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Path Components and Elementary Orbits of Um(2,R)

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PATH COMPONENTS AND ELEMENTARY ORBITS IN $Um(2, R)$

BY

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DEDICATION

For my students: past, present, and future.

"There's heroes and there's legends. Heroes get remembered, but legends never die. Follow your heart, kid, and you'll never go wrong"

–The Sandlot

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ABSTRACT

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For any commutative ring R with identity, $Um(2, R)$ is the set of all vectors $\alpha \in R^2$ such that $\alpha^T \beta = 1$ for some $\beta \in R^2$. Motivated by Hinson and Samuel, we endow $Um(2, R)$ with a pseudo-graph structure and introduce a family of functions on the general ring R defined in terms of graph theoretic distance from a designated base-point or base-set. We propose a particular connected base-set whose quasi-Euclidean function exhibits the most computationally convenient properties. Unlike in $Um(n, R)$, $n \geq 3$, the relationship between path components of $Um(2, R)$ and its orbits under elementary matrix action is complicated, and we develop tools to analyze this case. The main such tool uses closed paths in $Um(2, R)$ satisfying certain properties with respect to actions of elementary orthogonal matrices. Among our applications are: the equivalence of path-connectedness of $Um(2, R)$ and the GE_2 status of R; recovering Cohn's result that for F a field, $SL(2, F[x, y]) \neq E(2, F[x, y])$; and demonstrating $Um(2, F[x, y])$ has infinitely many distinct path components.

CHAPTER 1

BACKGROUND

For the purposes of this dissertation, unless otherwise noted, R will denote a commutative ring with multiplicative identity 1. Vectors will be denoted by Greek letters and will represent column vectors, and their transposes will be row vectors. In the context of paths, the vectors will not have specific column/row designation. We denote ε_i to be the vector with the ith component 1 and zeros elsewhere.

1.1 Unimodular vectors ($n = 2$ vs. $n \geq 3$)

Definition 1.1. *A vector $\alpha = (a_1, ..., a_n)^T$ is unimodular if there exists a vector $\beta = (b_1, ..., b_n)^T$ such that $\alpha^T\beta = \sum_{i=1}^{n} a_i b_i = 1$. Furthermore, denote the set of n-dimensional unimodular vectors over R as $Um(n, R)$ and set $N(\alpha) = \left\{\beta \in R^n \,\middle|\, \alpha^T\beta = 1\right\}$.*

For $n \geq 2$, a unimodular vector is *completable* if it occurs as a column of some matrix in the appropriate $GL(n, R)$; the set of completable unimodular vectors is denoted $Umc(n, R)$. When $n \geq 3$, if $Um(n, R) = Umc(n, R)$ then every finitely generated projective module over R is a free module, but this condition does not always hold. When $n = 2$, the situation is simpler: $Um(2, R) = Umc(2, R)$ for all rings R, that is to say that all unimodular vectors are completable. This is not the only situation where $n = 2$ and $n \geq 3$ exhibit different behavior. Andrei Suslin [8] proved that, for $n \geq 3$, $E(n, R) \trianglelefteq SL(n, R)$, but for $n = 2$, that is not necessarily true.

The following is a fundamental result relating paths to invertible matrices.

Proposition 1.2. *For* $a_1, a_2, b_1, b_2 \in R$, $\det \begin{bmatrix} a_1 & -b_2 \\ a_2 & b_1 \end{bmatrix} = 1$ *if and only if* $(b_1, b_2) \in N((a_1, a_2))$.

Proof. The proof is immediate by the definition of the determinant. □

We will now define elementary generators and elementary matrices.

Definition 1.3. *Suppose R is a ring. Then the set of* elementary generators *over R of $E(n, R)$ is the set of all $n \times n$ matrices that are identical to I_n except for one off-diagonal entry. In particular, the set of elementary 2×2 generators is*

$$\left\{ e_{ij}(r) \,\middle|\, r \in R - \{0\}, i + j = 3 \right\}.$$

An elementary matrix *is a product of elementary generators.*

Edward Hinson [4] developed the idea of assigning a pseudo-graph structure to $Um(2, R)$, $n \geq 2$: two vectors are joined by an edge (i.e are adjacent) if and only if their standard inner product is 1. This gives rise to a path structure in $Um(n, R)$. Notice that this is a pseudo-graph because there are vectors that are self-adjacent.

We also study elementary (matrix) actions on $Um(n, R)$. In $Um(2, R)$, the division algorithm $a = qb + r$ can be recreated using elementary generators: if $a = qb + r$, then

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ b \end{bmatrix}.$$

This provides a link between the division algorithm defined in Euclidean domains and elementary actions on $Um(2, R)$.

Hinson links the pseudo-graph structure and the lengths of radial paths (see Definition 2.20) to the word length of a matrix (and hence elementary actions) with the following theorem:

Theorem 1.4. ([4], Theorem 4.4). *Let P in $E(2,R)$ have first column β. Then*

$$\|\beta\| \leq \nu(P) \leq \|\beta\| + 1,$$

and there exists some completion Q of β in $E(2,R)$ with $\nu(Q) = \|\beta\|$.

In the theorem above, $\nu(P)$ represents the elementary word length of the matrix P, which is length of the shortest factorization of P into elementary generators. In the terms of this dissertation, $\|\beta\| = \|\beta\|_1$, representing the length of the radial path for β with respect to ε_1. This will be an important tool to help determine $\|\alpha\|_1$ in terms of $\|\beta\|_1$ when $\alpha = E\beta$ for E elementary.

The pseudo-graph structure also illuminates another difference between $n = 2$ and $n \geq 3$: Hinson proves ([4], Theorem 1.6) that for $n \geq 3$, for $\alpha \in Um(n,R)$, either $\langle\alpha\rangle = [\alpha]$ or $\langle\alpha\rangle = [\alpha] \cup [\beta]$ for some $\beta \in N(\alpha)$. The following result of Moshe Roitman further develops the relationship between $\langle\alpha\rangle$ and $[\alpha]$.

Proposition 1.5. ([6], Proposition 7) *Let α, β be in $Um(2k,R)$, $k \geq 2$. If $\alpha^T\beta$ is invertible in R, then $[\alpha] = [\beta]$*

Applying Proposition 1.5 yields the following result linking elementary orbits and path components for $Um(2k,R), k \geq 2$: if $\alpha \in Um(2k,R)$ for $k \geq 2$, then $\langle\alpha\rangle = [\alpha]$. We examine the "missing" – and quite different – even case $Um(2,R)$ in Chapter 4.

For $n = 2$, it is known that every unimodular vector is completable, that is to say that $\alpha \in Um(2,R)$ if and only if α appears as the first column of an invertible matrix A. In [3], Hinson further provides a link between invertibility of a matrix A and the existence of a particular closed path (in the terms of this paper, a loop) with specific additivity properties (see Proposition 1.10). This will provide a link between the loops developed in Chapter 3 and unimodular vectors, through their completions.

1.2 A least algorithm on Euclidean domains

Definition 1.6. *A Euclidean domain is an integral domain R paired with a norm function $\varphi : R - \{0\} \to W$ for a totally ordered set W whose initial segment is $\mathbb{N}$ and a division algorithm such that for any two ring elements $a, b \in R$ with $b \neq 0$ there exist elements $q, r \in R$ such that $a = qb + r$ with $r = 0$ or $\varphi(r) < \varphi(b)$.*

Pierre Samuel [7] proved in 1970 that for every Euclidean domain R there exists a Euclidean norm $\theta : R - \{0\} \to W$ for a totally ordered set W whose initial segment is $\mathbb{N}$ such that if φ is any Euclidean norm on R, then $\theta(a) \leq \varphi(a)$ for all $a \in R - \{0\}$. He developed a construction method for this least algorithm θ using transfinite recursion. Samuel's method produces the least algorithm over Euclidean domains, but fails to produce a well-defined function when the ring R is not Euclidean. It gives no indication as to how "close" a non-Euclidean ring is to being Euclidean.

How does Samuel's construction work? For a ring A, and a totally ordered set W whose initial set is $\mathbb{N}$, we set $A_0 = \{0\}$. For $k \geq 0$ in W, define A_k in the following way: the set ${A_k}' = \bigcup_{j<k} A_j$ is already defined and A_k is the union of $\{0\}$ and the set of all $b \in A$ such that ${A_k}' \to A/Ab$ is surjective.

How does it fail when R is not Euclidean? When R is Euclidean, $\bigcup_{i \in W} A_i = R$, that is to say that every element of $R - \{0\}$ belongs to A_i for some i. When R is not Euclidean, there exists some $m \in W$ such that $A_n = A_m$ for all $n > m$ and some $r \notin A_i$ for all $i \leq m$.

The ring R is Euclidean if and only if this sequence exhausts all of the ring R; in this case the smallest algorithm θ on R is defined by

$$\theta(r) = \alpha \in W \quad \iff \quad r \in A_\alpha - A'_\alpha.$$

The main goal of the first part of this work is to create a base-set $\varepsilon_\circ$ and to create a function $\varphi_\circ : R - \{0\} \to \mathbb{N}$ that can be applied to both Euclidean and non-Euclidean rings. To define such a function $\varphi_\circ$, we will examine the pseudo-graph structure of $Um(2, R)$. A choice must be made for a base-set from which to make measurements. The choice of base-set will affect the properties of the functions $\varphi_\circ$ as well as the ease of calculation of values for $\varphi_\circ$ for ring elements. Ultimately, the function $\varphi_\circ : R - \{0\} \to \mathbb{N}$ will be a function that can be applied to both Euclidean and non-Euclidean rings, and in the case of Euclidean rings, $\varphi_\circ$ can be compared to other Euclidean functions. Additionally, $\varphi_\circ$ will have properties similar to those of Euclidean functions and will also be (relatively) easy to calculate.

1.3 Generalized Euclidean rings

Every Euclidean domain R satisfies $E(n, R) = SL(n, R)$ for all $n \geq 2$.

Definition 1.7. *A commutative ring with identity is* generalized Euclidean for n *(i.e is* GE_n*) if and only if* $E(n, R) = SL(n, R)$*. A ring is* generalized Euclidean *if it is* GE_n *for all* $n \geq 2$*.*

This is of interest because rings that are not Euclidean can be generalized Euclidean.

Of particular interest is the set of rings that are not GE_2: rings R where there is a 2×2 matrix over R is determinant 1 but not elementary.

Paul Cohn [1] started from the observation that over a field F, every matrix in $SL(2, F)$ is the product of elementary generators. Passing to $F[x, y]$, Cohn demonstrates that the matrix

$$\begin{bmatrix} 1 - xy & -x^2 \\ y^2 & 1 + xy \end{bmatrix}$$

is not elementary, but has determinant one. This matrix, when viewed as a completion of the unimodular vector $\left(1 - xy, y^2\right)^T$ gives the opportunity for new analysis of the path components and elementary orbits of $Um(2, F[x, y])$.

Examples of non-GE_2 rings include $F[x, y]$ where F is field and $\mathbb{Z}\left[\sqrt{-5}\right]$ (e.g. [2]).

1.4 Additional mathematical tools

From [3], we have the following definitions and propositions. Definition 1.8 and Proposition 1.9 below are used in the definition of Proposition 1.10, which we use in Chapter 3.

Definition 1.8. ([3], Definition 1.1) *For* $A \in \mathscr{M}_{n\times n}(R)$*, write* $A = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 & \dots & \beta_n \end{bmatrix}^T$*. Define*

$$\chi(A) := \bigcap_{i=1}^{n} N(\alpha_i) \quad \text{and} \quad \rho(A) := \bigcap_{i=1}^{n} N(\beta_i).$$

An element of $\chi(A)$ *is a* central column of A *and an element of* $\rho(A)$ *is a* central row of A*.*

For any invertible matrix, the central column and central row are unique.

Proposition 1.9. ([3], Proposition 1.2) *If* $A \in GL(n, R)$*,* $n \geq 1$*, then* $|\chi(A)| = |\rho(A)| = 1$*.*

Proof. $\gamma \in \rho(A)$ if and only if $A\gamma = \varepsilon_*$. Since A is invertible, we must have $\gamma = A^{-1}\varepsilon_*$ and this γ is the unique element of $\rho(A)$. In a similar way, one shows that $\delta \in \chi(A)$ if and only if $\delta = (A^T)^{-1}\varepsilon_*$. □

Proposition 1.10. ([3], Proposition 4.1) *Let* $A = \begin{bmatrix} a_1 & e_1 \\ a_2 & e_2 \end{bmatrix} \in \mathcal{M}_{2x2}(R)$. *Then* $A \in GL(2, R)$ *if and only if there exists a closed path*

$$\langle (a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2), (e_1, e_2), (f_1, f_2), (a_1, a_2) \rangle$$

in $Um(2, R)$ *such that* $(c_1, c_2) = (a_1, a_2) + (e_1, e_2)$. *In this case,* $B = \begin{bmatrix} b_1 & b_2 \\ e_1 & e_2 \end{bmatrix}$ *is* A^{-1} *if and only if* $(f_1, f_2) = (b_1, b_2) + (d_1, d_2)$.

Proof. Suppose the given A is invertible. Set $(f_1, f_2)^T = \chi(A)$, $(c_1, c_2) = \rho(A^{-1})$, and (b_1, b_2) and (d_1, d_2) the first and second rows, respectively, of A^{-1}. These assignments produce the required closed path in $Um(2, R)$ and the condition $(c_1, c_2) = (a_1, a_2) + (e_1, e_2)$ follows from $\rho(A^{-1}) = A\varepsilon_*$, as $(c_1, c_2) \in \rho(A)$ if and only if $A(c_1, c_2)^T = \varepsilon_*$ and since A is invertible, $(c_1, c_2)^T = A^{-1}\varepsilon_*$. Conversely, suppose the existence of the closed path satisfying $(c_1, c_2) = (a_1, a_2) + (e_1, e_2)$. Observe that

$$Q = BA = \begin{bmatrix} b_1 & b_2 \\ e_1 & e_2 \end{bmatrix} \begin{bmatrix} a_1 & e_1 \\ a_2 & e_2 \end{bmatrix} = \begin{bmatrix} 1 & b_1e_1 + b_2e_2 \\ d_1a_1 + d_2a_2 & 1 \end{bmatrix}.$$

Using the hypothesis on (c_1, c_2), we see that

$$1 = (c_1, c_2)(d_1, d_2)^T = ((a_1, a_2) + (e_1, e_2))(d_1, d_2)^T = (a_1, a_2)(d_1, d_2)^T + 1,$$

which implies

$$Q = \begin{bmatrix} 1 & b_1e_1 + b_2e_2 \\ 0 & 1 \end{bmatrix}.$$

Therefore Q is invertible, and it follows that $A, B \in GL(2, R)$, proving the first assertion.

Now, supposing the conditions of the first assertion hold, suppose further that $B = A^{-1}$. Then $A = B^{-1}$ and applying the first assertion to B (instead of A) yields $(f_1, f_2) = (b_1, b_2) + (d_1, d_2)$. To show the converse, observe that

$$1 = (f_1, f_2)(e_1, e_2)^T = ((b_1, b_2) + (d_1, d_2))(e_1, e_2)^T = (b_1, b_2)(e_1, e_2)^T + 1,$$

and so $b_1e_1 + b_2e_2 = 0$. By substituting this into Q above, we obtain $Q = I_2$. Therefore, $B = A^{-1}$ as required. □

The following is classified as "mathematical folklore," but a specialized proof in the language of this paper has been included for completeness and convenience. In the proof of the next result (and throughout this work), $\widetilde{(a, b)} = (-b, a)$.

Note that we need to use Propositions 2.10 and 2.13, which are proven independently of this result. The Propositions can be used without difficulty.

Lemma 1.11. *Suppose R is a domain. Then no closed path of the form $\langle \alpha, \beta, \gamma, \beta', \alpha \rangle$ exists with $\alpha, \beta, \gamma, \beta'$ distinct in $Um(2, R)$.*

Proof. Suppose to the contrary that $\langle \alpha, \beta, \gamma, \beta', \alpha \rangle$ exists with all four vectors distinct. Then Proposition 2.13 allows us to express β' in terms of β and $\widetilde{\alpha}$: $\beta' = \beta + k\widetilde{\alpha}$ for some $k \in R$. But β' is adjacent to γ, so $\beta'^T\gamma = \beta^T\gamma + k(\widetilde{\alpha})^T\gamma = 1$. Since β and γ are adjacent, $\beta^T\gamma = 1$, thus $k(\widetilde{\alpha})^T\gamma = 0$. Since R is a domain and k is not equal to zero (since β, β' are distinct), $\widetilde{\alpha} \in \ker(\gamma)$.

Additionally, $\langle \alpha, \beta, \gamma \rangle$ is a path of length 3, so if $\alpha = (a_1, a_2)^T$, $\beta = (b_1, b_2)^T$, $\gamma = (c_1, c_2)^T$, then $a_1b_1 + a_2b_2 = 1$ and $b_1c_1 + b_2c_2 = 1$. Solving the system of equations yields

$$b_2(a_2c_1 - a_1c_2) = c_1 - a_1.$$

Since $\widetilde{\alpha} \in \ker(\gamma)$, the right hand side equals zero. Hence $c_1 - a_1 = 0$.

A similar argument yields that $c_2 - a_2 = 0$. Thus $\alpha = \gamma$, a contradiction. $\square$

CHAPTER 2

A QUASI-EUCLIDEAN FUNCTION

For a given Euclidean domain, there can be numerous Euclidean functions defined. Over the integers, the absolute value ($f(n) = |n|$) is a Euclidean norm, but so is one plus the number of binary digits of n (which can be thought of as $f(n) = 1 + \lfloor \log_2(|n|) \rfloor$). Other well known examples of Eulcidean functions are the Gaussian integers $\mathbb{Z}[i]$ with the norm defined by $f(a + bi) = a^2 + b^2$, and $K[x]$, the ring of polynomials over a field K with the degree function plus one (for $p(x) \in K[x]$, $f(p(x)) = \deg(p(x)) + 1$). Our goal is to construct a function $\phi_\circ$ that mirrors Euclidean functions, that is to say, create a function $\phi_\circ$ that can be applied to both Euclidean and non-Euclidean domains that can be used to judge "Euclidean-like" properties of non-Euclidean domains.

