ON NON-COMMUTATIVE CONTINUOUS FUNCTIONS AND ASYMPTOTIC SYMMETRIC GAUGE NORMS

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This thesis is dedicated to everyone who loves Mathematics.

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ABSTRACT

On Non-Commutative Continuous Functions and Asymptotic Symmetric Gauge Norms

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This dissertation has two parts. The first part deals with Don Hadwin's non-commutative continuous functions of countably many variables and shows that every separable C^*-algebra can be described in terms of countably many generators $x_1, x_2, \ldots$ and a single relation $\varphi(x_1, x_2, \ldots) = 0$ where φ is a non-commutative continuous function. The second involves representation of operator algebras on spaces of Banach space valued measurable functions and groups of measure preserving transformations. The main emphasis concerns describing asymptotic norms based on symmetric gauge norms on $L^\infty[0, 1]$.

CHAPTER 1

INTRODUCTION TO NON-COMMUTATIVE CONTINUOUS FUNCTIONS

We dedicate this first chapter to the preparation of the generalized Hadwin-Kaonga-Mathes theorem in the next chapter. The theoretical background of non-commutative continuous function is given here. This important concept is involved with the original Hadwin-Kaonga-Mathes theorem which will later be generalized and proven as our main result of this dissertation.

1.1 History of Non-Commutative Continuous Functions

The idea of non-commutative continuous function was first introduced and developed in 1970. Don Hadwin invented decomposable functions, which he later called non-commutative continuous functions of a single variable. In the early 1990's, Don's student, Llolsten Kaonga, extended these concepts to non-commutative continuous functions of arbitrarily many variables. Later in 2003, L. Kaonga, Hadwin, and B. Mathes wrote a paper on these functions and their properties. The nicest results held for functions of finitely many variables. Some brief notes about important notation, definitions and propositions are given below.

First, we let $\mathcal{X} \neq \emptyset$ be a set, we will view the elements of $\mathcal{X}$ as variables. If $x \in \mathcal{X}$ then x^* represents the adjoint of x. The set of $*$-polynomials denoted by $\mathbb{P}(\mathcal{X})$. This definition of $*$ extends to a unique involution $*$ on $\mathbb{P}(\mathcal{X})$ satisfying $z^* = \overline{z}$ for all complex number z, $(p+q)^* = p^* + q^*$ and $(pq)^* = q^* p^*$ for all $*$-polynomials p and q. For example, $p(x_1, x_2, x_3) = 3ix_1^* x_3 + 4x_2^2 x_3^*$ and its adjoint $p^*(x_1, x_2, x_3) = -3ix_3^* x_1 + 4x_3 x_2^{*2}$ are the elements in $\mathbb{P}(\mathcal{X})$.

Second, we let $\mathcal{H}$ be a Hilbert space and $\mathscr{B}(\mathcal{H})$ be the algebra of all operators on $\mathcal{H}$. We define $\mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$ as the set of all functions f from $\mathcal{X}$ to $\mathscr{B}(\mathcal{H})$. Then we also define $\mathcal{F}(\mathcal{X})$

to be the class (not a set) of all functions f in $\mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$. This means that if $f \in \mathcal{F}(\mathcal{X})$ then there exists a Hilbert space $\mathcal{H}_f$ such that $f \in \mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}_f))$. For $p \in \mathscr{P}(\mathcal{X})$ and $f \in \mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$, we intuitively give a definition of $p(f)$ as an evaluation map from $\mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$ to $\mathscr{B}(\mathcal{H})$ by replacing x by $f(x)$ and x^* by $f(x)^*$ for any x in $\mathcal{X}$. For example, $p(f) = p(f(x_1), f(x_2), f(x_3)) = 3i(f(x_1))^* f(x_3) + 4f(x_2)^2 (f(x_3))^*$ and $p^*(f) = p((f(x_1))^*, (f(x_2))^*, (f(x_3))^*) = 3i(f(x_3))^* f(x_1) + 4f(x_3)(f(x_2))^{*2}$. We define a seminorm on $\mathscr{P}(\mathcal{X})$ by

$$\|p\|_f = \|p(f)\| \tag{1.1}$$

for any $f \in \mathcal{F}(\mathcal{X})$.

Now we define the non-commutative continuous function as follows.

Definition 1.1.1. If there exists a net $\{p_\lambda\}$ in $\mathbb{P}(\mathcal{X})$ such that

$$\|p_\lambda(f) - \varphi(f)\| \to 0 \tag{1.2}$$

for every $f \in \mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$ then φ is called a **non-commutative continuous function**. The set of all non-commutative continuous functions on $\mathcal{X}$ is denoted by $\mathcal{C}\langle\mathcal{X}\rangle$.

A nice way to check whether φ is a non-commutative continuous function is stated as the theorem below.

Theorem 1.1.2. *If φ is well-defined on $\mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$ for any Hilbert space $\mathcal{H}$ and satisfies the following properties:*

1. $\varphi(f) \in C^*(\{f(x) : x \in \mathcal{X}\})$
2. $\varphi(f \oplus g) = \varphi(f) \oplus \varphi(g)$ *for all* $f, g \in \mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$
3. $\varphi(UfU^*) = U\phi(f)U^*$ *for all* $f \in \mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$ *and unitary* U

Then φ is a non-commutative continuous function.

Example 1.1.3. $\varphi(x) = |x| = \sqrt{x^*x} = \lim_{n\to\infty} p_n(x^*x)$ where $p_n \in \mathbb{R}[t]$ and $p_n(t) \to \sqrt{t}$ uniformly on compact subsets of $[0, \infty)$. Then φ is a non-commutative continuous function.

1.2 Hadwin-Kaonga-Mathes Theorem

We end this chapter with the Hadwin-Kaonga-Mathes theorem.

Theorem 1.2.1. *Suppose $\mathcal{A}$ is a unital C^*-algebra with finitely many generators, $a_1, a_2, \ldots, a_n$. Then there is a non-commutative continuous function φ of finitely many variables such that $\mathcal{A}$ is isomorphic to the universal C^*-algebra with generators $x_1, x_2, \ldots, x_n$ with the relation $\varphi(x_1, x_2, \ldots, x_n) = 0$.*

The proof of the theorem can be found in [3] It is often that we have a (possibly uncountable) family of relations to represent the C^*-algebra. The difficulty will be the construction of the single relation that will be represented such a C^*-algebra. Replacing finitely many relations with a single one relation is based on the following idea.

If $z_1, z_2, \ldots, z_n$ are complex-valued quantities. Consider the system of relations $z_1 = 0, z_2 = 0, \ldots, z_n = 0$ can be written as a single relation $\overline{z_1}z_1 + \overline{z_2}z_2 + \ldots + \overline{z_n}z_n = 0$. To see this, we use the fact that

$$\begin{cases} z = 0 \\ w = 0 \end{cases} \Longleftrightarrow \begin{cases} |z|^2 = 0 \\ |w|^2 = 0 \end{cases} \Longleftrightarrow |z|^2 + |w|^2 = 0 \Longleftrightarrow \overline{z}z + \overline{w}w = 0 \tag{1.3}$$

In operator algebra, we also have the similar idea. A simple hold s for operator-valued quantities $S_1, S_2, \ldots, S_n$. We have $S_1 = 0, S_2 = 0, \ldots, S_n = 0$ if and only if $S_1^*S_1 + S_2^*S_2 + \ldots + S_n^*S_n = 0$. To show that, we suppose S, T be operators with their adjoint S^*, T^* and apply the following properties and relationships of norm and inner product as follows

$$\begin{aligned} &\begin{cases} S = 0 \\ T = 0 \end{cases} \Longleftrightarrow \begin{cases} \|S\vec{x}\|^2 = 0 \\ \|T\vec{x}\|^2 = 0 \end{cases} \Longleftrightarrow \|S\vec{x}\|^2 + \|T\vec{x}\|^2 = 0 \\ &\Longleftrightarrow \langle S\vec{x}, S\vec{x}\rangle + \langle T\vec{x}, T\vec{x}\rangle = 0 \Longleftrightarrow \langle S\vec{x}, (S^*)^*\vec{x}\rangle + \langle T\vec{x}, (T^*)^*\vec{x}\rangle = 0 \\ &\Longleftrightarrow \langle S^*S\vec{x}, \vec{x}\rangle + \langle T^*T\vec{x}, \vec{x}\rangle = 0 \Longleftrightarrow \langle (S^*S + T^*T)\vec{x}, \vec{x}\rangle = 0 \\ &\Longleftrightarrow S^*S + T^*T = 0 \end{aligned} \tag{1.4}$$

Also, an inequality $|z| \leq r$ is equivalent to

$$
\begin{aligned}
r^2 - \overline{z}z \geq 0 &\Longleftrightarrow r^2 - \overline{z}z = \left|r^2 - \overline{z}z\right| = \sqrt{\left(r^2 - \overline{z}z\right)^2} \\
&\Longleftrightarrow \left(r^2 - \overline{z}z\right) - \sqrt{\left(r^2 - \overline{z}z\right)^2} = 0
\end{aligned}
\tag{1.5}
$$

For operator $\sqrt{S^*S}$ is defined and we have

$$
\|S\| \leq r \Longleftrightarrow r^2 - S^*S - \sqrt{\left(r^2 - S^*S\right)^* \left(r^2 - S^*S\right)} = 0 \tag{1.6}
$$

Thus, we have the similar result by the non-commutative continuous functions

$$
\begin{cases}
\varphi_1 = 0 \\
\varphi_2 = 0 \\
\quad\vdots \\
\varphi_n = 0
\end{cases}
\quad \Longleftrightarrow \quad \varphi_1^*\varphi_1 + \varphi_2^*\varphi_2 + \cdots + \varphi_n^*\varphi_n = 0 \tag{1.7}
$$

and

$$
\|\varphi\| \leq r \quad \Longleftrightarrow \quad \left(r^2 - \varphi^*\varphi\right) - \sqrt{\left(r^2 - \varphi^*\varphi\right)^* \left(r^2 - \varphi^*\varphi\right)} = 0 \tag{1.8}
$$

The following finitely generated C^*-algebras are the examples in which we apply the above Hadwin-Kaonga-Mathes theorem to represent them by a single relation using non-commutative function in finitely many variables.

Example 1.2.2. $\mathscr{C}(\overline{\mathbb{D}(0,1)})$ is generated by $f(z) = z$. The function f is generated by x such that $x^*x = xx^*$ and $\|x\| \leq 1$. By applying the construction of relation from the above paragraph, we can conclude that $\mathscr{C}(\overline{\mathbb{D}(0,1)})$ is isomorphic to $C^*\left(x | \varphi\left(x\right) = 0\right)$ where

$$
\varphi(x) = (x^*x - xx^*)^2 + [(1 - xx^*) - |1 - xx^*|]^2 \tag{1.9}
$$

Example 1.2.3. $\mathscr{C}(S^1)$ is again generated by an identity function satisfies $xx^* = 1$ and $x^*x = 1$. Therefore, $\mathscr{C}(S^1)$ is isomorphic to $C^*\left(x | \varphi\left(x\right) = 0\right)$ where

$$
\varphi(x) = (1 - x^*x)^2 + (1 - xx^*)^2 \tag{1.10}
$$

Similarly, $\mathscr{C}(S^1 \times S^1)$ is isomorphic to $C^*\left(x, y | \varphi\left(x, y\right) = 0\right)$ where

$$
\varphi(x, y) = (1 - x^*x)^2 + (1 - xx^*)^2 + (1 - y^*y)^2 + (1 - yy^*)^2 + (xy - yx)^*(xy - yx) \tag{1.11}
$$

Example 1.2.4. $M_{2\times 2}(\mathbb{C})$ is isomorphic to $C^*\left(x|\varphi\left(x\right)=0\right)$ where

$$\varphi(x) = (x^*)^2x^2 + [x^*x + xx^* - 1]^2 \tag{1.12}$$

CHAPTER 2

GENERALIZED HADWIN-KAONGA-MATHES THEOREM AND ITS PROOF

The statement of the generalized Hadwin-Kaonga-Mathes theorem is given in this chapter. Then we discuss the failure of technique in the proof from the original theorem to the proof of the generalized theorem. The key tool, truncation function, for proving the theorem together with its properties will be introduced. Lastly, we end this chapter with the proof of the generalized Hadwin-Kaonga-Mathes theorem.

2.1 Problem in Old Proof of Hadwin-Kaonga-Mathes Theorem

If $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ and $p \in \mathcal{C}\langle\mathcal{X}\rangle$. The topology on $\mathcal{C}\langle\mathcal{X}\rangle$ can be given by countably many seminorm $\nu_1, \nu_2, \ldots$. We define

$$\nu_N(p) = \sup_{\|T_1\|,\|T_2\|,\ldots,\|T_n\|\leq N} p(T_1, T_2, \ldots, T_n) \tag{2.1}$$

This enable us to define a metric between and $p, q \in \mathcal{C}\langle\mathcal{X}\rangle$

$$d(p,q) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\nu_N(p-q)}{1+\nu_N(p-q)} \tag{2.2}$$

Therefore, $\mathcal{C}\langle\mathcal{X}\rangle$ together with the metric defined in 2.2 is a metric space.

Next, we define the field $\mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$, the complex number with real and imaginary rational part to be the set of complex-rational number and we further define $\mathbb{P}_{\mathbb{Q}}(\mathcal{X})$ to be the set of $*$-polynomials with coefficient in $\mathbb{C}_{\mathbb{Q}}$. Then we can see that $\mathbb{P}_{\mathbb{Q}}(\mathcal{X})$ is countable and dense in $\mathcal{C}\langle\mathcal{X}\rangle$. Thus, $\mathcal{C}\langle\mathcal{X}\rangle$ is separable.

As a conclusion, the space of non-commutative continuous functions in n variables was proved to be separable and metrizable when $1 \leq n < \infty$. This is central to the proof of our desired result for finitely generated C^*-algebras.

