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BEURLING THEOREMS AND APPROXIMATE EQUIVALENCE IN VON NEUMANN ALGEBRAS

Wenjing Liu
University of New Hampshire, Durham

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BY

WENJING LIU

MS, Mathematics, Shaanxi Normal University, 2013
BS, Mathematics, Shaanxi Normal University, 2010

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This dissertation has been examined and approved in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics by:

Dissertation Director, Don Hadwin,
Professor of Mathematics
and Statistics

Rita A. Hibschweiler,
Professor of Mathematics
and Statistics

Junhao Shen,
Professor of Mathematics
and Statistics

Mehmet Orhon,
Senior Lecturer of Mathematics
and Statistics

Eric A. Nordgren,
Professor of Mathematics
and Statistics

on March 29, 2019.

Original approval signatures are on file with the University of New Hampshire Graduate School.

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ABSTRACT

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Wenjing Liu
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The main part of the thesis relates to generalized Beurling-Helson-Lowdenslager theorems on the unit disk, in multiply connected domains, and in finite von Neumann algebras. I have also obtained results on approximate unitary equivalence of representations of separable ASH C*-algebras in a semifinite von Neumann algebra, extending results of D. Voiculescu. In the last part of the thesis, I give two characterizations of tracially nuclear C*-algebras.

CHAPTER 1

INTRODUCTION

The thesis deals with operator theory, operator algebras, and analysis. One part of my thesis relates to generalized Beurling-Helson-Lowdenslager theorems on the unit disk, in multiply connected domains, and in finite and semifinite von Neumann algebras. I have also obtained results on tracially nuclear C*-algebras, and I have obtained results on approximate unitary equivalence of representations of separable ASH C*-algebras in a semifinite von Neumann algebra, extending results of D. Voiculescu.

1.1 Beurling Theorems

1.1.1 Commutative Results

The classical Beurling theorem characterizes closed linear subspaces of $L^p(\mathbb{T},\mu)$ that are invariant under multiplication by the variable z, where μ is Haar measure on the unit circle $\mathbb{T}$ in the complex plane, and $1 \leq p < \infty$. The invariant subspaces are those of the form $\chi_E L^p(\mathbb{T},\mu)$ for a Borel subset $E \subset \mathbb{T}$, or φH^p for a measurable φ with $|\varphi(z)| = 1$ a.e. (μ). Here H^p is the Hardy space, which is the $\| \|_p$-closure of H^∞ (the bounded analytic functions on the open unit disk). When $p = \infty$, the same characterization applies to z-invariant subspaces of $L^\infty(\mu)$ that are closed in the weak*-topology.

In 2015 Yanni Chen proved that the Beurling theorem holds when $\| \|_p$ $(1 \leq p < \infty)$ is replaced by any continuous normalized gauge norm α such that $\alpha \geq \| \|_1$. This is a vast collection of norms including those of Lorenz, Ky Fan, Orlicz, and Marcinkiewicz.

My first result in this area attacked the requirement $\alpha \geq \| \|_{1,\mu}$. I proved that, for any continuous normalized gauge norm α and any $0 < r < 1$, there is a probability Borel measure λ on $\mathbb{T}$ that is mutually absolutely continuous with respect to μ such that $\alpha \geq r \left\|\right\|_{1,\lambda}$. Moreover, we proved that Beurling's theorem holds if $\log(d\lambda/\mu) \in L^1(\mu)$.

Later I attacked the requirement that α be continuous. For any gauge norm α with $\alpha \geq \left\|\right\|_1$, I defined a topology $\mathcal{T}_\alpha$ on $L^\alpha(\mu)$, which, when α is continuous coincides with the weak topology, and when $\alpha = \left\|\right\|_{\infty,\mu}$ coincides with the weak*-topology. In this general setting we proved that the Beurling conditions characterize the $\mathcal{T}_\alpha$-closed z-invariant linear subspaces of $L^\alpha(\mu)$.

Yanni Chen, Don Hadwin, Eric Nordgren, and Zhe Liu extended Yanni Chen's Beurling theorem on the unit disk to an open set whose boundary is a finite disjoint union of smooth Jordan curves. With a lot more work, I have been able to prove my extensions of Chen's results in this setting.

1.1.2 Noncommutative results

William Arveson defined the analog of H^∞ in a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. There is an analogue $L^p(\mathcal{M},\tau)$ of the Lebesgue spaces L^p, $1 \leq p < \infty$, using the norm

$$\|a\|_p = \tau\left(|a|^p\right)^{1/p},$$

where $|a| = (a^*a)^{1/2}$. The completion of the noncommutative H^∞ with respect to $\left\|\right\|_p$ is called the noncommutative Hardy space $H^p(\mathcal{M},\tau)$. In 2008 D. Blecher and L. E. Labuschagne proved a version of the Beurling theorem in this noncommutative setting. Later, Yanni Chen, Don Hadwin and Junhao Shen extended the noncommutative Beurling theorem to the case where $\left\|\right\|_p$ is replaced with any continuous normalized unitarily invariant norm α satisfying $\alpha \geq \left\|\right\|_1$. We showed that for any continuous unitarily invariant norm α on $\mathcal{M}$, and any $0 < r < 1$, there is another faithful normal tracial state ρ on $\mathcal{M}$ such that

$$\alpha \geq r \left\|\right\|_{1,\rho}.$$

There is an analogue of a Radon-Nikodym derivative $a \geq 0$ of ρ with respect to τ which is an element of $L^1(\mathcal{M},\tau)$, and there is a Kadison-Fuglede determinant $\Delta(\log(a)) = h$ of the logarithm of a. We proved that if $h \in L^1(\mathcal{M}, \tau)$, then the noncommutative Beurling theorem is true. I also extended my results on the disk to this noncommutative setting.

With Lauren Sager, we extended the Beurling theorem results to semifinite von Neumann algebras where there is a faithful normal tracial weight (analogous to a measure space with an infinite measure).

1.2 Approximate Unitary Equivalence

Suppose $\mathcal{A}$ is a separable unital C*-algebra, H is a separable Hilbert space, and $\pi, \rho : \mathcal{A} \to B(H)$ are unital $*$-homomorphisms into the algebra $B(H)$ of all bounded linear operators on H. The representations π and ρ are approximately unitarily equivalent, if there is a sequence $\{U_n\}$ of unitary operators (i.e., Hilbert-space isomorphisms) such that, for every $a \in \mathcal{A}$,

$$\lim_{n\to\infty} \left\| U_n^{-1} \pi(a) U_n - \rho(a) \right\| = 0.$$

In 1976 Dan Voiculescu gave a beautiful, purely algebraic characterization of approximate equivalence. Later Don Hadwin simplified this characterization in terms of rank, i.e., π and ρ are approximately equivalent if and only if, for every $a \in \mathcal{A}$,

$$rank(\pi(a)) = rank(\rho(a)).$$

Also Hadwin proved that this rank condition characterizes approximate equivalence for nonseparable C*-algebras and nonseparable Hilbert spaces. Voiculescu also proved that when π and ρ are approximately equivalent, a sequence $\{U_n\}$ of unitary operators as above with the additional condition that, for every $a \in \mathcal{A}$ and every $n \geq 1$,

$$U_n^{-1} \pi(a) U_n - \rho(a) \text{ is a compact operator.}$$

Later Huiru Ding and Don Hadwin extended the notion of rank to an element T of a von Neumann algebra $\mathcal{M}$, denoted by $\mathcal{M}$-rank(T), and they proved for any separable AH C*-algebra

$\mathcal{A}$ and any von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space, two representations $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are approximately unitarily equivalent using unitary operators in $\mathcal{M}$ if and only if, for every $x \in \mathcal{A}$,

$$\mathcal{M}\text{-rank}\pi\left(x\right) = \mathcal{M}\text{-rank}\rho\left(x\right).$$

With Don Hadwin I have extended these results to a much larger class of C*-algebras when $\mathcal{M}$ is a finite von Neumann algebra acting on any Hilbert space. In a semifinite von Neumann algebra there is an analogue of the notion of a compact operator and Don Hadwin, Rui Shi, and Junsheng Fang proved analogues of the compact part of Voiculescu's theorem for AH C*-algebras. With Don Hadwin I have extended this result to the much larger class of ASH C*-algebras.

In 2013 A. Ciuperca, T. Giordano, P. W. Ng, and Z. Niu defined a notion of weak*-approximate equivalence for representations from a C*-algebra into a von Neumann algebra, and they proved that the above $\mathcal{M}$-rank characterization was always true if and only if the C*-algebra is nuclear. In the summer of 2018, Don Hadwin, Weihua Li and I proved that this is true for all finite von Neumann algebras if and only if the C*-algebra is tracially nuclear. We also gave a characterization of tracially nuclear in terms of the second dual of the C*-algebra.

CHAPTER 2

AN EXTENSION OF THE CHEN-BEURLING-HELSON-LOWDENSLAGER THEOREM

Yanni Chen [10] extended the classical Beurling-Helson-Lowdenslager theorem for Hardy spaces on the unit circle $\mathbb{T}$ defined in terms of continuous gauge norms on L^∞ that dominate $\|\cdot\|_1$. In the chapter, we extend Chen's result to a much larger class of continuous gauge norms. A key ingredient is our result that if α is a continuous normalized gauge norm on L^∞, then there is a probability measure λ, mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$, such that $\alpha \geq c\|\cdot\|_{1,\lambda}$ for some $0 < c \leq 1$.

2.1 Introduction

Let $\mathbb{T}$ be the unit circle, i.e., $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and let μ be Haar measure (i.e., normalized arc length) on $\mathbb{T}$. The classical and influential Beurling-Helson-Lowdenslager theorem (see [5],[41]) states that if W is a closed $H^\infty(\mathbb{T}, \mu)$-invariant subspace (or, equivalently, $zW \subseteq W$) of $L^2(\mathbb{T}, \mu)$, then $W = \varphi H^2$ for some $\varphi \in L^\infty(\mathbb{T}, \mu)$, with $|\varphi| = 1$ a.e.(μ) or $W = \chi_E L^2(\mathbb{T}, \mu)$ for some Borel set $E \subset \mathbb{T}$. If $0 \neq W \subset H^2(\mathbb{T}, \mu)$, then $W = \varphi H^2(\mathbb{T}, \mu)$ for some $\varphi \in H^\infty(\mathbb{T}, \mu)$ with $|\varphi| = 1$ a.e. (μ). Later, the Beurling's theorem was extended to $L^p(\mathbb{T}, \mu)$ and $H^p(\mathbb{T}, \mu)$ with $1 \leq p \leq \infty$, with the assumption that W is weak*-closed when $p = \infty$ (see [36],[39],[41],[43]). In [10], Yanni Chen extended the Helson-Lowdenslager-Beurling theorem for all continuous $\|\cdot\|_{1,\mu}$-dominating normalized gauge norms on $\mathbb{T}$.

In this chapter we extend the Helson-Lowdenslager-Beurling theorem for a much larger class of norms. We first extend Chen's results to the case of $c\|\cdot\|_{1,\mu}$-dominating continuous gauge norms. We then prove that for any continuous gauge norm α, there is a probability measure λ that is mutually absolutely continuous with respect to μ such that α is $c\|\cdot\|_{1,\lambda}$-dominating. We use this result to extend Chen's theorem. Our extension depends on Radon-Nikodym derivative $d\lambda/d\mu$. In particular, Chen's theorem extends exactly whenever $\log(d\lambda/d\mu) \in L^1(\mathbb{T},\mu)$.

2.2 Continuous Gauge Norms on Ω

Suppose (Ω,Σ,ν) is a probability space. A norm α on $L^\infty(\Omega,\nu)$ is a *normalized gauge norm* if

1. $\alpha(1) = 1$,

2. $\alpha(|f|) = \alpha(f)$ for every $f \in L^\infty(\Omega,\nu)$.

In addition we say α is *continuous* (*ν-continuous*) if

$$\lim_{\nu(E)\to 0} \alpha(\chi_E) = 0,$$

that is, whenever $\{E_n\}$ is a sequence in Σ and $\nu(E_n) \to 0$, we have $\alpha(\chi_{E_n}) \to 0$.

We say that a *normalized gauge norm* α is *$c\|\cdot\|_{1,\nu}$-dominating* for some $c > 0$ if

$$\alpha(f) \geq c\|f\|_{1,\nu}, \text{ for every } f \in L^\infty(\Omega,\nu).$$

It is easy to see the following facts

(1) The common norm $\|\cdot\|_{p,\nu}$ is a α norm for $1 \leq p \leq \infty$.

(2) If ν and λ are mutually absolutely continuous probability measures, then $L^\infty(\Omega,\nu) = L^\infty(\Omega,\lambda)$ and a normalized gauge norm is ν-continuous if and only if it is λ-continuous.

We can extend the normalized gauge norm α from $L^\infty(\Omega,\nu)$ to the set of all measurable functions, and define α for all measurable functions f on Ω by

$$\alpha(f) = \sup\{\alpha(s) : s \text{ is a simple function }, 0 \leq s \leq |f|\}.$$

It is clear that $\alpha(f) = \alpha(|f|)$ still holds.

Now we define $\mathcal{L}^\alpha(\Omega,\nu) = \{f : f \text{ is a measurable function on } \Omega \text{ with } \alpha(f) < \infty\}$, and

$L^{\alpha}(\Omega,\nu)=\overline{L^{\infty}(\nu)}^{\alpha}$, i.e., the α -closure of $L^{\infty}(\nu)$ in $\mathcal{L}^{\alpha}$.

Since $L^{\infty}(\Omega,\nu)$ with the norm α is dense in $L^{\alpha}(\Omega,\nu)$, they have the same dual spaces. We prove in the next lemma that the normed dual $(L^{\alpha}(\Omega,\nu),\alpha)^{\#}=(L^{\infty}(\Omega,\nu),\alpha)^{\#}$ can be viewed as a vector subspace of $L^{1}(\Omega,\nu)$. Suppose $w\in L^{1}(\Omega,\nu)$, we define the functional $\varphi_{w}:L^{\infty}(\Omega,\nu)\to\mathbb{C}$ by

$$\varphi_{w}(f)=\int_{\Omega}fwd\nu.$$

Lemma 1. *Suppose (Ω,Σ,ν) is a probability space and α is a continuous normalized gauge norm on $L^{\infty}(\Omega,\nu)$. Then*

(1) if $\varphi:L^{\infty}(\Omega,\nu)\to\mathbb{C}$ is an α-continuous linear functional, then there is a $w\in L^{1}(\Omega,\nu)$ such that $\varphi=\varphi_{w}$,

(2) if φ_{w} is α-continuous on $L^{\infty}(\Omega,\nu)$, then

(a) $\|w\|_{1,\mu}\leq\|\varphi_{w}\|=\left\|\varphi_{|w|}\right\|$,

(b) given φ in the dual of $L^{\alpha}(\Omega,\lambda)$, i.e., $\varphi\in(L^{\alpha}(\Omega,\lambda))^{\#}$, there exists a $w\in L^{1}(\Omega,\lambda)$, such that

$$\forall f\in L^{\infty}(\Omega,\lambda),\varphi(f)=\int_{\Omega}fwd\lambda \textit{ and } wL^{\alpha}(\Omega,\lambda)\subseteq L^{1}(\Omega,\lambda).$$

Proof. (1) If α is continuous, it follows that, whenever $\{E_{n}\}$ is a disjoint sequence of measurable sets,

$$\lim_{N\to\infty}\alpha\left(\chi_{\cup_{n=1}^{\infty}E_{n}}-\sum_{k=1}^{N}\chi_{E_{k}}\right)=\lim_{N\to\infty}\alpha\left(\chi_{\cup_{k=N+1}^{\infty}E_{k}}\right)=0,$$

since $\lim_{N\to\infty}\nu\left(\cup_{k=N+1}^{\infty}E_{k}\right)=0$. It follows that

$$\rho(E)=\varphi(\chi_{E})$$

defines a measure ρ and $\rho<<\nu$. It follows that if $w=d\rho/d\nu$, then

$$\|w\|_{1,\nu}=\sup\left\{\left|\int_{\Omega}wsd\nu\right|:s\text{ is simple, }\|s\|_{\infty}\leq1\right\}$$

$$=\sup\{|\varphi(s)|:s\text{ simple, }\|s\|_{\infty}\leq1\}\leq\|\varphi\|.$$

Hence $w \in L^1(\Omega, \nu)$. Also, since, for every $f \in L^\infty(\Omega, \nu)$

$$|\varphi(f)| \leq \|\varphi\| \, \alpha(f) \leq \|\varphi\| \, \|f\|_\infty,$$

we see that φ is $\|\cdot\|_\infty$-continuous on $L^\infty(\Omega, \nu)$, so it follows that $\varphi = \varphi_w$.

(2a) From (1) we will see $\|w\|_{1,\nu} \leq \|\varphi\|$.

(2b) For any measurable set $E \subseteq \Omega$, and for all $\varphi \in (L^\alpha(\lambda))^\#$, define $\rho(E) = \varphi(\chi_E)$. We can prove ρ is a measure as in Theorem 2, and $\rho \ll \lambda$. By Radon-Nikodym theorem, there exists a function $w \in L^1(\lambda)$ such that, for every measurable set $E \subseteq \Omega$, $\varphi(\chi_E) = \rho(E) = \int_\Omega \chi_E w d\lambda$. Thus $\forall f \in L^\infty(\Omega, \lambda)$, $\varphi(f) = \int_\Omega f w d\lambda = \int_\Omega f w g d\mu = \int_\Omega f w |h| d\mu = \int_\Omega f w u h d\mu = \int_\Omega f \widetilde{w} h d\mu$, where $\widetilde{w} = wu$, $|\widetilde{w}| = |w|$, here $\widetilde{w} \in L^1(\Omega, \lambda)$ and g, h as in Theorem 2, so $\widetilde{w} h \in L^1(\mu)$. Therefore, $\varphi(f) = \int_\Omega f \widetilde{w} h d\mu$ for all $f \in L^\alpha(\Omega, \lambda)$.

If $f \in L^\alpha(\Omega, \lambda)$, $f = u|f|$ and $|u| = 1$,then $|f| \in L^\alpha(\Omega, \lambda)$. There exists an increasing positive sequence s_n such that $s_n \to |f|$ a.e. (μ), thus $us_n \to u|f|$ a.e.(μ). $\forall w \in L^1(\Omega, \lambda), w = v|w|$, where $|v| = 1$, so we have $\overline{v} s_n \to \overline{v}|f|$ a.e. (μ), where $\overline{v}$ is the conjugate of v and $\alpha(\overline{v} s_n - \overline{v}|f|) \to 0$. Thus $\varphi(\overline{v} s_n) \to \varphi(\overline{v}|f|)$. On the other hand, we also have $\varphi(\overline{v} s_n) = \int_\Omega \overline{v} s_n w d\lambda \to \int_\Omega \overline{v}|f| w d\lambda = \int_\Omega |f||w| d\lambda$ by monotone convergence theorem. Thus $\int_\Omega |f||w| d\lambda = \int_\Omega |f| \overline{v} w d\lambda = \varphi(\overline{v}|f|) < \infty$. Therefore $fw \in L^1(\Omega, \lambda)$, i.e., $wL^\alpha(\Omega, \lambda) \subseteq L^1(\Omega, \lambda)$, where $w \in L^1(\Omega, \lambda)$. □

Theorem 2. *Suppose (Ω, Σ, ν) is a probability space, α is a continuous normalized gauge norm on $L^\infty(\Omega, \nu)$ and $\varepsilon > 0$. Then there exists a constant c with $1 - \varepsilon < c \leq 1$ and a probability measure λ on Σ that is mutually absolutely continuous with respect to ν such that α is $c\|\cdot\|_{1,\lambda}$-dominating.*

Proof. Let $M = \{\nu(h^{-1}((0,\infty))) : h \in L^1(\Omega, \nu), h \geq 0, \varphi_h \text{ is } \alpha\text{-continuous}\}$. It follows from Lemma 1 that $M \neq \varnothing$. Choose $\{h_n\}$ in $L^1(\Omega, \nu)$ such that $h_n \geq 0$, φ_{h_n} is α-continuous, and such that

$$\nu\left(h_n^{-1}((0,\infty))\right) \to \sup M.$$

Let

$$h_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\|\varphi_{h_n}\|} h_n.$$

Since $\|h_n\|_{1,\nu} \leq \|\varphi_{h_n}\|$, we see that $\|h_0\|_{1,\nu} \leq 1$. Also

$$\varphi_{h_0} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\|\varphi_{h_n}\|} \varphi_{h_n},$$

so φ_{h_0} is α-bounded and $\|\varphi_{h_0}\| \leq 1$. On the other hand $h_n^{-1}((0,\infty)) \subset h_0^{-1}((0,\infty))$ for $n \geq 1$, so we have

$$\nu\left(h_0^{-1}((0,\infty))\right) = \sup M.$$

Let $E = \Omega \backslash h_0^{-1}((0,\infty))$ and assume, via contradiction, that $\nu(E) > 0$. Then $\alpha(\chi_E) > 0$. Hence, by the Hahn-Banach theorem, there is a $g \in L^1(\Omega,\nu)$ such that $\|\varphi_g\| = 1$ and

$$\alpha(\chi_E) = \varphi_g(\chi_E) = \int_\Omega g\chi_E d\nu = \varphi_{g\chi_E}(\chi_E) \leq \varphi_{|g|\chi_E}(\chi_E).$$

It follows that $\nu\left((|g|\chi_E)^{-1}(0,\infty)\right) = \eta > 0$, and that if $h_1 = h_0 + |g|\chi_E$, then

$$\sup M \geq \nu\left(h_1^{-1}((0,\infty))\right) = \nu\left(h^{-1}((0,\infty))\right) + \eta = \sup M + \eta.$$

This contradiction shows that $\nu(E) = 0$, so we can assume that $h_0(\omega) > 0$ a.e. (ν). By replacing h_0 with $h_0 / \int_\Omega h_0 d\nu$, we can assume that $\int_\Omega h_0 d\nu = 1$.

If we define a probability measure $\lambda : \Sigma \to [0,1]$ by

$$\lambda(E) = \int_E h_0 d\nu,$$

then λ is a measure, $\lambda << \nu$ and $\nu << \lambda$ since $0 < h_0$ a.e. (ν). Also, we have for every $f \in L^\infty(\Omega,\nu)$,

$$\|f\|_{1,\lambda} = \int_\Omega |f|\, d\lambda = \int_\Omega |f|\, h_0 d\nu = \varphi_{h_0}(|f|) \leq \|\varphi_{h_0}\|\, \alpha(f).$$

Since $\varphi_{h_0}(1) = 1$, we know $\|\varphi_{h_0}\| \geq 1$. Hence, $0 < c_0 = 1/\|\varphi_{h_0}\| \leq 1$, and we see that α is $c_0\|\cdot\|_{1,\lambda}$-dominating on E. If we apply the Hahn-Banach theorem as above with $E = \Omega$, we can find a nonnegative function $k \in L^1(\Omega, \nu)$ such that

$$\|\varphi_k\| = 1 = \alpha(1) = \varphi_k(1) = \int_\Omega k1 d\nu.$$

For $0 < t < 1$ let $h_t = (1-t)k + th_0$. Then $\varphi_{h_t} = (1-t)\varphi_k + t\varphi_{h_0}$. Thus

$$\lim_{t\to 0^+} \|\varphi_{h_t}\| = \|\varphi_k\| = 1.$$

Choose t so that $\|\varphi_{h_t}\| < 1/(1-\varepsilon)$, so $1 - \varepsilon < c = 1/\|\varphi_{h_t}\| \leq 1$. If we define a probability measure $\lambda_t : \Sigma \to [0,1]$ by

$$\lambda_t(E) = \int_E h_t d\nu,$$

we see that $\lambda_t << \mu\nu$ and since $h_t \geq th_0 > 0$, we see $\nu << \lambda_t$. As above we see, for every $f \in L^\infty(\Omega, \mu)$ we have

$$c\|f\|_{1,\lambda_t} \leq \frac{1}{\|\varphi_{h_t}\|}\int_\Omega |f| h_t d\nu = \frac{1}{\|\varphi_{h_t}\|}\varphi_{h_t}(|f|) \leq \alpha(f).$$

Therefore, α is $c\|\cdot\|_{1,\lambda_t}$-dominating on Ω. □

If we take $\Omega = \mathbb{T}$, Theorem 2 holds for the probability space $(\Omega, \nu) = (\mathbb{T}, \mu)$. The L^p-version of the Helson-Lowdenslager theorem also holds, in a sense, on the circle $\mathbb{T}$ when μ is replaced with a mutually absolutely continuous probability measure λ. Here the role of $H^p(\mathbb{T}, \lambda)$ is replaced with $\left(1/g^{\frac{1}{p}}\right) H^p(\mathbb{T}, \mu)$. This result is well-known, we include a proof for completeness as the following corollary.