2.1 Unimodular vectors

We will be using the structure of the pseudo-graph of $Um(2, R)$ to define norms of vectors, and then use those norms to define the quasi-Euclidean function.

Recall Definition 1.1: A vector $\alpha = (a_1, ..., a_n)^T$ is *unimodular* if there exists a vector $\beta = (b_1, ..., b_n)^T$ such that $\alpha^T \beta = \sum_{i=1}^{n} a_i b_i = 1$. The set of n-dimensional unimodular vectors over R is denoted $Um(n, R)$.

We will use $Um(n, R)$ with the pseudo-graph structure endowed by Hinson [4] to use ideas from graph theory to uncover facts about the vectors in $Um(n, R)$. To do this, vertices (corresponding to unimodular vectors) and edges are defined as follows.

The following definition formalizes terminology used in Section 1.1.

Definition 2.1. *Suppose $\alpha \in Um\,(n, R)$, then β is a* neighbor *of α if and only if $\alpha^T \beta = 1$. The set of all neighbors of α is denoted $N(\alpha)$. Define a pseudo-graph structure of $Um\,(n, R)$ by equipping the vector set with the following edges: two vectors α and β are joined by in edge if and only if $\alpha \in N\,(\beta)$.*

Note that if $\alpha \in N\,(\beta)$, then $\beta \in N\,(\alpha)$, so the edges are not endowed with a specific direction.

Recall the idea of a path between two vectors and a path component.

Definition 2.2. *Suppose $\alpha, \beta \in Um\,(n, R)$. A* path *from α to β is an ordered sequence $p = \langle \gamma_0, \ldots, \gamma_k \rangle$ with $\gamma_i \in Um\,(n, R)$, $\gamma_0 = \alpha$, $\gamma_k = \beta$, and $\gamma_i \in N(\gamma_{i+1})$ for $0 \leq i \leq k-1$. The* path component of α *is* $\langle \alpha \rangle = \left\{ \beta \,\middle|\, \text{there exists a path from } \alpha \text{ to } \beta \right\}$.

Definition 2.3. *Suppose $p = \langle \gamma_0, \ldots, \gamma_k \rangle$ is a path. A* strand *s is a connected sub-path of p, i.e. $s = \langle \gamma_m, \gamma_{m+1}, \ldots, \gamma_{n-1}, \gamma_n \rangle$ for $0 \leq m < n \leq k$.*

Although the ideas of the pseudo-graph are applicable to all $n \geq 2$, this paper will specialize to the case $n = 2$.

Definition 2.4. *Let $\alpha = (a_1, a_2)^T$. Define $\widehat{\alpha} = (a_2, a_1)^T$ and define $\widetilde{\alpha} = (-a_2, a_1)^T$.*

We have the following properties, which can be quickly verified using Definition 2.4.

Proposition 2.5. *For $\alpha \in Um(2, R)$:*

a. $-(-\alpha) = \alpha$

b. $\widehat{(\widehat{\alpha})} = \alpha$.

c. $\widehat{(-\alpha)} = -\left(\widehat{\alpha}\right)$.

d. $\widetilde{(\widetilde{\alpha})} = -\alpha$.

e. $\widetilde{(-\alpha)} = -\left(\widetilde{\alpha}\right)$.

f. $\left(\widetilde{\widehat{\alpha}}\right) = -\left(\widehat{\widetilde{\alpha}}\right)$.

One can see that the operators are simply matrix multiplication on the vector.

Proposition 2.6. *For $\alpha \in Um(2, R)$:*

a. $-\alpha = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \alpha$

b. $\widehat{\alpha} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \alpha$.

c. $\widetilde{\alpha} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \alpha$.

Proposition 2.7. *The negation, caret, and tilde operators form a group under function composition. The group is isomorphic to the dihedral group D_4.*

Proof. We can represent the operators using the matrices from Proposition 2.6 and label them

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Taking powers of the elements yields $M^2 = C^2 = T^4 = I_2$, $T^2 = M$, and $T^3 = -T = MT$. Additionally, $MT = TM$, $MC = CM$, and $CT = -TC = MTC$. We can notate the set of all operations as $G = \left\{I_2, T, -I_2, T^3, C, CT, -C, -CT\right\} = \langle T, C\rangle = D_4$. □

These algebraic operators play an important role in parameterizing neighbors of α. These operators also preserve pseudo-graph edges.

Proposition 2.8. *Suppose $\alpha, \beta \in Um(2, R)$ with $\beta \in N(\alpha)$. Then:*

a. $-\beta \in N(-\alpha)$.

b. $\widehat{\beta} \in N(\widehat{\alpha})$.

c. $\widetilde{\beta} \in N(\widetilde{\alpha})$.

Proof. Suppose $\alpha, \beta \in Um(2, R)$ with $\beta \in N(\alpha)$. By definition, $\alpha^T \beta = \beta^T \alpha = 1$. Let $\alpha = (a_1, a_2)^T$ and $\beta = (b_1, b_2)^T$.

Examine $-\alpha$ and $-\beta$: by the properties of the inner product, $(-\alpha)^T(-\beta) = \alpha^T \beta = 1$. Thus $(-\beta) \in N(-\alpha)$.

Next, for $\widehat{\alpha} = (a_2, a_1)^T$, $\widehat{\beta} = (b_2, b_1)^T$, $(b_2, b_1)(a_2, a_1)^T = b_2 a_2 + b_1 a_1 = 1$. Thus $\widehat{\beta} \in N(\widehat{\alpha})$.

Finally, for $\widetilde{\alpha} = (-a_2, a_1)^T$, $\widetilde{\beta} = (-b_2, b_1)^T$, $(-b_2, b_1)(-a_2, a_1)^T = b_2 a_2 + b_1 a_1 = 1$. Thus $\widetilde{\beta} \in N(\widetilde{\alpha})$. □

Corollary 2.9. *If $p = \langle \alpha, \beta, ..., \omega \rangle$ is a path, so are $p_- = \langle -\alpha, -\beta, ..., -\omega \rangle$, $p_\wedge = \left\langle \widehat{\alpha}, \widehat{\beta}, ..., \widehat{\omega} \right\rangle$, and $p_\sim = \left\langle \widetilde{\alpha}, \widetilde{\beta}, ..., \widetilde{\omega} \right\rangle$.*

Lemma 2.10. *For $\alpha \in Um(2, R)$, $\widetilde{\alpha}^T \alpha = \alpha^T (\widetilde{\alpha}) = 0$.*

Proof. Let $\alpha = (a_1, a_2)^T \in Um(2, R)$, $\widetilde{\alpha} = (-a_2, a_1)^T$. Then $\widetilde{\alpha}^T \alpha = (-a_2)a_1 + a_1(a_2) = 0$ and $\alpha^T (\widetilde{\alpha}) = a_1(-a_2) + a_1 a_2 = 0$. □

Although Suslin proves the following proposition for all $n \geq 2$, it is useful to give a proof for the $n = 2$ case using the notation in this paper, which we do in Proposition 2.12.

Proposition 2.11 ([8], Lemma 1.3)**.** *Suppose that $\alpha = (a_1, a_2, ..., a_n)^T \in Um(n, R)$ and $\omega = (v_1, v_2, ..., v_n)^T \in R^n$ are such that $\omega^T \alpha = 0$. Then $\omega = \sum_{i<j} r_{ij} (a_j e_i - a_i e_j)$ for some $r_{ij} \in R$.*

Proposition 2.12. *Suppose $\alpha = (a_1, a_2)^T \in Um(2, R)$. Then $\ker(\alpha) = \left\{ k\widetilde{\alpha} \,\middle|\, k \in R \right\}$.*

Proof. Using Lemma 2.10, it is clear that $(k\widetilde{\alpha})^T\alpha = k(\widetilde{\alpha}^T\alpha) = k(0) = 0$. Thus $\left\{k\widetilde{\alpha} \,\middle|\, k \in R\right\} \subseteq \ker(\alpha)$.

Suppose next that $\omega = (w_1, w_2)^T \in \ker(\alpha)$. Then $w_1a_1 + w_2a_2 = 0$ and since $\alpha \in Um(2, R)$, there exists a vector $(b_1, b_2)^T$ such that $a_1b_1 + a_2b_2 = 1$. Viewing this as a system of equations in variables a_1, a_2 yields $w_1a_1 + w_2a_2 = 0$ (since $\omega \in \ker(\alpha)$) and $b_1a_1 + b_2a_2 = 1$ (since $\beta \in N(\alpha)$). Manipulating the system of equations yields

$$b_1(-w_2a_2) + w_1b_2a_2 = w_1 \implies (-b_1w_2 + w_1b_2)a_2 = w_1,$$

and

$$w_2b_1a_1 + b_2(-w_1a_1) = w_2 \implies (b_1w_2 - w_1b_2)a_1 = w_2.$$

If we let $g = b_1w_2 - w_1b_2$, then $\omega = (w_1, w_2)^T = (-ga_2, ga_1)^T = g\widetilde{\alpha}$. Thus $\omega \in \left\{k\widetilde{\alpha} \,\middle|\, k \in R\right\}$, and therefore $\ker(\alpha) \subseteq \left\{k\widetilde{\alpha} \,\middle|\, k \in R\right\}$.

Thus $\ker(\alpha) = \left\{k\widetilde{\alpha} \,\middle|\, k \in R\right\}$. □

The role of the vector $\widetilde{\alpha}$ is important in parameterizing the set of neighbors of $\alpha \in Um(2, R)$ in Definition 2.1.

Proposition 2.13. *Suppose $\alpha \in Um(2, R)$ and $\beta \in N(\alpha)$. Then $N(\alpha) = \left\{\beta + c\widetilde{\alpha} \,\middle|\, c \in R\right\}$.*

Proof. Suppose that $\beta \in N(\alpha)$ and let $\gamma = \beta + c\widetilde{\alpha}$ for $c \in R$. Then

$$\gamma^T\alpha = (\beta^T + (c\widetilde{\alpha})^T)\alpha = \beta^T\alpha + c\widetilde{\alpha}^T\alpha = 1,$$

since $\beta \in N(\alpha)$ and $\widetilde{\alpha}^T\alpha = 0$.

Next, suppose $\gamma \in N(\alpha)$. Then $\gamma^T\alpha = 1$. Notice that $\gamma - \beta \in \ker(\alpha)$, since

$$(\gamma - \beta)^T\alpha = (\gamma^T - \beta^T)\alpha = \gamma^T\alpha - \beta^T\alpha = 1 - 1 = 0.$$

Since $\ker(\alpha) = \left\{ c\widetilde{\alpha} \,\middle|\, c \in R \right\}$, $\gamma - \beta = k\widetilde{\alpha}$ for some $c \in R$. Therefore $\gamma = \beta + c\widetilde{\alpha}$ for some $c \in R$ and $\beta \in N(\alpha)$. □

What Proposition 2.13 tells us is that every neighbor of α can be expressed as the sum of a fixed neighbor of α and a ring multiple of $\widetilde{\alpha}$.

Proposition 2.13 can also be used to parameterize any path in terms of the first two vectors of the path.

Proposition 2.14. *Suppose $p = \langle \sigma_0 = \alpha, \sigma_1 = \beta, \sigma_2, \sigma_3, ...\rangle$. Then for all $k \in \mathbb{N}$, $\sigma_{2k} = r_k\alpha + s_k\widetilde{\beta}$ for some $r_k, s_k \in R$ and $\sigma_{2k+1} = t_k\beta + u_k\widetilde{\alpha}$ for some $t_k, u_k \in R$.*

Proof. Proceed by induction: first, when $k = 0$, $\sigma_0 = \alpha = \alpha + 0\widetilde{\beta}$ and $\sigma_1 = \beta = \beta + 0\widetilde{\alpha}$, which establishes the base case.

Now, suppose for all $n < k$, $\sigma_{2n} = r_n\alpha + s_n\widetilde{\beta}$ for $r_n, s_n \in R$ and $\sigma_{2n+1} = t_n\beta + u_n\widetilde{\alpha}$ for $t_n, u_n \in R$. By Proposition 2.13, since $\sigma_{2k} \in N(\sigma_{2k-1})$, we can express σ_{2k} as

$$\sigma_{2k} = \sigma_{2k-2} + c_{2k}\widetilde{\sigma_{2k-1}}$$

for some $c_{2k} \in R$. By the inductive hypothesis, $\sigma_{2k-2} = \sigma_{2(k-1)} = r_{k-1}\alpha + s_{k-1}\widetilde{\beta}$ and $\sigma_{2k-1} = \sigma_{2(k-1)+1} = t_{k-1}\beta + u_{k-1}\widetilde{\alpha}$. Substituting these into σ_{2k} yields

$$\begin{aligned}\sigma_{2k} &= \sigma_{2k-2} + c_{2k}\widetilde{\sigma_{2k-1}} \\ &= \left(r_{k-1}\alpha + s_{k-1}\widetilde{\beta}\right) + c_{2k}(t_{k-1}\widetilde{\beta + u_{k-1}}\widetilde{\alpha}) \\ &= \left(r_{k-1}\alpha + s_{k-1}\widetilde{\beta}\right) + c_{2k}(-u_{k-1}\alpha + t_{k-1}\widetilde{\beta}) \\ &= (r_{k-1} - c_{2k}u_{k-1})\,\alpha + (s_{k-1} + c_{2k}t_{k-1})\widetilde{\beta}\end{aligned}$$

obtaining $r_k = (r_{k-1} - c_{2k}u_{k-1})\,, s_k = (s_{k-1} + c_{2k}t_{k-1})$.

Similarly, applying Proposition 2.13 to $\sigma_{2k+1} \in N(\sigma_{2k})$, we can express σ_{2k+1} as

$$\sigma_{2k+1} = \sigma_{2k-1} + c_{2k+1}\widetilde{\sigma_{2k}}.$$

By the inductive hypothesis, $\sigma_{2k-1} = \sigma_{2(k-1)+1} = t_{k-1}\beta + u_{k-1}\widetilde{\alpha}$, and by the previous case, $\sigma_{2k-2} = \sigma_{2k} = r_k\alpha + s_k\widetilde{\beta}$. Substituting these into σ_{2k+1} yields

$$\begin{aligned}\sigma_{2k+1} &= \sigma_{2k-1} + c_{2k+1}\widetilde{\sigma_{2k}} \\ &= (t_{k-1}\beta + u_{k-1}\widetilde{\alpha}) + c_{2k+1}\left(r_k\widetilde{\alpha + s_k}\widetilde{\beta}\right) \\ &= (t_{k-1}\beta + u_{k-1}\widetilde{\alpha}) + c_{2k+1}(-s_k\beta + r_k\widetilde{\alpha}) \\ &= (t_{k-1} - c_{2k+1}s_k)\,\beta + (u_{k-1} + c_{2k+1}r_k)\widetilde{\beta}\end{aligned}$$

obtaining $t_k = (t_{k-1} - c_{2k+1}s_k)\,, u_k = (u_{k-1} + c_{2k+1}r_k) \in R$. □

Proposition 2.14 not only parameterizes the path p in terms of the initial vectors α, β, it also gives rise to a second associated path.

Definition 2.15. *For a path p as defined in Proposition 2.14 with $\sigma_{2k} = r_k\alpha + s_k\widetilde{\beta}$ and $\sigma_{2k+1} = t_k\beta + u_k\widetilde{\alpha}$ for $r_k, s_k, t_k, u_k \in R$, define $\gamma_k = (r_k, s_k)^T$ and $\delta_k = (t_k, u_k)^T$.*

The vectors γ_k and δ_k can now be defined recursively.

Proposition 2.16. *Let γ_k, δ_k be defined as in Definition 2.15. Then for $k > 1$, $\gamma_k = \gamma_{k-1} + a_k\widetilde{\delta_{k-1}}$ for some $a_k \in R$ and $\delta_k = \delta_{k-1} + b_k\widetilde{\gamma_k}$ for some $b_k \in R$.*

Proof. From the definitions, $\sigma_0 = \alpha + 0\widetilde{\beta}$ and $\sigma_1 = \beta + 0\widetilde{\alpha}$. It follows that for every path p, $\gamma_0 = (1,0)^T = \varepsilon_1$ and $\delta_0 = (1,0)^T = \varepsilon_1$.

From the proof of Proposition 2.14, we can see that

$$\begin{aligned}\gamma_k &= (r_{k-1} - c_{2k}u_{k-1}, s_{k-1} + c_{2k}t_{k-1})^T \\ &= (r_{k-1}, s_{k-1})^T + c_{2k}(-u_{k-1}, t_{k-1})^T \\ &= \gamma_{k-1} + c_{2k}\widetilde{\delta_{k-1}} && \text{with } c_{2k} \in R, \text{ and}\end{aligned}$$

$$\begin{aligned}\delta_k &= (t_{k-1} - c_{2k+1}s_k, u_{k-1} + c_{2k+1}r_k)^T \\ &= (t_{k-1}, u_{k-1})^T + c_{2k+1}(-s_k, r_k)^T \\ &= \delta_{k-1} + c_{2k+1}\widetilde{\gamma_k} && \text{with } c_{2k+1} \in R.\end{aligned}$$

□

More is true about the structure of the γ_k and δ_k vectors. They themselves form a path in $Um(2, R)$.

