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On the Images of Braid Group Representations Coming from Braided Fusion Categories

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ON THE IMAGES OF BRAID GROUP REPRESENTATIONS COMING FROM BRAIDED FUSION CATEGORIES

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ABSTRACT

ON THE IMAGES OF BRAID GROUP REPRESENTATIONS COMING FROM BRAIDED FUSION CATEGORIES

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Braided fusion categories are algebraic structures with strong ties to the representation theory of finite groups, Hopf algebras, and quantum groups. These structures also have strong connections with braid groups and low-dimensional topology. Recently, braid group representations coming from braided fusion categories have become a topic of interest in areas of condensed matter physics and topological quantum computation. Particularly interesting are the properties of the images of these representations.

Calculations to determine the finiteness of these images have been performed for a few cases. A class of braided fusion categories coming from finite groups (group-theoretical) has been shown to yield finite images. In this work, we show that the images of braid group representations coming from the larger class of weakly group-theoretical braided fusion categories are also finite. We then compute the images of the pure braid groups for some specific representations.

CHAPTER 1
INTRODUCTION

Braided fusion categories and their namesake the braid groups B_m, $m \geq 1$, are connected by a series of braid group representations. For any object X of a braided fusion category, there is a representation of B_m on the m-fold tensor product $X^{\otimes m}$ coming from the braiding of the category. The images of these representations are of interest in applications related to topological quantum computation, low-dimensional topology, and condensed matter physics.

In particular, determining the finiteness of the images is a major question. A conjecture of Naidu and Rowell [NR] states that the images coming from X are finite for all $m \geq 1$ if and only if the Frobenius-Perron dimension of X is the square root of an integer. In particular, this means that the braid group images are finite for every object in an integral braided fusion category.

While this conjecture is still open, some partial results are known. It was proved in [ERW] that it is true for group-theoretical braided fusion categories, which can be described in terms of finite groups and their cohomology. In [RW], this conjecture was verified for the categories $\mathrm{SO}(N)_2$. The latter are examples of *weakly group-theoretical* braided fusion categories. These categories can be obtained from the category of finite-dimensional vector spaces by a sequence of extensions and equivariantizations by finite groups, so they too can be described in terms of group cohomology. Unlike the group-theoretical ones, however, weakly group-theoretical categories can contain objects of non-integral Frobenius-Perron dimension.

This thesis contributes to the study of braid group representations coming from braided fusion categories by proving that the images of the representations coming from objects of a weakly group-theoretical braided fusion category are finite (Theorem 4.2.22). The idea is to describe restrictions of these representations to finite index subgroups of B_m in terms of representations coming from group-theoretical braided fusion categories. We also compute the images of representations coming from specific weakly group-theoretical braided fusion categories.

The thesis is organized as follows.

In Chapter 2, we recall some preliminary material regarding fusion categories. Fusion category structure is an amalgam of other structures, so we will introduce the necessary information about monoidal, rigid, abelian, and finite categories. We then discuss key fusion category constructions and properties.

The braid groups and braided fusion categories are discussed in Chapter 3. We also introduce the braid group representations coming from braided fusion categories and notation.

In Chapter 4, we give a sketch of the proof in [ERW] that group-theoretical braided fusion categories yield finite braid group images. The proof for weakly group-theoretical braided fusion categories is given, with required definitions and constructions. We then discuss the finiteness conjecture and its impact on the study of braided fusion categories.

Inspired by this finiteness conjecture, we compute the images of the pure braid groups for representations coming from examples of weakly group-theoretical braided fusion categories in Chapter 5. In particular, we are interested in the relationship between the Frobenius-Perron dimension of an object X and the sequence of pure braid group images coming from X. We begin with discussions of symmetric fusion categories and pointed braided fusion categories. These examples belong to the larger class of *projectively symmetric* fusion categories (Definition 5.3.4). We prove that an object which projectively centralizes itself has integer Frobenius-Perron dimension (Corollary 5.3.20).

The pure braid group images coming from the non-invertible object in a braided Ising category are described in terms of a family of matrix groups (Propositions 5.4.11 and 5.4.12). This is our first computation for an object of non-integer Frobenius-Perron dimension. We discuss the relationship between the Frobenius-Perron dimension and the orders of its pure braid group images.

Finally, we consider the pure braid group images coming from a particular simple object in the center of $\mathrm{Vec}_{D_{2p}}$, for D_{2p} the dihedral group of the regular p-gon, p an odd prime. These images can be viewed as matrix groups related to the Burau representations over the finite field $\mathbb{F}_p$ evaluated at $t = -1$, which in turn are closely related to finite symplectic groups (Propositions 5.5.16 and 5.5.18). This yields an example of a sequence of pure braid group images whose orders grow exponentially.

CHAPTER 2
PRELIMINARIES

As the title suggests, we will introduce the preliminary material about fusion categories. In particular, we present the major structures and properties of fusion categories and the notation we will use throughout this thesis. The main source for the material of this chapter is [EGNO]. While a discussion about fusion categories could start with an introduction of category theory, we will not concern ourselves with the subtleties of category theory and refer the reader to [L], [Ma], and [Ri], all great introductions of the subject. Categories discussed in this thesis are *essentially small* (equivalent to a category whose objects and morphisms form a set).

2.1 Monoidal Categories

The first structure to introduce is that of a monoidal category. The notion of a monoidal category is a categorification of the notion of a *monoid*, a set with an associative, unital binary operation.

Definition 2.1.1. A *monoidal category* is a sextuple $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, l, r)$, where:

(i) $\mathcal{C}$ is a category,

(ii) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called the *tensor product bifunctor*,

(iii) $\alpha : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ is a natural isomorphism:

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathcal{C} \tag{2.1}$$

called the *associativity isomorphism* or *associativity constraint*,

(iv) $\mathbb{1} \in \mathcal{C}$ is a distinguished object called the *unit object*,

(v) $l : \mathbb{1} \otimes - \xrightarrow{\sim} -$ is a natural isomorphism:

$$l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X, \qquad X \in \mathcal{C} \tag{2.2}$$

called the *left unit constraint*,

(vi) $r : - \otimes \mathbb{1} \xrightarrow{\sim} -$ is a natural isomorphism:

$$r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X, \qquad X \in \mathcal{C} \tag{2.3}$$

called the *right unit constraint*,

such that the following diagrams

(a)

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{\alpha_{W,X,Y} \otimes \mathrm{id}_Z} & & \searrow^{\alpha_{W \otimes X,Y,Z}} & \\ (W \otimes (X \otimes Y)) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow{\scriptstyle \alpha_{W,X \otimes Y,Z}} & & & & \downarrow{\scriptstyle \alpha_{W,X,Y \otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow[\mathrm{id}_W \otimes \alpha_{X,Y,Z}]{} & & W \otimes (X \otimes (Y \otimes Z)) \end{array} \tag{2.4}$$

(b)

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & X \otimes (\mathbb{1} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes l_Y} & \\ & X \otimes Y & \end{array} \tag{2.5}$$

hold for all $W, X, Y, Z \in \mathcal{C}$. Diagram (2.4) is referred to as the *Pentagon Axiom* for a monoidal category, and diagram (2.5) is referred to as the *Triangle Axiom* for a monoial category.

Remark 2.1.2. We will refer to a monoidal category $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, l, r)$ as simply $\mathcal{C}$ when the monoidal structure is understood in context.

Example 2.1.3. For a group G, we can define a monoidal category $\mathcal{C}_G$ in the following way. The underlying category of $\mathcal{C}_G$ has objects $g \in G$ and only identity morphisms. For $g, h \in G$, the tensor product is defined as the usual product in G: $g \otimes h = gh$. The unit object is the identity element $e \in G$. All axioms are satisfied trivially.

More generally, given an abelian group A and a 3-cocycle ω of G with values in A, we can define a monoidal category $\mathcal{C}_G^\omega(A)$ as follows. It has objects $g \in G$ with $\mathrm{Hom}_{\mathcal{C}_G^\omega(A)}(g, g) = A$ and $\mathrm{Hom}_{\mathcal{C}_G^\omega(A)}(g, h) = \emptyset$ for $g \neq h$. Composition of morphisms is given by the usual product in A. The tensor product of objects is given by the usual product in G, and the tensor product of morphisms is given by the usual product in A. The unit object is the identity element $e \in G$. Given $g, h, k \in G$, the associativity isomorphism $\alpha_{g,h,k}$ is defined as $\omega(g, h, k)\mathrm{id}_{ghk}$, and the left and right unit constraints are defined as $l_g = \omega(e, e, g)^{-1}\mathrm{id}_g$ and $r_g = \omega(g, e, e)\mathrm{id}_g$. The Pentagon and Triangle Axioms hold as a result of the 3-cocycle condition on ω.

The Pentagon Axiom shows us that given four ordered objects W, X, Y, Z in a monoidal category, there are five ways to group and take the tensor products of them as given by the five arrangements of parentheses. Due to this, we will use the notation $\bigotimes_{i=1}^m X_i$ for the tensor product of objects $X_1, \ldots, X_m \in \mathcal{C}$ with product starting on the left:

$$\bigotimes_{i=1}^{m} X_i := ((\cdots((X_1 \otimes X_2) \otimes X_3) \otimes \cdots) \otimes X_{m-1}) \otimes X_m.$$

We will also use the notation $X^{\otimes m}$ for the m-fold tensor product of X starting on the left.

Definition 2.1.4. A *monoidal subcategory* of a monoidal category $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, l, r)$ is a monoidal category $(\mathcal{D}, \otimes, \alpha, \mathbb{1}, l, r)$, where $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ closed under the tensor product of objects and morphisms and containing $\mathbb{1}$.

Definition 2.1.5. Let $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, l, r)$ and $(\mathcal{C}', \otimes', \alpha', \mathbb{1}', l', r')$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a triple (F, J, φ), where:

(i) $F : \mathcal{C} \to \mathcal{C}'$ is a functor between categories,

(ii) $J : F(-) \otimes' F(-) \xrightarrow{\sim} F(- \otimes -)$ is a natural isomorphism:

$$J_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y), \qquad X, Y \in \mathcal{C} \tag{2.6}$$

called the *monoidal structure* of the functor, and

(iii) $\varphi : \mathbb{1}' \to F(\mathbb{1})$ is an isomorphism,

such that the following diagrams

(a)

$$\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\downarrow {\scriptstyle J_{X,Y} \otimes' \mathrm{id}_{F(Z)}} & & \downarrow {\scriptstyle \mathrm{id}_{F(X)} \otimes' J_{Y,Z}} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
\downarrow {\scriptstyle J_{X \otimes Y, Z}} & & \downarrow {\scriptstyle J_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z)),
\end{array} \tag{2.7}$$

(b)

$$\begin{array}{ccc}
\mathbb{1}' \otimes' F(X) & \xrightarrow{l'_{F(X)}} & F(X) \\
\downarrow {\scriptstyle \varphi \otimes' \mathrm{id}_{F(X)}} & & \downarrow {\scriptstyle F(l_X)^{-1}} \\
F(\mathbb{1}) \otimes' F(X) & \xrightarrow[J_{\mathbb{1},X}]{} & F(\mathbb{1} \otimes X),
\end{array} \tag{2.8}$$

(c)

$$\begin{array}{ccc} F(X)\otimes' \mathbb{1}' & \xrightarrow{r'_{F(X)}} & F(X) \\ \big\downarrow{\scriptstyle \mathrm{id}_{F(X)}\otimes'\varphi} & & \big\downarrow{\scriptstyle F(r_X)^{-1}} \\ F(X)\otimes' F(\mathbb{1}) & \xrightarrow[J_{X,\mathbb{1}}]{} & F(X\otimes \mathbb{1}), \end{array} \tag{2.9}$$

commute for $X, Y, Z \in \mathcal{C}$. Diagram (2.7) is referred to as the *Monoidal Structure Axiom* of a monoidal functor. A *monoidal equivalence* of monoidal categories is a monoidal functor which is also an equivalence of categories.

Remark 2.1.6. We will refer to a monoidal functor (F, J, φ) as simply F when it is understood in context.

Definition 2.1.7. A monoidal category $\mathcal{C}$ is called *strict* if for all objects $X, Y, Z \in \mathcal{C}$, we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $X \otimes \mathbb{1} = X = \mathbb{1} \otimes X$, and the natural isomorphisms (2.1) - (2.3) are the identity morphisms.

Remark 2.1.8. From Example 2.1.3 it can be seen that $\mathcal{C}_G$ is strict, but $\mathcal{C}_G^{\omega}(A)$ is strict if and only if ω is the trivial 3-cocycle of G.

Remark 2.1.9. In a strict monoidal category, there is no need to parenthesize a large product of objects. As the definition suggests, the objects $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are the same and we can denote this object by simply $X \otimes Y \otimes Z$. This carries over to a product of m arbitrary objects $X_1, \ldots, X_m$. Any two parenthesizations of a product of these objects (in this order) are isomorphic via associativity isomorphisms. Therefore, the object $\bigotimes_{i=1}^{m} X_i$ is the unique object formed from a product of these objects in this order.

While not every monoidal category is strict, we do have the following result known as *MacLane's Strictness Theorem*.

Theorem 2.1.10. *Any monoidal category is monoidally equivalent to a strict monoidal category.*

The following is referred to in the literature as *MacLane's Coherence Theorem* for monoidal categories. It can be thought of as an extension of the Pentagon Axiom to arbitrary tensor products.

Theorem 2.1.11. *Let $X_1, \ldots, X_m \in \mathcal{C}$ a monoidal category. Let P, P' be any two parenthesized products of $X_1, \ldots, X_m$ (in this order) with arbitrary insertions of the unit object $\mathbb{1}$. Let $f, g : P \to P'$ be two isomorphisms, obtained by composing associativity and unit constraints and their inverses possibly tensored with identity morphisms. Then $f = g$.*

Corollary 2.1.12. *Let $f : \bigotimes_{i=1}^{m} X_i \to \bigotimes_{i=1}^{m} X_i$ be an isomorphism obtained by composing associativity and unit contstraints and their inverses possibly tensored with identity morphisms. Then f is the identity morphism.*

Definition 2.1.13. A category $\mathcal{C}$ is called *skeletal* if isomorphic objects of $\mathcal{C}$ are equal.

Example 2.1.14. It is clear from Example 2.1.3 that every $\mathcal{C}_G^{\omega}(A)$ is skeletal.

While not every monoidal category is skeletal, we will utilize the following useful result which follows from the Axiom of Choice.

Theorem 2.1.15. *Any monoidal category is monoidally equivalent to a skeletal monoidal category.*

Remark 2.1.16. While a monoidal category $\mathcal{C}$ is monoidally equivalent to both a strict monoidal category and a skeletal monoidal category, $\mathcal{C}$ is not monoidally equivalent to a strict, skeletal monoidal category in general.

2.2 Rigid Monoidal Categories

For the following, let $\mathcal{C} = (\mathcal{C}, \otimes, \alpha, \mathbb{1}, l, r)$ be a monoidal category.

Definition 2.2.1. For $X \in \mathcal{C}$, a *left dual* of X is a triple $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$, where

(i) X^* is an object in $\mathcal{C}$,

(ii) $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ is a morphism in $\mathcal{C}$ called the *left evaluation*,

(iii) $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$ is a morphism in $\mathcal{C}$ called the *left coevaluation*,

such that the compositions

$$X \xrightarrow{l_X^{-1}} \mathbb{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes \mathbb{1} \xrightarrow{r_X} X,$$

$$X^* \xrightarrow{r_{X^*}^{-1}} X^* \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha_{X^*,X,X^*}^{-1}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} \mathbb{1} \otimes X^* \xrightarrow{l_{X^*}} X^*$$

are the identity morphisms.

Definition 2.2.2. For $X \in \mathcal{C}$, a *right dual* of X is a triple $({}^*X, \mathrm{ev}'_X, \mathrm{coev}'_X)$, where

(i) ${}^*X$ is an object in $\mathcal{C}$,

(ii) $\mathrm{ev}'_X : X \otimes {}^*X \to \mathbb{1}$ is a morphism in $\mathcal{C}$ called the *right evaluation*,

(iii) $\mathrm{coev}'_X : \mathbb{1} \to {}^*X \otimes X$ is a morphism in $\mathcal{C}$ called the *right coevaluation*,

such that the compositions

$$X \xrightarrow{r_X^{-1}} X \otimes \mathbb{1} \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{\alpha_{X,{}^*X,X}^{-1}} (X \otimes {}^*X) \otimes X \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{id}_X} \mathbb{1} \otimes X \xrightarrow{l_X} X,$$

$${}^*X \xrightarrow{l_{{}^*X}^{-1}} \mathbb{1} \otimes {}^*X \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{id}_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{\alpha_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathrm{id}_{{}^*X} \otimes \mathrm{ev}'_X} {}^*X \otimes \mathbb{1} \xrightarrow{r_{{}^*X}} {}^*X$$

are the identity morphisms.

Remark 2.2.3. We often simply say X^* is a left dual of X and ignore the left evaluation and coevaluation morphisms when the context is clear. The same goes for right duals. It can be shown that for X with left and right duals, ${}^*(X^*) \cong X \cong ({}^*X)^*$. We also have that the unit object is a left and right dual of itself. In general, left and right duals are unique up to unique isomorphism.

Definition 2.2.4. An object in a moniodal category is called *rigid* if it has left and right duals. A monoidal category $\mathcal{C}$ is called *rigid* if every object of $\mathcal{C}$ is rigid.

Example 2.2.5. The monoidal categories $\mathcal{C}_G^\omega(A)$ are rigid, with ${}^*g = g^* = g^{-1}$ and the evaluation and coevaluation morphisms given by values of ω.

Now suppose that $\mathcal{C}$ is a rigid monoidal category.

Definition 2.2.6. An object $X \in \mathcal{C}$ is called *invertible* if all evaluation and coevaluation morphisms of X are isomorphisms.

Proposition 2.2.7. [EGNO, Proposition 2.11.3] *If X and Y are invertible objects in a rigid monoidal category $\mathcal{C}$, then X^* is invertible, $X \otimes Y$ is invertible, and $X^* \cong {}^*X$.*

Remark 2.2.8. It is clear from this proposition that the collection $\mathcal{C}_{\text{inv}}$ of isomorphism classes of invertible objects in a rigid monoidal category $\mathcal{C}$ has the structure of a group with respect to the tensor product of $\mathcal{C}$.

2.3 Abelian Categories

Definition 2.3.1. An *additive category* is a category $\mathcal{C}$ satisfying the following axioms:

(i) For every $X, Y \in \mathcal{C}$, the set of morphisms from X to Y, denoted $\operatorname{Hom}_{\mathcal{C}}(X, Y)$, is equipped with the structure of an abelian group (written additively) such that composition of morphisms is biadditive with respect to this structure, meaning for $X, Y, Z \in \mathcal{C}$, $f, f_1, f_2 \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ and $g, g_1, g_2 \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$, we have

$$g \circ (f_1 + f_2) = (g \circ f_1) + (g \circ f_2) \qquad \text{and} \qquad (g_1 + g_2) \circ f = (g_1 \circ f) + (g_2 \circ f)$$

in $\operatorname{Hom}_{\mathcal{C}}(X, Z)$.

(ii) There exists a zero object $0 \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(0, 0) = 0$.

(iii) For any objects $X_1, X_2 \in \mathcal{C}$, there exists an object $X_1 \oplus X_2$, called the *direct sum* of X_1 and X_2, and morphisms $p_1 : X_1 \oplus X_2 \to X_1$, $p_2 : X_1 \oplus X_2 \to X_2$, $i_1 : X_1 \to X_1 \oplus X_2$, and $i_2 : X_2 \to X_1 \oplus X_2$ such that

$$p_1 \circ i_1 = \mathrm{id}_{X_1}, \qquad p_2 \circ i_2 = \mathrm{id}_{X_2}, \qquad (i_1 \circ p_1) + (i_2 \circ p_2) = \mathrm{id}_{X_1 \oplus X_2}$$

Remark 2.3.2. The direct sum $X_1 \oplus X_2$ of X_1, X_2 in an additive category $\mathcal{C}$ is unique up to a unique isomorphism. With this, every additive category is equipped with a direct sum bifunctor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Example 2.3.3. For a fixed ring R, the category $\mathrm{Mod}(R)$ of left modules over R has the structure of an additive category. If we view elements of the direct sum of R-modules $M \oplus N$ as ordered pairs (m, n) for $m \in M$, $n \in N$, then the maps p_1, p_2, i_1, i_2 are the usual projection and inclusion maps

$$p_1(m, n) = m, \qquad p_2(m, n) = n, \qquad i_1(m) = (m, 0), \qquad i_2(n) = (0, n).$$

Definition 2.3.4. Let $\mathcal{C}, \mathcal{D}$ be two additive categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is *additive* if the associated maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for $X, Y \in \mathcal{C}$ are homomorphisms of abelian groups.

Let $\mathcal{C}$ be an additive category.

Definition 2.3.5. For $X, Y \in \mathcal{C}$, let $f : X \to Y$ be a morphism in $\mathcal{C}$. The *kernel* $\mathrm{Ker}(f)$ of f (if it exists) is a pair (K, k), with $K \in \mathcal{C}$ and $k : K \to X$, such that $f \circ k = 0$ in $\mathrm{Hom}_{\mathcal{C}}(K, Y)$, and if $k' : K' \to X$ is a morphism in $\mathcal{C}$ such that $f \circ k' = 0$ in $\mathrm{Hom}_{\mathcal{C}}(K', Y)$, then there exists a unique morphism $l : K' \to K$ in $\mathcal{C}$ such that $k \circ l = k'$.

Definition 2.3.6. For $X, Y \in C$, let $f : X \to Y$ be a morphism in $\mathcal{C}$. The *cokernel* $\mathrm{Coker}(f)$ of f (if it exists) is a pair (C, c), with $C \in \mathcal{C}$ and $c : Y \to C$, such that $c \circ f = 0$

in $\mathrm{Hom}_{\mathcal{C}}(X, C)$, and if $c' : Y \to C'$ is a morphism in $\mathcal{C}$ such that $c' \circ f = 0$ in $\mathrm{Hom}_{\mathcal{C}}(X, C')$, then there exists a unique morphism $l : C \to C'$ such that $l \circ c = c'$.

Remark 2.3.7. If $\mathrm{Ker}(f)$ or $\mathrm{Coker}(f)$ exist, then they are unique up to unique isomorphism. We often simply denote a kernel or cokernel by its underlying object when the context is clear.