Corollary 3. *Suppose λ is a probability measure on $\mathbb{T}$ and $\mu << \lambda$ and $\lambda << \mu$. Let $g = d\lambda/d\mu$ and suppose $1 \leq p < \infty$. Suppose W is a closed subspace of $L^p(\mathbb{T}, \lambda)$, and $zW \subset W$. Then $g^{\frac{1}{p}}W = \chi_E L^1(\mathbb{T}, \mu)$ for some Borel subset E of $\mathbb{T}$ or $g^{\frac{1}{p}}W = \varphi H^p(\mathbb{T}, \mu)$ for some unimodular function φ.*

Proof. Define $U : L^p(\mathbb{T},\lambda) \longrightarrow L^p(\mathbb{T},\mu)$ by $Uf = fg^{\frac{1}{p}}$, for $f \in L^p(\mathbb{T},\lambda)$. Clearly U is a surjective isometry, since

$$\|Uf\|_{p,\mu}^p = \int_{\mathbb{T}} \left|fg^{\frac{1}{p}}\right|^p d\mu = \int_{\mathbb{T}} |f|^p g d\mu = \int_{\mathbb{T}} |f|^p d\lambda = \|f\|_{p,\lambda}.$$

Define

$M_{z,\mu} : L^p(\mathbb{T},\mu) \longrightarrow L^p(\mathbb{T},\mu)$ by $M_{z,\mu}f = zf$ and $M_{z,\lambda} : L^p(\mathbb{T},\lambda) \longrightarrow L^p(\mathbb{T},\lambda)$ by $M_{z,\lambda}f = zf$.

Then

$$UM_{z,\lambda}f = U(zf) = g^{\frac{1}{p}}zf = zg^{\frac{1}{p}}f = M_{z,\mu}g^{\frac{1}{p}}f = M_{z,\mu}Uf,$$

so $UM_{z,\lambda} = M_{z,\mu}U$. It follows that W is a closed z-invariant subspace of $L^p(\mathbb{T},\lambda)$ if and only if $g^{\frac{1}{p}}W = U(W)$ is a z-invariant closed linear subspace of $L^p(\mathbb{T},\mu)$. The conclusion now follows from the classical Beurling theorem for $L^p(\mathbb{T},\mu)$. □

2.3 Continuous Gauge Norms on the Unit Circle

Suppose α is a continuous normalized gauge norm on $L^\infty(\mathbb{T},\mu)$, suppose that $c > 0$ and λ is a probability measure on $\mathbb{T}$ such that $\lambda << \mu$ and $\mu << \lambda$ and such that α is $c\|\cdot\|_{1,\lambda}$-dominating. We let $g = d\lambda/d\mu$ and $g > 0$. We consider two cases

(1) $\int |\log g|\, d\mu < \infty$,

(2) $\int |\log g|\, d\mu = \infty$.

We define $L^p(\mathbb{T},\lambda)$ to be the $\|\cdot\|_{p,\lambda}$-closure of $L^\infty(\mathbb{T},\lambda)$ and define $H^p(\mathbb{T},\lambda)$ to be $\|\cdot\|_{p,\lambda}$-closure of the polynomials for $1 \le p < \infty$. Denote $L^\infty(\mathbb{T},\mu) = L^\infty(\mu)$, $L^p(\mathbb{T},\mu) = L^p(\mu)$ and $H^p(\mathbb{T},\mu) = H^p(\mu)$.

Lemma 4. *The following are true:*

(1) $\int |\log g|\, d\mu < \infty \Leftrightarrow$ *there is an outer function* $h \in H^1(\mu)$ *with* $|h| = g$,

(2) $\int |\log g|\, d\mu = \infty \Leftrightarrow H^1(\lambda) = L^1(\lambda)$.

Proof. Clearly $H^1(\lambda)$ is a closed z-invariant subspace of $L^1(\lambda)$. Thus, by Corollary 3, either $gH^1(\lambda) = \varphi H^1(\mu)$ for some unimodular φ or $gH^1(\lambda) = \chi_E L^1(\mu)$ for some Borel set $E \subset \mathbb{T}$.

For (1), if $gH^1(\lambda) = \varphi H^1(\mu)$ for some unimodular φ, and $0 < g \in gH^1(\lambda)$, then $0 \neq \overline{\varphi} g \in H^1(\mu)$ which implies $\log g = \log |\overline{\varphi} g| \in L^1(\mu)$. It is a standard fact that if $g > 0$ and $\log g$ are in $L^1(\mu)$, then there exists an outer function $h \in H^1(\mu)$ with the same modulus as g,(i.e., $|h| = g$). Therefore, (1) is proved by Lemma 3.2 in [10].

For (2), Since $gH^1(\lambda) = \varphi H^1(\mu)$ if and only if $\int |\log g| \, d\mu < \infty$. Suppose $\int |\log g| \, d\mu = \infty$. Then $gH^1(\lambda) = \chi_E L^1(\mu)$. We have $g = \chi_E f$ for some $f \in L^1(\mu)$, which implies $\chi_E = 1$ since $g > 0$. Thus $gH^1(\lambda) = L^1(\mu) = gL^1(\mu)$, which implies $H^1(\lambda) = L^1(\lambda)$. Conversely, if $H^1(\lambda) = L^1(\lambda)$, then $gH^1(\lambda) = gL^1(\lambda) = L^1(\mu) = \chi_{\mathbb{T}} L^1(\mu)$, which means $gH^1(\lambda) \neq \varphi H^1(\mu)$, i.e., $\int |\log g| \, d\mu = \infty$. □

There is an important characterization of outer functions in $H^1(\mu)$.

Lemma 5. *A function f is an outer function in $H^1(\mu)$ if and only there is a real harmonic function u with harmonic conjugate $\overline{u}$ such that*

(1) $u \in L^1(\mu)$,

(2) $f = e^{u + i\overline{u}}$,

(3) $f \in L^1(\mu)$.

Through the remainder of following sections we assume

1. α is a continuous normalized gauge norm on $L^\infty(\mu)$.

2. and that $c > 0$ and λ is a probability measure on $\mathbb{T}$ such that $\lambda << \mu$ and $\mu << \lambda$ and such that α is $c\|\cdot\|_{1,\lambda}$-dominating.

3. $h \in H^1(\mu)$ is an outer function, η is unimodular and $\bar{\eta} h = g = d\lambda / d\mu$.

Since λ and μ are mutually absolutely continuous we have $L^\infty(\mu) = L^\infty(\lambda), L^\alpha(\mu) = L^\alpha(\lambda)$ and $H^\alpha(\mu) = H^\alpha(\lambda)$, we will use L^∞ to denote $L^\infty(\mu)$ and $L^\infty(\lambda)$, use L^α to denote $L^\alpha(\mu)$ and

$L^\alpha(\lambda)$, use H^α to denote $H^\alpha(\mu)$ and $H^\alpha(\lambda)$. It follows that $L^\alpha, L^\infty, H^\alpha$ do not depend on λ or μ. However, this notation slightly conflicts with the classical notation for $L^1(\mu) = L^{\|\cdot\|_{1,\mu}}$ or $H^1(\mu) = H^{\|\cdot\|_{1,\mu}}$, so we will add the measure to the notation when we are talking about L^p or H^p.

Theorem 6. *We have $hL^1(\lambda) = L^1(\mu)$ and $hH^1(\lambda) = H^1(\mu)$.*

Proof. We know from our assumption (3) that $hL^1(\lambda) = g\eta L^1(\lambda) = gL^1(\lambda) = L^1(\mu)$. By Lemma 4(1), we have $gH^1(\lambda) = \eta H^1(\mu)$, so

$$hH^1(\lambda) = \eta g H^1(\lambda) = \eta\eta H^1(\mu) = H^1(\mu).$$

□

Corollary 7. *$gH^1(\lambda) = \gamma H^1(\mu)$ for some unimodular $\gamma \Leftrightarrow \int_{\mathbb{T}} |\log g|\, d\mu < \infty$.*

Proof. Assume $gH^1(\lambda) = \gamma H^1(\mu)$. Since $1 \in H^1(\lambda), g \in gH^1(\lambda)$, $\exists \phi \in H^1(\mu)$ such that $g = \gamma\phi$. Since $\phi \in H^1(\mu), \phi = \psi h$, where ψ is an inner function and h is an outer function. Thus, $\int_{\mathbb{T}} |\log g|\, d\mu = \int_{\mathbb{T}} \log|g| d\mu = \int_{\mathbb{T}} \log|h| d\mu < \infty$, since h is an outer function.

Assume $\int_{\mathbb{T}} |\log g|\, d\mu < \infty, g$ and $\log g \in L^1(\mu), g > 0$. Thus there exists an outer function $h \in H^1(\mu)$, such that $|h| = |g| = g, |h| = \phi h, |\phi| = 1, g = \eta h$, Define $V : L^1(\lambda) \longrightarrow L^1(\mu)$ by $Vf = hf$, as in Theorem 6, we have $hH^1(\lambda) = H^1(\mu)$, so $gH^1(\lambda) = \eta h H^1(\lambda) = \eta H^1(\mu)$. Let $\gamma = \eta$, then $gH^1(\lambda) = \gamma H^1(\mu)$. □

We now get a Helson-Lowdenslager theorem when $\alpha = \|\cdot\|_{p,\lambda}$ and $\log g \in L^1(\mu)$.

Corollary 8. *Suppose $1 \le p < \infty$. If W is a closed subspace of $L^p(\lambda)$ and $zW \subseteq W$, then either $W = \gamma H^p(\lambda)$ for some unimodular function γ, or $W = \chi_E L^p(\lambda)$ for some Borel subset E of $\mathbb{T}$.*

The following theorem shows the relation between $H^\alpha, H^1(\lambda)$ and L^α. This result parallels a result of Y. Chen [10], which is a key ingredient in her proof of her general Beurling theorem. However, her result was for $H^1(\mu)$ instead of $H^1(\lambda)$.

Theorem 9. $H^\alpha = H^1(\lambda) \cap L^\alpha$.

Proof. Since α is continuous $c\|\cdot\|_{1,\lambda}$-dominating, α-convergence implies $\|\cdot\|_{1,\lambda}$-convergence, thus

$$H^{\alpha} = \overline{H^{\infty}}^{\alpha} \subseteq \overline{H^{\infty}}^{\|\cdot\|_{1,\lambda}} = H^{1}(\lambda).$$

Also,

$$H^{\alpha} = \overline{H^{\infty}(\lambda)}^{\alpha} \subset \overline{L^{\infty}}^{\alpha} = L^{\alpha}.$$

Thus $H^{\alpha} \subseteq H^{1}(\lambda) \cap L^{\alpha}$.

Since α-convergence implies $\|\cdot\|_{1,\lambda}$-convergence, $H^{1}(\lambda) \cap L^{\alpha}$ is an α-closed subspace of L^{α}. Suppose $\varphi \in (L^{\alpha})^{\#}$ such that $\varphi|H^{\infty} = 0$. It follows from Lemma 1 that there is a $w \in L^{1}(\lambda)$ such that $wL^{\alpha} \subset L^{1}(\lambda)$ and such that, for every $f \in L^{\alpha}$,

$$\varphi(f) = \int f\bar{\eta}wd\lambda = \int fwhd\mu.$$

Since $wL^{\alpha} \subset L^{1}(\lambda)$, we know that $whL^{\alpha} \subset L^{1}(\mu)$. Since $\varphi|_{H^{\infty}} = 0$, we have

$$\int_{\mathbb{T}} z^{n}hwd\mu = \varphi(z^{n}) = 0$$

for every integer $n \geq 0$. Thus $hw \in H_{0}^{1}(\mu)$.

Now suppose $f \in H^{1}(\lambda) \cap L^{\alpha}$. Then $hf \in H^{1}(\mu)$. We know that every function in $H^{1}(\mu)$ has a unique inner-outer factorization. Thus we can write

$$hf = \gamma_{1}h_{1}$$

with γ_{1} inner and h_{1} outer. Moreover, since $hw \in H_{0}^{1}(\mu)$, we can write

$$(hw)(z) = z\gamma_{2}(z)\,h_{2}(z)$$

with γ_{2} inner and h_{2} outer. By Lemma 5, we can find real harmonic functions $u, u_{1}, u_{2} \in L^{1}(\mu)$ such that

$$h = e^{u+i\overline{u}}, h_{1} = e^{u_{1}+i\overline{u}_{1}}, \text{ and } h_{2} = e^{u_{2}+i\overline{u}_{2}}.$$

Thus

$$hfw = hfhw/h = z\gamma_1\gamma_2 e^{(u_1+u_2-u)+i(\overline{u}_1+\overline{u}_2-\overline{u})} \in H^1(\mu).$$

It follows from Lemma 5 that

$$\varphi(f) = \int_{\mathbb{T}} hfw\,d\mu = (hfw)(0) = 0.$$

Hence every continuous linear functional on L^α that annihilates H^α also annihilates $H^1(\lambda) \cap L^\alpha$. It follows from the Hahn-Banach theorem that $H^1(\lambda) \cap L^\alpha \subset H^\alpha$. □

The following result is a factorization theorem for L^α.

Theorem 10. *If $k \in L^\infty$, $k^{-1} \in L^\alpha$, then there is a unimodular function $u \in L^\infty$ and an outer function $s \in H^\infty$ such that $k = us$ and $s^{-1} \in H^\alpha$.*

Proof. Recall that an outer function is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Since $k^{-1} \in L^\alpha \subseteq L^1(\lambda)$, we know that $\|k\|_\infty > 0$. Thus $\log|k| \leq \log\|k\|_\infty \in \mathbb{R}$. Moreover, $k^{-1} \in L^\alpha \subseteq L^1(\lambda)$ implies $hk^{-1} \in L^1(\mu)$, so

$$\log|h| - \log|k| = \log\left(\left|hk^{-1}\right|\right) \leq \left|hk^{-1}\right|.$$

Hence

$$\log|h| - \left|hk^{-1}\right| \leq \log|k| \leq \log\|k\|_\infty,$$

and since $\log|h|$, $|hk^{-1}|$ and $\log\|k\|_\infty$ are in $L^1(\mu)$, we see that $\log|k| \in L^1(\mu)$. Therefore, by the first statement of Lemma 4, there is an outer function $s \in H^1(\mu)$ such that $|s| = |k|$. It follows that $s \in H^\infty$. Hence there is a unimodular function u such that $k = us$.

We also know that

$$\left|\log\left|hk^{-1}\right|\right| = |\log(|h|) - \log|k|| \leq |\log(|h|)| + |\log|k|| \in L^1(\mu),$$

so there exists an outer function $f \in H^1(\mu)$ such that $|k^{-1}h| = |f|$. Thus sf is outer in $H^1(\mu)$ and $|h| = |sf|$, so $h = e^{it}sf$ for some real number t. Since $H^1(\mu) = hH^1(\lambda)$, we see that there exists a function $f_1 \in H^1(\lambda)$ such that $hf_1 = f = h\left(e^{-it}s^{-1}\right)$. It follows that $s^{-1} = e^{it}f_1 \in H^1(\lambda)$. Also, $|s^{-1}| = |k^{-1}|$, so $s^{-1} \in L^\alpha$. It follows from Theorem 9 that $s^{-1} \in H^1(\lambda) \cap L^\alpha = H^\alpha$. □

Lemma 11. *If M is a closed subspace of L^α and $zM \subseteq M$, then $H^\infty M \subseteq M$.*

Proof. Suppose $\varphi \in (L^\alpha)^\#$ and $\varphi|_M = 0$. It follows from Lemma 1 that there is a $w \in L^1(\lambda)$ such that $wL^\alpha \subset L^1(\lambda)$. For every $f \in L^\alpha$

$$\varphi(f) = \int_{\mathbb{T}} fw\bar{\eta}d\lambda = \int_T fwhd\mu.$$

Suppose $f \in M$. Then, for every integer $n \geq 0$, we have $z^n f \in M$, so

$$0 = \int_{\mathbb{T}} z^n fwhd\mu.$$

Since $fwh \in hL^1(\lambda) = L^1(\mu)$, it follows that $fwh \in H_0^1(\mu)$. Thus if $k \in H^\infty$, we have

$$0 = \int_{\mathbb{T}} kfwhd\mu = \varphi(kf).$$

Hence every $\varphi \in (L^\alpha)^\#$ that annihilates M must annihilate $H^\infty M$. It follows from the Hahn-Banach theorem that $H^\infty M \subset M$. □

We let $\mathbb{B} = \{f \in L^\infty : \|f\|_\infty \leq 1\}$ denote the closed unit ball in $L^\infty(\lambda)$.

Lemma 12. *Let α be a continuous norm on $L^\infty(\lambda)$, then*

(1) The α-topology, the $\|\cdot\|_{2,\lambda}$-topology, and the topology of convergence in λ-measure coincide on $\mathbb{B}$,

(2) $\mathbb{B} = \{f \in L^\infty(\lambda) : \|f\|_\infty \leq 1\}$ is α-closed.

Proof. For (1), since α is $c\|\cdot\|_{1,\lambda}$-dominating, α-convergence implies $\|\cdot\|_{1,\lambda}$-convergence, and $\|\cdot\|_{1,\lambda}$-convergence implies convergence in measure. Suppose $\{f_n\}$ is a sequence in $\mathbb{B}$, $f_n \to f$ in measure and $\varepsilon > 0$. If $E_n = \{z \in \mathbb{T} : |f(z) - f_n(z)| \geq \frac{\varepsilon}{2}\}$, then $\lim_{n\to\infty} \lambda(E_n) = 0$. Since α is continuous, we have $\lim_{n\to\infty} \alpha(\chi_{E_n}) = 0$, which implies that

$$\begin{aligned}
\alpha(f_n - f) &= \alpha((f - f_n)\chi_{E_n} + (f - f_n)\chi_{\mathbb{T}\setminus E_n}) \\
&\leq \alpha((f - f_n)\chi_{E_n}) + \alpha((f - f_n)\chi_{\mathbb{T}\setminus E_n}) \\
&< \alpha((|f - f_n|)\chi_{E_n}) + \frac{\varepsilon}{2} \leq \|f - f_n\|_\infty \alpha(\chi_{E_n}) + \frac{\varepsilon}{2} \\
&\leq 2\alpha(\chi_{E_n}) + \frac{\varepsilon}{2}.
\end{aligned}$$

Hence $\alpha(f_n - f) \to 0$ as $n \to \infty$. Therefore α-convergence is equivalent to convergence in measure on $\mathbb{B}$. Since α was arbitrary, letting $\alpha = \|\cdot\|_{2,\lambda}$, we see that $\|\cdot\|_{2,\lambda}$-convergence is also equivalent to convergence in measure. Therefore, the α-topology and the $\|\cdot\|_{2,\lambda}$-topology coincide on $\mathbb{B}$.

For (2), suppose $\{f_n\}$ is a sequence in $\mathbb{B}$, $f \in L^\alpha$ and $\alpha(f_n - f) \to 0$. Since $\|f\|_{1,\lambda} \leq \frac{1}{c}\alpha(f)$. it follows that $\|f_n - f\|_{1,\lambda} \to 0$, which implies that $f_n \to f$ in λ-measure. Then there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. (λ). Hence $f \in \mathbb{B}$. □

The following theorem and its corollary relate the closed invariant subspaces of L^α to the weak*-closed invariant subspaces of L^∞.

Theorem 13. *Let W be an α-closed linear subspace of L^α and M be a weak*-closed linear subspace of $L^\infty(\lambda)$ such that $zM \subseteq M$ and $zW \subseteq W$. Then*
(1) $M = \overline{M}^\alpha \cap L^\infty(\lambda)$,
(2) $W \cap L^\infty(\lambda)$ is weak-closed in $L^\infty(\lambda)$,*
(3) $W = \overline{W \cap L^\infty(\lambda)}^\alpha$.

Proof. For (1), it is clear that $M \subset \overline{M}^\alpha \cap L^\infty(\lambda)$. Assume, via contradiction, that $w \in \overline{M}^\alpha \cap L^\infty(\lambda)$ and $w \notin M$. Since M is weak*-closed, there is an $F \in L^1(\lambda)$ such that $\int_{\mathbb{T}} Fwd\lambda \neq 0$, but $\int_{\mathbb{T}} Frd\lambda = 0$ for every $r \in M$. Since $k = \frac{1}{|F|+1} \in L^\infty(\lambda)$, $k^{-1} \in L^1(\lambda)$, it follows from Theorem 10, that there is an $s \in H^\infty(\lambda), s^{-1} \in H^1(\lambda)$ and a unimodular function u such that $k = us$.

Choose a sequence $\{s_n\}$ in $H^\infty(\lambda)$ such that $\|s_n - s^{-1}\|_{1,\lambda} \to 0$. Since $sF = \overline{u}kF = \overline{u}\frac{F}{|F|+1} \in L^\infty(\lambda)$, we can conclude that $\|s_n sF - F\|_{1,\lambda} = \|s_n sF - s^{-1}sF\|_{1,\lambda} \le \|s_n - s^{-1}\|_{1,\lambda}\|sF\|_\infty \to 0$. For each $n \in \mathbb{N}$. For every $r \in M$, from Lemma 11, we know that $s_n sr \in H^\infty(\lambda)M \subset M$. Hence

$$\int_{\mathbb{T}} r s_n sF d\lambda = \int_{\mathbb{T}} s_n s r F d\lambda = 0, \forall r \in M.$$

Suppose $r \in \overline{M}^\alpha$. Then there is a sequence $\{r_m\}$ in M such that $\alpha(r_m - r) \to 0$ as $m \to \infty$. For each $n \in \mathbb{N}$, it follows from $s_n sF \in H^\infty(\lambda)L^\infty(\lambda)$ that

$$\begin{aligned}
|\int_{\mathbb{T}} r s_n sF d\lambda - \int_{\mathbb{T}} r_m s_n sF d\lambda| &\le \int_{\mathbb{T}} |(r - r_m) s_n sF| d\lambda \\
&\le \|s_n sF\|_\infty \int_{\mathbb{T}} |r - r_m| d\lambda = \|s_n sF\|_\infty \|r - r_m\|_{1,\lambda} \\
&\le \|s_n sF\|_\infty \alpha(r - r_m) \to 0.
\end{aligned}$$

$$\int_{\mathbb{T}} r s_n sF d\lambda = \lim_{m\to 0} \int_{\mathbb{T}} r_m s_n sF d\lambda = 0, \forall r \in \overline{M}^\alpha.$$

In particular, $w \in \overline{M}^\alpha \cap L^\infty(\lambda)$ implies that

$$\int_{\mathbb{T}} s_n sFw d\lambda = \int_{\mathbb{T}} w s_n sF d\lambda = 0.$$

Hence,

$$\begin{aligned}
0 \ne |\int_{\mathbb{T}} Fw d\lambda| &\le \lim_{n\to\infty} |\int_{\mathbb{T}} Fw - s_n sFw d\lambda| + \lim_{n\to\infty} |\int_{\mathbb{T}} s_n sFw d\lambda| \\
&\le \lim_{n\to\infty} \|F - s_n sF\|_{1,\lambda} \|w\|_\infty + 0 = 0.
\end{aligned}$$

We get a contradiction. Hence $M = \overline{M}^\alpha \cap L^\infty(\lambda)$.

For (2), to prove $W \cap L^\infty(\lambda)$ is weak*-closed in $L^\infty(\lambda)$, using the Krein-Smulian theorem, we only need to show that $W \cap L^\infty(\lambda) \cap \mathbb{B}$, i.e., $W \cap \mathbb{B}$, is weak*-closed. By Lemma 12, $W \cap \mathbb{B}$ is α-closed. Since α is $c\|\cdot\|_{1,\lambda}$-dominating, it follows from the Lemma 12, $W \cap \mathbb{B}$ is $\|\cdot\|_{2,\lambda}$ closed. The fact that $W \cap \mathbb{B}$ is convex implies $W \cap \mathbb{B}$ is closed in the weak topology on $L^2(\lambda)$. If $\{f_\lambda\}$ is a net in $W \cap \mathbb{B}$ and $f_\lambda \to f$ weak* in $L^\infty(\lambda)$, then, for every $w \in L^1(\lambda)$, $\int_{\mathbb{T}} (f_\lambda - f) w d\lambda \to 0$. Since $L^2(\lambda) \subset L^1(\lambda)$, $f_\lambda \to f$ weakly in $L^2(\lambda)$, so $f \in W \cap \mathbb{B}$. Hence $W \cap \mathbb{B}$ is weak*-closed in $L^\infty(\lambda)$.

For (3), since W is α-closed in L^α, it is clear that $W \supset \overline{W \cap L^\infty(\lambda)}^\alpha$, suppose $f \in W$ and let $k = \frac{1}{|f|+1}$. Then $k \in L^\infty(\lambda)$, $k^{-1} \in L^\alpha$. It follows from Theorem 10 that there is an $s \in H^\infty(\lambda)$, $s^{-1} \in H^\alpha$ and an unimodular function u such that $k = us$, so $sf = \overline{u} k s = \overline{u} \frac{f}{|f|+1} \in L^\infty(\lambda)$. There is a sequence $\{s_n\}$ in $H^\infty(\lambda)$ such that $\alpha(s_n - s^{-1}) \to 0$. For each $n \in \mathbb{N}$, it follows from Lemma 11 that $s_n s f \in H^\infty(\lambda) H^\infty(\lambda) W \subset W$ and $s_n s f \in H^\infty(\lambda) L^\infty(\lambda) \subset L^\infty(\lambda)$, which implies that $\{s_n s f\}$ is a sequence in $W \cap L^\infty(\lambda)$, $\alpha(s_n s f - f) \leq \alpha(s_n - s^{-1}) \|sf\|_\infty \to 0$. Thus $f \in \overline{W \cap L^\infty(\lambda)}^\alpha$. Therefore $W = \overline{W \cap L^\infty(\lambda)}^\alpha$. □

Corollary 14. *A weak*-closed linear subspace M of $L^\infty(\lambda)$ satisfies $zM \subset M$ if and only if $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of $\mathbb{T}$.*

Proof. If $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of $\mathbb{T}$, clearly, a weak*-closed linear subspace M of $L^\infty(\lambda)$ with $zM \subset M$. Conversely, since $zM \subset M$, and we have $z\overline{M}^{\|\cdot\|_{2,\lambda}} \subset \overline{M}^{\|\cdot\|_{2,\lambda}}$. Hence by Beurling-Helson-Lowdenslager theorem for $\|\cdot\|_{2,\lambda}$, we consider either $\overline{M}^{\|\cdot\|_{2,\lambda}} = \varphi H^2(\lambda)$ for some unimodular function φ, then $M = \overline{M}^{\|\cdot\|_{2,\lambda}} \cap L^\infty(\lambda) = \varphi H^2(\lambda) \cap L^\infty(\lambda)$; or $\overline{M}^{\|\cdot\|_{2,\lambda}} = \chi_E L^2(\lambda)$, for some Borel subset E of $\mathbb{T}$, in this case, $M = \overline{M}^{\|\cdot\|_{2,\lambda}} \cap L^\infty(\lambda) = \chi_E L^2(\lambda) \cap L^\infty(\lambda) = \chi_E L^\infty(\lambda)$, i.e., $M = \chi_E L^\infty(\lambda)$. □

Now we obtain our main theorem, which extends the Chen-Beurling Helson-Lowdenslager theorem.

