal{M}}) = I_{\mathcal{N}}$.

(ii). Notice that $\phi(Q) \perp \phi(I_{\mathcal{M}}-Q)$ and $\tau_{\mathcal{N}}(\phi(Q)+\phi(I_{\mathcal{M}}-Q)) = \tau_{\mathcal{M}}(Q+(I_{\mathcal{M}}-Q)) = 1$. It follows that $\phi(Q)+\phi(I_{\mathcal{M}}-Q) = I_{\mathcal{N}}$. If $P \leq Q$, then $P \perp (I_{\mathcal{M}}-Q)$. Hence $\phi(P) \perp \phi(I_{\mathcal{M}}-Q)$, and we conclude $\phi(P) \leq \phi(Q)$.

(iii). Assume that $P \perp Q$. Then $P, Q \leq P + Q$. By (ii) and the fact that ϕ is ortho-preserving we have $\phi(P) + \phi(Q) \leq \phi(P + Q)$. On the other hand,

$$\tau_{\mathcal{N}}(\phi(P) + \phi(Q)) = \tau_{\mathcal{N}}(\phi(P)) + \tau_{\mathcal{N}}(\phi(Q)) = \tau_{\mathcal{M}}(P) + \tau_{\mathcal{M}}(Q) = \tau_{\mathcal{M}}(P + Q) = \tau_{\mathcal{N}}(\phi(P + Q)).$$

It follows that $\phi(P + Q) = \phi(P) + \phi(Q)$.

(iv) and (v). Since $PQ = QP$, it is easy to verify there exist mutually orthogonal projections $E_1, E_2, E_3 \in \mathcal{P}(\mathcal{M})$ such that $P = E_1 + E_2$ and $Q = E_2 + E_3$, where $PQ = E_2$. By (iii) we have $\phi(P) = \phi(E_1) + \phi(E_2)$ and $\phi(Q) = \phi(E_2) + \phi(E_3)$, where $\phi(E_1), \phi(E_2), \phi(E_3)$ are also mutually orthogonal. Now $\phi(P)\phi(Q) = \phi(Q)\phi(P) = \phi(E_2) = \phi(PQ)$, which proves (iv). By (iii) and (iv) we find

$$\begin{aligned}
\phi(P) \vee \phi(Q) &= \phi(P) + \phi(Q) - \phi(P)\phi(Q) \\
&= \phi(E_1) + \phi(E_2) + \phi(E_3) \\
&= \phi(E_1 + E_2 + E_3) \\
&= \phi(P \vee Q).
\end{aligned}$$

□

5.1.3 Extension of φ from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ to $\mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$

Recall that φ is a map from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ into $\mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ satisfying Conditions (A) and (B). Before we extend the map φ from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, we will prove some additional properties of φ.

Lemma 5.1.9. *Let $k \geq 2$ and $E_1, F_1, E_2, F_2, \ldots, E_k, F_k$ be projections in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that*

(1) $E_1, F_1, E_2, F_2, \ldots, E_k, F_k$ are mutually commutative;

(2) $E_1 \vee F_1, E_2 \vee F_2, \ldots, E_k \vee F_k$ are mutually orthogonal;

(3) $E_1 \vee F_1 + E_2 \vee F_2 + \cdots + E_k \vee F_k = I_{\mathcal{M}}$.

Then

(i) $\varphi(E_1) \vee \varphi(F_1), \varphi(E_2) \vee \varphi(F_2), \ldots, \varphi(E_k) \vee \varphi(F_k)$ *are mutually orthogonal;*

(ii) $\varphi(E_i)\varphi(F_i) = \varphi(F_i)\varphi(E_i)$ *for all* $i \in \{1, 2, \ldots, k\}$*;*

(iii) $\varphi(E_1) \vee \varphi(F_1) + \varphi(E_2) \vee \varphi(F_2) + \cdots + \varphi(E_k) \vee \varphi(F_k) = I_{\mathcal{N}}$.

Proof. (i). Suppose $i, j \in \{1, 2, \ldots, k\}$ are distinct indices. By assumption (2) we have $E_i \perp E_j, F_j$. It follows from Condition (B) on φ that $\varphi(E_i) \perp \varphi(E_j), \varphi(F_j)$. Similarly $\varphi(F_i) \perp \varphi(E_j), \varphi(F_j)$. We conclude that $\varphi(E_i) \vee \varphi(F_i)$ is orthogonal to $\varphi(E_j) \vee \varphi(F_j)$ for each pair of distinct indices.

(ii). By (i), we have

$$1 = \tau_{\mathcal{N}}(I_{\mathcal{N}}) \geq \tau_{\mathcal{N}}(\varphi(E_1) \vee \varphi(F_1)) + \tau_{\mathcal{N}}(\varphi(E_2) \vee \varphi(F_2)) + \cdots + \tau_{\mathcal{N}}(\varphi(E_k) \vee \varphi(F_k)).$$

On the other hand, for each $i \in \{1, 2, \ldots, k\}$, by Lemma 3.7.2 and Condition (A) we obtain $\tau_{\mathcal{N}}(\varphi(E_i) \wedge \varphi(F_i)) \leq \tau_{\mathcal{N}}(\varphi(E_i)\varphi(F_i)) = \tau_{\mathcal{M}}(E_iF_i) = \tau_{\mathcal{M}}(E_i \wedge F_i)$. Hence, as a consequence of Kaplansky's formula, $\tau_{\mathcal{N}}(\varphi(E_i) \vee \varphi(F_i)) \geq \tau_{\mathcal{M}}(E_i \vee F_i)$. Therefore

$$\begin{aligned} 1 &\geq \tau_{\mathcal{N}}(\varphi(E_1) \vee \varphi(F_1)) + \tau_{\mathcal{N}}(\varphi(E_2) \vee \varphi(F_2)) + \cdots + \tau_{\mathcal{N}}(\varphi(E_k) \vee \varphi(F_k)) \\ &\geq \tau_{\mathcal{M}}(E_1 \vee F_1) + \tau_{\mathcal{M}}(E_2 \vee F_2) + \cdots + \tau_{\mathcal{M}}(E_k \vee F_k) = \tau_{\mathcal{M}}(I_{\mathcal{M}}) = 1. \end{aligned} \tag{5.1}$$

It follows that, for each $i \in \{1, 2, \ldots, k\}$, $\tau_{\mathcal{N}}(\varphi(E_i) \vee \varphi(F_i)) = \tau_{\mathcal{M}}(E_i \vee F_i)$ and

$$\tau_{\mathcal{N}}(\varphi(E_i) \wedge \varphi(F_i)) = \tau_{\mathcal{M}}(E_i \wedge F_i) = \tau_{\mathcal{M}}(E_iF_i) = \tau_{\mathcal{N}}(\varphi(E_i)\varphi(F_i)).$$

Therefore, by Lemma 3.7.2, $\varphi(E_i)\varphi(F_i) = \varphi(F_i)\varphi(E_i)$.

(iii). By (5.1) we have

$$\varphi(E_1) \vee \varphi(F_1) + \varphi(E_2) \vee \varphi(F_2) + \cdots + \varphi(E_k) \vee \varphi(F_k) = I_{\mathcal{N}}.$$

$\square$

Lemma 5.1.10. *For all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $EF = FE$ and $\tau_{\mathcal{M}}(E \vee F) \leq \tau_{\mathcal{M}}(I_{\mathcal{M}}) - c$, then $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$ and $\tau_{\mathcal{N}}(\varphi(E) \vee \varphi(F)) = \tau_{\mathcal{M}}(E \vee F)$.*

Proof. Since $\tau_{\mathcal{M}}(I_{\mathcal{M}} - E \vee F) \geq c$, there exist a nonnegative integer k and mutually commuting projections $P_1, P_2, \ldots, P_k, Q_1, Q_2$ in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $E \vee F, P_1, P_2, \ldots, P_k, Q_1 \vee Q_2$ mutually orthogonal such that

$$E \vee F + P_1 + P_2 + \cdots + P_k + Q_1 \vee Q_2 = I_{\mathcal{M}}.$$

Then $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$ follows from Lemma 5.1.9. Moreover,

$$\begin{aligned}
\tau_{\mathcal{N}}(\varphi(E) \vee \varphi(F)) &= \tau_{\mathcal{N}}(\varphi(E)) + \tau_{\mathcal{N}}(\varphi(F)) - \tau_{\mathcal{N}}(\varphi(E) \wedge \varphi(F)) \\
&= 2c - \tau_{\mathcal{N}}(\varphi(E)\varphi(F)) \\
&= 2c - \tau_{\mathcal{M}}(EF) \\
&= \tau_{\mathcal{M}}(E \vee F).
\end{aligned}$$

$\square$

Lemma 5.1.11. *Let $E_1, F_1, E_2, F_2, \ldots, E_m, F_m \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be mutually commuting and $E_1 \vee F_1, E_2 \vee F_2, \ldots, E_m \vee F_m$ mutually orthogonal. Suppose $\tau_{\mathcal{M}}(\sum_{1 \leq i \leq m} (E_i \vee F_i)) \leq \tau_{\mathcal{M}}(I_{\mathcal{M}}) - c$. If $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $P \leq \sum_{1 \leq i \leq m} (E_i \vee F_i)$, then $\varphi(P) \leq \sum_{1 \leq i \leq m} (\varphi(E_i) \vee \varphi(F_i))$.*

Proof. Since $\tau_{\mathcal{M}}(\sum_{1 \leq i \leq m} (E_i \vee F_i)) \leq \tau_{\mathcal{M}}(I_{\mathcal{M}}) - c$, there exist a nonnegative integer k and mutually commuting projections $P_1, P_2, \ldots, P_k, Q_1, Q_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that

$$\sum_{1\le i\le m} (E_i \vee F_i) + (\sum_{1\le i\le k} P_i) + Q_1 \vee Q_2 = I_{\mathcal{M}}.$$

It follows from Lemma 5.1.9 that

$$\sum_{1\le i\le m} (\varphi(E_i) \vee \varphi(F_i)) + (\sum_{1\le i\le k} \varphi(P_i)) + \varphi(Q_1) \vee \varphi(Q_2) = I_{\mathcal{N}}.$$

If $P \le \sum_{1\le i\le m} (E_i \vee F_i)$, by Condition (B) we have that $\varphi(P) \perp \varphi(P_i)$ for $1 \le i \le k$, and $\varphi(P) \perp \varphi(Q_j)$ for $j = 1, 2$. Thus $\varphi(P) \le \sum_{1\le i\le m} (\varphi(E_i) \vee \varphi(F_i))$. □

We now aim to extend φ to a mapping $\hat{\varphi} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ which has several useful properties. However, we will give the definition of $\hat{\varphi}$ in several steps. We first define $\hat{\varphi}$ on $\mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$. Recall that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = 1$.

Definition 5.1.12. Let $\mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid c \le \tau_{\mathcal{M}}(P) \le 1 - c\}$. We define

$$\hat{\varphi} : \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$$

by

$$\hat{\varphi}(P) = \bigvee \{\varphi(E) \mid E \le P \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

We will now show that $\hat{\varphi}$, as defined above, is a trace-order-ortho-preserving map which extends φ.

Lemma 5.1.13. *Let $\hat{\varphi}$ be the map constructed in Definition 5.1.12. The following statements are true.*

(i) For each $P \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$, if there exist a nonnegative integer k and mutually commuting projections $P_1, P_2, \ldots, P_k, Q_1, Q_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $P = \sum_{1\le i\le k} P_i + Q_1 \vee Q_2$,

then

$$\hat{\varphi}(P) = \sum_{1 \le i \le k} \varphi(P_i) + \varphi(Q_1) \vee \varphi(Q_2).$$

(ii) $\tau_{\mathcal{N}}(\hat{\varphi}(P)) = \tau_{\mathcal{M}}(P)$ *for all* $P \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$.

(iii) $\hat{\varphi}(P) = \varphi(P)$ *for all* $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, *and* $\hat{\varphi}(Q) = I_{\mathcal{N}} - \varphi(I_{\mathcal{M}} - Q)$ *for all* $Q \in \mathcal{P}_{1-c}(\mathcal{M}, \tau_{\mathcal{M}})$;

(iv) For all $P, Q \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$, *if* $P \le Q$, *then* $\hat{\varphi}(P) \le \hat{\varphi}(Q)$.

(v) For all $P, Q \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$, *if* $P \perp Q$, *then* $\hat{\varphi}(P) \perp \hat{\varphi}(Q)$.

Proof. (i). This property is easily verified using Lemma 5.1.11 and Definition 5.1.12.

(ii). Let $P \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$. Then there exists a nonnegative integer k and mutually commuting projections $P_1, P_2, \ldots, P_k, Q_1, Q_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P = \sum_{1 \le i \le k} P_i + Q_1 \vee Q_2$. By (i) and Lemma 5.1.10 we have $\tau_{\mathcal{N}}(\hat{\varphi}(P)) = \tau_{\mathcal{M}}(P)$.

(iii). It follows directly from Definition 5.1.12 that $\hat{\varphi}(P) = \varphi(P)$ for any $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Now let $Q \in \mathcal{P}_{1-c}(\mathcal{M}, \tau_{\mathcal{M}})$. Then for any $E \le Q$ with $E \in P_c(\mathcal{M})$, we have $\varphi(E) \perp \varphi(I_{\mathcal{M}} - Q)$ by Condition (B). Therefore $\hat{\varphi}(Q) \le I_{\mathcal{N}} - \varphi(I_{\mathcal{M}} - Q)$. Notice that by (ii) we have $\tau_{\mathcal{N}}(\hat{\varphi}(Q)) = 1 - c = \tau_{\mathcal{N}}(I_{\mathcal{N}} - \varphi(I_{\mathcal{M}} - Q))$. Hence $\hat{\varphi}(Q) = I_{\mathcal{N}} - \varphi(I_{\mathcal{M}} - Q)$.