Proposition 2.17. *Given a path p as in Definition 2.14 and γ_k, δ_k defined as in Definition 2.15, $p_{\gamma\delta} = \langle \gamma_0, \delta_0, \gamma_1, \delta_1, \gamma_2, \delta_2, ... \rangle$ is a path.*

Proof. It suffices to show that $\delta_k \in N(\gamma_k)$ and $\gamma_{k+1} \in N(\delta_k)$ for all $k \geq 0$.

We will induct on k: when $k = 0$, $\gamma_0 = \delta_0 = \varepsilon_1$, so $\delta_0 \in N(\gamma_0)$, and by Proposition 2.16, $\gamma_1 = \gamma_0 + a_0\widetilde{\delta_0}$. By Proposition 2.13, $\gamma_1 \in N(\delta_0)$.

Now suppose $\delta_{k-1} \in N(\gamma_{k-1})$ and $\gamma_k \in N(\delta_{k-1})$. By Proposition 2.16, $\delta_k = \delta_{k-1} + b_k\widetilde{\gamma_k}$ for some $b_k \in R$. Using Proposition 2.13 with $\beta = \delta_{k-1}$ and $\alpha = \gamma_k$, $\delta_{k-1} \in N(\gamma_{k-1})$

by the inductive assumption, so $N(\gamma_k) = \left\{\delta_{k-1} + c\widetilde{\gamma_k} \,\middle|\, c \in R\right\}$. Since $\delta_k = \delta_{k-1} + b_k\widetilde{\gamma_k}$, $\delta_k \in N(\gamma_k)$. Similarly, $\gamma_{k+1} = \gamma_k + a_{k+1}\widetilde{\delta_k}$ from Proposition 2.16. Using Proposition 2.13 again, but with $\alpha = \delta_k$ and $\beta = \gamma_k$ (note that $\delta_k \in N(\gamma_k)$ from above), yields that $N(\delta_k) = \left\{\gamma_k + c\widetilde{\delta_k} \,\middle|\, c \in R\right\}$. Since $\gamma_{k+1} = \gamma_k + a_{k+1}\widetilde{\delta_k}$, we have $\gamma_{k+1} \in N(\delta_k)$. □

Corollary 2.18. *For $p = \langle\sigma_0, \sigma_1, \sigma_2, ...\rangle$ and $p_{\gamma\delta} = \langle\gamma_0, \delta_0, \gamma_1, \delta_1, ...\rangle$ as in the previous propositions, $\sigma_{2m} = \sigma_0$ if and only if $\gamma_m = \gamma_0$.*

Proof. This follows from the fact that $\sigma_1 \in N(\sigma_0)$ implies σ_0 and $\widetilde{\sigma_1}$ are linearly independent.

□

Corollary 2.19. *For $p = \langle\sigma_0 = \alpha, \sigma_1 = \beta, \sigma_2, ...\rangle$ and $p_{\gamma\delta} = \langle\gamma_0, \delta_0, \gamma_1, \delta_1, ...\rangle$ as in the previous propositions, and $A = \begin{bmatrix} \alpha & \widetilde{\beta} \end{bmatrix}$, then $(A^T)^{-1} = \begin{bmatrix} \beta & \widetilde{\alpha} \end{bmatrix}$, and these satisfy*

a. $A\gamma_k = \sigma_{2k}$.

b. $A^{-1}\sigma_{2k} = \gamma_k$.

c. $(A^T)^{-1}\delta_k = \sigma_{2k+1}$.

d. $A^T\sigma_{2k+1} = \delta_k$.

Proof. These follow from the application of Proposition 2.14 and Definition 2.15. □

2.2 Norm functions

The distance between two vectors α and β in the same path component of the psuedo-graph of $Um(2, R)$ is the number of edges in a shortest path between α and β. Norms of $\alpha \in Um(2, R)$ will be defined relative to a chosen base-point.

We shall see that most useful choices will be $\varepsilon_* = (1, 1)^T$, $\varepsilon_1 = (1, 0)^T$, $\varepsilon_2 = (0, 1)^T$.

Definition 2.20. *Let* $\circ$ *denote the choice of base-point. A* radial path for α *with respect to a given base-point* $\varepsilon_\circ$ *is a path from* α *to the base-point that has minimal length. The* $\circ$-norm of α, *denoted* $\|\alpha\|_\circ$, *is that minimal length.*

Each of ε_*, ε_1, ε_2 has potential advantages. The first and second columns of the identity matrix are ε_1 and ε_2, and ε_* is the central column of the identity matrix (and hence the only common neighbor of ε_1 and ε_2). The roles of these vectors will be advantageous later when exploring how elementary actions act on paths.

The following propositions illustrate more properties of $\|-\|_*$, $\|-\|_1$, and $\|-\|_2$.

Proposition 2.21. *Assume* $\alpha \in Um(2, R)$*. It follows that* $\|\alpha\|_* = \|\widehat{\alpha}\|_*$.

Proof. Suppose $\|\alpha\|_* = n$. Then there exists a radial path with n edges: $\langle \alpha, \beta_1, \beta_2, ..., \beta_n = \varepsilon_* \rangle$. Recall that since this path is radial, no shorter path between α and ε_* exists.

Then we have a path, by Proposition 2.9, from $\widehat{\alpha}$ to ε_*: $\left\langle \widehat{\alpha}, \widehat{\beta_1}, \widehat{\beta_2}, ..., \widehat{\beta_n} = \widehat{\varepsilon_*} = \varepsilon_* \right\rangle$, thus $\|\widehat{\alpha}\|_* \leq n$.

Now, suppose there exists a shorter path from $\widehat{\alpha}$ to ε_*, i.e. $\|\widehat{\alpha}\|_* = m < n$. Then there would exists a radial path of length m: $\langle \widehat{\alpha}, \gamma_1, \gamma_2, ..., \gamma_m = \varepsilon_* \rangle$. But then we would also have the path $\left\langle \widehat{(\widehat{\alpha})} = \alpha, \widehat{\gamma_1}, \widehat{\gamma_2} \cdots, \widehat{\gamma_m} = \varepsilon_* \right\rangle$, which would be a path of length m, a contradiction since a path of length $n > m$ is radial. Therefore no such shorter path can exists, and thus $\|\widehat{\alpha}\|_* = n$. □

Using $\|\alpha\|_*$, it is easy to classify which vectors are "close" to ε_*.

Proposition 2.22. *Suppose* $\alpha \in Um(2, R)$*. Then* $\|\alpha\|_* = 1$ *if and only if* $\alpha = (1 - r, r)^T$ *for some* $r \in R$.

Proof. First, suppose $\|\alpha\|_* = 1$. Then $\alpha \in N(\varepsilon_*)$. If $\alpha = (a_1, a_2)^T$, $a_1 + a_2 = 1$. Therefore $a_1 = 1 - a_2$.

Next, suppose $\alpha = (1-r, r)^T$ for some $r \in R - \{0\}$. Then $(1-r)(1) + (r)(1) = 1$. Thus $\varepsilon_* \in N(\alpha)$, so $\|\alpha\|_* = 1$. □

Proposition 2.23. *Suppose $r \in R$ with $r \notin \{0, 1\}$. Then $\|(1, r)\|_* = 2$.*

Proof. Assume $r \notin \{0, 1\}$. It follows that we have the path $\langle (1, r), \varepsilon_1, \varepsilon_* \rangle$. Since $r \neq 0$, $\|(1, r)\|_* \neq 1$ by Proposition 2.22. Therefore $\|(1, r)\|_* = 2$.

One can see that if $r = 1$, $\|(1, 1)\|_* = 0$ and if $r = 0$, then $\|(1, 0)\|_* = 1$. □

Proposition 2.23 does not fully determine all vectors α with $\|\alpha\|_* = 2$: over the ring $\mathbb{Z}$, the vector $(9, 8) \in Um(2, \mathbb{Z})$ does not have ε_* norm one (by Proposition 2.22), and has the radial path $\langle (9, 8), (-7, 8), \varepsilon_* \rangle$, so $\|(9, 8)\|_* = 2$.

Proposition 2.24. *Suppose $u \in R^*$ with $u \neq 1$. If $r \notin \{0, 1-u\}$, then $2 \leq \|(u, r)\|_* \leq 3$.*

Proof. Suppose $r \notin \{0, 1-u\}$. Then $\left\langle (u, r), (u^{-1}, 0), (u, 1-u), \varepsilon_* \right\rangle$ is a path of length three. Notice that $\|(u, r)\|_* = 1$ would imply that $(u, r) \in N(\varepsilon_*)$, hence $u + r = 1$, and thus $r = 1 - u$, which it cannot by hypothesis. Thus $2 \leq \|(u, r)\|_* \leq 3$. □

Notice that the choice of base-point changes the properties of the norm.

Proposition 2.25. *Suppose $\alpha \in Um(2, R)$, then $\|\alpha\|_1 = \|\widehat{\alpha}\|_2$.*

Proof. Suppose $\alpha \in Um(2, R)$ with $\|\alpha\|_1 = n$. Then by definition, there exists a radial path of length n from α to ε_1. By Proposition 2.9, then there exists a path of length n from $\widehat{\alpha}$ to ε_2. Note that no shorter path can exist: suppose that a path of length $m < n$ exists between $\widehat{\alpha}$ and ε_2. Then Proposition 2.9 would give us a path of length m between α and ε_1, which is impossible, since $\|\alpha\|_1 = n$ by hypothesis.

Thus $\|\alpha\|_1 = \|\widehat{\alpha}\|_2$. □

Again, for a vector α close to ε_1 and ε_2, it is easy to calculate $\|\alpha\|_1$ and $\|\alpha\|_2$, respectively.

Proposition 2.26. *Suppose $\alpha \in Um(2, R)$. Then $\|\alpha\|_1 = 1$ if and only if $\alpha = (1, r)^T$ for $r \in R$ with $r \neq 0$.*

Proof. Suppose first that $\|\alpha\|_1 = 1$. Then $\alpha = (a_1, a_2)^T \in N(\varepsilon_1)$, so $\alpha^T \varepsilon_1 = 1$. Therefore $a_1 = 1$. In this case, $\alpha = (1, a_2)^T$ with $r \neq 0$. Then $\|\alpha\|_1 \neq 0$. But $\alpha \in N(\varepsilon_1)$, since $\alpha^T \varepsilon_1 = 1$. Thus $\|\alpha\|_1 = 1$.

Suppose next that $\alpha = (1, r)^T$ with $r \neq 0$. Then $\|\alpha\|_1 = 1$ by Proposition 2.26. □

The symmetry with respect to the caret operator between $\|\alpha\|_1$ and $\|\alpha\|_2$ for all $\alpha \in Um(2, R)$ (Proposition 2.25) leads to the corresponding result.

Proposition 2.27. *Suppose $\alpha \in Um(2, R)$. Then $\|\alpha\|_2 = 1$ if and only if $\alpha = (r, 1)^T$ for $r \in R$ with $r \neq 0$.*

Vectors near the various base-points have easy-to-determine norms with respect to the other base-points.

Proposition 2.28. *Suppose $\beta \in N(\varepsilon_*)$ with $\beta \notin \{\varepsilon_1, \varepsilon_2\}$. Then $\|\beta\|_1 = \|\beta\|_2 = 2$.*

Proof. Suppose $\beta \in N(\varepsilon_*)$. By Proposition 2.13, $\beta = \varepsilon_1 + k\widetilde{\varepsilon_*} = (1 - k, k)^T$ for some $k \in R$ with $k \neq 0$, since $\beta \neq \varepsilon_1$. Since $\beta \neq (1, r)^T$, $\|\alpha\|_1 \neq 1$ by Proposition 2.26. But since $\beta \in N(\varepsilon_*)$, there exists a path $\langle \beta, \varepsilon_*, \varepsilon_1 \rangle$ of length 2 connecting β and ε_1. Therefore $\|\beta\|_1 = 2$.

The corresponding result with respect to ε_2 is proven similarly. □

Again, Proposition 2.28 does not fully classify vectors of norm 2: over the ring $\mathbb{Z}$, $(-9, 5) \in Um(2, \mathbb{Z})$ is clearly not a neighbor of ε_*, yet $\|(-9, 5)\|_1 = 2$ since we have the path $\langle (-9, 5), (1, 2), \varepsilon_1 \rangle$ (and by Proposition 2.26, $\|(-9, 5)\|_1 \neq 1$).

To have symmetry with respect to the negation, caret, and tilde operators from Definition 2.4, it will be necessary to include more vectors into a base-point set or base-set, and to revise Definition 2.20.

Definition 2.29. *Let $S \subset \langle \varepsilon_* \rangle$. We refer to S as a* base-set ε_S, *and we define* $\|\alpha\|_S = \min\limits_{\omega \in \varepsilon_S} \{\|\alpha\|_\omega\}$.

Consider the base-set $\varepsilon_\bullet = \{\varepsilon_1, \varepsilon_*, \varepsilon_2\}$. The advantage of taking multiple points in a base-set is immediate.

Proposition 2.30. *The norm $\|-\|_\bullet$ is symmetric with respect to the caret operator, i.e.* $\|\alpha\|_\bullet = \|\widehat{\alpha}\|_\bullet$.

Proof. Suppose $\alpha \in Um(2, R)$ with $\|\alpha\|_\bullet = n$. By Definition 2.29,

$$\|\alpha\|_\bullet = \min\{\|\alpha\|_1, \|\alpha\|_*, \|\alpha\|_2\} = n.$$

By Proposition 2.21, $\|\alpha\|_* = \|\widehat{\alpha}\|_*$, and by Proposition 2.25, $\|\alpha\|_1 = \|\widehat{\alpha}\|_2$. Thus

$$\begin{aligned}\|\widehat{\alpha}\|_\bullet &= \min\{\|\widehat{\alpha}\|_1, \|\widehat{\alpha}\|_*, \|\widehat{\alpha}\|_2\}\\ &= \min\{\|\alpha\|_2, \|\alpha\|_*, \|\alpha\|_1\}\\ &= n.\end{aligned}$$

Therefore, $\|\alpha\|_\bullet = \|\widehat{\alpha}\|_\bullet$. □

Proposition 2.30 tells us that $\|-\|_\bullet$ is symmetric with respect to the caret operator, but is not so with respect to negation or the tilde operator: notice that $-\varepsilon_1 = \widetilde{\varepsilon_2}$, and $\|\varepsilon_1\|_\bullet = 0$, but $\|-\varepsilon_1\|_\bullet = 2$ (by Propositions 2.22, 2.26, and 2.27, and with a radial path $\langle -\varepsilon_1, (-1, 1), \varepsilon_2 \rangle$). Additionally, this example shows that $\|\varepsilon_2\|_\bullet = 0$, but $\|\widetilde{\varepsilon_2}\|_\bullet = 2$. So the base-set $\varepsilon_\bullet$ and the associated norm $\|-\|_\bullet$ are not well behaved with respect to negation and the tilde operator.

To capture symmetry with respect to negation, the base-point can be expanded to $\varepsilon_6 = \{\varepsilon_1, \varepsilon_*, \varepsilon_2, -\varepsilon_1, -\varepsilon_*, -\varepsilon_2\}$. This set now determines the norm $\|\alpha\|_6$ which is symmetric with respect to negation as well as the caret operator of Definition 2.4. However, the norm with respect to ε_6 is not symmetric with respect to the tilde operator, specifically for vectors whose radial paths connect to ε_* or $-\varepsilon_*$: notice that $\|(1-r,r)\|_6 = 1$ for $r \notin \{0,1\}$ since $(1-r,r) \in N(\varepsilon_*)$, but $\left\|\widetilde{(1-r,r)}\right\|_6 = \|(-r,1-r)\|_6 \neq 1$ by Propositions 2.22, 2.26, and 2.27. Since we have the path $\langle(-r,1-r),(-1,1),\varepsilon_1\rangle$, $\|(1-r,r)\|_6 = 2$.

We can repair the norm with respect to ε_6 to be symmetric with respect to the tilde operator by including the vectors $(-1,1)$ and $(1,-1)$. This has the additional advantage of making the new base-set path connected as well, easing the calculations by removing the necessity to check two non-connected sets for a minimal path. Further, we shall see in Chapter 3 that viewing a path connected base-set as a loop will provide further structure.

Definition 2.31. *We will define the base-set $\mathfrak{e}$ to be the set*

$$\mathfrak{e} = \{\varepsilon_1, \varepsilon_*, \varepsilon_2, (-1,1), -\varepsilon_1, -\varepsilon_*, -\varepsilon_2, (1,-1)\}$$

and let the norm $\|-\|_\mathfrak{e}$ be the norm determined by $\mathfrak{e}$ as in Definition 2.29.

The set $\mathfrak{e}$ is now a path-connected set which is closed under the negation, caret, and tilde operators. Moreover, the norm $\|-\|_\mathfrak{e}$ is symmetric with respect to these operators.

Proposition 2.32. *Suppose $\alpha \in Um(2,R)$. Then*

a. $\|-\alpha\|_\mathfrak{e} = \|\alpha\|_\mathfrak{e}$.

b. $\|\widehat{\alpha}\|_\mathfrak{e} = \|\alpha\|_\mathfrak{e}$.

c. $\|\widetilde{\alpha}\|_{\mathfrak{e}} = \|\alpha\|_{\mathfrak{e}}$.