Theorem 15. *Suppose μ is Haar measure on $\mathbb{T}$ and α is a continuous normalized gauge norm on $L^\infty(\mu)$. Suppose also that $c > 0$ and λ is a probability measure that is mutually absolutely* *continuous with respect to μ such that α is $c\|\|_{1,\lambda}$-dominating and $\log|d\lambda/d\mu| \in L^1(\mu)$. Then a closed linear subspace W of $L^\alpha(\mu)$ satisfies $zW \subset W$ if and only if either $W = \varphi H^\alpha(\mu)$ for some unimodular function φ, or $W = \chi_E L^\alpha(\mu)$, for some Borel subset E of $\mathbb{T}$. If $0 \neq W \subset H^\alpha(\mu)$, then $W = \varphi H^\alpha(\mu)$ for some inner function φ.*

Proof. Recall that $L^\infty(\mu) = L^\infty(\lambda), L^\alpha(\mu) = L^\alpha(\lambda)$ and $H^\alpha(\mu) = H^\alpha(\lambda)$.The only if part is obvious. Let $M = W \cap L^\infty(\lambda)$, and in Theorem 2, we have proved that there exists a measure λ such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and there exists $c > 0, \forall f \in L^\infty(\mu) = L^\infty(\lambda), \alpha(f) \geq c\|f\|_{1,\lambda}$. i.e., α is a continuous $c\|\cdot\|_{1,\lambda}$-dominating normalized gauge norm on $L^\infty(\lambda)$. It follows from the (2) in Theorem 13 that M is weak* closed in $L^\infty(\lambda)$. Since $zW \subset W$, it is easy to check that $zM \subset M$. Then by Corollary 14, we can conclude that either $M = \varphi H^\infty(\lambda)$ for some unimodular function φ or $M = \chi_E L^\infty(\lambda)$, for some Borel subset E of $\mathbb{T}$. By the (3) in Theorem 13, if $M = \varphi H^\infty(\lambda)$, $W = \overline{W \cap L^\infty(\lambda)}^\alpha = \overline{M}^\alpha = \overline{\varphi H^\infty(\lambda)}^\alpha = \varphi H^\alpha = \varphi H^\alpha(\mu)$, for some unimodular function φ. If $M = \chi_E L^\infty(\lambda), W = \overline{W \cap L^\infty(\lambda)}^\alpha = \overline{M}^\alpha = \overline{\chi_E L^\infty(\lambda)}^\alpha = \chi_E L^\alpha = \chi_E L^\alpha(\mu)$, for some Borel subset E of $\mathbb{T}$. The proof is completed. □

2.4 How do we determine whether such a good λ exists ?

In the preceding section we proved a version of Beurling's theorem for L^α when there is a probability measure λ on $\mathbb{T}$ that is mutually absolutely continuous with respect to μ, such that α is $c\|\cdot\|_{1,\lambda}$-dominating and $d\lambda/d\mu$ is log-integrable with respect to μ. How do we tell when such a good λ exists. Suppose ρ is a probability measure on $\mathbb{T}$ that is mutually absolutely continuous with respect to μ such that

$$\int_{\mathbb{T}} \log(d\rho/d\mu)\, d\mu = -\infty.$$

Here are some useful examples.

Example 1. *Let $\alpha = \frac{1}{2}(\|\cdot\|_{1,\mu} + \|\cdot\|_{1,\rho})$. Then α is a continuous gauge norm. If we let $\lambda_1 = \rho$ and $\lambda_2 = \mu$ we see that $\alpha \geq \frac{1}{2}\lambda_k$ for $k = 1, 2$ and*

$$\int_{\mathbb{T}} |\log(d\lambda_k/d\mu)|\, d\mu = \begin{cases} \infty & \textit{if } k=1 \\ 0 & \textit{if } k=2 \end{cases}.$$

Hence there is both a bad choice of λ *and a good choice.*

Example 2. *Suppose* ρ *is as in the preceding example and let* $\alpha = \|\cdot\|_{1,\rho}$. *Suppose* λ *is a probability measure that is mutually absolutely continuous with respect to* μ *and*

$$\|\cdot\|_{1,\rho} = \alpha \geq c\|\cdot\|_{1,\lambda} \textit{ for some constant } c.$$

It follows that $d\lambda/d\rho \leq c$ *a.e., and thus*

$$\int_{\mathbb{T}} \log(d\lambda/d\mu)\, d\mu = \int_{\mathbb{T}} \log(d\lambda/d\rho)\, d\mu + \int_{\mathbb{T}} \log(d\rho/d\mu)\, d\mu \leq \log\varepsilon + (-\infty) = -\infty.$$

In this case there is no good λ.

2.5 A special case.

Suppose λ is any probability measure that is mutually absolutely continuous with respect to μ and $\alpha = \|\cdot\|_{p,\lambda}$ for some p with $1 \leq p < \infty$. Assume λ is bad, i.e., $\int_{\mathbb{T}} \left|\log \frac{d\lambda}{d\mu}\right| d\mu = \infty$. In this case, we define a bijective isometry mapping $U : L^p(\lambda) \to L^p(\mu)$ by $Uf = g^{\frac{1}{p}} f$. Let $H^p(\lambda)$ be the α-closure of all polynomials. Then $H^p(\lambda)$ is a closed subspace of $L^p(\lambda)$ and $zH^p(\lambda) \subseteq H^p(\lambda)$. Therefore, $g^{\frac{1}{p}} H^p(\lambda)$ is a z-invariant closed subspace of $L^p(\mu)$. By Beurling-Helson-Lowdenslager theorem, we have

$$g^{\frac{1}{p}} H^p(\lambda) = \chi_E L^p(\mu) \text{ for some Borel set } E \subseteq \mathbb{T}, \text{ or } \varphi H^p(\mu), \text{ where } |\varphi| = 1.$$

If $g^{\frac{1}{p}} H^p(\lambda) = \chi_E L^p(\mu)$, then $H^p(\lambda) = L^p(\lambda)$, in this case, $\varphi H^p(\lambda) = \varphi L^p(\lambda)$, where $|\varphi| = 1$. If $M_0 = \frac{1}{g^{1/p}} H^p(\mu)$, then M_0 is a proper z-invariant closed subspace of $L^p(\lambda)$, and $M_0 \neq \chi_E L^p(\lambda)$. Therefore, Beurling-Helson-Lowdenslager theorem is not true for this case. However, we have the following theorem

Theorem 16. *Suppose* λ *is any probability measure that is mutually absolutely continuous with respect to* μ *and* $\alpha = \|\cdot\|_{p,\lambda}$ *for some* p *with* $1 \leq p < \infty$*. Also assume* $\int_{\mathbb{T}} \left|\log \frac{d\lambda}{d\mu}\right| d\mu = \infty$*. If* M *is a closed subspace of* $L^{\alpha}(\lambda)$*, then* $zM \subseteq M$ *if and only if*

(1) $M = \varphi M_0$ *for some unimodular function* φ*, where* $M_0 = \frac{1}{g^{1/p}} H^p(\mu)$*, or*

(2) $M = \chi_E L^{\alpha}(\lambda)$ *for some Borel subset* E *of* $\mathbb{T}$*.*

CHAPTER 3

AN EXTENSION OF THE BEURLING-CHEN-HADWIN-SHEN THEOREM FOR NONCOMMUTATIVE HARDY SPACES ASSOCIATED WITH FINITE VON NEUMANN ALGEBRAS

In 2015, Yanni Chen, Don Hadwin and Junhao Shen proved a noncommutative version of Beurling's theorems for a continuous unitarily invariant norm α on a tracial von Neumann algebra $(\mathcal{M},\tau)$ where α is $\|\cdot\|_1$-dominating with respect to τ. In the chapter, we first define a class of norms $N_\Delta(\mathcal{M},\tau)$ on $\mathcal{M}$, called determinant, normalized, unitarily invariant continuous norms on $\mathcal{M}$. If $\alpha \in N_\Delta(\mathcal{M},\tau)$, then there exists a faithful normal tracial state ρ on $\mathcal{M}$ such that $\rho(x) = \tau(xg)$ for some positive $g \in L^1(\mathcal{Z},\tau)$ and the determinant of g is positive. For every $\alpha \in N_\Delta(\mathcal{M},\tau)$, we study the noncommutative Hardy spaces $H^\alpha(\mathcal{M},\tau)$, then prove that the Chen-Hadwin-Shen theorem holds for $L^\alpha(\mathcal{M},\tau)$. The key ingredients in the proof of our result include a factorization theorem and a density theorem for $L^\alpha(\mathcal{M},\rho)$.

3.1 Introduction

It has long been of great importance to operator theorists and operator algebraists to study noncommutative Beurling's theorem[3],[6],[7],[12],[28],[48]. We recall some concepts in noncommutative Hardy spaces with finite von Neumann algebras. Given a finite von Neumann algebra $\mathcal{M}$ acting on a Hilbert space H, the set of possibly unbounded closed and densely defined operators on H which are affiliated to $\mathcal{M}$, form a topological algebra where the topology is the (noncommutative) topology of convergence in measure. We denote this algebra by $\widetilde{\mathcal{M}}$. The trace τ extends naturally from $\mathcal{M}$ to the positive operators in $\widetilde{\mathcal{M}}$. The important fact regarding this algebra, is that it is large enough to accommodate all the noncommutative L^p spaces corresponding to $\mathcal{M}$. Specifically, if

$1 \leq p < \infty$, then we define the space $L^p(\mathcal{M}, \tau) = \{x \in \widetilde{\mathcal{M}} : \tau(|x|^p) < \infty\}$, where the ambient norm is given by $\| \cdot \|_p = \tau(\| \cdot \|^p)^{1/p}$. The space $L^\infty(\mathcal{M}, \tau)$ is defined to be $\mathcal{M}$ itself. These spaces capture all the usual properties of L^p spaces, with the dual action of L^p on L^q (q conjugate to p) given by $(a, b) \to \tau(ab)$. For any subset S of $\mathcal{M}$, we write $[\mathcal{S}]_p$ for the p-norm closure of $\mathcal{S}$ in $L^p(\mathcal{M}, \tau)$, with the understanding that $[\mathcal{S}]_p$ will denote the weak* closure in the case $p = \infty$. W. Arveson [3] introduced a concept of maximal subdiagonal algebra in 1967, also known as a noncommutative H^∞ space, to study the analyticity in operator algebras. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let $\mathcal{A}$ be a weak* closed unital subalgebra of $\mathcal{M}$, and $\mathcal{A}$ is called a finite maximal subalgebra of $\mathcal{M}$ with respect to Φ if (i) $\mathcal{A} + \mathcal{A}^*$ is weak* dense in $\mathcal{M}$; (ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for $\forall x, y \in \mathcal{A}$; (iii) $\tau \circ \Phi = \tau$; and (iv) $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Such a finite maximal subdiagonal subalgebra $\mathcal{A}$ of $\mathcal{M}$ is also called an H^∞ space of $\mathcal{M}$. For each $1 \leq p \leq \infty$, let H^p be the completion of Arveson's noncommutative H^∞ with respect to $\|\cdot\|_p$. After Arveson's introduction of noncommutative H^p spaces, many researchers obtained Beurling theorems for invariant subspaces in noncommutative H^p spaces (for example, see [7],[12]).

Y. Chen, D. Hadwin, and J. Shen obtained a version of the Blecher-Labuschagne-Beurling invariant subspace theorem on H^∞-right invariant subspace in a noncommutative $L^\alpha(\mathcal{M}, \tau)$ space, where α is a normalized unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm.

In this chapter, we will extend Chen-Hadwin-Shen's result in [12] by dropping the condition that α is $\|\cdot\|_1$-dominating. By defining a generalized α norm, we have a version of Chen-Hadwin-Shen's result for noncommutative Hardy spaces.

THEOREM 37. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a determinant, normalized, unitarily invariant, continuous norm on $\mathcal{M}$. Then there exists a faithful normal tracial state ρ on $\mathcal{M}$ such that $\alpha \in N_1(\mathcal{M}, \rho)$. Let H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$, then there exists a closed subspace $\mathcal{Y}$ of $L^\alpha(\mathcal{M}, \tau)$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M}$ such that:*

(1) $u_\lambda^* \mathcal{Y} = 0$ for all $\lambda \in \Lambda$,

(2) $u_\lambda^* u_\lambda \in \mathcal{D}$ *and* $u_\lambda^* u_\mu = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$

(3) $\mathcal{Y} = [H_0^\infty \mathcal{Y}]_\alpha$

(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} u_\lambda H^\alpha)$

Many tools used in [12] are no longer available in an arbitrary $L^\alpha(\mathcal{M}, \tau)$ space and new techniques must be invented. First, we need using the Fuglede-Kadison determinant, and inner, outer factorization for noncommutative Hardy spaces, more details seen in [4]. Let Δ be Fuglede-Kadison determinant on $\mathcal{M}$ defined by

$$\Delta(x) = exp(\tau(\log|x|)) = exp(\int_0^\infty \log(t) d\nu_{|x|}(t)),$$

where $d\nu_{|x|}(t)$ denotes the probability measure on $\mathbb{R}_+$, Also, the definition of this determinant can be extended to the $*$-algebra $\widetilde{\mathcal{M}}$.

In order to prove our main result of the chapter, we first get the following theorem.

THEOREM 22. *Let* $\mathcal{M}$ *be a finite von Neumann algebra with a faithful, normal, tracial state* τ *and* α *be a normalized, unitarily invariant, continuous norm on* $(\mathcal{M}, \tau)$ *. Then there exists a positive* $g \in L^1(\mathcal{Z}, \tau)$ *such that (i)* $\rho(\cdot) = \tau(\cdot g)$ *is a faithful normal tracial state on* $\mathcal{M}$ *, (ii)* α *is* $c\,\|\cdot\|_{1,\rho}$*-dominating, for some* $c > 0$ *. (iii),* $\rho(x) = \tau(xg)$ *for every* $x \in L^1(\mathcal{M}, \rho)$*.*

THEOREM 56. *If* $\alpha \in N_\Delta(\mathcal{M}, \tau)$*, then there exists a faithful normal tracial state* ρ *such that* $H^\alpha(\mathcal{M}, \rho) = H^1(\mathcal{M}, \rho) \cap L^\alpha(\mathcal{M}, \rho)$*.*

Then we get a factorization theorem and a density theorem for $L^\alpha(\mathcal{M}, \tau)$ to get the main theorem.

THEOREM 32. *Suppose* $\alpha \in N_\Delta(\mathcal{M}, \tau)$ *,there exists a faithful normal tracial state* ρ *on* $\mathcal{M}$ *such that* $\rho(x) = \tau(xg)$ *for some positive* $g \in L^1(\mathcal{Z}, \tau)$ *and the determinant of* g *is positive. If* $x \in \mathcal{M}$ *and* $x^{-1} \in L^\alpha(\mathcal{M}, \rho)$ *, then there are unitary operators* $u_1, u_2 \in \mathcal{M}$ *and* $s_1, s_2 \in H^\infty$ *such that* $x = u_1 s_1 = s_2 u_2$ *and* $s_1^{-1}, s_2^{-1} \in H^\alpha(\mathcal{M}, \rho)$*.*

THEOREM 33. *Let* $\alpha \in N_\Delta(\mathcal{M}, \tau)$ *, then there exists a faithful normal tracial state* ρ on $\mathcal{M}$ *such that* $\rho(x) = \tau(xg)$ *for some positive* $g \in L^1(\mathcal{Z}, \tau)$ *and the determinant of* g *is positive. Also, if* $\mathcal{W}$ *is a closed subspace of* $L^\alpha(\mathcal{M}, \rho)$ *and* $\mathcal{N}$ *is a weak* closed linear subspace of* $\mathcal{M}$ *such that*

$\mathcal{W}H^\infty \subset \mathcal{W}$ and $\mathcal{N}H^\infty \subset \mathcal{N}$, then

(1) $\mathcal{N} = [\mathcal{N}]_\alpha \cap \mathcal{M}$,

(2) *$\mathcal{W} \cap \mathcal{M}$ is weak* closed in $\mathcal{M}$*,

(3) $\mathcal{W} = [\mathcal{W} \cap \mathcal{M}]_\alpha$,

(4) *If $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S}H^\infty \subset \mathcal{S}$, then $[\mathcal{S}]_\alpha = [\overline{\mathcal{S}}^{w*}]_\alpha$, where $\overline{\mathcal{S}}^{w*}$ is the weak*-closure of $\mathcal{S}$ in $\mathcal{M}$.*

The organization of the chapter is as follows. In section 2, we introduce determinant, normalized, unitarily invariant continuous norms. In section 3, we study the relations between noncommutative Hardy spaces $H^\alpha(\mathcal{M}, \rho)$ and $H^\alpha(\mathcal{M}, \tau)$. In section 4, we prove the main result of the chapter, a version of Chen-Hadwin-Shen's result for noncommutative Hardy spaces associated with new norm. In section 5, we get a generalized noncommutative Beurling's theorem for special von Neumann algebras.

3.2 Determinant, normalized, unitarily invariant continuous norms

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, the $\|\cdot\|_p$ is a mapping from $\mathcal{M}$ to $[0, \infty)$ defined by $\|x\|_p = (\tau(|x|^p))^{1/p}, \forall x \in \mathcal{M}, 0 < p < \infty$. It is known that $\|\cdot\|_p$ is a norm if $1 \le p < \infty$, and a quasi-norm if $0 < p < 1$.

Definition 17. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Assume $\alpha : \mathcal{M} \to [0, \infty)$ is a norm satisfying*

(1) $\alpha(I) = 1$, i.e., α is normalized,

(2) $\alpha(x) = \alpha(|x|)$ for all $x \in \mathcal{M}$ and $|x| = (x^*x)^{1/2}$, i.e., α is a gauge,

(3) $\alpha(u^*xu) = \alpha(x), u \in \mathcal{U}(\mathcal{M})$ and $x \in \mathcal{M}$, i.e., α is unitarily invariant,

(4) $\lim_{\tau(e)\to 0} \alpha(e) = 0$ as e ranges over the projections in $\mathcal{M}$. i.e., if $\{e_\lambda\}$ is a net of projections in $\mathcal{M}$ and $\tau(e_\lambda) \to 0$, then $\alpha(e_\lambda) \to 0$ which means α is continuous.

Then we call α a normalized unitarily invariant continuous norm. And we denote $N(\mathcal{M}, \tau)$ to be the collection of all such norms.

Definition 18. *We denote by* $N_1(\mathcal{M},\tau)$ *, the collection of all these norms* $\alpha : \mathcal{M} \to [0,\infty)$ *such that*

(1) $\alpha \in N(\mathcal{M},\tau)$*,*

(2) $\forall x \in \mathcal{M}$*,* $\alpha(x) \geq c\|x\|_1$*, for some* $c > 0$*.*

A norm α *in* $N_1(\mathcal{M},\tau)$ *is called a* normalized, unitarily invariant $\|\cdot\|_1$-dominating continuous *norm on* $\mathcal{M}$*.*

Definition 19. *We denote by* $N_\Delta(\mathcal{M},\tau)$*, the collection of all these norms* $\alpha : \mathcal{M} \to [0,\infty)$ *such that*

(1) $\alpha \in N(\mathcal{M},\tau)$*,*

(2) There exists a positive $g \in L^1(\mathcal{M},\tau)$ *such that* $\Delta(g) > 0$ *and* $\alpha(x) \geq c\tau(|x|g)$ *for some* $c > 0$*.*

A norm α *in* $N_\Delta(\mathcal{M},\tau)$ *is called a* determinant, normalized, unitarily invariant continuous *norm on* $\mathcal{M}$*.*

Example 3. *For the definition 19, if we take* $g = 1$*, then* $\alpha \in N_1(\mathcal{M},\tau)$*, i.e.,* $N_\Delta(\mathcal{M},\tau) \subset N_1(\mathcal{M},\tau)$*.*

Example 4. *Each* p*-norm* $\|\cdot\|_p$ *is in* $N(\mathcal{M},\tau)$*,* $N_1(\mathcal{M},\tau)$*, and* $N_\Delta(\mathcal{M},\tau)$ *for* $1 \leq p < \infty$*.*

Example 5. *Let* $\mathcal{M}$ *be a finite von Neumann algebra with a faithful, normal, tracial state* τ*. Let* $E(0,1)$ *be a symmetric Banach function space on* $(0,1)$ *and* $E(\tau)$ *be the noncommutative Banach function space with a norm* $\|\cdot\|_{E(\tau)}$ *corresponding to* $E(0,1)$ *and associated with* $(\mathcal{M},\tau)$*. If* $E(0,1)$ *is also order continuous, then the restriction of the norm* $\|\cdot\|_{E(\tau)}$ *to* $\mathcal{M}$ *lies in* $N(\mathcal{M},\tau)$ *and* $N_1(\mathcal{M},\tau)$*.*

In order to prove the first theorem in this chapter, we need the following lemmas. The first lemma is proved by H. Fan, D. Hadwin and W. Liu in [21].

Lemma 20. *Suppose* (X,Σ,μ) *is a probability space and* α *is a continuous normalized gauge norm on* $L^\infty(\mu)$*. Then there exists* $0 < c < 1$ *and a probability measure* λ *on* Σ *such that* $\lambda \ll \mu$ *and* $\mu \ll \lambda$*, such that* α *is* $c\|\cdot\|_{1,\lambda}$*-dominating.*

Before we state the next lemma, we first introduce the property of central valued traces in [44], and introduce a class of determinant, normalized, unitarily invariant continuous norms on finite von Neumann algebras and some interesting examples from this class. In the end of this section, we will obtain our first theorem.

Proposition 1. *If $\mathcal{M}$ is a finite von Neumann algebra with the center $\mathcal{Z}$ of $\mathcal{M}$, then there is a unique positive linear mapping φ from $\mathcal{M}$ into $\mathcal{Z}$ such that*

(1) $\varphi(xy) = \varphi(yx)$ for each x and y in $\mathcal{M}$,

(2) $\varphi(z) = z$ for each z in $\mathcal{Z}$,

(3) $\varphi(x) > 0$ if $x > 0$ for x in $\mathcal{M}$,

(4) $\varphi(zx) = z\varphi(x)$ for each z in $\mathcal{Z}$ and x in $\mathcal{M}$,

(5) $\|\varphi(x)\| \leq \|x\|$ for x in $\mathcal{M}$,

(6) φ is ultraweakly continuous,

(7) For any $x \in \mathcal{M}$, $\varphi(x)$ is the unique central element in the norm closure of the convex hull of $\{uxu^*|u \in \mathcal{U}(\mathcal{M})\}$,

(8) Every tracial state on $\mathcal{M}$ is of the form $\tau \circ \varphi$ where τ is a state on $\mathcal{Z}$, i.e. every state on the center $\mathcal{Z}$ of $\mathcal{M}$ extends uniquely to a tracial state on $\mathcal{M}$,

(9) φ is faithful.