(iv) and (v). Notice that the map $\hat{\varphi}$ given in Definition 5.1.12 satisfies the hypothesis of Lemma 5.1.6, whence $\hat{\varphi}$ is an order-ortho-preserving map. □

5.1.4 Extension when $0 < c \le \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{4}$

We have extended φ to $\hat{\varphi}$ on $\mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$ in subsection 5.1.3. For the extension of φ on $\mathcal{P}_{[1-c,1]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$, we consider different cases depending on the value of c. Recall that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = 1$.

Lemma 5.1.14. *Suppose* $c \in (0, \frac{1}{4}]$ *and* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *is not an empty set. Suppose* $P, Q \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$ *are two commuting projections with* $\tau_{\mathcal{M}}(P) = \tau_{\mathcal{M}}(Q) = 2c$. *If* $P \vee Q \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$, *then* $\hat{\varphi}(P \vee Q) = \hat{\varphi}(P) \vee \hat{\varphi}(Q)$.

Proof. Since $PQ = QP$, it is easily verified that there exist mutually commuting projections $E_1, E_2, F_1, F_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P = E_1 + E_2$ and $Q = F_1 + F_2$. It follows from Lemma 5.1.13(i) that $\hat{\varphi}(P) = \varphi(E_1) + \varphi(E_2)$ and $\hat{\varphi}(Q) = \varphi(F_1) + \varphi(F_2)$. Since $\hat{\varphi}$ is order-preserving by Lemma 5.1.13, we have $\hat{\varphi}(P), \hat{\varphi}(P) \leq \hat{\varphi}(P \vee Q)$. Hence

$$\hat{\varphi}(P) \vee \hat{\varphi}(Q) \leq \hat{\varphi}(P \vee Q). \tag{5.2}$$

By Lemma 5.1.10, $\varphi(E_1), \varphi(E_2), \varphi(F_1), \varphi(F_2)$ mutually commute. By Condition (A), we have $\tau_{\mathcal{N}}(\varphi(E_i)\varphi(F_j)) = \tau_{\mathcal{M}}(E_i F_j)$ for $i, j \in \{1, 2\}$. Therefore $\hat{\varphi}(P)\hat{\varphi}(Q) = \hat{\varphi}(Q)\hat{\varphi}(P)$ and

$$\tau_{\mathcal{N}}(\hat{\varphi}(P)\hat{\varphi}(Q)) = \tau_{\mathcal{N}}((\varphi(E_1)+\varphi(E_2))(\varphi(F_1)+\varphi(F_2))) = \tau_{\mathcal{M}}((E_1+E_2)(F_1+F_2)) = \tau_{\mathcal{M}}(PQ).$$

Since $\hat{\varphi}$ is trace-preserving by Lemma 5.1.13, it follows that

$$\tau_{\mathcal{N}}(\hat{\varphi}(P) \vee \hat{\varphi}(Q)) = 4c - \tau_{\mathcal{N}}(\hat{\varphi}(P)\hat{\varphi}(Q)) = 4c - \tau_{\mathcal{M}}(PQ) = \tau_{\mathcal{M}}(P \vee Q) = \tau_{\mathcal{N}}(\hat{\varphi}(P \vee Q)).$$

By the above equivalence of traces and (5.2), we conclude $\hat{\varphi}(P) \vee \hat{\varphi}(Q) = \hat{\varphi}(P \vee Q)$. $\square$

Lemma 5.1.15. *Suppose $c \in (0, \frac{1}{4}]$. Let $E, F, P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be such that $EF = FE$. If $E \wedge F \leq P$, then $\varphi(E) \wedge \varphi(F) \leq \varphi(P)$.*

Proof. Clearly, $\tau_{\mathcal{M}}(E \vee F \vee P) \leq 3c$. Hence there exists $Q \in \mathcal{P}(\mathcal{M})$ with $\tau_{\mathcal{M}}(Q) = 3c$ and $E \vee F \vee P \leq Q$. Clearly $E \wedge F \leq P$ implies $Q - E \wedge F \geq Q - P$, so we have $(Q - E) \vee (Q - F) \geq Q - P$. Note that $EF = FE$. We use the fact that $\hat{\varphi}$ is order-preserving by Lemma 5.1.13, together with Lemma 5.1.14, to obtain

$$\hat{\varphi}(Q - E) \vee \hat{\varphi}(Q - F) \geq \hat{\varphi}(Q - P). \tag{5.3}$$

Since $\hat{\varphi}$ is a trace-order-ortho-preserving extension of φ by Lemma 5.1.13, we have

$$\tau_{\mathcal{N}}(\hat{\varphi}(Q-E)) = 2c = \tau_{\mathcal{N}}(\hat{\varphi}(Q)) - \tau_{\mathcal{N}}(\varphi(E)),$$

$\hat{\varphi}(Q-E) \leq \hat{\varphi}(Q)$, and $\hat{\varphi}(Q-E) \perp \varphi(E)$. Therefore $\hat{\varphi}(Q-E) \leq \hat{\varphi}(Q) - \varphi(E)$ with equivalency of traces, whence $\hat{\varphi}(Q-E) = \hat{\varphi}(Q) - \varphi(E)$. The same argument holds for $\hat{\varphi}(Q-F)$ and $\hat{\varphi}(Q-P)$. Thus by (5.3) we have

$$(\hat{\varphi}(Q) - \varphi(E)) \vee (\hat{\varphi}(Q) - \varphi(F)) \geq \hat{\varphi}(Q) - \varphi(P).$$

Therefore $\hat{\varphi}(Q) - \varphi(E) \wedge \varphi(F) \geq \hat{\varphi}(Q) - \varphi(P)$. We conclude

$$\varphi(E) \wedge \varphi(F) \leq \varphi(P).$$

□

We are now prepared to define $\hat{\varphi}$ on $\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid 0 \leq \tau_{\mathcal{M}}(P) \leq c\}$ for $c \in (0, \frac{1}{4}]$.

Definition 5.1.16. Assume that $c \in (0, \frac{1}{4}]$. We define

$$\hat{\varphi} : \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$$

by

$$\hat{\varphi}(P) = \bigwedge\{\varphi(E) \mid P \leq E \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } \; P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

Lemma 5.1.17. *Assume that $c \in (0, \frac{1}{4}]$. Let $\hat{\varphi}$ be the map constructed in Definition* 5.1.16. *The following statements are true.*

(i) $\hat{\varphi}(0) = 0$ *and* $\hat{\varphi}(P) = \varphi(P)$ *for all* $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

(ii) For all $P, Q \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $PQ = QP$, then $\hat{\varphi}(P \wedge Q) = \varphi(P) \wedge \varphi(Q)$.

(iii) $\hat{\varphi}$ is order-preserving.

(iv) $\hat{\varphi}$ is trace-preserving.

(v) $\hat{\varphi}$ is ortho-preserving.

Proof. (i). It is clear that $\hat{\varphi}(P) = \varphi(P)$ for any $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Take two projections $E_1, E_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $E_1 \perp E_2$. By Condition (B) we have $\varphi(E_1) \perp \varphi(E_2)$. By Definition 5.1.16, $\hat{\varphi}(0) \leq \varphi(E_1) \wedge \varphi(E_2) = 0$ and thus $\hat{\varphi}(0) = 0$.

(ii). The proof is clear by Lemma 5.1.15.

(iii). It follows directly from Definition 5.1.16.

(iv). Suppose $P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$. Then there exist $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $EF = FE$ such that $P = EF = E \wedge F$. By Lemma 5.1.10, (ii), and Condition (A) we have

$$\tau_{\mathcal{N}}(\hat{\varphi}(P)) = \tau_{\mathcal{N}}(\varphi(E) \wedge \varphi(F)) = \tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF) = \tau_{\mathcal{M}}(P).$$

(v). Notice by (iii) $\hat{\varphi}$ satisfies the hypothesis of Lemma 5.1.5, hence $\hat{\varphi}$ is ortho-preserving.

□

Assume that $c \in (0, \frac{1}{4}]$. The definition of $\hat{\varphi}$ on

$$\mathcal{P}_{[1-c,1]}(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid 1 - c \leq \tau_{\mathcal{M}}(P) \leq 1\}$$

is given in the following as a complement of $\hat{\varphi}$ on $\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$.

Definition 5.1.18. Assume that $c \in (0, \frac{1}{4}]$. We define $\hat{\varphi}$ on $\mathcal{P}_{[1-c,1]}(\mathcal{M}, \tau_{\mathcal{M}})$ by

$$\hat{\varphi}(P) = I_{\mathcal{N}} - \hat{\varphi}(I_{\mathcal{M}} - P), \qquad \text{for all } P \in \mathcal{P}_{[1-c,1]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

Lemma 5.1.19. *Assume that* $c \in (0, \frac{1}{4}]$. *Let* $\hat{\varphi}$ *be the map constructed in Definition* 5.1.18. *The following statements are true.*

(i) $\hat{\varphi}(I_{\mathcal{M}}) = I_{\mathcal{N}}$, *and* $\hat{\varphi}(I_{\mathcal{M}} - E) = I_{\mathcal{N}} - \varphi(E)$ *for all* $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

(ii) $\hat{\varphi}$ *is order-preserving. For all* $P, Q \in \mathcal{P}_{[1-c,1]}(\mathcal{M}, \tau_{\mathcal{M}})$, *if* $P \leq Q$, *then* $\hat{\varphi}(P) \leq \hat{\varphi}(Q)$.

(iii) $\hat{\varphi}$ *is trace-preserving.* $\tau_{\mathcal{N}}(\hat{\varphi}(P)) = \tau_{\mathcal{M}}(P)$ *for all* $P \in \mathcal{P}_{[1-c,1]}(\mathcal{M}, \tau_{\mathcal{M}})$.

(iv) For each $P \in \mathcal{P}_{[1-c,1]}(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\hat{\varphi}(P) = \bigvee \{\hat{\varphi}(E) \mid E \leq P \text{ and } E \in \mathcal{P}_{1-c}(\mathcal{M}, \tau_{\mathcal{M}})\}.$$

Proof. These statements follow easily from Definition 5.1.18 and Lemma 5.1.17. □

Note that, for $c \in (0, \frac{1}{4}]$, we have obtained a mapping $\hat{\varphi} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$, as an extension of φ, from Definitions 5.1.12, 5.1.16, and 5.1.18. It is easy to verify from Lemmas 5.1.13, 5.1.17, and 5.1.19 that Definitions 5.1.12 and 5.1.16 coincide on $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, and Definitions 5.1.12 and 5.1.18 coincide on $\mathcal{P}_{1-c}(\mathcal{M}, \tau_{\mathcal{M}})$.

Proposition 5.1.20. *Let* $\hat{\varphi} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ *be the mapping defined in Definitions* 5.1.12, 5.1.16, *and* 5.1.18. *Then* $\hat{\varphi}$ *is a trace-order-preserving orthomorphism, extending* φ, *such that* $\hat{\varphi}(0) = 0$ *and* $\hat{\varphi}(I_{\mathcal{M}}) = I_{\mathcal{N}}$.

Proof. It is clear from Lemmas 5.1.13, 5.1.17, and 5.1.19 that $\hat{\varphi}$ is trace-preserving and extends φ. Further, by Lemma 5.1.7, $\hat{\varphi}$ must be order-ortho-preserving. Now by Lemma 5.1.8 we have that $\hat{\varphi}$ is an orthomorphism such that $\hat{\varphi}(0) = 0$ and $\hat{\varphi}(I_{\mathcal{M}}) = I_{\mathcal{N}}$. □

By Proposition 5.1.20 and Lemma 3.3.5, we obtain the following theorem.