Proof. Suppose first that $\|-\alpha\|_{\mathfrak{e}} < \|\alpha\|_{\mathfrak{e}} = n$. Then there exists a path $\langle -\alpha, ..., \delta\rangle$ of length $k < n$ for some $\delta \in \mathfrak{e}$. But $-\delta \in \mathfrak{e}$, and the path of negations $\langle \alpha, ..., -\delta\rangle$ must have length of at least n, since $\|\alpha\|_{\mathfrak{e}} = n$. Therefore $\|-\alpha\|_{\mathfrak{e}} \geq n$. We can further deduce that $\|-\alpha\|_{\mathfrak{e}} = n$: since $\|\alpha\|_{\mathfrak{e}} = n$, there exists a path $\langle \alpha, ..., \gamma\rangle$ of length n for some $\gamma \in \mathfrak{e}$. By Corollary 2.9, we have a corresponding path $\langle -\alpha, ..., -\gamma\rangle$ of length n. Since $\gamma \in \mathfrak{e}$, we know that $\|-\alpha\|_{\mathfrak{e}} \leq n$. Therefore $\|-\alpha\|_{\mathfrak{e}} = n$.

Since $\mathfrak{e}$ is also closed with respect to the caret and tilde operators, similar arguments, again using Corollary 2.9, show that $\|\widehat{\alpha}\|_{\mathfrak{e}} = \|\alpha\|_{\mathfrak{e}}$ and $\|\widetilde{\alpha}\|_{\mathfrak{e}} = \|\alpha\|_{\mathfrak{e}}$. □

2.3 ϕ-functions

For each vector $\alpha \in Um(2, R)$ and each choice of base-point $\varepsilon_\circ$, there is an associated norm $\|\alpha\|_\circ$. To access information about the behavior exhibited by the ring R, we will define a function $\phi_\circ$.

Definition 2.33. *For each $a \in R - \{0\}$ and base-set $\varepsilon_\circ$, define a function $\phi_\circ : R - \{0\} \to \mathbb{N}$ to be*

$$\phi_\circ(a) := \sup\left\{ \|(a,r)\|_\circ \;\middle|\; (a,r) \in \langle \varepsilon_* \rangle \right\}.$$

We will choose a base-point or base-set to work with that best measures the "Euclidean-like" behavior that the ring R exhibits. Each choice of base-point again has advantages and disadvantages: ε_1 is the first column of the identity matrix, which is helpful when working with elementary actions. However, the norm $\|\alpha\|_1$ lacks desired symmetry: $\|(a,r)\|_1 \neq \|(r,a)\|_1$ in most cases. Therefore, finding $\phi_1(a)$ can be burdensome, as every vector of the form (a, r) must be considered. A choice of ε_2 also lacks the desired symmetry of the associated norm, and will thus have similar weaknesses.

The base-point ε_* does have symmetry with respect to the norm, as $\|(a,r)\|_* = \|(r,a)\|_*$ by Proposition 2.21. In general, if a norm with respect $\varepsilon_\circ$ is symmetric with respect to the caret operator, it allows for the definition of the ϕ-function to be expressed in an easier-to-calculate way.

Proposition 2.34. *Suppose a base-set $\varepsilon_\circ$ has the property $\|\alpha\|_\circ = \|\widehat{\alpha}\|_\circ$ for all $\alpha \in Um(2,R)$. Then*

$$\phi_\circ(a) := \sup\left\{ \|\alpha\|_\circ \;\middle|\; \alpha \in \langle \varepsilon_* \rangle \textit{ and } a \textit{ is one of the entries of } \alpha \right\}.$$

Proof. Since $\|\alpha\|_\circ = \|\widehat{\alpha}\|_\circ$ for all $\alpha \in Um(2,R)$, the definitions are equivalent. □

So $\phi_*(a) = \sup\left\{ \|\alpha\|_* \;\middle|\; \alpha \in \langle \varepsilon_* \rangle \text{ and } a \text{ is one of the entries of } \alpha \right\}$. But ϕ_* lacks other properties that would ease calculations and make it similar to Euclidean functions. In general, $\phi_*(a)$ may not be related to $\phi_*(-a)$.

Using the "fatter" base-set $\varepsilon_\bullet$ maintains the caret operator symmetry: Proposition 2.30 tells us that $\|\alpha\|_\bullet = \|\widehat{\alpha}\|_\bullet$, so the definition of the ϕ-function can be expressed as

$$\phi_\bullet(a) := \sup\left\{ \|\alpha\|_\bullet \;\middle|\; \alpha \in \langle \varepsilon_* \rangle \text{ and } a \text{ is one of the entries of } \alpha \right\}$$

by Proposition 2.34. Unfortunately, for the base-set $\varepsilon_\bullet$, $\|\alpha\|_\bullet \neq \|-\alpha\|_\bullet$ in general, so $\phi_\bullet(a)$ and $\phi_\bullet(-a)$ may not be related.

Using the base-set $\mathfrak{e}$ maintains the desired symmetry that $\varepsilon_\bullet$, but also, since $\|\alpha\|_\mathfrak{e} = \|-\alpha\|_\mathfrak{e}$ for all α, $\phi_\mathfrak{e}(a) = \phi_\mathfrak{e}(-a)$ for all $a \in R - \{0\}$.

We can prove some properties about the various $\phi_\circ$ functions.

Proposition 2.35. *Suppose S, T are connected subsets of $Um(2,R)$ with $S \subseteq T$, and let $\varepsilon_S, \varepsilon_T$ be their associated base-sets. Then $\phi_T(a) \leq \phi_S(a)$ for all $a \in R$.*

Proof. Suppose S, T are connected subsets of $Um(2, R)$ with $S \subseteq T$, and let $\varepsilon_S, \varepsilon_T$ be their associated base-sets. Then for all $\alpha \in Um(2, R)$, $\|\alpha\|_T \leq \|\alpha\|_S$ by Definition 2.29. So for any $a \in R$, $\phi_T(a) \leq \phi_S(a)$. $\square$

The following definition will help us estimate $\phi_{\mathfrak{e}}$.

Definition 2.36. *For $a \in R - \{0\}$, define*

$$\nu(a) := \max \left\{ \nu(A) \;\middle|\; A \in E(2, R) \text{ and } a \text{ is an entry of } A \right\},$$

where $\nu(A)$ is the elementary word length of A.

Although it can be quite difficult to determine the value of $\phi_{\mathfrak{e}}(a)$, we are able to bound the value of $\phi_{\mathfrak{e}}(a)$ using the related, but quite distinct invariant $\nu(a)$. This is a reflection of the construction of Samuel, which used elementary generator action to realize the least algorithm over Euclidean domains. Our function $\phi_{\mathfrak{e}}$, however, is defined over all rings.

First, we need the following result.

Proposition 2.37. *For every $a \in R - \{0\}$, $\phi_{\mathfrak{e}}(a) = \phi_{\mathfrak{e}}(-a)$.*

Proof. Suppose $a \in R - \{0\}$, and let $\phi_{\mathfrak{e}}(a) = n$. Then there exists a vector α with $\|\alpha\|_{\mathfrak{e}} = n$ being the largest norm of all vectors with a as an entry. By Proposition 2.32, $\|-\alpha\|_{\mathfrak{e}} = n$, thus $\phi_{\mathfrak{e}}(-a) \geq n$.

Now, let $\phi_{\mathfrak{e}}(-a) = m$. A similar argument yields $\phi_{\mathfrak{e}}(a) \geq m$. Combining these facts shows that $\phi_{\mathfrak{e}}(a) = \phi_{\mathfrak{e}}(-a)$. $\square$

Theorem 2.38. *For $a \in R - \{0\}$, $\nu(a) - 3 \leq \phi_{\mathfrak{e}}(a) \leq \nu(a) - 1$.*

Proof. First, observe that for $A = \begin{bmatrix} \alpha & \widetilde{\beta} \end{bmatrix} \in E(2,R)$, if $\nu(a) = \nu(A)$ with a an entry of A, without loss of generality we can assume that a is an entry of α since

$$A = \begin{bmatrix} * & a \\ * & * \end{bmatrix} \implies A^T = \begin{bmatrix} * & * \\ a & * \end{bmatrix} \quad \text{with} \quad \nu(A) = \nu(A^T)$$

and

$$A = \begin{bmatrix} * & * \\ * & a \end{bmatrix} \implies A^{-1} = \begin{bmatrix} a & * \\ * & * \end{bmatrix} \quad \text{with} \quad \nu(A) = \nu(A^{-1})$$

allow us to realize the a in the first column while leaving the matrix word length unchanged.

Now, for $a \in R - \{0\}$, applying Proposition 1.4,

$$\begin{aligned}
\nu(a) &= \sup\left\{ \nu(A) \,\middle|\, A = \begin{bmatrix} \alpha & \widetilde{\beta} \end{bmatrix} \in E(2,R) \text{ and } a \text{ is an entry of } \alpha \right\} \\
&= 1 + \sup\left\{ \|\alpha\|_1 \,\middle|\, a \text{ is an entry of } \alpha \in [\varepsilon_1] \subseteq Um(2,R) \right\} && \text{by Proposition 1.4} \\
&\geq 1 + \sup\left\{ \|\alpha\|_{\mathfrak{e}} \,\middle|\, a \text{ is an entry of } \alpha \in [\varepsilon_1] \subseteq Um(2,R) \right\} && \|\alpha\|_1 \geq \|\alpha\|_{\mathfrak{e}} \text{ by definition} \\
&= 1 + \sup\left\{ \|\alpha\|_{\mathfrak{e}} \,\middle|\, \alpha = (a,x)^T \in [\varepsilon_1] \subseteq Um(2,R) \right\} && \text{since } \left\|(a,x)^T\right\|_{\mathfrak{e}} = \left\|(x,a)^T\right\|_{\mathfrak{e}} \\
& && \text{for all } (a,x)^T \in [\varepsilon_1] \\
&= 1 + \phi_{\mathfrak{e}}(a)
\end{aligned}$$

Therefore $\phi_{\mathfrak{e}}(a) \leq \nu(a) - 1$.

To obtain the lower bound, we first observe that if $\phi_{\mathfrak{e}}(a) = \|\gamma\|_{\mathfrak{e}}$ and $\delta \in Um(2,R)$ with a and entry of δ, then $\|\gamma\|_{\mathfrak{e}} \geq \|\delta\|_{\mathfrak{e}}$, as $\|\gamma\|_{\mathfrak{e}}$ is maximal. But also, $\|\delta\|_1 \leq \|\delta\|_{\mathfrak{e}} + 2$, as Proposition 2.37 allows us to restrict to the base-strand $\langle (1,-1), \varepsilon_1, \varepsilon_*, \varepsilon_2 \rangle$ where we see the path for $\|\delta\|_{\mathfrak{e}}$ enters no more than two edges from ε_1.

We wish to prove that $\|\gamma\|_{\mathfrak{e}} \geq \|\delta\|_1 - 2$. Suppose the contrary, that $\|\gamma\|_{\mathfrak{e}} < \|\delta\|_1 - 2$. Then $\|\gamma\|_{\mathfrak{e}} \leq \|\delta\|_1 - 2 < (\|\delta\|_{\mathfrak{e}} + 2) - 2 = \|\delta\|_{\mathfrak{e}}$ and thus $\|\delta\|_{\mathfrak{e}} > \|\gamma\|_{\mathfrak{e}}$, a contradiction, since $\|\gamma\|_{\mathfrak{e}}$ is maximal. Thus $\|\gamma\|_{\mathfrak{e}} \geq \|\delta\|_1 - 2$ for all $\delta \in Um(2,R)$ with a an entry of δ.

Thus

$$\phi_{\mathfrak{e}}(a) = \|\gamma\|_{\mathfrak{e}} \geq \max\left\{ \|\delta\|_1 \;\middle|\; a \text{ is an entry of } \delta \right\} - 2$$
$$= (\nu(a) - 1) - 2 = \nu(a) - 3$$

as $\phi_1(a) = \nu(a) - 1$ by Proposition 1.4. Therefore $\nu(a) - 3 \leq \phi_{\mathfrak{e}}(a)$.

Combining the inequalities, we have $\nu(a) - 3 \leq \phi_{\mathfrak{e}}(a) \leq \nu(a) - 1$. □

CHAPTER 3

LOOP THEORY

To examine how $\langle\alpha\rangle$ and $[\alpha]$ are related, we shall define loops in $Um(2,R)$. Recall Definition 2.2: a path in $Um(2,R)$ is an ordered sequence of vectors $\langle\alpha_0,\alpha_1,...,\alpha_k\rangle$ such that ${\alpha_n}^T\alpha_{n+1} = 1$ for $0 \leq n \leq k-1$.

3.1 Loops

Definition 3.1. *A* loop *is a (ordered) path* $\mathfrak{a} = \langle\alpha_0,\alpha_1,...,\alpha_k = \alpha_0\rangle$*; such a loop is called a* k-loop*. A* pointed loop *with distinguished vector* α_0 *(marked with an asterisk) is the loop* $\left\langle\overset{*}{\alpha}_0,\alpha_1,...,\alpha_k = \alpha_0\right\rangle$*. Two pointed* k*-loops are equal if and only if they contain the same vectors in the same order and the same distinguished vector.*

Definition 3.2. *Let* $\mathfrak{b}$ *be a pointed loop. Then* $\mathfrak{b}$ *is an* m-translate *of the loop* $\mathfrak{a} = \left\langle\overset{*}{\alpha}_0,\alpha_1,...,\alpha_k = \alpha_0\right\rangle$ *if and only if they have the same elements in the same order, translated by* m *positions, i.e.* $\mathfrak{b} = \left\langle\overset{*}{\alpha}_m,\alpha_{m+1},...,\alpha_k = \alpha_0,\alpha_1,\alpha_2,...,\alpha_{m-1},\alpha_m\right\rangle$.

Recall how an invertible matrix acts on a path (and hence how it acts on a loop).

Definition 3.3. *Let* $A \in GL(2,R)$ *and let* $p = \langle\sigma_0,\sigma_1,...,\sigma_k\rangle$ *be a path. Then* A acts on the path p *by producing a path* $Ap = \langle A\sigma_0,(A^T)^{-1}\sigma_1,A\sigma_2,(A^T)^{-1}\sigma_3,...,M\sigma_k\rangle$*, where* $M = A$ *if* k *is even,* $M = (A^T)^{-1}$ *if* k *is odd. We will call this action the* standard action of the matrix A on the path p.

It is useful to think about matrices acting on paths as a group action: for a path $p = \langle \sigma_0, \sigma_1, ..., \sigma_k \rangle$, the identity matrix acting on a path p returns the original path p, and for $A, B \in GL(2, R)$,

$$(AB)p = \left\langle (AB)\sigma_0, ((AB)^T)^{-1}\sigma_1, (AB)\sigma_2, ((AB)^T)^{-1}\sigma_3, ..., M\sigma_k \right\rangle,$$

where $M = AB$ if k is even, $M = ((AB)^T)^{-1}$ if k is odd. Since $((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1}$,

$$\begin{aligned}(AB)p &= \left\langle (AB)\sigma_0, (A^T)^{-1}(B^T)^{-1}\sigma_1, (AB)\sigma_2, (A^T)^{-1}(B^T)^{-1}\sigma_3, ..., M\sigma_k \right\rangle \\ &= \left\langle A(B\sigma_0), (A^T)^{-1}((B^T)^{-1}\sigma_1), A(B\sigma_2), (A^T)^{-1}((B^T)^{-1}\sigma_3), ..., M\sigma_k \right\rangle = A(Bp)\end{aligned}$$

where $M = AB$ if k is even, $M = (A^T)^{-1}(B^T)^{-1}$ if k is odd.

Recall the paths p and $p_{\gamma\delta}$ from Definition 2.15. This is an example of the action of the matrix $A = \begin{bmatrix} \alpha & \widetilde{\beta} \end{bmatrix}$ on the path $p_{\gamma\delta}$.

Proposition 3.4. *For the paths p and $p_{\gamma\delta}$ defined in Definition 2.15, $p = Ap_{\gamma\delta}$ with $A = \begin{bmatrix} \alpha & \widetilde{\beta} \end{bmatrix}$, where α, β are the first two vectors of the path p.*

Proof. For $A = \begin{bmatrix} \alpha & \widetilde{\beta} \end{bmatrix}$, $(A^T)^{-1} = \begin{bmatrix} \beta & \widetilde{\alpha} \end{bmatrix}$. By definition,

$$Ap_{\gamma\delta} = \left\langle A\gamma_0, (A^T)^{-1}\delta_0, A\gamma_1, (A^T)^{-1}\delta_1, ... \right\rangle.$$

By Corollary 2.19, $A\gamma_k = \sigma_{2k}$ and $(A^T)^{-1}\delta_k = \sigma_{2k+1}$, therefore

$$Ap_{\gamma\delta} = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_2, ... \rangle = p.$$

□

It is clear that, for a pair of adjacent unimodular vectors α and β, α and $\widetilde{\beta}$ are R-linearly independent (as they form the columns of a matrix from $SL(2,R)$). The following lemma addresses the linear independence of two distinct vectors in $N(\beta)$.

Lemma 3.5. *Suppose $\langle\alpha,\beta,\gamma\rangle$ is a strand in $Um(2,R)$ with $\alpha\neq\gamma$. Then α and γ are linearly independent.*

Proof. Recall that two vectors α, $\gamma\in Um(2,R)$ are linearly independent if and only if for $r_1,r_2\in R$, $r_1\alpha+r_2\gamma=0_2\in R^2$ implies $r_1=r_2=0$. Let $\alpha=(a_1,a_2)^T$ and $\beta=(b_1,b_2)^T$. Proposition 2.13 tells us that

$$\gamma=\alpha+k\widetilde{\beta}=(a_1,a_2)^T+k(-b_2,b_1)^T=(a_1-kb_2,a_2+kb_1)^T$$

with $k\neq 0$ since $\alpha\neq\gamma$. Suppose $r_1\alpha+r_2\gamma=0_2$, then we have the system of equations $r_1a_1+r_2(a_1-kb_2)=0$ and $r_1a_2+r_2(a_2+kb_1)=0$. Manipulating the system of equations yields $-kr_2(a_1b_1+a_2b_2)=0$, and since $\beta\in N(\alpha)$, $-kr_2=0$. Since $-kr_2=0$ for all $k\neq 0$, $r_2=0$. Substituting back into the original equations yields $r_1a_1=0$ and $r_1a_2=0$. Since $\alpha\in Um(2,R)$, a_1 and a_2 cannot both be zero. Multiplying $\alpha^T\beta=1$ by r_1 gives us

$$r_1=r_1\alpha^T\beta=r_1(a_1b_1+a_2b_2)=(r_1a_1)\,b_1+(r_1a_2)\,b_2=0b_1+0b_2=0.$$

Thus we can conclude that $r_1=0$. Therefore α, γ are R-linearly independent. □

For certain loops and certain matrices, the standard action of matrices on paths restricts to a well-defined action on loops.