Lemma 21. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Suppose $\alpha \in N(\mathcal{M}, \tau)$, then the central valued trace φ satisfy $\alpha(\varphi(x)) \leq \alpha(x)$, for every $x \in \mathcal{M}$.*

Proof. By proposition 1 (7), for any $x \in \mathcal{M}$, the central value trace $\varphi(x)$ is in the norm closure of the convex hull of $\{uxu^*|u \in \mathcal{U}(\mathcal{M})\}$, so there exists a net $\{x_\lambda\}_{\lambda\in\Lambda}$ in the convex hull of $\{uxu^*|u \in \mathcal{U}(\mathcal{M})\}$ such that x_λ converges to $\varphi(x)$. Since α is a continuous norm, $\alpha(x_\lambda - \varphi(x)) \to 0$, i.e., $\alpha(\varphi(x)) = \lim_\lambda \alpha(x_\lambda)$. Since x_λ is in the convex hull of $\{uxu^*|u \in \mathcal{U}(\mathcal{M})\}$, $\alpha(x_\lambda) \leq \alpha(x)$. Therefore, $\alpha(\varphi(x)) \leq \alpha(x)$. □

Theorem 22. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, continuous norm on $(\mathcal{M}, \tau)$. Then there exists a positive*

$g \in L^1(\mathcal{Z}, \tau)$ *such that (i)* $\rho(\cdot) = \tau(\cdot g)$ *is a faithful normal tracial state on* $\mathcal{M}$*, (ii)* α *is* $c\,\|\cdot\|_{1,\rho}$*-dominating for some* $c > 0$*. (iii),* $\rho(x) = \tau(xg)$ *for every* $x \in L^1(\mathcal{M}, \rho)$*.*

Proof. Since the center $\mathcal{Z}$ of $\mathcal{M}$ is an abelian von Neumann algebra, there is a compact subset X of $\mathbb{R}$ and a regular Borel probability measure on X such that the mapping π from $\mathcal{Z}$ to $L^\infty(X, \mu)$ is $*$-isomorphic and WOT-homeomorphic. Since α is a continuous normalized unitarily invariant norm on $(\mathcal{M}, \tau)$, it is easy to check $\overline{\alpha} = \alpha \circ \pi^{-1}$ satisfying

(i) $\overline{\alpha}(1) = \alpha \circ \pi^{-1}(1) = \alpha(\pi^{-1}(1)) = \alpha(I) = 1$,

(ii) $\overline{\alpha}(f) = \alpha \circ \pi^{-1}(f) = \alpha(u\pi^{-1}(f)) = \alpha(\pi^{-1}(wf)) = \alpha(\pi^{-1}(|f|)) = \overline{\alpha}(|f|)$, where $|f| = wf, |w| = 1$ and there is a unitary u such that $\pi(u) = w$,

(iii) For given borel sets $\{E_n\}_{n=1}^\infty \subseteq X$, there exist a sequence $\{e_n\} \subseteq \mathcal{Z}$ such that $\pi^{-1}(\chi_{E_n}) = e_n$ for every $n \in \mathbb{N}$. If $\mu(E_n) \to 0$, then $\tau(e_n) \to 0$. So $\alpha(e_n) \to 0$ since α is continuous. Thus $\lim_{n\to\infty} \overline{\alpha}(\chi_{E_n}) = \lim_{n\to\infty} \alpha \circ \pi^{-1}(\chi_{En}) = \lim_{n\to\infty} \alpha(e_n) \to 0$.

Thus $\overline{\alpha}$ is a continuous normalized gauge norm on $L^\infty(X, \mu)$.

By the Lemma 20, there exists a probability measure λ such that $\lambda \ll \mu$ and $\mu \ll \lambda$ and there exists $c > 0$ such that $\forall f \in L^\infty(X, \mu) = L^\infty(X, \lambda), \overline{\alpha}(f) \geq c\|f\|_{1,\lambda}$. Define $\rho_0(x) = \int_X \pi(x) d\lambda$, we check ρ_0 is a faithful normal tracial state on $\mathcal{Z}$.

(1) $\rho_0(I) = \int_X \pi(I) d\lambda = \int_X 1 d\lambda = 1$,

(2) $\rho_0(xy) = \int_X \pi(xy) d\lambda = \int_X \pi(yx) d\lambda = \rho_0(yx)$,

(3) Since $x_n \to x$ in WOT topology, $\pi(x_n) \to \pi(x)$ in weak* topology, i.e., $\int_X \pi(x_n) d\lambda = \int_X \pi(x_n) g d\mu \to \int_X \pi(x) g d\mu = \int_X \pi(x) d\lambda$. Thus $\rho_0(x_n) \to \rho_0(x)$. Therefore ρ_0 is normal.

(4) For every $x \in \mathcal{Z}, \rho_0(x^*x) = \int_X \pi(x^*x) d\lambda = \int_X \pi(x)^2 d\lambda = 0$, so $\pi(x)^2 = 0$ and $x = 0$, which means ρ_0 is faithful.

Now claim that α is $c\|\cdot\|_{1,\rho}$-dominating on $(\mathcal{M}, \rho)$. For some constant $c > 0$, $\forall x \in \mathcal{Z}$, $\alpha(x) = \overline{\alpha} \circ \pi(x) = \overline{\alpha}(\pi(x)) \geq c\|\pi(x)\|_{1,\lambda} = c\int_X |\pi(x)| d\lambda = c\int_X \pi(|x|) d\lambda = c\rho(|x|) = c\|x\|_{1,\rho}$. So we have $\alpha(x) \geq c\|x\|_{1,\rho}, \forall x \in \mathcal{Z}$. Also, we have $\mathcal{M} \overset{\varphi}{\to} \mathcal{Z} \overset{\rho_0}{\to} \mathbb{C}$, where φ is the mapping in Proposition 1. Let $\rho = \rho_0 \circ \varphi$, then ρ is a state on $\mathcal{M}$, and $\forall x \in \mathcal{M}, \alpha(x) \geq \alpha(\varphi(x)) \geq$

$c\|\varphi(x)\|_{1,\rho_0} = c\|\varphi(x)\|_{1,\rho} = c\|x\|_{1,\rho}$. Therefore, there exists a faithful normal tracial state ρ on $\mathcal{M}$ such that α is a $c\|\cdot\|_{1,\rho}$-dominating on $(\mathcal{M}, \rho)$.

Since $\rho(x) = \int_X \pi(x)d\lambda = \int_X \pi(x)hd\mu$, where $h = \frac{d\lambda}{d\mu} \in L^1(X, \mu)$, we can choose simple functions $\{h_i\}_{i=1}^{\infty}$ such that $0 \le h_1 \le h_2 \le \cdots$ and $h_n \to h$ as $n \to \infty$. And also we can choose $0 \le x_1 \le x_2 \le \cdots$ in $\mathcal{Z}$ so that $\pi(x_n) = h_n$ for each n. Therefore,

$$\rho(x) = \rho_0(\varphi(x)) = \lim_{n\to\infty} \tau(x_n\varphi(x)) = \lim_{n\to\infty} \tau(\varphi(x_n x)) = \lim_{n\to\infty} \tau(x_n x)) = \tau(xg),$$

where $g \in L^1(\mathcal{Z}, \tau)$. □

Example 6. *Given any finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ and $\alpha \in N(\mathcal{M}, \tau)$, by theorem 22, there exists a positive $g \in L^1(\mathcal{M}, \tau)$ such that $\alpha(x) \ge c\tau(|x|\, g)$ for some $c > 0$. If $\Delta(g) > 0$, then $\alpha \in N_\Delta(\mathcal{M}, \tau)$.*

3.3 Noncommutative Hardy spaces

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Given a von Neumann subalgebra $\mathcal{D}$ of $\mathcal{M}$, a conditional expectation Φ: $\mathcal{M} \to \mathcal{D}$ is a positive linear map satisfying $\Phi(I) = I$ and $\Phi(x_1 y x_2) = x_1\Phi(y)x_2$ for all $x_1, x_2 \in \mathcal{D}$ and $y \in \mathcal{M}$. There exists a unique conditional expectation Φ_τ: $\mathcal{M} \to \mathcal{D}$ satisfying $\tau \circ \Phi_\tau(x) = \tau(x)$ for every $x \in \mathcal{M}$. Now we recall noncommutative classical Hardy spaces H^∞ in [3].

Definition 23. *Let $\mathcal{A}$ be a weak* closed unital subalgebra of $\mathcal{M}$, and let Φ_τ be the unique faithful normal trace preserving conditional expectation from $\mathcal{M}$ onto the diagonal von Neumann algebra $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Then $\mathcal{A}$ is called a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ with respect to Φ_τ if*

(1) $\mathcal{A} + \mathcal{A}^*$ is weak dense in $\mathcal{M}$,*

(2) $\Phi_\tau(xy) = \Phi_\tau(x)\Phi_\tau(y)$ for all $x, y \in \mathcal{A}$.

Such $\mathcal{A}$ will be denoted by H^∞, and $\mathcal{A}$ is also called a noncommutative Hardy space.

Example 7. *Let* $\mathcal{M} = L^\infty(\mathbb{T}, \mu)$*, and* $\tau(f) = \int f d\mu$ *for all* $f \in L^\infty(\mathbb{T}, \mu)$*. Let* $\mathcal{A} = H^\infty(\mathbb{T}, \mu)$*, then* $\mathcal{D} = H^\infty(\mathbb{T}, \mu) \cap H^\infty(\mathbb{T}, \mu)^* = \mathbb{C}$*. Let* Φ_τ *be the mapping from* $L^\infty(\mathbb{T}, \mu)$ *onto* $\mathbb{C}$ *defined by* $\Phi_\tau(f) = \int f d\mu$*. Then* $H^\infty(\mathbb{T}, \mu)$ *is a finite, maximal subdiagonal subalgebra of* $L^\infty(\mathbb{T}, \mu)$*.*

Example 8. *Let* $\mathcal{M} = \mathcal{M}_n(\mathbb{C})$ *with the usual trace* τ*. Let* $\mathcal{A}$ *be the subalgebra of lower triangular matrices, now* $\mathcal{D}$ *is the diagonal matrices and* Φ_τ *is the natural projection onto the diagonal matrices. Then* $\mathcal{A}$ *is a finite maximal subdiagonal subalgebra of* $\mathcal{M}_n(\mathbb{C})$*.*

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, Φ_τ be the conditional expectation and α be a determinant, normalized, unitarily invariant, continuous norm on $\mathcal{M}$. Let $L^\alpha(\mathcal{M}, \tau)$ be the α closure of $\mathcal{M}$, i.e., $L^\alpha(\mathcal{M}, \tau) = [\mathcal{M}]_\alpha$. Similarly, $H^\alpha(\mathcal{M}, \tau) = [H^\infty(\mathcal{M}, \tau)]_\alpha$, $H_0^\infty(\mathcal{M}, \tau) = \ker(\Phi_\tau) \cap H^\infty(\mathcal{M}, \tau)$ and $H_0^\alpha(\mathcal{M}, \tau) = \ker(\Phi_\tau) \cap H^\alpha(\mathcal{M}, \tau)$. If we take $\alpha = \|\cdot\|_p$, then $L^p(\mathcal{M}, \tau) = [\mathcal{M}]_p$, $H^p(\mathcal{M}, \tau) = [H^\infty(\mathcal{M}, \tau)]_p$. Recall ρ is a faithful normal tracial state on $\mathcal{M}$ satisfying all three conditions in Theorem 22. We define the noncommutative Hardy spaces $H^1(\mathcal{M}, \rho)$ and $H_0^1(\mathcal{M}, \rho)$ by $H^1(\mathcal{M}, \rho) = \overline{H^\infty(\mathcal{M}, \tau)}^{\|\cdot\|_{1,\rho}}$ and $H_0^1(\mathcal{M}, \rho) = \overline{H_0^\infty(\mathcal{M}, \tau)}^{\|\cdot\|_{1,\rho}}$. In [56], K. S. Saito characterized the noncommutative Hardy spaces $H^p(\mathcal{M}, \tau)$ and $H_0^p(\mathcal{M}, \tau)$. Recall $H^p(\mathcal{M}, \tau) = \{x \in L^p(\mathcal{M}, \tau), \tau(xy) = 0$, for all $y \in H_0^\infty\}$ for $1 \leq p < \infty$, also we have $H_0^p(\mathcal{M}, \tau) = \{x \in L^p(\mathcal{M}, \tau), \tau(xy) = 0, \forall y \in H^\infty\}$. In this chapter, we get similar result for noncommutative Hardy spaces $H^p(\mathcal{M}, \rho)$ and $H_0^p(\mathcal{M}, \rho)$ by using the inner-outer factorization and the properties of outer functions in noncommutative Hardy spaces from papers [6] and [7]. Let Δ be Fuglede-Kadison determinant on $\mathcal{M}$ defined by

$$\Delta(x) = exp(\tau(\log|x|)) = exp(\int_0^\infty \log(t) d\nu_{|x|}(t)),$$

where $d\nu_{|x|}(t)$ denotes the probability measure on $\mathbb{R}_+$, Also, the definition of this determinant can be extended to the $*$-algebra $\widetilde{\mathcal{M}}$.

Definition 24. *Let* $1 \leq p \leq \infty$*. An element* $x \in H^p(\mathcal{M}, \tau)$ *is outer if* $I \in [xH^p(\mathcal{M}, \tau)]_p$*, and* $x \in H^p(\mathcal{M}, \tau)$ *is strongly outer if* x *is outer and* $\Delta(x) > 0$*. An element* u *is inner if* $u \in H^\infty(\mathcal{M}, \tau)$ *and* u *is unitary.*

Lemma 25. *(from [7]) If H^∞ is a maximal subdiagonal algebra, then $x \in H^p(\mathcal{M},\tau)$ with $\Delta(x) > 0$ iff $x = uy$ for an inner u and a strongly outer $y \in H^p(\mathcal{M},\tau)$ for $1 \le p \le \infty$. The factorization is unique up to a unitary in $\mathcal{D}$.*

Lemma 26. *(from [7]) Let Φ_τ be the conditional expectation on $\mathcal{M}$. Then $x \in H^p(\mathcal{M},\tau)$ is outer if and only if $\Phi_\tau(x)$ is outer in $L^p(\mathcal{D})$ and $\overline{xH_0^\infty(\mathcal{M},\tau)}^{\|\cdot\|_{p,\tau}} = H_0^p(\mathcal{M},\tau)$ for $1 \le p \le \infty$.*

Lemma 27. *If $\alpha \in N_\Delta(\mathcal{M},\tau)$, then there exits a faithful tracial state ρ and a strongly outer h in $H^1(\mathcal{M},\tau)$ such that $g = |h|$,where g as in Theorem 22 and $hH^1(\mathcal{M},\rho) = H^1(\mathcal{M},\tau)$.*

Proof. Since $\alpha \in N_\Delta(\mathcal{M},\tau)$, $\Delta(g) > 0$. By Lemma 25, $g = |h|$ for a strongly outer $h \in H^1(\mathcal{M},\tau)$. Let $\rho(x) = \tau(xg)$, $\forall x \in \mathcal{M}$, by Theorem 22, ρ is a faithful normal tracial state on $\mathcal{M}$. Then we define $U : L^1(\mathcal{M},\rho) \longrightarrow L^1(\mathcal{M},\tau)$ by $Ux = hx$, which is a surjective isometry:

$$\|U(x)\|_{1,\tau} = \|xg\|_{1,\tau} = \tau(|xg|) = \tau(|x|\, g) = \rho(|x|) = \|x\|_{1,\rho}.$$

Since $g \in gH^1(\mathcal{M},\rho)$ and $H^1(\mathcal{M},\tau) \subseteq H^1(\mathcal{M},\rho)$, $gH^\infty(\mathcal{M},\tau) \subseteq gH^1(\mathcal{M},\rho)$. Since $g = |h|$, $g = vh$, where v is modular. Thus $vhH^\infty(\mathcal{M},\tau) \subseteq gH^1(\mathcal{M},\rho) = vhH^1(\mathcal{M},\rho) = hH^1(\mathcal{M},\rho)$. Since h is a strongly outer in $H^1(\mathcal{M},\tau)$,we have $hH^1(\mathcal{M},\rho) = H^1(\mathcal{M},\tau)$. □

Corollary 28. *Let Φ_τ be the conditional expectation on $\mathcal{M}$. If $\alpha \in N_\Delta(\mathcal{M},\tau)$, then there exists a faithful normal tracial state ρ such that*

(1) $H^1(\mathcal{M},\rho) = \{x \in L^1(\mathcal{M},\rho) : \rho(xy) = 0$ for all $y \in H_0^\infty\}$,

(2) $H_0^1(\mathcal{M},\rho) = \{x \in L^1(\mathcal{M},\rho) : \rho(xy) = 0$ for all $y \in H^\infty\}$,

(3) $H_0^1(\mathcal{M},\rho) = \{x \in H^1(\mathcal{M},\rho) : \Phi_\tau(xh) = 0\}$.

Proof. Since $\alpha \in N_\Delta(\mathcal{M},\tau)$, there exists a positive $g \in L^1(\mathcal{M},\tau)$ and $\Delta(g) > 0$ such that $\alpha(x) \ge c\tau(|x|\, g)$ for some $c > 0$. We define $\rho(x) = \tau(xg), \forall x \in \mathcal{M}$, ρ is a faithful normal tracial state on $\mathcal{M}$. By lemma 27 and $H^1(\mathcal{M},\tau) = \{x \in L^1(\mathcal{M},\tau), \tau(xy) = 0$ for all $y \in H_0^\infty\}$, we have (1). For (2), We know $\overline{H_0^\infty(\mathcal{M},\tau)}^{\|\cdot\|_{1,\rho}} = H_0^1(\mathcal{M},\rho)$, and $hH_0^1(\mathcal{M},\rho) = h\overline{H_0^\infty(\mathcal{M},\tau)}^{\|\cdot\|_{1,\rho}} = \overline{hH_0^\infty(\mathcal{M},\tau)}^{\|\cdot\|_{1,\tau}} = H_0^1(\mathcal{M},\tau)$ since h is outer in $H^1(\mathcal{M},\tau)$. The last statement is clearly by [56]. □

Proposition 2. *If $\alpha \in N_{\Delta}(\mathcal{M},\tau)$, then there exists a faithful normal tracial state ρ such that*

$$H^{\alpha}(\mathcal{M},\rho)=\{x\in L^{\alpha}(\mathcal{M},\rho):\rho(xy)=0\,\textit{for all}\ y\in H_0^1(\mathcal{M},\rho)\cap(L^{\alpha}(\mathcal{M},\rho))^{\#}\},$$

where $(L^{\alpha}(\mathcal{M},\rho))^{\#}$ is the dual space of $L^{\alpha}(\mathcal{M},\rho)$.

Proof. Since $\alpha \in N_{\Delta}(\mathcal{M},\tau)$, then there exists a faithful normal tracial state ρ on $\mathcal{M}$ such that $\rho(x)=\tau(xg)$ for some positive $g\in L^1(\mathcal{Z},\tau)$ and the determinant of g is positive, which means $\alpha\in N_1(\mathcal{M},\rho)$. Let $\mathcal{J}=\{x\in L^{\alpha}(\mathcal{M},\rho):\rho(xy)=0$ for all $y\in H_0^1(\mathcal{M},\rho)\cap(L^{\alpha}(\mathcal{M},\rho))^{\#}\}$. Suppose $x\in H^{\infty}(\mathcal{M},\rho)$. If $y\in H_0^1(\mathcal{M},\rho)\cap(L^{\alpha}(\mathcal{M},\rho))^{\#}\subseteq H_0^1(\mathcal{M},\rho)$, then it follows from Corollary 28 that $\rho(xy)=0$, for all $x\in\mathcal{J}$, and so $H^{\infty}(\mathcal{M},\rho)\subseteq\mathcal{J}$. We claim that $\mathcal{J}$ is α-closed in $L^{\alpha}(\mathcal{M},\rho)$. In fact, suppose $\{x_n\}$ is a sequence in $\mathcal{J}$ and $x\in L^{\alpha}(\mathcal{M},\rho)$ such that $\alpha(x_n-x)\to 0$. If $y\in H_0^1(\mathcal{M},\rho)\cap(L^{\alpha}(\mathcal{M},\rho))^{\#}$, then by the generalized Holder's inequality in [12], we have

$$|\rho(xy)-\rho(x_ny)|=|\rho((x-x_n)y)|\le\alpha(x-x_n)\alpha'\to 0.$$

It follows that $\rho(xy)=\lim_{n\to\infty}\rho(x_ny)=0$ for all $y\in H_0^1(\mathcal{M},\rho)\cap(L^{\alpha}(\mathcal{M},\rho))^{\#}$. By the definition of $\mathcal{J}$, we know $x\in\mathcal{J}$. Hence $\mathcal{J}$ is closed in $L^{\alpha}(\mathcal{M},\rho)$. Therefore, $H^{\alpha}(\mathcal{M},\rho)=[H^{\infty}(\mathcal{M},\rho)]_{\alpha}\subseteq\mathcal{J}$.

Next, we show that $H^{\alpha}(\mathcal{M},\rho)=\mathcal{J}$. Assume, via contradiction, that $H^{\alpha}(\mathcal{M},\rho)\subsetneq\mathcal{J}\subseteq L^{\alpha}(\mathcal{M},\rho)$. By the Hahn-Banach theorem, there is a linear functional $\phi\in(L^{\alpha}(\mathcal{M},\rho))^{\#}$ and $x\in\mathcal{J}$ such that

(a) $\phi(x)\neq 0$,

(b) $\phi(y)=0$ for all $y\in H^{\alpha}(\mathcal{M},\rho)$.

In the beginning of this proof, we know $\alpha\in N_1(\mathcal{M},\rho)$, which means α is normalized, unitarily invariant $\|\cdot\|_1$-dominating, continuous norm on $(\mathcal{M},\rho)$. It follows from [12] that there exists a $\xi\in(L^{\alpha}(\mathcal{M},\rho))^{\#}$ such that

(c) $\phi(z)=\rho(z\xi)$ for all $z\in L^{\alpha}(\mathcal{M},\rho)$.

Hence from (b) and (c) we can conclude that

(d) $\rho(y\xi) = \phi(y) = 0$ for every $y \in H^\infty(\mathcal{M}, \rho) \subseteq H^\alpha(\mathcal{M}, \rho) \subseteq L^\alpha(\mathcal{M}, \rho)$.

Since $\phi \in (L^\alpha(\mathcal{M}, \rho))^\# \subseteq L^1(\mathcal{M}, \rho)$, so $\xi \in H^1_0(\mathcal{M}, \rho)$, which means $\xi \in H^1_0(\mathcal{M}, \rho) \cap (L^\alpha(\mathcal{M}, \rho))^\#$. Combining with the fact that $x \in \mathcal{J} = \{x \in L^\alpha(\mathcal{M}, \rho) : \rho(xy) = 0, \forall y \in H^1_0(\mathcal{M}, \rho) \cap (L^\alpha(\mathcal{M}, \rho))^\#\}$, we obtain that $\rho(x\xi) = 0$. Note, again, that $x \in \mathcal{J} \subseteq L^\alpha(\mathcal{M}, \rho)$. From (a) and (c), it follows that $\rho(x\xi) = \phi(x) \neq 0$. This is a contradiction. Therefore

$H^\alpha(\mathcal{M}, \rho) = \{x \in L^\alpha(\mathcal{M}, \rho) : \rho(xy = 0) \text{ for all } y \in H^1_0(\mathcal{M}, \rho) \cap (L^\alpha(\mathcal{M}, \rho))^\#\}$. ☐

Lemma 29. *(from [4]) The conditional expectation Φ_τ is multiplicative on Hardy spaces. More precisely, $\Phi_\tau(xy) = \Phi_\tau(x)\Phi_\tau(y)$ for $x \in H^p(\mathcal{M}, \tau)$, $y \in H^q(\mathcal{M}, \tau)$ and $xy \in H^r(\mathcal{M}, \tau)$ with $0 < p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

Theorem 30. *If $\alpha \in N_\Delta(\mathcal{M}, \tau)$, then there exists a faithful normal tracial state ρ such that $H^\alpha(\mathcal{M}, \rho) = H^1(\mathcal{M}, \rho) \cap L^\alpha(\mathcal{M}, \rho)$.*

Proof. Since $\alpha \in N_\Delta(\mathcal{M}, \tau)$, there exists a positive $g \in L^1(\mathcal{M}, \tau)$ and $\Delta(g) > 0$ such that $\alpha(x) \geq c\tau(|x| \cdot g)$ for some $c > 0$. We define a faithful normal tracial state $\rho(x) = \tau(xg)$, $\forall x \in \mathcal{M}$. Since $\alpha(x) \geq c\tau(|x| \cdot g) = c\rho(|x|) = c\|x\|_{1,\rho}$, α is $\|\cdot\|_{1,\rho}$-dominating, so α-convergence implies $\|\cdot\|_{1,\rho}$-convergence, thus $H^\alpha(\mathcal{M}, \rho) = \overline{H^\infty(\mathcal{M}, \rho)}^\alpha \subseteq \overline{H^\infty(\mathcal{M}, \rho)}^{\|\cdot\|_{1,\rho}} = H^1(\mathcal{M}, \rho)$. Also, $H^\alpha(\mathcal{M}, \rho) = \overline{H^\infty(\mathcal{M}, \rho)}^\alpha \subseteq L^\alpha(\mathcal{M}, \rho)$. Therefore, $H^\alpha(\mathcal{M}, \rho) \subseteq H^1(\mathcal{M}, \rho) \cap L^\alpha(\mathcal{M}, \rho)$.

To prove $H^1(\mathcal{M}, \rho) \cap L^\alpha(\mathcal{M}, \rho) \subseteq H^\alpha(\mathcal{M}, \rho)$. Suppose $x \in H^1(\mathcal{M}, \rho) \cap L^\alpha(\mathcal{M}, \rho)$, then $x \in L^\alpha(\mathcal{M}, \rho)$. Assume that $y \in H^1_0(\mathcal{M}, \rho) \cap (L^\alpha(\mathcal{M}, \rho))^\#$. So $\Phi_\tau(hy) = 0$. Note that $hx \in H^1(\mathcal{M}, \tau)$, $hy \in H^1_0(\mathcal{M}, \tau)$ and $hxhy \in H^1(\mathcal{M}, \tau)H^1_0(\mathcal{M}, \tau) \subseteq H^{\frac{1}{2}}(\mathcal{M}, \tau)$. From theorem 2.1 in [4], and lemma 29 we know that $\Phi_\tau(hxhy) \in L^{\frac{1}{2}}(\mathcal{D}, \tau)$ and $\Phi_\tau(hxhy) = \Phi_\tau(hx)\Phi_\tau(hy) = 0$. Moreover, $x \in L^\alpha(\mathcal{M}, \rho)$ and $y \in (L^\alpha(\mathcal{M}, \rho))^\#$, we know $xy \in L^\alpha(\mathcal{M}, \rho) \subseteq L^1(\mathcal{M}, \rho)$. So $hxy \in L^1(\mathcal{M}, \tau)$, and $\Phi_\tau(hxy)$ is also in $L^1(\mathcal{M}, \tau)$. Thus $\rho(xy) = \tau(hxy) = \tau(\Phi_\tau(hxhy)) = \tau(0) = 0$.