Definition 2.3.8. An additive category $\mathcal{C}$ is *abelian* if for every morphism $f : X \to Y$ in $\mathcal{C}$, there exists a sequence

$$\mathrm{Ker}(f) \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} \mathrm{Coker}(f)$$

in $\mathcal{C}$ such that $j \circ i = f$ and $I = \mathrm{Coker}(k) = \mathrm{Ker}(c)$. Such a sequence is called a *canonical decomposition* of f, and the object I is called the *image* of f and is denoted by $\mathrm{Im}(f)$.

Example 2.3.9. For a fixed associative $\Bbbk$-algebra A, the category $\mathrm{Mod}(A)$ of left modules is an abelian category. In this case, the notions of 'kernel', 'cokernel', and 'image' are the same as those in linear algebra.

Let $\mathcal{C}$ be an abelian category.

Definition 2.3.10. A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be a *monomorphism* if $\mathrm{Ker}(f) = 0$, and is said to be an *epimorphism* if $\mathrm{Coker}(f) = 0$.

Definition 2.3.11. A *subobject* of an object Y in an abelian category $\mathcal{C}$ is a pair (X, i), where $X \in \mathcal{C}$ and $i : X \to Y$ is a monomorphism. A *quotient object* of Y is a pair (Z, p), where $Z \in \mathcal{C}$ and $p : Y \to Z$ is an epimorphism.

Remark 2.3.12. We often just say that X is a subobject of Y, or that Z is a quotient object of Y when the context is clear.

Definition 2.3.13. A nonzero object $X \in \mathcal{C}$ is called *simple* if 0 and X are the only subobjects of X. An object X is called *semisimple* if it is a direct sum of simple objects, and $\mathcal{C}$ is called *semisimple* if every object of $\mathcal{C}$ is semisimple.

2.4 Finite Abelian Categories

For the following, let $\mathcal{C}$ be an abelian category and $\Bbbk$ be a field.

Definition 2.4.1. The category $\mathcal{C}$ is said to be $\Bbbk$-*linear* (or defined over $\Bbbk$) if for any objects $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is equipped with a structure of a vector space over $\Bbbk$, such that composition of morphisms is $\Bbbk$-bilinear with respect to this structure, meaning for $X, Y, Z \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, and $\lambda \in \Bbbk$, we also have

$$g \circ (\lambda f) = (\lambda g) \circ f = \lambda(g \circ f)$$

in $\mathrm{Hom}_{\mathcal{C}}(X, Z)$.

Definition 2.4.2. Let $\mathcal{C}, \mathcal{D}$ be two $\Bbbk$-linear abelian categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is $\Bbbk$-*linear* if the associated maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for $X, Y \in \mathcal{C}$ are $\Bbbk$-linear.

Definition 2.4.3. A $\Bbbk$-linear abelian category $\mathcal{C}$ is said to be *finite* if the following three conditions are satisfied:

(i) for any two objects $X, Y \in \mathcal{C}$, the vector space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is finite-dimensional,

(ii) every object of $\mathcal{C}$ is isomorphic to a finite direct sum of simple objects, and

(iii) there are only finitely many isomorphism classes of simple objects of $\mathcal{C}$.

Example 2.4.4. The categories $\mathrm{Mod}(A)$ in Example 2.3.9 are $\Bbbk$-linear in general, but finite if and only if A is finite-dimensional.

We will denote the finite set of isomorphism classes of simple objects in a finite abelian category $\mathcal{C}$ by $\mathcal{O}(\mathcal{C})$. A useful tool for identifying simple objects is given by the following lemma, known in some contexts as *Schur's Lemma*:

Lemma 2.4.5. *In any locally finite abelian category $\mathcal{C}$ over an algebraically closed field $\Bbbk$, we have $Hom_{\mathcal{C}}(X, Y) = 0$ if X, Y are simple and non-isomorphic and $Hom_{\mathcal{C}}(X, X) = \Bbbk$ for any simple object $X \in \mathcal{C}$.*

2.5 Fusion Categories

For what follows, let $\mathcal{C}$ be a finite abelian rigid monoidal category over an algebraically closed field $\Bbbk$ of characteristic zero.

Definition 2.5.1. We call $\mathcal{C}$ a *fusion category* over $\Bbbk$ if the tensor product bifunctor $\otimes$: $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms, the unit object $\mathbb{1}$ is simple, and $\mathcal{C}$ is semisimple.

We refer the reader to [ENO1] for the basic facts about fusion categories. Here, we recall some terminology and constructs that will be of importance in this thesis.

Example 2.5.2. For a finite group G, the category $\text{Rep}(G)$ of finite-dimensional complex representations of G has the structure of a fusion category. The direct sum and tensor product of objects in $\text{Rep}(G)$ correspond to the usual notion of direct sum and tensor product of representations. Simple objects in this fusion category are the *irreducible* representations of G with the trivial representation as the unit object.

More generally, let H be a semisimple Hopf algebra over $\Bbbk$. The category $\text{Rep}(H)$ of finite-dimensional representations of H has the structure of a fusion category.

Definition 2.5.3. A *fusion subcategory* of $\mathcal{C}$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ closed under taking subquotients, tensor products, and duality.

Remark 2.5.4. The smallest fusion subcategory of $\mathcal{C}$ is generated by the unit object of $\mathcal{C}$ and is equivalent to Vec, the fusion category of finite-dimensional vector spaces over $\Bbbk$.

Let $X_i \in \mathcal{O}(\mathcal{C})$, $1 \leq i \leq n$, be a set of representatives for the isomorphism classes of simple objects. For any object $X \in \mathcal{C}$, we denote by N^X the $n \times n$ matrix over $\mathbb{Z}_{\geq 0}$ defined by $X \otimes X_i \cong \bigoplus_{i=1}^{n} N_{i,j}^X X_j$. The *Frobenius-Perron Theorem* of [Ga] gives us that N^X has nonnegative real eigenvalues the largest of which $\lambda(N^X)$ dominates the absolute values of all other eigenvalues of N^X.

Definition 2.5.5. The *Frobenius-Perron dimension* of $X \in \mathcal{C}$, denoted $\mathrm{FPdim}(X)$, is the maximal nonnegative real eigenvalue $\lambda(N^X)$. The *Frobenius-Perron dimension* of $\mathcal{C}$ is the sum $\mathrm{FPdim}(\mathcal{C}) = \sum_{X_i \in \mathcal{O}(\mathcal{C})} \mathrm{FPdim}(X_i)^2$.

We say that $\mathcal{C}$ is *integral* if $\mathrm{FPdim}(X) \in \mathbb{Z}$ for all objects X of $\mathcal{C}$. We say that $\mathcal{C}$ is *weakly integral* if $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$.

Definition 2.5.6. A *grading* of a fusion category $\mathcal{C}$ by a group G is a map $\deg : \mathcal{O}(\mathcal{C}) \to G$ such that for all $X, Y, Z \in \mathcal{O}(\mathcal{C})$ one has $\deg(X)\deg(Y) = \deg(Z)$ when Z is contained in $X \otimes Y$. In this case we have a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g,$$

where $\mathcal{C}_g$ is the full additive subcategory of $\mathcal{C}$ generated by all objects of degree $g \in G$. The subcategory $\mathcal{C}_e$ corresponding to the identity element $e \in G$ is a fusion subcategory of $\mathcal{C}$ and is called the *trivial component* of the grading. A grading is *faithful* if the map $\deg : \mathcal{O}(\mathcal{C}) \to G$ is surjective. In this case we say that $\mathcal{C}$ is a *G-extension* (or, simply, an *extension*) of $\mathcal{C}_e$.

Example 2.5.7. The simplest example of a graded fusion category is a *pointed* fusion category, i.e., a category $\mathcal{C}$ in which every simple object is invertible. Indeed, in this case $G = \mathcal{O}(\mathcal{C})$ has a structure of a finite group and $\mathcal{C}$ is a G-extension of Vec. Such extensions are described as follows.

For the group G and a 3-cocycle ω of G with values in $\Bbbk^\times$, consider the category Vec_G^ω of finite-dimensional G-graded vector spaces. Objects of this category are vector spaces V with decomposition $V = \oplus_{g \in G} V_g$. Morphisms in this category are linear maps which respect the grading. The isomorphism classes of simple objects may be represented by the elements $g \in G$ where g is the one-dimensional vector space with decomposition $(g)_g = \Bbbk$. The tensor product is defined on simple objects such that $g \otimes h \cong gh$, and the unit object is the identity element $e \in G$. The associativity constraint $\alpha_{g,h,k}$, for $g, h, k \in G$, is defined as $\omega(g,h,k)\mathrm{id}_{ghk}$. This can be thought of as some sort of 'linear' version of the monoidal category $\mathcal{C}_G^\omega(\Bbbk^\times)$.

Any pointed fusion category is equivalent to some Vec_G^ω.

Proposition 2.5.8. *Let $\mathcal{C}$ be a weakly integral fusion category.*

(i) $FPdim(X)^2 \in \mathbb{Z}$ for any $X \in \mathcal{O}(\mathcal{C})$.

(ii) The map $deg : \mathcal{O}(\mathcal{C}) \to \mathbb{Q}^\times_{>0}/(\mathbb{Q}^\times_{>0})^2$ that takes $X \in \mathcal{O}(\mathcal{C})$ to the image of $FPdim(X)^2$ in $\mathbb{Q}^\times_{>0}/(\mathbb{Q}^\times_{>0})^2$ is a grading of $\mathcal{C}$.

PROOF: (i) was proved in [ENO1, Proposition 8.27] and (ii) was proved in [GN, Theorem 3.10]. □

Corollary 2.5.9. *Let $\mathcal{C}$ be a weakly integral fusion category. There is a canonical faithful grading of $\mathcal{C}$ by an elementary Abelian 2-group $G(\mathcal{C})$,*

$$\mathcal{C} = \bigoplus_{g\in G(\mathcal{C})} \mathcal{C}_g,$$

such that $\mathcal{C}_e$ is generated by all simple objects of integral Frobenius-Perron dimension, and there are square free integers N_g such that $FPdim(X_g) \in \mathbb{Z}\sqrt{N_g}$ for all $X_g \in \mathcal{C}_g$, $g \in G(\mathcal{C})$.

Definition 2.5.10. (i) A fusion category is *group-theoretical* [ENO1] if it is categorically Morita equivalent to a pointed fusion category.

(ii) A fusion category is *nilpotent* [GN] if it can be obtained from Vec by a sequence of extensions.

(iii) A fusion category is *weakly group-theoretical* [ENO3] if it is categorically Morita equivalent to a nilpotent fusion category.

CHAPTER 3
BRAID GROUPS AND BRAIDED FUSION CATEGORIES

3.1 Braid Groups

The braid groups B_m are defined and studied in different areas of mathematics. We recall some of these definitions from [B]:

- In topology, we may consider a closed disc D and let $p_1, \dots, p_m$ be a list of distinct interior points of the disc D. The space $D_m = D \setminus \{p_1, \dots, p_m\}$ is a disc with m punctures, and we may define the group B_m as the mapping class group of D_m. So braids may be thought of as equivalence classes of homeomorphisms from D_m to itself which fix the boundary of the disk.

- Consider the disc D and interior points $p_1, \dots, p_m \in D$ again. Define a *strand* in $D \times [0,1]$ to be a path in $D \times [0,1]$ with endpoints in $\{p_1, \dots, p_m\} \times \{0,1\}$ which intersects each of $D \times 0$ and $D \times 1$ exactly once. One can then define a *geometric braid* (Emil Artin's original motivation for defining the braid groups) as a disjoint union of m such strands. Two geometric braids are considered equivalent if it is possible to deform one to the other through a continuous family of geometric braids. We may define B_m as the group of these equivalence classes, and the group structure comes from "stacking" two geometric braids on top of each other and then deforming this new geometric braid to fit inside $D \times [0,1]$ again. In this way, we sometimes refer to B_m as the group of braids on m strands.

For the purposes of this dissertation, we will mostly focus on a third definition for B_m as a finitely presented group. Let $m \geq 2$ be an integer.

Definition 3.1.1. The *mth braid group* B_m is the group with generators $\sigma_1, \ldots, \sigma_{m-1}$ (which we will refer to as *simple braids*) and relations:

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$, and
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq m - 2$.

Remark 3.1.2. We often define a braid group B_1 as the trivial group. All other braid groups B_m for $m \geq 2$ are infinite.

Remark 3.1.3. B_2 is defined simply as $\langle \sigma_1 \rangle$, and thus $B_2 \cong \mathbb{Z}$. Otherwise, the braid groups B_m for $m \geq 3$ are non-abelian.

Fix $m \geq 2$. There is a natural action of B_m on any ordered set of size m. For $A = \{1, 2, \ldots, m\}$, the action of $\sigma_i \in B_m$, $1 \leq i \leq m - 1$, is defined by

$$\sigma_i(i) = i + 1, \qquad \sigma_i(i + 1) = i, \qquad \sigma_i(j) = j \text{ for } j \neq i, i + 1$$

There is also a natural action of the symmetric group S_m on A, such that the actions of $\sigma_i \in B_m$ and $(i,\ i + 1) \in S_m$ agree. This allows us define a group homomorphism $\varphi_m : B_m \to S_m$, given on simple braids by $\varphi(\sigma_i) = (i,\ i + 1)$.

Definition 3.1.4. For $m \geq 2$, the *mth pure braid group* P_m is the kernel of the homomorphism $\varphi_m : B_m \to S_m$.

According to [KT], the pure braid group can be generated by conjugates of squares of simple braids in the following way.

Proposition 3.1.5. *For $m \geq 2$, P_m is generated by braids of the form*

$$\sigma_{i,j} = \sigma_j \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j^{-1}$$

for $1 \leq i \leq j \leq m - 1$.

Remark 3.1.6. As the kernel of φ_m, it is clear that P_m is a finite index subgroup of B_m with $[B_m : P_m] = m!$, $m \geq 2$.

3.2 Braided Fusion Categories

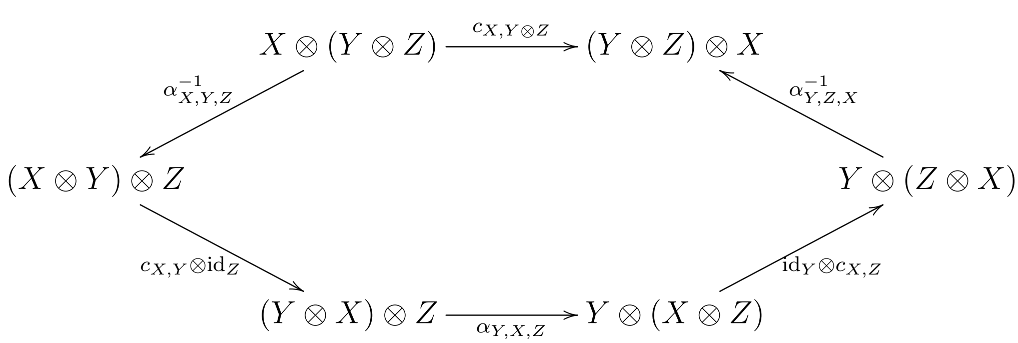
Braided fusion categories can be thought of as the 'commutative' counterparts to fusion categories. They possess a chosen collection of isomorphisms that allow for objects to commute across the tensor product, satisfying certain coherence conditions. Let $\mathcal{B}$ be a fusion category with associativity constraint α.

Definition 3.2.1. A *braiding* on $\mathcal{B}$ is a family of natural isomorphisms

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \qquad X, Y \in \mathcal{B} \tag{3.1}$$

such that the following diagrams

(a)



(3.2)

and

(b)

$$\begin{tikzcd}[column sep=small]
& (X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] \arrow[dl, "\alpha_{X,Y,Z}"'] & Z \otimes (X \otimes Y) & \\
X \otimes (Y \otimes Z) \arrow[dr, "\mathrm{id}_X \otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \arrow[ul, "\alpha_{Z,X,Y}"'] \\
& X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z}\otimes \mathrm{id}_Y"'] &
\end{tikzcd} \tag{3.3}$$

commute for all objects $X, Y, Z \in \mathcal{B}$. Diagrams (3.2) - (3.3) are referred to as the *Hexagon Axioms* for a braiding.

Definition 3.2.2. A *braided* fusion category is a pair consisting of a fusion category $\mathcal{B}$ and a braiding c on $\mathcal{B}$.

Remark 3.2.3. A braiding is a *structure* on a fusion category, meaning a fusion category may have no braidings or several different braidings.

Example 3.2.4. For a finite group G, there is a braiding on the fusion category $\text{Rep}(G)$. For representations $V, W \in \text{Rep}(G)$ the 'flip' map $\tau_{V,W} : V \otimes W \to W \otimes V$ defined on simple tensors by $\tau_{V,W}(v \otimes w) = w \otimes v$ defines a braiding $c_{V,W} := \tau_{V,W}$ on $\text{Rep}(G)$.

Definition 3.2.5. Let $\mathcal{B}^1$ and $\mathcal{B}^2$ be braided fusion categories with braidings c^1 and c^2, respectively. A tensor functor $(F, J, \varphi) : \mathcal{B}^1 \to \mathcal{B}^2$ is called *braided* if the following diagram

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "c^2_{F(X),F(Y)}"] \arrow[d, "J_{X,Y}"'] & F(Y) \otimes F(X) \arrow[d, "J_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F\left(c^1_{X,Y}\right)"'] & F(Y \otimes X)
\end{tikzcd} \tag{3.4}$$

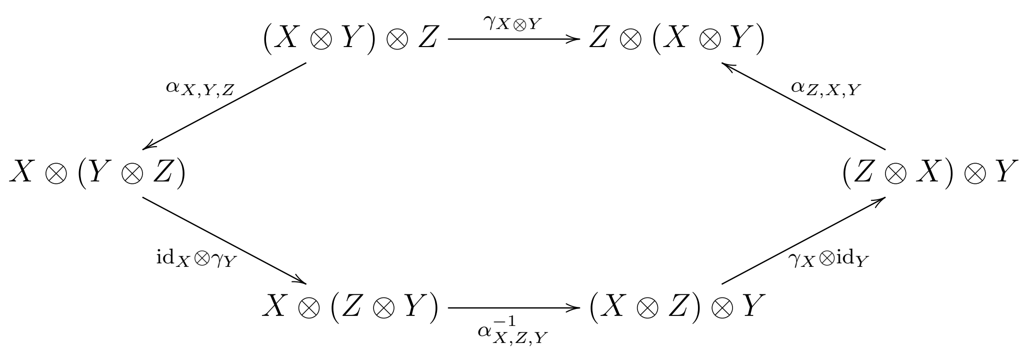
commutes for all $X, Y \in \mathcal{B}^1$. A *braided tensor equivalence* of braided fusion categories is a braided tensor functor which is also an equivalence of categories.

Let $\mathcal{C}$ be a fusion category with associativity constraint α.

Definition 3.2.6. The *(Drinfeld) center* of $\mathcal{C}$ is the category $\mathcal{Z}(\mathcal{C})$ defined as follows. Objects in $\mathcal{Z}(\mathcal{C})$ are pairs (Z, γ) such that $Z \in \mathcal{C}$ and γ is a collection of natural isomorphisms

$$\gamma_X : X \otimes Z \xrightarrow{\sim} Z \otimes X, \qquad X \in \mathcal{C},$$

such that the following diagram



is commutative for all $X, Y \in \mathcal{C}$. A morphism from (Z, γ) to (Z', γ') is a morphism $f : Z \to Z'$ in $\mathcal{C}$ such that for each $X \in \mathcal{C}$ we have $\gamma'_Z \circ (\mathrm{id}_X \otimes f) = (f \otimes \mathrm{id}_X) \circ \gamma_X$.

The center $\mathcal{Z}(\mathcal{C})$ of a fusion category $\mathcal{C}$ has the structure of a braided fusion category. The tensor product is defined by $(Z, \gamma) \otimes (Z', \gamma') := (Z \otimes Z', \overline{\gamma})$, where

$$\overline{\gamma}_X = \alpha^{-1}_{Z,Z',X} \circ (\mathrm{id}_Z \otimes \gamma'_X) \circ \alpha_{Z,X,Z'} \circ (\gamma_X \otimes \mathrm{id}_{Z'}) \circ \alpha^{-1}_{X,Z,Z'}.$$

The associativity isomorphism $\alpha_{(Z,\gamma),(Z',\gamma'),(Z'',\gamma'')}$ is given by $\alpha_{Z,Z',Z''}$. The braiding $c_{(Z,\gamma),(Z',\gamma')}$ is defined by γ'_Z. In particular, centers of fusion categories are *non-degenerate* braided fusion categories [DGNO2, Definition 2.28]. The Frobenius-Perron dimension of $\mathcal{Z}(\mathcal{C})$ is $\mathrm{FPdim}(\mathcal{C})^2$.

For a braided fusion category $\mathcal{B}$, there is a natural inclusion functor $\mathcal{B} \hookrightarrow \mathcal{Z}(\mathcal{B})$ defined on objects by $X \mapsto (X, c_{-,X})$. We may view a braided fusion category as a subcategory of its center in this way.

3.3 Braid Group Representations Coming from Braided Fusion Categories

With the proper definitions presented, we can now describe the braid group representations that come from braided fusion categories.