Proposition 3.6. *Suppose $A\in GL(2,R)$ and let $\mathfrak{a}=\langle\sigma_0,\sigma_1,...,\sigma_k=\sigma_0\rangle$ be a loop. Then $A\mathfrak{a}$ is a loop if and only if k is even or $A=(A^T)^{-1}$.*

Proof. Suppose first that A acting on a k-loop $\mathfrak{a}$ produces a loop. Then $A\sigma_0 = M\sigma_k$, with $M = A$ if k is even, $M = (A^T)^{-1}$ if k is odd. If k is even, this direction is complete. If k is odd, then $M\sigma_k = (A^T)^{-1}\sigma_k = A\sigma_0$. Since $\mathfrak{a}$ is a loop, it follows that $\sigma_0 = \sigma_k$, and therefore, since k is odd, $(A^T)^{-1}\sigma_0 = A\sigma_0$. Thus $\sigma_0 = (A^TA)\sigma_0$. Employing a similar calculation on the 2-translate of $\mathfrak{a}$ yields $\sigma_2 = (A^TA)\sigma_2$. By Lemma 3.5, σ_0 and σ_2 are linearly independent, and hence an R-basis of R^2. Therefore A^TA acts as the identity of the R-basis, so $A^TA = I_2$. Thus $(A^T)^{-1} = A$.

Next, suppose $A = (A^T)^{-1}$. Then A acting on a k-loop $\mathfrak{a}$ produces the path

$$\left\langle A\sigma_0, (A^T)^{-1}\sigma_1, A\sigma_2, (A^T)^{-1}\sigma_3, ..., (A^T)^{-1}\sigma_k \right\rangle.$$

Since $\mathfrak{a}$ is a loop, $\sigma_0 = \sigma_k$, and because $A = (A^T)^{-1}$, $A\sigma_0 = (A^T)^{-1}\sigma_k$. Therefore $A = (A^T)^{-1}$ acting on a k-loop $\mathfrak{a}$ produces a loop. □

Definition 3.7. *For a given path $p = \langle \sigma_0, \sigma_1, ..., \sigma_k \rangle$, define the ordered set $A * p :=$ $\left\{ A\left(\overset{*}{\sigma}_0\right), A\sigma_1, ..., A\sigma_{k-1}, A\sigma_k \right\}$. Note that in general, this is not always a path.*

Definition 3.8. *Let $\mathfrak{a} = \langle \sigma_0, \sigma_1, ..., \sigma_k = \sigma_0 \rangle$ be a loop. Define $G_{\mathfrak{a}}$ is the set of matrices $A \in GL(2, R)$ such that $A * \mathfrak{a}$ is a path, and hence a loop, since $A\sigma_0 = A\sigma_k$.*

3.2 Special Loops

Definition 3.9. *We define a pointed loop $\mathfrak{a} = \left\langle \overset{*}{\sigma}_0, \sigma_1, ..., \sigma_{k-1}, \sigma_k = \sigma_0 \right\rangle$ to be* special *if there exists a non-identity matrix $S \in E(2, R)$ and $m \in \mathbb{N}$ such that*

1. *$S\sigma_0 = \sigma_m$, and*

2. *$S * \mathfrak{a}$ is an m-translate of $\mathfrak{a}$, i.e.*

$$S * \mathfrak{a} = \left\langle \overset{*}{\sigma}_m, \sigma_{m+1}, ..., \sigma_{k-1}, \sigma_0, ..., \sigma_{m-1}, \sigma_m \right\rangle.$$

We say that the matrix S is co-special *to $\mathfrak{a}$.*

Proposition 3.10. *Suppose $\mathfrak{a}$ is a special loop. If S is a co-special matrix to $\mathfrak{a}$, then $S = (S^T)^{-1}$.*

Proof. Suppose S is a co-special matrix for the k-loop $\mathfrak{a} = \left\langle \overset{*}{\alpha_0}, \alpha_1, ..., \alpha_{k-1}, \alpha_k = \alpha_0 \right\rangle$. Then by Definition 3.9, $S * \mathfrak{a}$ produces the loop

$$S * \mathfrak{a} = \left\langle S\overset{*}{\alpha_0}, S\alpha_1, ..., S\alpha_{k-1}, S\alpha_k = S\alpha_0 \right\rangle$$

On the other hand, the standard action of S on the loop $\mathfrak{a}$ yields the loop

$$S\mathfrak{a} = \left\langle S\overset{*}{\alpha_0}, (S^T)^{-1}\alpha_1, S\alpha_2, ..., M\alpha_k \right\rangle$$

with $M = S$ if k is even, $M = (S^T)^{-1}$ if k is odd.

Note that the first three elements $S * \mathfrak{a}$ and $S\mathfrak{a}$ each forms a 3-strand with $S\alpha_0$ and $S\alpha_2$ as end-points, with the middle elements different. We can construct the 4-loop: $\langle S\alpha_0, S\alpha_1, S\alpha_2, (S^T)^{-1}\alpha_1, S\alpha_0 \rangle$. Since R is a domain, by Lemma 1.11, $S\alpha_1 = (S^T)^{-1}\alpha_1$, thus $(S^TS)\alpha_1 = \alpha_1$.

Similarly, the third, fourth, and fifth elements of $S * \mathfrak{a}$ and $S\mathfrak{a}$ also each forms a 3-strand with $S\alpha_2$ and $S\alpha_4$ as end-points, with the middle elements different. We can construct the 4-loop: $\langle S\alpha_2, S\alpha_3, S\alpha_4, (S^T)^{-1}\alpha_3, S\alpha_2 \rangle$. Since R is a domain, by Lemma 1.11, $S\alpha_3 = (S^T)^{-1}\alpha_3$, thus $(S^TS)\alpha_3 = \alpha_3$.

By Lemma 3.5, α_1 and α_3 are linearly independent, so $A = \begin{bmatrix} \alpha_1 & \alpha_3 \end{bmatrix}$ is invertible, and $(S^TS)A = A$. Thus S^TS acts as the identity on the R-basis $\{\alpha_1, \alpha_3\}$ of R^2, so $S^TS = I_2$. Therefore $S = (S^T)^{-1}$. □

Proposition 3.10 means that if S is a co-special matrix for special loop $\mathfrak{a}$, then $S * \mathfrak{a}$ and the standard action $S\mathfrak{a}$ produce the same loop. This restriction on co-special matrices allows us to further describe the set $G_{\mathfrak{a}}$ from Definition 3.8.

Corollary 3.11. *The set $G_{\mathfrak{a}}$ is precisely the set of co-special matrices for the loop $\mathfrak{a}$.*

Proposition 3.12. *Suppose that $\mathfrak{a}$ is a special k-loop. Then the only matrix whose action is a k-translate is I_2.*

Proof. Suppose $\mathfrak{a} = \left\langle \overset{*}{\alpha}_0, \alpha_1, ..., \alpha_{k-1}, \alpha_k = \alpha_0 \right\rangle$ and let B be a co-special matrix for $\mathfrak{a}$ such that

$$B * \mathfrak{a} = \left\langle \overset{*}{B\alpha_0}, B\alpha_1, ..., B\alpha_{k-1}, B\alpha_k = B\alpha_0 \right\rangle = \mathfrak{a}.$$

Then $B\alpha_i = \alpha_i$ for all $0 \leq i \leq k$. By Lemma 3.5, α_0 and α_2 are linearly independent. Specifically, $B\alpha_0 = \alpha_0$ and $B\alpha_2 = \alpha_2$, so

$$B \begin{bmatrix} \alpha_0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_2 \end{bmatrix}.$$

Since α_0 and α_2 are linearly independent by Lemma 3.5, B is acting as the identity on a basis for R^2, so $B = I_2$. □

Proposition 3.13. *Suppose a loop $\mathfrak{a}$ is special with respect to co-special matrix S and is also special with respect to co-special matrix T. Then $\mathfrak{a}$ is special with respect to co-special matrix ST.*

Proof. Let $\mathfrak{a}$ be special with respect to co-special matrices S and T, and assume $S * \mathfrak{a}$ is an m-translate of $\mathfrak{a}$ and $T * \mathfrak{a}$ is an n-translate of $\mathfrak{a}$. Therefore

$$
\begin{aligned}
(ST) * \mathfrak{a} &= \left\langle (ST)\left(\overset{*}{\sigma_0}\right), (ST)\sigma_1, ..., (ST)\sigma_{k-1}, (ST)\sigma_k \right\rangle \\
&= \left\langle ((\overset{*}{ST})\sigma_0), S(T\sigma_1), ..., S(T\sigma_{k-1}), S(T\sigma_k) \right\rangle \\
&= \left\langle (\overset{*}{S}\sigma_n), S(\sigma_{n+1}), ..., S(\sigma_{k-1}), S(\sigma_0), S(\sigma_1), ..., S(\sigma_{n-1}), S(\sigma_n) \right\rangle \\
&= \left\langle (\overset{*}{\sigma}_{m+n}), \sigma_{m+n+1}, ..., \sigma_{m+k-1}, \sigma_m, \sigma_{m+1}, ..., \sigma_{m+n-1}, \sigma_{m+n}) \right\rangle
\end{aligned}
$$

with each subscript of σ taken modulo k. Thus ST produces an $m+n$ translate of $\mathfrak{a}$ and $\mathfrak{a}$ is special with respect to ST. □

Corollary 3.14. *Suppose a loop $\mathfrak{a}$ is special with respect to co-special matrix S. Then $\mathfrak{a}$ is also special with respect to S^k for all integers $k \geq 1$.*

Proof. Proceeding by induction on k; if $k = 1$, then $\mathfrak{a}$ is special with respect to co-special matrix S by hypothesis.

Suppose $\mathfrak{a}$ is special with respect to co-special matrix S^{k-1}. Then applying Proposition 3.13 with $S = S$ and $T = S^{k-1}$ gives us that $\mathfrak{a}$ is special with respect to co-special matrix S^k. □

Proposition 3.15. *Suppose a loop $\mathfrak{a}$ is special with respect to co-special matrix S. Then the order of S in the group $GL(2, R)$ is finite.*

Proof. Suppose $\mathfrak{a}$ is a special k-loop, and S is co-special to $\mathfrak{a}$ with $S*\mathfrak{a}$ an m-translate. Then S^i is a special matrix for all integers i by Corollary 3.14. Let $j = \text{lcm}(k, m)$, then $S^j = S^{ak}$ for some integer a. Therefore S^j produces a k-translate on $\mathfrak{a}$, since $ak \equiv k \text{ mod}(k)$. By Proposition 3.12, $S^j = I_2$, and hence $|S|$, the order of S, is finite in the group $GL(2, R)$ and divides j. □

Proposition 3.16. *Suppose a loop $\mathfrak{a}$ is special with respect to co-special matrix S. Then the loop $\mathfrak{a}$ is also special with respect to co-special matrix S^{-1}.*

Proof. By Proposition 3.15, $|S| = h$ for some finite h. Proposition 3.14 tells us that $\mathfrak{a}$ is special with respect to co-special matrix S^k for all $k \geq 1$, in particular, $S^{h-1} = S^{-1}$. □

Theorem 3.17. *Given a special loop $\mathfrak{a}$, $G_{\mathfrak{a}}$, the set of all matrices co-special to $\mathfrak{a}$, is a subgroup of the group of 2×2 orthogonal matrices over R.*

Proof. Suppose $\mathfrak{a}$ is a special loop, and let $S, T \in G_{\mathfrak{a}}$. Then Proposition 3.13 tells us that ST is a co-special matrix, so $ST \in G_{\mathfrak{a}}$. Proposition 3.16 tells us that S^{-1} is a co-special matrix to $\mathfrak{a}$, so $S^{-1} \in G_{\mathfrak{a}}$. Also, Proposition 3.12 tells us that $I_2 \in G_{\mathfrak{a}}$. Hence the $G_{\mathfrak{a}}$ is a group with the operation matrix multiplication. □

Recall the set $\mathfrak{e}$ from Chapter 2. It is an 8-loop as presented (and is special with respect the co-special matrix $S = -I_2$). It can be augmented into a the 12-loop $\mathfrak{e}_+$ below, which is special with respect to the matrices associated with the negation and tilde operators from Proposition 2.6, which the reader can easily verify.

Definition 3.18. *We define the* base-8-loop *to be the 8-loop*

$$\mathfrak{e} = \langle \varepsilon_1, \varepsilon_*, \varepsilon_2, (-1,1), -\varepsilon_1, -\varepsilon_*, -\varepsilon_2, (1,-1), \varepsilon_1 \rangle \, .$$

We will define the augmented base-loop *to be the 12-loop obtained from the base-8-loop augmented by the four self-adjacent vectors $\varepsilon_1, \varepsilon_2, -\varepsilon_1, -\varepsilon_2$:*

$$\mathfrak{e}_+ = \langle \varepsilon_1, \varepsilon_1, \varepsilon_*, \varepsilon_2, \varepsilon_2, (-1,1), -\varepsilon_1, -\varepsilon_1, -\varepsilon_*, -\varepsilon_2, -\varepsilon_2, (1,-1), \varepsilon_1 \rangle \, .$$

Note that there is no point in further augmentation by self adjacency: if a loop $\mathfrak{a}$ has more than two self-adjacent vectors appearing consecutively, then any matrix acting on $\mathfrak{a}$ will produce a self-intersecting path: the self-adjacent portion will be a trivial 2-loop, so we will disregard such loops.

Proposition 3.19. *Each neighbor pair α, β appears in a special 12-loop with distinguished point α and second vector β. Moreover, such a special 12-loop can be constructed using a co-special matrix of order 4.*

Proof. Suppose that $\alpha \in Um(2,R)$ with $\beta \in N(\alpha)$. By Proposition 2.13, every neighbor of β can be expressed as $\gamma = \alpha + k\widetilde{\beta}$. In order for $S^4\alpha = \alpha$, to hold, $\gamma = \alpha + k\widetilde{\beta} \in N(\beta)$ must have k with $k^2 = 1$. We will use the co-special matrix

$$S = k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

to construct a special 12-loop. For this existence proof, we will use $k = 1$ (other choices k with $k^2 = 1$ may produce distinct 12-loops, as seen below). Examine the path

$$\mathfrak{p} = \left\langle \alpha, \beta, \gamma, S\alpha, S\beta, S\gamma, S^2\alpha, S^2\beta, S^2\gamma, S^3\alpha, S^3\beta, S^3\gamma, \alpha \right\rangle .$$

The strand $\langle \alpha, \beta, \gamma \rangle$ is a path by construction. Also, $\langle S\alpha, S\beta, S\gamma \rangle$, is a strand, since $S = (S^T)^{-1}$ by Proposition 3.10 and Proposition 3.6. Similar arguments show that $\left\langle S^2\alpha, S^2\beta, S^2\gamma \right\rangle$ and $\left\langle S^3\alpha, S^3\beta, S^3\gamma \right\rangle$ are also strands. The only edges that remain to be checked are γ to $S\alpha$, $S\gamma$ to $S^2\alpha$, $S^2\gamma$ to $S^3\alpha$, and $S^3\gamma$ to α. Note that left multiplication by S is same as the tilde operator ($S\sigma = \widetilde{\sigma}$ for all σ). So the edges that need to be checked are γ to $\widetilde{\alpha}$, $\widetilde{\gamma}$ to $\widetilde{\widetilde{\alpha}} = -\alpha$, $\widetilde{\widetilde{\gamma}} = -\gamma$ to $\widetilde{-\alpha}$, and $\widetilde{-\gamma}$ to α. Examine $\gamma^T\widetilde{\alpha} = ((\alpha)^T + (\widetilde{\beta})^T)(\widetilde{\alpha}) = \alpha^T\widetilde{\alpha} + \widetilde{\beta}^T\widetilde{\alpha} = 1$ since $\widetilde{\alpha} \in \ker(\alpha)$ by Proposition 2.10 and $\widetilde{\beta} \in N(\widetilde{\alpha})$ by Proposition 2.8. Notice that the other three are simply the adjacent pair α, β acted on by powers of S (and $(S^T)^{-1} = S$), hence $\mathfrak{p}$ is a loop, and it is special by construction. □

So for any neighbor pair $\beta \in N(\alpha)$, we can constuct a loop

$$\mathfrak{p} = \left\langle \alpha, \beta, \alpha + \widetilde{\beta}, \widetilde{\alpha}, \widetilde{\beta}, -\beta + \widetilde{\alpha}, -\alpha, -\beta, -\alpha - \widetilde{\beta}, -\widetilde{\alpha}, -\widetilde{\beta}, \beta - \widetilde{\alpha}, \alpha \right\rangle .$$

Using the paramaterization from Proposition 2.14, we can see that $\mathfrak{p}$ has an associated path $\mathfrak{p}_{\gamma\delta}$ (as in Proposition 2.15), which is a 12-loop:

$$\mathfrak{p}_{\gamma\delta} = \langle \varepsilon_1, \varepsilon_1, \varepsilon_*, \varepsilon_2, \varepsilon_2, (-1,1), -\varepsilon_1, -\varepsilon_1, -\varepsilon_*, -\varepsilon_2, -\varepsilon_2, (1,-1), \varepsilon_1 \rangle = \mathfrak{e}_+.$$

Observe that in the proof of Proposition 3.19, if $k = -1$ is chosen, then the constructed loop is

$$\mathfrak{p}' = \left\langle \alpha, \beta, \alpha - \widetilde{\beta}, -\widetilde{\alpha}, -\widetilde{\beta}, -\beta - \widetilde{\alpha}, -\alpha, -\beta, -\alpha + \widetilde{\beta}, \widetilde{\alpha}, \widetilde{\beta}, \beta + \widetilde{\alpha}, \alpha \right\rangle.$$

Using the paramaterization from Proposition 2.14, we can see that $\mathfrak{p}'$ has an associated path $\mathfrak{p}'_{\gamma\delta}$ (as in Proposition 2.15), which is a 12-loop:

$$\mathfrak{p}'_{\gamma\delta} = \langle \varepsilon_1, \varepsilon_1, (1,-1), -\varepsilon_2, -\varepsilon_2, -\varepsilon_*, -\varepsilon_1, -\varepsilon_1, (-1,1), \varepsilon_2, \varepsilon_2, \varepsilon_*, \varepsilon_1 \rangle,$$

which is the loop $\mathfrak{e}_+$ with direction reversed, beginning with two copies of ε_1. Note that, passing from ı to $p_{\gamma\delta}$ does not quite commute with reversing the direction of the path; they commute modulo a 1-translate, as by definition, $p_{\gamma\delta}$ must begin with two copies of ε_1 for any path p.