Now we check $\Phi_\tau(hxy) = 0$. Since h is strongly outer in $H^1(\mathcal{M}, \rho)$, there is a sequence $\{a_n\}$ in H^∞ such that $a_n h \to 1$ in $\|\cdot\|_1$ norm. Therefore, $\|hxyha_n - hxy\|_{\frac{1}{2}} = \|hxy(ha_n - 1)\|_{\frac{1}{2}} \leq$

$\|hxy\|_1\|ha_n - 1\|_1 \to 0$ as $n \to \infty$. And by theorem 2.1 in [4], $\Phi_\tau(hxyha_n) \to \Phi_\tau(hxy)$. Also, we have $\Phi_\tau(hxyha_n) = \Phi_\tau(hx)\Phi_\tau(hy)\Phi_\tau(a_n) = 0$, so $\Phi_\tau(hxy) = 0$. By the definition of $\mathcal{J}$ in proposition 2, we conclude that $x \in \mathcal{J}$. Therefore $H^1(\mathcal{M},\rho) \cap L^\alpha(\mathcal{M},\rho) \subseteq \mathcal{J} = H^\alpha(\mathcal{M},\rho)$. □

3.4 Beurling's invariant subspace theorem

In this section, we extend the Chen-Hadwin-Shen theorem for continuous normalized unitarily invariant norms on $(\mathcal{M},\tau)$.

First, we will prove the factorization theorem. In order to do this, we need the following lemma.

Lemma 31. *(from [28]) Let $x \in L^p(\mathcal{M},\tau)$, $p > 0$, then we have*
(1) $\Delta(x) = \Delta(x^*) = \Delta(|x|)$,
(2) $\Delta(xy) = \Delta(x)\Delta(y) = \Delta(yx)$ for any $y \in L^s(\mathcal{M},\tau)$, $s > 0$.

Theorem 32. *Suppose $\alpha \in N_\Delta(\mathcal{M},\tau)$, there exists a faithful normal tracial state ρ on $\mathcal{M}$ such that $\rho(x) = \tau(xg)$ for some positive $g \in L^1(\mathcal{Z},\tau)$ and the determinant of g is positive. If $x \in \mathcal{M}$ and $x^{-1} \in L^\alpha(\mathcal{M},\rho)$, then there are unitary operators $u_1, u_2 \in \mathcal{M}$ and $s_1, s_2 \in H^\infty$ such that $x = u_1 s_1 = s_2 u_2$ and $s_1^{-1}, s_2^{-1} \in H^\alpha(\mathcal{M},\rho)$.*

Proof. Since $\alpha \in N_\Delta(\mathcal{M},\tau)$, the first statement is clear from theorem 22. Suppose $x \in \mathcal{M}$ with $x^{-1} \in L^\alpha(\mathcal{M},\rho)$. Assume that $x = v|x|$ is the polar decomposition of x in $\mathcal{M}$, where v is a unitary in $\mathcal{M}$ and $|x| \in \mathcal{M}$. Since $log(|x|) \le |x|$, $log(|h|) - log(|x|) = log(|h||x|^{-1}) \le |h||x|$ and $-\log(|x|) \le |h||x| - log(|h|)$, $|log(|x|)| \le |x| + (|h||x| - log(|h|))$, so $\Delta(|x|) = e^{\tau(\log|x|)} > 0$ and $|x| \in L^1(\mathcal{M})^+$. By corollary 4.17 in [7], there exists a strongly outer $s \in H^1(\mathcal{M},\tau)$ and $s = u_1|s|$ is the polar decomposition of s such that $|x| = |s|$. Since $|x| \in \mathcal{M}$, $|s| \in \mathcal{M}$, therefore, $s \in \mathcal{M}$ and $s \in H^1(\mathcal{M},\tau)$ implies $s \in H^\infty(\mathcal{M},\tau)$. Also, we have $|x| = u_1^* s$, so $x = vu_1^* s = us$, where $u = vu_1^*$.

Now we check $s^{-1} \in H^\alpha(\mathcal{M},\rho)$. First, $x^{-1} \in L^\alpha(\mathcal{M},\rho) \subseteq L^1(\mathcal{M},\rho)$, so $hx^{-1} \in L^1(\mathcal{M},\tau)$. Since $x^{-1} = |x|^{-1}v^* \in L^\alpha(\mathcal{M},\rho)$, $|x|^{-1} \in L^\alpha(\mathcal{M},\rho) \subseteq L^1(\mathcal{M},\rho)$ and $|h|\,|x|^{-1} \in L^1(\mathcal{M},\tau)$. $\Delta(|h|\,|x|^{-1}) = \Delta(|h|)\Delta(|x|^{-1}) > 0$ by lemma 31. Then there exists a strongly outer $f \in$

$H^1(\mathcal{M},\tau)$ such that $|h|\,|x|^{-1} = |f|$. Since $H^1(\mathcal{M},\tau) = hH^1(\mathcal{M},\rho)$, there exists $f_1 \in H^1(\mathcal{M},\rho)$ such that $f = hf_1$. Since $\Delta(fs) = \Delta(f)\Delta(s) > 0$, by lemma 25, fs is outer. And $|f|\,|s| = |h|\,|x|^{-1}\,|s|$, so $|h| = |f|\,|s|$. Therefore, $|h| = u_2^* f u_1^* s$, i.e., $h = u_3^* u_2^* f u_1^* s$, $hf_1 = f = u_2 u_3 h s^{-1} u_1$, so $s^{-1} = h^{-1} u_3^* u_2^* h f_1 u_1^* = u_3^* u_2^* f_1 u_1^* \in H^1(\mathcal{M},\rho)$. Also, we know $s^{-1} \in L^\alpha(\mathcal{M},\rho)$. Therefore, by theorem 30, $s^{-1} \in L^\alpha(\mathcal{M},\rho) \cap H^1(\mathcal{M},\rho) = H^\alpha(\mathcal{M},\rho)$. □

The following density theorem also plays an important role in the proof of our main result of the chapter.

Theorem 33. *Let $\alpha \in N_\Delta(\mathcal{M},\tau)$, then there exists a faithful normal tracial state ρ on $\mathcal{M}$ such that $\rho(x) = \tau(xg)$ for some positive $g \in L^1(\mathcal{Z},\tau)$ and the determinant of g is positive. Also, if $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\rho)$ and $\mathcal{N}$ is a weak* closed linear subspace of $\mathcal{M}$ such that $\mathcal{W}H^\infty \subset \mathcal{W}$ and $\mathcal{N}H^\infty \subset \mathcal{N}$, then*

(1) $\mathcal{N} = [\mathcal{N}]_\alpha \cap \mathcal{M}$,

(2) $\mathcal{W} \cap \mathcal{M}$ is weak closed in $\mathcal{M}$,*

(3) $\mathcal{W} = [\mathcal{W} \cap \mathcal{M}]_\alpha$,

(4) If $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S}H^\infty \subset \mathcal{S}$, then $[\mathcal{S}]_\alpha = [\overline{\mathcal{S}}^{w*}]_\alpha$, where $\overline{\mathcal{S}}^{w*}$ is the weak-closure of $\mathcal{S}$ in $\mathcal{M}$.*

Proof. Since $\alpha \in N_\Delta(\mathcal{M},\tau)$, clearly, there exists a faithful normal tracial state ρ on $\mathcal{M}$ by theorem 22. For (1), it is clear that $\mathcal{N} \subseteq [\mathcal{N}]_\alpha \cap \mathcal{M}$. Assume, via contradiction, that $\mathcal{N} \subsetneq [\mathcal{N}]_\alpha \cap \mathcal{M}$. Note that $\mathcal{N}$ is a weak* closed linear subspace of $\mathcal{M}$ and $L^1(\mathcal{M},\rho)$ is the predual space of $(\mathcal{M},\rho)$. It follows from the Hahn-Banach theorem that there exists $\xi \in L^1(\mathcal{M},\rho)$ and an $x \in [\mathcal{N}]_\alpha \cap \mathcal{M}$ such that

(a) $\rho(\xi x) \neq 0$ and (b) $\rho(\xi y) = 0$ for all $y \in \mathcal{N}$.

We claim that there exists $z \in \mathcal{M}$ such that

(a′) $\rho(zx) \neq 0$ and (b′) $\rho(zy) = 0$ for all $y \in \mathcal{N}$. Actually assume that $\xi = |\xi^*|v$ is the polar decomposition of $\xi \in L^1(\mathcal{M},\rho)$, where v is a unitary element in $\mathcal{M}$ and $|\xi^*|$ is in $L^1(\mathcal{M},\rho)$ is positive. Let f be a function on $[0,\infty)$ defined by the formula $f(t) = 1$ for $0 \le t \le 1$ and $f(t) = 1/t$

for $t > 1$. We define $k = f(|\xi^*|)$ by the functional calculus. Then by the construction of f, we know that $k \in \mathcal{M}$ and $k^{-1} = f^{-1}(|\xi^*|) \in L^1(\mathcal{M}, \rho)$. It follows from theorem 32 that there exist a unitary operator $u \in \mathcal{M}$ and $s \in H^\infty$ such that $k = us$ and $s^{-1} \in H^1(\mathcal{M}, \rho)$. Therefore, we can further assume that $\{t_n\}_{n=1}^\infty$ is a sequence of elements in H^∞ such that $\|s^{-1} - t_n\|_{1,\rho}$. Observe that

(i) Since s, t_n are in H^∞, for each $y \in \mathcal{N}$ we have that $yt_ns \in \mathcal{N}H^\infty \subseteq \mathcal{N}$ and $\rho(t_n s\xi y) = \rho(\xi y t_n s) = 0$,

(ii) We have $s\xi = (u * u)s(|\xi^*|v) = u * (k|\xi^*|)v \in \mathcal{M}$, by the definition of k,

(iii) From (a) and (i), we have $0 \neq \rho(\xi x) = \rho(s^{-1}s\xi x) = \lim_{n\to\infty} \rho(t_n s\xi x)$.

Combining (i), (ii) and (iii), we are able to find an $N \in \mathbb{Z}$ such that $z = t_N s\xi \in \mathcal{M}$ satisfying

(a′) $\rho(zx) \neq 0$ and (b′) $\rho(zy) = 0$ for all $y \in \mathcal{N}$.

Recall that $x \in [\mathcal{N}]_\alpha$. Then there is a sequence $\{x_n\} \subseteq \mathcal{N}$ such that $\alpha(x - x_n) \to 0$. We have

$|\rho(zx_n) - \rho(zx)| = |\rho(x - x_n)| \leq \|x - x_n\|_{1,\rho}\|z\| \to 0$.

Combining with (b′) we conclude that $\rho(zx) = \lim_{n\to\infty} \rho(zx_n) = 0$. This contradicts with the result (a′). Therefore, $\mathcal{N} = [\mathcal{N}]_\alpha \cap \mathcal{M}$.

For (2), let $\overline{\mathcal{W} \cap \mathcal{M}}^{w*}$ be the weak*-closure of $\mathcal{W} \cap \mathcal{M}$ in $\mathcal{M}$. In order to show that $\mathcal{W} \cap \mathcal{M} = \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$, it suffices to show that $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subseteq \mathcal{W}$. Assume, to the contrary, that $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \nsubseteq \mathcal{W}$. Thus there exists an element x in $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subset \mathcal{M} \subseteq L^\alpha(\mathcal{M}, \rho)$, but $x \notin \mathcal{W}$. Since $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \rho)$, by the Hahn-Banach theorem, there exists a $\xi \in L^1(\mathcal{M}, \rho)$ such that $\rho(\xi x) \neq 0$ and $\rho(\xi y) = 0$ for all $y \in \mathcal{W}$. Since $\xi \in L^1(\mathcal{M}, \rho)$, the linear mapping $\rho_\xi : \mathcal{M} \to \mathbb{C}$, defined by $\rho_\xi(a) = \rho(\xi a)$ for all $a \in \mathcal{M}$ is weak*-continuous. Note that $x \in \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$ and $\rho(\xi y) = 0$ for all $y \in \mathcal{W}$. We know that $\rho(\xi x) = 0$, which contradicts with the assumption that $\rho(\xi x) \neq 0$. Hence $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subseteq \mathcal{W}$, so $\mathcal{W} \cap \mathcal{M} = \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$.

For (3), since $\mathcal{W}$ is α-closed, it is easy to see $[\mathcal{W} \cap \mathcal{M}]_\alpha \subseteq \mathcal{W}$. Now we assume $[\mathcal{W} \cap \mathcal{M}]_\alpha \subsetneq \mathcal{M} \subseteq L^\alpha(\mathcal{M}, \rho)$. By the Hahn-Banach theorem, there exists an $x \in \mathcal{W}$ and $\xi \in L^1(\mathcal{M}, \rho)$ such that $\rho(\xi x) \neq 0$ and $\rho(\xi y) = 0$ for all $y \in [\mathcal{W} \cap \mathcal{M}]_\alpha$. Let $x = v|x|$ be the polar decomposition of x in $L^\alpha(\mathcal{M}, \rho)$, where v is a unitary element in $\mathcal{M}$. Let f be a function on $[0, \infty)$ defined by the formula $f(t) = 1$ for $0 \leq t \leq 1$ and $f(t) = 1/t$ for $t > 1$. We define $k = f(|x|)$ by the functional calculus. Then by the construction of f, we know that $k \in \mathcal{M}$ and $k^{-1} = f^{-1}(|x|) \in L^{\alpha}(\mathcal{M}, \rho)$. It follows from theorem 32 that there exist a unitary operator $u \in \mathcal{M}$ and $s \in H^{\infty}$ such that $k = su$ and $s^{-1} \in H^{\alpha}(\mathcal{M}, \rho)$. A little computation shows that $|x|k \in \mathcal{M}$ which implies that $xs = xsuu^* = xku^* = v(|x|k)u^* \in \mathcal{M}$. Since $s \in H^{\infty}$, we know $xs \in \mathcal{W}H^{\infty} \subseteq \mathcal{W}$ and thus $xs \in \mathcal{W} \cap \mathcal{M}$. Furthermore, note that $(\mathcal{W} \cap \mathcal{M})H^{\infty} \subseteq \mathcal{W} \cap \mathcal{M}$. Thus, if $t \in H^{\infty}(\mathcal{M}, \rho)$ we see $xst \in \mathcal{W} \cap \mathcal{M}$, and $\rho(\xi xst) = 0$. Since $H^{\infty}(\mathcal{M}, \rho)$ is dense in $H^{\alpha}(\mathcal{M}, \rho)$ and $\xi \in L^{1}(\mathcal{M}, \rho)$, it follows that $\rho(\xi xst) = 0$ for all $t \in H^{\alpha}(\mathcal{M}, \rho)$. Since $s^{-1} \in H^{\alpha}(\mathcal{M}, \rho)$, we see that $\rho(\xi x) = \rho(\xi xss^{-1}) = 0$. This contradicts the assumption that $\rho(\xi x) \neq 0$. Therefore $\mathcal{W} = [\mathcal{W} \cap \mathcal{M}]_{\alpha}$.

For (4), assume that $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S}H^{\infty}(\mathcal{M}, \rho) \subset \mathcal{S}$ and $\overline{\mathcal{S}}^{w*}$ is weak*-closure of $\mathcal{S}$ in $\mathcal{M}$. Then $[\mathcal{S}]_{\alpha}H^{\infty}(\mathcal{M}, \rho) \subseteq [\mathcal{S}]_{\alpha}$. Note that $\mathcal{S} \subseteq [\mathcal{S}]_{\alpha} \cap \mathcal{M}$. From (2), we know that $[\mathcal{S}]_{\alpha} \cap \mathcal{M}$ is weak*-closed. Therefore, $\overline{\mathcal{S}}^{w*} \subseteq [\mathcal{S}]_{\alpha} \cap \mathcal{M}$. Hence $[\overline{\mathcal{S}}^{w*}]_{\alpha} \subseteq [\mathcal{S}]_{\alpha}$ and $[\mathcal{S}]_{\alpha} = [\overline{\mathcal{S}}^{w*}]_{\alpha}$. □

Before we obtain our main result in the chapter, we recall the definitions of internal column sum of a family of subspaces, and the lemma in [6].

Definition 34. *Let X be a closed subspace of $L^{\alpha}(\mathcal{M}, \tau)$ with $\alpha \in N_{\Delta}(\mathcal{M}, \tau)$. Then X is called an internal column sum of a family of closed subspaces $\{X_{\lambda}\}_{\lambda \in \Lambda}$ of $L^{\alpha}(\mathcal{M}, \tau)$, denoted by $X = \bigoplus_{\lambda \in \Lambda}^{col} X_{\lambda}$ if*

(1) $X_{\mu}^{*}X_{\lambda} = \{0\}$ for all distinct $\lambda, \mu \in \Lambda$, and

(2) $X = [span\{X_{\lambda} : \lambda \in \Lambda\}]_{\alpha}$.

Definition 35. *Let X be a weak*-closed subspace of $\mathcal{M}$ and $\alpha \in N_{\Delta}(\mathcal{M}, \tau)$. Then X is called an internal column sum of a family of weak*-closed subspaces $\{X_{\lambda}\}_{\lambda \in \Lambda}$ of $L^{\alpha}(\mathcal{M}, \tau)$, denoted by $X = \bigoplus_{\lambda \in \Lambda}^{col} X_{\lambda}$ if*

(1) $X_{\mu}^{*}X_{\lambda} = \{0\}$ for all distinct $\lambda, \mu \in \Lambda$, and

(2) $X = \overline{span\{X_{\lambda} : \lambda \in \Lambda\}}^{w*}$.

Lemma 36. *(from [6]) Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant $\|\cdot\|_{1,\tau}$-dominating continuous norm on $\mathcal{M}$. Let*

H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{W} \subseteq \mathcal{M}$ is a weak-closed subspace such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$. Then there exists a weak*-closed subspace $\mathcal{Y}$ of $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$of partial isometries in $\mathcal{M}$ such that*
(1) $u_\lambda^*\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,
(2) $u_\lambda^* u_\lambda \in \mathcal{D}$ and $u_\lambda^* u_\mu = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,
(3) $\mathcal{Y} = \overline{H_0^\infty \mathcal{Y}}^{w*}$,
(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} u_\lambda H^\infty)$.

Now we are ready to prove our main result of the chapter, an extension of the Chen-Hadwin-Shen theorem for noncommutative Hardy spaces associated with finite von Neumann algebras.

Theorem 37. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a determinant, normalized, unitarily invariant, continuous norm on $\mathcal{M}$. Then there exists a faithful normal tracial state ρ on $\mathcal{M}$ such that $\alpha \in N_1(\mathcal{M}, \rho)$. Let H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$, then there exists a closed subspace $\mathcal{Y}$ of $L^\alpha(\mathcal{M}, \tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that*
(1) $u_\lambda^*\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,
(2) $u_\lambda^* u_\lambda \in \mathcal{D}$ and $u_\lambda^* u_\mu = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,
(3) $\mathcal{Y} = [H_0^\infty \mathcal{Y}]_\alpha$,
(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} u_\lambda H^\alpha)$.

Proof. Suppose $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $\mathcal{W}H^\infty \subset \mathcal{W}$. Then it follows from part(2) of the theorem 33 that $\mathcal{W} \cap \mathcal{M}$ is weak* closed in $(\mathcal{M}, \tau) = (\mathcal{M}, \rho)$, we also notice $L^\infty(\mathcal{M}, \tau) = \mathcal{M} = L^\infty(\mathcal{M}, \rho)$, $L^\alpha(\mathcal{M}, \tau) = L^\alpha(\mathcal{M}, \rho)$ and $H^\alpha(\mathcal{M}, \tau) = H^\alpha(\mathcal{M}, \rho)$. It follows from the lemma 36 that

$$\mathcal{W} \cap \mathcal{M} = \mathcal{Y}_1 \bigoplus^{col} (\bigoplus_{i\in\mathcal{I}}^{col} u_i H^\infty),$$

where $\mathcal{Y}_1$ is a closed subspace of $L^\infty(\mathcal{M}, \rho)$ such that $\mathcal{Y}_1 = \overline{\mathcal{Y}_1 H_0^\infty}^{w*}$, and where u_i are partial isometries in $\mathcal{W}\cap\mathcal{M}$ with $u_j^* u_i = 0$ if $i \neq j$ and with $u_i^* u_i \in \mathcal{D}$. Moreover, for each i, $u_i^*\mathcal{Y}_1 = \{0\}$, left multiplication by the $u_iu_i^*$ are contractive projections from $\mathcal{W}\cap\mathcal{M}$ onto the summands u_iH^∞, and left multiplication by $I-\sum_i u_iu_i^*$ is a contractive projection from $\mathcal{W}\cap\mathcal{M}$ onto $\mathcal{Y}_1$.

Let $\mathcal{Y} = [\mathcal{Y}_1]_\alpha$. It is not hard to verify that for each $i, u_i^*\mathcal{M} = \{0\}$. We also claim that $[u_iH^\infty]_\alpha = u_iH^\alpha$. In fact it is obvious that $[u_iH^\infty]_\alpha \supseteq u_iH^\alpha$. We will need only to show that $[u_iH^\infty]_\alpha \subseteq u_iH^\alpha$. Suppose $x\in[u_iH^\infty]_\alpha$, there is a net $\{x_n\}_{n=1}^\infty \subseteq H^\infty$ such that $\alpha(u_ix_n - x)\to 0$. By the choice of u_i, we know that $u_iu \in \mathcal{D} \subseteq H^\infty$, so $u_iux_n \in H^\infty$ for each $n\geq 1$. Combining with the fact that $\alpha(u_i^*u_ix_n - u_i^*x) \leq \alpha(u_ix_n - x)\to 0$, we obtain that $u_i^*x\in H^\alpha$. Again from the choice of u_i, we know that $u_iu_i^*u_ix_n = u_ix_n$ for each $n\geq 1$. This implies that $x = u_i(u_i^*x)\in u_iH^\alpha$. Thus we conclude that $[u_iH^\infty]_\alpha \subseteq u_iH^\alpha$, so $[u_iH^\infty]_\alpha = u_iH^\alpha$. Now from parts (3) and (4) of the theorem 33 and from the definition of internal column sum, it follows that

$$\begin{aligned}\mathcal{W} &= [\mathcal{W}\cap\mathcal{M}]_\alpha = [\overline{span\{\mathcal{Y}_1, u_iH^\infty : i\in\mathcal{I}\}}^{w*}]_\alpha = [span\{\mathcal{Y}_1, u_iH^\infty : i\in\mathcal{I}\}]_\alpha.\\ &= [span\{\mathcal{Y}, u_iH^\alpha : i\in\mathcal{I}\}]_\alpha = \mathcal{Y}\bigoplus^{col}(\bigoplus_{i\in\mathcal{I}}^{col} u_iH^\alpha).\end{aligned}$$

Next, we will verify that $\mathcal{Y} = [\mathcal{Y}H_0^\infty]_\alpha$. Recall that $\mathcal{Y} = [\mathcal{Y}_1]_\alpha$. It follows from part (1) of the theorem 33, we have

$$[\mathcal{Y}_1H_0^\infty]_\alpha \cap \mathcal{M} = \overline{\mathcal{Y}_1H_0^\infty}^{w*} = \mathcal{Y}_1.$$

Hence from part (3) of the theorem 33 we have that

$$\mathcal{Y} \supseteq [\mathcal{Y}H_0^\infty]_\alpha \supseteq [\mathcal{Y}_1H_0^\infty]_\alpha = [[\mathcal{Y}_1H_0^\infty]_\alpha\cap\mathcal{M}]_\alpha = [\mathcal{Y}_1]_\alpha = \mathcal{Y}.$$

Thus $\mathcal{Y} = [\mathcal{Y}H_0^\infty]_\alpha$. Moreover, it is not difficult to verify that for each i, left multiplication by the $u_iu_i^*$ are contractive projections from $\mathcal{K}$ onto the summands u_iH^α, and left multiplication by $I-\sum_i u_iu_i^*$ is a contractive projection from $\mathcal{W}$ onto $\mathcal{Y}$. Now the proof is completed. □

If we consider α as some specific norms, then we have some corollaries. If we take α be a unitarily invariant, $\|\cdot\|_{1,\tau}$-dominating, continuous norm, then we have Chen-Hadwin-Shen's result in [12].

Corollary 38. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, $\|\|_{1,\tau}$-dominating, continuous norm on $\mathcal{M}$. Let H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $H^\infty\mathcal{W} \subseteq \mathcal{W}$. Then there exists a closed subspace $\mathcal{Y}$ of $L^\alpha(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{W} \cap \mathcal{M}$ such that*
(1) $u_\lambda^*\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,
(2) $u_\lambda^* u_\lambda \in \mathcal{D}$ and $u_\lambda^* u_\mu = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,
(3) $\mathcal{Y} = [H_0^\infty \mathcal{Y}]_\alpha$,
(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} H^\alpha u_\lambda)$.

If we take $\alpha = \|\cdot\|_p$, then we have D. Blecher and L. E. Labuschagne's result in [6].