If $\mathcal{A} = C^*(a_1, a_2, \ldots, a_n)$, then

$$\mathcal{J} = \{\varphi \in \mathcal{C}\langle \mathcal{X} \rangle : \varphi(a_1, a_2, \ldots, a_n) = 0\} \tag{2.3}$$

is a closed ideal of $\mathcal{C}\langle \mathcal{X} \rangle$. Thus, there is a countable dense subset $\{\varphi_1, \varphi_2, \ldots\}$ of $\mathcal{J}$. So $C^*(a_1, a_2, \ldots, a_n)$ is the universal C^*-algebra generated by $x_1, x_2, \ldots, x_n$ with the relations $\varphi_1(x_1, x_2, \ldots, x_n) = 0, \varphi_2(x_1, x_2, \ldots, x_n) = 0, \ldots$. We replace these many relations by a single relation

$$\varphi = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{1}{\nu_N(\varphi_N^* \varphi_N)} \varphi_N^* \varphi_N \tag{2.4}$$

This series converges for every $T_1, T_2, \ldots, T_n$ since $N \geq \max\{\|T_1\|, \|T_2\|, \ldots, \|T_n\|\}$.

$$\left\| \frac{1}{2^N} \frac{1}{\nu_N(\varphi_N^* \varphi_N)} \varphi_N^* \varphi_N \right\| \leq \frac{1}{2^N} \tag{2.5}$$

When $\mathcal{X} = \{x_1, x_2, \ldots\}$, the topology on $\mathcal{C}\langle \mathcal{X} \rangle$ is not given by countably many seminorms, and there is no analogue of ν_N which is used to get the countable dense subset and the convergence factors. The metrizability is not true when $n = \aleph_0$ and thus the Hadwin-Kaonga-Mathes theorem cannot be applied in this case. This means that we have to come up with a new approach to proof for the countably infinite many variables case.

2.2 Truncation Function

To generalize the original Hadwin-Kaonga-Mathes theorem, we introduce the idea of truncation of function.

Definition 2.2.1. Suppose $0 < r < \infty$. The **r-truncation function**, τ_r is defined by

$$\tau_r(x) = x h_r\left((x^* x)^{1/2}\right), \tag{2.6}$$

where $h_r : [0, \infty) \to [0, r)$ is defined by

$$h_r(t) = \begin{cases} r & \text{if } 0 \le t \le r \\ \frac{r^2}{t} & \text{if } t > r \end{cases} . \tag{2.7}$$

Example 2.2.2. For any complex number z,

$$\tau_r(z) = \begin{cases} z & \text{if } |z| \le r \\ r\frac{z}{|z|} & \text{if } |z| > r \end{cases} \tag{2.8}$$

Theorem 2.2.3. *For every noncommutative continuous function φ of any number of variables, the r-truncation of φ, is defined as $\tau_r \circ \varphi$. Then we have*

1. $\|(\tau_r \circ \varphi)(\vec{x})\| \le r$ *for all* $\vec{x}$
2. $\varphi(\vec{x}) = (\tau_r \circ \varphi)(\vec{x})$ *if and only if* $\|\varphi(\vec{x})\| \le r$
3. $(\tau_r \circ \varphi)(\vec{x}) = 0$ *if and only if* $\varphi(\vec{x}) = 0$

Proof. The proof is followed by definition of trucation function and properties of non-commutative continuous functions. ■

Lemma 2.2.4. *If* $\sum_{n=1}^{\infty} r_n < \infty$ *with* $r_n > 0$ *and* $\varphi_1, \varphi_2, \ldots$ *are non-commutative continuous functions then*

$$\varphi = \sum_{n=1}^{\infty} \tau_{r_n} \circ \varphi_n \tag{2.9}$$

is also a non-commutative continuous function.

Proof. We will prove this by using theorem 1.1.2. Take $f, g \in \mathcal{F}(\mathcal{X}, \mathscr{B}(\mathcal{H}))$ and an unitary operator U.

$$\begin{aligned} \varphi(f \oplus g) &= \sum_{n=1}^{\infty} (\tau_{r_n} \circ \varphi_n)(f \oplus g) \\ &= \sum_{n=1}^{\infty} (\tau_{r_n} \circ \varphi_n)(f) \oplus (\tau_{r_n} \circ \varphi_n)(g) \\ &= \sum_{n=1}^{\infty} (\tau_{r_n} \circ \varphi_n)(f) \oplus \sum_{n=1}^{\infty} (\tau_{r_n} \circ \varphi_n)(g) = \varphi(f) \oplus \varphi(g) \end{aligned}$$

$$
\begin{aligned}
\varphi\left(UfU^{*}\right) &= \sum_{n=1}^{\infty}\left(\tau_{r_n}\circ\varphi_n\right)\left(UfU^{*}\right) \\
&= \sum_{n=1}^{\infty}U\left(\tau_{r_n}\circ\varphi_n\right)(f)\,U^{*} \\
&= U\sum_{n=1}^{\infty}\left(\tau_{r_n}\circ\varphi_n\right)(f)\,U^{*} = U\varphi(f)\,U^{*}
\end{aligned}
$$

■

2.3 Generalized Hadwin-Kaonga-Mathes Theorem

In this section, we will give a complete proof of the Generalized Hadwin-Kaonga-Mathes theorem. We need the following important well-known lemma.

Lemma 2.3.1. *Suppose $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are countably infinitely many generators for unital C^*-algebra $\mathcal{A}$ and $\mathcal{B}$, respectively. The following are equivalent.*

1. *There is a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that $\pi(a_k) = b_k$ for all $k = 1, 2, \ldots$*
2. *$\|p(b_1, b_2, \ldots)\| \leq \|p(a_1, a_2, \ldots)\|$ for every complex coefficient polynomials p.*
3. *$\|p(b_1, b_2, \ldots)\| \leq \|p(a_1, a_2, \ldots)\|$ for every complex-rational coefficient polynomials p.*

We note that the field of complex-rational number is $\mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$, the complex number with rational real part and imaginary part.

Here is the statement of the Generalized Hadwin-Kaonga-Mathes theorem.

Theorem 2.3.2. *Suppose $\mathcal{A}$ is a unital C^*-algebra with countably infinitely many generators, $a_1, a_2, \ldots$. Then there is a non-commutative continuous function φ of countably infinitely many variables such that $\mathcal{A}$ is isomorphic to the universal C^*-algebra with generators $x_1, x_2, \ldots$ with the relation $\varphi(x_1, x_2, \ldots) = 0$.*

Proof of Generalized Hadwin-Kaonga-Mathes Theorem. Fix $n \in \mathbb{N}$. By Lemma 2.3.1, we will prove that for any complex-rational coefficient polynomials p.

$$\|p(x_1, x_2, \ldots)\| \leq \|p(a_1, a_2, \ldots)\|$$

Take $r_n = \|p(a_1, a_2, \ldots)\|$. We have $\|p(x_1, x_2, \ldots)\| \le r_n^{\frac{1}{2}}$. Equivalently, we obtain the non-commutative continuous functions

$$\varphi_n = r_n^2 - p^*p - \sqrt{(r_n^2 - p^*p)^*(r_n^2 - p^*p)} = 0$$

By Theorem 2.2.3, we truncate this non-commutative continuos function by $\frac{1}{2^n}$ and sum them up to get

$$\varphi = \sum_n \tau_{\frac{1}{2^n}} \circ \varphi_n = 0$$

■

Example 2.3.3. Recall that $\mathscr{C}(S^1 \times S^1)$ is a unital C^*-algebra generated by x and y with the relations $x^*x = xx^* = 1, y^*y = yy^* = 1$ and $xy = yx$. After combining the relations into single relation, it is isomorphic to $C^*(x, y|\varphi(x, y) = 0)$ where

$$\begin{aligned}
\varphi(x,y) &= (1 - x^*x)^*(1 - x^*x) + (1 - xx^*)^*(1 - xx^*) \\
&\quad + (1 - y^*y)^*(1 - y^*y) + (1 - yy^*)^*(1 - yy^*) \\
&\quad + (xy - yx)^*(xy - yx) \\
&= (1 - x^*x^{**})(1 - x^*x) + (1 - x^{**}x^*)(1 - xx^*) \\
&\quad + (1 - y^*y^{**})(1 - y^*y) + (1 - y^{**}y^*)(1 - yy^*) \\
&\quad + (xy - yx)^*(xy - yx) \\
&= (1 - x^*x)(1 - x^*x) + (1 - xx^*)(1 - xx^*) \\
&\quad + (1 - y^*y)(1 - y^*y) + (1 - yy^*)(1 - yy^*) \\
&\quad + (xy - yx)^*(xy - yx) \\
&= (1 - x^*x)^2 + (1 - xx^*)^2 + (1 - y^*y)^2 + (1 - yy^*)^2 \\
&\quad + (xy - yx)^*(xy - yx)
\end{aligned} \tag{2.10}$$

We can see that each circle will provide us one generator. Inductively for countably infinite case, $\mathscr{C}(S^1 \times S^1 \times S^1 \times \ldots)$ is a unital C^*-algebra generated by x_i for all $i = 1, 2, 3, \ldots$ with the relations

$$x_i^* x_i = x_i x_i^* = 1 \text{ and } x_i x_j = x_j x_i \text{ for all } i, j \in \{1, 2, 3, \ldots\} \tag{2.11}$$

We have $\mathscr{C}(S^1 \times S^1 \times S^1 \times \dots)$ is isomorphic to C^*-algebra generated by $x_1, x_2, x_3, \dots$ with the relation $\varphi\left(x_1, x_2, x_3, \dots\right) = 0$ where

$$\begin{aligned}\varphi\left(x_1, x_2, x_3, \dots\right) = &\sum_{i=1}^{\infty} \tau_{\frac{1}{2^i}} \circ (1 - x_i^* x_i)^2 + \sum_{j=1}^{\infty} \tau_{\frac{1}{2^j}} \circ (1 - x_j x_j^*)^2 \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tau_{\frac{1}{2^{i+j}}} \circ (x_i x_j - x_j x_i)^* (x_i x_j - x_j x_i)\end{aligned} \tag{2.12}$$

CHAPTER 3

APPLICATION OF GENERALIZED HADWIN-KAONGA-MATHES THEOREM

The success of generalized Hadwin-Kaonga-Mathes theorem for the case of countably infinitely many variables from the previous chapter will lead us nicely to characterize the classes of separable C^*-algebras by some certain properties, semiprojectivity and tracial stability. The corollaries stated with the assumption of countably infinite variables are similar to the proposition founded in the old works. Technically, we get the result in this chapter for free by following the proof for finitely many variables case.

3.1 Semiprojectivity

All of the definitions and theorems about semiprojectivity can be founded in [9] and [10] with much more details. We will only state the definitions that only use in the dissertation.

Definition 3.1.1. A C^*-algebra $\mathcal{A}$ is called **semiprojective** if for every $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}/\overline{\bigcup I_n}$, where I_n are increasing ideals in $\mathcal{B}$ i.e. $I_1 \subset I_2 \subset \ldots$ and with $q_N : \mathcal{B}/I_N \to \mathcal{B}/\overline{\bigcup I_n}$ the natural quotient map, there exists, for some N, a $*$-homomorphism $\overline{\pi} : \mathcal{A} \to \mathcal{B}/I_N$ such that $q_N \circ \overline{\pi} = \pi$.

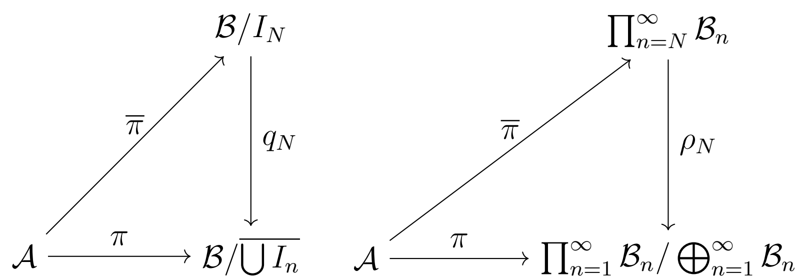
The next definition is a special case of semiprojectivity of C^*-algebra. Given C^*-algebras $\mathcal{B}_1, \mathcal{B}_2, \ldots$, we can construct a new C^*-algebra $\prod_{n=1}^{\infty} \mathcal{B}_n$ as a tensor product of $\mathcal{B}_n$, with its ideal $\bigoplus_{n=1}^{\infty} \mathcal{B}_n$ as the direct sum of $\mathcal{B}_n$. Next, we define the quotient map ρ_N from $\prod_{n=1}^{\infty} \mathcal{B}_n$ to $\prod_{n=1}^{\infty} \mathcal{B}_n / \bigoplus_{n=1}^{\infty} \mathcal{B}_n$ for some $N \in \mathbb{N}$ by

$$\rho_N : (b_N, b_{N+1}, \ldots) \longmapsto [(0, 0, \ldots, 0, b_N, b_{N+1}, \ldots)]$$

where $b_N \in \mathcal{B}_N, b_{N+1} \in \mathcal{B}_{N+1}, \ldots$.

Definition 3.1.2. A C^*-algebra $\mathcal{A}$ is called **weakly semiprojective** if for every $*$-homomorphism $\pi : \mathcal{A} \to \prod_{n=1}^{\infty} \mathcal{B}_n / \bigoplus_{n=1}^{\infty} \mathcal{B}_n$, and $\rho_N : \prod_{n=N}^{\infty} \mathcal{B}_n \to \prod_{n=1}^{\infty} \mathcal{B}_n / \bigoplus_{n=1}^{\infty} \mathcal{B}_n$ defined as above, there exists, for some N, a $*$-homomorphism $\overline{\pi} : \mathcal{A} \to \prod_{n=N}^{\infty} \mathcal{B}_n$ such that $\rho_N \circ \overline{\pi} = \pi$.

These two definitions can be stated as the lifting properties of $*$-homomorphisms that make the each of the following diagrams commute.



Another way to define weakly semiprojectivity is as follows.

Definition 3.1.3. Suppose, for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a unital C^*-algebra. We say that a C^*-algebra $\mathcal{A}$ is called weakly semiprojective if, for every $*$-homomorphism $\pi : \mathcal{A} \to \prod_{n=1}^{\infty} \mathcal{B}_n \,/ \bigoplus_{n=1}^{\infty} \mathcal{B}_n$ there is an $N \in \mathbb{N}$ and $*$-homomorphisms $\pi_n : \mathcal{A} \to \mathcal{B}_n$ for all $n \geq N$, such that, for every $a \in \mathcal{A}$,

$$\pi(a) = \rho\left((0, \ldots, 0, \pi_N(a), \pi_{N+1}(a), \ldots)\right).$$

From this point, we let $\mathcal{A}$ be a universal C^*-algebra generated by $x_1, x_2, \ldots$ with the relation $\varphi(x_1, x_2, \ldots) = 0$ where φ is a non-commutative continuous function.

Since the generalized Hadwin-Kaonga-Mathes holds, we obtain these two following corollaries.