Proposition 1.10 provides a method to create a 6-loop from an invertible 2×2 matrix, so it can be used to create loops from a given unimodular vector and a fixed neighbor. We can, in fact, prove a stronger assertion than originally stated in Proposition 1.10: B is always A^{-1}.

Proposition 3.20. *With A, B as in Proposition 1.10, suppose the equivalent conditions are true. Then $B = A^{-1}$, and therefore $(f_1, f_2) = (b_1, b_2) + (d_1, d_2)$.*

Proof. Suppose $A = \begin{bmatrix} a_1 & e_1 \\ a_2 & e_2 \end{bmatrix}$ is an invertible matrix, with the associated closed path

$$\langle (a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2), (e_1, e_2), (f_1, f_2), (a_1, a_2) \rangle$$

with $(c_1, c_2) = (a_1, a_2) + (e_1, e_2)$, and let $B = \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix}$. Then

$$BA = \begin{bmatrix} a_1b_1 + a_2b_2 & e_1b_1 + e_2b_2 \\ a_1d_1 + a_2d_2 & e_1d_1 + e_2d_2 \end{bmatrix}.$$

Since $(a_1, a_2) \in N(b_1, b_2)$, $a_1b_1 + a_2b_2 = 1$; also, $(e_1, e_2) \in N(d_1, d_2)$, so $e_1d_1 + e_2d_2 = 1$. Now, $(c_1, c_2) \in N(b_1, b_2)$, so by Proposition 2.13, $(c_1, c_2) = (a_1, a_2) + k(-b_2, b_1)$ for some $k \in R$. Since $(c_1, c_2) = (a_1, a_2) + (e_1, e_2)$, $(e_1, e_2) = k(-b_2, b_1)$, we have that $(e_1, e_2) \in \ker(b_1, b_2)$. Similarly, $(c_1, c_2) \in N(d_1, d_2)$, so by Proposition 2.13, $(c_1, c_2) = (e_1, e_2) + k(-d_2, d_1)$ for some $k \in R$. Since $(c_1, c_2) = (a_1, a_2) + (e_1, e_2)$, $(a_1, a_2) = k(-d_2, d_1)$, $(a_1, a_2) \in \ker(d_1, d_2)$. Thus

$$BA = \begin{bmatrix} a_1b_1 + a_2b_2 & e_1b_1 + e_2b_2 \\ a_1d_1 + a_2d_2 & e_1d_1 + e_2d_2 \end{bmatrix} = I_2.$$

Hence $B = A^{-1}$. □

Proposition 3.21. *Suppose* $A = \begin{bmatrix} \sigma_1 & \sigma_5 \end{bmatrix}$ *is an invertible matrix, with corresponding 6-loop* $\mathfrak{h} = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_1 \rangle$*, where* $\sigma_3 = \sigma_1 + \sigma_5$*,* $A^{-1} = \begin{bmatrix} {\sigma_2}^T \\ {\sigma_4}^T \end{bmatrix}$*, and* $\sigma_6 = \sigma_2 + \sigma_4$ *as in Proposition 1.10. Then any five vectors in* $\mathfrak{h}$ *appear as five consecutive vectors of a special 12-loop. The number of associated co-special matrices to such a 12-loop depends on the choice of vectors.*

Proof. There are six cases, depending on which vector is omitted from the given 6-loop.

Case 1. *Omitting σ_1: consider the 5-strand $\langle \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \rangle$.*

We will start by defining a matrix B_1: let

$$B_1 = (A^{-1})^T \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_2 & \sigma_4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (\sigma_2 + \sigma_4) & -\sigma_2 \end{bmatrix} = \begin{bmatrix} \sigma_6 & -\sigma_2 \end{bmatrix}.$$

Since B_1 is invertible, it must have a unique central column $\omega_1 = \chi(B_1)$.

First, we show that $\mathfrak{a}_1 = \langle \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \omega_1, -\sigma_2, -\sigma_3, -\sigma_4, -\sigma_5, -\sigma_6, -\omega_1, \sigma_2 \rangle$ is a 12-loop. Since we have the 5-strand above, the only strands that need to be verified are the 3-strands $\langle \sigma_6, \omega_1, -\sigma_2 \rangle$ and $\langle -\sigma_6, -\omega_1, \sigma_2 \rangle$. Since ω_1 is the central column of an invertible matrix B_1, it must be adjacent to both columns of B_1. Hence $\omega_1 \in N(\sigma_6) \cap N(-\sigma_2)$. Also, $-\omega_1 \in N(-\sigma_6) \cap N(\sigma_2)$ by Proposition 2.8. Finally, $\mathfrak{a}_1$ is special with respect to $S = -I_2$, and S induces a 6-translate.

Additionally, there cannot be a co-special matrix of order 4 (which would have an associated 3-translate). Since $\sigma_2 \in \ker(\sigma_5)$, $\sigma_2 = k\widetilde{\sigma_5}$ for some $k \in R$, any such order 4 co-special matrix would be of the form $S = k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In this case, we would have $S^2 = -I_2$, since the induced 6-translate (associated with S^2) must send σ_2 to $-\sigma_2$. Thus we would have $k^2 = 1$. If S were to be co-special, then $S\sigma_4 = \omega_1$. But $S\sigma_4 = k\widetilde{\sigma_4} \in \ker(\sigma_4)$, so $\omega_1 \in \ker(\sigma_4)$. But also, $\sigma_1 \in \ker(\sigma_4)$, so $\omega_1 = r\sigma_1$ for some $r \in R$. Since $\omega_1 \in N(\sigma_6) \cap N(-\sigma_2)$ and $\sigma_1 \in N(\sigma_6)$, we would conclude that $r = 1$. However, σ_1 cannot be in $N(\sigma_2) \cap N(-\sigma_2)$ (which is empty), therefore no such co-special matrix S can exist.

Case 2. *Omitting σ_2: consider the 5-strand $\langle \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_1 \rangle$.*

We will start by defining a matrix B_2: let

$$B_2 = A \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_1 & -(\sigma_1 + \sigma_5) \end{bmatrix} = \begin{bmatrix} \sigma_1 & -\sigma_3 \end{bmatrix}.$$

Since B_2 is invertible, it must have a unique central column $\omega_2 = \chi(B_2)$.

First, we show that $\mathfrak{a}_2 = \langle \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_1, \omega_2, -\sigma_3, -\sigma_4, -\sigma_5, -\sigma_6, -\sigma_1, -\omega_2, \sigma_3 \rangle$ is a 12-loop. Since we have the 5-strand above, the only strands that need to be verified are the 3-strands $\langle \sigma_1, \omega_2, -\sigma_3 \rangle$ and $\langle -\sigma_1, -\omega_2, \sigma_3 \rangle$. Since ω_2 is the central column of an invertible matrix B_2, it must be adjacent to both columns of B_2. Hence $\omega_2 \in N(\sigma_1) \cap N(-\sigma_3)$. Also, $-\omega_2 \in N(-\sigma_1) \cap N(\sigma_3)$ by Proposition 2.8. Finally, $\mathfrak{a}_2$ is special with respect to $S = -I_2$, and S induces a 6-translate.

Additionally, there cannot be a co-special matrix of order 4 (which would have an associated 3-translate). Since $\sigma_4 \in \ker(\sigma_1)$, $\sigma_4 = k\widetilde{\sigma_1}$ for some $k \in R$, any such order 4 matrix would be of the form $S = k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In this case, we would have $S^2 = -I_2$, since the induced 6-translate (associated with S^2) must send σ_3 to $-\sigma_3$. Thus we would have $k^2 = 1$. If S were to be co-special, then $S\sigma_5 = \omega_2$. But $S\sigma_5 = k\widetilde{\sigma_5} \in \ker(\sigma_5)$, so $\omega_2 \in \ker(\sigma_5)$. But also, $\sigma_2 \in \ker(\sigma_5)$, so $\omega_2 = r\sigma_2$ for some $r \in R$. Since $\omega_2 \in N(\sigma_1) \cap N(-\sigma_3)$, and $\sigma_2 \in N(\sigma_1)$, we would conclude that $r = 1$. However, ω_2 cannot be in $N(\sigma_1) \cap N(-\sigma_1)$ (which is empty), therefore no such co-special matrix S can exist.

Case 3. *Omitting σ_3: consider the 5-strand $\langle \sigma_4, \sigma_5, \sigma_6, \sigma_1, \sigma_2 \rangle$.*

We will start by defining a matrix B_3: let

$$B_3 = (A^{-1})^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_2 & \sigma_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_2 & -\sigma_4 \end{bmatrix}.$$

Since B_3 is invertible, it must have a unique central column $\omega_3 = \chi(B_3)$.

First, we show that $\mathfrak{a}_3 = \langle \sigma_4, \sigma_5, \sigma_6, \sigma_1, \sigma_2, \omega_3, -\sigma_4, -\sigma_5, -\sigma_6, -\sigma_1, \sigma_2, -\omega_3, \sigma_4 \rangle$ is a 12-loop. Since we have the 5-strand above, the only strands that need to be verified are the 3-strands $\langle \sigma_2, \omega_3, -\sigma_4 \rangle$ and $\langle -\sigma_2, -\omega_3, \sigma_4 \rangle$. Since ω_3 is the central column of an invertible matrix B_3, it must be adjacent to both columns of B_3. Hence $\omega_3 \in N(\sigma_2) \cap N(-\sigma_4)$. Also, $-\omega_3 \in N(-\sigma_2) \cap N(\sigma_4)$ by Proposition 2.8. Finally, $\mathfrak{a}_3$ is special with respect to $S = -I_2$, and S induces a 6-translate.

Moreover, since $\omega_3 \in N(\sigma_2)$, by Proposition 2.13, $\omega_3 = \sigma_1 + j\alpha$ for some $\alpha \in \ker(\sigma_2)$. Also, $\omega_3 \in N(-\sigma_4)$, hence $\omega_3 = -\sigma_5 + k\alpha'$ for some $\alpha' \in \ker(-\sigma_4)$. Since $\sigma_5 \in \ker(\sigma_2)$ and $-\sigma_1 \in \ker(-\sigma_4)$, we can express $\omega_3 = \sigma_1 - \sigma_5$.

Additionally, $\mathfrak{a}_3 = \langle \sigma_4, \sigma_5, \sigma_6, \sigma_1, \sigma_2, \omega_3, -\sigma_4, -\sigma_5, -\sigma_6, -\sigma_1, \sigma_2, -\omega_3, \sigma_4 \rangle$ is special with respect to the co-special matrix $S = k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, for k any unit with $k^2 = 1$, which induces a 3-translate. Since $\sigma_1 \in \ker(\sigma_4)$, $\sigma_4 = k\widetilde{\sigma_1}$ for some $k \in R$. But also, $-\sigma_4 \in \ker(\sigma_1)$, so $\sigma_1 = k'\widetilde{(-\sigma_4)} = -k'\widetilde{\sigma_4}$ for some $k' \in R$. Combining yields $\sigma_1 = -k'\widetilde{\sigma_4} = -k'\widetilde{k\widetilde{\sigma_1}} = -k'k(-\sigma_1) = k'k\sigma_1$. So $k'k = 1$, thus k' and k are units with $k' = k^{-1}$. For S to induce a 3-translate, $k = k'$.

Case 4. *Omitting σ_4: consider the 5-strand $\langle \sigma_5, \sigma_6, \sigma_1, \sigma_2, \sigma_3 \rangle$.*

We will start by defining a matrix B_4: let

$$B_4 = A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (\sigma_1 + \sigma_5) & -\sigma_5 \end{bmatrix} = \begin{bmatrix} \sigma_3 & -\sigma_5 \end{bmatrix}.$$

Since B_4 is invertible, it must have a unique central column $\omega_4 = \chi(B_4)$.

First, we show that $\mathfrak{a}_4 = \langle \sigma_5, \sigma_6, \sigma_1, \sigma_2, \sigma_3, \omega_4, -\sigma_5, -\sigma_6, -\sigma_1, -\sigma_2, -\sigma_3, -\omega_4, \sigma_5 \rangle$ is a 12-loop. Since we have the 5-strand above, the only strands that need to be verified are the 3-strands $\langle \sigma_3, \omega_4, -\sigma_5 \rangle$ and $\langle -\sigma_3, -\omega_4, \sigma_5 \rangle$. Since ω_4 is the central column of an invertible matrix B_4, it must be adjacent to both columns of B_4. Hence $\omega_4 \in N(\sigma_3) \cap N(-\sigma_5)$. Also, $-\omega_4 \in N(-\sigma_3) \cap N(\sigma_5)$ by Proposition 2.8. Finally, $\mathfrak{a}_4$ is special with respect to $S = -I_2$, and S induces a 6-translate.

Additionally, there cannot be a co-special matrix of order 4 (which would have an associated 3-translate). Since $\sigma_2 \in \ker(\sigma_5)$, $\sigma_2 = k\widetilde{\sigma_5}$ for some $k \in R$, any such order 4 matrix would be of the form $S = k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In this case, we would have $S^2 = -I_2$, since the induced 6-translate (associated with S^2) must send σ_5 to $-\sigma_5$. Thus we would have that $k^2 = 1$. If S is were to be co-special, then $S\sigma_1 = \omega_4$. But $S\sigma_1 = k\widetilde{\sigma_1} \in \ker(\sigma_1)$, so $\omega_4 \in \ker(\sigma_1)$. But also, $\sigma_4 \in \ker(\sigma_1)$, so $\omega_4 = r\sigma_4$ for some $r \in R$. Since $\omega_4 \in N(\sigma_3) \cap N(-\sigma_5)$, and $\sigma_4 \in N(\sigma_3)$, we would conclude that $r = 1$. However, ω_4 cannot be in $N(\sigma_3) \cap N(-\sigma_3)$ (which is empty), therefore no such co-special matrix S can exist.

Case 5. *Omitting σ_5: consider the 5-strand $\langle \sigma_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$.*

We will start by defining a matrix B_5: let

$$B_5 = (A^{-1})^T \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_2 & \sigma_4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_4 & -(\sigma_2 + \sigma_4) \end{bmatrix} = \begin{bmatrix} \sigma_4 & -\sigma_6 \end{bmatrix}.$$

Since B_5 is invertible, it must have a unique central column $\omega_5 = \chi(B_5)$.

First, we show that $\mathfrak{a}_5 = \langle \sigma_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \omega_5, -\sigma_6, -\sigma_1 - \sigma_2, -\sigma_3, -\sigma_4, -\omega_5, \sigma_6 \rangle$ is a 12-loop. Since we have the 5-strand above, the only strands that need to be verified are the 3-strands $\langle \sigma_4, \omega_5, -\sigma_6 \rangle$ and $\langle -\sigma_4, -\omega_5, \sigma_6 \rangle$. Hence $\omega_5 \in N(\sigma_4) \cap N(-\sigma_6)$. Since ω_5 is the central column of an invertible matrix B_5, it must be adjacent to both columns of B_5. Also, $-\omega_5 \in N(-\sigma_4) \cap N(\sigma_6)$ by Proposition 2.8. Finally, $\mathfrak{a}_1$ is special with respect to $S = -I_2$, and S induces a 6-translate.

Additionally, there cannot be a co-special matrix of order 4 (which would have an associated 3-translate). Since $\sigma_1 \in \ker(\sigma_4)$, $\sigma_1 = k\widetilde{\sigma_4}$ for some $k \in R$, any such order 4 matrix would be of the form $S = k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In this case, we would have $S^2 = -I_2$, since the induced 6-translate (associated with S^2) would send σ_3 to $-\sigma_3$. Thus $k^2 = 1$. If S were to be co-special, then $S\sigma_2 = \omega_5$. But $S\sigma_2 = k\widetilde{\sigma_2} \in \ker(\sigma_2)$, hence $\omega_5 \in \ker(\sigma_2)$. But also, $\sigma_5 \in \ker(\sigma_2)$, so $\omega_5 = r\sigma_5$ for some $r \in R$. Since $\omega_5 \in N(\sigma_4) \cap N(-\sigma_6)$, and $\sigma_5 \in N(\sigma_4)$, we would conclude that $r = 1$. However, ω_5 cannot be in $N(\sigma_4) \cap N(-\sigma_4)$ (which is empty), therefore no such co-special matrix S can exist.