Corollary 39. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{W}$ is a closed subspace of $L^p(\mathcal{M},\tau), 1 \leq p \leq \infty$ such that $H^\infty\mathcal{W} \subseteq \mathcal{W}$. Then there exists a closed subspace $\mathcal{Y}$ of $L^p(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{W} \cap \mathcal{M}$ such that*
(1) $u_\lambda^*\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,
(2) $u_\lambda^* u_\mu \in \mathcal{D}$ and $u_\lambda^* u_\mu = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,
(3) $\mathcal{Y} = [H_0^\infty \mathcal{Y}]_p$,
(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} H^p u_\lambda)$.

3.5 Generalized Beurling theorem for special von Neumann algebras

In theorem 37, let $\mathcal{M}$ be classical Hardy space on unit circle $\mathbb{T}$ with haar measure, i.e., $\mathcal{M} = L^\infty(\mathbb{T},\mu)$, $H^\infty = H^\infty(\mathbb{T},\mu)$, then $\mathcal{D} = H^\infty \cap (H^\infty)^* = \mathbb{C}$ and the center $\mathcal{Z}$ of $\mathcal{M} = L^\infty(\mathbb{T},\mu)$ is itself. So $\mathcal{Z} \nsubseteq \mathcal{D} = \mathbb{C}$. However, for a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ, let H^{∞} be a finite subdiagonal subalgebra of $\mathcal{M}$, $\mathcal{D} = H^{\infty} \cap (H^{\infty})^{*}$, if the center $\mathcal{Z} \subseteq \mathcal{D}$, then generalized Beurling theorem holds for normalized, unitarily invariant, continuous norms on $(\mathcal{M}, \tau)$.

Theorem 40. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let H^{∞} be a finite subdiagonal subalgebra of $\mathcal{M}$, $\mathcal{D} = H^{\infty} \cap (H^{\infty})^{*}$, and the center $\mathcal{Z} \subseteq \mathcal{D}$. Let α be a normalized, unitarily invariant, continuous norm on $(\mathcal{M}, \tau)$. If $\mathcal{W}$ is a closed subspace of $L^{\alpha}(\mathcal{M}, \tau)$ such that $\mathcal{W}H^{\infty} \subseteq \mathcal{W}$, then there exists a closed subspace $\mathcal{Y}$ of $L^{\alpha}(\mathcal{M}, \tau)$ and a family $\{u_{\lambda}\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M}$ such that*
(1) $u_{\lambda}^{*}\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,
(2) $u_{\lambda}^{*}u_{\lambda} \in \mathcal{D}$ and $u_{\lambda}^{*}u_{\mu} = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,
(3) $\mathcal{Y} = [H_{0}^{\infty}\mathcal{Y}]_{\alpha}$,
(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda \in \Lambda}^{col} u_{\lambda} H^{\alpha})$.

Proof. By theorem 22, there exists a faithful normal tracial state ρ on $\mathcal{M}$ and a $c > 0$ such that α is a continuous normalized unitarily invariant $c\|\cdot\|_{1,\rho}$-dominating norm on $(\mathcal{M}, \rho)$. First, recall the definition of conditional expectation $\Phi_{\mathcal{D},\tau}$. We know that $\Phi_{\mathcal{D},\tau}$ is multiplicative on H^{∞}. In general, $\Phi_{\mathcal{D},\rho}$ won't be multiplicative on H^{∞}, however, the condition $\mathcal{Z} \subset \mathcal{D}$ makes sure $\Phi_{\mathcal{D},\rho}$ is multiplicative on H^{∞}, because we can choose $0 \leq x_1 \leq x_2 \leq \cdots$ in $\mathcal{Z}$ such that, for every $x \in \mathcal{M}$,

$$\rho(x) = \lim_{n \to \infty} \tau(x_n x) = \lim_{n \to \infty} \tau(\Phi_{\mathcal{D},\tau}(x_n x)).$$

Since $\mathcal{Z} \subset \mathcal{D}$, $\Phi_{\mathcal{D},\tau}(x_n x) = x_n \Phi_{\mathcal{D},\tau}(x)$. Thus

$$\rho(x) = \lim_{n \to \infty} \tau(x_n \Phi_{\mathcal{D},\tau}(x)) = \rho(\Phi_{\mathcal{D},\tau}(x)).$$

It follows that $\Phi_{\mathcal{D},\tau} = \Phi_{\mathcal{D},\rho}$. This now reduces to the $c\|\cdot\|_1$-dominating version of the Chen-Hadwin-Shen theorem in [12]. □

CHAPTER 4

A GENERALIZED BEURLING THEOREM IN FINITE VON NEUMANN ALGEBRAS

In 2016 and 2017, Haihui Fan, Don Hadwin and Wenjing Liu proved a commutative and noncommutative version of Beurling's theorems for a continuous unitarily invariant norm α on $L^\infty(\mathbb{T},\mu)$ and tracial finite von Neumann algebras $(\mathcal{M},\tau)$, respectively. In the chapter, we study unitarily $\|\|_1$-dominating invariant norms α on finite von Neumann algebras. First we get a Beurling theorem in commutative von Neumann algebras by defining $H^\alpha(\mathbb{T},\mu) = \overline{H^\infty(\mathbb{T},\mu)}^{\sigma(L^\alpha(\mathbb{T}),\mathcal{L}^{\alpha'}(\mathbb{T}))} \cap L^\alpha(\mathbb{T},\mu)$, then prove that the generalized Beurling theorem holds. Moreover, we get similar result in noncommutative case. The key ingredients in the proof of our result include a factorization theorem and a density theorem for $L^\alpha(\mathcal{M},\tau)$.

4.1 Introduction

Let $\mathbb{T}$ be the unit circle, i.e., $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and let μ be Haar measure (i.e., normalized arc length) on $\mathbb{T}$. The classical and influential Beurling-Helson-Lowdenslager theorem (see [5],[39],[41]) states that if W is a closed $H^\infty(\mathbb{T},\mu)$-invariant subspace (or, equivalently, $zW \subseteq W$) of $L^2(\mathbb{T},\mu)$, then $W = \varphi H^2$ for some $\varphi \in L^\infty(\mathbb{T},\mu)$, with $|\varphi| = 1$ a.e.(μ) or $W = \chi_E L^2(\mathbb{T},\mu)$ for some Borel set $E \subset \mathbb{T}$. If $0 \neq W \subset H^2(\mathbb{T},\mu)$, then $W = \varphi H^2(\mathbb{T},\mu)$ for some $\varphi \in H^\infty(\mathbb{T},\mu)$ with $|\varphi| = 1$ a.e. (μ). Later, the Beurling's theorem was extended to $L^p(\mathbb{T},\mu)$ and $H^p(\mathbb{T},\mu)$ with $1 \leq p \leq \infty$, with the assumption that W is weak*-closed when $p = \infty$ (see [36],[39],[41],[43]). In [10], Yanni Chen extended the Helson-Lowdenslager-Beurling theorem for all continuous $\|\|_1$-dominating normalized gauge norms on $\mathbb{T}$. In [21], [22] Haihui Fan, Don Hadwin and Wenjing Liu proved a commutative and noncommutative version of Beurling's theorems for a continuous unitarily invariant norm α on $L^\infty(\mathbb{T},\mu)$ and a tracial finite von Neumann algebra $(\mathcal{M},\tau)$, respectively. Later, Lauren Sager and Wenjing Liu got a similarly result for semifinite von Neumann algebras in [55].

In this chapter, we first extend the Helson-Lowdenslager-Beurling theorem for a much larger class of norms, $\|\|_1$-dominating normalized gauge norms on $L^\infty(\mathbb{T},\mu)$. For each such norm α, we define the dual norm α', let $\mathcal{L}^\alpha(\mathbb{T},\mu) = \{f : f \text{ is a measurable function on } \mathbb{T} \text{ with } \alpha(f) < \infty\}$, and $L^\alpha(\mathbb{T},\mu) = \overline{L^\infty(\mu)}^\alpha$, i.e., the α-closure of $L^\infty(\mu)$ in $\mathcal{L}^\alpha(\mathbb{T},\mu)$. We have Banach space $L^\alpha(\mathbb{T},\mu) = \overline{L^\infty(\mathbb{T},\mu)}^\alpha$ and a Hardy space $H^\alpha = \overline{H^\infty(\mathbb{T},\mu)}^{\sigma(L^\alpha(\mathbb{T}),\mathcal{L}^{\alpha'}(\mathbb{T}))} \cap L^\alpha(\mathbb{T},\mu)$ with $L^\infty(\mathbb{T},\mu) \subset L^\alpha(\mathbb{T},\mu) \subset L^1(\mathbb{T},\mu)$ and $H^\infty(\mathbb{T},\mu) \subset H^\alpha(\mathbb{T},\mu) \subset H^1(\mathbb{T},\mu)$. In this new setting, we prove the following Beurling-Helson-Lowdenslager theorem, which is the main result of this chapter.

THEOREM 53 Suppose μ is Haar measure on $\mathbb{T}$ and α is a normalized gauge norm on $L^\infty(\mathbb{T})$ with $\alpha(.) \geq \|.\|_1$. Let W be an $\sigma(L^\alpha(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed linear subspace of $L^\alpha(\mathbb{T})$ with $zW \subseteq W$ if and only if either $W = \varphi H^\alpha(\mu)$ for some unimodular function φ, or $W = \chi_E L^\alpha(\mu)$, for some Borel subset E of $\mathbb{T}$. If $0 \neq W \subset H^\alpha(\mu)$, then $W = \varphi H^\alpha(\mu)$ for some inner function φ.

To prove THEOREM 53. we need the following technical theorems in Section 3. THEOREM 48 Let α be a normalized gauge norm on $L^\infty(\mathbb{T})$ with $\alpha(.) \geq \|.\|_1$. If $k \in L^\infty$, $k^{-1} \in L^\alpha$, then there is a unimodular function $u \in L^\infty$ and an outer function $s \in H^\infty$ such that $k = us$ and $s^{-1} \in H^\alpha$.
THEOREM 51 Suppose α is a normalized gauge norm on $L^\infty(\mathbb{T})$ with $\alpha(.) \geq \|.\|_1$. Let M be an $\sigma(L^\alpha(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed linear subspace of $L^\alpha(\mathbb{T})$ with $zM \subseteq M$. Then
(1) $M \cap L^\infty(\mathbb{T})$ is weak*-closed in $L^\infty(\mathbb{T})$,
(2) $M = \overline{M \cap L^\infty(\mathbb{T})}^{\sigma(L^\alpha(\mathbb{T}),L^{\alpha'}(\mathbb{T}))}$.

In noncommutative case, we obtain similarly result. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful, normal, tracial state τ, Φ_τ be the conditional expectation and α is a normalized, unitarily invariant $\|\|_1$-dominating norm on $\mathcal{M}$. Let $L^\alpha(\mathcal{M},\tau)$ be the α closure of $\mathcal{M}$,i.e., $L^\alpha(\mathcal{M},\tau) = [\mathcal{M}]_\alpha$. Similarly, $H^\alpha(\mathcal{M},\tau) = \overline{H^\infty(\mathcal{M},\tau)}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau)$,

$H_0^\infty(\mathcal{M},\tau) = \ker(\Phi_\tau) \cap H^\infty(\mathcal{M},\tau)$ and $H_0^\alpha(\mathcal{M},\tau) = \ker(\Phi_\tau) \cap H^\alpha(\mathcal{M},\tau)$. Then we get the following generalized Beurling theorem in finite von Neumann algebras.

THEOREM 61 Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, $\|\|_1$-dominating norm on $\mathcal{M}$.Let H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$, then there exists a $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ closed subspace $\mathcal{Y}$ of $L^\alpha(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that

(1) $u_\lambda^*\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,

(2) $u_\lambda^* u_\lambda \in \mathcal{D}$ and $u_\lambda^* u_\mu = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,

(3) $\mathcal{Y} = \overline{H_0^\infty \mathcal{Y}}^{\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))}$,

(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} u_\lambda H^\alpha)$.

The organization of the chapter is as follows. In section 2, we introduce $\|\|_1$-dominating normalized, unitarily invariant norms. In section 3, we study the relations between commutative Hardy spaces $H^\alpha(\mathbb{T},\mu)$ and get the generalized Beurling theorem in the commutative von Neumann algebras setting. In section 4, using similar techniques as in section 3, we prove a version of the generalized noncommutative Beurling's theorem for finite von Neumann algebras.

4.2 Gauge Norms on the Unit Circle

A norm α on $L^\infty(\mathbb{T},\mu)$ is a *normalized gauge norm* if

1. $\alpha(1) = 1$,

2. $\alpha(|f|) = \alpha(f)$ for every $f \in L^\infty(\mathbb{T},\mu)$.

We say that a *normalized gauge norm* α is $\|\cdot\|_{1,\mu}$*-dominating* if there exists $c \in R^+$ such that

(3) $\alpha(f) \geq c\|f\|_{1,\mu},$ for every $f \in L^\infty(\mathbb{T},\mu)$.

For example, it is easy to see the following fact that

1. The common norm $\|\cdot\|_{p,\mu}$ is a α norm for $1 \leq p \leq \infty$.

2. If $1 \leq p_n < \infty$ for $n \geq 1$, $\sum_{n=1}^{\infty} \frac{1}{2^n} \| \cdot \|_{p_n,\mu}$ is a α norm, which is not equivalent to any $\| \cdot \|_{p,\mu}$.

We can extend the normalized gauge norm α from $L^{\infty}(\mathbb{T}, \mu)$ to the set of all measurable functions, and define α for all measurable functions f on $\mathbb{T}$ by

$$\alpha(f) = \sup\{\alpha(s) : s \text{ is a simple function } , 0 \leq s \leq |f|\}.$$

It is clear that $\alpha(f) = \alpha(|f|)$ still holds.

Define the following two spaces.

$$\mathcal{L}^{\alpha}(\mathbb{T}, \mu) = \{f : f \text{ is a measurable function on } \mathbb{T} \text{ with } \alpha(f) < \infty\},$$

$$L^{\alpha}(\mathbb{T}, \mu) = \overline{L^{\infty}(\mu)}^{\alpha}, \text{ i.e., the } \alpha \text{ -closure of } L^{\infty}(\mu) \text{ in } \mathcal{L}^{\alpha}(\mathbb{T}, \mu).$$

The following are some properties of α norm in []

Lemma 41. *Suppose $f, g : \mathbb{T} \to \mathbb{C}$ are measurable. Let α be a $\|\cdot\|_{1,\mu}$-dominating normalized gauge norm. Then the following statements are true*

(1) If $|f| \leq |g|$, then $\alpha(f) \leq \alpha(g)$;

(2) $\alpha(fg) \leq \alpha(f) \|g\|_{\infty}$;

(3) $\alpha(g) \leq \|g\|_{\infty}$;

(4) $L^{\infty}(\mathbb{T}, \mu) \subset L^{\alpha}(\mathbb{T}, \mu) \subset \mathcal{L}^{\alpha}(\mathbb{T}, \mu) \subset L^{1}(\mathbb{T}, \mu)$.

Let α be a $\|\cdot\|_{1,\mu}$-dominating normalized gauge norm on $L^{\infty}(\mathbb{T}, \mu)$. We define the dual norm $\alpha^{'} : L^{\infty}(\mathbb{T}, \mu) \to [0, \infty]$ by

$$\alpha^{'}(f) = sup\{|\int_{\mathbb{T}} fhd\mu| : h \in L^{\infty}(\mathbb{T}, \mu), \alpha(h) \leq 1\}$$

$$= sup\{\int_{\mathbb{T}} |fh|d\mu : h \in L^{\infty}(\mathbb{T}, \mu), \alpha(h) \leq 1\}$$

Lemma 42. *Let α be a $\|\|_1$-dominating normalized gauge norm on $L^{\infty}(\mathbb{T},\mu)$. Then the dual norm $\alpha^{'}$ is also a $\|\|_1$-dominating normalized gauge norm on $L^{\infty}(\mathbb{T},\mu)$.*

We also can define the dual spaces of $\mathcal{L}^{\alpha}(\mathbb{T},\mu)$ and $L^{\alpha}(\mathbb{T},\mu)$.

$$\mathcal{L}^{\alpha^{'}}(\mathbb{T},\mu) = \{f : f \text{ is a measurable function on } \mathbb{T} \text{ with } \alpha^{'}(f) < \infty\}.$$

$$L^{\alpha^{'}}(\mathbb{T},\mu) = \overline{L^{\infty}(\mu)}^{\alpha^{'}}, \text{ i.e., the } \alpha^{'}\text{-closure of } L^{\infty}(\mathbb{T},\mu) \text{ in } \mathcal{L}^{\alpha^{'}}(\mathbb{T},\mu).$$

By lemma 2.3 in [10], we have

$$L^{\infty}(\mathbb{T},\mu) \subset L^{\alpha^{'}}(\mathbb{T},\mu) \subset \mathcal{L}^{\alpha^{'}}(\mathbb{T},\mu) \subset L^{1}(\mathbb{T},\mu)$$

Now we consider the $\sigma(L^{\alpha}(\mathbb{T}), \mathcal{L}^{\alpha^{'}}(\mathbb{T}))$ topology on $L^{\alpha}(\mathbb{T},\mu)$ space. Since $L^{\infty}(\mathbb{T},\mu) \subset L^{\alpha}(\mathbb{T},\mu)$, $\overline{L^{\infty}(\mathbb{T},\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),\mathcal{L}^{\alpha^{'}}(\mathbb{T}))} \subset \overline{L^{\alpha}(\mathbb{T},\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),\mathcal{L}^{\alpha^{'}}(\mathbb{T}))} = L^{\alpha}(\mathbb{T},\mu)$. Thus we have the following result.

Lemma 43. $\overline{L^{\infty}(\mathbb{T},\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),\mathcal{L}^{\alpha^{'}}(\mathbb{T}))} = L^{\alpha}(\mathbb{T},\mu)$.

Proof. As we show above, $\overline{L^{\infty}(\mathbb{T},\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),\mathcal{L}^{\alpha^{'}}(\mathbb{T}))} \subset L^{\alpha}(\mathbb{T},\mu)$. Additionally, by properties of norm and weak closure, we have $L^{\alpha}(\mathbb{T},\mu) \subset \overline{L^{\infty}(\mathbb{T},\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),\mathcal{L}^{\alpha^{'}}(\mathbb{T}))}$. □

Since $L^{\infty}(\mathbb{T},\mu)$ with the norm α is dense in $L^{\alpha}(\mathbb{T},\mu)$, they have the same dual spaces. i.e. the normed dual $(L^{\alpha}(\mathbb{T},\mu),\alpha)^{\#} = (L^{\infty}(\mathbb{T},\mu),\alpha)^{\#}$. By the following lemma, we can view the dual space as a vector space, a vector subspace of $L^{1}(\mathbb{T},\mu)$. Suppose $w \in L^{1}(\mathbb{T},\mu)$, we define the functional $\varphi_w : L^{\infty}(\mathbb{T},\mu) \to \mathbb{C}$ by

$$\varphi_w(f) = \int_{\mathbb{T}} f w d\mu.$$

Lemma 44. *Let α be a $\|\cdot\|_1$-dominating normalized gauge norm on $L^{\infty}(\mathbb{T},\mu)$ and $\alpha^{'}$ be its dual norm . Then*

(1) if $\varphi : L^{\infty}(\mathbb{T},\mu) \to \mathbb{C}$ is an $\sigma(L^{\alpha}(\mathbb{T}), \mathcal{L}^{\alpha^{'}}(\mathbb{T}))$-continuous linear functional, then there is a $w \in L^1(\mathbb{T},\mu)$ such that $\varphi = \varphi_w$, where $\varphi_w(f) = \int_{\mathbb{T}} fwd\mu$.

(2) if φ_w is $\sigma(L^{\alpha}(\mathbb{T}), \mathcal{L}^{\alpha^{'}}(\mathbb{T}))$-continuous on $L^{\infty}(\mathbb{T},\mu)$, then

(a) $\|w\|_{1,\mu} \le \|\varphi_w\| = \left\|\varphi_{|w|}\right\|$,

(b) given φ in the dual of $L^{\alpha}(\mathbb{T},\mu)$, i.e., $\varphi \in \left(L^{\alpha}(\mathbb{T},\mu), \sigma(L^{\alpha}(\mathbb{T}), \mathcal{L}^{\alpha^{'}}(\mathbb{T}))\right)^{\#}$, there exists a $w \in L^1(\mathbb{T},\mu)$, such that

$$\forall f \in L^{\infty}(\mathbb{T},\mu), \varphi(f) = \int_{\mathbb{T}} fwd\mu \text{ and } wL^{\alpha}(\mathbb{T},\mu) \subseteq L^1(\mathbb{T},\mu)$$

(3) we have $L^{\alpha^{'}}(\mathbb{T},\mu) \subseteq \left(L^{\alpha}(\mathbb{T},\mu), \sigma(L^{\alpha}(\mathbb{T}), \mathcal{L}^{\alpha^{'}}(\mathbb{T}))\right)^{\#}$.

Proof. For (1), It's easy to check by the definition of $\sigma(L^{\alpha}(\mathbb{T}), \mathcal{L}^{\alpha^{'}}(\mathbb{T}))$-continuous linear functional

For (2a), From (1), we have

$$\|w\|_{1,\mu} = \sup\left\{\left|\int_{\mathbb{T}} wsd\mu\right| : s \text{ is simple, } \|s\|_{\infty} \le 1\right\}$$

$$= \sup\{|\varphi(s)| : s \text{ simple, } \|s\|_{\infty} \le 1\} \le \|\varphi\|.$$

We will see $\|w\|_{1,\mu} \le \|\varphi\|$.

(2b) Suppose $f \in L^{\alpha}(\mathbb{T},\mu)$, $f = u|f|, |u| = 1$. $|f| \in L^{\alpha}(\mathbb{T},\mu)$. There exists an increasing positive sequence s_n such that $s_n \to |f|$ a.e. (μ), thus $us_n \to u|f|$ a.e.(μ). $\forall w \in L^1(\mathbb{T},\mu), w = v|w|$, where $|v| = 1$, so we have $\overline{v}s_n \to \overline{v}|f|$ a.e. (μ), where $\overline{v}$ is the conjugate of v and $\alpha(\overline{v}s_n - \overline{v}|f|) \to 0$. Thus we have $\varphi(\overline{v}s_n) \to \varphi(\overline{v}|f|)$. On the other hand, we also have $\varphi(\overline{v}s_n) = \int_{\mathbb{T}} \overline{v}s_n wd\mu \to \int_{\mathbb{T}} \overline{v}|f|wd\mu = \int_{\mathbb{T}} |f||w|d\mu$ by monotone convergence theorem. Thus $\int_{\mathbb{T}} |f||w|d\mu = \int_{\mathbb{T}} |f|\overline{v}wd\mu = \varphi(\overline{v}|f|) < \infty$, therefore $fw \in L^1(\mathbb{T},\mu)$, i.e., $wL^{\alpha}(\mathbb{T},\mu) \subseteq L^1(\mathbb{T},\mu)$, where $w \in L^1(\mathbb{T},\mu)$.

For (3), By (2b) we know that if $\varphi \in L^\alpha(\mathbb{T},\mu)$, then there exists $w \in L^1(\mathbb{T},\mu)$ such that $\varphi(f) = \varphi_w(f), \forall f \in L^\infty(\mathbb{T},\mu)$. By (1), $\varphi(f) = \varphi_w(f)$ implies φ is an $\sigma(L^\alpha(\mathbb{T}), \mathcal{L}^{\alpha'}(\mathbb{T}))$-continuous linear functional.

□

4.3 The Extension of Beurling Theorem in Commutative von Neumann Algebras

Let α be a $\|\cdot\|_1$-dominating normalized gauge norm on $L^\infty(\mathbb{T},\mu)$. We define $H^\alpha(\mathbb{T},\mu) = \overline{H^\infty(\mathbb{T},\mu)}^{\sigma(L^\alpha(\mathbb{T}),\mathcal{L}^{\alpha'}(\mathbb{T}))} \cap L^\alpha(\mathbb{T},\mu)$, from the definition, we first extend the classical $L^p(\mathbb{T},\mu)$ spaces.

Example 9. *If we take α to be p-norm, then $H^p(\mathbb{T},\mu) = \overline{H^\infty(\mathbb{T},\mu)}^{\sigma(L^p(\mathbb{T}),\mathcal{L}^q(\mathbb{T}))} \cap L^p(\mathbb{T},\mu)$.*

In addition, in the classical Hardy space, we have $H^p(\mathbb{T},\mu) = H^1(\mathbb{T},\mu) \cap L^p(\mathbb{T},\mu)$, now we have similar result in the following theorem.

Theorem 45. $H^\alpha(\mathbb{T},\mu) = H^1(\mathbb{T},\mu) \cap L^\alpha(\mathbb{T},\mu)$.

Proof. By the definition of H^α, we know that

$$H^\alpha = \overline{H^\infty(\mu)}^{\sigma(L^\alpha(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} \cap L^\alpha(\mathbb{T},\mu) \subset \overline{L^\infty}^{\sigma(L^\alpha(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} = L^\alpha(\mathbb{T},\mu).$$

For every $f \in H^\alpha = \overline{H^\infty(\mu)}^{\sigma(L^\alpha(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} \cap L^\alpha(\mathbb{T},\mu) \subsetneq L^1(\mathbb{T},\mu)$, there is a sequence f_n in H^∞ such that $f_n \to f$ in $\sigma(L^\alpha(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$ topology. Thus, for every $g \in L^{\alpha'(\mathbb{T})}$, $\int_\mathbb{T}(f_n g)d\mu \to \int_\mathbb{T}(fg)d\mu$. Therefore, we have

$$c_{-m} = \int_\mathbb{T} f z^m d\mu = \lim_{n\to\infty} \int_\mathbb{T} f_n z^m d\mu = \lim_{n\to\infty} c_{-mn} = \lim_{n\to\infty} 0 = 0, m \geq 0$$

So $f \in H^1(\mathbb{T},\mu)$. Thus $H^\alpha(\mathbb{T},\mu) \subseteq H^1(\mathbb{T},\mu) \cap L^\alpha(\mathbb{T},\mu)$.