Corollary 3.1.4. *Suppose $\|x_k\| < r_k$ for all $k \in \mathbb{N}$. $\mathcal{A}$ is weakly semiprojective if and only if for $\epsilon > 0$, there is $\delta > 0$ and non-commutative continuous functions $\psi_m(t_1, t_2, \ldots)$ for all $m \in \mathbb{N}$ such that for any unital C^*-algebra $\mathcal{B}$ and $b_1, b_2, \ldots \in \mathcal{B}$ with $\|b_k\| \leq r_k$ for all $k \in \mathbb{N}$, if $\varphi \|(b_1, b_2, \ldots)\| < \delta$ then*

1. *$\psi_n = \tau_{r_n} \circ \psi_n$ for all $n \in \mathbb{N}$ where τ_{r_n} is a truncation.*

2. *Take* $\vec{t} = (t_1, t_2, \ldots)$ *and* $\vec{b} = (b_1, b_2, \ldots)$. *We get*

$$\varphi\left(\psi_1\left(\vec{t}\right), \left(\psi_2\left(\vec{t}\right)\right), \ldots\right) = 0 \text{ and } \sum_{n=1}^{\infty} \frac{1}{2^{n+1} r_n} \left\| b_n - \psi_n\left(\vec{b}\right)\right\| < \epsilon.$$

Corollary 3.1.5. *Suppose* $\|x_k\| < r_k$ *for all* $k \in \mathbb{N}$. $\mathcal{A}$ *is semiprojective if and only if for* $\epsilon > 0$, *there is* $\delta > 0$ *and non-commutative continuous functions* $\psi_m(t_1, t_2, \ldots)$ *for all* $m \in \mathbb{N}$ *such that for any unital* C^**-algebra* $\mathcal{B}$ *and* $b_1, b_2, \ldots \in \mathcal{B}$ *with* $\|b_k\| \leq r_k$ *for all* $k \in \mathbb{N}$, *if* $\varphi \|(b_1, b_2, \ldots)\| < \delta$ *then*

1. $\psi_n = \tau_{r_n} \circ \psi_n$ *for all* $n \in \mathbb{N}$ *where* τ_{r_n} *is a truncation.*

2. *Take* $\vec{t} = (t_1, t_2, \ldots)$ *and* $\vec{b} = (b_1, b_2, \ldots)$. *We get*

$$\varphi\left(\psi_1\left(\vec{t}\right), \left(\psi_2\left(\vec{t}\right)\right), \ldots\right) = 0 \text{ and } \sum_{n=1}^{\infty} \frac{1}{2^{n+1} r_n} \left\| b_n - \psi_n\left(\vec{b}\right)\right\| < \epsilon.$$

moreover, if $\varphi(b_1, b_2, \ldots) = 0$ *then* $\psi_n\left(\vec{b}\right) = b_n$.

We can find the similar proof in [3].

Note that these two corollaries confirms the fact that if $\mathcal{A}$ is semiprojective then $\mathcal{A}$ is weakly semiprojective.

3.2 Tracial Stability

The tracial stability property was in [11]. There are many formulations for determining the tracial stability. We might treat the next theorem as our own $\epsilon - N$ definition of tracial stability in this dissertation. Its statement and proof are also in [11]. Let $\mathcal{S}$ be a class of unital C^*-algebra that closed under isomorphisms.

Theorem 3.2.1. $\mathcal{A}$ *is* ***tracially stable*** *if and only if for every* $\epsilon > 0$ *for every* $n \in \mathbb{N}$, *and for every tracial state* ζ *on* $\mathcal{A}$, *there is* $N \in \mathbb{N}$ *such that if* $\mathcal{B} \in \mathcal{S}$ *is a and* ξ *is a tracial state on* $\mathcal{B}$ *and* $b_1, b_2, \ldots, b_N \in \mathcal{B}$ *such that* $\|b_k\| \leq 1 + \|x_k\|$ *for* $1 \leq k \leq N$ *and*

$$\left|\zeta\left(m\left(x_1, x_2, \ldots, x_N\right)\right) - \xi\left(m\left(b_1, b_2, \ldots, b_N\right)\right)\right| < \frac{1}{N}$$

for all $*$-monomials $m(t_1, t_2, \ldots, t_N)$ with degree at most N then there is a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that

$$\sum_{k=1}^{n} \|\pi(x_k) - b_k\|_{2,\xi} < \epsilon$$

The above properties imply the whatever C^*-properties that $\mathcal{B}$ has then $\mathcal{A}$ will be tracially stable with that C^*-properties.

By generalized Hadwin-Kaonga-Mathes theorem, we similarly let

$$\mathcal{A} = C^* \langle x_1, x_2, \ldots | \varphi(x_1, x_2, \ldots) = 0 \rangle$$

as in previous section. Other two corollaries about characterization of C^*-algebra by tracial stability are below.

Corollary 3.2.2. *Suppose $\|x_k\| < r_k$ for all $k \in \mathbb{N}$. $\mathcal{A}$ is tracially stable if and only if for $\epsilon > 0$, there is $\delta > 0$ and non-commutative continuous functions $\psi_m(t_1, t_2, \ldots)$ for all $m \in \mathbb{N}$ such that for any unital C^*-algebra $\mathcal{B}$ with a tracial state ζ and $b_1, b_2, \ldots \in \mathcal{B}$ with $\|b_k\| \leq r_k$ for all $k \in \mathbb{N}$, if $\varphi \|(b_1, b_2, \ldots)\|_{2,\zeta} < \delta$ then*

1. *$\psi_n = \tau_{r_n} \circ \psi_n$ for all $n \in \mathbb{N}$ where τ_{r_n} is a truncation.*
2. *Take $\vec{t} = (t_1, t_2, \ldots)$ and $\vec{b} = (b_1, b_2, \ldots)$. We get*

$$\varphi\left(\psi_1\left(\vec{t}\right), \left(\psi_2\left(\vec{t}\right)\right), \ldots\right) = 0 \text{ and } \textstyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1} r_n} \left\|b_n - \psi_n\left(\vec{b}\right)\right\|_{2,\zeta} < \epsilon.$$

Corollary 3.2.3. *Suppose $\|x_k\| < r_k$ for all $k \in \mathbb{N}$. $\mathcal{A}$ is matrically tracially stable if and only if for $\epsilon > 0$, there is $\delta > 0$ and non-commutative continuous functions $\psi_m(t_1, t_2, \ldots)$ for all $m \in \mathbb{N}$ such that for $b_1, b_2, \ldots \in M_{n \times n}(\mathbb{C})$ with $\|b_k\| \leq r_k$ for all $k \in \mathbb{N}$, if $\varphi \|(b_1, b_2, \ldots)\|_{2,\zeta_n} < \delta$ then*

1. *$\psi_n = \tau_{r_n} \circ \psi_n$ for all $n \in \mathbb{N}$ where τ_{r_n} is a truncation.*
2. *Take $\vec{t} = (t_1, t_2, \ldots)$ and $\vec{b} = (b_1, b_2, \ldots)$. We get*

$$\varphi\left(\psi_1\left(\vec{t}\right), \left(\psi_2\left(\vec{t}\right)\right), \ldots\right) = 0 \text{ and } \textstyle\sum_{n=1}^{\infty} \frac{1}{2^{n+1} r_n} \left\|b_n - \psi_n\left(\vec{b}\right)\right\|_{2,\zeta_n} < \epsilon.$$

CHAPTER 4

MEASURE-PRESERVING TRANSFORMATIONS AND NORMALIZED SYMMETRIC GAUGE NORMS

We begin this chapter by defining what a measure-preserving transformation is. This transformation plays a big role in the Hadwin-Hoover theorem in chapter 5. Then we introduce new norms called normalized symmetric gauge norms. Many nice and useful properties of such norms will be given. A lot of common norms such as p-norms, Lorentz norms, etc., are all this type of norm. Finally, the evaluation of the limit norms of normalized symmetric gauge norms will be presented as theorems and lemmas. These calculations and computations are key ingredients in proving the Hadwin-Hoover theorem and our work, the generalized Hadwin-Hoover theorem in the last chapter.

4.1 Measure-Preserving Transformations

Definition 4.1.1. Suppose (Ω, Σ, μ) is a finite measure space. We say that $\gamma : \Omega \to \Omega$ is an **invertible measure preserving transformation** if and only if

1. γ is bijective.
2. γ, γ^{-1} are measurable.
3. for every $E \in \Sigma, \mu(E) = \mu(\gamma(E)) = \mu(\gamma^{-1}(E))$.

Also, we let $\mathbb{MP}(\Omega, \mu)$ be the group under composition of all invertible measure preserving transformations.

Definition 4.1.2. Suppose G is a countable discrete group of invertible measure preserving transformations on Ω. We say that G is **freely acting** if and only if

for each $E \in \Sigma$ with $\mu(E) > 0$ and each finite subset $H \subset G$, there is an $E_0 \subset E$ with $\mu(E_0) > 0$ such that $\{\gamma(E_0) : \gamma \in H\}$ is disjoint.

Example 4.1.3. Suppose $\Omega = \mathbb{R}$ and $\mu =$Lesbegue measure. Let G be a countable subgroup of $\mathbb{R}^+$ and for each $r \in G$ define $\gamma_r : \mathbb{R} \to \mathbb{R}$ by $\gamma_r(x) = x + r$.

4.2 Normalized Symmetric Gauge Norms on $[0, 1]$

Let μ be the Lebesgue measure on $[0, 1]$ and let $\mathbb{MP}([0, 1], \mu) = \mathbb{MP}([0, 1])$

Definition 4.2.1. α is a **normalized symmetric gauge norm** of f on $L^\infty[0, 1]$ denoted by $\alpha(f)$ if and only if

1. $\alpha(1) = 1$
2. $\alpha(f) = \alpha(|f|)$
3. $\alpha(f \circ \gamma) = \alpha(f)$ for $\gamma \in \mathbb{MP}[0, 1]$

We sometimes call $\alpha(f)$, an α-norm of f

We give some basic properties of normalized symmetric gauge norms. These properties will be applied later in this chapter.

Theorem 4.2.2. *Suppose $f, g : [0, 1] \to \mathbb{C}$ are measurable functions. Let α be the normalized symmetric gauge norm. Some basic properties of normalized symmetric gauge norms are given below*

1. $\alpha(fg) \leq \alpha(f) \|g\|_\infty$
2. $|f| \leq |g| \Rightarrow \alpha(f) \leq \alpha(g)$
3. $\|f\|_1 \leq \alpha(f) \leq \|f\|_\infty$
4. *The set of nomalized symmetric gauge norms on $L^\infty[0, 1]$ is a convex set that is compact under the topology of pointwise convergence on $L^\infty[0, 1]$ and closed under suprema.*

The proof in details of theorem 4.2.2 can be found on [6]. We are now going to present a lot of examples of normalized symmetric gauge norms. The limit norms will be defined and computed in the next section.

Example 4.2.3 (Mean of p-Norms)**.** Let $1 \leq p < \infty$. The easiest and most obvious normalized symmetric gauge norm is the usual p-norm of any measurable function f in L^p-space defined as follows

$$\|f\|_p = \left(\int_{[0,1]} |f|^p \, d\mu \right)^{\frac{1}{p}} \tag{4.1}$$

Let $1 \leq p \leq q < \infty$. The mean of two p-norms

$$\frac{1}{2} \left(\|f\|_p + \|f\|_q \right) \tag{4.2}$$

is a normalized symmetric gauge norm.

Let $1 \leq p_1 \leq p_2 \leq \ldots \leq p_n < \infty$. The mean of p_i-norms

$$\frac{\|f\|_{p_1} + \|f\|_{p_2} + \ldots + \|f\|_{p_n}}{n} \tag{4.3}$$

is also a normalized symmetric gauge norm.

Example 4.2.4 (Convex Combinations of p-Norms)**.** We can extend the idea from the previous example. The mean of p_i-norms is actually a convex combination of all p_i-norms. If $\sum_{i=1}^{n} c_i = 1$ then the convex combination of p_i-norms is given by

$$\sum_{i=1}^{n} c_i \|f\|_{p_i} = c_1 \|f\|_{p_1} + c_2 \|f\|_{p_2} + \ldots + c_n \|f\|_{p_n} \tag{4.4}$$

is again a normalized symmetric gauge norm.

According to the above examples, we construct the new normalized symmetric gauge norms by the usual p-norm. In the next few examples, we introduce the norms which are much more complicated. Nevertheless, all of them are also normalized symmetric gauge norms. At the start, some important way to rewrite a complex-valued function f with a measure μ will be presented.

Definition 4.2.5. The function $\mu_f : [0, \infty) \to [0, 1]$ defined by

$$\mu_f(\lambda) = \mu\left(\{x : |f(x)| > \lambda\}\right) \text{ for } \lambda \geq 0 \tag{4.5}$$

is called a **distribution function** of f.

The function $f^{\star} : [0, 1] \to [0, \infty]$ defined by

$$f^{\star}(s) = \inf\{\lambda : \mu_f(\lambda) \leq s\} \tag{4.6}$$

is called a **non-increasing rearrangement** of f.

Example 4.2.6. Let χ_A be the characteristic function on a measurable set A.

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ \\ 0, & x \notin A \end{cases} \tag{4.7}$$

The non-increasing rearrangement of a characteristic function is

$$\chi_A^{\star}(s) = \chi_{[0,\mu(A))}(s) \tag{4.8}$$

Next, we let $f = \sum_{k=1}^{n} a_k \chi_{E_k}$ be a simple function. We can find $u : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ so that $a_{u(1)} \geq a_{u(2)} \geq \ldots \geq a_{u(n)} \geq 0$. Then, the non-increasing rearrangement of a simple function is

$$f^{\star} = a_{u(1)}\chi_{\left[0,\mu\left(E_{u(1)}\right)\right]} + a_{u(2)}\chi_{\left[\mu\left(E_{u(1)}\right),\mu\left(E_{u(2)}\right)\right]} + \ldots + a_{u(n)}\chi_{\left[\mu\left(E_{u(n-1)}\right),\mu\left(E_{u(n)}\right)\right]} \tag{4.9}$$

Example 4.2.7 (Lorentz norms)**.** Suppose β is normalized gauge norm (not symmetric). We define the α-norm

$$\alpha_\beta(f) = \sup_{\gamma \in \mathbb{MP}[0,1]} \beta(f \circ \gamma). \tag{4.10}$$

Note that α_β is a normalized symmetric gauge norm. Let ν be a probability Borel measure and assume ν is absolute continuity with respect to μ, i.e., $\nu << \mu$. The Radon-Nikodym theorem says $\nu << \mu$ if and only if there exists a measurable function $g \geq 0$ such that for all measurable function $f \geq 0$, we have

$$\int_{[0,1]} f \, d\nu = \int_{[0,1]} fg \, d\mu \tag{4.11}$$

Let $1 \leq p < 0$ and $\beta(f) = \|\cdot\|_p$ with respect to measure ν. Then

$$\beta(f) = \left(\int_{[0,1]} |f|^p \, d\nu\right)^{\frac{1}{p}} \tag{4.12}$$

is a gauge norm on $L^\infty[0,1]$. We obtain the α-norm as

$$\alpha_\beta(f) = \sup_{\gamma \in \mathbb{MP}[0,1]} \left(\int_{[0,1]} |f \circ \gamma|^p \, g \, d\mu\right)^{\frac{1}{p}} \tag{4.13}$$

Now we can see that the norm is defined to be a supremum. In other words, we want to calculate the α-norm, we must maximize the integral. The maximum of the above integral occurs when we apply the product of the non-increasing rearrangement of $|f|$ and g as the integrand.

$$\alpha_\beta(f) = \left(\int_{[0,1]} (|f|^\star)^p \, g^\star \, d\mu\right)^{\frac{1}{p}} \tag{4.14}$$

If g is decreasing then $g = g^\star$ and thus the equation 4.15 becomes

$$\alpha_\beta(f) = \left(\int_{[0,1]} (|f|^\star)^p \, g \, d\mu\right)^{\frac{1}{p}} \tag{4.15}$$

We call this norm the **Lorentz norm** denote by $\lambda_{p,g}$.