Theorem 5.1.21. *Suppose* $\mathcal{M}$ *is a finite von Neumann algebra with a faithful normal tracial state* $\tau_{\mathcal{M}}$. *We further assume that* $\mathcal{M}$ *has no type* I_2 *direct summand, and that* $\mathcal{M}$ *is either diffuse or atomic such that each minimal projection in* $\mathcal{M}$ *has the same trace. Let* $\mathcal{N}$ *be* *a finite von Neumann algebra with faithful normal tracial state* $\tau_{\mathcal{N}}$*. If* $c \in (0, \frac{1}{4}]$ *and* $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *is a map satisfying Condition (A), then* φ *can be extended to a trace-preserving Jordan* $*$*-homomorphism* $\rho : \mathcal{M} \to \mathcal{N}$*.*

Proof. Note that $c \leq \frac{1}{4}$ and φ satisfies Condition (A). Hence, as observed in Remark 5.1.2, φ also satisfies Condition (B). Thus, by Proposition 5.1.20 and Lemma 3.3.5, there exists a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that $\rho(P) = \varphi(P)$ for all $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. □

5.1.5 Extension when $\frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{4} < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$

In this subsection we will consider the case where $\frac{1}{4} < c < \frac{1}{2}$. Recall that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = 1$. For any $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, denote $J^{\perp} = I_{\mathcal{M}} - J$ and $\varphi(J)^{\perp} = I_{\mathcal{N}} - \varphi(J)$. Clearly $\tau_{\mathcal{M}}(J^{\perp}) = \tau_{\mathcal{N}}(\varphi(J)^{\perp}) = 1 - c$. We let

$$\mathcal{P}(J^{\perp}\mathcal{M}J^{\perp}) = \{P \in \mathcal{P}(\mathcal{M}) \mid P \perp J\}$$

and

$$\mathcal{P}_c(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(J^{\perp}\mathcal{M}J^{\perp}) \mid \tau_{\mathcal{M}}(P) = c\}.$$

Observe that $\mathcal{P}_c(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}}) \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

Proposition 5.1.22. *Assume that* $\frac{1}{4} < c \leq \frac{1}{3}$*. Then, for each* $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*, there exists a map* $\varphi_J : \mathcal{P}(J^{\perp}\mathcal{M}J^{\perp}) \to \mathcal{P}(\varphi(J)^{\perp}\mathcal{N}\varphi(J)^{\perp})$ *such that*

(i) $\varphi_J(0) = 0$ *and* $\varphi_J(J^{\perp}) = \varphi(J)^{\perp}$*;*

(ii) $\varphi_J(E) = \varphi(E)$ *for all* $E \in \mathcal{P}_c(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$*;*

(iii) φ_J *is trace-preserving;*

(iv) φ_J *is order-preserving;*

(v) φ_J *is an orthomorphism. For all* $P, Q \in \mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$*, if* $P \perp Q$*, then* $\varphi_J(P) \perp \varphi_J(Q)$ *and* $\varphi_J(P+Q) = \varphi_J(P) + \varphi_J(Q)$*.*

Proof. Let $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Note that Definition 5.1.12 gives an extension $\hat{\varphi}$ of φ on the set $\mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}})$, for $c \in (\frac{1}{4}, \frac{1}{3}]$, defined by

$$\hat{\varphi}(P) = \bigvee \{\varphi(E) \mid E \leq P \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

By Lemma 5.1.13, $\hat{\varphi}$ is trace-order-ortho-preserving. Moreover

$$\hat{\varphi}(J^\perp) = I_{\mathcal{N}} - \varphi(I_{\mathcal{M}} - J^\perp) = I_{\mathcal{N}} - \varphi(J) = \varphi(J)^\perp.$$

We define $\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$ by

$$\varphi_J(P) = \begin{cases} \hat{\varphi}(P) & \text{if } P \in \mathcal{P}_{[c,1-c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \\ \hat{\varphi}(J)^\perp - \hat{\varphi}(J^\perp - P) & \text{if } P \in \mathcal{P}_{[0,c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \end{cases}.$$

Note that $\hat{\varphi}$ is an orthomorphism by Lemma 3.3.3, whence

$$\hat{\varphi}(J)^\perp = \hat{\varphi}(J^\perp) = \hat{\varphi}(E + (J^\perp - E)) = \varphi(E) + \hat{\varphi}(J^\perp - E)$$

for any $E \in \mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$. Therefore φ_J is well-defined. Again, by Lemma 5.1.13, φ_J is trace-order-ortho-preserving on the set $\mathcal{P}_{[c,1-c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$ and consequently on $\mathcal{P}_{[0,c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$ as well. Thus, by Lemma 5.1.7, φ_J is trace-order-ortho-preserving on $\mathcal{P}(J^\perp \mathcal{M} J^\perp)$. Now the result follows from Lemma 5.1.8. □

Proposition 5.1.23. *Assume that* $\frac{1}{3} < c < \frac{1}{2}$*. Then, for each* $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*, there exists a map* $\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$ *such that*

(i) $\varphi_J(0) = 0$ *and* $\varphi_J(J^\perp) = \varphi(J)^\perp$*;*

(ii) $\varphi_J(E) = \varphi(E)$ *for all* $E \in \mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$*;*

(iii) φ_J *is trace-preserving;*

(iv) φ_J *is order-preserving;*

(v) φ_J *is an orthomorphism.*

Proof. Suppose $c \in (\frac{1}{3}, \frac{1}{2})$ and $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. Let $x = 1 - 2c$. Therefore $1 = 2c + x$ and $0 < x < c$. For $0 \le a \le b \le c + x$, we denote

$$\mathcal{P}_{[a,b]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid P \perp J \text{ and } a \le \tau_{\mathcal{M}}(P) \le b\} \subseteq \mathcal{P}_{[a,b]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

By Definition 5.1.12, we can extend φ to a trace-order-ortho-preserving map $\hat{\varphi}$ on the set $\mathcal{P}_{[c,c+x]}(\mathcal{M}, \tau_{\mathcal{M}})$ by

$$\hat{\varphi}(P) = \bigvee \{\varphi(E) \mid E \le P \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{[c,c+x]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

The restriction of this map to $\mathcal{P}_{[c,c+x]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$ is also trace-order-ortho-preserving and satisfies, for each $P \in \mathcal{P}_{[c,c+x]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$,

$$\hat{\varphi}(P) = \bigvee \{\varphi(E) \mid E \le P \text{ and } E \in \mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})\}. \tag{5.4}$$

It is now easily verified that $\tilde{\varphi}$, given by

$$\tilde{\varphi}(P) = \varphi(J)^\perp - \hat{\varphi}(J^\perp - P), \qquad \text{for all } P \in \mathcal{P}_{[0,x]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}), \tag{5.5}$$

is a trace-order-preserving map on $\mathcal{P}_{[0,x]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$.

In the following we will present an extension of φ on $\mathcal{P}_{[x,c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$. Recall that

$$\mathcal{P}_x(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid P \perp J \text{ and } \tau_{\mathcal{M}}(P) = x\}$$

$$\mathcal{P}_x(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp, \tau_{\mathcal{N}}) = \{Q \in \mathcal{P}(\mathcal{N}) \mid Q \perp \varphi(J) \text{ and } \tau_{\mathcal{N}}(Q) = x\}.$$

Define $\varphi_x : \mathcal{P}_x(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \to \mathcal{P}_x(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp, \tau_{\mathcal{N}})$ by setting

$$\varphi_x(P) = \varphi(J)^\perp - \varphi(J^\perp - P), \qquad \text{for all } P \in \mathcal{P}_x(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}).$$

It is easy to verify that φ_x is well-defined and is a restriction of $\tilde{\varphi}$.

Claim 5.1.23.1. *For all* $P_1, P_2 \in \mathcal{P}_x(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$*, if* $P_1 P_2 = P_2 P_1$*, then*

$$\tau_{\mathcal{N}}(\varphi_x(P_1)\varphi_x(P_2)) = \tau_{\mathcal{M}}(P_1 P_2).$$

Proof of Claim 5.1.23.1. Let $E_1 = J^\perp - P_1$ and $E_2 = J^\perp - P_2$. Then $\tau_{\mathcal{M}}(E_1) = \tau_{\mathcal{M}}(E_2) = c$ and $E_1 E_2 = E_2 E_1$. Note that $\tau_{\mathcal{M}}(E_1 \vee E_2) \le \tau_{\mathcal{M}}(J^\perp) = 1 - c$. By Condition (B) we have that $E_1, E_2 \perp J$ implies $\varphi(E_1), \varphi(E_2) \le \varphi(J)^\perp$. We conclude

$$\begin{aligned}
\tau_{\mathcal{N}}(\varphi_x(P_1)\varphi_x(P_2)) &= \tau_{\mathcal{N}}((\varphi(J)^\perp - \varphi(E_1))(\varphi(J)^\perp - \varphi(E_2))) \\
&= \tau_{\mathcal{N}}(\varphi(J)^\perp - \varphi(E_1) - \varphi(E_2) + \varphi(E_1)\varphi(E_2)) \\
&= 1 - 3c + \tau_{\mathcal{M}}(E_1 E_2) \qquad \text{(by Condition (A))} \\
&= \tau_{\mathcal{M}}(J^\perp - E_1 - E_2 + E_1 E_2) \\
&= \tau_{\mathcal{M}}(P_1 P_2).
\end{aligned}$$

This finishes the proof of the claim.

We observe, further, that by Claim 5.1.23.1 φ_x is ortho-preserving. Thus $\tilde{\varphi}$ is ortho-preserving.

Notice that $J^\perp \mathcal{M} J^\perp$ and $\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp$ are finite von Neumann algebras with faithful normal tracial states $\frac{1}{c+x}\tau_{\mathcal{M}}$ and $\frac{1}{c+x}\tau_{\mathcal{N}}$, respectively. Therefore, by Claim 5.1.23.1, Definition 5.1.12, and Lemma 5.1.13, the map $\varphi_x : \mathcal{P}_x(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \to \mathcal{P}_x(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp, \tau_{\mathcal{N}})$ can be extended to a trace-order-ortho-preserving map

$$\hat{\varphi}_x : \mathcal{P}_{[x,c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \to \mathcal{P}_{[x,c]}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp, \tau_{\mathcal{N}})$$

defined by

$$\hat{\varphi}_x(P) = \bigvee \{\varphi(E) \mid E \leq P \text{ and } E \in \mathcal{P}_x(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})\} \tag{5.6}$$

for each $P \in \mathcal{P}_{[x,c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$.

Claim 5.1.23.2. *For all $P \in \mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$, $\hat{\varphi}_x(P) = \varphi(P)$.*

Proof of Claim 5.1.23.2. By Lemma 5.1.13 we find

$$\hat{\varphi}_x(P) = \varphi(J)^\perp - \varphi_x(J^\perp - P) = \varphi(J)^\perp - (\varphi(J)^\perp - \varphi(J^\perp - (J^\perp - P))) = \varphi(P).$$

From equations (5.4), (5.5), and (5.6), we define a mapping

$$\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$$

by

$$\varphi_J(P) = \begin{cases} \hat{\varphi}(P) & \text{if } P \in \mathcal{P}_{[c,c+x]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \\ \hat{\varphi}_x(P) & \text{if } P \in \mathcal{P}_{[x,c]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \\ \tilde{\varphi}(P) & \text{if } P \in \mathcal{P}_{[0,x]}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}}) \end{cases}.$$

By Claim 5.1.23.2, we have that φ_J is an extension of φ. Thus (ii) holds.

From Lemma 5.1.13 and Lemma 5.1.7, the map $\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$ is trace-ortho-preserving, whence (iii) is true. Now properties (i), (iv), and (v) follow from Lemma 5.1.8. □

A combination of Proposition 5.1.22 and Proposition 5.1.23 gives us the following corollary.

Corollary 5.1.24. *Suppose $\frac{1}{4} < c < \frac{1}{2}$ and a map $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ satisfies Conditions (A) and (B). For each $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, there exists a trace-preserving orthomorphism $\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$ extending φ on $\mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$.*

Lemma 5.1.25. *Suppose $\frac{1}{4} < c < \frac{1}{2}$ and $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ is a map satisfying Conditions (A) and (B). Let $\Lambda = \{\tau_{\mathcal{M}}(P) \mid P \in \mathcal{M}\}$ and $y_1 \in (\frac{1}{4}, c] \cap \Lambda$. Suppose $\varphi_1 : \mathcal{P}_{y_1}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{y_1}(\mathcal{N}, \tau_{\mathcal{N}})$ is a map such that φ and φ_1 satisfy the following two conditions.*

(i) For each $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, there exists a trace-preserving orthomorphism

$$\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$$

extending φ on $\mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$.

(ii) For each $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if

$$\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$$

is a trace-preserving orthomorphism extending φ on $\mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$, then φ_J extends φ_1 on $\mathcal{P}_{y_1}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$.

Then, for any $y_2 \in [\frac{3y_1+c-1}{2}, y_1) \cap \Lambda$, there exists a mapping $\varphi_2 : \mathcal{P}_{y_2}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{y_2}(\mathcal{N}, \tau_{\mathcal{N}})$ such that, for each $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if

$$\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$$

is a trace-preserving orthomorphism extending φ on $\mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$, then φ_J extends φ_2 on $\mathcal{P}_{y_2}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$.

Proof. We first prove the following claim.

Claim 5.1.25.1. *For all P_1, P_2, Q in $\mathcal{P}_{y_1}(\mathcal{M}, \tau_{\mathcal{M}})$ with $P_1P_2 = P_2P_1$, $P_1 \wedge P_2 \leq Q$, and $\tau_{\mathcal{M}}(P_1 \wedge P_2) = y_2$, we have $\varphi_1(P_1) \wedge \varphi_1(P_2) \leq \varphi_1(Q)$ and $\tau_{\mathcal{N}}(\varphi_1(P_1) \wedge \varphi_1(P_2)) = y_2$.*

Proof of Claim 5.1.25.1. Note that $Q = P_1P_2 + (Q - P_1P_2)$, and $P_i = P_1P_2 + (P_i - P_1P_2)$ for $i \in \{1, 2\}$. Therefore

$$\tau_{\mathcal{M}}(P_1 \vee P_2 \vee Q) \leq y_2 + 3(y_1 - y_2) = 3y_1 - 2y_2 \leq 3y_1 - (3y_1 + c - 1) = 1 - c.$$

Hence there exists $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P_1 \vee P_2 \vee Q \leq J^{\perp}$. By condition (i), there exists a trace-preserving orthomorphism $\varphi_J : \mathcal{P}(J^{\perp}\mathcal{M}J^{\perp}) \to \mathcal{P}(\varphi(J)^{\perp}\mathcal{N}\varphi(J)^{\perp})$ extending φ on $\mathcal{P}_c(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$. It follows from condition (ii) that $\varphi_J(P) = \varphi_1(P)$ for all $P \in \mathcal{P}_{y_1}(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$. By Lemma 5.1.8,

$$\varphi_1(P_1) \wedge \varphi_1(P_2) = \varphi_J(P_1) \wedge \varphi_J(P_2) = \varphi_J(P_1)\varphi_J(P_2) = \varphi_J(P_1P_2) \leq \varphi_J(Q) = \varphi_1(Q).$$

Also,

$$\tau_{\mathcal{N}}(\varphi_1(P_1) \wedge \varphi_1(P_2)) = \tau_{\mathcal{N}}(\varphi_J(P_1)\varphi_J(P_2)) = \tau_{\mathcal{N}}(\varphi_J(P_1P_2)) = \tau_{\mathcal{M}}(P_1P_2) = y_2.$$

This completes the proof of the claim.