Let $\mathcal{B}$ be a braided fusion category with braiding c and let $X \in \mathcal{B}$. The braiding structure of $\mathcal{B}$ affords a natural action of the braid group B_m on $X^{\otimes m}$, or a representation $\rho_{m,X} : B_m \to \mathrm{End}(X^{\otimes m})$. When $\mathcal{B}$ is strict, this action is defined on the simple braids σ_i, $1 \leq i \leq m-1$, by

$$\rho_{m,X}(\sigma_i) = \mathrm{id}_{X^{\otimes(i-1)}} \otimes c_{X,X} \otimes \mathrm{id}_{X^{\otimes(m-i+1)}}$$

When $\mathcal{B}$ is not strict, the object $X^{\otimes m}$ has a specific parenthesization as an m-fold tensor product. The images of our simple braids will need to incorporate associativity constraints. Associate to each $i = 1, 2, \ldots, m-1$, a parenthesization of an m-fold tensor product of X, denoted $X^{\otimes(m,i)}$, wherein the product begins with the $(i, i+1)$ copies of X, moves all the way to the left, and then all the way to the right. For example, $X^{\otimes(m,1)} = X^{\otimes m}$ and for $m = 4$ we have

$$X^{\otimes(4,1)} = ((X \otimes X) \otimes X) \otimes X,$$

$$X^{\otimes(4,2)} = (X \otimes (X \otimes X)) \otimes X,$$

$$X^{\otimes(4,3)} = X \otimes (X \otimes (X \otimes X)).$$

We define the image $\rho_{m,X}(\sigma_i)$ to be the automorphism

$$X^{\otimes m} = X^{\otimes(m,1)} \overset{\alpha_{(X,m,i)}}{\longrightarrow} X^{\otimes(m,i)} \overset{c_{X,X}}{\longrightarrow} X^{\otimes(m,i)} \overset{\alpha^{-1}_{(X,m,i)}}{\longrightarrow} X^{\otimes(m,1)} = X^{\otimes m} \tag{3.5}$$

where $\alpha_{(X,m,i)}$ is any composition of associativity isomorphisms from $X^{(m,1)}$ to $X^{(m,i)}$ (which is unique by MacLane's Coherence Theorem) and $c_{X,X}$ here means the appropriate parenthesization of $c_{X,X}$ on the $(i, i+1)$ copies of X with identity morphisms. For example, when

$m = 4$ the images of $\sigma_1, \sigma_2, \sigma_3$ in $\mathrm{End}(X^{\otimes 4})$ are

$$\rho_{4,X}(\sigma_1) = (c_{X,X} \otimes \mathrm{id}_X) \otimes \mathrm{id}_X$$

$$\rho_{4,X}(\sigma_2) = (\alpha^{-1}_{X,X,X} \circ (\mathrm{id}_X \otimes c_{X,X}) \circ \alpha_{X,X,X}) \otimes \mathrm{id}_X$$

$$\rho_{4,X}(\sigma_3) = \alpha^{-1}_{X\otimes X,X,X} \circ \alpha^{-1}_{X,X,X\otimes X} \circ (\mathrm{id}_X \otimes (\mathrm{id}_X \otimes c_{X,X})) \circ \alpha_{X,X,X\otimes X} \circ \alpha_{X\otimes X,X,X}$$

We also have an action of the pure braid group P_m on products $\bigotimes_{i=1}^m X_i$ for objects $X_1, \ldots, X_m \in \mathcal{B}$. It is defined using similar parenthesizations and compositions of associativity isomorphisms and braidings.

Using braid group representations, we are able to study braided fusion categories through the lens of group theory and linear algebra. Of particular interest are three questions:

1. If a braided fusion category $\mathcal{B}$ has a property P, must the braid group images coming from $\mathcal{B}$ have a corresponding property P'? For example if $\mathcal{B}$ is weakly integral, must the braid group images coming from $\mathcal{B}$ be finite?

2. If we can imagine a particular sequence of braid group images, is there an object X which yields this sequence?

3. If two objects X and X' yield the same braid group images, what else do these objects have in common?

CHAPTER 4
BRAID GROUP IMAGES AND FINITENESS

In this chapter, we will discuss the braid group representations coming from two classes of braided fusion categories: group-theoretical and, more generally, weakly group-theoretical braided fusion categories. The data associated to these categories stem from finite group theory, and as a result the braid group images from these braided fusion categories are proven to be finite. A characterization of braided fusion categories with finite braid group images has been conjectured, motivated by examples and applications.

4.1 From Group-theoretical Braided Fusion Categories

Finiteness properties of braid group and pure braid group images coming from group-theoretical braided fusion categories were studied in [ERW]. We give a brief review of their findings, which will prove useful in the sequel.

Let $\mathcal{B}$ be a group-theoretical braided fusion category with braiding c. The following is a result from [ERW].

Theorem 4.1.1. *Let X be an object in $\mathcal{B}$. Then $\rho_{m,X}(B_m)$ is finite for $m \geq 2$.*

While the proof will not be replicated here, there are a handful of observations that should prove useful later.

Any braided fusion category $\mathcal{B}$ can be realized as a fusion subcategory of its center $\mathcal{Z}(\mathcal{B})$ (Definition 3.2.6). Indeed, there is a braided tensor functor $F : \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ which sends an object $X \in \mathcal{B}$ to $F(X) := (X, c_{-,X}) \in \mathcal{Z}(\mathcal{B})$. To show that the braid group images from $\mathcal{B}$ are finite, it is enough to show that the braid group images from $\mathcal{Z}(\mathcal{B})$ are finite.

When $\mathcal{B}$ is group-theoretical, its center $\mathcal{Z}(\mathcal{B})$ is equivalent to the representation category of an algebraic structure known as the *twisted quantum double* [DPR]. For a finite group G and a 3-cocycle $\omega \in Z^3(G, \Bbbk^\times)$, the twisted quantum double $D^\omega(G)$ of G with respect to ω is a $|G|^2$-dimensional, quasitriangular quasi-Hopf algebra over $\Bbbk$. With these structures on $D^\omega(G)$, the category $\mathrm{Rep}(D^\omega(G))$ of finite-dimensional representations of $D^\omega(G)$ has the structure of a braided fusion category. Thus, showing that the braid group images from $\mathcal{Z}(\mathcal{B})$ are finite is equivalent to showing that the braid group images from $\mathrm{Rep}(D^\omega(G))$ are finite.

There is a notion of a regular representation R of $D^\omega(G)$ which is finite-dimensional and, thus, an object in $\mathrm{Rep}(D^\omega(G))$. Any irreducible representation of $D^\omega(G)$ is a subrepresentation of R, and so any simple object in $\mathrm{Rep}(D^\omega(G))$ is a subobject of R. By extension, any object in $\mathrm{Rep}(D^\omega(G))$ is a subobject of a direct sum of the form $R \oplus \cdots \oplus R$. To show that the braid group images in $\mathrm{Rep}(D^\omega(G))$ are finite, it is enough to show by way of [ERW, Lemma 2.1] that the braid group images $\rho_{m,R}(B_m)$ are finite groups, for $m \geq 2$.

Fix $m \geq 2$. We may assume that this representation category is strict, so that the only morphisms involved in $\rho_{m,R}(B_m)$ are the identity morphisms and braiding $c_{R,R}$ of $\mathrm{Rep}(D^\omega(G))$. With respect to the basis of $R \otimes R$, it can be shown that $c_{R,R}$ can be represented by an $|G|^4 \times |G|^4$ *monomial* matrix, i.e. a matrix with a single nonzero entry in any row or column. In this case the nonzero entries of $c_{R,R}$ are given by products of ω and its inverses. This is attributed to the fact that $c_{R,R}$ permutes the basis of $R \otimes R$ up to scalars coming from ω.

In general, the values of the 3-cocycle ω are elements in $\Bbbk^\times$. The following is a well known result and we include it for completeness.

Lemma 4.1.2. *Let G be a finite group. Let $\beta \in Z^n(G, \Bbbk^\times)$ be an n-cocycle, $n \geq 1$. Then there exists an n-cocycle β' cohomologous to β such that the values of β' are $|G|$th roots of unity.*

Proof. Let $g_0, g_1, \ldots, g_n \in G$. The n-cocycle condition for β is

$$\beta(g_1, \ldots, g_n)\beta(g_0g_1, \ldots, g_n)^{-1} \cdots \beta(g_0, \ldots, g_{n-1}g_n)^{(-1)^n}\beta(g_0, \ldots, g_{n-1})^{(-1)^{n+1}} = 1.$$

Taking a product over every $g_0 \in G$ we get

$$\prod_{g_0 \in G} \beta(g_1, \ldots, g_n)\beta(g_0g_1, \ldots, g_n)^{-1} \cdots \beta(g_0, \ldots, g_{n-1}g_n)^{(-1)^n}\beta(g_0, \ldots, g_{n-1})^{(-1)^{n+1}} = 1. \tag{4.1}$$

Let $m = |G|$. Choose a function $r : G^{n-1} \to \Bbbk^\times$ such that

$$r(g_1, \ldots, g_{n-1})^{-m} = \prod_{g_0 \in G} \beta(g_0, g_1, \ldots, g_{n-1}).$$

Then (4.1) can be rewritten as

$$\beta(g_1, \ldots, g_n)^m r(g_2, \cdots, g_n)^m \cdots r(g_1, \ldots, g_{n-1}g_n)^{(-1)^{n-1}m} r(g_1, \ldots, g_{n-1})^{(-1)^n m} = 1.$$

Take $\beta' := \beta \cdot d(r)$, then $\beta'(g_1, \ldots, g_n)^m = 1$. ☐

Since the representation category of $D^\omega(G)$ depends just on the cohomology class of ω [DPR], we may assume that the values of ω are $|G|$th roots of unity by Lemma 4.1.2. Thus, the image $\rho_{m,R}(B_m)$ can be realized as a subgroup of the matrix group of all $|G|^{2m} \times |G|^{2m}$ monomial matrices whose nonzero entries are $|G|$th roots of unity. This group is a finite group, and thus the image $\rho_{m,R}(B_m)$ is a finite group.

With this result in hand, one can prove that a braided fusion category $\mathcal{B}$ has finite braid group images if it can be shown that $\mathcal{B}$ is group-theoretical. In [NR], some criteria for group-theoreticity were discussed and so we can make a short-list of some braided fusion categories $\mathcal{B}$ which have finite group images.

- Integral $\mathcal{B}$ such that $\mathrm{FPdim}(\mathcal{B}) = p^n$ for some prime p.

- Integral $\mathcal{B}$ such that $\text{FPdim}(\mathcal{B}) = pq$ for distinct primes p, q.
- Integral $\mathcal{B}$ such that $\text{FPdim}(\mathcal{B}) = pqr$ for distinct primes p, q, r.
- Integral modular $\mathcal{B}$ of Frobenius-Perron dimension less than 36.
- Integral $\mathcal{B}$ such that $\text{FPdim}(X) \in \{1, 2\}$ for any simple object X, and all such objects are *self-dual*, i.e. $X \cong X^*$
- The trivial component $\mathcal{Z}(\mathcal{TY}(A, \chi, \tau))_0$ of the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{Z}(\mathcal{TY}(A, \chi, \tau))$.

While determining finiteness of braid group images from $\text{Rep}(D^\omega(G))$ was the main result of [ERW], another result about the structure of braid group images is also of interest.

Theorem 4.1.3. [ERW, Theorem 4.5] *Suppose that G is a finite p-group and $X \in Rep(D^\omega(G))$. Then $\rho_{m,X}(P_m)$ is a p-group for all $m \geq 2$.*

Corollary 4.1.4. *Suppose that $\mathcal{B}$ is a group-theoretical braided fusion category such that $FPdim(\mathcal{B}) = p^n$ for a prime p. For $X \in \mathcal{B}$, $\rho_{m,X}(P_m)$ is a p-group for all $m \geq 2$.*

Proof. Since $\mathcal{B}$ embeds into its center $\mathcal{Z}(\mathcal{B})$, it is enough to show that the pure braid group images from $\mathcal{Z}(\mathcal{B})$ are p-groups. If $\text{FPdim}(\mathcal{B}) = p^n$, then $\text{FPdim}(\mathcal{Z}(\mathcal{B})) = p^{2n}$. The center $\mathcal{Z}(\mathcal{B})$ is equivalent to $\text{Rep}(D^\omega(G))$ for some p-group G and 3-cocycle ω, so the pure braid group images from $\mathcal{Z}(\mathcal{B})$ are p-groups by Theorem 4.1.3. □

4.2 From Weakly Group-theoretical Braided Fusion Categories

Group-theoretical braided fusion categories belong to a larger class of categories which are characterized by finite group theory: weakly group-theoretical braided fusion categories. In this section, we aim to prove that braid group images coming from these braided fusion categories are finite. We begin by introducing the key fusion category constructions utilized in the main theorem

4.2.1 Braided G-Crossed Fusion Categories

Let G be a finite group. The following notion is due to Turaev.

Definition 4.2.1. A *braided G-crossed fusion category* is a fusion category $\mathcal{C}$ equipped with the following structures:

(i) an action of G on $\mathcal{C}$, i.e., a collection of tensor autoequivalences g of $\mathcal{C}$ along with natural isomorphisms

$$\mu_g(X,Y) : g(X) \otimes g(Y) \xrightarrow{\sim} g(X \otimes Y) \quad \text{and} \quad \gamma_{g,h}(X) : g(h(X)) \xrightarrow{\sim} gh(X) \tag{4.2}$$

for all $X, Y \in \mathcal{C}$, $g, h \in G$, satisfying monoidal functor structure axioms;

(ii) a (not necessarily faithful) grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$;

(iii) a natural isomorphism

$$c_{X,Y} : X \otimes Y \simeq g(Y) \otimes X, \qquad X \in \mathcal{C}_g,\, Y \in \mathcal{C}, \tag{4.3}$$

called a *G-crossed braiding*.

These data should satisfy the following conditions:

(a) the diagram

$$\begin{array}{ccc} g(X) \otimes g(Y) & \xrightarrow{c_{g(X),g(Y)}} & ghg^{-1}(g(Y)) \otimes g(X) \\ \uparrow{\scriptstyle \mu_g(X,Y)^{-1}} & & \downarrow{\scriptstyle \gamma_{ghg^{-1},g}(Y) \otimes \mathrm{id}_{g(X)}} \\ g(X \otimes Y) & & gh(Y) \otimes g(X) \\ \downarrow{\scriptstyle g(c_{X,Y})} & & \uparrow{\scriptstyle \gamma_{g,h}(Y) \otimes \mathrm{id}_{g(X)}} \\ g(h(Y) \otimes X) & \xrightarrow{\mu_g(h(Y),X)^{-1}} & g(h(Y)) \otimes g(X), \end{array} \tag{4.4}$$

commutes for all $g, h \in G$ and objects $X \in \mathcal{C}_h$, $Y \in \mathcal{C}$,

(b) the diagram

$$
\begin{array}{ccccc}
 & & (X \otimes Y) \otimes Z & & \\
 & \swarrow^{\alpha_{X,Y,Z}} & & \searrow^{c_{X,Y} \otimes \mathrm{id}_Z} & \\
X \otimes (Y \otimes Z) & & & & (g(Y) \otimes X) \otimes Z \\
\downarrow{\scriptstyle c_{X,Y \otimes Z}} & & & & \downarrow{\scriptstyle \alpha_{g(Y),X,Z}} \\
g(Y \otimes Z) \otimes X & & & & g(Y) \otimes (X \otimes Z) \\
\downarrow{\scriptstyle \mu_g(Y,Z)^{-1} \otimes \mathrm{id}_X} & & & & \downarrow{\scriptstyle \mathrm{id}_{g(Y)} \otimes c_{X,Z}} \\
(g(Y) \otimes g(Z)) \otimes X & & \xrightarrow{\alpha_{g(Y),g(Z),X}} & & g(Y) \otimes (g(Z) \otimes X)
\end{array}
\tag{4.5}
$$

commutes for all $g \in G$ and objects $X \in \mathcal{C}_g, Y, Z \in \mathcal{C}$, and

(c) the diagram

$$
\begin{array}{ccccc}
 & & X \otimes (Y \otimes Z) & & \\
 & \nearrow^{\alpha_{X,Y,Z}} & & \searrow^{\mathrm{id}_X \otimes c_{Y,Z}} & \\
(X \otimes Y) \otimes Z & & & & X \otimes (h(Z) \otimes Y) \\
\uparrow{\scriptstyle c^{-1}_{X \otimes Y,Z}} & & & & \downarrow{\scriptstyle \alpha^{-1}_{X,h(Z),Y}} \\
gh(Z) \otimes (X \otimes Y) & & & & (X \otimes h(Z)) \otimes Y \\
\uparrow{\scriptstyle \gamma_{g,h}(Z) \otimes \mathrm{id}_{X \otimes Y}} & & & & \downarrow{\scriptstyle c_{X,h(Z)} \otimes \mathrm{id}_Y} \\
g(h(Z)) \otimes (X \otimes Y) & & \xrightarrow{\alpha^{-1}_{g(h(Z)),X,Y}} & & (g(h(Z)) \otimes X) \otimes Y.
\end{array}
\tag{4.6}
$$

commutes for all $g, h \in G$ and objects $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h, Z \in \mathcal{C}$.

Here α denotes the associativity constraint of $\mathcal{C}$.

Remark 4.2.2. The trivial component $\mathcal{C}_e$ of a braided G-crossed fusion category $\mathcal{C}$ is a braided fusion category and G acts on it by braided autoequivalences.

Definition 4.2.3. We say that a braided G-crossed fusion category $\mathcal{C}$ is *non-degenerate* if its grading is faithful and $\mathcal{C}_e$ is a non-degenerate braided fusion category.

Definition 4.2.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be braided G-crossed fusion categories. A *braided G-crossed tensor functor* $(F, J, \varphi) : \mathcal{C} \to \mathcal{C}'$ is a tensor functor preserving the G-grading along with a natural isomorphism of tensor functors

$$\eta_g : F \circ g \to g \circ F, \qquad g \in G, \tag{4.7}$$

such that the diagrams

(a)

$$\begin{array}{ccc} F(g(h(X))) & \xrightarrow{F(\gamma_{g,h}(X))} & F(gh(X)) \\ {\scriptstyle \eta_g(h(X))}\downarrow & & \\ g(F(h(X))) & & \downarrow{\scriptstyle \eta_{gh}(X)} \\ {\scriptstyle g(\eta_h(X))}\downarrow & & \\ g(h(F(X))) & \xrightarrow{\gamma'_{g,h}(F(X))} & gh(F(X)), \end{array} \tag{4.8}$$

and

(b)

$$\begin{array}{ccccc} F(g(X)) \otimes F(g(Y)) & \xrightarrow{J_{g(X),\, g(Y)}} & F(g(X) \otimes g(Y)) & \xrightarrow{F((\mu_g)_{X,Y})} & F(g(X \otimes Y)) \\ {\scriptstyle \eta_g(X) \otimes \eta_g(Y)}\downarrow & & & & \downarrow{\scriptstyle \eta_g(X \otimes Y)} \\ g(F(X)) \otimes g(F(Y)) & \xrightarrow{(\mu'_g)_{F(X),F(Y)}} & g(F(X) \otimes F(Y)) & \xrightarrow{g(J_{X,Y})} & g(F(X \otimes Y)), \end{array} \tag{4.9}$$

commute for all $g, h \in G$ and $X, Y \in \mathcal{C}$ and the diagram

(c)

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{c'_{F(X),F(Y)}} & g(F(Y)) \otimes F(X) \\ & & \downarrow{\scriptstyle \eta_g(Y)^{-1} \otimes \mathrm{id}_{F(X)}} \\ {\scriptstyle J_{X,Y}}\downarrow & & F(g(Y)) \otimes F(X) \\ & & \downarrow{\scriptstyle J_{g(Y),X}} \\ F(X \otimes Y) & \xrightarrow{F(c_{X,Y})} & F(g(Y) \otimes X) \end{array} \tag{4.10}$$

commutes for all $X \in \mathcal{C}_g$, $g \in G$, and $Y \in \mathcal{C}$.

Here γ, μ, c (respectively, γ', μ', c') denote the structure isomorphisms of $\mathcal{C}$ (respectively,

$\mathcal{C}'$).

Example 4.2.5. Let G be a finite group and $\omega \in Z^3(G, \Bbbk^\times)$ be a 3-cocycle. There is a canonical braided G-crossed category structure on Vec_G^ω defined as follows. The action of $g \in G$ is by $g(x) = gxg^{-1}$, for simple $x \in \mathrm{Vec}_G^\omega$, with the tensor functor structure of g given by

$$\mu_g(y, z) = \frac{\omega(gyg^{-1}, gzg^{-1}, g)\omega(g, y, z)}{\omega(gyg^{-1}, g, z)} \,\mathrm{id}_{gyzg^{-1}} : g(y) \otimes g(z) \to g(y \otimes z), \tag{4.11}$$

the monoidal functor structure on the functor $G \to \mathrm{Aut}(\mathrm{Vec}_G^\omega)$ given by

$$\gamma_{g,h}(x) = \frac{\omega(g, hxh^{-1}, h)}{\omega(g, h, x)\omega(ghxh^{-1}g^{-1}, g, h)} \,\mathrm{id}_{ghxh^{-1}g^{-1}} : g(h(x)) \to (gh)(x), \tag{4.12}$$

and the crossed braiding given by

$$c_{g,x} = \mathrm{id}_{gx} : g \otimes x \to g(x) \otimes g, \tag{4.13}$$

for all $x, y, z, g, h \in G$.

That the above maps μ and γ determine an action of G on Vec_G^ω which follows from the identities

$$\frac{\mu_g(xy, z)\mu_g(x, y)}{\mu_g(x, yz)\mu_g(y, z)} = \frac{\omega(gxg^{-1}, gyg^{-1}, gzg^{-1})}{\omega(x, y, z)}, \tag{4.14}$$

$$\gamma_{gh,f}(x)\gamma_{g,h}(fxf^{-1}) = \gamma_{g,hf}(x)\gamma_{h,f}(x), \tag{4.15}$$

$$\frac{\mu_{gh}(x, y)}{\mu_h(x, y)\mu_g(hxh^{-1}, hyh^{-1})} = \frac{\gamma_{g,h}(xy)}{\gamma_{g,h}(x)\gamma_{g,h}(y)}, \tag{4.16}$$

for all $x, y, z, f, g, h \in G$. Diagram (4.8) commutes thanks to the identity

$$\frac{\mu_g(x, y)}{\mu_g(xyx^{-1}, x)} = \frac{\gamma_{gxg^{-1},g}(y)}{\gamma_{g,x}(y)}, \tag{4.17}$$

for all $g, x, y \in G$, while the diagrams (4.9) and (4.10) are the definitions of μ and γ. The above identities are consequences the 3-cocycle equation for ω.

Proposition 4.2.6. *The braided G-crossed category structure μ, γ, c on Vec_G^ω defined by formulas* (4.11) - (4.13) *in Example 4.2.5 is unique up to a braided G-crossed equivalence.*

Proof. If μ', γ', c' is another braided G-crossed category structure on Vec_G^ω then the identity tensor functor $\mathrm{id}_{\mathrm{Vec}_G^\omega}$ (with the tensor structure $J_{x,y} = \mathrm{id}_{xy}$) equipped with the natural isomorphism

$$\eta_g(x) = \frac{c'_{g,x}}{c_{g,x}} \mathrm{id}_{gxg^{-1}} : g(x) \to g(x) \tag{4.18}$$

establishes an equivalence between these braided G-crossed categories. Indeed, comparing diagrams (4.5) and (4.6) for them we get

$$\frac{\gamma'_{g,h}(x)}{\gamma_{g,h}(x)} = \frac{\eta_{gh}(x)}{\eta_g(hxh^{-1})\eta_h(x)} \quad \text{and} \quad \frac{\mu'_g(x,y)}{\mu_g(x,y)} = \frac{\eta_g(xy)}{\eta_g(x)\eta_g(y)}$$

for all $x, y, g, h \in G$, which gives commutativity of diagrams (4.8) and (4.9). The commutativity of (4.10) is immediate from the definition (4.18) of η. □

Corollary 4.2.7. *The braided G-crossed fusion category Vec_G^ω is equivalent to one in which all scalars ω, μ, γ, and c corresponding to the structure maps are $|G|$th roots of unity in* $\Bbbk$.