Case 6. *Omitting σ_6: consider the 5-strand $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \rangle$.*

We will start by defining a matrix B_6: let

$$B_6 = A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_5 & -\sigma_1 \end{bmatrix}.$$

Since B_6 is invertible, it must have a unique central column $\omega_6 = \chi(B_6)$.

First, we show that $\mathfrak{a}_6 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \omega_6, -\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4, -\sigma_5, -\omega_6, \sigma_1 \rangle$ is a 12-loop. Since we have the 5-strand above, the only strands that need to be verified are the 3-strands $\langle \sigma_5, \omega_6, -\sigma_1 \rangle$ and $\langle -\sigma_5, -\omega_6, \sigma_1 \rangle$. Since ω_6 is the central column of an invertible matrix B_6, it must be adjacent to both columns of B_6. Hence $\omega_6 \in N(\sigma_5) \cap N(-\sigma_1)$. Also, $-\omega_6 \in N(-\sigma_5) \cap N(\sigma_1)$ by Proposition 2.8. Finally, $\mathfrak{a}_6$ is special with respect to $S = -I_2$, and S induces a 6-translate.

Moreover, since $\omega_6 \in N(\sigma_5)$, by Proposition 2.13, $\omega_6 = \sigma_4 + j\alpha$ for some $\alpha \in \ker(\sigma_5)$. Also, $\omega_6 \in N(-\sigma_1)$, hence $\omega_6 = -\sigma_2 + k\alpha'$ for some $\alpha' \in \ker(-\sigma_1)$. Since $\sigma_2 \in \ker(\sigma_5)$ and $-\sigma_4 \in \ker(-\sigma_1)$, we can express $\omega_3 = \sigma_4 - \sigma_2$.

Additionally, $\mathfrak{a}_6 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \omega_6, -\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4, -\sigma_5, -\omega_6, \sigma_1 \rangle$ is special with respect to the co-special matrix $S = k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, for k any unit with $k^2 = 1$, which induces a 3-translate. Since $\sigma_4 \in \ker(\sigma_1)$, $\sigma_1 = k\widetilde{\sigma_4}$ for some $k \in R$. But also, $-\sigma_1 \in \ker(\sigma_4)$, so $\sigma_4 = k'\widetilde{(-\sigma_1)} = -k'\widetilde{\sigma_1}$ for some $k' \in R$. Combining yields $\sigma_4 = -k'\widetilde{\sigma_1} = -k'\widetilde{k\widetilde{\sigma_4}} =$

$-k'k(-\sigma_4) = k'k\sigma_4$. Therefore $k'k = 1$, thus k' and k are units with $k' = k^{-1}$. For S to induce a 3-translate, $k = k'$. □

CHAPTER 4

PATH COMPONENTS AND ELEMENTARY ORBITS

For this chapter, we will assume R is an integral domain unless otherwise stated. Recall that if R is an integral domain, then $R[x]$ is an integral domain as well. Additionally, unless otherwise stated, F will be a field with $F[x, y]$ the usual polynomial ring in indeterminates x and y. Observe that $F[x, y]$ is not a Euclidean domain, whereas $F[x]$ is.

4.1 Degree difference and path components

In general, for a ring R and $n \geq 3$, $Umc(n, R) \subseteq Um(n, R)$. Recall that for $n = 2$ the relationship between completable unimodular vectors and all unimodular vectors is straightforward: $Umc(2, R) = Um(2, R)$. This allows us to study $Um(2, R)$ through a different lens: since $\alpha \in Um(2, R)$ is completable, it can be viewed as the first column of a determinant one matrix.

Definition 4.1. *For $\alpha \in Um\left(2, R\left[x\right]\right)$ with $\alpha = \left(f, g\right)^T$, define the* degree difference of α*,*

$$\Delta\left(\alpha\right) := deg\left(f\right) - deg\left(g\right),$$

where $deg(f)$ denotes polynomial degree of the function f, with the degree of a non-zero constant being zero, and the degree of the zero polynomial being $-\infty$. A unimodular vector α is called equidegree *if and only if $\Delta(\alpha) = 0$.*

Note that $\Delta\left(\alpha\right)$ can be positive, negative, zero, or undefined, since $\Delta\left(\alpha\right)$ is well defined if and only if α does not have the zero polynomial as an entry.

Proposition 4.2. *Suppose* $\alpha \in Um(2, R[x])$, *with* $\alpha = (f, g)^T$. *If* $\Delta(\alpha) = n$ *and* $\beta \in N(\alpha)$ *with* $\Delta(\beta)$ *defined, then* $\Delta(\beta) = -n$.

Proof. Suppose $\alpha = (f, g)^T$ with $\Delta(\alpha) = n$, and let $\beta \in N(\alpha)$ have defined degree difference. Label $\beta = (a, b)^T$. Since $\beta \in N(\alpha)$, it follows by definition that $fa + gb = 1$. Hence $deg(fa + gb) = 0$ and $deg(fa) = deg(gb)$. Therefore $deg(f) + deg(a) = deg(g) + deg(b)$. Since $\Delta(\alpha) = n$, $deg(f) = deg(g) + n$, we have that $[deg(g) + n] + deg(a) = deg(g) + deg(b)$, therefore $n + deg(a) = deg(b)$. Thus $deg(a) - deg(b) = -n$, and therefore $\Delta(\beta) = -n$. $\square$

Note that the fact that R is an integral domain ensures that $deg(fa) = deg(f) + deg(a)$ as the leading coefficients of f and a cannot multiply together to zero.

Corollary 4.3. *Suppose* $\alpha \in Um(2, R[x])$ *and* $\langle \gamma_0 = \alpha, \gamma_1, \gamma_2, ..., \gamma_k = \beta \rangle$ *is a path with* $\Delta(\gamma_i)$ *defined for* $0 \leq i \leq k$. *Then* $\Delta(\alpha) = \Delta(\beta)$ *when* k *is even and* $\Delta(\beta) = -\Delta(\alpha)$ *when* k *is odd.*

Corollary 4.4. *Suppose* $\alpha \in Um(2, R[x])$ *is equidegree. Then if* $\beta \in N(\alpha)$ *has defined degree difference,* β *is equidegree.*

Notice that ε_* is equidegree, therefore all its neighbors with defined degree difference must also be equidegree. Hence the only way to build a path between an equidegree unimodular vector (in particular, ε_*) and a vector with non-zero degree difference is to go through a vector with undefined degree difference, i.e. a vector with a zero entry, which must be a unit multiple of ε_1 or ε_2 (i.e. the only strands close to ε_* that can change the degree difference are of the form $\langle ..., (u^{-1}, 0), (u, 1-u), \varepsilon_* \rangle$ and $\langle ..., (0, u^{-1}), (1-u, u), \varepsilon_* \rangle$, with the second to last vector omitted when $u = 1$).

Corollary 4.5. *Suppose $\alpha \in Um(2, R[x])$, with $\alpha = (f, g)^T$. If α is not equidegree and f and g are non-constant polynomials, then any radial path for α with respect to base-point ε_* must travel through either a unit multiple of ε_1 or a unit multiple of ε_2.*

It is useful to see how the Δ function interacts with the operators from Definition 2.4 and negation. The proofs of the parts of Proposition 4.6 are immediate from Definition 2.4.

Proposition 4.6. *For $\alpha \in Um(2, R[x])$, if $\Delta(\alpha) = n$, then:*

a. $\Delta(-\alpha) = n$.

b. $\Delta(\widehat{\alpha}) = -n$.

c. $\Delta(\widetilde{\alpha}) = -n$.

Proposition 4.7. *If $\alpha \in Um(2, R[x])$ is of the form $\alpha = (f, u)^T$ where u is a unit in $R[x]$, then α has neighbors with defined degree difference and neighbors with undefined degree difference. Furthermore, $\|\alpha\|_* \leq 3$.*

Proof. Suppose $\alpha \in Um(2, R[x])$ is of the form $\alpha = (f, u)^T$ where u is a unit in $R[x]$. Then $\beta \in N(\alpha)$ must be of the form $\beta = (0, u^{-1})^T + k(x)\widetilde{\alpha}$, where $k(x) \in R[x]$ by Proposition 2.13. If $k(x) = 0$, then $\Delta(\beta)$ is undefined; if $k(x) \neq 0$, then $\Delta(\beta) = \Delta(\widetilde{\alpha}) = -\Delta(\alpha)$.

A path to ε_* is $\langle \alpha, (0, u^{-1}), (1 - u, u), \varepsilon_* \rangle$, thus $\|\alpha\|_* \leq 3$. □

4.2 Spokes

Although $\langle \varepsilon_* \rangle$ is by definition connected, Corollary 4.3 tells us that if $|\Delta(\alpha)| \neq |\Delta(\beta)|$, then any path between α and β must go through a unit multiple of ε_1, ε_2, or ε_*.

Definition 4.8. *Recall $\mathfrak{e} = \{\varepsilon_1, \varepsilon_*, \varepsilon_2, (-1, 1), -\varepsilon_1, -\varepsilon_*, -\varepsilon_2, (1, -1)\}$. We will define the* punctured graph *of $Um(2, R)$ to be $Um(2, R) - R^*\mathfrak{e}$, where R^* is the group of units for the ring R, and $R^*\mathfrak{e} = \left\{ u\gamma \,\middle|\, u \in R^* \text{ and } \gamma \in \mathfrak{e} \right\}$.*

As we will see in Proposition 4.13, the punctured graph in many cases is not connected (as in the case of $Um(2, R[x])$ for an integral domain R) by application of Corollary 4.3. However, Corollary 4.3 depends on the degree difference function, which is not necessarily defined for all rings (for example, over $\mathbb{Z}$).

Definition 4.9. *Suppose $\alpha \in Um(2, R) - R^*\mathfrak{e}$. The* spoke containing α, *denoted $\mathscr{S}(\alpha)$, is the set of all vectors $\beta \in \langle \alpha \rangle$ where there exists a path between α and β that does not travel through $R^*\mathfrak{e}$.*

Proposition 4.10. *Suppose $\alpha, \beta \in Um(2, R) - R^*\mathfrak{e}$. Define the relation $\alpha \sim \beta$ if and only if there exists a path $\langle \alpha, \sigma_1, ..., \sigma_k, \beta \rangle$ with $\sigma_i \notin \mathfrak{e}$ for $1 \leq i \leq k$. Then the relation $\sim$ is an equivalence relation on the set $Um(2, R) - R^*\mathfrak{e}$.*

Proof. Suppose $\alpha, \beta, \gamma \in Um(2, R) - R^*\mathfrak{e}$. To verify that $\sim$ is an equivalence relation, it must be reflexive, symmetric, and transitive.

First, $\sim$ is reflexive, as the trivial path $\langle \alpha \rangle$ does not contain any vectors from $R^*\mathfrak{e}$ since $\alpha \in Um(2, R) - R^*\mathfrak{e}$.

Next, suppose $\alpha \sim \beta$. Then there exists a path $\langle \alpha, \sigma_1, ..., \sigma_k, \beta \rangle$ with $\sigma_i \notin \mathfrak{e}$ for $1 \leq i \leq k$. Reversing the path gives us the path $\langle \beta, \sigma_k, ..., \sigma_1, \alpha \rangle$ with $\sigma_i \notin \mathfrak{e}$ for $1 \leq i \leq k$, which is a path from β to α not traveling through $R^*\mathfrak{e}$. Thus $\beta \sim \alpha$ and $\sim$ is symmetric.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. Then there exists paths $\langle \alpha, \sigma_1, ..., \sigma_k, \beta \rangle$ with $\sigma_i \notin R^*\mathfrak{e}$ for $1 \leq i \leq k$ and $\langle \beta, \tau_1, ..., \tau_l, \gamma \rangle$ with $\tau_i \notin \mathfrak{e}$ for $1 \leq i \leq l$. Concatenating these paths yields the path $\langle \alpha, \sigma_1, ..., \sigma_k, \beta, \tau_1, ..., \tau_l, \gamma \rangle$ with $\sigma_i \notin R^*\mathfrak{e}$ for $1 \leq i \leq k$ and $\tau_i \notin \mathfrak{e}$ for $1 \leq i \leq l$. Therefore $\alpha \sim \gamma$ and $\sim$ is transitive. □

Notice that Proposition 4.10 and Definition 4.9 identify the same spokes. Proposition 4.10 has immediate corollaries.

Corollary 4.11. *$Um(2,R) - R^*\mathfrak{e}$ is the union of disjoint spokes. That is, each $\alpha \in Um(2,R) - R^*\mathfrak{e}$ is in exactly one spoke.*

Corollary 4.12. *Suppose $\alpha, \beta \in Um(2,R)$. Then $\mathscr{S}(\alpha) = \mathscr{S}(\beta)$ or $\mathscr{S}(\alpha) \cap \mathscr{S}(\beta) = \emptyset$.*

For a generic ring R, it is not immediately clear how many spokes $Um(2,R)$ has. In some cases, it is more obvious.

Proposition 4.13. *Suppose R is an integral domain. For $\alpha, \beta \in Um(2,R[x])$, if $\Delta(\alpha) \neq \pm\Delta(\beta)$, then $\mathscr{S}(\alpha) \neq \mathscr{S}(\beta)$.*

Proof. Let $\alpha, \beta \in Um(2,R[x])$ with $\Delta(\alpha) \neq \pm\Delta(\beta)$. By Corollary 4.3, at least one vector on the path between α and β must have undefined degree difference. Hence any path between α and β intersects $R^*\mathfrak{e}$. Therefore $\mathscr{S}(\alpha) \neq \mathscr{S}(\beta)$. $\square$

Corollary 4.14. *There are infinitely many spokes in $Um(2,R[x])$.*

Proof. One can see that there is a unimodular vector of every degree difference n: let $\alpha = \left(1 - x^{n+1}, x\right)^T$. Then $\alpha \in Um(2,R[x])$, as α has neighbor $(1,x^n)^T$, and $\alpha \in \langle \varepsilon_* \rangle$ as we have the path $\langle \alpha, (1,x^n)^T, \varepsilon_1, \varepsilon_* \rangle$. Therefore, for each n, there is a distinct spoke. $\square$

It is unclear in $R[x]$ if there is a single spoke for each α with $\Delta(\alpha) = n$, or if we can have two vectors $\alpha, \beta \in Um(2,R)$ with $\Delta(\alpha) = \Delta(\beta)$ and $\mathscr{S}(\alpha) \neq \mathscr{S}(\beta)$. However, we can say something about situations where $Um(2,R[x])$ is not path-connected.

Proposition 4.15. *If $Um(2,R[x])$ is not path-connected, then there exists at least one n and $\alpha, \beta \in Um(2,R[x])$ such that $n = \Delta(\alpha) = \Delta(\beta)$ with $\mathscr{S}(\alpha) \neq \mathscr{S}(\beta)$.*

Proof. Choose $\beta \notin \langle \varepsilon_* \rangle$ and let $n = \Delta(\beta)$. Then $\alpha = (1-x^{n+1}, x)^T \in Um(2,R)$ as displayed in the proof of Corollary 4.14 has $\Delta(\alpha) = n$, but since $\alpha \in \langle \varepsilon_* \rangle$, $\mathscr{S}(\alpha) \neq \mathscr{S}(\beta)$. $\square$

Over other rings, however, determining how many spokes exist is a potentially difficult question. Over $\mathbb{Z}$, for example, it is unknown whether $Um(2, \mathbb{Z}) - \mathbb{Z}^*\mathfrak{e}$ is connected or disconnected.

We will re-visit this idea in section 4.3, when we will see more about how spokes extend over pseudo-graphs $Um(2, R)$ that are not themselves connected.

4.3 Properties of $F[x, y]$

It is known that the set $Um(n, R)$ is not the same as the set of completable unimodular vectors: there exists rings R and values of $n \geq 3$ for which $Um(n, R) - Umc(n, R) \neq \emptyset$. In the case $n = 2$, every $\alpha \in Um(2, R)$ is completable.

We shall explore the conditions under which this completion of α is not the product of elementary generators, and hence R is not GE_2.

Recall a ring R is GE_2 if every 2×2 matrix with determinant one is the product of elementary generators. Cohn [1] exhibited that for a field F, $F[x, y]$ is not GE_2: there exists matrices in $SL(2, F[x, y])$ that are non-elementary. In particular, he presented the example

$$\begin{bmatrix} 1 - xy & -x^2 \\ y^2 & 1 + xy \end{bmatrix}$$

and proved that this determinant one matrix is not the product of elementary matrices using the following proposition.

Proposition 4.16. ([1], Proposition 7.3) *If R is a k-ring with degree function which is also a GE_2-ring, then [given] any two elements of the same degree which form a regular row, each is R-dependent on the other.*

For any field F, $F[x,y]$ is a k-ring, and Cohn notes that the first row $(1-xy, x^2)$ is regular and has the same (total) degree, and that the elements are not R-dependent, therefore $F[x,y]$ cannot be GE_2.

With Hinson's pseudo-graph structure and the theorems presented in this paper, this fact can be proven along with others relating the structure of the pseudo-graph and how it relates to properties of $F[x,y]$.

First, we need a pair of lemmas to prove the result.