We now define a mapping $\varphi_2 : \mathcal{P}_{y_2}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$ by

$$\varphi_2(P) = \bigwedge\{\varphi_1(E) \mid P \leq E \text{ and } E \in \mathcal{P}_{y_1}(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{y_2}(\mathcal{M}, \tau_{\mathcal{M}}).$$

Claim 5.1.25.2. *If $P_1, P_2 \in \mathcal{P}_{y_1}(\mathcal{M}, \tau_{\mathcal{M}})$ are such that $P_1P_2 = P_2P_1$ and $\tau_{\mathcal{M}}(P_1P_2) = y_2$, then $\varphi_2(P_1P_2) = \varphi_1(P_1) \wedge \varphi_1(P_2)$ and $\tau_{\mathcal{N}}(\varphi_2(P_1P_2)) = y_2$.*

Proof of Claim 5.1.25.2. It follows easily from the definition of φ_2 and Claim 5.1.25.1.

For each $P \in \mathcal{P}_{y_2}(\mathcal{M}, \tau_{\mathcal{M}})$ we can find commuting projections $P_1, P_2 \in \mathcal{P}_{y_1}(\mathcal{M}, \tau_{\mathcal{M}})$ with $P = P_1P_2$. By Claim 5.1.25.2, we then have $\varphi_2(P) \in \mathcal{P}_{y_2}(\mathcal{N}, \tau_{\mathcal{N}})$. Let $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and $\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$ be a trace-preserving orthomorphism which extends φ on $\mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$. To complete the proof, it remains to show that φ_J extends φ_2 on $\mathcal{P}_{y_2}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$. By condition (ii), φ_J extends φ_1 on $\mathcal{P}_{y_1}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$. Let $Q \in \mathcal{P}_{y_2}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$. Then there exist $P_1, P_2 \in \mathcal{P}_{y_1}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$ such that $P_1P_2 = P_2P_1$ and $P_1 \wedge P_2 = Q$. By Claim 5.1.25.2 and Lemma 5.1.8, we have

$$\varphi_2(Q) = \varphi_2(P_1 \wedge P_2) = \varphi_1(P_1) \wedge \varphi_1(P_2) = \varphi_J(P_1)\varphi_J(P_2) = \varphi_J(P_1P_2) = \varphi_J(Q).$$

Hence φ_J extends φ_2 on $\mathcal{P}_{y_2}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$. This ends the proof. $\square$

For a diffuse von Neumann algebra $\mathcal{M}$ we have the following.

Theorem 5.1.26. *Assume $\frac{1}{4} < c < \frac{1}{2}$. Suppose $\mathcal{M}, \mathcal{N}$ are finite von Neumann algebras with faithful normal tracial states $\tau_{\mathcal{M}}, \tau_{\mathcal{N}}$ respectively. Assume that $\mathcal{M}$ is diffuse and has no direct summand of type I_2. Let $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ be a map satisfying Conditions (A) and (B). Then φ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$.*

Proof. We construct a sequence $\{y_n\}$ by setting $y_1 = c$ and $y_{n+1} = \frac{3y_n + c - 1}{2}$ for $n \geq 1$. It is easy to verify that $\{y_n\}$ is a strictly decreasing sequence. Notice, too, that $y_n - y_{n+1} \geq \frac{1}{2} - c$ for any $n \geq 1$ and $y_n - y_{n+1} < \frac{1}{4}$ if $y_n > \frac{1}{4}$. Hence there exists some $m \in \mathbb{N}$ such that $0 < y_m \leq \frac{1}{4} < y_{m-1}$. By Corollary 5.1.24 and Lemma 5.1.25, there exists a mapping $\varphi_m : \mathcal{P}_{y_m}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{y_m}(\mathcal{N}, \tau_{\mathcal{N}})$ such that, if $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and $\varphi_J : \mathcal{P}(J^\perp \mathcal{M} J^\perp) \to \mathcal{P}(\varphi(J)^\perp \mathcal{N} \varphi(J)^\perp)$ is a trace-preserving orthomorphism extending φ on $\mathcal{P}_c(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$, then φ_J extends φ_m on $\mathcal{P}_{y_m}(J^\perp \mathcal{M} J^\perp, \tau_{\mathcal{M}})$.

Claim 5.1.26.1. *$\varphi_m : \mathcal{P}_{y_m}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{y_m}(\mathcal{N}, \tau_{\mathcal{N}})$ satisfies Conditions (A) and (B) with c replaced by y_m.*

Proof of Claim 5.1.26.1. Take $E_1, E_2 \in \mathcal{P}_{y_m}(\mathcal{M}, \tau_{\mathcal{M}})$ with $E_1E_2 = E_2E_1$. Clearly we have

$\tau_{\mathcal{M}}(E_1 \vee E_2) \leq 2y_m < 1-c$. Hence there exists $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E_1 \vee E_2 \leq J^{\perp}$. By Corollary 5.1.24, there exists a trace-preserving orthomorphism $\varphi_J : \mathcal{P}(J^{\perp}\mathcal{M}J^{\perp}) \to \mathcal{P}(\varphi(J)^{\perp}\mathcal{N}\varphi(J)^{\perp})$ that extends φ on $\mathcal{P}_c(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$. It follows from the construction of φ_m that φ_J extends φ_m on $\mathcal{P}_{y_m}(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$. Therefore $\varphi_m(E_1) = \varphi_J(E_1)$ and $\varphi_m(E_2) = \varphi_J(E_2)$. By Lemma 5.1.8,

$$\tau_{\mathcal{N}}(\varphi_m(E_1)\varphi_m(E_2)) = \tau_{\mathcal{N}}(\varphi_J(E_1)\varphi_J(E_2)) = \tau_{\mathcal{N}}(\varphi_J(E_1E_2)) = \tau_{\mathcal{M}}(E_1E_2).$$

Hence Condition (A) is satisfied. Note that $y_m \leq \frac{1}{4}$, whence Condition (B) is also satisfied.

By Claim 5.1.26.1 and Proposition 5.1.20, φ_m can be extended to a trace-preserving orthomorphism $\psi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$.

Claim 5.1.26.2. $\psi(E) = \varphi(E)$ *for all* $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

Proof of Claim 5.1.26.2. Note that $\tau_{\mathcal{M}}(E) = c < 1-c$, so there exists $J \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E \perp J$. By Corollary 5.1.24, there exists a trace-preserving orthomorphism $\varphi_J : \mathcal{P}(J^{\perp}\mathcal{M}J^{\perp}) \to \mathcal{P}(\varphi(J)^{\perp}\mathcal{N}\varphi(J)^{\perp})$ that extends φ on $\mathcal{P}_c(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$. By the construction of φ_m, φ_J extends φ_m on $\mathcal{P}_{y_m}(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$. Note that $E \leq J^{\perp}$. We may assume that $E = F_1 + F_2 + \cdots + F_k + G_1 \vee G_2$, where $F_1, F_2, \ldots, F_k, G_1, G_2$ are mutually commuting in $\mathcal{P}_{y_m}(J^{\perp}\mathcal{M}J^{\perp}, \tau_{\mathcal{M}})$. Therefore

$$\begin{aligned}
\varphi(E) = \varphi_J(E) &= \varphi_J(F_1 + F_2 + \cdots + F_k + G_1 \vee G_2) && (\varphi_J \text{ extends } \varphi)\\
&= \varphi_J(F_1) + \varphi_J(F_2) + \cdots + \varphi_J(F_k) + \varphi_J(G_1) \vee \varphi_J(G_2) && (\text{by Lemma 5.1.8})\\
&= \varphi_m(F_1) + \varphi_m(F_2) + \cdots + \varphi_m(F_k) + \varphi_m(G_1) \vee \varphi_m(G_2) && (\varphi_J \text{ extends } \varphi_m)\\
&= \psi(F_1) + \psi(F_2) + \cdots + \psi(F_k) + \psi(G_1) \vee \psi(G_2) && (\psi \text{ extends } \varphi_m)\\
&= \psi(F_1 + F_2 + \cdots + F_k + G_1 \vee G_2) && (\text{by Lemma 5.1.8})\\
&= \psi(E).
\end{aligned}$$

This ends the proof of the claim.

By Lemma 5.1.8 and Lemma 3.3.5, there is a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that, for all $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, $\rho(E) = \psi(E)$. Thus, by Claim 5.1.26.2, ρ extends φ on $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. $\square$

When $\mathcal{M}$ is atomic, we have the following result.

Theorem 5.1.27. *Let $\mathcal{M} = \oplus_{i=1}^{k} M_{n_i}(\mathbb{C})$, where $n = n_1 + n_2 + \cdots + n_k \geq 3$ and each $n_i \neq 2$. Assume that $\tau_{\mathcal{M}}$ is a tracial state of $\mathcal{M}$ such that $\tau_{\mathcal{M}}(P) = \frac{1}{n}$ for every rank one projection $P \in \mathcal{M}$. Let m be a positive integer with $2m < n$ and $\mathcal{N}$ be a finite von Neumann algebra with a faithful normal tracial state $\tau_{\mathcal{N}}$. Suppose a map $\varphi : \mathcal{P}_{\frac{m}{n}}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_{\frac{m}{n}}(\mathcal{N}, \tau_{\mathcal{N}})$ satisfies Conditions (A) and (B). Then φ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$.*

Proof. Recall that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = 1$. Denote $c = \frac{m}{n}$. Since $n > 2m$, we must have either $n = 2m + 1$ or $n \geq 2m + 2$.

Assume that $n \geq 2m + 2$. Let $y_j = \frac{m-j+1}{n}$ for $j \geq 1$. Observe that $\mathcal{P}_{y_j}(\mathcal{M}, \tau_{\mathcal{M}})$ is non-empty for all $j \in \{1, \ldots, m\}$. Note, too, that

$$y_j - \frac{3y_j + c - 1}{2} = \frac{1 - c - y_j}{2} = \frac{n - 2m + j - 1}{2n} \geq \frac{1}{n}.$$

This implies $y_{j+1} \in [\frac{3y_j+c-1}{2}, y_j)$. Let p be a positive integer such that $y_p \in (0, \frac{1}{4}]$. The rest of the proof is then similar to that of Theorem 5.1.26 and is omitted.

Now we assume that $n = 2m + 1$ and will prove the result by considering two cases: $m = 1$ and $m \geq 2$.

Case (1). If $m = 1$, then $n = 2m + 1 = 3$ and $c = \frac{1}{3}$. By Definition 5.1.12 and Lemma 5.1.13, φ can be extended to a trace-ortho-preserving map $\hat{\varphi} : \mathcal{P}_{[c,1-c]}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$. Further defining $\hat{\varphi}(0) = 0$ and $\hat{\varphi}(I_{\mathcal{M}}) = I_{\mathcal{N}}$, we conclude by Lemma 5.1.8 and Lemma 3.3.5 that $\hat{\varphi}$, and hence φ, extends to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$.

Case (2). Now assume that $m \geq 2$. Note that $n = 2m + 1 \geq 5$. Before completing the proof of the theorem, we must first prove a few claims.

Claim 5.1.27.1. *For all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, *if* $EF = FE$ *and* $\tau_{\mathcal{M}}(E \wedge F) = c - \frac{1}{n}$, *then* $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$ *and* $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = c - \frac{1}{n}$.

Proof of Claim 5.1.27.1. Observe that, by Lemma 3.2.2 and the assumption $n = 2m + 1$,

$$\tau_{\mathcal{M}}(E \vee F) = \tau_{\mathcal{M}}(E) + \tau_{\mathcal{M}}(F) - \tau_{\mathcal{M}}(E \wedge F) = c + \frac{1}{n} = 1 - c.$$

By Lemma 5.1.10, $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$. The proof of the claim now follows from Condition (A).

Claim 5.1.27.2. *For mutually commutative projections* P, E, F *in* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *such that* $E \perp F$, *if* $P \leq E + F$ *and* $\tau_{\mathcal{M}}(P \wedge E) = c - \frac{1}{n}$, *then* $\varphi(P) \leq \varphi(E) + \varphi(F)$.

Proof of Claim 5.1.27.2. Let $Q = I_{\mathcal{M}} - P \vee E$. Observe that, by the commutativity of E and P, we have

$$Q = I_{\mathcal{M}} - P - E + PE.$$

Therefore $\tau_{\mathcal{M}}(Q) = 1 - 2c + (c - \frac{1}{n}) = c$ and

$$\tau_{\mathcal{M}}(F \wedge Q) = \tau_{\mathcal{M}}(F - PF) = \tau_{\mathcal{M}}(F - P + PE) = c - \frac{1}{n}.$$

It is easily verified that $QF = FQ$ and $Q \perp E, P$. By Claim 5.1.27.1 we obtain $\varphi(E)\varphi(P) = \varphi(P)\varphi(E)$, $\varphi(F)\varphi(Q) = \varphi(Q)\varphi(F)$, and

$$\tau_{\mathcal{M}}(\varphi(E)\varphi(P)) = c - \frac{1}{n} = \tau_{\mathcal{M}}(\varphi(Q)\varphi(F)).$$

Let $H = \varphi(Q) - \varphi(Q)\varphi(F)$. Then $\tau_{\mathcal{N}}(H) = \frac{1}{n}$. Moreover, by Condition (B), $\varphi(Q) \perp \varphi(E), \varphi(P)$ so that $H \perp \varphi(E)$ and $H \perp \varphi(P)$. It is also clear that $\varphi(F) \perp H$. Therefore

$\tau_{\mathcal{N}}(H + \varphi(E) + \varphi(F)) = 1$. It follows that $\varphi(E) + \varphi(F) = I_{\mathcal{N}} - H$. Since $H \perp \varphi(P)$, we conclude $\varphi(P) \leq \varphi(E) + \varphi(F)$.