Example 4.2.8 (Ky Fan Norm and Marcinkiewicz Norm)**.** From the previous example, take

$$g = \frac{1}{t}\chi_{[0,t)} \tag{4.16}$$

then the norm

$$\lambda_{1,g}(f) = \frac{1}{t}\int_0^t |f(x)|^\star \, dx \tag{4.17}$$

is called **Ky Fan norm** denoted by KF_t. In other word, the Ky Fan norm is the average over $(0,t)$ of the non-increasing rearrangement of $|f|$.

Another norm related to the Ky Fan norm is called the **Marcinkiewicz norm**. Let $u : (0,1] \to (0,1]$ be any function such that $v(t) = t/u(t)$ is concave and increasing, $v(0) = 0, v(1) = 1$ and $\lim_{t\to 0^+} \frac{t}{v(t)} = 0$. We define the Marcinkiewicz norm by

$$\mathfrak{M}_u(f) = \sup_{t\in(0,1]} \frac{u(t)}{t}\int_0^t |f(x)|^\star \, dx \tag{4.18}$$

We can see that of $\mathfrak{M}_u = \sup_{0<t\leq 1} u(t)\, KF_t$. Ky Fan norm and Marcinkiewicz norm are both normalized symmetric gauge norms.

Example 4.2.9 (Other Norms)**.** We can obtain many new normalized symmetric gauge norms by taking a mean or a linear combination of the norms from the previous examples. Here are some norms from this procedure.

$$\frac{1}{2}\left(\|f\|_p + \lambda_{p,g}(f)\right) \tag{4.19}$$

$$\frac{1}{3}\left(\text{Lorentz norm} + \text{Ky-Fan norm} + \text{Marcinkiewicz norm}\right) \tag{4.20}$$

$$c_1\lambda_{p,g_1}(f) + c_2\lambda_{p,g_2}(f) + c_3\lambda_{p,g_3}(f) + c_4\lambda_{p,g_4}(f) \text{ where } c_1+c_2+c_3+c_4 = 1 \tag{4.21}$$

4.3 Computation of the Limit Norms

A key part of the proof of the Hadwin-Hoover theorem is based on the following idea.

Suppose $n \in \mathbb{N}$ and $\{E_1, E_2, \ldots, E_n\}$ is a disjoint collection of measurable subsets on $\Omega = [0,1]$ such that $\mu(E_1) = \cdots = \mu(E_n) = t$.

Let χ_{E_k} be a characteristic function on E_k. Then $\|\chi_{E_k}\|_p = t^{1/p}$ for $k \in \{1, 2, \ldots, n\}$ and

$$\frac{1}{\|\chi_{E_k}\|_p}\chi_{E_k} = \frac{1}{t^{1/p}}\chi_{E_k}. \tag{4.22}$$

Then for $a_1, a_2, \ldots, a_n \in \mathbb{C}$,

$$\left\|\sum_{k=1}^{n} a_k\left(\frac{1}{s^{1/p}}\chi_{E_k}\right)\right\|_{L^p(\mu)} = \|(a_1, a_2, \ldots, a_n)\|_{\ell^p} \tag{4.23}$$

If we let $t \to 0$ and then $n \to \infty$, we end up with the ℓ^p norm.

This means that we need to know what will happen to the term of the left hand side on the equation 4.23. Precisely, we have to compute the limit norm as $n \to \infty$ and $t \to 0$. For the different type of norms, there is no universal way to make this computations. We will focus on the specific normalized symmetric gauge norm from the previous section. This main idea will be used to provide the proof of our work, the generalized Hadwin-Hoover theorem. In order to make the proof more concise, we replace the p-norm with other normalized symmetric gauge norms. The computation of the limit norms will be stated as lemmas and theorems and will be applied in the final chapter. To begin with, we define the term that we will use later in the limit computation

Let $a_1, a_2, \ldots, a_n \in \mathbb{C}$. Initially, we set up

$$\alpha_{n,t}(a_1, a_2, \ldots, a_n) = \alpha\left(\sum_{k=1}^{n} a_k \frac{\chi_{E_k}}{\alpha(\chi_{E_k})}\right) \tag{4.24}$$

We can make the above equation much simpler using the fact that there is always a invertible measure preserving transformation γ on $[0,1]$ such that $\gamma^{-1}(E_k) = [0, \mu(E_k))$ and hence $\alpha(\chi_{E_k}) = \alpha\left(\chi_{[0,\mu(E_k))}\right)$. Since $\mu(E_k) = t$, we subdivide the interval $[0,1]$ into subintervals

$[(k-1)\,t, kt)$ and $\mu\left(E_k\right) = \mu\left([(k-1)\,t, kt)\right) = t$ for $k \in \{1, 2, \ldots, n\}$ and $0 \leq t \leq \frac{1}{n}$. Then we define

$$\alpha_{n,t}(a_1, a_2, \ldots, a_n) = \alpha\left(\sum_{k=1}^{n} a_k \frac{\chi_{[(k-1)t,kt)}}{\alpha\left(\chi_{[(k-1)t,kt)}\right)}\right) \tag{4.25}$$

The next theorem is about the properties of $\alpha_{n,t}$.

Theorem 4.3.1. *Let α be a normalized symmetric gauge norm. The following properties hold.*

1. $\alpha_{n,t}\left(1, 0, \ldots, 0\right) = 1$
2. $\left\|(a_1, a_2, \ldots, a_n)\right\|_\infty \leq \alpha_{n,t}\left(a_1, a_2, \ldots, a_n\right) \leq \left\|(a_1, a_2, \ldots, a_n)\right\|_1$
3. $\alpha_{n,t}\left(a_1, a_2, \ldots, a_n\right) = \alpha_{n,t}\left(\left|a_1\right|, \left|a_2\right|, \ldots, \left|a_n\right|\right)$
4. $\alpha_{n,t}\left(a_1, a_2, \ldots, a_n\right) = \alpha_{n,t}\left(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}\right)$ *for every permutation σ on $\{1, 2, \ldots, n\}$*

Proof. 1. We evaluate $\alpha_{n,t}\left(1, 0, \ldots, 0\right)$ directly.

$$\begin{aligned}
\alpha_{n,t}\left(1, 0, \ldots, 0\right) &= \alpha\left(1 \cdot \frac{\chi_{[0,t)}}{\alpha\left(\chi_{[0,t)}\right)} + 0 \cdot \frac{\chi_{[t,2t)}}{\alpha\left(\chi_{[t,2t)}\right)} + \ldots + 0 \cdot \frac{\chi_{[(n-1)t,nt)}}{\alpha\left(\chi_{[(n-1)t,nt)}\right)}\right) \\
&= \alpha\left(\frac{\chi_{[0,t)}}{\alpha\left(\chi_{[0,t)}\right)} + 0 + \ldots + 0\right) \\
&= \frac{\alpha\left(\chi_{[0,t)}\right)}{\alpha\left(\chi_{[0,t)}\right)} \\
&= 1
\end{aligned}$$

2. Recall that $\left\|(a_1, a_2, \ldots, a_n)\right\|_\infty = \max_{1 \leq i \leq n} |a_i|$ and $\left\|(a_1, a_2, \ldots, a_n)\right\|_1 = \sum_{i=1}^{n} |a_i|$. It is easy to see that

$$\max_{1 \leq i \leq n} |a_i| \leq \alpha\left(\sum_{k=1}^{n} a_k \frac{\chi_{[(k-1)t,kt)}}{\alpha\left(\chi_{[(k-1)t,kt)}\right)}\right) \leq \sum_{i=1}^{n} |a_i|$$

3. Consider

$$\alpha_{n,t}\left(a_1, a_2, \ldots, a_n\right) = \alpha\left(\sum_{k=1}^{n} a_k \frac{\chi_{[(k-1)t,kt)}}{\alpha\left(\chi_{[(k-1)t,kt)}\right)}\right)$$

$$\alpha_{n,t}\left(\left|a_1\right|, \left|a_2\right|, \ldots, \left|a_n\right|\right) = \alpha\left(\sum_{k=1}^{n} \left|a_k\right| \frac{\chi_{[(k-1)t,kt)}}{\alpha\left(\chi_{[(k-1)t,kt)}\right)}\right)$$

4. Suppose σ is a permutation on $\{1, 2, \ldots, n\}$. By commutativity of addition, we have

$$\sum_{k=1}^{n} a_k \frac{\chi_{[(k-1)t,kt)}}{\alpha\left(\chi_{[(k-1)t,kt)}\right)} = \sum_{k=1}^{n} a_{\sigma(k)} \frac{\chi_{[(\sigma(k)-1)t,\sigma(k)t)}}{\alpha\left(\chi_{[(\sigma(k)-1)t,\sigma(k)t)}\right)}$$

Then we immediately get the equality of these normalized symmetric gauge norms. ■

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$. We denote the set of all norms α' in $\mathbb{C}^n$ that satisfy properties $(1) - (4)$ in Theorem 4.3.1 by $\mathcal{NSG}_n$. Since each $\alpha' \in \mathcal{NSG}_n$ is a norm, α' is determined by its value on $U_n = \{a \in \mathbb{C}^n : \|a\|_1 \leq 1\}$, i.e.,

$$\alpha'(a) = \|a\|_1 \alpha' \left(\frac{1}{\|a\|_1} a \right) \tag{4.26}$$

We want to topologize $\mathcal{NSG}_n$ with the topology of pointwise convergence. A net $\{\alpha'_\lambda\}$ in $\mathcal{NSG}_n$ converges pointwise on $\mathbb{C}^n$ to a $\alpha' \in \mathcal{NSG}_n$ if and only if, for every $a \in U_n$,

$$\lim_{\lambda} |\alpha'_\lambda(a) - \alpha'(a)| = 0. \tag{4.27}$$

Also if $a \in U_n$, then, for every $\alpha' \in \mathcal{NSG}_n$, we have

$$0 \leq \alpha'(a) \leq \|a\|_1 \leq 1, \tag{4.28}$$

which means that $\alpha' : U_n \to [0, 1]$. We obtain a beautiful theorem about $\mathcal{NSG}_n$.

Theorem 4.3.2. *$\mathcal{NSG}_n$ with the topology of pointwise convergence is a compact metric space.*

Proof. From the previous paragraph, we know that $\alpha' \in \mathcal{NSG}_n$ takes the value between 0 and 1 on U_n by 4.26 and 4.28. This is the case because we can always normalize the vector to be a unit vector by dividing all components by its magnitude and the norm α' is always less than the 1-norm. This implies that

$$\mathcal{NSG}_n \subset \{f | f : U_n \to [0, 1]\} \tag{4.29}$$

Moreover, $\{f | f : U_n \to [0, 1]\} \cong \prod_{U_n} [0, 1]$. Consider the usual topology on $\mathbb{R}$, the set $[0, 1]$ is compact as a subspace of $\mathbb{R}$ because it is closed and bounded. We note that the product of compact spaces is compact so $\prod_{U_n} [0, 1]$ is compact. Clearly, $\prod_{U_n} [0, 1]$ is metrizable as a subspace of metrizable space $\mathbb{R}$. Therefore, $\mathcal{NSG}_n$ is a compact metric space. ■

The set of all $\alpha_{n,t}$ in $\mathcal{NSG}_n$ is denoted by $\mathcal{NSG}_n(\alpha)$. Precisely, this means that

$$\mathcal{NSG}_n(\alpha) = \{\alpha_{n,t} : \alpha_{n,t} \in \mathcal{NSG}_n\} \tag{4.30}$$

Clearly, $\mathcal{NSG}_n(\alpha) \subset \mathcal{NSG}_n$. If $0 \leq s \leq \frac{1}{n}$ then we further define

$$\mathcal{NSG}_n(\alpha, s) = \left\{ \alpha_{n,t} : \alpha_{n,t} \in \mathcal{NSG}_n \text{ and } 0 \leq t \leq s \leq \frac{1}{n} \right\} \tag{4.31}$$

Now we are ready to define the **limit -** α **norm** or **limit norm** in short.

Definition 4.3.3. The set of all limit norms is defined to be

$$\lim \mathcal{NSG}_n(\alpha) = \bigcap_{0 \leq s \leq \frac{1}{n}} \overline{\mathcal{NSG}_n(\alpha, s)} \tag{4.32}$$

Immediately from Theorem 4.3.2, we have the following corollary.

Theorem 4.3.4. $\lim \mathcal{NSG}_n(\alpha)$ *as a subspace of* $\mathcal{NSG}_n$ *is a nonempty compact metric space.*

Proof. As a subspace of a metric space $\mathcal{NSG}_n$, $\lim \mathcal{NSG}_n(\alpha)$ inherits the same metric so it is again a metric space. For compactness, we know from the previous theorem that $\mathcal{NSG}_n$ is compact and recall the fact that every closed subset of a compact space is also compact. We define the set of all limit norms as a intersection of a closure of $\mathcal{NSG}_n(\alpha, s)$ and every closure is a closed set. It follows that $\lim \mathcal{NSG}_n(\alpha)$ is a compact metric space and lastly by the finite intersection properties, we clearly show $\lim \mathcal{NSG}_n(\alpha)$ is nonempty. ■

According to the above definition, we define the limit norm abstractly. We cannot calculate the limit norm explicitly and numerically but the following lemma will help us to understand more about computation of the limit norm.