Proof. By Lemma 4.1.2, we can assume that values of ω are $|G|$th roots of 1. The result follows since the values of μ and γ are products of values of ω and its inverses. □

It was shown in [K], [Mü] (see also [T, Appendix 5]; [DGNO2, Sec 4.4.3]) that the *equivariantization* construction $\mathcal{C} \mapsto \mathcal{C}^G$ gives rise to a 2-equivalence between the 2-category of braided G-crossed fusion categories and the 2-category of braided fusion categories containing $\mathrm{Rep}(G)$.

The inverse to the equivariantization construction is called *de-equivariantization.* It proceeds as follows. Let $\mathcal{B}$ be a braided fusion category containing a Tannakian fusion subcategory $\mathcal{E} = \mathrm{Rep}(G)$ in the sense of [DGNO2]. The algebra $\mathrm{Fun}(G)$ of $\Bbbk$-valued functions on G is a commutative algebra in $\mathcal{E}$ (and, hence, in $\mathcal{B}$). The category $\mathcal{B}_G$ of $\mathrm{Fun}(G)$-modules in $\mathcal{B}$ has a canonical structure of a braided G-crossed fusion category.

One has canonical equivalences $(\mathcal{C}^G)_G \cong \mathcal{C}$ of braided G-crossed fusion categories and $(\mathcal{B}_G)^G \cong \mathcal{B}$ of braided fusion categories.

Given a braided G-crossed fusion category $\mathcal{C}$ with the G-crossed braiding c the braiding $\tilde{c}$ on $\mathcal{C}^G$ is defined as follows. Let X and Y be objects in $\mathcal{C}^G$. Let $\{v_g^Y : g(Y) \to Y\}_{g\in G}$ denote the G-equivariant structure on Y and let $X = \oplus_{g\in G} X_g$ be the decomposition of X with respect to the grading of $\mathcal{C}$. Set

$$\tilde{c}_{X,Y} : X \otimes Y = \bigoplus_{g\in G} X_g \otimes Y \xrightarrow{\oplus\, c_{X_g,Y}} \bigoplus_{g\in G} g(Y) \otimes X_g \xrightarrow{\oplus\, v_g^Y \otimes \mathrm{id}_{X_g}} \bigoplus_{g\in G} Y \otimes X_g = Y \otimes X. \tag{4.19}$$

Let $F_{\mathcal{C}} : \mathcal{C}^G \to \mathcal{C}$ be the tensor functor forgetting the G-equivariant structure. Its right adjoint $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^G$ is the induction that can be explicitly described as follows:

$$I_{\mathcal{C}}(X) = \bigoplus_{g\in G} g(X) \tag{4.20}$$

for any $X \in \mathcal{C}$ with the G-equivariant structure coming from the permutation of direct summands of X by $h \in G$:

$$v_h^{I_{\mathcal{C}}(X)} : h(I_{\mathcal{C}}(X)) = \bigoplus_{g\in G} h(g(X)) \xrightarrow{\oplus\gamma_{h,g}(X)} \bigoplus_{g\in G} hg(X) = I_{\mathcal{C}}(X), \qquad h \in G.$$

For $X \in \mathcal{C}_x$, $x \in G$, and $Y \in \mathcal{C}$ the braiding between the induced objects $I_{\mathcal{C}}(X)$ and $I_{\mathcal{C}}(Y)$ is given by

$$\begin{aligned} \tilde{c}_{I_{\mathcal{C}}(X),I_{\mathcal{C}}(Y)} : I_{\mathcal{C}}(X) \otimes I_{\mathcal{C}}(Y) = \bigoplus_{g,h\in G} g(X) \otimes h(Y) &\xrightarrow{\oplus c_{g(X),h(Y)}} \bigoplus_{g,h\in G} gxg^{-1}(h(Y)) \otimes g(X) \\ &\xrightarrow{\oplus_g\, v_{gxg^{-1}}^{I_{\mathcal{C}}(Y)} \otimes \mathrm{id}_{g(X)}} \bigoplus_{g,h'\in G} h'(Y) \otimes g(X) = I_{\mathcal{C}}(Y) \otimes I_{\mathcal{C}}(X). \end{aligned} \tag{4.21}$$

where $h' = gxg^{-1}h$.

Let $\mathcal{C}$ be a braided G-crossed fusion category. The notion of the *reverse* category of $\mathcal{C}$ was considered in [S]. This braided G-crossed category $\mathcal{C}^{\text{rev}}$ is defined as follows. As an Abelian category, $\mathcal{C}^{\text{rev}} = \mathcal{C}$ with the same action of G. For $X \in \mathcal{C}_g$, $Y \in \mathcal{C}$, the tensor product in $\mathcal{C}^{\text{rev}}$ is $X \otimes^{\text{rev}} Y := X \otimes g^{-1}(Y)$ with obvious associativity and unit constraints. The G-grading on $\mathcal{C}^{\text{rev}}$ is given by $(\mathcal{C}^{\text{rev}})_g = \mathcal{C}_{g^{-1}}$. The G-crossed braiding is given by

$$c^{\text{rev}}_{X,Y} := c^{-1}_{g^{-1}(Y),g^{-1}h^{-1}g(X)} : X \otimes^{\text{rev}} Y = X \otimes g^{-1}(Y) \to g^{-1}(Y) \otimes g^{-1}h^{-1}g(X) = g^{-1}(Y) \otimes^{\text{rev}} X$$

for all $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$.

In the special case when G is trivial the above notion coincides with that of the reverse of a braided fusion category.

Proposition 4.2.8. *There is a canonical braided equivalence $(\mathcal{C}^{rev})^G \cong (\mathcal{C}^G)^{rev}$ or, equivalently, a braided G-crossed equivalence $\mathcal{C}^{rev} \cong ((\mathcal{C}^G)^{rev})_G$.*

Proof. As Abelian k-linear categories $(\mathcal{C}^{\text{rev}})^G$ and $(\mathcal{C}^G)^{\text{rev}}$ are equal to $\mathcal{C}^G$. We define a tensor functor $F : (\mathcal{C}^{\text{rev}})^G \to (\mathcal{C}^G)^{\text{rev}}$ as the identity functor equipped with the tensor structure

$$X \otimes^{\text{rev}} Y = \bigoplus_{g\in G} X_g \otimes g^{-1}(Y) \xrightarrow{\oplus_g \mathrm{id}_{X_g} \otimes v^Y_{g^{-1}}} \bigoplus_{g\in G} X_g \otimes Y = X \otimes Y \qquad (4.22)$$

for all $X, Y \in \mathcal{C}^G$, where $X = \oplus_{g\in G} X_g$ with $X_g \in \mathcal{C}_g$.

Using the definition of the tensor products of G-equivariant objects and naturality of the associativity and braiding constraints one can check directly that F is a braided tensor equivalence. □

Proposition 4.2.9. *Let $\mathcal{A}$ be a weakly group-theoretical fusion category. There is a braided G-crossed fusion category*

$$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$$

such that the trivial component $\mathcal{C}_e$ is pointed and $\mathcal{Z}(\mathcal{A}) \cong \mathcal{C}^G$.

Proof. Let $\mathcal{E} = \text{Rep}(G)$ be a maximal Tannakian subcategory of $\mathcal{Z}(\mathcal{A})$. The corresponding de-equivariantization $\mathcal{C} = \mathcal{Z}(\mathcal{A})_G$ is a braided G-crossed fusion category and $\mathcal{Z}(\mathcal{A}) \cong \mathcal{C}^G$. The trivial component $\mathcal{C}_e$ is called the *core* of $\mathcal{Z}(\mathcal{A})$ and its braided equivalence class is independent of the choice of $\mathcal{E}$ [DGNO2]. We claim that $\mathcal{C}_e$ is pointed.

Note that $\mathcal{Z}(\mathcal{A})$ is weakly group-theoretical by [ENO3]. It was shown in [Na] that the core of a weakly group-theoretical braided fusion category is either pointed or is the Deligne product of a pointed braided fusion category and an Ising braided fusion category. Thus, $\mathcal{C}_e$ must have one of these forms. Let $\xi(\mathcal{M}) \in \Bbbk^\times$ denote the multiplicative central charge of a modular category $\mathcal{M}$ [DGNO2, Sec. 6.2]. Note that $\mathcal{Z}(\mathcal{A})$ is non-degenerate and weakly integral and, hence, is modular (with respect to the canonical spherical structure on the weakly integral category $\mathcal{Z}(\mathcal{A})$ [ENO1]). Since the central charge is invariant under taking the core, we have $\xi(\mathcal{C}_e) = \xi(\mathcal{Z}(\mathcal{A})) = 1$. This implies that an equivalence $\mathcal{C}_e \cong \mathcal{P} \boxtimes \mathcal{I}$, where $\mathcal{P}$ is pointed and $\mathcal{I}$ is an Ising category, is impossible. Indeed, $\xi(\mathcal{P})$ is an 8th root of 1 [DGNO2, Proposition A.7] while $\xi(\mathcal{I})$ is a primitive 16th root of 1 [DGNO2, Corollary B.16], so that $\xi(\mathcal{C}_e) = \xi(\mathcal{P})\xi(\mathcal{I}) \neq 1$. Therefore, $\mathcal{C}_e$ is pointed. □

4.2.2 Determinants in Graded Fusion Categories

Let $\mathcal{C}$ be an integral fusion category. For any object X in $\mathcal{C}$ let $d(X)$ denote the Frobenius-Perron dimension of X.

Definition 4.2.10. Given an automorphism $\phi : X \to X$ its *determinant* is

$$\det(\phi) = \prod_{Z \in \mathcal{O}(C)} \det(\phi|_{\mathrm{Hom}_{\mathcal{C}}(Z,X)})^{d(Z)} \in \Bbbk^\times.$$

Remark 4.2.11. Determinants can be defined for automorphisms in an arbitrary (i.e., not necessarily integral) fusion category. In general, they take values in $\mathbb{A} \otimes_{\mathbb{Z}} \Bbbk^\times$, where $\mathbb{A}$ is the ring of algebraic integers in $\mathbb{R}$.

Determinants have the following familiar properties [E, Proposition 2.1].

Proposition 4.2.12. *For all objects X, Y in $\mathcal{C}$, automorphisms $\phi, \psi : X \to X$, $\zeta : Y \to Y$, and $\lambda \in \Bbbk^\times$ we have*

(i) $\det(\phi \circ \psi) = \det(\phi)\det(\psi)$,

(ii) $\det(\phi \oplus \zeta) = \det(\phi)\det(\zeta)$,

(iii) $\det(\lambda \cdot id_X) = \lambda^{d(X)}$,

(iv) $\det(\phi \otimes id_Y) = \det(id_Y \otimes \phi) = \det(\phi)^{d(Y)}$.

Let G be a finite group and let

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

be an integral G-graded fusion category. Let $D = \mathrm{FPdim}(\mathcal{C}_e)$.

Let $R_g = \oplus_{X \in \mathcal{O}(\mathcal{C}_g)} d(X)\, X$ be the regular object in $\mathcal{C}_g$. We have $d(R_g) = D$ and

$$R_f \otimes R_g = D\, R_{fg}. \tag{4.23}$$

Denote by $\mathcal{C}_{reg}$ the abelian category generated by R_g, $g \in G$. We may and will assume that $\mathcal{C}_{reg}$ is skeletal.

Let $g_1, \ldots, g_n,\, h_1, \ldots, h_n \in G$ be such that $g_1 \cdots g_n = h_1 \cdots h_n$. Any isomorphism

$$\phi_{R_{g_1}, \ldots, R_{g_n}} : R_{g_1} \otimes \cdots \otimes R_{g_n} \to R_{h_1} \otimes \cdots \otimes R_{h_n}$$

is identified with an automorphism of $D^{n-1} R_{g_1 \cdots g_n}$.

Let $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ denote the associativity constraint in $\mathcal{C}$.

Proposition 4.2.13. *The function $\alpha^{\mathcal{C}} : G^3 \to \Bbbk^\times$ defined by*

$$\alpha^{\mathcal{C}}(f, g, h) = \det(\alpha_{R_f, R_g, R_h})^D, \qquad f, g, h \in G, \tag{4.24}$$

is a 3*-cocycle on* G *with values in* $\Bbbk^\times$*. Its class in* $H^3(G,\,\Bbbk^\times)$ *is an invariant of the* G*-graded fusion category* $\mathcal{C}$*. That is, if* $\mathcal{C}'$ *is a* G*-graded fusion category equivalent to* $\mathcal{C}$ *by a grading preserving tensor equivalence then* $\alpha^{\mathcal{C}'}$ *and* $\alpha^{\mathcal{C}}$ *are cohomologous.*

Proof. The 3-cocycle condition for $\alpha^{\mathcal{C}}$ follows from taking the determinants of both sides of the pentagon equation

$$\alpha_{R_f,R_g,R_h\otimes R_i}\circ\alpha_{R_f\otimes R_g,R_h,R_i}=(\mathrm{id}_{R_f}\otimes\alpha_{R_g,R_h,R_i})\circ\alpha_{R_f,R_g\otimes R_h,R_i}\circ(\alpha_{R_f,R_g,R_h}\otimes\mathrm{id}_{R_i}),$$

where $f,g,h,i\in G$, and using (4.23) along with the identities (ii) and (iv) from Proposition 4.2.12.

Let $F:\mathcal{C}\to\mathcal{C}'$ be a grading preserving tensor equivalence with a tensor functor structure $J_{X,Y}:F(X)\otimes F(Y)\xrightarrow{\sim}F(X\otimes Y)$. Let $R'_g=F(R_g)$, $g\in G$, denote the homogeneous regular objects of $\mathcal{C}'$. From the definition of a tensor functor we have

$$J_{R_f,R_g\otimes R_h}\circ(\mathrm{id}_{R_f}\otimes J_{R_g,R_h})\circ\alpha_{R'_f,R'_g,R'_h}=F(\alpha_{R_f,R_g,R_h})\circ J_{R_f\otimes R_g,R_h}\circ(J_{R_f,R_g}\otimes\mathrm{id}_{R_h}),$$

for all $f,g,h\in G$. Taking determinants of both sides of this equation we get

$$\frac{\alpha^{\mathcal{C}}(f,g,h)}{\alpha^{\mathcal{C}'}(f,g,h)}=\frac{\det(\alpha_{R_f,R_g,R_h})^D}{\det(\alpha_{R'_f,R'_g,R'_h})^D}=\frac{\det(J_{R_f,R_{gh}})^{D^2}\det(J_{R_g,R_h})^{D^2}}{\det(J_{R_{fg},R_h})^{D^2}\det(J_{R_f,R_g})^{D^2}},$$

i.e., $\alpha^{\mathcal{C}}$ and $\alpha^{\mathcal{C}'}$ are cohomologous. □

Remark 4.2.14. Given a G-graded fusion category $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ and a 3-cocycle $\omega\in Z^3(G,\,k^\times)$ denote by $\mathcal{C}(\omega)$ a new G-graded fusion category constructed by multiplying the associativity constraint on homogeneous objects by values of ω [ENO2]. Using Proposition 4.2.12(iii), we have

$$\alpha^{\mathcal{C}(\omega)}(f,\,g,\,\,h)=\det(\omega(f,\,g,\,\,h)\,\alpha_{R_f,R_g,R_h})^D=\omega(f,\,g,\,\,h)^{d(R_f\otimes R_g\otimes R_h)\,D}\alpha^{\mathcal{C}}(f,\,g,\,\,h)$$

for all $f,\ g,\ h \in G$. Thus, $\alpha^{\mathcal{C}(\omega)} = \alpha^{\mathcal{C}}\omega^{D^4}$.

Let $\mathcal{C} = \oplus_{g\in G}\,\mathcal{C}_g$ be an integral braided G-crossed fusion category with structure isomorphisms μ, γ, and c, as defined in (4.2) and (4.3). Let $D = \mathrm{FPdim}(\mathcal{C}_e)$. Consider the following functions:

$$\mu_g^{\mathcal{C}}(x,y) = \det(\mu_g(R_x, R_y))^{D^2}, \tag{4.25}$$

$$\gamma_{g,h}^{\mathcal{C}}(x) = \det(\gamma_{g,h}(R_x))^{D^3}, \tag{4.26}$$

$$c_{g,x}^{\mathcal{C}} = \det(c_{R_g,R_x})^{D^2}. \tag{4.27}$$

It follows from axioms (4.4) - (4.6) that they define a braided G-crossed category structure on $\mathrm{Vec}_G^{\alpha^{\mathcal{C}}}$.

Remark 4.2.15. By Lemma 4.1.2, there is $r : G^2 \to \Bbbk^\times$ such that the values of $\alpha^{\mathcal{C}}d(r)$ are $|G|$th roots of 1. Choose a function $t : G^2 \to \Bbbk^\times$ such that $t^D = r$. We can replace $\mathcal{C}$ by an equivalent braided G-crossed fusion category $\mathcal{C}'$ with the associativity constraint $\alpha'_{X,Y,Z} = d(t)(f,g,h)\,\alpha_{X,Y,Z}$ for $X \in \mathcal{C}_f$, $Y \in \mathcal{C}_g$, $Z \in \mathcal{C}_h$, $f,g,h \in G$, so that $\alpha^{\mathcal{C}'} = \alpha^{\mathcal{C}}d(r)$. Thus, we may assume that the values of $\alpha^{\mathcal{C}}$ are $|G|$th roots of 1. Similarly, by Proposition 4.2.6 and Corollary 4.2.7 we may assume that the values of functions (4.25) - (4.27) are $|G|$th roots of 1.

For $x, y, g, h \in G$ consider the composition

$$\sigma^{g,h}_{R_x,R_y} : g(R_x)\otimes h(R_y) \xrightarrow{c_{R_{gxg^{-1}},R_{hyh^{-1}}}} gxg^{-1}(h(R_y))\otimes g(R_x) \xrightarrow{\gamma_{gxg^{-1},h}(R_y)\otimes \mathrm{id}} gxg^{-1}h(R_y)\otimes g(R_x). \tag{4.28}$$

viewed as an automorphism of $D\,R_{gxg^{-1}hyh^{-1}}$. These compositions are the components of the braiding on the induced object of $\mathcal{C}^G$, see (4.21).

Corollary 4.2.16. *$\mathcal{C}$ is equivalent to a braided G-crossed fusion category in which*

$$\det\left(\sigma^{g,h}_{R_x,R_y}\right)^{|G|D^2} = 1$$

for all $x, y, g, h \in G$.

4.2.3 The Fiber Product of Braided G-Crossed Fusion Categories

The following notion was introduced in [Ni].

Definition 4.2.17. Let $\mathcal{C}^1, \mathcal{C}^2$ be fusion categories graded by the same group G. The *fiber product* of $\mathcal{C}^1$ and $\mathcal{C}^2$ is the G-graded fusion category $\mathcal{C}^1 \boxtimes_G \mathcal{C}^2$ with grading defined by

$$(\mathcal{C}^1 \boxtimes_G \mathcal{C}^2)_g = \mathcal{C}^1_g \boxtimes \mathcal{C}^2_g \tag{4.29}$$

for $g \in G$ with trivial component $\mathcal{C}^1_e \boxtimes \mathcal{C}^2_e$. Here, $\boxtimes$ denotes Deligne's tensor product of Abelian categories.

Remark 4.2.18. It can be seen that $\mathcal{C}^1 \boxtimes_G \mathcal{C}^2$ is a fusion subcategory of $\mathcal{C}^1 \boxtimes \mathcal{C}^2$. However, neither of $\mathcal{C}^1$ nor $\mathcal{C}^2$ are fusion subcategories of the fiber product $\mathcal{C}^1 \boxtimes_G \mathcal{C}^2$ in general.

Suppose that $\mathcal{C}^1$ and $\mathcal{C}^2$ are braided G-crossed fusion categories with G-braiding c^1, c^2, respectively. Then the fiber product $\mathcal{C}^1 \boxtimes_G \mathcal{C}^2$ is also a braided G-crossed fusion category, with action defined by

$$g(X_1 \boxtimes X_2) := g(X_1) \boxtimes g(X_2)$$

for $X_1 \in \mathcal{C}^1_x$, $X_2 \in \mathcal{C}^2_x$, $g, x \in G$, and G-braiding c defined by

$$c_{X_1 \boxtimes X_2, Y_1 \boxtimes Y_2} := c^1_{X_1,Y_1} \boxtimes c^2_{X_2,Y_2}.$$

Recall that a Tannakian subcategory $\mathcal{E}$ of a non-degenerate braided fusion category $\mathcal{B}$ is *Lagrangian* if $\mathrm{FPdim}(\mathcal{E})^2 = \mathrm{FPdim}(\mathcal{B})$. If a group G acts on $\mathcal{B}$ by autoequivalences, we say that $\mathcal{E}$ is *G-stable* if $g(\mathcal{E}) = \mathcal{E}$ for all $g \in G$.

Lemma 4.2.19. *Let $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ be a non-degenerate braided G-crossed fusion category. Suppose that $\mathcal{C}_e$ contains a G-stable Lagrangian subcategory. Then $\mathcal{C}^G$ is group-theoretical.*

Proof. Let $\mathcal{E}$ be a G-stable Lagrangian subcategory of $\mathcal{C}_e$. Then $\mathcal{E}^G$ is a Lagrangian subcategory of $\mathcal{C}^G$ and the statement follows from [DGNO2, Theorem 4.64]. $\square$

Proposition 4.2.20. *Let $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ be a non-degenerate braided G-crossed fusion category such that $\mathcal{C}_e$ is pointed. Then $(\mathcal{C} \boxtimes_G \mathcal{C}^{rev})^G$ is group-theoretical.*

Proof. The subcategory of $\mathcal{C}_e \boxtimes \mathcal{C}_e^{\mathrm{rev}}$ spanned by $\{X \boxtimes X \mid X \in \mathcal{O}(\mathcal{C}_e)\}$ is Lagrangian and G-stable. So the result follows from Lemma 4.2.19. $\square$

4.2.4 The Finiteness Result

Fix $m \geq 1$ and let G be a finite group. The following action of B_m on G^{2m} was considered in [ERW, Sec. 4]:

$$\pi_{m,G}(\sigma_i)\left((g_1,h_1),\ldots,(g_m,h_m)\right)$$

$$= \left((g_1,h_1),\ldots,(g_{i-1},h_{i-1}),(g_ih_ig_i^{-1}g_{i+1},h_{i+1}),(g_i,h_i),(g_{i+2},h_{i+2}),\ldots,(g_m,h_m)\right), \quad (4.30)$$

for all $g_j, h_j \in G$, $1 \leq j \leq m$. Let $K_{m,G}$ denote the kernel of $\pi_{m,G}$. It can be seen that $K_{m,G} < P_m$ and $[B_m : K_{m,G}] < \infty$.