Claim 5.1.27.3. *For mutually commutative projections P, E, F in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E \perp F$, if $P \leq E + F$, then $\varphi(P) \leq \varphi(E) + \varphi(F)$.*

Proof of Claim 5.1.27.3. There exists a positive integer t and a family $\{P_0, P_1, P_2, \ldots, P_t\}$ of subprojections of $E + F$ in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E = P_0, P_1, P_2, \ldots, P_t = P, F$ are mutually commuting and $\tau_{\mathcal{M}}(P_i P_{i+1}) = c - \frac{1}{n}$ for $0 \leq i \leq t - 1$. Let $Q_i = E + F - P_i$ for $0 \leq i \leq t$. Then $\tau_{\mathcal{M}}(Q_i Q_{i+1}) = \tau_{\mathcal{M}}((E + F - P_i)(E + F - P_{i+1})) = c - \frac{1}{n}$ for $0 \leq i \leq t - 1$. By Claim 5.1.27.2, $\varphi(P_{i+1}) \leq \varphi(P_i) + \varphi(Q_i)$ and $\varphi(Q_{i+1}) \leq \varphi(P_i) + \varphi(Q_i)$ for $0 \leq i \leq t - 1$. Hence

$$\varphi(P) = \varphi(P_t) \leq \varphi(P_t) + \varphi(Q_t) \leq \cdots \leq \varphi(P_0) + \varphi(Q_0) = \varphi(E) + \varphi(F).$$

Claim 5.1.27.4. *For all projections P, E, F in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E \perp F$ and $P \leq E + F$, we have $\varphi(P) \leq \varphi(E) + \varphi(F)$.*

Proof of Claim 5.1.27.4. Recall the assumption $m \geq 2$. The claim now follows directly from Claim 5.1.27.3 and Proposition 3.6.3.

We are now in position to complete the proof of the current theorem. For each $P \in \mathcal{P}_{\frac{1}{n}}(\mathcal{M}, \tau_{\mathcal{M}})$ there exist a pair $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E + F + P = I_{\mathcal{M}}$. Define $\varphi_1 : \mathcal{P}_{\frac{1}{n}}(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}(\mathcal{N})$ by

$$\varphi_1(P) = I_{\mathcal{N}} - (\varphi(E) + \varphi(F)), \qquad \text{for all } P \in \mathcal{P}_{\frac{1}{n}}(\mathcal{M}, \tau_{\mathcal{M}}),$$

where E, F are in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and $E + F + P = I_{\mathcal{M}}$. Clearly $\varphi_1(P) \in \mathcal{P}_{\frac{1}{n}}(\mathcal{N}, \tau_{\mathcal{N}})$ for any $P \in \mathcal{P}_{\frac{1}{n}}(\mathcal{M}, \tau_{\mathcal{M}})$, and by Claim 5.1.27.4 φ_1 is well-defined.

We claim that, for all $P \in \mathcal{P}_{\frac{1}{n}}(\mathcal{M}, \tau_{\mathcal{M}})$ and $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $P \leq E$, then $\varphi_1(P) \leq \varphi(E)$. In fact, let $F_1, F_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be commuting projections, orthogonal to E, such that $E + F_1 \vee F_2 = I_{\mathcal{M}}$. By Lemma 5.1.9, $\varphi(E) + \varphi(F_1) \vee \varphi(F_2) = I_{\mathcal{N}}$. Notice that $P \perp (F_1 \vee F_2)$, hence by the definition of φ_1 we have $\varphi_1(P) \perp (\varphi(F_1) \vee \varphi(F_2))$. It follows that $\varphi_1(P) \leq \varphi(E)$.

We will prove that, for all $P_1, P_2 \in \mathcal{P}_{\frac{1}{n}}(\mathcal{M}, \tau_{\mathcal{M}})$, if $P_1P_2 = P_2P_1$, then $\tau_{\mathcal{N}}(\varphi_1(P_1)\varphi_1(P_2)) = \tau_{\mathcal{N}}(P_1P_2)$. In fact, since P_1, P_2 are minimal projections in $\mathcal{M}$, $P_1P_2 = P_2P_1$ if and only if $P_1 = P_2$ or $P_1P_2 = 0$. We need only verify the case for $P_1P_2 = 0$. Then there exists $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P_1 \leq E$ and $P_2 \perp E$. By the definition of φ_1, we have $\varphi_1(P_2) \perp \varphi(E)$. The discussion of the previous paragraph concludes that $\varphi_1(P_1) \leq \varphi(E)$. It follows that $\varphi_1(P_1) \perp \varphi_1(P_2)$.

Recall that $n = 2m + 1 \geq 5$. By Theorem 5.1.21, φ_1 can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$. Note that, for each $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, there exist mutually orthogonal projections $P_1, P_2, \ldots, P_m$ in $\mathcal{P}_{\frac{1}{n}}(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E = P_1 + P_2 + \cdots + P_m$. The arguments in the previous two paragraphs prove that $\varphi_1(P_i) \leq \varphi(E)$ and $\varphi_1(P_i) \perp \varphi_1(P_j)$ for all distinct $i, j \in \{1, 2, \ldots, m\}$. Thus

$$\varphi(E) \geq \varphi_1(P_1) + \varphi_1(P_2) + \cdots + \varphi_1(P_m).$$

Actually, since both sides have the same trace, we find

$$\varphi(E) = \varphi_1(P_1) + \varphi_1(P_2) + \cdots + \varphi_1(P_m) = \rho(E).$$

Hence ρ extends φ on $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. This completes the proof of the theorem. □

5.1.6 Proof of Theorem 5.1.4

It is clear that Theorem 5.1.4 is a consequence of Theorems 5.1.21, 5.1.26, and 5.1.27.

5.2 Extension for maps on semi-finite von Neumann algebras

5.2.1 Assumptions and statement of main result

In this section, we assume $\mathcal{M}$ and $\mathcal{N}$ are semi-finite von Neumann algebras with faithful normal semi-finite tracial weights $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively.

For a positive number c, let φ be a map from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ into $\mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$.

Assumption 5.2.1. *We assume the following are true in this section.*

(i) $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = \infty$.

(ii) $\mathcal{M}$ *is either diffuse or atomic, and* $\mathcal{M}$ *has no type* I_2 *direct summand.*

(iii) For an atomic von Neumann algebra $\mathcal{M}$, *we assume that*

(iii_1) $\tau_{\mathcal{M}}$ *is the canonical tracial weight satisfying*

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(iii_2) c *is a positive integer with* $c \geq 2$.

(iv) The map $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *satisfies, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

- *Condition (SA): If* $EF = FE$, *then* $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF)$.

Remark 5.2.2. Notice that Condition (SA) implies that, for all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $EF = 0$ then $\varphi(E)\varphi(F) = 0$. That is, φ is ortho-preserving. Moreover, any map satisfying Condition (SA) also satisfies both Condition (A) and Condition (B) of Assumption 5.1.1.

This section is devoted to show the following result, whose proof will be presented later.

Theorem 5.2.3. *Under Assumption* 5.2.1, φ *can be extended to a trace-preserving Jordan* $*$*-homomorphism* $\rho : \mathcal{M} \to \mathcal{N}$ *such that* $\rho(E) = \varphi(E)$ *for all* $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$.

5.2.2 Technical lemmas

In the following we present a number of technical lemmas.

Lemma 5.2.4. *Let E, F, G be mutually commutative projections in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E \perp F$. If $G \leq E + F$, then $\varphi(G) \leq \varphi(E) + \varphi(F)$.*

Proof. Notice that φ satisfies Condition (SA). By Remark 5.2.2, $\varphi(E) \perp \varphi(F)$. Moreover,

$$\tau_{\mathcal{N}}(\varphi(G)(\varphi(E) + \varphi(F))) = \tau_{\mathcal{N}}(\varphi(G)\varphi(E) + \varphi(G)\varphi(F)) = \tau_{\mathcal{M}}(GE + GF) = \tau_{\mathcal{M}}(G) = c.$$

Therefore $\varphi(G) \leq \varphi(E) + \varphi(F)$. □

Lemma 5.2.5. *Let $P, Q_1, Q_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be such that $Q_1 \perp Q_2$ and $P \leq Q_1 + Q_2$. Then $\varphi(P) \leq \varphi(Q_1) + \varphi(Q_2)$.*

Proof. We consider two cases: when $\mathcal{M}$ is diffuse and when $\mathcal{M}$ is atomic. When $\mathcal{M}$ is diffuse, the result follows from Lemma 5.2.4 and Proposition 3.5.3. When $\mathcal{M}$ is atomic, since $c \geq 2$, the result follows from Lemma 5.2.4 and Proposition 3.6.3. □

Lemma 5.2.6. *Let $P, Q_1, Q_2, \ldots, Q_m \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be such that $Q_1, Q_2, \ldots, Q_m$ are mutually orthogonal and $P \leq Q_1 + Q_2 + \cdots + Q_m$. Then $\varphi(P) \leq \varphi(Q_1) + \varphi(Q_2) + \cdots + \varphi(Q_m)$.*

Proof. Note that $P \leq Q_1 + Q_2 + \cdots + Q_m$. There exists $E_1 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P \leq Q_1 + E_1$ and $E_1 \leq Q_2 + Q_3 + \cdots + Q_m$. By Lemma 5.2.5, $\varphi(P) \leq \varphi(Q_1) + \varphi(E_1)$. Likewise, there exists $E_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E_1 \leq Q_2 + E_2$ and $E_2 \leq Q_3 + Q_4 + \cdots + Q_m$. Similarly we have $\varphi(E_1) \leq \varphi(Q_2) + \varphi(E_2)$. Continuing this process, we obtain $E_1, E_2, \ldots, E_{m-2}$ such that $E_i \leq Q_{i+1} + E_{i+1}$ for $1 \leq i \leq m-3$ and $E_{m-2} \leq Q_{m-1} + Q_m$. We conclude, by

Lemma 5.2.5, that

$$\begin{aligned}\varphi(P) &\le \varphi(Q_1) + \varphi(E_1)\\ &\le \varphi(Q_1) + \varphi(Q_2) + \varphi(E_2)\\ &\le \dots\\ &\le \varphi(Q_1) + \varphi(Q_2) + \cdots + \varphi(Q_{m-2}) + \varphi(E_{m-2})\\ &\le \varphi(Q_1) + \varphi(Q_2) + \cdots + \varphi(Q_{m-2}) + \varphi(Q_{m-1}) + \varphi(Q_m)\end{aligned}$$

□

Lemma 5.2.7. *Let $P \in \mathcal{P}(\mathcal{M})$ be a projection with finite trace $\tau_{\mathcal{M}}(P) < \infty$. Then there exists a von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ such that*

(i) $\tau_{\mathcal{M}}(I_{\mathcal{M}_1}) = nc$ for some positive integer $n \ge 4$.

(ii) $P\mathcal{M}P \subseteq \mathcal{M}_1$.

(iii) There exists a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M}_1 \to \mathcal{N}$ such that $\rho(E) = \varphi(E)$ for all $E \in \mathcal{P}_c(\mathcal{M}_1, \tau_{\mathcal{M}})$.

Proof. By Proposition 3.5.4 or Proposition 3.6.4, there exists a von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$, diffuse or atomic, and with no direct summand of type I_2, which satisfies conditions (i) and (ii). Therefore there exist $E_1, E_2, \dots, E_n \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that

$$I_{\mathcal{M}_1} = E_1 + E_2 + \cdots + E_n.$$

Let $H = \varphi(E_1) + \varphi(E_2) + \cdots + \varphi(E_n)$. By Condition (SA) in Assumption 5.2.1, we find $\varphi(E_1), \varphi(E_2), \dots, \varphi(E_n)$ are mutually orthogonal and thus H is a projection in $\mathcal{N}$ with $\tau_{\mathcal{N}}(H) = nc$.

It is easy to verify that $\mathcal{M}_1$ and $H\mathcal{N}H$ are finite von Neumann algebras with tracial states $\frac{1}{nc}\tau_{\mathcal{M}}$ and $\frac{1}{nc}\tau_{\mathcal{N}}$, respectively. By Lemma 5.2.6, for any $Q \le E_1 + E_2 + \cdots + E_n = I_{\mathcal{M}_1}$

with $\tau_{\mathcal{M}}(Q) = c$, we have $\varphi(Q) \le \varphi(E_1) + \varphi(E_2) + \cdots + \varphi(E_n) = H$. Therefore $\varphi|_{\mathcal{P}_c(\mathcal{M}_1,\tau_{\mathcal{M}})} : \mathcal{P}_c(\mathcal{M}_1, \tau_{\mathcal{M}}) \to \mathcal{P}_c(H\mathcal{N}H, \tau_{\mathcal{N}})$ is a map which satisfies Condition (SA). By Theorem 5.1.4, $\varphi|_{\mathcal{P}_c(\mathcal{M}_1,\tau_{\mathcal{M}})}$ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M}_1 \to H\mathcal{N}H$ such that $\rho(E) = \varphi(E)$ for all $E \in \mathcal{P}_c(\mathcal{M}_1, \tau_{\mathcal{M}})$. □

Lemma 5.2.8. *Let $P, E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be such that $EF = FE$. Then*

(i) $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$;

(ii) $\tau_{\mathcal{N}}(\varphi(P)(\varphi(E) \vee \varphi(F))) = \tau_{\mathcal{M}}(P(E \vee F))$;

(iii) if $E \wedge F \le P$, then $\varphi(E) \wedge \varphi(F) \le \varphi(P)$.