Corollary 4.3.5. *Let* α *be a normalized symmetric gauge norm. Then we have*

1. *If* $\alpha' \in \lim \mathcal{NSG}_n(\alpha)$ *then there is a decreasing sequence* $\frac{1}{n} > t_1 > t_2 > \ldots$ *and* $\lim_{m\to\infty} t_m = 0$ *such that* $\alpha_{n,t_m} \to \alpha'$ *in* $\mathcal{NSG}_n$
2. *If* $\frac{1}{n} > s_1 > s_2 > \ldots$ *and* $\lim_{m\to\infty} s_m = 0$*, then there is a subsequence* $\{s_{m_l}\}$ *and a* $\alpha' \in \lim \mathcal{NSG}_n(\alpha)$ *such that* $\alpha_{n,s_{m_l}} \to \alpha'$ *in* $\mathcal{NSG}_n$
3. $\lim \mathcal{NSG}_n(\alpha) = \{\alpha'\}$ *if and only if, for every* $a \in \mathbb{C}^n$,

$$\lim_{t\to 0^+} \alpha_{n,t}(a) = \alpha'(a) \tag{4.33}$$

Proof. By Theorem 4.3.4, we have $\lim \mathcal{NSG}_n(\alpha)$ is compact and metrizable. The convergence is sequential convergence and every convergent sequence has a convergent subsequence. Therefore, 1., 2. and 3. hold. ■

We summarize the existence of the limit norm by the following equation

$$\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots, a_n) = \begin{cases} \lim\limits_{m\to\infty} \alpha_{n,t_m}(a_1, a_2, \ldots, a_n) \\ \\ \lim\limits_{l\to\infty} \alpha_{n,s_{m_l}}(a_1, a_2, \ldots, a_n) \end{cases} \tag{4.34}$$

The limit norms are now defined completely and nicely on $\mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \ldots \mathbb{C}}_{n \text{ terms}}$ which we identify with

$$\mathbb{C}^n = \{(a_1, a_2, \ldots, a_n, 0, 0, \ldots) : a_1, a_2, \ldots, a_n \in \mathbb{C}\}. \tag{4.35}$$

We have already known how to compute and calculate the limit norm on $\mathbb{C}^n$ by the Lemma 4.3.5. The examples of the limit norm based on the previous examples in this chapter are presented now. We notice this important and useful fact

$$\alpha\left(\chi_{[(k-1)t,kt)}\right) = \alpha\left(\chi_{[0,t)}\right) \tag{4.36}$$

Example 4.3.6 (Mean of p-Norms)**.** Let $1 \leq p_1 \leq p_2 \leq \ldots \leq p_n < \infty$. We recall the normalized symmetric gauge norm to be the mean of p-norms from equation 4.3 and apply the equation 4.36. We obtain the α-norm of the mean of p-norms of $\chi_{[0,t)}$ as follow

$$\frac{\left\|\chi_{[0,t)}\right\|_{p_1} + \left\|\chi_{[0,t)}\right\|_{p_2} + \ldots + \left\|\chi_{[0,t)}\right\|_{p_n}}{n} = \frac{t^{1/p_1} + t^{1/p_2} + \ldots + t^{1/p_n}}{n} \tag{4.37}$$

We compute the limit norm by use of Lemma 4.3.5.

$$\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = \lim_{t\to 0^+} \frac{n}{\sum\limits_{i=1}^{n} t^{1/p_i}} \cdot \frac{1}{n} \sum_{i=1}^{n} \left(\int_0^1 \left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)} \right)^{p_i} dx \right)^{1/p_i} \tag{4.38}$$

We split the integral on the interval $[0,1]$ into the integral from $(k-1)\,t$ to kt. Observe that if $x \notin [(k-1)\,t, kt)$ then the integral will be zero. The integral term is simplified to be ℓ_{p_i}-norms multiplied with t^{1/p_i} as follows.

$$\left(\int_0^1 \left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right)^{p_i} dx\right)^{1/p_i} = \left(a_1^{p_i} + a_2^{p_i} + \ldots + a_n^{p_i}\right)^{1/p_i} t^{1/p_i} \tag{4.39}$$

for all $i = 1, 2, \ldots, n$. It follows that

$$\begin{aligned}\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) &= \lim_{t\to 0^+} \frac{\left(a_1^{p_1} + a_2^{p_1} + \ldots + a_n^{p_1}\right)^{1/p_1} t^{1/p_1} + \ldots + \left(a_1^{p_n} + a_2^{p_n} + \ldots + a_n^{p_n}\right)^{1/p_n} t^{1/p_n}}{t^{1/p_1} + \ldots + t^{1/p_n}} \times \frac{\frac{1}{t^{1/p_n}}}{\frac{1}{t^{1/p_n}}} \\ &= \lim_{t\to 0^+} \frac{\left(a_1^{p_n} + a_2^{p_n} + \ldots + a_n^{p_n}\right)^{1/p_n} + \ldots + \left(a_1^{p_1} + a_2^{p_1} + \ldots + a_n^{p_1}\right)^{1/p_1} t^{1/p_1 - 1/p_n}}{1 + \ldots + t^{1/p_1 - 1/p_n}}\end{aligned} \tag{4.40}$$

Since $1/p_i - 1/p_n > 0$ for all $i = 1, 2, \ldots, n-1$, the equation 4.33 becomes

$$\begin{aligned}\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) &= \frac{\left(a_1^{p_n} + a_2^{p_n} + \ldots + a_n^{p_n}\right)^{1/p_n} + 0 + 0 + \ldots + 0}{1 + 0 + 0 + \ldots + 0} \\ &= \left(a_1^{p_n} + a_2^{p_n} + \ldots + a_n^{p_n}\right)^{1/p_n}\end{aligned} \tag{4.41}$$

We have the p_n-norm of $(a_1, a_2, \ldots)$ as the limit norm of mean of p-norms.

Example 4.3.7 (Convex Combination of p-Norms)**.** Let $1 \leq p_1 \leq p_2 \leq \ldots \leq p_n < \infty$. Take the α-norm to be the convex combination of p-norm defined as equation 4.4. Let us first calculate the α-norm of $\chi_{[0,t)}$, we get

$$c_1 \left\|\chi_{[0,t)}\right\|_{p_1} + c_2 \left\|\chi_{[0,t)}\right\|_{p_2} + \ldots + c_n \left\|\chi_{[0,t)}\right\|_{p_n} = c_1 t^{1/p_1} + c_2 t^{1/p_2} + \ldots + c_n t^{1/p_n} \tag{4.42}$$

By equation 4.33 in Lemma 4.3.5, we have

$$\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = \lim_{t\to 0^+} \frac{1}{\sum_{i=1}^{n} c_i t^{1/p_i}} \sum_{i=1}^{n} c_i \left(\int_0^1 \left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right)^{p_i} dx\right)^{1/p_i} \tag{4.43}$$

Likewise, we apply the same technique by splitting the integral, we have

$$\left(\int_0^1 \left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right)^{p_i} dx\right)^{1/p_i} = \left(a_1^{p_i} + a_2^{p_i} + \ldots + a_n^{p_i}\right)^{1/p_i} t^{1/p_i} \tag{4.44}$$

for all $i = 1, 2, \ldots, n$. The equation 4.43 becomes

$$\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = \lim_{t\to 0^+} \frac{c_1 \left(a_1^{p_1} + a_2^{p_1} + \ldots + a_n^{p_1}\right) t^{1/p_1} + \ldots + c_n \left(a_1^{p_n} + a_2^{p_n} + \ldots + a_n^{p_n}\right)^{1/p_n} t^{1/p_n}}{c_1 t^{1/p_1} + \ldots + c_n t^{1/p_n}} \tag{4.45}$$

We end up again with

$$\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = (a_1^{p_n} + a_2^{p_n} + \ldots + a_n^{p_n})^{1/p_n} \tag{4.46}$$

The limit norm of mean of convex combination of p-norms equals the p_n-norm of $(a_1, a_2, \ldots)$.

Now we will calculate the limit norm of non-increasing rearrangement function. Without loss of generality, suppose $a_1 \geq a_2 \geq \ldots \geq a_n$ and write $f = \sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}$. Here are the examples.

Example 4.3.8 (Lorentz Norm)**.** By the definition of Lorentz norm, we compute the limit norm depending on the boundedness of the function g in two cases. First, we calculate

$$\lambda_{p,g}\left(\chi_{[0,t)}\right) = \left(\int_0^t g\,dx\right)^{1/p} \tag{4.47}$$

By Lemma 4.3.5, we have

$$\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = \lim_{t\to 0^+} \frac{1}{\left(\int_0^t g\,dx\right)^{1/p}} \left(\int_0^1 \left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right)^p g\,dx\right)^{1/p} \tag{4.48}$$

Consider on the interval $[(k-1)\,t, kt)$, we get

$$\left(\int_0^1 \left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right)^p g\,dx\right)^{1/p} = \left(\sum_{k=1}^{n} a_k^p \int_{(k-1)t}^{kt} g\,dx\right)^{1/p} \tag{4.49}$$

Now the equation 4.48 can be written as

$$\begin{aligned} \lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) &= \lim_{t\to 0^+} \left(\frac{\sum_{k=1}^{n} a_k^p \int_{(k-1)t}^{kt} g\,dx}{\int_0^t g\,dx}\right)^{1/p} \\ &= \lim_{t\to 0^+} \left(\sum_{k=1}^{n} a_k^p \left(\frac{\int_{(k-1)t}^{kt} g\,dx}{\int_0^t g\,dx}\right)\right)^{1/p} \end{aligned} \tag{4.50}$$

Case g is bounded.

Since g is bounded, we have the inequality $g(kt) \cdot t \leq \int_{(k-1)t}^{kt} g\,dx \leq g((k-1)\,t) \cdot t$ for all $k = 1, 2, 3, \ldots$ and in particular $g(0)$ exists. Let's apply the inequality to the equation 4.50

$$\lim_{t\to 0^+}\left(\sum_{k=1}^{n} a_k^p\left(\frac{g\left(kt\right)}{g\left(0\right)}\right)\right)^{1/p} < \lim_{t\to 0^+}\left(\sum_{k=1}^{n} a_k^p\left(\frac{\displaystyle\int_{(k-1)t}^{kt} g\,dx}{\displaystyle\int_0^t g\,dx}\right)\right)^{1/p} < \lim_{t\to 0^+}\left(\sum_{k=1}^{n} a_k^p\left(\frac{g\left(\left(k-1\right)t\right)}{g\left(t\right)}\right)\right)^{1/p} \tag{4.51}$$

By sandwich theorem, we have

$$\lim_{t\to 0^+}\left(\sum_{k=1}^{n} a_k^p\left(\frac{\displaystyle\int_{(k-1)t}^{kt} g\,dx}{\displaystyle\int_0^t g\,dx}\right)\right)^{1/p} = \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} = \left(a_1^p + a_2^p + \ldots + a_n^p\right)^{1/p} \tag{4.52}$$

We obtain ℓ_p-norm as the limit norm. Up to this point, all of the limit norm is in form of ℓ_p-norm for some p but this is not always the case. The next case will be the first limit norm that is not in this form.

<u>Case g is unbounded.</u>

To compute the limit norm in this case, since g is unbounded $\left(\lim_{x\to 0} g\left(x\right) = \infty\right)$, we have to know the specific function of g in order to integrate. The inequality from the previous case can no longer be applied. For example, take $g\left(x\right) = \frac{1}{2\sqrt{x}}$. The equation 4.50 becomes

$$\begin{aligned}\lim_{t\to 0^+}\left(\sum_{k=1}^{n} a_k^p\left(\frac{\displaystyle\int_{(k-1)t}^{kt} g\,dx}{\displaystyle\int_0^t g\,dx}\right)\right)^{1/p} &= \left(\sum_{k=1}^{n} a_k^p\left(\lim_{t\to 0^+}\frac{\displaystyle\int_{(k-1)t}^{kt} \frac{1}{2\sqrt{x}}\,dx}{\displaystyle\int_0^t \frac{1}{2\sqrt{x}}\,dx}\right)\right)^{1/p}\\ &= \left(\sum_{k=1}^{n} a_k^p\left(\lim_{t\to 0^+}\frac{\sqrt{kt}-\sqrt{\left(k-1\right)t}}{\sqrt{t}}\right)\right)^{1/p}\\ &= \left(\sum_{k=1}^{n}\left(\sqrt{k}-\sqrt{k-1}\right)a_k^p\right)^{1/p}\end{aligned} \tag{4.53}$$

Clearly, this is not the form of any ℓ_p-norm. We note that the equation 4.50 is in the indeterminate form $\left(\frac{0}{0}\right)$. Thus, we use L'hopital rule.

$$\left(\sum_{k=1}^{n} a_k^p\left(\lim_{t\to 0^+}\frac{\displaystyle\int_{(k-1)t}^{kt} g\,dx}{\displaystyle\int_0^t g\,dx}\right)\right)^{1/p} \overset{\left(\frac{0}{0}\right)}{=} \left(\sum_{k=1}^{n} a_k^p\left(\lim_{t\to 0^+}\frac{g\left(kt\right)\cdot k - g\left(\left(k-1\right)t\right)\cdot\left(k-1\right)}{g\left(t\right)}\right)\right)^{1/p} \tag{4.54}$$

According to the analysis of these two cases, we gain some theorems in the above example. For the case when function g is bounded, a basic definition and nice theorems about the characterization of normalized symmetric gauge norm are listed below.