Lemma 4.17. *Every completion of ε_1 to $SL(2,R)$ is an elementary generator.*

Proof. Suppose A is a completion of ε_1 with $\det(A) = 1$. Then $A = \begin{bmatrix} \varepsilon_1 & \beta \end{bmatrix}$ for some β with $\beta = (b_1, 1)^T$, as $A \in SL(2,R)$. Hence A is an elementary generator. □

Lemma 4.18. *For $\alpha \in \langle \varepsilon_1 \rangle$, every completion of α to $SL(2,R)$ is elementary.*

Proof. Suppose to the contrary that there exists a completion of $\alpha \in \langle \varepsilon_1 \rangle$ to $B \in SL(2,R)$ that is not elementary. Since $\langle \varepsilon_1 \rangle = [\varepsilon_1]$ and $\alpha \in \langle \varepsilon_1 \rangle$, then there exists an elementary matrix $A \in E(2,R)$ such that A is a completion of α. Since B and A are both completions of α, one can compute that BA^{-1} has first column ε_1. Since $B, A \in SL(2,R)$, $BA^{-1} \in SL(2,R)$, and by Lemma 4.17, BA^{-1} is an elementary generator E. Hence $B = EA$ is elementary, a contradiction. Therefore B must be elementary. □

Theorem 4.19. *A ring R is GE_2 if and only if $Um\,(2,R)$ is connected.*

Proof. Suppose first that R is non-GE_2. Then there exists a matrix A such that $A \in SL\,(2,R)$ and $A \notin E\,(2,R)$. Write this A as $\begin{bmatrix} \alpha & \widetilde{\beta} \end{bmatrix}$. Recall that $\det(A) = 1$ if and only if $\beta \in N(\alpha)$ in $Um(2,R)$ by Proposition 1.2. It is easy to see that $\alpha \notin \langle \varepsilon_* \rangle$, since by Lemma 4.18, $\alpha \in \langle \varepsilon_* \rangle$ would imply $A \in E(2,R)$.

Suppose next that the graph of $Um\,(2,R)$ is not connected. Then there exists a vector γ such that $\gamma \in Um\,(2,R)$ with $\gamma \notin \langle \varepsilon_* \rangle$. Then no completion of γ is elementary (since $[\varepsilon_1] = \langle \varepsilon_1 \rangle$). For $\delta \in N(\gamma)$, let $A_\gamma = \begin{bmatrix} \gamma & \widetilde{\delta} \end{bmatrix}$. Then $\det(A_\gamma) = 1$, but A_γ is not elementary. Therefore $A_\gamma \in SL\,(2,R)$ and $A_\gamma \notin E\,(2,R)$. $\square$

Proving that $F[x,y]$ is non-GE_2 is now equivalent to proving that $Um(2, F[x,y])$ is disconnected.

Definition 4.20. *For a non-zero $f\,(x,y) \in F\,[x,y]$, $deg_x\,(f(x,y))$ is the largest power of x (including 0) that appears in any term of $f(x,y)$. If $f(x,y) = 0$, set $deg_x\,(f(x,y)) = -\infty$. Define $deg_y\,(f(x,y))$ similarly.*

Note that we will view polynomials as they appear in extensions; for example, $f(x,y) = y^2$ has $deg_x(f) = 0$ and $deg_y(f) = 2$. The x-degree of the function in $F[x,y]$ equals the x-degree of the function when viewed as a function of $(F[y])[x]$, and similarly for y-degree.

Definition 4.21. *For $\alpha \in Um\,(2, F\,[x,y])$ with $\alpha = (f(x,y), g(x,y))^T$, the* x-degree difference of α *is*

$$\Delta_x\,(\alpha) := deg_x\,(f(x,y)) - deg_x\,(g(x,y))\,.$$

Define $\Delta_y\,(\alpha)$ similarly. Notice that $\Delta_x\,(\alpha)$, $\Delta_y\,(\alpha)$ are well defined if and only if α does not have the zero polynomial as an entry.

For example, for $\alpha = (1 - xy, y^2)^T$, $deg_x\,(1 - xy) = 1$, $deg_x\left(y^2\right) = 0$, and $\Delta_x\,(\alpha) = 1$. Similarly, $deg_y\,(1 - xy) = 1$, $deg_y\left(y^2\right) = 2$, and $\Delta_y\,(\alpha) = -1$.

If $\Delta_x\,(\alpha)$ is defined, so is $\Delta_y\,(\alpha)$, and any combination of positive, negative, and zero can occur (for example, there exist unimodular vectors with positive x-degree difference and zero y-degree difference, and any other combination).

Suppose $\alpha = (f(x,y), f(x,y)+1)^T$ and $f(x,y) \notin \{0,-1\}$. Then $\alpha \in Um(2, F[x,y])$ (with neighbor $(-1,1)^T$) and $\Delta_x(\alpha) = \Delta_y(\alpha) = 0$ (if $f(x,y) \in \{0,-1\}$, then $\Delta_x(\alpha), \Delta_y(\alpha)$ are undefined). We have seen that $\Delta_x\left((1-xy, y^2)^T\right) = 1$ and $\Delta_y\left((1-xy, y^2)^T\right) = -1$. Finally, suppose $\beta = (x^2y^2+1, xy)^T$. Then $\beta \in Um(2, F[x,y])$ (with neighbor $(1, -xy)^T$), and $\Delta_x(\alpha) = \Delta_y(\alpha) = 1$.

It is interesting to see how degree differences and the relation of being neighbors are related, especially away from the base points.

Proposition 4.22. *Suppose $\alpha, \beta \in Um(2, F[x,y])$ with $\beta \in N(\alpha)$. Then when defined, $\Delta_x(\alpha) = -\Delta_x(\beta)$ and $\Delta_y(\alpha) = -\Delta_y(\beta)$.*

Proof. If we view $F[x,y]$ as $(F[x])[y]$, then $\Delta_y(\alpha) = -\Delta_y(\beta)$ by Proposition 4.2. Similarly, if we view $F[x,y]$ as $(F[y])[x]$, then $\Delta_x(\alpha) = -\Delta_x(\beta)$. □

Proposition 4.22 is very useful; it states that for any set of neighbors, their x-degree differences and y-degree differences "flip" signs when defined, i.e. away from the base-point, every vector $\gamma \in \langle\alpha\rangle$ has either $\Delta_x(\gamma) = \Delta_x(\alpha)$ and $\Delta_y(\gamma) = \Delta_y(\alpha)$ or $\Delta_x(\gamma) = -\Delta_x(\alpha)$ and $\Delta_y(\gamma) = -\Delta_y(\alpha)$.

Proposition 4.23. *Suppose $\alpha \in Um(2, F[x,y])$. Then every vector $\gamma \in \langle\alpha\rangle$ connected to α by a path not traveling through $R^*\mathfrak{e}$ has $\Delta_x(\gamma) = \pm\Delta_x(\alpha)$ and $\Delta_y(\gamma) = \pm\Delta_y(\alpha)$, with $\Delta_x(\gamma) = \Delta_x(\alpha)$, $\Delta_y(\gamma) = \Delta_y(\alpha)$ corresponding to paths of even length between γ and α, and $\Delta_x(\gamma) = -\Delta_x(\alpha)$, $\Delta_y(\gamma) = -\Delta_y(\alpha)$ corresponding to paths of odd length.*

Proof. By applying Proposition 4.22, we can see that $\gamma \in \langle\alpha\rangle$ connected to α by an even length path not traveling through $R^*\mathfrak{e}$ results in Proposition 4.22 being applied an even number of times, thus $\Delta_x(\gamma) = \Delta_x(\alpha)$, $\Delta_y(\gamma) = \Delta_y(\alpha)$. Applying the Proposition 4.22 to an odd length path not traveling through $R^*\mathfrak{e}$ results in $\Delta_x(\gamma) = -\Delta_x(\alpha)$, $\Delta_y(\gamma) = -\Delta_y(\alpha)$. □

Vectors can now be classified based on their x-degree differences and y-degree differences.

Definition 4.24. *A vector $\alpha \in Um(2, F[x,y])$ is called* heterogeneous *if $\Delta_x(\alpha)$, $\Delta_y(\alpha)$ exist and differ in sign (i.e. one strictly positive, one strictly negative). No vector α with either $\Delta_x(\alpha) = 0$ or $\Delta_y(\alpha) = 0$ is heterogeneous, even if the other variable's degree difference is non-zero.*

What is of particular interest about heterogeneous unimodular vectors? The heterogeneity of α ensures the heterogeneity of every neighbor of α, which would imply that every vector in $\langle\alpha\rangle$ is heterogeneous.

Lemma 4.25. *Suppose $\alpha \in Um(2, F[x,y])$ is heterogeneous. Then for every $\beta \in N(\alpha)$, $\Delta_x(\beta)$ and $\Delta_y(\beta)$ are defined.*

Proof. Let $\alpha = (f(x,y), g(x,y))^T$ with $\Delta_x(\alpha) = n$, $\Delta_y(\alpha) = -m$ for $n, m > 0$. Suppose to the contrary that $\beta \in N(\alpha)$ and $\Delta_x(\beta)$ is undefined. Then such a β would be of the form $(0, j(x,y))^T$ or $(h(x,y), 0)^T$.

Suppose first that $\beta = (0, j(x,y))^T$. Then since $\beta \in N(\alpha)$, $g(x,y)j(x,y) = 1$. Since $\Delta_y(\alpha) = -m$, $deg_y(g(x,y)) \geq m$. But $deg_y(g(x,y)j(x,y)) = deg_y(g(x,y)) + deg_y(j(x,y)) = 0$, therefore no such $j(x,y)$ can exist.

Similarly, if $\beta = (h(x,y), 0)^T$, $f(x,y)h(x,y) = 1$. Since $\Delta_x(\alpha) = n$, $deg_x(f(x,y)) \geq n$. But $deg_x(f(x,y)h(x,y)) = deg_x(f(x,y)) + deg_x(h(x,y)) = 0$, therefore no such $h(x,y)$ can exist.

A similar argument shows that $\Delta_y(\beta)$ must be defined as well. □

Corollary 4.26. *Suppose $\alpha \in Um(2, F[x,y])$ is heterogeneous. Then every $\beta \in N(\alpha)$ is also heterogeneous. Moreover, every $\beta \in \langle\alpha\rangle$ is also heterogeneous.*

Proof. Follows from Proposition 4.23 and Lemma 4.25. □

Proposition 4.27. *Suppose $\alpha \in Um\left(2, F\left[x, y\right]\right)$ is heterogeneous. Then $\alpha \notin \langle \varepsilon_* \rangle$.*

Proof. Note that $\Delta_x\left(\varepsilon_*\right) = 0 = \Delta_y\left(\varepsilon_*\right)$. Thus ε_* is equidegree, and not heterogeneous. Thus $\varepsilon_* \notin \langle \alpha \rangle$, hence $\alpha \notin \langle \varepsilon_* \rangle$. □

Corollary 4.28. *The vector $\alpha = (1 - xy, y^2)^T$ is a completable unimodular vector, but no path from α to ε_1 exists. Consequently, $Um(2, F[x, y])$ is not path connected.*

Proposition 4.29. *For any field F, $F[x, y]$ is not GE_2.*

Proof. By Corollary 4.28, we have a completable unimodular vector $\alpha = (1 - xy, y^2)^T$ with $\alpha \notin \langle \varepsilon_1 \rangle$. Therefore, by Theorem 4.19, $F[x, y]$ is not GE_2. □

Revisiting the idea of spokes, it is clear that the non-base path components are not changed by puncturing of the pseudo-graph $Um(2, F[x, y])$.

But an analagous question remains: how many distinct path components exist? Corollary 4.26 and Proposition 4.23 can be combined into the following result: suppose α is a heterogeneous unimodular vector, and let the ordered pair $(n, -m)$ be the x-degree difference and y-degree difference. Since α is heterogeneous, $n, -m$ differ in signs (i.e. either $n, m > 0$ or $n, m < 0$). Every vector in $\langle \alpha \rangle$ has degree differences $(n, -m)$ or $(-n, m)$.

Proposition 4.30. *There are infinitely many distinct path components in $Um(2, F[x, y])$.*

Proof. It suffices to show that for every pair of integers $n, m > 0$, there exists some $\alpha \notin \langle \varepsilon_* \rangle$ with $\Delta_x(\alpha) = n$ and $\Delta_y(\alpha) = -m$. Proposition 4.23 insures that distinct such pairs (up to negation) are in distinct path components. Given $n, m > 0$, let $\alpha = (1 - x^n y^m, y^{2m})^T$. We observe that $\alpha \in N((1 + x^n y^m, x^{2n}))$, and $\Delta(\alpha) = n$, $\Delta_y(\alpha) = -m$ as required.

□

4.4 Relationship between elementary orbits and path components

Much is known about the base-path component. By definition, any two vectors in the base-path component can be connected by a path. Additionally, for all commutative rings, $\langle \varepsilon_* \rangle = [\varepsilon_*]$. We shall see in this section that the behavior outside the base-path component can be very different.

Proposition 4.31. *Suppose α, $\beta \in Um(2, F[x,y])$ with $\alpha, \beta \notin \langle \varepsilon_* \rangle$. If $\langle \alpha \rangle = \langle \beta \rangle$, then either*

1. $\Delta_x(\alpha) = \Delta_x(\beta)$ *and* $\Delta_y(\alpha) = \Delta_y(\beta)$, *or*

2. $\Delta_x(\alpha) = -\Delta_x(\beta)$ *and* $\Delta_y(\alpha) = -\Delta_y(\beta)$.

Proof. Suppose $\langle \alpha \rangle = \langle \beta \rangle$, then there exists a path of length k $\langle \gamma_0 = \alpha, \gamma_1, \gamma_2, ..., \gamma_k = \beta \rangle$ between α and β. Viewing $F[x,y]$ as $(F[y])[x]$, we can use Corollary 4.3: $\Delta_x(\alpha) = \Delta_x(\beta)$ when k is even. Viewing $F[x,y]$ as $(F[x])[y]$, we can use Corollary 4.3 again: $\Delta_y(\alpha) = \Delta_y(\beta)$. A similar argument with k odd yields $\Delta_x(\alpha) = -\Delta_x(\beta)$ and $\Delta_y(\alpha) = -\Delta_y(\beta)$. □

Proposition 4.32. *Suppose $\alpha \in Um(2, F[x,y])$ with $\alpha \notin \langle \varepsilon_* \rangle$. If E is an elementary matrix then $E\alpha \notin \langle \varepsilon_* \rangle$.*

Proof. Suppose $\alpha \notin \langle \varepsilon_* \rangle$ and suppose to the contrary that $E\alpha \in \langle \varepsilon_* \rangle$. Then since $\langle \varepsilon_* \rangle = [\varepsilon_*]$, $E\alpha \in [\varepsilon_*]$. But then we would have $E^{-1}(E\alpha) = \alpha \in [\varepsilon_*] = \langle \varepsilon_* \rangle$ a contradiction. Thus $E\alpha \notin \langle \varepsilon_* \rangle$. □

Proposition 4.32 tells us that for any $\alpha \notin \langle \varepsilon_* \rangle$ and any $\beta \in [\alpha]$, $\beta \notin \langle \varepsilon_* \rangle$. For all commutative rings, $\langle \varepsilon_* \rangle = [\varepsilon_*]$; how are $[\alpha]$ and $\langle \alpha \rangle$ related when $\alpha \notin \langle \varepsilon_* \rangle$?

Proposition 4.33. *Suppose $\alpha \in Um(2, F[x,y])$ with $\alpha \notin \langle \varepsilon_* \rangle$. Then there exists $\sigma \in Um(2, F[x,y])$ such that $\sigma \in [\alpha]$, with $\sigma \notin \langle \alpha \rangle$.*

Proof. Suppose $\alpha = (f, g)^T \in Um(2, F[x, y])$. Let

$$\sigma = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \alpha = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f + gh \\ g \end{pmatrix}.$$

If h is chosen with $deg_x(h) > \Delta_x(\alpha)$, then $\Delta_x(\sigma) \neq \pm\Delta_x(\alpha)$, and by Proposition 4.23, $\sigma \notin \langle\alpha\rangle$. □

Corollary 4.34. *For $\alpha \in Um(2, F[x, y])$ with $\alpha \notin \langle\varepsilon_*\rangle$, $\langle\alpha\rangle \neq [\alpha]$.*

Observe that having matching degree differences may not imply matching path components: there may exist non-connected vectors outside $\langle\varepsilon_*\rangle$ that have the same degree differences. Proposition 4.33 tells us that $[\alpha] \not\subseteq \langle\alpha\rangle$. Even though $\langle\alpha\rangle \neq [\alpha]$, they will have common elements other than α itself.

Proposition 4.35. *Suppose $\alpha \in Um(2, F[x, y])$ for $\alpha \notin \langle\varepsilon_*\rangle$, then $|\langle\alpha\rangle \cap [\alpha]| \geq 4$.*

Proof. Suppose $\alpha \in Um(2, F[x, y])$ with $\alpha \notin \langle\varepsilon_*\rangle$ and let $\beta \in N(\alpha)$. Using Proposition 3.19 with $k = 1$, we can construct the loop $\mathfrak{a} = \big\langle\alpha, \beta, \gamma, S\alpha, S\beta, S\gamma, S^2\alpha, S^2\beta, S^2\gamma, S^3\alpha, S^3\beta, S^3\gamma, \alpha\big\rangle$ with

$$\gamma = \alpha - \widetilde{\beta}, \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Notice that S is elementary, as

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

thus $S\alpha, S^2\alpha, S^3\alpha \in [\alpha]$. But since $\mathfrak{a} \subset \langle\alpha\rangle$, we get that $S\alpha, S^2\alpha, S^3\alpha \in \langle\alpha\rangle$. Thus $\big\{\alpha, S\alpha, S^2\alpha, S^3\alpha\big\} \subset \langle\alpha\rangle \cap [\alpha]$. □

We can now conclude the following: for every $\alpha \in Um(2, R)$, we know that $|\langle \alpha \rangle \cap [\alpha]| = |[\alpha]|$ when $\alpha \in \langle \varepsilon_* \rangle$ and $|\langle \alpha \rangle \cap [\alpha]| \geq 4$ when $\alpha \notin \langle \varepsilon_* \rangle$. It is still unknown, and seems to be a quite difficult problem, whether the interection $|\langle \alpha \rangle \cap [\alpha]|$ is finite or infinite when $\alpha \notin \langle \varepsilon_* \rangle$.

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