Proof. By Lemma 5.2.7, there exists a von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ such that E, F, and P are in $\mathcal{M}_1$, and there exists a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M}_1 \to \mathcal{N}$ such that $\rho(Q) = \varphi(Q)$ for all $Q \in \mathcal{P}_c(\mathcal{M}_1, \tau_{\mathcal{M}})$. The three desired properties now follow easily from Lemma 3.2.3.

(i). It follows from Lemma 3.2.3 that

$$\varphi(E)\varphi(F) = \rho(E)\rho(F) = \rho(EF) = \rho(F)\rho(E) = \varphi(F)\varphi(E).$$

(ii). Moreover,

$$\begin{aligned}
\tau_{\mathcal{N}}(\varphi(P)(\varphi(E) \vee \varphi(F))) &= \tau_{\mathcal{N}}(\rho(P)(\rho(E) \vee \rho(F))) && (\rho \text{ extends } \varphi \text{ on } \mathcal{P}_c(\mathcal{M}_1, \tau_{\mathcal{M}})) \\
&= \tau_{\mathcal{N}}(\rho(P)\rho(E \vee F)) && (\text{by Lemma 3.2.3}) \\
&= \tau_{\mathcal{M}}(P(E \vee F)). && (\text{again by Lemma 3.2.3})
\end{aligned}$$

(iii). We have observed $\varphi(E)\varphi(F) = \rho(EF)$. If $E \wedge F \le P$, then again by Lemma 3.2.3 we have that $\rho(EF)\rho(P) = \rho(EFP) = \rho(EF)$. Thus

$$\varphi(E) \wedge \varphi(F) = \rho(EF) \le \rho(P) = \varphi(P).$$

This finishes the proof. □

A direct consequence of Lemma 5.2.8 is the following result.

Corollary 5.2.9. *Let* $\{E_\lambda, F_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *be a family of mutually commuting projections such that*

$$(E_\lambda \vee F_\lambda) \perp (E_\mu \vee F_\mu) \qquad \textit{for all } \lambda \neq \mu \textit{ in } \Lambda.$$

Then

$$\tau_{\mathcal{N}}(\sum_{\lambda\in\Lambda} \varphi(E_\lambda) \vee \varphi(F_\lambda)) = \tau_{\mathcal{M}}(\sum_{\lambda\in\Lambda} E_\lambda \vee F_\lambda).$$

Proof. By Lemma 5.2.8 and Condition (SA), for each $\lambda \in \Lambda$,

$$\begin{aligned}\tau_{\mathcal{N}}(\varphi(E_\lambda) \vee \varphi(F_\lambda)) &= 2c - \tau_{\mathcal{N}}(\varphi(E_\lambda) \wedge \varphi(F_\lambda)) \\ &= 2c - \tau_{\mathcal{N}}(\varphi(E_\lambda)\varphi(F_\lambda)) \\ &= 2c - \tau_{\mathcal{M}}(E_\lambda F_\lambda) = \tau_{\mathcal{M}}(E_\lambda \vee F_\lambda).\end{aligned}$$

Since $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$ are normal we have that

$$\tau_{\mathcal{N}}(\sum_{\lambda\in\Lambda} \varphi(E_\lambda) \vee \varphi(F_\lambda)) = \sum_{\lambda\in\Lambda} \tau_{\mathcal{N}}(\varphi(E_\lambda) \vee \varphi(F_\lambda)) = \sum_{\lambda\in\Lambda} \tau_{\mathcal{M}}(E_\lambda \vee F_\lambda) = \tau_{\mathcal{M}}(\sum_{\lambda\in\Lambda} E_\lambda \vee F_\lambda).$$

The proof is complete. □

Lemma 5.2.10. *Let* $\{E_\lambda, F_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *be a family of mutually commuting projections such that*

$$(E_\lambda \vee F_\lambda) \perp (E_\mu \vee F_\mu) \qquad \textit{for all } \lambda \neq \mu \textit{ in } \Lambda.$$

For each $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*, if* $P \leq \sum_{\lambda\in\Lambda} E_\lambda \vee F_\lambda$*, then*

$$\varphi(P) \leq \sum_{\lambda\in\Lambda} \varphi(E_\lambda) \vee \varphi(F_\lambda).$$

Proof. By Lemma 5.2.8, $\tau_{\mathcal{M}}(P(E_\lambda \vee F_\lambda)) = \tau_{\mathcal{N}}(\varphi(P)(\varphi(E_\lambda) \vee \varphi(F_\lambda)))$ for each $\lambda \in \Lambda$. Since $P \leq \sum_{\lambda\in\Lambda} E_\lambda \vee F_\lambda$, and since both $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$ are normal, it follows that

$$\begin{aligned}
\tau_{\mathcal{N}}(\sum_{\lambda\in\Lambda} \varphi(P)(\varphi(E_\lambda) \vee \varphi(F_\lambda))) &= \sum_{\lambda\in\Lambda} \tau_{\mathcal{N}}(\varphi(P)(\varphi(E_\lambda) \vee \varphi(F_\lambda))) \\
&= \sum_{\lambda\in\Lambda} \tau_{\mathcal{M}}(P(E_\lambda \vee F_\lambda)) \\
&= \tau_{\mathcal{M}}(\sum_{\lambda\in\Lambda} P(E_\lambda \vee F_\lambda)) = \tau_{\mathcal{M}}(P) = \tau_{\mathcal{N}}(\varphi(P)),
\end{aligned}$$

which implies, by Lemma 3.2.1, that

$$\varphi(P) \leq \sum_{\lambda\in\Lambda} \varphi(E_\lambda) \vee \varphi(F_\lambda).$$

□

In the following we will extend φ to a trace-preserving orthomorphism on $\mathcal{P}(\mathcal{M})$.

Definition 5.2.11. Let $\mathcal{P}_{[c,\infty]}(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \geq c\}$ and $\mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}) = \{P \in \mathcal{P}(\mathcal{M}) \mid \tau_{\mathcal{M}}(P) \leq c\}$. Define $\hat{\varphi} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ by

$$\hat{\varphi}(P) = \bigvee\{\varphi(E) \mid E \leq P \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{[c,\infty]}(\mathcal{M}, \tau_{\mathcal{M}})$$

and

$$\hat{\varphi}(P) = \bigwedge\{\varphi(E) \mid P \leq E \text{ and } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}, \qquad \text{for all } P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}}).$$

It follows from Definition 5.2.11 that $\hat{\varphi}$ coincides with φ on $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ and thus is well-defined. The following result follows directly from Lemma 5.2.10.

Lemma 5.2.12. *Let $\hat{\varphi} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ be the map constructed in Definition 5.2.11. For each $P \in \mathcal{P}_{[c,\infty]}(\mathcal{M}, \tau_{\mathcal{M}})$, if $P = \sum_{\lambda\in\Lambda}(E_\lambda \vee F_\lambda)$ for a family $\{E_\lambda, F_\lambda\}_{\lambda\in\Lambda}$ of mutually* *commuting projections in* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ *such that* $(E_\lambda \vee F_\lambda) \perp (E_\mu \vee F_\mu)$ *for all* $\lambda \neq \mu$ *in* Λ, *then*

$$\hat{\varphi}(P) = \sum_{\lambda \in \Lambda} (\varphi(E_\lambda) \vee \varphi(F_\lambda)).$$

We end the subsection with the next lemma.

Lemma 5.2.13. *Let* $\hat{\varphi} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N})$ *be the map constructed in Definition* 5.2.11. *The following statements are true.*

(i) $\hat{\varphi}(0) = 0$, *and* $\hat{\varphi}(I_{\mathcal{M}}) = \bigvee \{\varphi(E) \mid E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})\}$.

(ii) $\hat{\varphi}$ *is trace-preserving. For each* $P \in \mathcal{P}(\mathcal{M})$, $\tau_{\mathcal{N}}(\hat{\varphi}(P)) = \tau_{\mathcal{M}}(P)$.

(iii) $\hat{\varphi}$ *is order-preserving. For all* $P, Q \in \mathcal{P}(\mathcal{M})$, *if* $P \leq Q$, *then* $\hat{\varphi}(P) \leq \hat{\varphi}(Q)$.

(iv) $\hat{\varphi}$ *is an orthomorphism. For all* $P, Q \in \mathcal{P}(\mathcal{M})$, *if* $P \perp Q$, *then* $\hat{\varphi}(P) \perp \hat{\varphi}(Q)$ *and* $\hat{\varphi}(P + Q) = \hat{\varphi}(P) + \hat{\varphi}(Q)$.

Proof. (i). The proof is clear from Definition 5.2.11.

(ii). If $P \in \mathcal{P}_{[c,\infty]}(\mathcal{M}, \tau_{\mathcal{M}})$, then we can find a family $\{E_\lambda, F_\lambda\}_{\lambda \in \Lambda}$ of mutually commuting projections in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $(E_\lambda \vee F_\lambda) \perp (E_\mu \vee F_\mu)$ for all $\lambda \neq \mu$ in Λ and

$$P = \sum_{\lambda \in \Lambda} (E_\lambda \vee F_\lambda)$$

It follows from Lemma 5.2.12 and Corollary 5.2.9 that $\tau_{\mathcal{N}}(\hat{\varphi}(P)) = \tau_{\mathcal{M}}(P)$.

If $P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$, then there exist commuting projections $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P = E \wedge F$. It follows from Lemma 5.2.8 and Definition 5.2.11 that $\hat{\varphi}(P) = \varphi(E) \wedge \varphi(F)$, and that $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$. By Condition (SA) we conclude

$$\tau_{\mathcal{N}}(\hat{\varphi}(P)) = \tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF) = \tau_{\mathcal{M}}(P).$$

(iii). It is easy to verify directly by Definition 5.2.11.

(iv). First we will show that $\hat{\varphi}$ is ortho-preserving. Let $P, Q \in \mathcal{P}(\mathcal{M})$ be given such that $P \perp Q$. We will consider the following three cases:

Case (1): $P, Q \in \mathcal{P}_{[c,\infty]}(\mathcal{M}, \tau_{\mathcal{M}})$. Then $\hat{\varphi}(P) \perp \hat{\varphi}(Q)$ by Definition 5.2.11.

Case (2): $P, Q \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$. Then there exist $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E \perp F$, $P \leq E$, and $Q \leq F$. By (iii) and Condition (SA) we have $\hat{\varphi}(P) \leq \hat{\varphi}(E)$, $\hat{\varphi}(Q) \leq \hat{\varphi}(F)$, and $\hat{\varphi}(E) \perp \hat{\varphi}(F)$. It follows that $\hat{\varphi}(P) \perp \hat{\varphi}(Q)$.

Case (3): $P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $Q \in \mathcal{P}_{[c,\infty]}(\mathcal{M}, \tau_{\mathcal{M}})$. Then for any $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $E \leq Q$, we have $\hat{\varphi}(P) \perp \varphi(E)$ by Case (2). Hence $\hat{\varphi}(P) \perp \hat{\varphi}(Q)$.

Having shown that $\hat{\varphi}$ is ortho-preserving, it only remains to show that if $P, Q \in \mathcal{P}(\mathcal{M})$ are orthogonal, then $\hat{\varphi}(P+Q) = \hat{\varphi}(P) + \hat{\varphi}(Q)$. If $P, Q \in \mathcal{P}_{[c,\infty]}(\mathcal{M}, \tau_{\mathcal{M}})$, then $\hat{\varphi}(P+Q) = \hat{\varphi}(P) + \hat{\varphi}(Q)$ follows directly from Definition 5.2.11 and Lemma 5.2.12. If $\tau_{\mathcal{M}}(P), \tau_{\mathcal{M}}(Q)$ are both finite, then $\hat{\varphi}(P+Q) = \hat{\varphi}(P) + \hat{\varphi}(Q)$ follows from (ii) and (iii). The only remaining case to address is when $P \in \mathcal{P}_{[0,c]}(\mathcal{M}, \tau_{\mathcal{M}})$ and $\tau_{\mathcal{M}}(Q) = \infty$. Then there exist a pair of orthogonal projections, $Q_1, Q_2 \in \mathcal{P}(\mathcal{M})$, with $\tau_{\mathcal{M}}(Q_1) = c$ and $Q = Q_1 + Q_2$. It follows from Lemma 5.2.12 that $\hat{\varphi}(Q) = \hat{\varphi}(Q_1) + \hat{\varphi}(Q_2)$ and $\hat{\varphi}(P+Q) = \hat{\varphi}(P+Q_1+Q_2) = \hat{\varphi}(P+Q_1) + \hat{\varphi}(Q_2)$. Since $\tau_{\mathcal{M}}(P), \tau_{\mathcal{M}}(Q_1)$ are finite, $\hat{\varphi}(P+Q_1) = \hat{\varphi}(P) + \hat{\varphi}(Q_1)$. Thus

$$\hat{\varphi}(P+Q) = \hat{\varphi}(P+Q_1) + \hat{\varphi}(Q_2) = \hat{\varphi}(P) + \hat{\varphi}(Q_1) + \hat{\varphi}(Q_2) = \hat{\varphi}(P) + \hat{\varphi}(Q).$$

□

5.2.3 Proof of Theorem 5.2.3

We are now ready to present the proof of the main result of this section.