Definition 4.3.9. Given $\|\cdot\|_1$ and $\|\cdot\|_2$ any two norms, we say that $\|\cdot\|_1$-norm is equivalent to $\|\cdot\|_2$-norm if and only if there exists $r > 0$ such that

$$\frac{1}{r}\|\cdot\|_2 < \|\cdot\|_1 < r\|\cdot\|_2 . \tag{4.55}$$

Theorem 4.3.10. *If α_1-norm and α_2-norm are equivalent then the limit of α_1-norm is equivalent to the limit of α_2-norm and vice versa.*

Proof. Suppose α_1-norm and α_2-norm are equivalent. It follows that there exists $r > 0$ such that $\frac{1}{r}\alpha_2(f) < \alpha_1(f) < r\alpha_2(f)$. Particularly, we take $f = \chi_{[(k-1)t,kt)}$ and we get

$$\frac{1}{r}\alpha_2\left(\chi_{[(k-1)t,kt)}\right) < \alpha_1\left(\chi_{[(k-1)t,kt)}\right) < r\alpha_2\left(\chi_{[(k-1)t,kt)}\right).$$

Then flip the inequality reciprocally, we obtain

$$\frac{1}{r}\frac{1}{\alpha_2\left(\chi_{[(k-1)t,kt)}\right)} < \frac{1}{\alpha_1\left(\chi_{[(k-1)t,kt)}\right)} < r\frac{1}{\alpha_2\left(\chi_{[(k-1)t,kt)}\right)}$$

We again apply the hypothesis with the simple function $f = \sum_{k=1}^{n} a_k\chi_{[(k-1)t,kt)}$

$$\frac{1}{r}\alpha_2\left(\sum_{k=1}^{n} a_k\chi_{[(k-1)t,kt)}\right) < \alpha_1\left(\sum_{k=1}^{n} a_k\chi_{[(k-1)t,kt)}\right) < r\alpha_2\left(\sum_{k=1}^{n} a_k\chi_{[(k-1)t,kt)}\right).$$

Finally, we multiply the last two inequalities and take the limit as t approaches 0. Therefore,

$$\frac{1}{r^2}\alpha_{2,n,t}(a_1, a_2, \ldots, a_n) < \alpha_{1,n,t}(a_1, a_2, \ldots, a_n) < r^2\alpha_{2,n,t}(a_1, a_2, \ldots, a_n).$$

■

We state the helpful theorem about computation of the limit norm of Lorentz norm when g is bounded.

Corollary 4.3.11. *Let g be a bounded function and defined in the Lorentz norm. The limit norm of Lorentz norm equals to the ℓ_p-norm.*

Proof. Suppose g is bounded. Then there exists a constant M such that $|g| \leq M$. It is obvious to see that by the inequality 4.51, the Lorentz norm is less than ℓ_p-norm. Moreover, we know that every α-norm is greater than ℓ_1-norm by theorem 4.2.2. Moreover, $1 < p < \infty$ implies ℓ_p-norm is less than ℓ_1-norm. Thus, the Lorentz norm is equivalent to the ℓ_p-norm. By Theorem 4.3.10, their limit norms are also equivalent. Indeed, the limit norm of Lorentz norm in this case is equivalent to ℓ_p-norm. Since both sides of the last inequality in the proof of Theorem 4.3.10 in this special case are independent of t, the limit norm of Lorentz norm is precisely equal to the ℓ_p-norm. The proof is complete. ∎

For the unbounded function g, we conclude the limit norm of the Lorentz norm as the following therom.

Theorem 4.3.12. *Let g be an unbounded function and defined in the Lorentz norm. If $g(x) = f'(x)$ for some differentiable function f then the limit norm of Lorentz norm is of the form*

$$\left(\sum_{k=1}^{n} h(k)\, a_k^p\right)^{1/p} \quad \textit{where } h(k) = \lim_{t \to 0^+} \frac{f(kt) - f((k-1)t)}{f(t) - f(0)} \tag{4.56}$$

Proof. From the equation 4.50, we have the limit norm as

$$\lim_{t \to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = \left(\sum_{k=1}^{n} a_k^p \left(\lim_{t \to 0^+} \frac{\int_{(k-1)t}^{kt} g\, dx}{\int_0^t g\, dx}\right)\right)^{1/p}$$

Since $g = f'$. We apply fundamental theorem of calculus.

$$\lim_{t \to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = \left(\sum_{k=1}^{n} \left(\lim_{t \to 0^+} \frac{f(kt) - f((k-1)t)}{f(t) - f(0)}\right) a_k^p\right)^{1/p}$$

∎

Example 4.3.13 (Other Norms)**.** Other new normalized symmetric gauge norms can be obtained by combining norm from the above examples as we mentioned in previous section. Let's take the mean of p-norm and Lorentz norm. We are now calculate its limit norm. Recall equation 4.19 and compute the α-norm of $\chi_{[0,1)}$ and $\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}$.

$$\frac{1}{2}\left(\left\|\chi_{[0,1)}\right\|_p + \lambda_{p,g}\left(\chi_{[0,1)}\right)\right) = \frac{1}{2}\left(t^{1/p} + \left(\int_0^t g\,dx\right)^{1/p}\right) \tag{4.57}$$

$$\frac{1}{2}\left(\left\|\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right\|_p + \lambda_{p,g}\left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right)\right) = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p} + \left(\sum_{k=1}^{n} a_k^p \int_{(k-1)t}^{kt} g\,dx\right)^{1/p}\right) \tag{4.58}$$

The above equations hold by the equations 4.37,4.38 and 4.47,4.49. Now we have the limit norm as

$$\lim_{t\to 0^+} \alpha_{n,t}\left(a_1, a_2, \ldots\right) = \lim_{t\to 0^+} \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p} + \left(\sum_{k=1}^{n} a_k^p \int_{(k-1)t}^{kt} g\,dx\right)^{1/p}}{t^{1/p} + \left(\int_0^t g\,dx\right)^{1/p}} \tag{4.59}$$

Let's do the easier case when g is bounded. We have the inequality

$$\lim_{t\to 0^+} \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p} + \left(\sum_{k=1}^{n} a_k^p \int_{(k-1)t}^{kt} g\,dx\right)^{1/p}}{t^{1/p} + \left(\int_0^t g\,dx\right)^{1/p}} < \lim_{t\to 0^+} \frac{\left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p g\left(\left(k-1\right)t\right)\right)^{1/p}\right) t^{1/p}}{\left(1 + \left(g\left(t\right)\right)^{1/p}\right) t^{1/p}} \tag{4.60}$$

The right hand side of the inequality is equal to

$$\begin{aligned}
\lim_{t\to 0^+} \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p g\left(\left(k-1\right)t\right)\right)^{1/p}}{1 + \left(g\left(t\right)\right)^{1/p}} &= \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p g\left(0\right)\right)^{1/p}}{1 + \left(g\left(0\right)\right)^{1/p}} \\
= \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(g\left(0\right)\right)^{1/p}}{1 + \left(g\left(0\right)\right)^{1/p}} &= \frac{1 + \left(g\left(0\right)\right)^{1/p}}{1 + \left(g\left(0\right)\right)^{1/p}} \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} = \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}
\end{aligned} \tag{4.61}$$

On the other hand, we also have

$$\lim_{t\to 0^+} \frac{\left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p g\left(kt\right)\right)^{1/p}\right) t^{1/p}}{\left(1+\left(g\left(0\right)\right)^{1/p}\right) t^{1/p}} < \lim_{t\to 0^+} \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p} + \left(\sum_{k=1}^{n} a_k^p \int_{(k-1)t}^{kt} g\, dx\right)^{1/p}}{t^{1/p} + \left(\int_0^t g\, dx\right)^{1/p}} \tag{4.62}$$

The left hand side of the inequality is equal to

$$\begin{aligned}\lim_{t\to 0^+} \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p g\left(kt\right)\right)^{1/p}}{1+\left(g\left(0\right)\right)^{1/p}} &= \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p g\left(0\right)\right)^{1/p}}{1+\left(g\left(0\right)\right)^{1/p}} \\ = \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(g\left(0\right)\right)^{1/p}}{1+\left(g\left(0\right)\right)^{1/p}} &= \frac{1+\left(g\left(0\right)\right)^{1/p}}{1+\left(g\left(0\right)\right)^{1/p}} \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} = \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\end{aligned} \tag{4.63}$$

Therefore, the limit norm of the mean of p-norm and Lorentz norm is ℓ_p-norm. Things get much more interesting for unbounded case, we can extend the idea by letting $g\left(x\right) = sx^{s-1}$ for $0 < s < 1$ from the previous example.

$$\frac{1}{2}\left(\left\|\chi_{[0,1)}\right\|_p + \lambda_{p,g}\left(\chi_{[0,1)}\right)\right) = \frac{1}{2}\left(t^{1/p} + \left(\int_0^t sx^{s-1}\, dx\right)^{1/p}\right) = \frac{1}{2}\left(t^{1/p} + t^{s/p}\right) \tag{4.64}$$

$$\begin{aligned}&\frac{1}{2}\left(\left\|\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right\|_p + \lambda_{p,g}\left(\sum_{k=1}^{n} a_k \chi_{[(k-1)t,kt)}\right)\right) \\ &\quad = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p} + \left(\sum_{k=1}^{n} a_k^p \int_{(k-1)t}^{kt} sx^{s-1}\, dx\right)^{1/p}\right) \\ &\quad = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p} + \left(\sum_{k=1}^{n} \left(k^s - \left(k-1\right)^s\right)^{1/p} a_k^p\right)^{1/p} t^{s/p}\right)\end{aligned} \tag{4.65}$$

Take the limit as t approaches 0 from the right of the quotient of above equations, the limit norm is equal to

$$\lim_{t\to 0^+} \alpha_{n,t}(a_1, a_2, \ldots) = \lim_{t\to 0^+} \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p} + \left(\sum_{k=1}^{n} (k^s - (k-1)^s)^{1/p} a_k^p\right)^{1/p} t^{s/p}}{t^{1/p} + t^{s/p}}$$
$$= \lim_{t\to 0^+} \frac{\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} t^{1/p-s/p} + \left(\sum_{k=1}^{n} (k^s - (k-1)^s)^{1/p} a_k^p\right)^{1/p}}{1 + t^{1/p-s/p}} \tag{4.66}$$

Since $s < 1$, this implies that $s/p < 1/p$ or $1/p - s/p > 0$.

Hence, the limit norm will be $\left(\sum_{k=1}^{n} (k^s - (k-1)^s)^{1/p} a_k^p\right)^{1/p}$.

The same result holds for the linear combination of p-norm and Lorentz norm.

We have so far defined limit norms on each $\mathbb{C}^n$. We let c_{00} be the set of all sequences in $\mathbb{C}$ that are eventually 0. If we identify $\mathbb{C}^n$ with the elements $x = (x_1, x_2, \ldots) \in c_{00}$ such that $x_k = 0$ for all $k > n$, we see that

$$\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset c_{00} \tag{4.67}$$

and

$$c_{00} = \bigcup_{n=1}^{\infty} \mathbb{C}^n. \tag{4.68}$$

If β is a norm on $\mathbb{C}^n$, we can view β as a seminorm on c_{00} by

$$\beta((x_1, x_2, \ldots)) = \beta((x_1, x_2, \ldots, x_n, 0, 0, \cdots)). \tag{4.69}$$

We let $\mathcal{NG}_\infty$ denote the set of all seminorms on c_{00} such that

1. $\beta((1, 0, 0, \ldots)) = 1$
2. $\beta((x_1, x_2, \ldots)) = \beta((|x_1|, |x_2|, \ldots))$,
3. $\beta(\vec{x}) \le \|\vec{x}\|_1$ for every $\vec{x} \in c_{00}$

For each positive integer n we let $\mathcal{NG}_n$ denote the $\beta \in \mathcal{NG}_\infty$ such that, for every permutation σ of $\{1, \ldots, n\}$, we have

$$\beta((x_1, x_2, \ldots)) = \beta\left(\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}, x_{n+2}, \ldots\right)\right). \tag{4.70}$$

Definition 4.3.14. Suppose α is a normalized symmetric gauge norm on $L^{\infty}[0,1]$. We define

$$\mathcal{NSG}_{\infty} = \bigcap_{n=1}^{\infty} \overline{\left[\bigcup_{k=n}^{\infty} \mathcal{NG}_n \right]}. \tag{4.71}$$

and

$$\lim \mathcal{NSG}_{\infty}(\alpha) = \bigcap_{n=1}^{\infty} \overline{\left[\bigcup_{k=n}^{\infty} \lim \mathcal{NSG}_k(\alpha) \right]}. \tag{4.72}$$

It is clear that, with the topology of pointwise convergence, $\mathcal{NG}_n$ is compact for $1 \leq n \leq \infty$, so $\mathcal{NSG}_{\infty}$ is also compact.

Theorem 4.3.15. $\lim \mathcal{NSG}_{\infty}(\alpha)$ *is a nonempty compact subset of* $\mathcal{NSG}_{\infty}$.

Proof. Recall that $\lim \mathcal{NSG}_k(\alpha)$ is compact. Use the fact that every closed subspace of a compact space is compact. Thus, $\lim \mathcal{NSG}_{\infty}(\alpha)$ is also compact. The non-emptyness is followed by the finite intersection property. ■

Theorem 4.3.16. *Suppose* α *is a normalized symmetric gauge norm on* $L^{\infty}[0,1]$. *Then*

1. *For every* $\eta \in \mathcal{NSG}_{\infty}$

   (a) $\|\vec{x}\|_{\infty} \leq \eta(\vec{x})$

   (b) $\|(x_1, x_2, \ldots)\| = \left\|\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots\right)\right\|$ *for every bijective* $\sigma : \mathbb{N} \to \mathbb{N}$.

2. *A norm* η *on* c_{00} *is in* $\mathcal{NSG}_{\infty}$ *if and only if there is a sequence* $\{n_k\}$ *in* $\mathbb{N}$ *and a decreasing sequence* $\{t_l\}$ *in* $(0,1]$ *such that*

   (a) $\lim_{k\to\infty} t_k = 0$,

   (b) $\lim_{k\to\infty} n_k = \infty$, *and*

   (c) $\lim_{k\to\infty} \alpha_{n_k,t_k} = \eta$ *pointwise on* c_{00}.

Proof. This is a consequence of the Theorem 4.3.15. We apply that fact that every convergent sequence has a convergent subsequence on the compact space. ■

Definition 4.3.17. Suppose $\eta \in \mathcal{NSG}_{\infty}$. We define $\ell^{\eta}(\mathbb{N})$ to be the completion of c_{00} with respect to the norm η.

Remark. It is clear that if $\eta \in \mathcal{NSG}_\infty$, we have

$$\ell^1 \subset \ell^\eta(\mathbb{N}) \subset c_0. \tag{4.73}$$

Remark. Suppose K is a countably infinite set and $\sigma : K \to \mathbb{N}$ is a bijection. There is a linear bijective map $\rho_\sigma : c_{00} \to c_{00}(K)$ defined by$\rho_\sigma(f) = f \circ \sigma$. If $\eta \in \mathcal{NSG}_\infty$, we can define a norm, which we still denote by η, by

$$\eta(\rho_\sigma(f)) = \eta(f). \tag{4.74}$$

Note that if $\sigma_1 : K \to \mathbb{N}$ is a bijection, then $\sigma \circ \sigma_1^{-1} : \mathbb{N} \to \mathbb{N}$, which implies, for every $f \in c_{00}$, that

$$\eta\left(f \circ \left(\sigma \circ \sigma_1^{-1}\right)\right) = \eta(f). \tag{4.75}$$

Thus

$$\eta(\rho_{\sigma_1}(f)) = \eta\left(\rho_{\sigma_1}\left(f \circ \left(\sigma \circ \sigma_1^{-1}\right)\right)\right) = \eta(\rho_\sigma(f)). \tag{4.76}$$

Thus, the norm on $c_{00}(K)$ induced by η is independent of σ. Thus we can unambiguously define $\ell^\eta(K)$ as the completion of $c_{00}(K)$ with respect to η. Clearly the map ρ_σ extends uniquely to a linear isometric isomorphism from $\ell^\eta(K)$ to ℓ^η.