Now since $H^{\alpha}(\mathbb{T},\mu)$ is an $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed subspace of $L^{\alpha}(\mathbb{T},\mu)$, for every $f \in L^{\alpha}(\mathbb{T},\mu)$ and $f \notin H^{\alpha}(\mathbb{T},\mu)$, there is a $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-continuous functional φ on $L^{\alpha}(\mathbb{T},\mu)$ such that $\varphi(H^{\alpha}(\mathbb{T},\mu)) = 0$ and $\varphi(f) \neq 0$. Also, there is a $g \in L^{\alpha'}(\mathbb{T},\mu)$ such that $\varphi(h) = \int_{\mathbb{T}} hg d\mu$ for all $h \in L^{\alpha}(\mathbb{T},\mu)$. And we know $g \in L^{\alpha'}(\mathbb{T},\mu) \subset L^{1}(\mathbb{T},\mu)$, so we can write $g(z) = \sum_{n=-\infty}^{\infty} c_n z^n$. Since $\phi(H^{\alpha}(\mathbb{T},\mu)) = 0$,we have

$$c_{-n} = \int_{\mathbb{T}} gz^n d\mu = \varphi(z^n) = 0, n \geq 0.$$

Thus g is analytic and $g(0) = 0$.

Take $w \in H^{1}(\mathbb{T},\mu) \cap L^{\alpha}(\mathbb{T},\mu)$,then wg is analytic and $wg \in L^{1}(\mathbb{T},\mu)$. Hence

$$\varphi(w) = \int_{\mathbb{T}} wg d\mu = w(0)g(0) = 0$$

Since for every $f \in L^{\alpha}(\mathbb{T},\mu)$ and $f \notin H^{\alpha}(\mathbb{T},\mu)$, there is a φ is $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-continuous functional on $L^{\alpha}(\mathbb{T},\mu)$ such that $\varphi(H^{\alpha}(\mathbb{T},\mu)) = 0$ and $\varphi(f) \neq 0$, $w \in H^{\alpha}(\mathbb{T},\mu)$ by Hahn-Banach theorem, which implies $H^{1}(\mathbb{T},\mu) \cap L^{\alpha}(\mathbb{T},\mu) \subset H^{\alpha}(\mathbb{T},\mu)$.

□

Lemma 46. *Let α be a $\|\cdot\|_1$-dominating normalized gauge norm on $L^{\infty}(\mathbb{T},\mu)$. If W is an $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed linear subspace of $L^{\alpha}(\mathbb{T},\mu)$ with $zW \subseteq W$, then $H^{\infty}(\mu)W \subset W$.*

Proof. Let $P^{+} = \{e_n : n \in \mathbb{N}\}$ denote the class of all polynomials in $H^{\infty}(\mathbb{T},\mu)$, where $e_n(z) = z^n$ for all z in the unit circle $\mathbb{T}$. Since $zW \subseteq W$, we see $p(z)W \subseteq W$ for any polynomial $p \in P^{+}$. To complete the proof, it suffices to show that $fh \in W$ for every $h \in W$and every $f \in H^{\infty}(\mathbb{T},\mu)$. Now we assume that u is a nonzero element in $L^{\alpha'}(\mathbb{T},\mu)$, then it follows from lemma 44 (2b) $hu \in WL^{\alpha'}(\mathbb{T},\mu) \subset L^{\alpha}(\mathbb{T}) L^{\alpha'}(\mathbb{T}) \subset L^{1}(\mathbb{T})$. Since $f \in H^{\infty}$, define $\varphi(h) = \int_{\mathbb{T}} hg d\mu$ for all $h \in L^{\alpha}(\mathbb{T},\mu)$, now we have $c_{-n} = \int_{\mathbb{T}} fz^n d\mu = \varphi(z^n) = 0$, for all $n > 0$, which implies that the partial sums $S_n(f) = \sum_{-n}^{n} c_n e^n = \sum_{0}^{n} c_n e^n \in P+$ for all $n > 0$. Hence the Cesaro means

$$\sigma_n(f) = \frac{S_0(f) + S_1(f) + ... + S_n(f)}{n+1} \in P^{+}$$

Moreover, we know that $\sigma_n(f) \to f$ in the weak* topology. Since $hu \in L^1(\mathbb{T})$ we have

$$\int_{\mathbb{T}} \sigma_n(f) hud\mu \to \int_{\mathbb{T}} fhud\mu$$

Observe that $\sigma_n(f)h \in P^+W \subset W$ and $u \in L^{\alpha'}(\mathbb{T})$, it follows that $\sigma_n(f)h \to fh$ in $\sigma(L^\alpha(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$ topology. Since W is $\sigma(L^\alpha(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed, $fh \in W$. This completes the proof. □

A key ingredient is based on the following result that uses the Herglotz kernel in [9].

Lemma 47. $\{|h| : 0 \neq h \in H^1(\mathbb{T}, \mu)\} = \{f \in L^1(\mathbb{T}, \mu) : f \geq 0$ *and* $\log f \in L^1(\mathbb{T}, \mu)\}$ *, In fact, if* $f \geq 0$ *and* $\phi, \log \phi \in L^1(\mathbb{T}, \mu)$*, then*

$$f(z) = exp \int_{\mathbb{T}} \frac{w+z}{w-z} \log f(w) d\mu(w)$$

defines an outer function h *on* $\mathbb{D}$ *and* $|h| = f$ *on* $\mathbb{T}$.

The following result is a factorization theorem for $L^\alpha(\mathbb{T}, \mu)$.

Theorem 48. *Let* α *be a* $\|\cdot\|_1$*-dominating normalized gauge norm on* $L^\infty(\mathbb{T}, \mu)$*. If* $k \in L^\infty(\mathbb{T}, \mu)$*,* $k^{-1} \in L^\alpha(\mathbb{T}, \mu)$*, then there is a unimodular function* $u \in L^\infty(\mathbb{T}, \mu)$ *and an outer function* $s \in H^\infty(\mathbb{T}, \mu)$ *such that* $k = us$ *and* $s^{-1} \in H^\alpha(\mathbb{T}, \mu)$*.*

Proof. Recall that an outer function is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Suppose $k \in L^\infty(\mathbb{T}, \mu)$, $k^{-1} \in L^\alpha(\mathbb{T}, \mu)$, on the circle we have

$$-|k| < -\log|k| = \log|k^{-1}| \leq |k^{-1}|$$

It follows from $k \in L^\infty(\mathbb{T}, \mu)$, and $k^{-1} \in L^\alpha(\mathbb{T}) \subset L^1(\mathbb{T})$ that

$$-\infty < \int_{\mathbb{T}} -|k| d\mu \leq \int_{\mathbb{T}} \log|k^{-1}| d\mu \leq \int_{\mathbb{T}} |k^{-1}| d\mu < \infty$$

Hence $|k^{-1}|$ is log integrable, by lemma 47,there is an outer function $h \in H^1(\mathbb{T}, \mu)$ such that $|h| = |k^{-1}|$ on $\mathbb{T}$. If we let $s = h^{-1}$ and $u = kh$, we know h is outer, $s = h^{-1}$ is analytic on $\mathbb{D}$, also, $|s| = |h^{-1}| = |k| \in L^\infty$, so $s \in H^\infty$ such that $k = us$ where u is unimodular. Further, since $h \in H^1(\mu)$ and $uk^{-1} \in L^\alpha(\mathbb{T},\mu)$, it follows that $s^{-1} = h = uk^{-1} \in H^1(\mathbb{T},\mu) \cap L^\alpha(\mathbb{T},\mu) = H^\alpha(\mathbb{T},\mu)$.

□

We let $\mathbb{B} = \{f \in L^\infty(\mathbb{T},\mu) : \|f\|_\infty \le 1\}$ denote the closed unit ball in $L^\infty(\mathbb{T},\mu)$.

Lemma 49. *Let α be a $\|\cdot\|_1$-dominating normalized gauge norm on $L^\infty(\mathbb{T},\mu)$. Then $\mathbb{B} = \{f \in L^\infty(\mu) : \|f\|_\infty \le 1\}$ is α-closed.*

Proof. Suppose $\{f_n\}$ is a sequence in $\mathbb{B}$, $f \in L^\alpha$ and $\alpha(f_n - f) \to 0$. Since $\|f\|_1 \le \alpha(f)$. it follows that $\|f_n - f\|_1 \to 0$, which implies that $f_n \to f$ in μ-measure. Then there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. (μ). Hence $f \in \mathbb{B}$. □

The following theorem and its corollary relate the closed invariant subspaces of $L^\alpha(\mathbb{T},\mu)$ to the weak*-closed invariant subspaces of L^∞.

The following lemma is the Krein-Smulian theorem from [15].

Lemma 50. *Let X be a Banach space. A convex set in $X^\#$ is weak* closed if and only if its intersection with $\mathbb{B} = \{\phi : \|\phi\| \le 1\}$ is weak* closed.*

Theorem 51. *Let α be a $\|\cdot\|_1$-dominating normalized gauge norm on $L^\infty(\mathbb{T},\mu)$. Let M be an $\sigma(L^\alpha(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed linear subspace of $L^\alpha(\mathbb{T},\mu)$ with $zM \subseteq M$. Then*
(1) $M \cap L^\infty(\mathbb{T})$ is weak-closed in $L^\infty(\mathbb{T})$,*
(2) $M = \overline{M \cap L^\infty(\mathbb{T})}^{\sigma(L^\alpha(\mathbb{T}),L^{\alpha'}(\mathbb{T}))}$.

Proof. For (1), to prove $M \cap L^\infty(\mathbb{T},\mu)$ is weak*-closed in $L^\infty(\mathbb{T},\mu)$, using the Krein-Smulian theorem, we only need to show that $M \cap L^\infty(\mathbb{T},\mu) \cap \mathbb{B}$, i.e., $M \cap \mathbb{B}$, is weak*-closed. If $\{f_\lambda\}$ is a net in $M \cap \mathbb{B}$ and $f_\lambda \to f$ weak* in $L^\infty(\mathbb{T},\mu)$, then, for every $g \in L^1(\mathbb{T},\mu)$, $\int_{\mathbb{T}}(f_\lambda - f)gd\mu \to 0$. Since $\alpha' \ge \|cdot.\|_1$, $\mathbb{L}\alpha'(\mathbb{T},\mu) \subset L^1(\mathbb{T},\mu)$ and we have $f_\lambda \to f$in $\sigma(L^\alpha(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$ topology, so $f \in M$. Since $\mathbb{B}$ is weak* closed, $f \in \mathbb{B}$,thus $f \in W \cap \mathbb{B}$. Hence $M \cap \mathbb{B}$ is weak*-closed in $L^\infty(\mu)$.

For (2), since M is $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed linear subspace of $L^{\alpha}(\mathbb{T},\mu)$, it is clear that $M \supset \overline{M \cap L^{\infty}(\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))}$. Suppose $f \in M$ and let $k = \frac{1}{|f|+1}$. Then $k \in L^{\infty}(\mathbb{T},\mu)$, $k^{-1} \in L^{\alpha}(\mathbb{T},\mu)$. It follows from theorem 48 that there is an $s \in H^{\infty}(\mathbb{T},\mu), s^{-1} \in H^{\alpha}(\mathbb{T},\mu)$ and an unimodular function u such that $k = us$, so $sf = \overline{u}kf = \overline{u}\frac{f}{|f|+1} \in L^{\infty}(\mathbb{T},\mu)$. There is a sequence $\{s_n\}$ in $H^{\infty}(\mathbb{T},\mu)$ such that $s_n \to s^{-1}$ in $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$ topology. For each $n \in \mathbb{N}$, it follows from lemma 46 that $s_n sf \in H^{\infty}(\mu)H^{\infty}(\mu)M \subset M$ and $s_n sf \in H^{\infty}(\mu)L^{\infty}(\mu) \subset L^{\infty}(\mu)$, which implies that $\{s_n sf\}$ is a sequence in $M \cap L^{\infty}(\mu)$. For every $g \in L^{\alpha'(\mathbb{T})}$, $\int_{\mathbb{T}}(s_n sf - f)gd\mu = \int_{\mathbb{T}}(s_n - s^{-1})sfgd\mu$. Since $s_n \to s^{-1}$ in $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$ topology, and $sfg \in L^{\alpha'(\mathbb{T})}$, $\int_{\mathbb{T}}(s_n - s^{-1})sfgd\mu \to 0$. Thus $f \in \overline{M \cap L^{\infty}(\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))}$. Therefore $M = \overline{M \cap L^{\infty}(\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))}$.

□

Lemma 52. *A weak*-closed linear subspace M of $L^{\infty}(\mathbb{T},\mu)$ satisfies $zM \subset M$ if and only if $M = \varphi H^{\infty}(\mathbb{T},\mu)$ for some unimodular function φ or $M = \chi_E L^{\infty}(\mathbb{T},\mu)$, for some Borel subset E of $\mathbb{T}$.*

Theorem 53. *Suppose μ is Haar measure on $\mathbb{T}$ and tet α be a $\|\cdot\|_1$-dominating normalized gauge norm on $L^{\infty}(\mathbb{T},\mu)$. Let W be an $\sigma(L^{\alpha}(\mathbb{T}), L^{\alpha'}(\mathbb{T}))$-closed linear subspace of $L^{\alpha}(\mathbb{T})$ with $zW \subseteq W$ if and only if either $W = \varphi H^{\alpha}(\mu)$ for some unimodular function φ, or $W = \chi_E L^{\alpha}(\mu)$, for some Borel subset E of $\mathbb{T}$. If $0 \neq W \subset H^{\alpha}(\mu)$, then $W = \varphi H^{\alpha}(\mu)$ for some inner function φ.*

Proof. Let $M = W \cap L^{\infty}(\mathbb{T},\mu)$, it follows from the (1) in theorem 51 that M is weak* closed in $L^{\infty}(\mathbb{T},\mu)$. Since $zW \subset W$, it is easy to check that $zM \subset M$. Then by lemma 52, we can conclude that either $M = \varphi H^{\infty}(\mathbb{T},\mu)$ for some unimodular function φ or $M = \chi_E L^{\infty}(\mathbb{T},\mu)$, for some Borel subset E of $\mathbb{T}$. By the (2) in theorem 51, if $M = \varphi H^{\infty}(\mu)$, $W = \overline{W \cap L^{\infty}(\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} = \overline{M}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} = \overline{\varphi H^{\infty}(\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} = \varphi H^{\alpha}(\mathbb{T},\mu)$, for some unimodular function φ. If $M = \chi_E L^{\infty}(\mu)$, $W = \overline{W \cap L^{\infty}(\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} = \overline{M}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} = \overline{\chi_E L^{\infty}(\mu)}^{\sigma(L^{\alpha}(\mathbb{T}),L^{\alpha'}(\mathbb{T}))} = \chi_E L^{\alpha}(\mathbb{T},\mu)$, for some Borel subset E of $\mathbb{T}$. The proof is completed. □

4.4 The Extension of Beurling Theorem in Finite von Neumann Algebras

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Given a von Neumann subalgebra $\mathcal{D}$ of $\mathcal{M}$, a conditional expectation Φ: $\mathcal{M} \to \mathcal{D}$ is a positive linear map satisfying $\Phi(I) = I$ and $\Phi(x_1 y x_2) = x_1 \Phi(y) x_2$ for all $x_1, x_2 \in \mathcal{D}$ and $y \in \mathcal{M}$. There exists a unique conditional expectation Φ_τ: $\mathcal{M} \to \mathcal{D}$ satisfying $\tau \circ \Phi_\tau(x) = \tau(x)$ for every $x \in \mathcal{M}$. Now we recall noncommutative Hardy spaces $H^\infty(\mathcal{M}, \tau)$ in [3].

Definition 54. *Let $\mathcal{A}$ be a weak* closed unital subalgebra of $\mathcal{M}$, and let Φ_τ be the unique faithful normal trace preserving conditional expectation from $\mathcal{M}$ onto the diagonal von Neumann algebra $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Then $\mathcal{A}$ is called a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ with respect to Φ_τ if*

(1) $\mathcal{A} + \mathcal{A}^*$ is weak dense in $\mathcal{M}$,*

(2) $\Phi_\tau(xy) = \Phi_\tau(x)\Phi_\tau(y)$ for all $x, y \in \mathcal{A}$.

Such $\mathcal{A}$ will be denoted by $H^\infty(\mathcal{M}, \tau)$, and $\mathcal{A}$ is also called a noncommutative Hardy space.

Example 10. *Let $\mathcal{M} = L^\infty(\mathbb{T}, \mu)$, and $\tau(f) = \int f d\mu$ for all $f \in L^\infty(\mathbb{T}, \mu)$. Let $\mathcal{A} = H^\infty(\mathbb{T}, \mu)$, then $\mathcal{D} = H^\infty(\mathbb{T}, \mu) \cap H^\infty(\mathbb{T}, \mu)^* = \mathbb{C}$. Let Φ_τ be the mapping from $L^\infty(\mathbb{T}, \mu)$ onto $\mathbb{C}$ defined by $\Phi_\tau(f) = \int f d\mu$. Then $H^\infty(\mathbb{T}, \mu)$ is a finite, maximal subdiagonal subalgebra of $L^\infty(\mathbb{T}, \mu)$.*

Example 11. *Let $\mathcal{M} = \mathcal{M}_n(\mathbb{C})$ be with the usual trace τ. Let $\mathcal{A}$ be the subalgebra of lower triangular matrices, now $\mathcal{D}$ is the diagonal matrices and Φ_τ is the natural projection onto the diagonal matrices. Then $\mathcal{A}$ is a finite maximal subdiagonal subalgebra of $\mathcal{M}_n(\mathbb{C})$.*

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, Φ_τ be the conditional expectation and α be a normalized, unitarily invariant $\|\|_1$-dominating norm on $\mathcal{M}$. Let $L^\alpha(\mathcal{M}, \tau)$ be the α closure of $\mathcal{M}$,i.e., $L^\alpha(\mathcal{M}, \tau) = [\mathcal{M}]_\alpha$ and $(L^\alpha(\mathcal{M}, \tau))'$ be the dual space of $L^\alpha(\mathcal{M}, \tau)$, more details about the dual space of $L^\alpha(\mathcal{M}, \tau)$ is in [12]. Similarly, we define $H^\alpha(\mathcal{M}, \tau) = \overline{H^\infty(\mathcal{M}, \tau)}^{\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M}, \tau)$, $H_0^\infty(\mathcal{M}, \tau) = \ker(\Phi_\tau) \cap H^\infty(\mathcal{M}, \tau)$ and $H_0^\alpha(\mathcal{M}, \tau) = \ker(\Phi_\tau) \cap H^\alpha(\mathcal{M}, \tau)$.

Example 12. *Let* $\alpha = \|\cdot\|_p$*, then* $L^p(\mathcal{M},\tau) = [\mathcal{M}]_p$*,* $H^p(\mathcal{M},\tau) = \overline{H^\infty(\mathcal{M},\tau)}^{\sigma(L^p(\mathcal{M},\tau),L^q(\mathcal{M},\tau))} \cap L^p(\mathcal{M},\tau)$.

In [56], K. S. Saito characterized the noncommutative Hardy spaces $H^p(\mathcal{M},\tau)$ and $H_0^p(\mathcal{M},\tau)$.

Lemma 55. *Let* $\mathcal{M}$ *be a finite von Neumann algebra with a faithful, normal, tracial state* τ*, and then*

(1) $H^1(\mathcal{M},\tau) = \{x \in L^1(\mathcal{M},\tau) : \tau(xy) = 0 \text{ for all } y \in H_0^\infty\}$*,*

(2) $H_0^1(\mathcal{M},\tau) = \{x \in L^1(\mathcal{M},\tau) : \tau(xy) = 0 \text{ for all } y \in H^\infty\}$*,*

(3) $H_0^1(\mathcal{M},\tau) = \{x \in H^1(\mathcal{M},\tau) : \Phi_\tau(xh) = 0\}$.

Theorem 56. *Let* $\mathcal{M}$ *be a finite von Neumann algebra with a faithful, normal, tracial state* τ *and* α *be a normalized, unitarily invariant* $\|\cdot\|_{1,\tau}$*-dominating norm on* $\mathcal{M}$*. Let* H^∞ *be a finite subdiagonal subalgebra of* $\mathcal{M}$*. Then there exists a faithful normal tracial state* τ *such that* $H^\alpha(\mathcal{M},\tau) = H^1(\mathcal{M},\tau) \cap L^\alpha(\mathcal{M},\tau)$.

Proof. By the definition of $H^\alpha(\mathcal{M},\tau)$, we have $H^\alpha(\mathcal{M},\tau) = \overline{H^\infty(\mathcal{M},\tau)}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau) \subseteq L^\alpha(\mathcal{M},\tau)$. For every $x \in H^\alpha(\mathcal{M},\tau) = \overline{H^\infty(\mathcal{M},\tau)}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau)$, there exists a net x_n in $H^\infty(\mathcal{M},\tau)$ such that $x_n \to x$ in $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ topology. Since $x_n \in H^\infty(\mathcal{M},\tau) \subseteq H^1(\mathcal{M},\tau)$, $\tau(x_n y) = 0$ for all $y \in H^\infty(\mathcal{M},\tau), \Phi(y) = 0$. We know $x_n \to x$ in $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ topology, so $\forall y \in H^\infty(\mathcal{M},\tau), \tau(x_n y) \to \tau(xy)$. Therefore, for all $y \in H^\infty(\mathcal{M},\tau), \Phi(y) = 0, \tau(xy) = 0$. Thus $H^\alpha(\mathcal{M},\tau) \subseteq H^1(\mathcal{M},\tau)$. Therefore, $H^\alpha(\mathcal{M},\tau) \subseteq H^1(\mathcal{M},\tau) \cap L^\alpha(\mathcal{M},\tau)$.

Next, we show that $H^\alpha(\mathcal{M},\tau) = H^1(\mathcal{M},\tau) \cap L^\alpha(\mathcal{M},\tau)$.

Assume, via contradiction, that $H^\alpha(\mathcal{M},\tau) \subsetneq H^1(\mathcal{M},\tau) \cap L^\alpha(\mathcal{M},\tau)$. By the Hahn-Banach theorem, there is a $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$-continuous functional Φ on $L^\alpha(\mathcal{M},\tau)$ and $x \in H^1(\mathcal{M},\tau) \cap L^\alpha(\mathcal{M},\tau)$ such that $\Phi(x) = 0$ and $\Phi(y) = 0$ for $\forall y \in H^\alpha(\mathcal{M},\tau)$. Since $\xi \in L^{\alpha'}(\mathcal{M},\tau)$ such that $\Phi(z) = \tau(z\xi), \forall z \in L^\alpha(\mathcal{M},\tau)$, we have $\Phi(y) = \tau(y\xi), \forall y \in H^\alpha(\mathcal{M},\tau) \subseteq L^\alpha(\mathcal{M},\tau)$. Because $\xi \in L^{\alpha'}(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau)$ and $\Phi(y) = \tau(y\xi), \forall y \in H^\infty(\mathcal{M},\tau)$, $\xi \in H^1(\mathcal{M},\tau)_0$.

Since $x \in H^1(\mathcal{M},\tau), \tau(x\xi_n) = 0, \forall \xi_n \in H^\infty(\mathcal{M},\tau)_0$. There exists a net $\xi_n \in H^\infty(\mathcal{M},\tau)_0$ such that $\xi_n \to \xi$ in $\|\|_1$ topology, so $\tau(\xi_n) \to \tau(\xi)$. By lemma 3.4 in [12], $\tau(x\xi_n) \to \tau(x\xi)$. Therefore, $\Phi(x) = \tau(x\xi) = 0$, which contradicts $\Phi(x) = 0$. Thus, $H^\alpha(\mathcal{M},\tau) = H^1(\mathcal{M},\tau) \cap L^\alpha(\mathcal{M},\tau)$. □

Theorem 57. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, $\|\|_1$-dominating norm on $\mathcal{M}$. Let H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$. If $k \in \mathcal{M}$ and $k^{-1} \in L^\alpha(\mathcal{M},\tau)$, then there are unitary operators $u_1, u_2 \in \mathcal{M}$ and $s_1, s_2 \in H^\infty$ such that $k = u_1 s_1 = s_2 u_2$ and $s_1^{-1}, s_2^{-1} \in H^\alpha(\mathcal{M},\tau)$.*

Proof. Suppose $k \in \mathcal{M}$ with $k^{-1} \in L^\alpha(\mathcal{M},\tau)$. Assume that $k = v|k|$ is the polar decomposition of k in $\mathcal{M}$, where v is a unitary in $\mathcal{M}$. Then from the assumption that $k^{-1} = |k|^{-1} v^*$, so we have $|k|^{-1} \in L^\alpha(\mathcal{M},\tau) \subset L^1(\mathcal{M},\tau)$. Since $|k|$ in $\mathcal{M}$ positive, we have $|k|^{-\frac{1}{2}} \in L^2(\mathcal{M},\tau)$ and $|k|^{\frac{1}{2}} \in \mathcal{M}$. There exists a unitary operator$u_1 \in \mathcal{M}$and $s_1 \in H^\infty$ such that

□

The following density theorem also plays an important role in the proof of our main result of the chapter.