Proof of Theorem 5.2.3. By Lemma 5.2.13 and Lemma 3.3.5, $\hat{\varphi}$ given in Definition 5.2.11 can be extended to a Jordan $*$-homomorphism $\rho : \mathcal{M} \to \hat{\varphi}(I_{\mathcal{M}})\mathcal{N}\hat{\varphi}(I_{\mathcal{M}})$. Next we will show that ρ is trace-preserving.

Assume that $T \geq 0$ is a positive operator in $\mathcal{M}$. Let $\mathcal{A}$ be a maximal self-adjoint abelian von Neumann subalgebra of $\mathcal{M}$ such that $T \in \mathcal{A}$. By Assumption 5.2.1, there exists a family $\{E_\lambda\}$ of mutually orthogonal projections and two commuting projections F_1, F_2 in $\mathcal{P}_c(\mathcal{A}, \tau_{\mathcal{M}})$ such that

$$(\sum_\lambda E_\lambda) \perp (F_1 \vee F_2) \tag{5.7}$$

and

$$I_{\mathcal{M}} = (\sum_\lambda E_\lambda) + F_1 \vee F_2. \tag{5.8}$$

It follows from Lemma 5.2.12 that $\rho(F_1 \vee F_2) = \rho(F_1) \vee \rho(F_2)$,

$$(\sum_\lambda \rho(E_\lambda)) \perp \rho(F_1 \vee F_2), \tag{5.9}$$

and

$$\rho(I_{\mathcal{M}}) = (\sum_\lambda \rho(E_\lambda)) + \rho(F_1 \vee F_2). \tag{5.10}$$

We can now conclude

$$\begin{aligned}
\tau_{\mathcal{M}}(T) &= \tau_{\mathcal{M}}\left(\sum_\lambda E_\lambda T E_\lambda + (F_1 \vee F_2)T(F_1 \vee F_2)\right) && \text{(by (5.7) and (5.8))}\\
&= \sum_\lambda \tau_{\mathcal{M}}(E_\lambda T E_\lambda) + \tau_{\mathcal{M}}((F_1 \vee F_2)T(F_1 \vee F_2)) && \text{(because } \tau_{\mathcal{M}} \text{ is normal)}\\
&= \sum_\lambda \tau_{\mathcal{N}}(\rho(E_\lambda T E_\lambda)) + \tau_{\mathcal{N}}(\rho((F_1 \vee F_2)T(F_1 \vee F_2))) && \text{(by Lemma 3.3.5)}\\
&= \sum_\lambda \tau_{\mathcal{N}}(\rho(E_\lambda)\rho(T)\rho(E_\lambda)) + \tau_{\mathcal{N}}(\rho(F_1 \vee F_2)\rho(T)\rho(F_1 \vee F_2)) && \text{(by Lemma 2 in [12])}\\
&= \tau_{\mathcal{N}}\left(\sum_\lambda \rho(E_\lambda)\rho(T)\rho(E_\lambda) + \rho(F_1 \vee F_2)\rho(T)\rho(F_1 \vee F_2)\right) && \text{(because } \tau_{\mathcal{N}} \text{ is normal)}\\
&= \tau_{\mathcal{N}}(\rho(T)). && \text{(by (5.9) and (5.10))}
\end{aligned}$$

We now have that $\rho : \mathcal{M} \to \mathcal{N}$ is a trace-preserving Jordan $*$-homomorphism which extends $\hat{\varphi}$. Since $\hat{\varphi}$ extends φ, the proof is complete. □

5.3 Extension for maps from finite von Neumann algebras

5.3.1 Assumptions

In this section, we assume $\mathcal{M}$ and $\mathcal{N}$ are semi-finite von Neumann algebras with faithful normal semi-finite tracial weights $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively.

For a positive number c, let φ be a map from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ into $\mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$.

Assumption 5.3.1. *We assume the following are true in this section.*

(i) $\tau_{\mathcal{M}}(I_{\mathcal{M}}) < \tau_{\mathcal{N}}(I_{\mathcal{N}})$ *and* $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$.

(ii) $\mathcal{M}$ *is either diffuse or atomic, and* $\mathcal{M}$ *has no type* I_2 *direct summand.*

(iii) For an atomic von Neumann algebra $\mathcal{M}$*, we assume that*

(iii_1) $\tau_{\mathcal{M}}$ *is the canonical tracial weight satisfying*

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(iii_2) c *is a positive integer with* $c \geq 2$.

(iv) The map $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *satisfies, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

- *Condition (SA): If* $EF = FE$*, then* $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF)$.

5.3.2 Technical lemmas

Lemma 5.3.2. *Let* P, E, F *be projections in* $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$*. The following statements are true.*

(i) If $EF = 0$*, then* $\varphi(E)\varphi(F) = 0$.

(ii) If $EF = 0$*,* $P \leq E + F$*, and* P, E, F *are commutative, then* $\varphi(P) \leq \varphi(E) + \varphi(F)$.

(iii) If $EF = 0$ *and* $P \leq E + F$*, then* $\varphi(P) \leq \varphi(E) + \varphi(F)$.

(iv) If $EF = FE$, then $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$ and $\tau_{\mathcal{N}}(\varphi(E) \vee \varphi(F)) = \tau_{\mathcal{M}}(E \vee F)$.

Proof. (i). This follows directly from Condition (SA) on φ, as noted in Remark 5.2.2.

(ii). By (i) we have that $\varphi(E) \perp \varphi(F)$. Now a quick calculation made possible by Condition (SA) yields

$$\begin{aligned}\tau_{\mathcal{N}}(\varphi(P)(\varphi(E) + \varphi(F))) &= \tau_{\mathcal{N}}(\varphi(P)\varphi(E)) + \tau_{\mathcal{N}}(\varphi(P)\varphi(F)) \\ &= \tau_{\mathcal{M}}(PE) + \tau_{\mathcal{M}}(PF) \\ &= \tau_{\mathcal{M}}(P(E + F)) = \tau_{\mathcal{M}}(P) = \tau_{\mathcal{N}}(\varphi(P)).\end{aligned}$$

condition (ii) now follows from Lemma 3.2.1.

(iii) and (iv). Note that (iii) follows directly from (ii) and Proposition 3.5.3 or Proposition 3.6.3. The proof of Theorem 3.1 in [22] gives (iv). □

Lemma 5.3.3. *Let m be a positive integer and $E_1, \ldots, E_m, F_1, F_2$ be mutually commutative projections in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E_1, \ldots, E_m, F_1 \vee F_2$ are mutually orthogonal. If P is a projection in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $P \leq E_1 + \ldots + E_m + F_1 \vee F_2$, then*

$$\varphi(P) \leq \varphi(E_1) + \ldots + \varphi(E_m) + \varphi(F_1) \vee \varphi(F_2).$$

Proof. Our method will be similar to the proof of Lemma 5.2.6. From the assumptions on $P, E_1, \ldots, E_m, F_1, F_2$, there exist $Q_1, \ldots, Q_m$ in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that

$$P \leq E_1 + Q_1 \leq E_1 + E_2 + Q_2 \leq \ldots \leq E_1 + E_2 + \ldots + E_m + Q_m \leq E_1 + E_2 + \ldots + E_m + F_1 \vee F_2.$$

Or, equivalently,

$$P \leq E_1 + Q_1, \; Q_1 \leq E_2 + Q_2, \; \ldots, Q_{m-1} \leq E_m + Q_m, \text{ and } Q_m \leq F_1 \vee F_2.$$

By Lemma 5.3.2, we obtain

$$\varphi(P) \leq \varphi(E_1) + \varphi(Q_1),\ \varphi(Q_1) \leq \varphi(E_2) + \varphi(Q_2),\ \ldots,\ \varphi(Q_{m-1}) \leq \varphi(E_m) + \varphi(Q_m). \quad (5.11)$$

Note that E_m, F_1, F_2 are mutually commutative and $E_m(F_1 \vee F_2) = 0$. There exists a projection F_3 in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $(F_2 - F_1F_2) \leq F_3 \leq (F_2 - F_1F_2) + E_m$. Clearly $(F_2 - F_1F_2) + E_m$ is orthogonal to F_1, whence $F_3 \perp F_1$. By the choice of F_3 we find

$$F_2 \leq F_1 + F_3, \qquad Q_m \leq F_1 \vee F_2 \leq F_1 + F_3, \qquad \text{and} \qquad F_3 \leq F_2 + E_m.$$

By Lemma 5.3.2, we now have

$$\varphi(Q_m) \leq \varphi(F_1) + \varphi(F_3) \leq \varphi(F_1) \vee \varphi(F_2) + \varphi(E_m). \quad (5.12)$$

From equations (5.11) and (5.12), we conclude

$$\varphi(P) \leq \varphi(E_1) + \ldots + \varphi(E_m) + \varphi(F_1) \vee \varphi(F_2).$$

□

5.3.3 Main result

We are ready to present the main result of this section.

Theorem 5.3.4. *Under Assumption* 5.3.1, *φ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$.*

Proof. Since $\tau_{\mathcal{M}}(I_{\mathcal{M}}) < \infty$, there exist mutually commutative projections $E_1, \ldots, E_m, F_1, F_2$ in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $E_1, \ldots, E_m, F_1 \vee F_2$ are mutually orthogonal and

$$I_{\mathcal{M}} = E_1 + \ldots + E_m + F_1 \vee F_2.$$

Let

$$H = \varphi(E_1) + \ldots + \varphi(E_m) + \varphi(F_1) \vee \varphi(F_2).$$

By Lemma 5.3.2,

$$\begin{aligned}\tau_{\mathcal{N}}(H) &= \tau_{\mathcal{N}}(\varphi(E_1) + \ldots + \varphi(E_m) + \varphi(F_1) \vee \varphi(F_2)) \\ &= \tau_{\mathcal{M}}(E_1 + \ldots + E_m + F_1 \vee F_2) \\ &= \tau_{\mathcal{M}}(I_{\mathcal{M}}).\end{aligned}$$

Thus $\mathcal{M}$ and $H\mathcal{N}H$ are finite von Neumann algebras with tracial states $\frac{1}{\tau_{\mathcal{M}}(I_{\mathcal{M}})}\tau_{\mathcal{M}}$ and $\frac{1}{\tau_{\mathcal{N}}(H)}\tau_{\mathcal{N}}$, respectively. Moreover, from Lemma 5.3.3 we have that $\varphi(P) \leq H$ for all $P \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. In other words, φ is a map from $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ into $\mathcal{P}_c(H\mathcal{N}H, \tau_{\mathcal{N}})$, and φ satisfies Condition (SA). Therefore φ also satisfies Conditions (A) and (B), and by Theorem 5.1.4 there exists a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that $\rho(E) = \varphi(E)$ for all $E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. This completes the proof. □

5.4 Transition probability preserving maps

Recall that a map $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ is said to be *transition probability preserving* if $\tau_{\mathcal{M}}(EF) = \tau_{\mathcal{N}}(\varphi(E)\varphi(F))$ for all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$. A combination of Theorems 5.1.4, 5.2.3, and 5.3.4 proves the following result on maps which preserve transition probability *for commuting projections.*

Theorem 5.4.1. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra without a direct summand of type I_2 and with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. Let $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$. Suppose the following are true.*

(a) $\mathcal{M}$ is either diffuse or atomic.

(b) For an atomic von Neumann algebra $\mathcal{M}$, we assume that

(b_1) $\tau_{\mathcal{M}}$ *is the canonical tracial weight satisfying*

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b_2) c *is a positive integer with* $c \geq 2$.

Let $\mathcal{N}$ *be a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight* $\tau_{\mathcal{N}}$.