Proposition 4.2.21. *Let $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ and $\mathcal{D} = \oplus_{g\in G}\mathcal{D}_g$ be integral braided G-crossed fusion categories. If the objects of $(\mathcal{C} \boxtimes_G \mathcal{D})^G$ yield finite braid group images then so do the objects of $\mathcal{C}^G$ and $\mathcal{D}^G$.*

Proof. To show that $\mathcal{C}^G$ and $\mathcal{D}^G$ have finite braid group images, it is enough to show that the braid group images coming from their regular objects are finite. Let

$$R = \bigoplus_{h\in G} R_h, \qquad \text{where} \qquad R_h = \bigoplus_{X\in\mathcal{O}(\mathcal{C}_h)} d(X)X, \quad (4.31)$$

$$S = \bigoplus_{h\in G} S_h, \qquad \text{where} \qquad S_h = \bigoplus_{X\in\mathcal{O}(\mathcal{D}_h)} d(X)X, \quad (4.32)$$

be regular objects for $\mathcal{C}$ and $\mathcal{D}$, respectively.

We can induce from $R \in \mathcal{C}$ the object $I_{\mathcal{C}}(R) \in \mathcal{C}^G$ (4.20). The object $I_{\mathcal{C}}(R)$ is a regular object of $\mathcal{C}^G$, so it is enough to show that the braid group images $\rho_{m,I_{\mathcal{C}}(R)}(B_m)$ are finite. Similarly for $S \in \mathcal{D}$, we will show that the braid group images $\rho_{m,I_{\mathcal{D}}(S)}(B_m)$ are finite.

Note that showing $\rho_{m,I_{\mathcal{C}}(R)}(K_{m,G})$ (resp. $\rho_{m,I_{\mathcal{D}}(S)}(K_{m,G})$) is finite is enough because $K_{m,G}$ is a finite index subgroup of B_m.

Define $Z_h = R_h \boxtimes S_h$ and $Z = \oplus_{h\in G} Z_h \in \mathcal{C} \boxtimes_G \mathcal{D}$. By the hypothesis, the induced object $I_{\mathcal{C}\boxtimes_G\mathcal{D}}(Z) \in (\mathcal{C} \boxtimes_G \mathcal{D})^G$ yields finite braid group images. So the image of

$$\rho_{m,I_{\mathcal{C}\boxtimes_G\mathcal{D}}(Z)} : B_m \to \mathrm{End}(I_{\mathcal{C}\boxtimes_G\mathcal{D}}(Z)^{\otimes m}) = \mathrm{End}\left(\bigoplus_{g_1,\dots,g_m,h_1,\dots,h_m\in G} g_1(Z_{h_1}) \otimes \cdots \otimes g_m(Z_{h_m})\right) \tag{4.33}$$

is finite. For each simple braid $\sigma_i \in B_m$, $1 \le i \le m-1$, the automorphism $\rho_{m,I_{\mathcal{C}\boxtimes_G\mathcal{D}}(Z)}(\sigma_i)$ maps the summand

$$g_1(Z_{h_1}) \otimes \cdots \otimes g_i(Z_{h_i}) \otimes g_{i+1}(Z_{h_{i+1}}) \otimes \cdots \otimes g_m(Z_{h_m})$$

in the direct sum in (4.33) to the summand

$$g_1(Z_{h_1}) \otimes \cdots \otimes g_i h_i g_i^{-1} g_{i+1}(Z_{h_{i+1}}) \otimes g_i(Z_{h_i}) \otimes \cdots \otimes g_m(Z_{h_m})$$

for all $g_1, \dots, g_m, h_1, \dots, h_m \in G$. Thus each summand is stable under $K_{m,G}$. Note that these summands are, in general, objects of $\mathcal{C} \boxtimes_G \mathcal{D}$, but not of $(\mathcal{C} \boxtimes_G \mathcal{D})^G$. Let

$$\rho^{(g_1,h_1),\dots,(g_m,h_m)}_{m,I_{\mathcal{C}\boxtimes_G\mathcal{D}}(Z)} : K_{m,G} \to \mathrm{End}_{\mathcal{C}\boxtimes_G\mathcal{D}}\left(g_1(Z_{h_1}) \otimes \cdots \otimes g_m(Z_{h_m})\right) \tag{4.34}$$

denote the corresponding restrictions. The images of $K_{m,G}$ under these restrictions are finite.

We have

$$\begin{aligned} &\mathrm{End}_{\mathcal{C}\boxtimes_G\mathcal{D}}\left(g_1(Z_{h_1}) \otimes \cdots \otimes g_m(Z_{h_m})\right) \\ &= \mathrm{End}_{\mathcal{C}}(g_1(R_{h_1}) \otimes \cdots \otimes g_m(R_{h_m})) \otimes_{\Bbbk} \mathrm{End}_{\mathcal{D}}(g_1(S_{h_1}) \otimes \cdots \otimes g_m(S_{h_m})) \end{aligned} \tag{4.35}$$

and

$$\rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{C}\boxtimes_G\mathcal{D}}(Z)}(\sigma) = \rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{C}}(R)}(\sigma) \ \otimes_{\Bbbk} \ \rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{D}}(S)}(\sigma) \tag{4.36}$$

for each $\sigma \in K_{m,G}$. The image of $K_{m,G}$ in this tensor product is finite, but that is not enough to claim that either individual image of $K_{m,G}$ is finite. Define

$$\Delta(K_{m,G}) = \left\{ \left(\rho^{(g_1,h_2),\ldots,(g_m,h_m)}_{m,I_{\mathcal{C}}(R)}(\sigma), \rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{D}}(S)}(\sigma) \right) \mid \sigma \in K_{m,G} \right\},$$

the diagonal subgroup of $\rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{C}}(R)}(K_{m,G}) \times \rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{D}}(S)}(K_{m,G})$. We claim that $\Delta(K_{m,G})$ is finite.

Equation (4.36) defines a surjective group homomorphism

$$p : \Delta(K_{m,G}) \to \rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{C}\boxtimes_G\mathcal{D}}(Z)}(K_{m,G}) \tag{4.37}$$

The image of p is finite, so $\Delta(K_{m,G})$ is finite if and only if the kernel of p is finite. The kernel of p is contained in

$$L = \{(\lambda \cdot \mathrm{id}_{g_1(R_{h_1})\otimes\cdots\otimes g_m(R_{h_m})}, \lambda^{-1} \cdot \mathrm{id}_{g_1(S_{h_1})\otimes\cdots\otimes g_m(S_{h_m})}) \mid \lambda \in \Bbbk^{\times}\} \cap \Delta(K_{m,G}),$$

since the Kronecker product of two linear operators equals the identity if and only if the factors are reciprocal scalar multiples of the identity. Thus, it suffices to show that L is finite.

For $\sigma \in K_{m,G}$, we can write σ as a product of simple braids in B_m. Let $\sigma = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}$ be one such presentation. By definition (4.21), each $\rho_{m,I_{\mathcal{C}}(R)}(\sigma_{i_k})$, $k = 1,\ldots,n$, is a block permutation matrix whose blocks are obtained by tensoring maps of the form (4.28) with identity and conjugating by the associativity isomorphisms. By Corollary 4.2.16 the determinant of each block of $\rho_{m,I_{\mathcal{C}}(R)}(\sigma_{i_k})$ is an Nth root of unity, where $N = |G|D^2$ and $D = \mathrm{FPdim}(C_e)$.

Hence, the determinant of each diagonal block $\rho^{(g_1,h_1),\ldots,(g_m,h_m)}_{m,I_{\mathcal{C}}(R)}(\sigma)$ of $\rho_{m,I_{\mathcal{C}}(R)}(\sigma)$ is an Nth root of unity. But if $\rho_{m,I_{\mathcal{C}}(R)}^{(g_1,h_1),\ldots,(g_m,h_m)}(\sigma) = \lambda\, \mathrm{id}_{g_1(R_{h_1})\otimes\cdots\otimes g_m(R_{h_m})}$ then, by Proposition 4.2.12,

$$\det\left(\rho_{m,I_{\mathcal{C}}(R)}^{(g_1,h_1),\ldots,(g_m,h_m)}(\sigma)\right) = \lambda^{D^m}.$$

Hence, $\lambda^{|G|D^{m+2}} = 1$ and L is finite. So $\Delta(K_{m,G})$ is finite.

It follows that the images $\rho_{m,I_{\mathcal{C}}(R)}^{(g_1,h_1),\ldots,(g_m,h_m)}(K_{m,G})$ and $\rho_{m,I_{\mathcal{D}}(S)}^{(g_1,h_1),\ldots,(g_m,h_m)}(K_{m,G})$ are finite for all $g_j, h_j \in G$, $j = 1, \ldots, m$. Hence, $\rho_{m,I_{\mathcal{C}}(R)}(K_{m,G})$ and $\rho_{m,I_{\mathcal{D}}(S)}(K_{m,G})$ are finite. □

Theorem 4.2.22. *The braid group images coming from a weakly group-theoretical braided fusion category are finite.*

Proof. First, let us prove this theorem in the case when $\mathcal{B}$ is integral. As we noted in Section 4.1, we can embed $\mathcal{B}$ within its center $\mathcal{Z}(\mathcal{B})$ which is also integral and weakly group-theoretical by [ENO3]. So it is enough to show that the braid group images from $\mathcal{Z}(\mathcal{B})$ are finite.

By Proposition 4.2.9, there is a braided G-crossed fusion category $\mathcal{C}$ with pointed trivial component such that $\mathcal{Z}(\mathcal{B}) \simeq \mathcal{C}^G$. By Proposition 4.2.20, $(\mathcal{C} \boxtimes_G \mathcal{C}^{\mathrm{rev}})^G$ is group-theoretical and by [ERW] it has finite braid group images. The statement follows from Proposition 4.2.21.

Now suppose that $\mathcal{B}$ is an arbitrary weakly group-theoretical braided fusion category. Recall that $\mathcal{B}$ can be graded as a weakly integral fusion category

$$\mathcal{B} = \bigoplus_{g\in G(\mathcal{B})} \mathcal{B}_g$$

over an elementary abelian 2-group $G(\mathcal{B})$ by Corollary 2.5.9. The fiber product $\mathcal{B} \boxtimes_{G(\mathcal{B})} \mathcal{B}$ is an integral weakly group-theoretical braided fusion category, and so would yield finite braid group images. To show that $\mathcal{B}$ itself has finite braid group images, we mimic our proof of Proposition 4.2.21.

Let $a_1, \ldots, a_m \in G(\mathcal{B})$, $X_i, Y_i \in \mathcal{B}_{a_i}$, and let $Z_i = X_i \boxtimes Y_i$, $1 \le i \le m$. It suffices to show that $\otimes_{i=1}^m X_i$ yields finite pure braid group images. As in Proposition 4.2.21 we have

$$\mathrm{End}_{\mathcal{B}\boxtimes_{G(\mathcal{B})}\mathcal{B}}\left(\bigotimes_{i=1}^m Z_i\right) = \mathrm{End}_{\mathcal{B}}\left(\bigotimes_{i=1}^m X_i\right) \otimes_{\Bbbk} \mathrm{End}_{\mathcal{B}}\left(\bigotimes_{i=1}^m Y_i\right)$$

and $\rho_{m,Z_1,\dots,Z_m}(\sigma) = \rho_{m,X_1,\dots,X_m}(\sigma) \otimes_{\Bbbk} \rho_{m,Y_1,\dots,Y_m}(\sigma)$ for every $\sigma \in P_m$. Define

$$\Delta(P_m) = \{(\rho_{m,X_1,\dots,X_m}(\sigma), \rho_{m,Y_1,\dots,Y_m}(\sigma)) \mid \sigma \in P_m\}$$

To prove that $\Delta(P_m)$ is a finite group, we consider the surjective homomorphism

$$p : \Delta(P_m) \to \rho_{m,Z_1,\dots,Z_m}(P_m) \tag{4.38}$$

$$(\rho_{m,X_1,\dots,X_m}(\sigma), \rho_{m,Y_1,\dots,Y_m}(\sigma)) \mapsto \rho_{m,X_1,\dots,X_m}(\sigma) \otimes_{\Bbbk} \rho_{m,Y_1,\dots,Y_m}(\sigma)$$

and note that the kernel of this homomorphism lies in

$$L = \{(\lambda \cdot \mathrm{id}_{\bigotimes_{i=1}^m X_i}, \lambda^{-1} \cdot \mathrm{id}_{\bigotimes_{i=1}^m Y_i}) \mid \lambda \in \Bbbk^\times\} \cap \Delta(P_m).$$

Since $\rho_{m,Z_1,\dots,Z_m}(P_m)$ is finite, showing $\Delta(P_m)$ is finite amounts to showing that L is finite.

Let $I = \rho_{m,X_1,\dots,X_m}(P_m) < \mathrm{End}_{\mathcal{B}}(\bigotimes_{i=1}^m X_i)$. If $\bigotimes_{i=1}^m X_i$ has a decomposition over d simple summands, then we can identify elements of I with $d \times d$ matrices over $\Bbbk$. We claim there are only finitely many scalar matrices in I.

Note that the determinant of any matrix in the commutator subgroup $[I, I]$ would necessarily be 1 since this subgroup is generated by elements of the form $ABA^{-1}B^{-1}$ for $A, B \in I$ and $\det(ABA^{-1}B^{-1}) = 1$. Note also that $I/[I, I]$ is an abelian group.

The morphism $\rho_{m,X_1,\dots,X_m}(\sigma_i^2)$, $1 \leq i \leq m-1$, in $\mathrm{End}(\bigotimes_{i=1}^m X_i)$ has finite order, because the square of a braiding in a braided fusion category has finite order [E]. Since P_m is finitely generated by conjugates of these σ_i^2 by Proposition 3.1.5, $I/[I, I]$ is a finitely generated abelian group whose generators have finite order, i.e. finite. Let N be the exponent of

$I/[I,I]$. If s is a scalar matrix $\lambda \cdot I_d \in I$, then $s^N \in [I,I]$ and has determinant 1. So $\lambda^{dN} = 1$ and L must be finite. □

We next extend the result of Corollary 4.1.4 to the weakly group-theoretical case.

Proposition 4.2.23. *Suppose that $\mathcal{B}$ is a weakly group-theoretical braided fusion category such that FPdim$(\mathcal{B}) = p^n$ for a prime p. For $X \in \mathcal{B}$, the images $\rho_{m,X}(P_m)$ are p-groups for all $m \geq 2$.*

Proof. In the case when p is an odd prime, $\mathcal{B}$ is integral by [DGNO2, Corollary 2.22]. In general, any integral fusion category of dimension p^n for prime p is group-theoretical by [DGNO1, Corollary 6.8]. So if p is odd or if $\mathcal{B}$ is integral, then the result follows from Corollary 4.1.4.

Now suppose that $\mathcal{B}$ is a weakly group-theoretical braided fusion category of dimension 2^n. Recall that $\mathcal{B}$ can be graded as a weakly integral fusion category

$$\mathcal{B} = \bigoplus_{g \in G(\mathcal{B})} \mathcal{B}_g$$

over an elementary abelian 2-group $G(\mathcal{B})$ by Corollary 2.5.9. The fiber product $\mathcal{B} \boxtimes_{G(\mathcal{B})} \mathcal{B}$ is an integral weakly group-theoretical braided fusion category, and so would yield pure braid group images which are 2-groups. To show that $\mathcal{B}$ itself would yield 2-groups of pure braid group images, we mimic the proof of Theorem 4.2.22.

Let $a_1, \ldots, a_m \in G(\mathcal{B})$, $X_i \in \mathcal{B}_{a_i}$, and let $Z_i = X_i \boxtimes X_i$, $1 \leq i \leq m$. It suffices to show that $\otimes_{i=1}^m X_i$ yields pure braid group images which are 2-groups. As in Proposition 4.2.21 we have

$$\mathrm{End}_{\mathcal{B} \boxtimes_{G(\mathcal{B})} \mathcal{B}} \left(\bigotimes_{i=1}^m Z_i \right) = \mathrm{End}_{\mathcal{B}} \left(\bigotimes_{i=1}^m X_i \right) \otimes_{\Bbbk} \mathrm{End}_{\mathcal{B}} \left(\bigotimes_{i=1}^m X_i \right)$$

and $\rho_{m,Z_1,\ldots,Z_m}(\sigma) = \rho_{m,X_1,\ldots,X_m}(\sigma) \otimes_{\Bbbk} \rho_{m,X_1,\ldots,X_m}(\sigma)$ for every $\sigma \in P_m$. Define

$$\Delta(P_m) = \{(\rho_{m,X_1,\ldots,X_m}(\sigma), \rho_{m,X_1,\ldots,X_m}(\sigma)) \mid \sigma \in P_m\}$$

To prove that $\Delta(P_m)$ is a 2-group, we consider the surjective homomorphism

$$p : \Delta(P_m) \to \rho_{m,Z_1,\ldots,Z_m}(P_m)$$

$$(\rho_{m,X_1,\ldots,X_m}(\sigma), \rho_{m,X_1,\ldots,X_m}(\sigma)) \mapsto \rho_{m,X_1,\ldots,X_m}(\sigma) \otimes_{\Bbbk} \rho_{m,X_1,\ldots,X_m}(\sigma)$$

and note that the kernel of this homomorphism lies in

$$L = \{(\lambda \cdot \mathrm{id}_{\bigotimes_{i=1}^m X_i}, \lambda^{-1} \cdot \mathrm{id}_{\bigotimes_{i=1}^m X_i}) \mid \lambda \in \Bbbk^\times\} \cap \Delta(P_m).$$

Since $\rho_{m,Z_1,\ldots,Z_m}(P_m)$ is a 2-group, showing $\Delta(P_m)$ is a 2-group amounts to showing that L is a 2-group. The coordinates for a pair in $\Delta(P_m)$ are the same, so L can only consist of pairs with $\lambda = \lambda^{-1}$, or $\lambda^2 = 1$. Thus L is either trivial or $\mathbb{Z}/2\mathbb{Z}$. $\square$

We can use this result to discuss the pure braid group images coming from braided *nilpotent* fusion categories (Definition 2.5.10.ii). Braided nilpotent fusion categories have a decomposition into fusion categories of prime power dimension similar to the Sylow decomposition for nilpotent groups.

Proposition 4.2.24. [DGNO1, Theorem 6.12] *Let $\mathcal{B}$ be a braided nilpotent fusion category. Then $\mathcal{B}$ has a unique decomposition as a Deligne tensor product of braided fusion categories of prime power dimension.*

Corollary 4.2.25. *Let $\mathcal{B}$ be a braided nilpotent fusion category. For $X \in \mathcal{B}$, the images $\rho_{m,X}(P_m)$ are nilpotent groups for all $m \geq 2$.*

Proof. Let $\mathcal{B} = \boxtimes_p \mathcal{B}_p$ be the decomposition of $\mathcal{B}$ from Proposition 4.2.24 with $\mathrm{FPdim}(\mathcal{B}_p)$ a power of p. The pure braid group image $\rho_{m,X}(P_m)$ coming from $\mathcal{B}$ decomposes as a direct product of pure braid group images coming from each $\mathcal{B}_p$. The result follows from Proposition 4.2.23 as the pure braid group image coming from $\mathcal{B}_p$ is the Sylow p-subgroup of $\rho_{m,X}(P_m)$. $\square$

4.3 The Property F Conjecture

It appears that the first goal in understanding braid group images in braided fusion categories is determining finiteness. The proofs so far have mostly worked toward this goal. Other sources such as [ERW] and [NR] reference infinite braid group images coming from other braided categories. These braided categories are closely related to quantum groups and polynomial link invariants and have less to do with finite group theory. What makes finiteness interesting?

One major motivation is in the theory of topological quantum computation. According to [Ro], (modular) braided fusion categories possess the data required to build models for topological quantum computers. Certain aspects of the categories correspond to certain computational properties. For example, images of braid group representations are closely related to the computational power of the corresponding topological quantum computer. The more complicated the braid group images, the greater the computational power. Topological quantum computers coming from braided fusion categories whose objects yield finite braid group images are generally inefficient in even storing information!

This interest motivates the following definitions, found in [ERW] and [GRR]. Let $\mathcal{B}$ be a braided fusion category.

Definition 4.3.1. An object X in $\mathcal{B}$ is said to be a *property F object* if the braid group images coming from X are all finite. $\mathcal{B}$ is said to have *property F* if every object of $\mathcal{B}$ is a property F object.

It is clear from Theorems 4.1.1 and 4.2.22 that every group-theoretical and, more generally, every weakly group-theoretical braided fusion category has property F. It is unclear if there are any braided fusion categories $\mathcal{B}$ with property F which are not weakly group-theoretical. Determining that an object X in $\mathcal{B}$ is not a property F object tends to be a simpler task as one need only prove that the image of B_3 is infinite. Based on the evidence at the time, a conjecture was made in [NR] regarding property F:

Conjecture 4.3.2. A braided fusion category $\mathcal{B}$ has property F if and only if $\mathcal{B}$ is weakly integral.

This is a fascinating conjecture because it attempts to relate a property of a braided fusion category (finiteness of the braid group images) to a property of its underlying fusion category (the Frobenius-Perron dimension of the category). Weakly group-theoretical fusion categories are weakly integral so Theorem 4.2.22 supports one half of this conjecture. There is a question in fusion category theory which, if answered in the negative, would lend greater support to the property F conjecture:

Question 4.3.3. [ENO3, Question 2] Does there exist a weakly integral fusion category which is not weakly group-theoretical?