CHAPTER 5

HADWIN-HOOVER THEOREM

In this chapter, the Hadwin-Hoover theorem will be stated at the end. Our generalization of this theorem (Theorem ... in chapter 6) is another main theorem of this dissertation. Throughout this chapter

1. (Ω, Σ, μ) is a finite measure space.

2. Y is a separable Banach space whose norm dual $Y^{\#}$ is also separable.

3. $1 \leq p < \infty$

4. G is a countable group of invertible measure preservaing transformations, i.e., $G \leq \mathbb{MP}(\Omega, \mu)$

We first define the important spaces built from μ, Y and p. Then we define important operators on these spaces.

5.1 New Spaces from Old and Important Operators

Let $1 \leq p \leq \infty$. We talk briefly about the L^p-space and its discrete version, the ℓ^p-space. We might say that the L^p-space is a space of measurable functions f such that their p-norm, $\left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$ is finite or for which the p-th power of the absolute value of f is Lebesgue integrable. For the case $p = \infty$, we define the infinity norm, $\|f\|_\infty$ to be $\inf\{C : |f| \leq C \text{ a.e.}\}$. On the other hand, the ℓ^p-space is a space of sequences $(a_1, a_2, \ldots)$ that its ℓ^p-norm, $(\sum |a_i|^p)^{\frac{1}{p}} < \infty$ for $1 \leq p < \infty$ or its ℓ^∞-norm, $\sup |a_i| < \infty$ for $p = \infty$. Both of these spaces are Banach spaces with the usual addition and scalar multiplication. Now we will extend this idea and define new function spaces.

If X is a normed space and $k : \Omega \to X$ is a function we define the function $|k| : \Omega \to [0, \infty)$ by

$$|k|(\omega) = \|k(\omega)\|. \tag{5.1}$$

If k is measurable, then $|k| = \|\| \circ k$ is measurable.

We define the space $L^p(\mu, Y)$ to be the space of all measurable functions $f : \Omega \to Y$ such that

$$\|f\|_p = \left(\int_\Omega \|f(\omega)\|^p \, d\mu \right)^{1/p} < \infty. \tag{5.2}$$

This means that a measurable function $f : \Omega \to Y$ is in $L^p(\mu, Y)$ if and only if $|f| \in L^p(\mu)$ and

$$\|f\|_p = \||f|\|_p. \tag{5.3}$$

As usual, we identify two functions in $L^p(\mu, Y)$ that are equal almost everywhere with respect to μ.

Next, we define $\ell^p(G, Y)$ to be all of the functions $h : G \to Y$ such that

$$\|h\|_p = \left(\sum_{\gamma \in G} \|h(\gamma)\|^p \right)^{1/p} < \infty. \tag{5.4}$$

It is well-known that $L^p(\mu, Y)$ and $\ell^p(G, Y)$ are Banach spaces.

We define $L^\infty(\mu, \mathscr{B}(Y))$ to be the set of bounded functions φ that are measurable with respect to the hstrong operator topology on $\mathscr{B}(Y)$, again identifying two functions that are equal almost everywhere with respect to μ. Since $\|\| : \mathscr{B}(Y) \to [0, \infty)$ is strong-operator measurable, we see that $|\varphi| = \|\| \circ \varphi$ is measurable on Ω. We define

$$\|\varphi\|_\infty = \||\varphi|\|_\infty. \tag{5.5}$$

We can view the elements of $L^\infty(\mu, \mathscr{B}(Y))$ as operators on $L^p(\mu, Y)$. If $f \in L^p(\mu, Y)$ and $\varphi \in L^\infty(\mu, \mathscr{B}(Y))$ we define $M_\varphi f \in L^p(\mu, Y)$ by

$$(M_\varphi(f))(\omega) = \varphi(\omega)(f(\omega)). \tag{5.6}$$

This means that M_φ is an operator on $L^p(\mu, Y)$. It is easy to see that $\|M_\varphi\| \leq \|\varphi\|_\infty$. In fact, it is true that $\|M_\varphi\| = \|\varphi\|_\infty$.

Note: We choose $\varphi \in L^{\infty}(\mu, \mathscr{B}(Y))$, we can change φ on a set of measure 0 and assume that $\|\varphi(\omega)\| \leq \|\varphi\|_{\infty}$ for every $\omega \in \Omega$.

Suppose $\gamma \in G$. We define an operator U_γ on $L^p(\mu, Y)$ by

$$U_\gamma f = f \circ \gamma^{-1}. \tag{5.7}$$

We see that

$$\|U_\gamma f\|_p = \left\|f \circ \gamma^{-1}\right\|_p = \left\|\left|f \circ \gamma^{-1}\right|\right\|_p = \left\||f| \circ \gamma^{-1}\right\|_p = \|f\|_p. \tag{5.8}$$

Thus U_γ is an invertible isometry on $L^p(\mu, Y)$.

We now define the discrete analogues on $\ell^p(G, Y)$. We define $\ell^{\infty}(G, \mathscr{B}(Y))$ to be the set of all bounded functions ψ from G to $\mathscr{B}(Y)$ and define

$$\|\psi\|_{\infty} = \sup_{\gamma \in G} \|\psi(\gamma)\|. \tag{5.9}$$

We define the multiplication M_ψ on $\ell^p(G, Y)$ by

$$(M_\psi h)(\gamma) = \psi(\gamma)(h(\gamma)). \tag{5.10}$$

It is easy to see that

$$\|M_\psi\| = \|\psi\|_{\infty}. \tag{5.11}$$

If $\gamma_0 \in G$, we define an operator V_{γ_0} on $\ell^p(G, Y)$ by

$$(V_{\gamma_0} h)(\gamma) = h\left(\gamma \gamma_0^{-1}\right). \tag{5.12}$$

Again we see that V_{γ_0} is an invertible isometry on $\ell^p(G, Y)$.

If $\varphi \in L^{\infty}(\mu, B(Y))$ and $\|\varphi(\omega)\| \leq \|\varphi\|_{\infty}$ for all $\omega \in [0, 1]$, and if $\omega \in [0, 1]$, we define $\varphi_\omega \in \ell^{\infty}(G, B(Y))$ by

$$\varphi_\omega(\gamma) = \varphi(\gamma(\omega)). \tag{5.13}$$

Then $M_{\varphi_\omega} \in B\left(\ell^{\beta}(G, Y)\right)$ for every $\omega \in [0, 1]$.

Example 5.1.1. G is isomorphic to $(\mathbb{Z}, +)$

$$G = \langle\gamma\rangle = \{\gamma^n : n \in \mathbb{Z}\},$$

i.e. $\gamma \longleftrightarrow 1$ and $\gamma^n \longleftrightarrow n$.

Let $Y = \mathscr{B}(Y) = \mathbb{C}$ and

$$\ell^p(G,Y) = \ell^p(G) = \{f = \big(\ldots, a_{-1}, \underline{a_0}, a_1, \ldots\big) : \textstyle\sum_{n\in\mathbb{Z}} |a_n|^p < \infty\},$$

If $h \in \ell^\infty(G, \mathscr{B}(Y)) = \ell^\infty(G)$ then we have

$$\tilde{M}_h = \text{ diagonal operator } = diag\,(\ldots, h(-1), h(0), h(1), \ldots)$$

$$\tilde{M}_h f = (\ldots, h(-1)a_{-1}, h(0)a_0, h(1)a_1, \ldots)$$

$$V_\gamma = \text{ bilateral shift operator}$$

$$V_\gamma f = \Big(\ldots, \underline{a_{-1}}, a_0, a_1, \ldots\Big)$$

5.2 Hadwin-Hoover Theorem

Now, this is the time to state the Hadwin-Hoover theorem.

Theorem 5.2.1. *Suppose G is a countable group of measure preserving transformations on Ω that is freely acting. Suppose Y and $Y^\#$ are separable Banach spaces. Suppose further that $\mathcal{F} \subset L^\infty(\mu, \mathscr{B}(Y))$ is finite.*

Let $\mathcal{A}$ be the unital inverse-closed and norm-closed algebra generated by

$$\{U_\gamma : \gamma \in G\} \cup \{M_\varphi : \varphi \in \mathcal{F}\} \subset \mathscr{B}\left(L^p\left(\mu, Y\right)\right). \tag{5.14}$$

Then for almost every $\omega \in \Omega$, there is a norm-one unital algebra homomorphism $\pi_\omega : \mathcal{A} \to \mathscr{B}(\ell^p(G,Y))$ such that, for every $\gamma \in G$

$$\pi_\omega(U_\gamma) = V_\gamma, \tag{5.15}$$

and for $\varphi \in \mathcal{F}$,

$$(\pi_\omega(M_\varphi)f)(\gamma) = \varphi(\gamma(\omega))f(\gamma). \tag{5.16}$$

In other words, $\pi_\omega(M_\varphi) = \tilde{M}_{\varphi_\omega}$ where $\varphi_\omega(\gamma) = \varphi(\gamma(\omega))$.

The proof of the theorem and further details can be found in [2].

Example 5.2.2. $\Omega = \{z \in \mathbb{C} : |z| = 1\}$ with $\mu = \frac{1}{2\pi}$arclength

$G \cong \mathbb{Z}$ and $Y = \mathscr{B}(Y) = \mathbb{C}$.

$L^p(\mu, Y) = L^p(\mu)$ and $L^\infty(\mu, Y) = L^\infty(\mu)$.

Suppose $\gamma(z) = e^{2\pi i\theta} z$ for $\theta \in [0,1], \theta$ is irrational. We can see that γ is a measure preserving transformation. Precisely, γ is a rotation of z by $2\pi\theta$

Suppose $f \in L^p(\mu)$ and $\varphi \in L^\infty(\mu)$

$$(M_\varphi f)(z) = \varphi(z) f(z) \text{ and } (U_\gamma f)(z) = f(e^{-2\pi i\theta} z)$$

Take $\omega_0 \in \Omega$.

By Hadwin-Hoover theorem,

$$\pi_{\omega_0}(M_\varphi) = diag\left(\varphi\left(\gamma^n\left(\omega_0\right)\right)\right) = diag\left(\varphi\left(e^{2n\pi i\theta}\left(\omega_0\right)\right)\right)$$

$$\pi_{\omega_0}(U_\gamma) = V_\gamma = \text{ bilateral shift operator}$$

We end this chapter by introducing one key idea in the proof of the Hadwin-Hoover theorem. It involves the familiar concept of essential range and the less familiar notion of essential domain of a measurable map into a separable metric space.

Lemma 5.2.3. *Suppose (Ω, Σ, μ) is a measure space and Y is a separable metric space, and $\Gamma : \Omega \to Y$ is a measurable map. Then*

1. *If $U = \bigcup \{V \subset Y : V \text{ is open and } \mu\left(\Gamma^{-1}\left(V\right)\right) = 0\}$, then U is open, and $\mu\left(\Gamma^{-1}\left(U\right)\right) = 0$*

2. *If $K = Y \backslash U$, then K is closed and W is an open subset of Y with $W \cap K \neq \varnothing$, then $\mu\left(\Gamma^{-1}\left(W\right)\right) > 0$.*

The set K is the preceding Lemma is called the **essential range** of Γ, denoted by essran(Γ), and $\Gamma^{-1}\left(K\right)$ is called the **essential domain** of Γ, denoted by essdom(Γ). It is clear that

$$\mu\left(\Omega \backslash \text{essdom}\left(\Gamma\right)\right) = 0 \tag{5.17}$$

and

$$\Gamma\left(\omega\right) \in \text{essran}\left(\Gamma\right) \text{ a.e. } \left(\mu\right). \tag{5.18}$$

One important fact in the Hadwin-Hoover theorem is that the assumption that $Y^{\#}$ is separable implies, from the Radon-Nikodym property that

$$L^p(\mu, Y)^{\#} = L^q\left(\mu, Y^{\#}\right). \tag{5.19}$$

The assumption that $Y^{\#}$ is norm separable also implies that the unit balls of $\mathscr{B}(L^p(Y))$ and $\mathscr{B}\left(Y^{\#}\right)$ are complete separable metric spaces in the strong operator topology.

CHAPTER 6

GENERALIZED HADWIN-HOOVER THEOREM AND ITS PROOF

We end this dissertation and obtain our goal, the generalize the Hadwin-Hoover theorem based on normalized symmetric gauge norms. The proof of the theorem will be presented and shown in rigorous details.

6.1 L^{α}-Spaces

We want to replace the norms $\|\|_p$ with symmetric gauges norms. This forces us to assume, throughout this section, that $\Omega = [0,1]$ and μ is Lebesgue measure. We still assume that Y is a separable Banach space whose dual space $Y^{\#}$ is separable. Instead of assuming that $1 \leq p < \infty$, we assume that α is a normalized symmetric gauge norm on $L^{\infty}[0,1]$ and $\eta \in \mathcal{NSG}_{\infty}(G)$.

We define $L^{\alpha}(\mu, Y)$ to be the set of all measurable functions $f : [0,1] \to Y$ such that $|f| \in L^{\alpha}(\mu)$, and we define

$$\alpha(f) = \alpha(|f|). \tag{6.1}$$

Note that when $\alpha = \|\|_p$, we have

$$L^{\alpha}(\mu, Y) = L^{p}(\mu, Y). \tag{6.2}$$

As usual we identify two functions that are equal almost everywhere with respect to μ.

If $\varphi \in L^{\infty}(\mu, \mathscr{B}(Y))$, we define the operator M_{φ} on $L^{\alpha}(\mu, Y)$ by

$$(M_{\varphi}(f))(\omega) = \varphi(\omega)(f(\omega)). \tag{6.3}$$

We still have $\|M_{\varphi}\| = \|\varphi\|_{\infty}$.

If $\gamma \in G$, we define the operator U_γ on $L^\alpha(\mu, Y)$ by

$$(U_\gamma f) = f \circ \gamma^{-1}. \tag{6.4}$$

Since α is a symmetric norm, we see that U_γ is an invertible isometry on $L^\alpha(\mu, Y)$.