Suppose $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *is a map such that, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\text{if } EF = FE, \text{ then } \tau_{\mathcal{M}}(EF) = \tau_{\mathcal{N}}(\varphi(E)\varphi(F)).$$

Then there exists a trace-preserving Jordan $*$*-homomorphism* $\rho : \mathcal{M} \to \mathcal{N}$ *such that*

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

Proof. We claim

$$\tau_{\mathcal{M}}(I_{\mathcal{M}}) \leq \tau_{\mathcal{N}}(I_{\mathcal{N}}).$$

Since $\tau_{\mathcal{M}}$ is a faithful normal semi-finite tracial weight, we may assume

$$I_{\mathcal{M}} = \sum_{\lambda} E_{\lambda} + (F_1 \vee F_2)$$

for a family $\{E_{\lambda}\}$ of mutually orthogonal projections and two commuting projections F_1, F_2

in $\mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ such that $(\sum_\lambda E_\lambda) \perp (F_1 \vee F_2)$. By a calculation

$$\begin{aligned}
\tau_{\mathcal{M}}(I_{\mathcal{M}}) &= \tau_{\mathcal{M}}(\sum_\lambda E_\lambda + (F_1 \vee F_2)) \\
&= \sum_\lambda \tau_{\mathcal{M}}(E_\lambda) + \tau_{\mathcal{M}}(F_1 \vee F_2) && \text{(because } \tau_{\mathcal{M}} \text{ is normal)} \\
&= \sum_\lambda \tau_{\mathcal{M}}(E_\lambda) + (2c - \tau_{\mathcal{M}}(F_1 F_2)) && \text{(because } F_1F_2 = F_2F_1\text{)} \\
&= \sum_\lambda \tau_{\mathcal{N}}(\varphi(E_\lambda)) + (2c - \tau_{\mathcal{N}}(\varphi(F_1)\varphi(F_2))) && \text{(by hypothesis)} \\
&\leq \sum_\lambda \tau_{\mathcal{N}}(\varphi(E_\lambda)) + (2c - \tau_{\mathcal{N}}(\varphi(F_1) \wedge \varphi(F_2))) && \text{(by Lemma 3.7.2)} \\
&= \sum_\lambda \tau_{\mathcal{N}}(\varphi(E_\lambda)) + \tau_{\mathcal{N}}(\varphi(F_1) \vee \varphi(F_2)) && \text{(by Lemma 3.2.2)} \\
&= \tau_{\mathcal{N}}(\sum_\lambda \varphi(E_\lambda) + \varphi(F_1) \vee \varphi(F_2)) && \text{(because } \tau_{\mathcal{N}} \text{ is normal)} \\
&\leq \tau_{\mathcal{N}}(I_{\mathcal{N}}).
\end{aligned}$$

Thus $\tau_{\mathcal{M}}(I_{\mathcal{M}}) \leq \tau_{\mathcal{N}}(I_{\mathcal{N}})$ as claimed. If $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) < \infty$, then Theorem 5.4.1 follows from Theorem 5.1.4. If $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) = \infty$, then Theorem 5.4.1 is a consequence of Theorem 5.2.3. If $\tau_{\mathcal{M}}(I_{\mathcal{M}}) < \tau_{\mathcal{N}}(I_{\mathcal{N}})$, then Theorem 5.4.1 is a consequence of Theorem 5.3.4.

□

5.5 Wigner's theorem for L^p-isometries on von Neumannn algebras

We assume $\mathcal{M}$ and $\mathcal{N}$ are semi-finite von Neumann algebras with faithful normal semi-finite tracial weights $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively.

Lemma 5.5.1. *Suppose that* $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$ *and* $0 < p < \infty$*. Let* $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ *be a map such that, for all* $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\textit{if } EF = FE, \textit{ then } \|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p.$$

The following statements are true.

(i) For all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $E \perp F$, then $\varphi(E) \perp \varphi(F)$.

(ii) Assume that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) < \infty$. For all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $EF = FE$ and $\tau_{\mathcal{M}}(E \vee F) \leq \tau_{\mathcal{M}}(I_{\mathcal{M}}) - c$, then $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF)$.

(iii) Assume $0 < p \leq 2$. For all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $EF = FE$, then $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF)$.

Proof. (i). This has actually been shown in the proof of Theorem 4.4.1. If $E \perp F$, then $\|\varphi(E) - \varphi(F)\|_p^p = \|E - F\|_p^p = 2c$ by Lemma 3.7.2 and the assumption on φ. Also by Lemma 3.7.2, $\|\varphi(E) - \varphi(F)\|_p^p \leq 2c - 2\tau_{\mathcal{N}}(\varphi(E) \wedge \varphi(F))$. Hence $\tau_{\mathcal{N}}(\varphi(E) \wedge \varphi(F)) = 0$ and the equality holds. Therefore $\varphi(E)\varphi(F) = \varphi(F)\varphi(E) = \varphi(E) \wedge \varphi(F)$. It follows that $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = 0$, which implies $\varphi(E) \perp \varphi(F)$.

(ii). Assume that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) < \infty$. Let $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ be given such that $EF = FE$ and $\tau_{\mathcal{M}}(E \vee F) \leq \tau_{\mathcal{M}}(I_{\mathcal{M}}) - c$. Then $\tau_{\mathcal{M}}(I_{\mathcal{M}} - E \vee F) \geq c$, and thus there exist mutually commuting projections $P_1, P_2, \ldots, P_k, Q_1, Q_2 \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ with $P_1, P_2, \ldots, P_k, Q_1 \vee Q_2$ mutually orthogonal such that $I_{\mathcal{M}} - E \vee F = P_1 + P_2 + \cdots + P_k + Q_1 \vee Q_2$. It follows from (i) that $\varphi(E) \vee \varphi(F), \varphi(P_1), \varphi(P_2), \ldots, \varphi(P_k), \varphi(Q_1) \vee \varphi(Q_2)$ are mutually orthogonal. Therefore

$$\tau_{\mathcal{N}}(\varphi(E) \vee \varphi(F)) + \tau_{\mathcal{N}}(P_1) + \tau_{\mathcal{N}}(P_2) + \cdots + \tau_{\mathcal{N}}(P_k) + \tau_{\mathcal{N}}(\varphi(Q_1) \vee \varphi(Q_2)) \leq \tau_{\mathcal{N}}(I_{\mathcal{N}}). \quad (5.13)$$

From Lemma 3.7.2 and Lemma 3.2.2 we obtain

$$2\tau_{\mathcal{N}}(\varphi(E) \vee \varphi(F)) - 2c \geq \|\varphi(E) - \varphi(F)\|_p^p = \|E - F\|_p^p = 2\tau_{\mathcal{M}}(E \vee F) - 2c.$$

Therefore $\tau_{\mathcal{N}}(\varphi(E) \vee \varphi(F)) \geq \tau_{\mathcal{M}}(E \vee F)$. Similarly $\tau_{\mathcal{N}}(\varphi(Q_1) \vee \varphi(Q_2)) \geq \tau_{\mathcal{M}}(Q_1 \vee Q_2)$. By

(5.13) we have

$$\begin{aligned}\tau_{\mathcal{N}}(I_{\mathcal{N}}) &\geq \tau_{\mathcal{N}}(\varphi(E)\vee\varphi(F)) + \tau_{\mathcal{N}}(\varphi(P_1)) + \cdots + \tau_{\mathcal{N}}(\varphi(P_k)) + \tau_{\mathcal{N}}(\varphi(Q_1)\vee\varphi(Q_2)) \\ &\geq \tau_{\mathcal{M}}(E\vee F) + \tau_{\mathcal{M}}(P_1) + \cdots + \tau_{\mathcal{M}}(P_k) + \tau_{\mathcal{M}}(Q_1\vee Q_2) = \tau_{\mathcal{M}}(I_{\mathcal{M}}).\end{aligned}$$

It follows that $\tau_{\mathcal{N}}(\varphi(E)\vee\varphi(F)) = \tau_{\mathcal{M}}(E\vee F)$ and hence

$$\|\varphi(E)-\varphi(F)\|_p^p = 2\tau_{\mathcal{N}}(\varphi(E)\vee\varphi(F)) - 2c.$$

Therefore, by Lemma 3.7.2, $\varphi(E)\varphi(F) = \varphi(F)\varphi(E)$ and

$$\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = 2c - \tau_{\mathcal{N}}(\varphi(E)\vee\varphi(F)) = 2c - \tau_{\mathcal{M}}(E\vee F) = \tau_{\mathcal{M}}(EF).$$

(iii). Assume that $0 < p \leq 2$. First, we show that, for any $P, Q \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, if $PQ = QP$, then $\tau_{\mathcal{N}}(\varphi(P)\varphi(Q)) \geq \tau_{\mathcal{M}}(PQ)$. In fact, since $PQ = QP$, we have

$$\|\varphi(P)-\varphi(Q)\|_p^p = \|P-Q\|_p^p = 2c - 2\tau_{\mathcal{M}}(PQ)$$

by Lemma 3.7.2. On the other hand, for $0 < p \leq 2$, Lemma 3.7.2 shows that

$$\|\varphi(P)-\varphi(Q)\|_p^p \geq 2c - 2\tau_{\mathcal{N}}(\varphi(P)\varphi(Q)).$$

Therefore

$$\text{for all } P, Q \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}),\text{ if } PQ = QP,\text{ then } \tau_{\mathcal{N}}(\varphi(P)\varphi(Q)) \geq \tau_{\mathcal{M}}(PQ). \tag{5.14}$$

Now suppose $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$ are such that $EF = FE$. There exists $G \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$, orthogonal to F, which commutes with E and satisfies $E \leq F + G$. From (i), $\varphi(F) \perp \varphi(G)$. Now, by (5.14), we have

$$c \geq \tau_{\mathcal{N}}(\varphi(E)(\varphi(F)+\varphi(G))) = \tau_{\mathcal{N}}(\varphi(E)\varphi(F)) + \tau_{\mathcal{N}}(\varphi(E)\varphi(G)) \geq \tau_{\mathcal{M}}(EF) + \tau_{\mathcal{M}}(EG) = c.$$

Hence $\tau_{\mathcal{N}}(\varphi(E)\varphi(F)) = \tau_{\mathcal{M}}(EF)$. This ends the proof of the lemma. $\square$

We are ready to prove the remaining main results of this work.

Theorem 5.5.2. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra without a direct summand of type I_2 and with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. Let $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$. Suppose that the following are true.*

(a) $\mathcal{M}$ is either diffuse or atomic.

(b) For an atomic von Neumann algebra $\mathcal{M}$, we assume that

(b_1) $\tau_{\mathcal{M}}$ is the canonical tracial weight satisfying

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b_2) c is a positive integer.

Let $\mathcal{N}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight $\tau_{\mathcal{N}}$.

Assume that $\tau_{\mathcal{M}}(I_{\mathcal{M}}) = \tau_{\mathcal{N}}(I_{\mathcal{N}}) < \infty$ and $0 < p < \infty$. Suppose $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ is a map such that, for all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\text{if } EF = FE, \text{ then } \|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p .$$

Then there exists a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

Proof. The proof follows from Lemma 5.5.1 and Theorem 5.1.4. □

Example 5.5.3. Let $M_n(\mathbb{C})$ be a complex matrix algebra with $n \geq 3$ and τ be the canonical trace of $M_n(\mathbb{C})$. Let c be a positive integer such that $2c < n$ and $\mathcal{P}_c$ be the collection of all projections in $M_n(\mathbb{C})$ with trace c. Let $0 < p < \infty$. By Theorem 5.5.2, a map $\varphi : \mathcal{P}_c \to \mathcal{P}_c$ satisfying

$$\|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p$$

for all E and F in $\mathcal{P}_c$ with $EF = FE$ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : M_n(\mathbb{C}) \to M_n(\mathbb{C})$.

Example 5.5.4. Let $\mathcal{R}$ be a factor of type II_1 with tracial state τ. Let $0 < c < \frac{1}{2}$ and $0 < p < \infty$. By Theorem 5.5.2, a map $\varphi : \mathcal{P}_c(\mathcal{R}, \tau) \to \mathcal{P}_c(\mathcal{R}, \tau)$ satisfying

$$\|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p$$

for all $E, F \in \mathcal{P}_c(\mathcal{R}, \tau)$ with $EF = FE$ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{R} \to \mathcal{R}$.

Theorem 5.5.5. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra without a direct summand of type I_2 and with a faithful normal semi-finite tracial weight $\tau_{\mathcal{M}}$. Let $0 < c < \frac{\tau_{\mathcal{M}}(I_{\mathcal{M}})}{2}$. Suppose that the following are true.*

(a) $\mathcal{M}$ is either diffuse or atomic.

(b) For an atomic von Neumann algebra $\mathcal{M}$, we assume that

(b_1) $\tau_{\mathcal{M}}$ is the canonical tracial weight satisfying

$$\tau_{\mathcal{M}}(H) = 1, \text{ for every minimal projection } H \text{ in } \mathcal{M};$$

(b_2) c is a positive integer with $c \geq 2$.

Let $\mathcal{N}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight $\tau_{\mathcal{N}}$.

Assume that $0 < p \leq 2$. Suppose $\varphi : \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{P}_c(\mathcal{N}, \tau_{\mathcal{N}})$ is a map such that, for all $E, F \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}})$,

$$\text{if } EF = FE, \text{ then } \|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p .$$

Then there exists a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that

$$\rho(E) = \varphi(E) \text{ for all } E \in \mathcal{P}_c(\mathcal{M}, \tau_{\mathcal{M}}).$$

Proof. The proof follows directly from Lemma 5.5.1 and Theorem 5.4.1. □

Example 5.5.6. Let $\mathcal{R}$ be a factor of type I_∞ and τ the canonical tracial weight of $\mathcal{R}$. Assume that c is a positive integer with $c \geq 2$.

Let $0 < p \leq 2$. By Theorem 5.5.5, a map $\varphi : \mathcal{P}_c(\mathcal{R}, \tau) \to \mathcal{P}_c(\mathcal{R}, \tau)$ satisfying

$$\|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p$$

for all $E, F \in \mathcal{P}_c(\mathcal{R}, \tau)$ with $EF = FE$ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{R} \to \mathcal{R}$.

Example 5.5.7. Let $\mathcal{R}$ be a factor of type II_∞ and τ a faithful normal semi-finite tracial weight of $\mathcal{R}$. Let c be a positive number.

Let $0 < p \leq 2$. By Theorem 5.5.5, a map $\varphi : \mathcal{P}_c(\mathcal{R}, \tau) \to \mathcal{P}_c(\mathcal{R}, \tau)$ satisfying

$$\|\varphi(E) - \varphi(F)\|_p = \|E - F\|_p$$

for all $E, F \in \mathcal{P}_c(\mathcal{R}, \tau)$ with $EF = FE$ can be extended to a trace-preserving Jordan $*$-homomorphism $\rho : \mathcal{R} \to \mathcal{R}$.

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