CHAPTER 5

EXAMPLES OF IMAGES OF SPECIFIC BRAID GROUP REPRESENTATIONS

Inspired by the property F conjecture, this chapter aims to slowly chip away at a larger question: Given an object X in a braided fusion category $\mathcal{B}$, what is the relationship between $\mathrm{FPdim}(X)$ and the images of the representations of the pure braid groups that come from X? Each of the following examples contribute to a strategy for how to address this question in general.

5.1 From Symmetric Fusion Categories

Definition 5.1.1. A braided fusion category $\mathcal{B}$ with braiding c is called *symmetric* if

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y} \tag{5.1}$$

for all $X, Y \in \mathcal{B}$.

Example 5.1.2. Let G be a finite group. The braided fusion category $\mathrm{Rep}(G)$ is a symmetric fusion category with braiding given in Example 3.2.4

Let X be an object in a symmetric fusion category $\mathcal{B}$ with braiding c. Recall that each pure braid group P_m, $m \geq 2$, is generated by conjugates of σ_i^2, $1 \leq i \leq m-1$, according to Proposition 3.1.5. The image of each σ_i^2 in $\mathrm{End}(X^{\otimes m})$ is $c_{X,X} \circ c_{X,X} = \mathrm{id}_{X\otimes X}$ (with appropriate identity morphisms). Conjugates of an identity morphism are themselves identity morphisms, so the image of any braid of P_m in $\mathrm{End}(X^{\otimes m})$ is the identity morphism.

This yields the following result.

Proposition 5.1.3. *For an object X in a symmetric fusion category $\mathcal{B}$, $\rho_{m,X}(P_m)$ is trivial for every $m \geq 2$.*

This is straightforward from the definition of a symmetric fusion category. It is not immediately obvious, however, what properties an object X possesses in a general braided fusion category $\mathcal{B}$ with braiding c if $c_{X,X} \circ c_{X,X} = \mathrm{id}_{X \otimes X}$. We can utilize the following definition.

Definition 5.1.4. We say that two objects X, Y in a braided fusion category $\mathcal{B}$ with braiding c *centralize* each other if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$.

Now suppose that $\mathcal{B}$ is a braided fusion category with braiding c and $X \in \mathcal{B}$ such that X centralizes itself. Let $\mathcal{B}[X]$ be the fusion subcategory of $\mathcal{B}$ generated by X, i.e. the smallest full fusion subcategory of $\mathcal{B}$ containing X. $\mathcal{B}[X]$ inherits the braiding structure from $\mathcal{B}$.

Proposition 5.1.5. *$\mathcal{B}[X]$ is a symmetric fusion category.*

Proof. Let $Y, Z \in \mathcal{B}[X]$. It is enough to show that Y and Z centralize each other when they are both simple, so we suppose Y, Z are as such. Since X generates $\mathcal{B}[X]$ as a fusion category, there exist nonnegative integers p, q such that Y, Z are subobjects of $X^{\otimes p}, X^{\otimes q}$, respectively. To show that Y, Z centralize each other it is enough to show that $X^{\otimes p}$ and $X^{\otimes q}$ centralize each other. This can be shown using the Hexagon Axioms and the fact that X centralizes itself. □

If an object X in a braided fusion category $\mathcal{B}$ centralizes itself, then the generated subcategory $\mathcal{B}[X]$ is symmetric. Symmetric fusion categories have the following property.

Proposition 5.1.6. [DGNO2, Corollary 2.46] *A symmetric fusion category is integral.*

Corollary 5.1.7. *For an object X in a braided fusion category $\mathcal{B}$, if $\rho_{m,X}(P_m)$ is trivial for all $m \geq 2$, then FPdim(X) is an integer.*

Proof. If $\rho_{2,X}(P_2)$ is trivial, then X centralizes itself. By Proposition 5.1.5, X is an object of a symmetric fusion category $\mathcal{B}[X]$. Thus X has integral Frobenius-Perron dimension by Proposition 5.1.6. □

Remark 5.1.8. When an object X has the simplest possible pure braid group images (all trivial), we have $\mathrm{FPdim}(X)$ is an integer. It is clear that the images $\rho_{m,X}(P_m)$ for $m \geq 2$ do not yield any extra information about the specific Frobenius-Perron dimension of X in this case. For example, they cannot possibly tell us anything about the size of the dimension. Indeed, the category $\mathrm{Rep}(S_m)$ for $m \geq 2$ has a symmetric braiding and a simple object of Frobenius-Perron dimension $m-1$. So there are objects of any positive integer dimension which have trivial pure braid group images.

5.2 From Pointed Braided Fusion Categories

For a pointed fusion category $\mathcal{C}$, the set $G = \mathcal{O}(\mathcal{C})$ of isomorphism classes of simple objects of $\mathcal{C}$ has the structure of a finite group. If $\mathcal{C}$ is braided, then the group G is abelian. To understand the braid group images coming from pointed braided fusion categories, we first must discuss quadratic forms on abelian groups.

Definition 5.2.1. Let G be an abelian group. A *quadratic form* on G with values in $\Bbbk$ is a function $q : G \to \Bbbk^\times$ such that $q(g^n) = q(g)^{n^2}$ for any $g \in G$ and the function $b_q : G \times G \to \Bbbk^\times$ defined by

$$b_q(g,h) := \frac{q(gh)}{q(g)q(h)}$$

is a symmetric bicharacter of G, meaning

$$b_q(g,h) = b_q(h,g), \qquad b_q(g,h_1h_2) = b_q(g,h_1)b_q(g,h_2)$$

for all $g, h, h_1, h_2 \in G$.

Remark 5.2.2. It is well known that for a bicharacter b of a group G, $b(g,g)$ is a $|g|$th root of unity or, more generally, that $b(g,h)$ is a $|G|$th root of unity, for every $g,h \in G$. For a quadratic form $q : G \to \Bbbk^\times$ with its associated bicharacter b_q, $q(g)^2$ is a $|g|$th root of unity for every $g \in G$ as well:

$$q(g)^2 = \frac{q(g)^4}{q(g)q(g)} = \frac{q(g^2)}{q(g)q(g)} = b_q(g,g).$$

Let $\mathcal{B}$ be a pointed braided fusion category with braiding c and let G be the finite abelian group of the isomorphism classes of its simple objects. If X is a simple invertible object of $\mathcal{B}$ whose isomorphism class is represented by $g \in G$, the object $X \otimes X$ is simple with isomorphism class $g^2 \in G$. The braiding morphism $c_{X,X}$ is an isomorphism of a simple object to itself, so it is a scalar mutliple of the identity, say $q(g) \cdot \mathrm{id}_{X \otimes X}$.

Proposition 5.2.3. [EGNO, Lemma 8.4.2] *The function $q : G \to \Bbbk^\times$ is a quadratic form.*

Remark 5.2.4. If b_q is the symmetric bicharacter assocciated to the quadratic form q above, then for simple objects X, Y with isomorphism classes g, h, respectively, $b_q(g,h)$ is the scalar such that $c_{Y,X} \circ c_{X,Y} = b_q(g,h)\mathrm{id}_{X \otimes Y}$.

This means that for simple $X \in \mathcal{B}$ with isomorphism class $g \in G$, the images of P_m in $\mathrm{End}(X^{\otimes m})$ are easily described for $m \geq 2$. Indeed, the image of a braid of the form σ_i^2, $1 \leq i \leq m-1$, is $q(g)^2 \cdot \mathrm{id}_{X \otimes X}$ possibly tensored with identity morphisms. Scalar morphisms in $\mathrm{End}(X^{\otimes m})$ are preserved under conjugation by endomorphisms of $X^{\otimes m}$, so $\rho_{m,X}(P_m)$ is generated by $q(g)^2\mathrm{id}_{X \otimes X}$ as a result of Proposition 3.1.5.

Proposition 5.2.5. *Given a pointed braided fusion category $\mathcal{B}$ whose isomorphism classes of simple objects form the group G and whose braiding defines a quadratic form $q : G \to \Bbbk^\times$, let X be a simple object corresponding to class $g \in G$. For every $m \geq 2$, the image $\rho_{m,X}(P_m)$ is a cyclic group of order $|q(g)^2|$.*

Knowing a bit more about quadratic forms and their associated bicharacters, we can say the following.

Proposition 5.2.6. *For any simple object X in a pointed braided fusion category, $\rho_{m,X}(P_m)$ is a cyclic group whose order divides the order of G, the group of isomorphism classes of simple objects.*

Proof. According to Proposition 5.2.5, the order of the cyclic group $\rho_{m,X}(P_m)$ is $|q(g)^2|$, where $g \in G$ is the isomorphism class of X. Remark 5.2.4 shows that $q(g)^2 = b_q(g, g)$, for b_q the associated bicharacter of q, and the latter is a $|G|$th root of unity. □

We now have examples of objects whose pure braid group images are all the same (potentially non-trivial) cyclic group. This feels like the next simplest case to work with when trying to relate the images of the pure braid groups coming from an object X to the Frobenius-Perron dimension of X. For a simple object X in a pointed braided fusion category we have $\mathrm{FPdim}(X) = 1$, and this gives more evidence to suggest that objects with 'basic' pure braid group images should have integral dimension.

5.3 From Projectively Symmetric Fusion Categories

The pure braid group images coming from the simple objects of symmetric fusion categories and pointed braided fusion categories are easy to find because the square of the braiding on a simple object is a scalar multiple of the identity morphism. For any object X in a symmetric fusion category, X centralizes itself. For a simple object Y in a pointed braided fusion category, Y centralizes itself up to a scalar. This behavior has its own terminology.

Definition 5.3.1. We say that two objects X, Y in a braided fusion category $\mathcal{B}$ with braiding c *projectively centralize* each other if there is a scalar $\lambda_{X,Y} \in \Bbbk^\times$ such that $c_{Y,X} \circ c_{X,Y} = \lambda_{X,Y} \cdot \mathrm{id}_{X \otimes Y}$.

Example 5.3.2.

- It is clear from the definition (5.1.1) that a braided fusion category $\mathcal{B}$ is symmetric if and only if $\lambda_{X,Y} = 1$ for all $X, Y \in \mathcal{B}$.
- According to Remark 5.2.4, any two simple objects X, Y in a pointed braided fusion category $\mathcal{B}$ projectively centralize each other. Indeed, if X, Y have isomorphism classes

$g, h \in G$, respectively, and the braiding on $\mathcal{B}$ yields the quadratic form q, then $\lambda_{X,Y} = b_q(g,h)$ for b_q the associated bicharacter of q.

Lemma 5.3.3. [DGNO2, Lemma 3.15] *Let X, Y, Z be objects in a braided fusion category $\mathcal{B}$.*

(i) If X and Y projectively centrzlize each other, then

$$\lambda_{Y,X} = \lambda_{X,Y}. \tag{5.2}$$

(ii) If X, Y projectively centralize each other and X, Z projectively centralize each other, then X and $Y \otimes Z$ projectively centralize each other such that

$$\lambda_{X,Y\otimes Z} = \lambda_{X,Y}\lambda_{X,Z}. \tag{5.3}$$

(iii) If X, Y projectively centralize each other, then X, Y^* projectively centralize each other such that

$$\lambda_{X,Y^*} = \lambda_{X,Y}^{-1}. \tag{5.4}$$

Definition 5.3.4. A braided fusion category $\mathcal{B}$ with braiding c is called *projectively symmetric* if for every pair of simple objects X, Y in $\mathcal{B}$, there exists a nonzero scalar $\lambda_{X,Y} \in \Bbbk^\times$ such that

$$c_{Y,X} \circ c_{X,Y} = \lambda_{X,Y} \cdot \mathrm{id}_{X\otimes Y},$$

i.e. every pair of simple objects projectively centralize each other.

Projectively symmetric fusion categories arise in the study of graded symmetric fusion categories. In particular, the process of zesting a graded symmetric fusion category by an abelian 3-cocycle yields a projectively symmetric fusion category.

Definition 5.3.5. Let A be an abelian group. An *abelian 3-cocycle* on A is a pair (ω, c) of functions $\omega : A \times A \times A \to \Bbbk^\times$ and $c : A \times A \to \Bbbk^\times$ satisfying the equalities

$$\omega(a_1a_2, a_3, a_4)\omega(a_1, a_2, a_3a_4) = \omega(a_1, a_2, a_3)\omega(a_1, a_2a_3, a_4)\omega(a_2, a_3, a_4), \tag{5.5}$$

$$c(a_1, a_2a_3) = \omega(a_1, a_2, a_3)^{-1}c(a_1, a_2)\omega(a_2, a_1, a_3)c(a_1, a_3)\omega(a_2, a_3, a_1)^{-1}, \tag{5.6}$$

$$c(a_1a_2, a_3) = \omega(a_1, a_2, a_3)c(a_2, a_3)\omega(a_1, a_3, a_2)^{-1}c(a_1, a_3)\omega(a_3, a_1, a_2). \tag{5.7}$$

Let $\mathcal{B}$ be a symmetric fusion category graded by an abelian group A:

$$\mathcal{B} = \bigoplus_{a \in A} \mathcal{B}_a. \tag{5.8}$$

Let α and c be the associativity constraint and braiding for $\mathcal{B}$, respectively. As referred to in the literature, the *zesting* of $\mathcal{B}$ by an abelian 3-cocycle (ω, c) on A is the category $\tilde{\mathcal{B}}$ with associativity constraint $\tilde{\alpha}$ and braiding $\tilde{c}$ defined by

$$\tilde{\alpha}_{X,Y,Z} := \omega(a_1, a_2, a_3) \cdot \alpha_{X,Y,Z}, \qquad X \in \mathcal{B}_{a_1},\ Y \in \mathcal{B}_{a_2},\ Z \in \mathcal{B}_{a_3}, \tag{5.9}$$

$$\tilde{c}_{X,Y} := c(a_1, a_2) \cdot c_{X,Y}, \qquad X \in \mathcal{B}_{a_1},\ Y \in \mathcal{B}_{a_2}. \tag{5.10}$$

It can be seen that (5.9) and (5.10) give $\tilde{\mathcal{B}}$ the structure of a braided fusion category as a result of the Pentagon and Hexagon Axioms for α, c and equalities (5.5) - (5.7).

Proposition 5.3.6. *$\tilde{\mathcal{B}}$ is a projectively symmetric fusion category.*

Proof. For simple objects $X \in \mathcal{B}_{a_1}$, $Y \in \mathcal{B}_{a_2}$, we have

$$\tilde{c}_{Y,X} \circ \tilde{c}_{X,Y} = c(a_1, a_2)c(a_2, a_1) \cdot c_{Y,X} \circ c_{X,Y}.$$

Since $\mathcal{B}$ is symmetric, the result follows from (5.1). □

The pure braid group images coming from the simple objects of a projectively symmetric fusion category are considered as 'basic' as those coming from our previous two examples. Take a simple object X in a projectively symmetric fusion category $\mathcal{B}$. For $m \geq 2$, the image of a braid σ_i^2, $1 \leq i \leq m-1$, from P_m in $\mathrm{End}(X^{\otimes m})$ is $\lambda_{X,X} \cdot \mathrm{id}_{X \otimes X}$ possibly tensored with identity morphisms. From our discussions in Section 5.2, we can say the following.

Proposition 5.3.7. *For an object X in a projectively symmetric fusion category $\mathcal{B}$, $\rho_{m,X}(P_m)$ is a cyclic group of order $|\lambda_{X,X}|$ for every $m \geq 2$.*

Our next goal is to determine whether a projectively symmetric fusion category is necessarily integral. To do so, we require a few results and definitions from [DGNO2] and [ENO1].

Lemma 5.3.8. [DGNO2, Proposition 3.22] *For any simple objects X, Y in a braided fusion category $\mathcal{B}$, the following conditions are equivalent:*

1. *X centralizes $Y \otimes Y^*$;*
2. *$X \otimes X^*$ centralizes Y;*
3. *X and Y projectively centralize each other.*

Every pair of simple objects in a projectively symmetric fusion category $\mathcal{B}$ projectively centralize each other. Thus the objects $X \otimes X^*$, for X simple in $\mathcal{B}$, centralize every simple object in $\mathcal{B}$ by Lemma 5.3.8 and, by extension, every object in $\mathcal{B}$. There is a name for the collection of objects X in a braided fusion category $\mathcal{B}$ which centralize every object in $\mathcal{B}$:

Definition 5.3.9. For a braided fusion category $\mathcal{B}$ with braiding c, we denote by $\mathcal{B}'$ the *centralizer* of $\mathcal{B}$, the full braided fusion subcategory of $\mathcal{B}$ of objects that centralize every object of $\mathcal{B}$.

Remark 5.3.10. It follows from the definition that the centralizer of a symmetric fusion category is itself.

In a projectively symmetric fusion category $\mathcal{B}$, the subcategory generated by objects $X \otimes X^*$, $X \in \mathcal{O}(\mathcal{B})$, is a subcategory of $\mathcal{B}'$. This generated category has its own name:

Definition 5.3.11. For a fusion category $\mathcal{C}$, the smallest fusion subcategory of $\mathcal{C}$ containing all objects $X \otimes X^*$, for $X \in \mathcal{O}(\mathcal{C})$, is denoted $\mathcal{C}_{ad}$ and called the *adjoint subcategory* of $\mathcal{C}$.

Definition 5.3.12. For a fusion category $\mathcal{C}$, any two faithful gradings of $\mathcal{C}$ have a common refinement and we denote by $U_{\mathcal{C}}$ the *universal grading group* of $\mathcal{C}$.

Lemma 5.3.13. [DGNO2, Proposition 2.3.ii] *The trivial component of the universal grading equals $\mathcal{C}_{ad}$.*

Lemma 5.3.14. [DGNO2, Corollary 2.6, Remark 2.7] *There is a one-to-one correspondence between equivalence classes of faithful gradings of a braided fusion category $\mathcal{B}$ and fusion subcategories $\mathcal{D} \subset \mathcal{B}$ containing $\mathcal{B}_{ad}$. Namely, one associates to $\mathcal{D}$ the universal grading of $\mathcal{C}$ trivial on $\mathcal{D}$; one associates to a grading its trivial component.*

For a projectively symmetric fusion category $\mathcal{B}$, we have $\mathcal{B}_{ad} \subset \mathcal{B}'$. Thus by Lemma 5.3.14, there is an abelian group $A \subset U_{\mathcal{B}}$ such that $\mathcal{B} = \bigoplus_{a \in A} \mathcal{B}_a$ with trivial component $\mathcal{B}_e = \mathcal{B}'$.

Consider the function $b : A \times A \to \mathbb{k}^\times$ defined as $b(a_1, a_2) := \lambda_{X,Y}$ for $X \in \mathcal{B}_{a_1}$, $Y \in \mathcal{B}_{a_2}$.

Lemma 5.3.15. *The function b is a non-degenerate symmetric bicharacter.*

Proof. We must first show that b is well-defined. Let $X, X' \in \mathcal{B}_{a_1}$ and $Y, Y' \in \mathcal{B}_{a_2}$. The objects $X \otimes (X')^*$ and $Y \otimes (Y')^*$ lie in $\mathcal{B}_e = \mathcal{B}'$, so they centralize every object in $\mathcal{B}$. Since $X \otimes (X')^*$ centralizes Y, (5.2) - (5.4) give us $\lambda_{X,Y} = \lambda_{X',Y}$. Similarly, X' and $Y \otimes (Y')^*$ centralize each other, so $\lambda_{X',Y} = \lambda_{X',Y'}$. This proves b is well-defined.

The non-degeneracy of b follows from $\mathcal{B}_e = \mathcal{B}'$, and (5.2) and (5.3) give that b is symmetric and a bicharacter. □

Let $q : A \to \mathbb{k}^\times$ be a quadratic form such that $q(a_1 a_2) = b(a_1, a_2) q(a_1) q(a_2)$. It is a result of Eilenberg and MacLane that q corresponds to an abelian 3-cocycle (ω, c) such that

$c(a_1,a_2)c(a_2,a_1) = b(a_1,a_2)$. Using the abelian 3-cocycle (ω^{-1}, c^{-1}), we may 'zest' $\mathcal{B}$ to get a braided fusion category $\tilde{\mathcal{B}}$.

Lemma 5.3.16. *$\tilde{\mathcal{B}}$ is a symmetric fusion category.*

Proof. For $X \in \mathcal{B}_{a_1}$ and $Y \in \mathcal{B}_{a_2}$,

$$\tilde{c}_{Y,X} \circ \tilde{c}_{X,Y} = c(a_2,a_1)^{-1}c(a_1,a_2)^{-1} \cdot c_{Y,X} \circ c_{X,Y} = b(a_1,a_2)^{-1}\lambda_{X,Y} \cdot \mathrm{id}_{X\otimes Y} = \mathrm{id}_{X\otimes Y}.$$

□

This lemma shows that all projectively symmetric fusion categories can be found as zestings of graded symmetric fusion categories.

Proposition 5.3.17. *Projectively symmetric fusion categories are integral.*

Proof. A zesting $\tilde{\mathcal{B}}$ of a graded symmetric fusion category $\mathcal{B}$ has the same objects and fusion rules as $\mathcal{B}$. As a result, the Frobenius-Perron dimension of an object in $\tilde{\mathcal{B}}$ is the same as the Frobenius-Perron dimension of the object in $\mathcal{B}$. The result follows from Proposition 5.1.6. □

Now take a general braided fusion category $\mathcal{B}$ with braiding c and a simple object $X \in \mathcal{B}$ such that X projectively centralizes itself. We wish to prove that the Frobenius-Perron dimension of X is an integer, and we can follow the steps taken in Section 5.1. Namely, consider the category $\mathcal{B}[X]$.

Proposition 5.3.18. *$\mathcal{B}[X]$ is a projectively symmetric fusion category.*

Proof. Let $Y, Z \in \mathcal{O}(\mathcal{B}[X])$. Since X generates $\mathcal{B}[X]$ as a fusion category, there exist non-negative integers p, q such that Y, Z are subobjects of $X^{\otimes p}, X^{\otimes q}$, respectively. To show that Y, Z projectively centralize each other it is enough to show that $X^{\otimes p}$ and $X^{\otimes q}$ projectively centralize each other. This can be shown using the Hexagon Axioms and the fact that X projectively centralizes itself. □

Remark 5.3.19. It can be shown that if X projectively centralizes itself with associated scalar $\lambda_{X,X}$, then $X^{\otimes p}$ and $X^{\otimes q}$ projectively centralize each other with associated scalar $(\lambda_{X,X})^{pq}$.