We define $\ell^\beta(G, Y)$ to be the set of all functions $h : G \to Y$ such that $|h| \in \ell^\beta(G)$. We define $c_{00}(G, Y)$ to be the functions $h : G \to Y$ such that $\{\gamma \in G : h(\gamma) \neq 0\}$ is finite. It is clear from the definition of $\ell^\beta(G)$ that $c_{00}(G, Y)$ is dense in $\ell^\beta(Y)$.

If $\psi \in \ell^\infty(G, B(Y))$, we define M_ψ on $\ell^\beta(G, Y)$ by

$$(M_\psi h)(\gamma) = \psi(\gamma)(h(\gamma)). \tag{6.5}$$

We have

$$\|M_\psi\|_\infty = \|\psi\|_\infty. \tag{6.6}$$

Also, if $\gamma_0 \in G$ we define an invertible isometry V_{γ_0} on $\ell^\beta(G, Y)$ by

$$(V_{\gamma_0} h)(\gamma) = h\left(\gamma \gamma_0^{-1}\right). \tag{6.7}$$

If $\varphi \in L^\infty(\mu, B(Y))$ and $\|\varphi(\omega)\| \leq \|\varphi\|_\infty$ for all $\omega \in [0,1]$, and if $\omega \in [0,1]$, we define $\varphi_\omega \in \ell^\infty(G, B(Y))$ by

$$\varphi_\omega(\gamma) = \varphi(\gamma(\omega)). \tag{6.8}$$

Then $M_{\varphi_\omega} \in B\left(\ell^\beta(G, Y)\right)$ for every $\omega \in [0,1]$.

6.2 Generalized Hadwin-Hoover Theorem

We are now ready to state our generial Hadwin-Hoover Theorem.

Theorem 6.2.1. *Suppose α is a normalized symmetric gauge norm on $L^\infty[0,1]$ and $\eta \in \lim \mathcal{NSG}_\infty(\alpha)$. Suppose G is a countable group of measure-preserving transformations on Ω that is freely acting. Suppose Y and $Y^\#$ are separable Banach spaces. Suppose further that $\mathcal{F} \subset L^\infty(\mu, \mathscr{B}(Y))$ is countable.*

Let $\mathcal{A}(\mathcal{F}, G, \alpha)$ be the unital norm-closed algebra generated by

$$\{U_\gamma : \gamma \in G\} \cup \{M_\varphi : \varphi \in \mathcal{F}\} \subset \mathscr{B}(L^\alpha(\mu, Y)). \tag{6.9}$$

Then for almost every $\omega \in \Omega$, there is a norm-one unital algebra homomorphism $\pi_\omega : \mathcal{A}(\mathcal{F}, G, \alpha) \to \mathscr{B}(\ell^\eta(G, Y))$ such that, for every $\gamma \in G$

$$\pi_\omega(U_\gamma) = V_\gamma, \tag{6.10}$$

and for $\varphi \in \mathcal{F}$,

$$(\pi_\omega(M_\varphi)f)(\gamma) = \varphi(\gamma(\omega))f(\gamma). \tag{6.11}$$

In other words, $\pi_\omega(M_\varphi) = \tilde{M}_{\varphi_\omega}$ where $\varphi_\omega(\gamma) = \varphi(\gamma(\omega))$.

Remark. In the original Hadwin-Hoover theorem $\mathcal{A}$ was defined to be the unital norm-closed *inverse-closed* algebra generated by $\{M_\varphi : \varphi \in \mathcal{F}\} \cup \{U_\gamma : \gamma \in G\}$. The prove of the inverse-closed part uses the fact that, since $Y^\#$ is separable, the dual space of $L^p(\mu, Y)$ is $L^q(\mu, Y^\#)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Although, for each symmetric gauge norm α there is a dual norm α', we do not always have $L^\alpha(\mu)^\# = L^{\alpha'}(\mu)$. Even when equality holds, it is unknown whether $L^\alpha(\mu, Y)^\# = L^{\alpha'}(\mu, Y^\#)$ holds when $Y^\#$ is separable. Hence we are forced to omit the inverse-closed part of the definition of $\mathcal{A}$.

We introduce one more lemma that will be applied in the proof of generalized Hadwin-Hoover theorem.

Lemma 6.2.2. *Suppose X and Y are normed spaces and $\{W_\lambda\}$ is a net of linear maps from X to Y such that for every $x \in X$,*

$$\lim_\lambda \|W_\lambda x\| = \|x\|. \tag{6.12}$$

Let $\mathcal{A}$ be the set of all operators $T \in \mathscr{B}(Y)$ for which there is an $S \in \mathscr{B}(X)$ such that, for the set $D(T, S)$ of all $x \in X$ for which

$$\lim_\lambda \|TW_\lambda x - W_\lambda Sx\| = 0 \tag{6.13}$$

is dense in X. Then

1. *If $T \in \mathscr{B}(Y)$ and $S \in \mathscr{B}(X)$, then $D(T,S)$ is closed, so if it is dense it is all of X.*

2. *For every $T \in \mathcal{A}$ there is a unique $S = \pi(T) \in \mathscr{B}(X)$ such that*

$$\lim_{\lambda} \|TW_\lambda x - W_\lambda Sx\| = 0.$$

3. *$\mathcal{A}$ is a norm closed unital algebra $\pi : \mathcal{A} \to \mathscr{B}(X)$ is a unital homomorphism and $\|\pi\| \leq 1$.*

Proof. 1. and 2. Suppose, for every $x \in X$,

$$\lim_{\lambda} \|TW_\lambda x - W_\lambda Sx\| = \lim_{\lambda} \|TW_\lambda x - W_\lambda S'x\| = 0.$$

Then, by the hypothesis, for every $x \in X$,

$$\begin{aligned}\|Sx - S'x\| &= \lim_{\lambda} \|W_\lambda Sx - W_\lambda S'x\| \\ &\leq \lim_{\lambda} \|TW_\lambda x - W_\lambda Sx\| + \|TW_\lambda x - W_\lambda S'x\| = 0.\end{aligned}$$

3. If $T_1, T_2 \in \mathcal{A}$ and $z \in \mathbb{C}$, then

$$\begin{aligned}&\lim_{\lambda} \|(T_1 + zT_2) W_\lambda - W_\lambda [\pi(T_1) + z\pi(T_2)]\| \\ &\qquad \leq \lim_{\lambda} \|T_1 W_\lambda x - W_\lambda \pi(T_1) x\| + |z| \|T_2 W_\lambda x - W_\lambda \pi(T_2) x\| = 0.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{\lambda} \|T_1 T_2 W_\lambda x - W_\lambda \pi(T_1) \pi(T_2) x\| &\leq \lim_{\lambda} \|T_1 [T_2 W_\lambda x - W_\lambda \pi(T_2) x]\| \\ &\quad + \|T_1 W_\lambda [\pi(T_2) x] - W_\lambda \pi(T_1) [\pi(T_2) x]\| = 0.\end{aligned}$$

This proves that $\mathcal{A}$ is an algebra and $\pi : \mathcal{A} \to \mathscr{B}(X)$ is a homomorphism. Clearly, $1 \in \mathcal{A}$ and $\pi(1) = 1$. Also if $x \in X$ and $T \in \mathcal{A}$ we have

$$\|\pi(T) x\| = \lim_{\lambda} \|W_\lambda \pi(T) x\| = \lim_{\lambda} \|TW_\lambda x\| \leq \lim_{\lambda} \|T\| \|W_\lambda x\| = \|T\| \|x\|.$$

Thus $\|\pi(T)\| \leq \|T\|$ for every $T \in \mathcal{A}$. It follows that π extends uniquely to a homomorphism $\hat{\pi} : \mathcal{A}^{-\|\|} \to \mathscr{B}(X)$. Suppose $A \in \mathcal{A}^{-\|\|}$ and $\varepsilon > 0$. We can choose $T \in \mathcal{A}$ so that $\|A - T\| < \varepsilon/3$. Thus, for every $x \in X$ and every λ, we have

$$
\begin{aligned}
\|AW_\lambda x - W_\lambda \pi(A)x\| &\le \|(A-T)W_\lambda x\| + \|TW_\lambda x - W_\lambda \pi(T)x\| \\
&\le \|(A-T)W_\lambda x\| + \|TW_\lambda x - W_\lambda \pi(T)x\| + \|W_\lambda \pi(T-A)x\| \\
&\le \frac{\varepsilon}{3}\|W_\lambda x\| + \|TW_\lambda x - W_\lambda \pi(T)x\| + \frac{\varepsilon}{3}\|x\| \\
&= \frac{2\varepsilon}{3}\|x\| + \|TW_\lambda x - W_\lambda \pi(T)x\|.
\end{aligned}
$$

Since $T \in \mathcal{A}$, there is a λ_0 such that whenever $\lambda \ge \lambda_0$, we have

$$\|TW_\lambda x - W_\lambda \pi(T)x\| < \frac{\varepsilon}{3}\|x\|,$$

which implies

$$\|AW_\lambda x - W_\lambda \pi(A)x\| < \varepsilon\|x\|.$$

It follows that

$$\lim_\lambda \|AW_\lambda x - W_\lambda \pi(A)x\| = 0.$$

Hence $A \in \mathcal{A}$, so $\mathcal{A}$ is norm closed. ■

Proof of Generalized Hadwin-Hoover Theorem. Write $G = \{\gamma_1, \gamma_2, \ldots\}$ with $\gamma_1 = e$ (the identity in G). Let

$$P = \prod_{(\gamma,\varphi)\in G\times\mathcal{F}} \text{ball}(\mathscr{B}(Y))$$

with the product SOT topology. We can assume that $\|\varphi\|_\infty \le 1$ for every $\varphi \in \mathcal{F}$. Consider the map $\Gamma : [0,1] \to \prod_{(\gamma,\varphi)\in G\times\mathcal{F}}$ ball($\mathscr{B}(Y)$) with the product SOT topology, by

$$\Gamma(\gamma,\varphi)(\omega) = \varphi(\gamma(\omega)).$$

Then, since ball($\mathscr{B}(Y)$) is a complete separable metric space with respect to the SOT, we see that $P = \prod_{\gamma,\varphi\in G\times\mathcal{F}}$ ball($\mathscr{B}(Y)$) is a separable metric space with metric d and that Γ is Borel measurable. Let $E =$ ess-Domain(Γ). Then $\mu([0,1]\setminus E) = 0$.

Suppose $\eta \in \lim \mathcal{NSG}_\infty(\alpha)$ and ω_0 is in the essential domain of Γ. We write $\mathcal{F} = \{\varphi_1, \varphi_2, \ldots\}$. Le Λ be the set of all (F,n) with $F \subset Y\setminus\{0\}$ finite and $n \in \mathbb{N}$. We make Λ into a directed set with the relation $\le\, = (\subset, \le)$, i.e., $(E,n) \le (F,m) \Leftrightarrow E \subset F$ and $n \le m$.

Suppose $\lambda = (E_\lambda, n_\lambda) \in \Lambda$. Let E be the set of all $\omega \in \Omega$ such that

$$\|\varphi_k(\omega)\, y - \varphi_k(\omega_0)\, y\| < \|y\|/n^2 \text{ for } 1 \leq k \leq n \text{ and } y \in F.$$

Since ω_0 is in the essential range of Γ, we have $\mu(E) > 0$. Since G is freely acting, there is an $E' \subset E$ with $\mu(E') > 0$ such that $\{\gamma_k(E) : 1 \leq k \leq n\}$ is disjoint. We now choose t such that $0 < t < \mu(E')$ a nd an $E_\lambda \subset E'$ with $\mu(E_\lambda) = t$ such that

$$|\alpha_{n,t}(a_1, \ldots a_n) - \eta(a_1, \ldots, a_n, 0, 0, \ldots)| < 1/n$$

for all $a_1, \ldots, a_n \in \{\|y\| : y \in E\}$. We define $W_\lambda : c_{00}(G, Y) \to L^\alpha(\mu, Y)$ by

$$W_\lambda f = \sum_{k=1}^{n} f(\gamma_k)\, \chi_{\gamma_k^{-1}(E_\lambda)}.$$

It is clear that, for every $f \in c_{00}(G, Y)$ that

$$\lim_\lambda \alpha(W_\lambda f) = \eta(f).$$

If we let $\mathcal{A}$ be as in Lemma 6.2.2, we merely need to show that $\mathcal{A}(\mathcal{F}, G, \alpha) \subset \mathcal{A}$. Thus we need to show that the condition of Lemma 6.2.2 holds when $T = M_{\varphi_k}$ and when $T = U_{\gamma_k}$ for each k in $\mathbb{N}$ and we are done. ■

BIBLIOGRAPHY

[1] Donald Hadwin. *Continuous Functions of Operators: A Functional Calculus*. Indiana University Mathematics Journal, 27(1): 113-125, 1978.

[2] Donald Hadwin and Thomas B. Hoover. *Representations of Weight Translation Algebras*. Houston Journal of Mathematics, 18(2): 1283-1305, 1992.

[3] Don Hadwin, Llolsten Kaonga and Ben Mathes. *Noncommutative Continuous Functions*. Journal of the Korean Mathematical Society, 40(5): 789-830, 2003.

[4] Junsheng Fang, Don Hadwin, Eric Nordgren, Junhao Shen. *Tracially Gauge Norms on Finite Von Neumann Algebras Satisfying the Weak Dixmier Property*. Journal of Functional Analysis, 255: 142-148, 2008.

[5] Yanni Chen. *Lebesgue and Hardy Spaces for Symmetric Norms I*. arXiv:1407.7920v1 [math.OA], 1-14, 2014.

[6] Yanni Chen. *A General Beurling-Helson-Lowdenslager Theorm on the Disk*. Advances in Applied Mathematics, 87: 1-15, 2017.

[7] Terry A. Loring. *C^*-Algebras Generated by Stable Relations*. Journal of Functional Analysis, 112: 159-203, 1993.

[8] Terry A. Loring. *Projective C^*-Algebras*. Mathematica Scandinavica, 73: 274-280, 1993.

[9] Terry A. Loring. *Stable Relations II: Corona Semiprojectivity and Dimension-Drop C^*-Algebras*. Pacific Journal of Mathematics, 172: 461-475, 1996.

[10] Terry A. Loring. *C^*-Algebras that are Only Weakly Semiprojective*. Proceeding of the American Mathematical Society. 126: 2713-2715, 1998.

[11] Donald Hadwin and Thomas Shulman. *Tracial Stability for C^*-Algebras*. Integral Equations and Operator Theory, 20: 1-35, 2018.