Corollary 5.3.20. *For an object X in a braided fusion category $\mathcal{B}$, if X projectively centralizes itself then $\rho_{m,X}(P_m)$ are all isomorphic to the same cyclic group for $m \geq 2$ and $FPdim(X)$ is an integer.*

Proof. If X projectively centralizes itself, then it is an object of the projectively symmetric fusion category $\mathcal{B}[X]$ by Proposition 5.3.18. This category is integral by Proposition 5.3.17. □

5.4 From Braided Ising Categories

We now turn our attention toward a class of braided fusion categories which are not integral, but weakly integral: braided Ising categories. Given the discussion in the previous sections, we might anticipate that braided Ising categories yield more complicated pure braid group images. An object in a projectively symmetric fusion category has images which form a constant sequence consisting of a cyclic group. Images for a certain simple object in braided Ising categories will be proven non-abelian in general and increasing in size. This is due to a more complex braiding and monoidal structure.

We borrow from [DGNO2, Appendix B] the key notions and results required for future computations, and refer the reader for a more general treatment of Ising categories.

Definition 5.4.1. Let $\lambda, \zeta \in \Bbbk$ with $\lambda^2 = 2$ and $\zeta^8 = -1$. We define the *braided Ising category* $\mathcal{I} = \mathcal{I}(\lambda, \zeta)$ as follows:

(i) $\mathcal{I}$ is a fusion category with three simple objects: the unit object δ_0, an invertible object δ_1 not isomorphic to δ_0, and a non-invertible simple object X.

(ii) The tensor product is defined, for $a, b, a+b \in \mathbb{Z}/2\mathbb{Z}$, as

$$\delta_a \otimes \delta_b = \delta_{a+b}, \qquad \delta_a \otimes X = X \otimes \delta_a = X, \qquad X \otimes X = \delta_0 \oplus \delta_1 \tag{5.11}$$

(iii) The associativity isomorphism α is defined such that for $a, b, c \in \mathbb{Z}/2\mathbb{Z}$,

$$\alpha_{\delta_a, X, \delta_b} = (-1)^{ab} \cdot \mathrm{id}_X, \tag{5.12}$$

$$\alpha_{X,\delta_a,X} = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^a \end{pmatrix} : \delta_0 \oplus \delta_1 \xrightarrow{\sim} \delta_0 \oplus \delta_1, \tag{5.13}$$

$$\alpha_{X,X,X} = \begin{pmatrix} \lambda^{-1} & \lambda^{-1} \\ \lambda^{-1} & -\lambda^{-1} \end{pmatrix} : X \oplus X \xrightarrow{\sim} X \oplus X, \tag{5.14}$$

and $\alpha_{\delta_a,\delta_b,\delta_c}$, $\alpha_{\delta_a,\delta_b,X}$, $\alpha_{X,\delta_a,\delta_b}$, $\alpha_{\delta_a,X,X}$, and α_{X,X,δ_a} are identity morphisms.

(iv) The braiding c is defined such that for $a, b, a+b \in \mathbb{Z}/2\mathbb{Z}$,

$$c_{\delta_a,\delta_b} = (-1)^{ab} \cdot \mathrm{id}_{\delta_{a+b}}, \qquad c_{\delta_a,X} = c_{X,\delta_a} = \zeta^{4a} \cdot \mathrm{id}_X, \tag{5.15}$$

$$c_{X,X} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-3} \end{pmatrix} : \delta_0 \oplus \delta_1 \xrightarrow{\sim} \delta_0 \oplus \delta_1 \tag{5.16}$$

Proposition 5.4.2. [DGNO2, Proposition B.3] *For the braided Ising category $\mathcal{I}$, FPdim(δ_0) = FPdim$(\delta_1) = 1$, FPdim$(X) = \sqrt{2}$, and FPdim$(\mathcal{I}) = 4$.*

Since the dimension of $\mathcal{I}$ is a power of 2, Proposition 4.2.23 gives us that the pure braid group images coming from $\mathcal{I}$ are 2-groups.

Remark 5.4.3. Note that according to (5.15), δ_a centralizes itself for $a \in \mathbb{Z}/2\mathbb{Z}$, so the pure braid group images coming from δ_a are all trivial.

Consider the non-invertible simple object X. The image $\rho_{2,X}(P_2)$ is generated by the double braiding $c_{X,X} \circ c_{X,X}$. The order of this morphism is the order of ζ^2, which is 8. Thus $\rho_{2,X}(P_2) \cong \mathbb{Z}/8\mathbb{Z}$.

For $m \geq 3$, finding the image $\rho_{m,X}(P_m)$ requires a bit more effort. This is the first case where the square of the braiding $c_{X,X}$ (5.16) is not a scalar multiple of the identity morphism, so the associativity isomorphisms play a role. Since we have matrix representations (5.12) - (5.16) for the associativity isomorphisms and braidings on simple objects, we aim to discuss the pure braid group images as matrix groups.

The matrices we will see in the images $\rho_{m,X}(P_m)$ have a particular structure related to a family of matrix groups G_{2^m}, $m \geq 1$, over $\Bbbk$ which we now define. Let G_2 be the multiplicative matrix group generated by the matrices

$$z = \zeta^2 I_2 = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.17}$$

The matrix z is central in G_2 and $TS = -ST = z^4 ST$. In general, every element of G_2 can be described uniquely as a product of a power of z and exactly one of the following matrices:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad ST = -TS = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.18}$$

Remark 5.4.4. The choice of denoting the matrices as S and T is that the former represents a change in the Sign of the entries, and the latter represents a Translation of the entries from diagonal to anti-diagonal, but with both matrices having nonzero entry 1 in the first row.

Remark 5.4.5. It is clear from (5.16) and (5.17) that $\rho_{2,X}(P_2) < G_2$ with generator zS.

Definition 5.4.6. For $m > 1$, we define the matrix groups G_{2^m} inductively as follows. G_{2^m}

is the group of $2^m \times 2^m$ matrices of the form

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}, \quad \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}, \quad \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix}$$

with $M \in G_{2^{m-1}}$. Namely, every matrix M' of G_{2^m} is described uniquely by a block $M \in G_{2^{m-1}}$, whether M' is block diagonal or block anti-diagonal, and by whether the blocks have the same sign or not in M'.

Remark 5.4.7. Another way of describing G_{2^m} is that it consists of all matrices of the form $I_2 \otimes M$, $S \otimes M$, $T \otimes M$, and $ST \otimes M$ for $M \in G_{2^{m-1}}$, where $\otimes$ denotes the Kronecker product of matrices.

Proposition 5.4.8. *For $m \geq 1$, the group G_{2^m} has generators S_i, T_i for $0 \leq i \leq m-1$ and $z = \zeta^2 I_{2^m}$ such that $z^8 = S_i^2 = T_i^2 = I_{2^m}$, $T_i S_i = z^4 S_i T_i$, and all other pairs of generators commute.*

Proof. We prove the proposition with induction on m. This has been shown for the case when $m = 1$ with $S_0 = S$ and $T_0 = T$. Suppose this proposition holds for some G_{2^n} in general. The matrices in $G_{2^{n+1}}$ are described uniquely as one of $I_2 \otimes M$, $S \otimes M$, $T \otimes M$, or $ST \otimes M$ for some $M \in G_{2^n}$. Given the generators $z, S_0, \ldots, S_{n-1}, T_0, \ldots, T_{n-1}$ of G_{2^n}, define

$$z' := I_2 \otimes z = \zeta^2 I_{2^{n+1}}, \qquad S_i' := I_2 \otimes S_i, \qquad T_i' := I_2 \otimes T_i.$$

Also define $S_n' = S \otimes I_{2^{n+1}}$ and $T_n' = T \otimes I_{2^{n+1}}$.

It is clear from the definition of $G_{2^{n+1}}$ and the properties of the Kronecker product that $z', S_0', \ldots, S_n', T_0', \ldots, T_n'$ generate $G_{2^{n+1}}$. Definitions (5.17) give us $(S_n')^2 = (T_n')^2 = I_{2^{n+1}}$ and $T_n' S_n' = -S_n' T_n' = (z')^4 S_n' T_n'$. Finally S_n' and T_n' commute with every matrix of the form $I_2 \otimes M$ for $M \in G_{2^n}$. $\square$

Remark 5.4.9. With the description of the groups G_{2^m} coming from Proposition 5.4.8 it is clear that for any m, we have injections $G_{2^m} \hookrightarrow G_{2^{m+1}}$. The injection sends a matrix $M \in G_{2^m}$ to $I_2 \otimes M \in G_{2^{m+1}}$. This also gives us a helpful description for the generators $S_i, T_i \in G_{2^m}$, $1 \leq i \leq m-1$, when doing computations later: $S_i = I_{2^{m-i-1}} \otimes S \otimes I_{2^i}$ and $T_i = I_{2^{m-i-1}} \otimes T \otimes I_{2^i}$.

For a fixed m, it can be seen from (5.11) - (5.14) and (5.16) that the images $\rho_{2m,X}(P_{2m})$ and $\rho_{2m+1,X}(P_{2m+1})$ are represented by matrices in $\mathrm{GL}_{2^m}(\Bbbk)$. In general, isomorphisms between $2m$-fold tensor products of X or between $(2m-1)$-fold tensor products of X are also represented by matrices in $\mathrm{GL}_{2^m}(\Bbbk)$ by (5.11). For $0 \leq r \leq m-1$, let A_r be the matrix

$$A_r = I_{2^{m-r-1}} \otimes \begin{pmatrix} \lambda^{-1} & \lambda^{-1} \\ \lambda^{-1} & -\lambda^{-1} \end{pmatrix} \otimes I_{2^r}.$$

The matrix A_r represents the associativity isomorphism of the form $\alpha_{X,X^{\otimes(2m-2r-1,2m-2r-2)},X}$ (where $\alpha_{X,X^{\otimes(1,0)},X} := \alpha_{X,X,X}$), tensored appropriately by identity morphisms. For $0 \leq s \leq m-2$, let B_s be the matrix

$$B_s = I_{2^{m-s-2}} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes I_{2^s}.$$

The matrix B_s represents the associativity isomorphism $\alpha_{X,X^{\otimes(2m-2s-2,2m-2s-3)},X}$, tensored appropriately by identity morphisms. Define the matrix C as

$$C = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-3} \end{pmatrix} \otimes I_{2^{m-1}}.$$

The matrix C represents the braiding (5.16), tensored appropriately by identity morphisms.

It can be shown that every A_r and B_s has order 2.

The group G_{2^m} has some useful properties as a subgroup of $\mathrm{GL}_{2^m}(\Bbbk)$. We will be interested in whether G_{2^m} is closed under conjugation by the matrices A_r, B_s and C, as generators of pure braid group images are described as conjugates involving associativity isomorphisms and braidings.

Lemma 5.4.10. *Fix $m \geq 1$ and consider the group G_{2^m}. Let $0 \leq r \leq m-1$ and $0 \leq s \leq m-2$.*

(i) $A_r S_r A_r^{-1} = T_r$*,* $A_r T_r A_r^{-1} = S_r$*, and* A_r *commutes with all other generators of* G_{2^m}

(ii) $B_s T_s B_s^{-1} = T_s S_{s+1}$*,* $B_s T_{s+1} B_s^{-1} = S_s T_{s+1}$*, and* B_s *commutes with all other generators of* G_{2^m}

(iii) $C T_{m-1} C^{-1} = z^2 S_{m-1} T_{m-1}$ *and* C *commutes with all other generators of* G_{2^m}

In particular, G_{2^m} *is invariant under conjugation by the matrices* A_r*,* B_s*, and* C*.*

Proof. To simplify computations, denote

$$A = \begin{pmatrix} \lambda^{-1} & \lambda^{-1} \\ \lambda^{-1} & -\lambda^{-1} \end{pmatrix}, \qquad B = \mathrm{diag}\{1,1,1,-1\}, \qquad C' = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-3} \end{pmatrix}$$

Using the properties of the Kronecker product of matrices, it is enough (and easy) to show that

$$A^{-1} S A = T, \qquad A^{-1} T A = S,$$

$$B^{-1}(I_2 \otimes T)B = S \otimes T, \qquad B^{-1}(T \otimes I_2)B = T \otimes S,$$

$$C' T (C')^{-1} = \zeta^4 S T.$$

□

We are now ready to find the pure braid group images $\rho_{m,X}(P_m)$ for $m \geq 3$, but we will consider the cases when m is odd and even separately.

Proposition 5.4.11. *For $m \geq 1$, $\rho_{2m+1,X}(P_{2m+1}) \cong G_{2^m}$.*

Proof. Using the definitions (5.13) - (5.14) and (5.16), it can be shown that

$$\rho_{2m+1,X}(\sigma_1^2) = zS_{m-1} \in G_{2^m},$$

$$\rho_{2m+1,X}(\sigma_{2i-1}^2) = zS_{m-i}S_{m-i+1} \in G_{2^m}, \qquad \text{for } 1 \leq i \leq m,$$

$$\rho_{2m+1,X}(\sigma_{2i}^2) = zT_{m-i} \in G_{2^m}, \qquad \text{for } 1 \leq i \leq m,$$

$$\rho_{2m+1,X}(\sigma_2\sigma_1^2\sigma_2^{-1}) = z^7 S_{m-1}T_{m-1} \in G_{2^m}.$$

Since $\rho_{2m+1,X}(\sigma_1^2)\rho_{2m+1,X}(\sigma_2\sigma_1^2\sigma_2^{-1})\rho_{2m+1,X}(\sigma_2^2) = z$, the image $\rho_{2m+1,X}(P_{2m+1})$ contains all the generators of G_{2^m} and $G_{2^m} < \rho_{2m+1,X}(P_{2m+1})$.

We must show that all other generators of P_{2m+1} get mapped into G_{2^m}. Note that each $\rho_{2m+1,X}(\sigma_i^2)$ is a conjugate of the double braiding $c_{X,X} \circ c_{X,X}$ (tensored with identity morphisms) by associativity isomorphisms (possibly tensored with identity morphisms). Thus the image of any generator of P_{2m+1} is a conjugate of this double braiding by associativity morphisms and single braidings. The double braiding is represented by $zS_{m-1} \in G_{2^m}$, so the proposition is proven if we can show that conjugation by associativity isomorphisms and single braidings, when represented by matrices in $\mathrm{GL}_{2^m}(\Bbbk)$, leaves G_{2^m} invariant.

The associativity isomorphisms can be chosen so they are of the form $\alpha_{X,X^{\otimes(k,k-1)},X}$, for $1 \leq k \leq 2m-1$, tensored appropriately with identity morphisms, as a result of MacLane's Coherence Theorem. Conjugating by associativity isomorphism of these types is the same as conjugating in $\mathrm{GL}_{2^m}(\Bbbk)$ by the matrices A_r, B_s, $0 \leq r \leq m-1$, $0 \leq s \leq m-2$. Conjugating by a single braiding $c_{X,X}$, tensored by identity morphisms, is the same as conjugating in $\mathrm{GL}_{2^m}(\Bbbk)$ by the matrix C. The proposition follows from Lemma 5.4.10. □

Proposition 5.4.12. *For $m \geq 2$, $\rho_{2m,X}(P_{2m}) \cong G_{2^m}/\langle T_0 \rangle$.*

Proof. Using the definitions (5.13) - (5.14) and (5.16), it can be shown that

$$\rho_{2m,X}(\sigma_1^2) = zS_{m-1} \in G_{2^m},$$

$$\rho_{2m,X}(\sigma_{2i-1}^2) = zS_{m-i}S_{m-i+1} \in G_{2^m}, \qquad \text{for } 1 \leq i \leq m,$$

$$\rho_{2m,X}(\sigma_{2i}^2) = zT_{m-i} \in G_{2^m}, \qquad \text{for } 1 \leq i \leq m-1,$$

$$\rho_{2m,X}(\sigma_2\sigma_1^2\sigma_2^{-1}) = z^7 S_{m-1}T_{m-1} \in G_{2^m}.$$

Since $\rho_{2m,X}(\sigma_1^2)\rho_{2m,X}(\sigma_2\sigma_1^2\sigma_2^{-1})\rho_{2m,X}(\sigma_2^2) = z$, the image $\rho_{2m,X}(P_{2m})$ contains all the generators of G_{2^m} except possibly T_0.

We will show that all other generators of P_{2m} get mapped into G_{2^m} and can be written as a product of generators not including T_0. Similar to the proof of Proposition 5.4.11, it is enough to show that the subgroup of G^{2^m} generated by $z, S_0, \ldots, S_{m-1}, T_1, \ldots, T_{m-1}$ is invariant under conjugation by the appropriate associativity isomorphisms and single braidings, when represented as matrices in $\mathrm{GL}_{2^m}(\Bbbk)$.

The associativity isomorphisms can be chosen so they are of the form $\alpha_{X,X^{\otimes(k,k-1)},X}$, for $1 \leq k \leq 2m-2$, tensored appropriately with identity morphisms, as a result of MacLane's Coherence Theorem. Conjugating by associativity isomorphism of these types is the same as conjugating in $\mathrm{GL}_{2^m}(\Bbbk)$ by the matrices A_r, B_s, $1 \leq r \leq m-1$, $0 \leq s \leq m-2$. Conjugating by a single braiding $c_{X,X}$, tensored by identity morphisms, is the same as conjugating in $\mathrm{GL}_{2^m}(\Bbbk)$ by the matrix C. The proposition follows from Lemma 5.4.10. □

This gives us a description for all pure braid group images coming from $X \in \mathcal{I}$. Where does this case fit into the bigger picture? The object X has Frobenius-Perron dimension $\sqrt{2}$, so this is the first case so far of pure braid group images coming from a simple object with non-integer dimension. The pure braid group images coming from X are certainly more complex than those in the previous sections. Indeed, the images for $m > 2$ are not cyclic groups, and they are not all isomorphic. Instead, they are non-abelian groups of increasing size. The sizes of these images can be found easily:

Proposition 5.4.13. *Let X be the non-invertible simple object in $\mathcal{I}$. Then $\rho_{2,X}(P_2) \cong \mathbb{Z}/8\mathbb{Z}$ and for $m > 2$, $\rho_{m,X}(P_m)$ is isomorphic to a central extension of $(\mathbb{Z}/2\mathbb{Z})^{m-1}$ by $\mathbb{Z}/8\mathbb{Z}$ and $|\rho_{m,X}(P_m)| = 2^{m+2}$.*

Proof. The group G_{2^n} for $n \geq 1$ has a cyclic center generated by z of order 8. Modulo this center, G_{2^n} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2n}$ as the generators S_i, T_i, $0 \leq i \leq n-1$, pairwise commute up to a potential sign. The result follows from Propositions 5.4.11 and 5.4.12. □

Unlike the objects coming from projectively symmetric fusion categories, there may be a relationship between the Frobenius-Perron dimension of X and the orders of the pure braid group images: the dimension is $\sqrt{2}$, and the orders of the images are powers of 2. In fact, the ratio of $|\rho_{m+1,X}(P_{m+1})|$ to $|\rho_{m,X}(P_m)|$, for $m > 2$, is always 2.

5.5 From the Center of $\mathbf{Vec}_{D_{2p}}$

Our final example comes from an object with integral Frobenius-Perron dimension, but with pure braid group images of a much higher complexity than seen in the projectively symmetric fusion category case. We will discuss any relationship between the pure braid group images and the dimension of the object, and find connections similar to those in the braided Ising category case.

Let p be an odd prime, and denote by D_{2p} the dihedral group with $2p$ elements (i.e. the symmetry group of the regular p-gon with rotations r^i and reflections sr^i, $0 \leq i \leq p-1$). Consider the braided fusion category $\mathcal{B} = \mathcal{Z}(\mathrm{Vec}_{D_{2p}})$, the center of the category of D_{2p}-graded finite-dimensional vector spaces. We will focus on the pure braid group images coming from the object $X = (V, \gamma)$, where $V = \bigoplus_{i=0}^{p-1} sr^i$ and for $g \in D_{2p}$, $\gamma_g : g \otimes V \xrightarrow{\sim} V \otimes g$ is given by the permutation of summands $g \otimes sr^i \cong sr^j \otimes g$, where $sr^j = g \cdot sr^i \cdot g^{-1}$. It can be seen that X is simple in $\mathcal{B}$.

The braiding is defined as $c_{X,X} = \gamma_V$, where the summand $sr^i \otimes sr^j$ of $V \otimes V$ is mapped to $sr^{2i-j} \otimes sr^i$ (since $sr^i \cdot sr^j \cdot (sr^i)^{-1} = sr^{2i-j}$). We can view the action of a simple braid on $V \otimes V$ as an action on $\mathbb{F}_p^2$ which sends the pair (i, j) to $(2i - j, i)$, represented by the matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

over the field $\mathbb{F}_p$. Thus for a fixed $m \geq 2$, the image $\rho_{m,X}(\sigma_i)$, $1 \leq i \leq m-1$, can be represented by the $m \times m$ matrix

$$u_i = \mathrm{Id}_{i-1} \oplus \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \oplus \mathrm{Id}_{m-i-1} \tag{5.19}$$

with elements in $\mathbb{F}_p$.

These generators (5.19) are related to another set of representations of the braid group B_m: the Burau representations.

Definition 5.5.1. For a fixed $m \geq 2$, the *unreduced Burau representation* of B_m over a field $\Bbbk$ with respect to an indeterminate t is defined such that the image of a simple braid σ_i, $1 \leq i \leq m-1$, is the $m \times m$ matrix

$$u_{i,t} = \mathrm{Id}_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus \mathrm{Id}_{m-i-1} \tag{5.20}$$

over $\Bbbk[t, t^{-1}]$.

The generators (5.19) are exactly the generators (5.20) for the image of the unreduced Burau representation of B_m with $\Bbbk = \mathbb{F}_p$ and $t = -1$. Thus, $\rho_{m,X}(B_m)$ is isomorphic to the image of B_m under this reduced Burau representation. We will refer to both groups as $\rho_{m,X}(B_m)$.

There is another Burau representation whose image is a quotient of the image of the unreduced Burau representation:

Definition 5.5.2. For a fixed $m \geq 3$, the *reduced Burau representation* of B_m over a field $\Bbbk$ with respect to an indeterminate t is defined such that the images of the simple braids are given by the $(m-1) \times (m-1)$ matrices

$$\sigma_1 \mapsto r_{1,t} = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus \mathrm{Id}_{m-3},$$

$$\sigma_i \mapsto r_{i,t} = \mathrm{Id}_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathrm{Id}_{m-i-2}, \qquad 2 \leq i \leq m-2,$$

$$\sigma_{m-1} \mapsto r_{m-1,t} = \mathrm{Id}_{m-3} \oplus \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}, \tag{5.21}$$

over $\Bbbk[t, t^{-1}]$. For $m = 2$, it is defined so that $\sigma_1 \mapsto r_{1,t} = (-t)$.

Remark 5.5.3. We will refer to the image of σ_i under the reduced Burau representation over $\mathbb{F}_p$ with $t = -1$ simply by $r_i = r_{i,-1}$.

Let us begin computing the braid group images coming from X. To start, $\rho_{2,X}(B_2)$ is generated by the matrix $u_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ and it can be shown that this matrix has order p. Thus $\rho_{2,X}(B_2) \cong \mathbb{Z}/p\mathbb{Z}$.

Remark 5.5.4. Up until now, we have not shifted our focus to the images of the pure braid groups. This is intentional, and it is due to the fact that the image of any simple braid σ_i has odd order in this case. If the image of σ_i^2 is in $\rho_{m,X}(P_m)$, then

$$\rho_{m,X}(\sigma_i^2)^{\frac{p+1}{2}} = \rho_{m,X}(\sigma_i)^{p+1} = \rho_{m,X}(\sigma_i) \in \rho_{m,X}(P_m).$$

Thus, $\rho_{m,X}(P_m)$ can be generated by the images of the simple braids and $\rho_{m,X}(P_m) \cong \rho_{m,X}(B_m)$. We do not gain any information by focusing on the images of the pure braid groups in this case, so we will keep with the entire braid groups B_m.

We now focus on the images $\rho_{2m+1,X}(B_{2m+1})$ for $m \geq 1$. First, an observation about the Burau representations over $\mathbb{F}_p$ with $t = -1$.

Proposition 5.5.5. *For $m \geq 1$, the images of the unreduced Burau representation and reduced Burau representation of B_{2m+1} over $\mathbb{F}_p$ with $t = -1$ are isomorphic.*

Proof. Suppose that the matrices $u_1, \dots, u_{2m}$ coming from the unreduced Burau representation are defined over a basis $\{e_1, \dots, e_{2m+1}\}$. Consider the following vectors:

$$f_0 = \sum_{j=1}^{2m+1} e_j, \qquad f_k = (-1)^{k+1} e_k + (-1)^{k+1} e_{k+1}, \quad 1 \leq k \leq 2m.$$

The set $F = \{f_0, \dots, f_{2m}\}$ is linearly independent, and we have $u_i = \mathrm{Id}_1 \oplus r_i$ over F for $1 \leq i \leq 2m$. Thus the unreduced and reduced Burau images of B_{2m+1} are isomorphic over $\mathbb{F}_p$ with $t = -1$. □

To find $\rho_{2m+1,X}(B_{2m+1})$, it is enough to find the group generated by $r_1, \dots, r_{2m}$:

$$r_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \mathrm{Id}_{2m-2},$$

$$r_i = \mathrm{Id}_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathrm{Id}_{2m-i-1}, \qquad 2 \leq i \leq 2m-1,$$

$$r_{2m} = \mathrm{Id}_{2m-2} \oplus \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{5.22}$$

over $\mathbb{F}_p$. They form a subgroup of $\mathrm{GL}_{2m}(\mathbb{F}_p)$. To motivate the remainder of this discussion, we introduce another subgroup of $\mathrm{GL}_{2m}(\mathbb{F}_p)$: the symplectic group. We refer the reader to [Gr] for a more in-depth look at classical matrix groups such as this.

Let W be a $2m$-dimensional vector space over a field $\Bbbk$ of characteristic not equal to 2, and B a non-degenerate skew-symmetric bicharacter on W with values in $\Bbbk$.

Definition 5.5.6. We say that a pair of linearly independent vectors $u, v \in W$ form a *hyperbolic pair of* B if $B(u, v) = 1$.

Definition 5.5.7. If there exists a basis $\{u_1, v_1, \dots, u_m, v_m\}$ of W such that each pair (u_i, v_i) is a hyperbolic pair of B, we call this basis a *symplectic basis* for W and we say that W is a *symplectic vector space.*

Definition 5.5.8. An invertible linear operator τ on a symplectic vector space W with associated bicharacter B is said to be *symplectic* if $B(\tau(u), \tau(v)) = B(u, v)$ for all $u, v \in W$. The group of all such symplectic transformations of W is denoted $\mathrm{Sp}(W)$.

Remark 5.5.9. If we replace B with some other non-degenerate skew-symmetric bicharacter B' on W, the symplectic group with respect to B' is conjugate in $\mathrm{GL}(W)$ to the symplectic group with respect to B. So relative to appropriately chosen bases, the two groups would be represented by the same matrices in $\mathrm{GL}_{2m}(\Bbbk)$.

This remark allows us to define a matrix group $\mathrm{Sp}_{2m}(\Bbbk)$ as *the* symplectic group of size $2m$ over the field $\Bbbk$.

Definition 5.5.10. Let B be a non-degenerate skew-symmetric $2m \times 2m$ matrix over a field $\Bbbk$. The symplectic group $\mathrm{Sp}_{2m}(\Bbbk)$ is the group of all matrices $M \in \mathrm{GL}_{2m}(\Bbbk)$ such that $M^t B M = B$. If $\Bbbk$ is a finite field of order q, we denote the symplectic group as $\mathrm{Sp}_{2m}(q)$.

Proposition 5.5.11. [Gr, Theorem 3.12] *If $\Bbbk$ is a finite field of order q, then*

$$|Sp_{2m}(q)| = \prod_{i=1}^{m} q^{2i-1}(q^{2i} - 1) = q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1)$$

Remark 5.5.12. While B can be any non-degenerate skew-symmetric $2m \times 2m$ matrix, there are a couple of common choices for B when performing computations. The first is

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

The other, which we will denote by Ω, is defined such that $\Omega_{i,i+1} = 1$ and $\Omega_{i+1,i} = -1$ for $1 \leq i \leq 2m-1$.

The definition we have for $\mathrm{Sp}_{2m}(\mathbb{k})$ makes it relatively easy to determine whether a matrix M is symplectic. What is still difficult is determining whether a general matrix group of symplectic matrices is the entire symplectic group. It would help to know how to generate symplectic groups, and we do have some information regarding this.

Definition 5.5.13. For a non-identity symplectic transformation $\tau \in \mathrm{Sp}(W)$, we say that τ is a *symplectic transvection* with fixed one-dimensional subspace $U \subset W$ if $\tau|_U = \mathrm{id}_U$ and $\tau(v) - v \in U$ for all $v \in W$.

We now return to determining the structure of $\rho_{2m+1,X}(B_{2m+1})$ as generated by the matrices (5.22). Let Ω be the $2m \times 2m$ matrix defined above. This is a matrix whose entries are zero except entries of 1 on the main superdiagonal (the diagonal above the main diagonal) and entries of -1 on the main subdiagonal (the diagonal below the main diagonal). It can be shown that for each i, $1 \leq i \leq 2m$, we have $r_i^t \Omega r_i = \Omega$. Thus, the group generated by $r_1, \ldots, r_{2m}$ is a subgroup of some conjugate of $\mathrm{Sp}_{2m}(p)$ in $\mathrm{GL}_{2m}(\mathbb{F}_p)$.

Let W be a symplectic space with basis $\{f_1, \ldots, f_{2m}\}$ such that $r_i \in \mathrm{Sp}(W)$ for $1 \leq i \leq 2m$.

Proposition 5.5.14. *The matrices r_i are symplectic transvections.*

Proof. For each i, $1 \leq i \leq 2m$, let U_i be the one-dimensional subspace of W on the basis element f_i. Note that

$$r_i(f_i) = f_i, \qquad 1 \leq i \leq 2m,$$

$$r_i(f_{i-1}) - f_{i-1} = -f_i, \qquad 2 \le i \le 2m,$$

$$r_i(f_{i+1}) - f_{i+1} = f_i, \qquad 1 \le i \le 2m-1,$$

$$r_i(f_j) - f_j = 0, \qquad 1 \le j \le 2m, \quad |i-j| > 1.$$

Thus $r_i|_{U_i} = \mathrm{id}_{U_i}$ and $r_i(v) - v \in U_i$ for all $v \in W$. □

The generators (5.22) for $\rho_{2m+1,X}(B_{2m+1})$ are symplectic transvections. It is also clear from the previous proposition that the group $\rho_{2m+1,X}(B_{2m+1})$ is *irreducible*, meaning the only invariant subspaces of W under the action of $\rho_{2m+1,X}(B_{2m+1})$ are the trivial subspace and itself. The following theorem is applicable.

Theorem 5.5.15. [SZ, Main Theorem 4.11] *Suppose $G < GL_n(\Bbbk)$ is an irreducible group generated by transvections. Suppose also that $\Bbbk$ is a finite field of characteristic $p > 2$ and that $n > 2$. Then G is conjugate in $GL_n(\Bbbk)$ to one of the groups $SL_n(q)$, $Sp_n(q)$, or $SU_n(q)$, where q is a subfield of $\Bbbk$.*

Proposition 5.5.16. *For $m \ge 1$, $\rho_{2m+1,X}(B_{2m+1}) \cong Sp_{2m}(p)$.*

Proof. Proposition 5.5.14 proves that $\rho_{2m+1,X}(B_{2m+1})$ is generated by symplectic transvections and is irreducible. Since $\rho_{2m+1,X}(B_{2m+1}) < \mathrm{GL}_{2m}(p)$, the result follows from Theorem 5.5.15. □

Moving on to the images of the even braid groups, we stop to notice that we have a chain of groups

$$\rho_{2,X}(B_2) < \rho_{3,X}(B_3) < \cdots < \rho_{2m-1,X}(B_{2m-1}) < \rho_{2m,X}(B_{2m}) < \rho_{2m+1,X}(B_{2m+1}) < \cdots$$

$$\mathbb{Z}/p\mathbb{Z} < \mathrm{Sp}_2(p) < \cdots < \mathrm{Sp}_{2m-2}(p) < \rho_{2m,X}(B_{2m}) < \mathrm{Sp}_{2m}(p) < \cdots$$

So $\rho_{2m,X}(B_{2m})$ for $m \ge 2$ contains an isomorphic copy of $\mathrm{Sp}_{2m-2}(p)$ as a subgroup, and is isomorphic to a subgroup of $\mathrm{Sp}_{2m}(p)$.

Fix some $m \geq 2$. Here is a way to visualize $\mathrm{Sp}_{2m-2}(p)$ as a subgroup of $\rho_{2m,X}(B_{2m})$. Take the generators (5.19) over the basis $e_1, ..., e_{2m}$ as before. Define a new basis

$$g_0 = \sum_{j=1}^{2m-1} e_j, \qquad g_k = (-1)^{k+1} e_k + (-1)^{k+1} e_{k+1}, \quad 1 \leq k \leq 2m-2, \qquad g_{2m-1} = e_1$$

Note that this is not quite the same basis as the f_i we had earlier. In particular, we have to set the final basis element g_{2m-1} not to be $e_{2m-1} + e_{2m}$ since this vector is the same as $g_0 - g_1 - g_3 - ... - g_{2m-3}$. Over this basis, our generators look like

$$u_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & & & & 1 \\ 0 & & & & 0 \\ \vdots & & r_1 & & \vdots \\ 0 & & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$u_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad 2 \leq i \leq 2m-2,$$

$$u_{2m-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 & 0 \\ 0 & & & & & 0 \\ \vdots & & A & & & \vdots \\ 0 & & & & & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \tag{5.23}$$

where A is the $(2m-2)\times(2m-2)$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The matrices $r_1,\ldots,r_{2m-2}$ generate $\rho_{2m-1,X}(B_{2m-1})\cong \mathrm{Sp}_{2m-2}(p)$ by (5.22) and Proposition 5.5.16. It can also be shown that $A\in\langle r_1,...,r_{2m-2}\rangle$ so every element in $\rho_{2m,X}(B_{2m})$ can be represented, over a the basis $\{g_0,\ldots,g_{2m-1}\}$, by a matrix with a 'core' which is an element of $\mathrm{Sp}_{2m-2}(p)$.

For $y\in\mathbb{F}_p$ and $\vec{x},\vec{z}\in\mathbb{F}_p^{2m-2}$, denote by $(M,\vec{x},y,\vec{z})$ the matrix

$$\begin{pmatrix} 1 & \vec{x} & y \\ 0 & M & \vec{z} \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.24}$$

where $M\in\mathrm{Sp}_{2m-2}(p)$. The product of two such matrices is

$$(M_1,\vec{x}_1,y_1,\vec{z}_1)\cdot(M_2,\vec{x}_2,y_2,\vec{z}_2)=(M_1M_2,\vec{x}_2+\vec{x}_1M_2,y_1+\vec{x}_1\bullet\vec{z}_2+y_2,M_1\vec{z}_2+\vec{z}_1), \tag{5.25}$$

where $\vec{x}_1\bullet\vec{z}_2$ is the usual inner product. Denote by Z_{2m} the group of possible matrices $(I_{2m-2},\vec{x},y,\vec{z})\in\rho_{2m,X}(B_{2m})$. We turn our attention to learning more about this Z_{2m}.

Lemma 5.5.17. *Any matrix of the form $(I_{2m-2},\vec{x},y,\vec{z})$ has order p.*

Proof. It can be shown that

$$(I_{2m-2}, \vec{x}, y, \vec{z})^k = \left(I_{2m-2}, k\vec{x}, ky + \frac{k(k-1)}{2}\vec{x} \bullet \vec{z}, k\vec{z}\right)$$

so $(I_{2m-2}, \vec{x}, y, \vec{z})^p = (I_{2m-2}, \vec{0}, 0, \vec{0}) = I_{2m} \in \rho_{2m,V}(B_{2m})$. □

Thus Z_{2m} is a p-group with exponent p. Given that $\rho_{2m,V}(B_{2m}) < \mathrm{Sp}_{2m}(p)$, we can use the order of $\mathrm{Sp}_{2m}(p)$ and $\mathrm{Sp}_{2m-2}(p)$ from Proposition 5.5.11 to determine that Z_{2m} is a p-group of order at most p^{2m-1}.

Proposition 5.5.18. *Z_{2m} is isomorphic to a central extension of $\mathbb{Z}/p\mathbb{Z}$ by $(\mathbb{Z}/p\mathbb{Z})^{2m-2}$. Specifically, Z_{2m} is a non-abelian p-group of order p^{2m-1} with center $\mathbb{Z}/p\mathbb{Z}$ and exponent p.*

Proof. To prove this, denote by α_i, $1 \leq i \leq 2m-2$, the matrix $(I_{2m-2}, e_i, 0, \vec{z}_i)$, where

$$\vec{z}_{2j-1} = \sum_{k=j}^{m-1} -e_{2j}, \qquad \vec{z}_{2j} = \sum_{k=1}^{j} e_{2j-1}, \qquad 1 \leq j \leq m-1. \tag{5.26}$$

Denote by γ the matrix $(I_{2m-2}, \vec{0}, 1, \vec{0})$. From (5.26), we have that γ commutes with every α_i, and every pair α_i, α_j either commute or $[\alpha_i, \alpha_j] = \gamma^{\pm 2}$. So the group $\langle \alpha_1, \ldots, \alpha_{2m-2}, \gamma \rangle$ is a non-abelian p-group isomorphic to an extension of $\mathbb{Z}/p\mathbb{Z}$ by $(\mathbb{Z}/p\mathbb{Z})^{2m-2}$ and has exponent p. We will show that the group generated by the α_i and γ is a subgroup of Z_{2m}, thus it must be equal.

To do this, we need to show that every α_i and γ are in $\rho_{2m,X}(B_{2m})$. For $1 \leq i \leq j \leq 2m-1$ define $q(i,j)$ such that $q(i,i) = u_i$ and for $i < j$,

$$q(i,j) = u_i(u_{i+1}u_i)(u_{i+2}u_{i+1}u_i)\cdots(u_j u_{j-1}\cdots u_i).$$

Note that $q(i,j)^2$ generates the center of the group generated by $u_i, u_{i+1}, \ldots, u_j$. Brute force computations show that for k, $1 \le k \le m-1$, we have

$$q(1, 2k-1)^{p+1} q(2k+1, 2m-1)^{-p-1} = \alpha_{2k},$$

$$q(1, 2m-1)^{p+1} = \gamma.$$

So $\rho_{2m,X}(B_{2m})$ contains γ and α_{2k} for $1 \le k \le m-1$. All we need now are α_{2k-1} for $1 \le k \le m-1$. More brute force computations give us the following equalities modulo some powers of γ:

$$q(1, 2m-3)^{p+1} u_{2m-2} q(1, 2m-3)^{p+1} u_{2m-2}^{-1} u_{2m-1}^{-1} u_{2m-2}^{-1} = \alpha_{2m-3}\alpha_{2m-2},$$

$$q(1, 2k-1)^{p+1} u_{2k} q(1, 2k-1)^{p+1} u_{2k}^{-1} q(2k+1, 2m-1)^{-p-1} u_{2k}^{-1} = \alpha_{2k-1}\alpha_{2k}\alpha_{2k+1}^{-1}, \quad 1 \le k \le m-2.$$

Since our braid group image contains γ and α_{2m-2}, this first equation gets us α_{2m-3}. The last set of equations would then imply that every α_{2k-1} is in $\rho_{2m,V}(B_{2m})$ for $1 \le k \le m-1$.

Thus γ and α_i, $1 \le i \le 2m-2$, are elements of Z_{2m}. Since Z_{2m} can have order at most p^{2m-1} and the group $\langle \alpha_1, \ldots, \alpha_{2m-2}, \gamma \rangle$ has order p^{2m-1}, they must be equal. □

Proposition 5.5.19. *For $m \ge 2$, $\rho_{2m,X}(B_{2m}) \cong Z_{2m} \rtimes Sp_{2m-2}(p)$.*

Proof. By (5.23) and (5.24), every element of $\rho_{2m,X}(B_{2m})$ can be represented as a matrix of the form $(M, \vec{x}, y, \vec{z})$ with $M \in \mathrm{Sp}_{2m-2}(p)$. Consider the following sequence

$$1 \to Z_{2m} \overset{\iota}{\hookrightarrow} \rho_{2m,X}(B_{2m}) \overset{\pi}{\to} \mathrm{Sp}_{2m-2}(p) \to 1$$

with ι the inclusion map and $\pi(M, \vec{x}, y, \vec{z}) := M$. The map π is a homomorphism as a result of (5.25) and it can be seen that this sequence is a short exact sequence of groups.

Define a map $\theta : \mathrm{Sp}_{2m-2}(p) \to \rho_{2m,X}(B_{2m})$ on the generators $r_1, \ldots, r_{2m-2} \in \mathrm{Sp}_{2m-2}(p)$ by $\theta(r_1) = u_1 = (r_1, \vec{0}, 0, e_1)$ and $\theta(r_i) = u_i = (r_i, \vec{0}, 0, \vec{0})$ for $2 \le i \le 2m-2$. In general, the image of an element $M \in \mathrm{Sp}_{2m-2}(p)$ under θ is a matrix of the form $(M, \vec{0}, 0, \vec{z})$. So for $M_1, M_2 \in \mathrm{Sp}_{2m-2}(p)$, $\theta(M_1M_2)\theta(M_2)^{-1}\theta(M_1)^{-1}$ is a matrix of the form $(I_{2m-2}, \vec{0}, 0, \vec{z})$ by (5.25). From Proposition 5.5.18, the only matrix in $\rho_{2m,X}(B_{2m})$ of this form is I_{2m}, so $\theta(M_1M_2) = \theta(M_1)\theta(M_2)$ and θ is a homomorphism. It is clear that $\pi \circ \theta = \mathrm{id}_{\mathrm{Sp}_{2m-2}(p)}$ and this sequence splits. □

Proposition 5.5.20. *Let p be an odd prime, and let $X = (V, \gamma)$ be the simple object in $\mathcal{Z}(Vec_{D_{2p}})$ with $V = \bigoplus_{i=0}^{p-1} sr^i$ and for $g \in D_{2p}$, γ_g is given by permutation of summands. Then $\rho_{2,X}(B_2) \cong \mathbb{Z}/p\mathbb{Z}$, $\rho_{2m+1,X}(B_{2m+1}) \cong Sp_{2m-2}(p)$ for $m \geq 1$, and $\rho_{2m,X}(B_{2m}) \cong Z_{2m} \rtimes Sp_{2m-2}(p)$ for $m \geq 2$.*

Proof. The image $\rho_{2,X}(B_2)$ is generated by the matrix u_1 of (5.19) and has order p. The result follows from Propositions 5.5.16 and 5.5.19. □

The orders of the braid group images for this object X can be computed using Proposition 5.5.11. Of interest is the ratio between the orders of consecutive braid group images. For the non-invertible object in a braided Ising category, the ratio was fixed. For the object $X \in \mathcal{B}$ here, we have

$$\frac{|\rho_{2m+1,X}(B_{2m+1})|}{|\rho_{2m,X}(B_{2m})|} = p^{2m} - 1, \qquad \frac{|\rho_{2m+2,X}(B_{2m+2})|}{|\rho_{2m+1,X}(B_{2m+1})|} = p^{2m+1}, \qquad m \geq 1.$$

The Frobenius-Perron dimension of X is p. Unlike for braided Ising categories, the orders of the braid group images from X are not powers of $\mathrm{FPdim}(X)$. The ratios above are powers of p half the time, and for large m they are approximately a power of p the other half.

Remark 5.5.21. The category $\mathcal{Z}(\mathrm{Vec}_{D_{2p}})$ is an example of a weakly group-theoretical fusion category which does not have prime power dimension and is not nilpotent, but is *solvable* in the sense of [ENO3]. We have proven that weakly group-theoretical braided fusion categories which have prime power dimension p^n yield pure braid group images which are p-groups (Proposition 4.2.23) and that nilpotent braided fusion categories yield pure braid group images which are nilpotent (Corollary 4.2.25). This example shows that pure braid group images coming from solvable fusion categories are not, in general, solvable groups as finite symplectic groups are generally simple.

The presence of the symplectic groups as images of braid group representations coming from braided fusion categories inspires the following question:

Question 5.5.22. What simple groups appear as the images of braid group representations coming from braided fusion categories?

In conclusion, there is much to do with regards to objects in braided fusion categories and their pure braid group images. With the sizes of the pure braid group images coming from some objects, attacking this problem by hand or even with computer algebra programs is not possible. We need more concrete evidence before any general theories can be formulated.

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