Theorem 58. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, $\|\|_1$-dominating norm on $\mathcal{M}$. If $\mathcal{W}$ is a $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$-closed subspace of $L^\alpha(\mathcal{M},\tau)$ and $\mathcal{N}$ is a weak* closed linear subspace of $\mathcal{M}$ such that $\mathcal{W}H^\infty \subset \mathcal{W}$ and $\mathcal{N}H^\infty \subset \mathcal{N}$, then*
(1) $\mathcal{N} = \overline{\mathcal{N}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$,
(2) $\mathcal{W} \cap \mathcal{M}$ is weak closed in $\mathcal{M}$,*
(3) $\mathcal{W} = \overline{\mathcal{W} \cap \mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$,
(4) If $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S}H^\infty \subset \mathcal{S}$, then $\overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} = \overline{\overline{\mathcal{S}}^{w*}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$, where $\overline{\mathcal{S}}^{w*}$ is the weak-closure of $\mathcal{S}$ in $\mathcal{M}$.*

Proof. For (1), it is clear that $\mathcal{N} \subseteq \overline{\mathcal{N}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$. Assume, via contradiction, that $\mathcal{N} \subsetneq \overline{\mathcal{N}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$. Note that $\mathcal{N}$ is a weak* closed linear subspace of $\mathcal{M}$ and $L^1(\mathcal{M},\tau)$ is the predual space of $(\mathcal{M},\tau)$. It follows from the Hahn-Banach theorem that there exist a $\xi \in$

$L^1(\mathcal{M},\tau)$ and an $x \in \overline{\mathcal{N}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$ such that

(a) $\tau(\xi x) \neq 0$ and (b) $\tau(\xi y) = 0$ for all $y \in \mathcal{N}$.

We claim that there exists a $z \in \mathcal{M}$ such that

(a′) $\tau(zx) \neq 0$ and (b′) $\tau(zy) = 0$ for all $y \in \mathcal{N}$. Actually assume that $\xi = |\xi^*|v$ is the polar decomposition of $\xi \in L^1(\mathcal{M},\tau)$, where v is a unitary element in $\mathcal{M}$ and $|\xi^*|$ is in $L^1(\mathcal{M},\tau)$ is positive. Let f be a function on $[0,\infty)$ defined by the formula $f(t) = 1$ for $0 \leq t \leq 1$ and $f(t) = 1/t$ for $t > 1$. We define $k = f(|\xi^*|)$ by the functional calculus. Then by the construction of f, we know that $k \in \mathcal{M}$ and $k^{-1} = f^{-1}(|\xi^*|) \in L^1(\mathcal{M},\tau)$. It follows from theorem 57 that there exist a unitary operator $u \in \mathcal{M}$ and $s \in H^\infty$ such that $k = us$ and $s^{-1} \in H^1(\mathcal{M},\tau)$. Therefore, we can further assume that $\{t_n\}_{n=1}^\infty$ is a sequence of elements in H^∞ such that $\|s^{-1} - t_n\|_{1,\tau} \to 0$. Observe that

(i) Since s, t_n are in H^∞, for each $y \in \mathcal{N}$ we have that $yt_ns \in \mathcal{N}H^\infty \subseteq \mathcal{N}$ and $\tau(t_ns\xi y) = \tau(\xi y t_n s) = 0$,

(ii) We have $s\xi = (u * u)s(|\xi^*|v) = u * (k|\xi^*|)v \in \mathcal{M}$, by the definition of k,

(iii) From (a) and (i), we have $0 \neq \tau(\xi x) = \tau(s^{-1}s\xi x) = \lim_{n\to\infty} \tau(t_ns\xi x)$.

Combining (i), (ii) and (iii), we are able to find an $N \in \mathbb{Z}$ such that $z = t_N s\xi \in \mathcal{M}$ satisfying

(a′) $\tau(zx) \neq 0$ and (b′) $\tau(zy) = 0$ for all $y \in \mathcal{N}$.

Recall that $x \in \overline{\mathcal{N}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. Then there is a sequence $\{x_n\} \subseteq \mathcal{N}$ such that $x_n \to x$ in $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ topology. We have

$|\tau(zx_n) - \tau(zx)| \to 0.$

Combining with (b′) we conclude that $\tau(zx) = \lim_{n\to\infty} \tau(zx_n) = 0$. This contradicts with the result (a′). Therefore, $\mathcal{N} = \overline{\mathcal{N}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$.

For (2), let $\overline{\mathcal{W} \cap \mathcal{M}}^{w*}$ be the weak*-closure of $\mathcal{W} \cap \mathcal{M}$ in $\mathcal{M}$. In order to show that $\mathcal{W} \cap \mathcal{M} = \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$, it suffices to show that $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subseteq \mathcal{W}$. Assume, to the contrary, that $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \nsubseteq \mathcal{W}$. Thus there exists an element x in $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subset \mathcal{M} \subseteq L^\alpha(\mathcal{M},\tau)$, but $x \notin \mathcal{W}$. Since $\mathcal{W}$ is a $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$-closed subspace of $L^\alpha(\mathcal{M},\tau)$, by the Hahn-Banach theorem, there exists a $\xi \in L^1(\mathcal{M},\tau)$ such that $\tau(\xi x) \neq 0$ and $\tau(\xi y) = 0$ for all $y \in \mathcal{W}$. Since $\xi \in L^1(\mathcal{M},\tau)$, the linear mapping $\tau_\xi : \mathcal{M} \to \mathbb{C}$, defined by $\tau_\xi(a) = \tau(\xi a)$ for all $a \in \mathcal{M}$ is weak*-continuous. Note that $x \in \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$ and $\tau(\xi y) = 0$ for all $y \in \mathcal{W}$. We know that $\tau(\xi x) = 0$, which contradicts with the assumption that $\tau(\xi x) \neq 0$. Hence $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subseteq \mathcal{W}$, so $\mathcal{W} \cap \mathcal{M} = \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$.

For (3), since $\mathcal{W}$ is $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$-closed, we have $\overline{\mathcal{W} \cap \mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \subseteq \mathcal{W}$. Now we assume $\overline{\mathcal{W} \cap \mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \subsetneq \mathcal{W} \subseteq L^\alpha(\mathcal{M},\tau)$. By the Hahn-Banach theorem, there exists an $x \in \mathcal{W}$ and $\xi \in L^1(\mathcal{M},\tau)$ such that $\tau(\xi x) \neq 0$ and $\tau(\xi y) = 0$ for all $y \in \overline{\mathcal{W} \cap \mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. Let $x = v|x|$ be the polar decomposition of x in $L^\alpha(\mathcal{M},\tau)$, where v is a unitary element in $\mathcal{M}$. Let f be a function on $[0,\infty)$ defined by the formula $f(t) = 1$ for $0 \leq t \leq 1$ and $f(t) = 1/t$ for $t > 1$. We define $k = f(|x|)$ by the functional calculus. Then by the construction of f, we know that $k \in \mathcal{M}$ and $k^{-1} = f^{-1}(|x|) \in L^\alpha(\mathcal{M},\tau)$. It follows from theorem 57 that there exist a unitary operator $u \in \mathcal{M}$ and $s \in H^\infty$ such that $k = su$ and $s^{-1} \in H^\alpha(\mathcal{M},\tau)$. A little computation shows that $|x|k \in \mathcal{M}$ which implies that $xs = xsuu^* = xku^* = v(|x|k)u^* \in \mathcal{M}$. Since $s \in H^\infty$, we know $xs \in \mathcal{W}H^\infty \subseteq \mathcal{W}$ and thus $xs \in \mathcal{W} \cap \mathcal{M}$. Furthermore, note that $(\mathcal{W} \cap \mathcal{M})H^\infty \subseteq \mathcal{W} \cap \mathcal{M}$. Thus, if $t \in H^\infty$ we see $xst \in \mathcal{W} \cap \mathcal{M}$, and $\tau(\xi xst) = 0$. Since $H^\alpha(\mathcal{M},\tau) = \overline{H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau)$, $\forall \in H^\alpha(\mathcal{M},\tau)$ and there is a net t_n in H^∞ such that $t_n \to t$ in $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ topology. We have $\xi xs \in L^{\alpha'}(\mathcal{M},\tau)$ because $\alpha'(\xi xs) \leq \alpha'(\xi)\|xs\|$. Therefore, $\tau(\xi xst_n) \to \tau(\xi xst)$, which follows that $\tau(\xi xst) = 0$ for all $t \in H^\alpha(\mathcal{M},\tau)$. Since $s^{-1} \in H^\alpha(\mathcal{M},\tau)$, we see that $\tau(\xi x) = \tau(\xi xss^{-1}) = 0$. This contradicts with the assumption that $\tau(\xi x) \neq 0$. Therefore $\mathcal{W} = \overline{\mathcal{W} \cap \mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$.

For (4), assume that $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S}H^\infty \subset \mathcal{S}$ and $\overline{\mathcal{S}}^{w*}$ is weak*-closure of $\mathcal{S}$ in $\mathcal{M}$. Then $\overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} H^\infty \subseteq \overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. Note that $\mathcal{S} \subseteq \overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$. From (2), we know that $\overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$ is weak*-closed. Therefore, $\overline{\mathcal{S}}^{w*} \subseteq \overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}$.
Since $\overline{\overline{\mathcal{S}}^{w*}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \subseteq \overline{\overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap \mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} = \overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$, we have $\overline{\mathcal{S}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} = \overline{\overline{\mathcal{S}}^{w*}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. $\square$

Before we obtain our main result in the chapter, we call the definitions of internal column sum of a family of subspaces, and the lemma in [6].

Definition 59. *(from [6]) Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, $\|\centerdot\|_1$-dominating norm. Suppose X be a closed subspace of $L^\alpha(\mathcal{M},\tau)$ with $\alpha \in N_\Delta(\mathcal{M},\tau)$. Then X is called an internal column sum of a family of closed subspaces $\{X_\lambda\}_{\lambda\in\Lambda}$ of $L^\alpha(\mathcal{M},\tau)$, denoted by $X = \bigoplus_{\lambda\in\Lambda}^{col} X_\lambda$ if*

(1) $X_\mu^* X_\lambda = \{0\}$ for all distinct $\lambda, \mu \in \Lambda$, and

(2) $X = \overline{span\{X_\lambda : \lambda \in \Lambda\}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$.

Lemma 60. *(from [6]) Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant $\|\cdot\|_1$-dominating norm on $\mathcal{M}$. Let H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{W} \subseteq \mathcal{M}$ is a weak*-closed subspace such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$. Then there exists a weak*-closed subspace $\mathcal{Y}$ of $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$of partial isometries in $\mathcal{M}$ such that*

(1) $u_\lambda^*\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,

(2) $u_\lambda^* u_\lambda \in \mathcal{D}$ and $u_\lambda^* u_\mu = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,

(3) $\mathcal{Y} = \overline{H_0^\infty \mathcal{Y}}^{w*}$,

(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus_{\lambda\in\Lambda}^{col} u_\lambda H^\infty)$.

Now we are ready to prove our main result of the chapter, the generalized Beurling Theorem for noncommutative Hardy spaces associated with finite von Neumann algebras.

Theorem 61. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ and α be a normalized, unitarily invariant, $\|\|_1$-dominating norm on $\mathcal{M}$.Let H^∞ be a finite subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$, then there exists a $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ closed subspace $\mathcal{Y}$ of $L^\alpha(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that*

(1) $u_\lambda^*\mathcal{Y} = 0$ for all $\lambda \in \Lambda$,

(2) $u_\lambda^* u_\lambda \in \mathcal{D}$ and $u_\lambda^* u_\mu = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$,

(3) $\mathcal{Y} = \overline{H_0^\infty \mathcal{Y}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$,

(4) $\mathcal{W} = \mathcal{Y} \oplus^{col} (\oplus^{col}_{\lambda\in\Lambda} u_\lambda H^\alpha)$.

Proof. Suppose $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{W}H^\infty \subset \mathcal{W}$. Then it follows from part(2) of the theorem 58 that $\mathcal{W}\cap\mathcal{M}$ is weak* closed in $(\mathcal{M},\tau)$, we also notice $L^\infty(\mathcal{M},\tau) = \mathcal{M}$, and $H^\alpha(\mathcal{M},\tau) = \overline{H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau)$. It follows from the lemma 60 that

$$\mathcal{W}\cap\mathcal{M} = \mathcal{Y}_1 \bigoplus^{col}(\bigoplus^{col}_{i\in\mathcal{I}} u_i H^\infty),$$

where $\mathcal{Y}_1$ is a closed subspace of $L^\infty(\mathcal{M},\tau)$ such that $\mathcal{Y}_1 = \overline{\mathcal{Y}_1 H_0^\infty}^{w*}$, and where u_i are partial isometries in $\mathcal{W}\cap\mathcal{M}$ with $u_j^* u_i = 0$ if $i \neq j$ and with $u_i^* u_i \in \mathcal{D}$. Moreover, for each i, $u_i^*\mathcal{Y}_1 = \{0\}$, left multiplication by the $u_i u_i^*$ are contractive projections from $\mathcal{W}\cap\mathcal{M}$ onto the summands $u_i H^\infty$, and left multiplication by $I - \sum_i u_i u_i^*$ is a contractive projection from $\mathcal{W}\cap\mathcal{M}$ onto $\mathcal{Y}_1$.

Let $\mathcal{Y} = \overline{\mathcal{Y}_1}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. It is not hard to verify that for each i, $u_i^*\mathcal{M} = \{0\}$. We also claim that $\overline{u_i H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau) = u_i H^\alpha = u_i(\overline{H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau))$. In fact it is obvious that $\overline{u_i H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau) \supseteq u_i H^\alpha$. We will need only to show that $\overline{u_i H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau) \subseteq u_i H^\alpha$. Suppose $x \in \overline{u_i H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau)$, there is a net $\{x_n\}_{n=1}^\infty \subseteq H^\infty$ such that $u_i x_n \to x$ in $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ topology. By the choice of u_i, we know that $u_i^* u_i \in \mathcal{D} \subseteq H^\infty$, so $u_i^* u_i x_n \in H^\infty$ for each $n \geq 1$. So $u_i^* u_i x_n \to u_i^* x$ in $\sigma(L^\alpha(\mathcal{M},\tau), L^{\alpha'}(\mathcal{M},\tau))$ topology, we obtain that $u_i^* x \in \overline{H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \cap L^\alpha(\mathcal{M},\tau) = H^\alpha(\mathcal{M},\tau)$. Again from the choice of u_i, we know that $u_i u_i^* u_i x_n = u_i x_n$ for each $n \geq 1$. This implies that $x = u_i(u_i^* x) \in u_i H^\alpha$. Thus we conclude that $\overline{u_i H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \subseteq u_i H^\alpha$. So $\overline{u_i H^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} = u_i H^\alpha$. Now from parts (3) and (4) of the theorem 58 and from the definition of internal column sum, it follows that

$$\begin{aligned}\mathcal{W} &= \overline{\mathcal{W}\cap\mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} = \overline{\overline{span\{\mathcal{Y}_1, u_iH^\infty : i\in\mathcal{I}\}}^{*}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \\ &= \overline{span\{\mathcal{Y}_1, u_iH^\infty : i\in\mathcal{I}\}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \\ &= \overline{span\{\mathcal{Y}, u_iH^\alpha : i\in\mathcal{I}\}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \\ &= \mathcal{Y}\bigoplus^{col}(\bigoplus^{col}_{i\in\mathcal{I}} u_iH^\alpha).\end{aligned}$$

Next, we will verify that $\mathcal{Y} = \overline{\mathcal{Y}H_0^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. Recall that $\mathcal{Y} = \overline{\mathcal{Y}_1}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. It follows from part (1) of the theorem 58, we have

$$\overline{\mathcal{Y}_1H_0^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}\cap\mathcal{M} = \overline{\mathcal{Y}_1H_0^\infty}^{w*} = \mathcal{Y}_1.$$

Hence from part (3) of the theorem 58 we have that

$$\begin{aligned}\mathcal{Y} &\supseteq \overline{\mathcal{Y}H_0^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \supseteq \overline{\mathcal{Y}_1H_0^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \\ &= \overline{\overline{\mathcal{Y}_1H_0^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}\cap\mathcal{M}}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \\ &= \overline{\mathcal{Y}_1}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))} \\ &= \mathcal{Y}.\end{aligned}$$

Thus $\mathcal{Y} = \overline{\mathcal{Y}H_0^\infty}^{\sigma(L^\alpha(\mathcal{M},\tau),L^{\alpha'}(\mathcal{M},\tau))}$. Moreover, it is not difficult to verify that for each i, left multiplication by the $u_iu_i^*$ are contractive projections from $\mathcal{K}$ onto the summands u_iH^α, and left multiplication by $I-\sum_i u_iu_i^*$ is a contractive projection from $\mathcal{W}$ onto $\mathcal{Y}$. Now the proof is completed. □

CHAPTER 5

A GENERALIZED BEURLING THEOREM FOR HARDY SPACES ON MULTIPLY CONNECTED DOMAINS

In [11], Yanni Chen, Don Hadwin, Zhe Liu and Eric Nordgren proved a version of the Beurling-Helson-Lowdenslager invariant subspace theorem for operators on certain Banach spaces of functions on multiply connected domains in $\mathbb{C}$. The norms for these spaces are either the usual Lebesgue and Hardy space norms or certain continuous gauge norms. In this chapter, we study the norms for Hardy spaces on a multiply connected domain are gauge norms without $\| \|_1$-dominating, then characterize the Hardy spaces for such norms. Finally we prove that the generalized Beurling theorem holds in this context. It also extended the results in chapter 2 from simple connected domains to multiply connected domains. The key ingredients in the proof of our result include a factorization theorem and a density theorem.

5.1 Introduction

Let Ω be a finitely connected domain in $\mathbb{C}$ with analytic boundary curves Γ. The Lebesgue spaces are defined relative to harmonic measure ω corresponding to an arbitrarily chosen point $\widehat{w}$ in Ω. The domain Ω has an analytic covering map τ from the unit disk $\mathbb{D}$ onto Ω which induces a measure preserving transformation from the unit circle $\mathbb{T}$ onto Γ, and consequently an isometric composition operator C_τ from the Lebesgue space $L^p(\Gamma, \omega)$ for $1 \leq p \leq \infty$ into its counterpart $L^p(\mathbb{T}, m)$ on the circle, where m is normalized Lebesgue measure on $\mathbb{T}$. The Hardy spaces on Ω were introduced by Parreau [50] and Rudin [52] as consisting of analytic functions f with $|f|^p$ dominated by some harmonic function. As on the disk, these functions have boundary limits, and hence the spaces $H^p(\Omega)$ may be identified with isometrically isomorphic subspaces $H^p(\Gamma)$

of $L^p(\Gamma, \omega)$. More background about Hardy spaces on multiply connected domains seen in [11], [63],[24] and [25]. Also, In [51] and [47] , we can find more information about harmonic functions, periods, and harmonic conjugate.

In [5], [40], they got different versions of the Beurling theorems in this context. Later Sarason [58], Hasumi [38], Voichick [65], and Rudol [54] Further studied the area, although by using Royden's definition of inner function (see [51]), there are the simpler more traditional form of theorems (see Theorems 75). Their version is modeled on the one obtained by Royden [51] for Hardy spaces on a multiply connected domain. It describes the invariant subspaces of the set of all multiplication operators induced by bounded analytic functions but does not address the more difficult question of the invariant subspaces of "multiplication by z" alone which was attacked by Royden [51], Hitt [42], and Aleman and Richter [1, 2]. In [11], Chen, Hadwin, Liu and Nordgren proved a version of the Beurling-Helson-Lowdenslager invariant subspace theorem for operators on certain Banach spaces of functions on a multiply connected domain in $\mathbb{C}$. The norms for these spaces are either the usual Lebesgue and Hardy space norms or certain continuous gauge norms.

The Hardy space theory of the unit circle has been extended by W. Liu, H.Fan and D. Hadwin in chapter 2 and the first author [9, 10]by considering norms that are more general than the Lebesgue norms. We further study the gauge norms in section 2. In addition to the continuous gauge norms α on $L^\infty(\Gamma, \omega)$ with $\|\|_1$-dominating property, we consider more general continous gauge norms α without $\|\|_1$-dominating property. This leads us to general Lebesgue spaces $L^\alpha(\Gamma, \omega)$ and Hardy spaces $H^\alpha(\Gamma)$ where we obtain a general Beurling theorem (see Theorem 75) in Section 3 by using a slight modification of the proof of the first author in [10] (see also [26, 27]).

5.2 Gauge norms on Γ

In [9] the first author introduced the study of Hardy spaces on $\mathbb{T}$ under a family of norms that properly includes the p-norms. You also see more information about gauge norms in section 2 in chapter 2. Since our interest is in the space Γ with the measure ω, we will introduce norms of this type in a more general setting. Let ω be a nonatomic probability measure on a σ-algebra in a set Ω, and let α be a norm on $L^\infty(\Omega,\omega)$. Suppose (Ω,Σ,ω) is a probability space. A norm α on $L^\infty(\Omega,\omega)$ is a *normalized gauge norm* if

1. $\alpha(1)=1$,
2. $\alpha(|f|)=\alpha(f)$ for every $f\in L^\infty(\Omega,\omega)$.

In addition we say α is *continuous* (*ω-continuous*) if

$$\lim_{\omega(E)\to 0}\alpha(\chi_E)=0,$$

that is, whenever $\{E_n\}$ is a sequence in Σ and $\omega(E_n)\to 0$, we have $\alpha(\chi_{E_n})\to 0$.

We say that a *normalized gauge norm* α is *$c\|\cdot\|_{1,\omega}$-dominating* for some $c>0$ if

$$\alpha(f)\geq c\|f\|_{1,\omega},\ \text{for every } f\in L^\infty(\Omega,\omega).$$

A gauge norm α may be extended to all measurable complex functions f on Ω by

$$\alpha(f)=\sup\left\{\alpha(s): s \text{ is a simple function and } 0\leq s\leq |f|\right\}.$$

Let $\mathcal{L}^\alpha(\Omega,\omega)$ consist of all measurable functions f such that $\alpha(f)<\infty$. If α is a continuous dominating gauge norm on $L^\infty(\Omega,\omega)$, then its extension to $\mathcal{L}^\alpha(\Omega,\omega)$ has the same properties. The space $\mathcal{L}^\alpha(\Omega,\omega)$ is a Banach space, and we define $L^\alpha(\Omega,\omega)$ to be the closure of $L^\infty(\Omega,\omega)$ in $\mathcal{L}^\alpha(\Omega,\omega)$.

Let α be a dominating, gauge norm, and define its *dual norm* α' on $L^\infty(\Omega,\omega)$ by

$$\alpha'(f)=\sup\left\{\left|\int_\Omega fh\,\mathrm{d}\mu\right| : h\in L^\infty(\Omega,\omega) \text{ and } \alpha(h)\leq 1\right\}.$$

The following are Lemma 2.6 and Proposition 2.7 of [9].

Lemma 62. *The dual norm α' of a dominating gauge norm α is also a dominating gauge norm.*

Proposition 3. *Suppose α' is the dual norm of a dominating gauge norm α on $L^\infty(\Omega,\omega)$. The dual space $\left(L^\alpha(\Omega,\omega)\right)^\#$ is $\mathcal{L}^{\alpha'}(\Omega,\omega)$ in the sense that if φ is a continuous linear functional on $L^\alpha(\Omega,\omega)$, then there exists a unique $F \in \mathcal{L}^{\alpha'}(\Omega,\omega)$ satisfying $\|\varphi\| = \alpha'(F)$ such that for all $f \in L^\alpha(\Omega,\omega)$, $fF \in L^1(\Omega,\omega)$ and*

$$\varphi(f) = \int_\Omega fF d\omega.$$

Throughout the rest of the chapter, without explicit assumption to the contrary, α will be assumed to be a continuous, normalized gauge norm on $L^\alpha(\Gamma,\omega)$. The set of these norms constitute a set that we will label $\mathfrak{N}$. Also $\mathfrak{N}_\infty$ will be $\mathfrak{N}$ with the essential supremum norm adjoined.

Hardy spaces in this context are obtained by defining $H^\alpha(\Gamma)$ to be the subspace of $L^\alpha(\Gamma,\omega)$ obtained by taking the α-norm closure of $H^\infty(\Gamma)$. Since $L^\alpha(\Gamma,\omega)$ is a closed subspace of $L^1(\Gamma,\omega)$, $H^\alpha(\Gamma)$ is a closed subspace of $H^1(\Gamma)$. Thus we may define $H^\alpha(\Omega)$ as the subspace of $H^1(\Omega)$ consisting of those functions whose boundary functions are in $H^\alpha(\Gamma)$. For $f \in H^\alpha(\Omega)$ and $w \in \Omega$ we have $f(w) = \int_\Gamma f \, \mathrm{d}\omega_w$, and since ω_w is boundedly absolutely continuous with respect to ω, it follows from the dominating property that point evaluations are continuous linear functionals on $H^\alpha(\Omega)$ and by extension on $H^\alpha(\Gamma)$. Thus $H^\alpha(\Omega)$ is a functional Banach space, and provides an equivalent but different view to $H^\alpha(\mathbb{T})$.

Since $L^\infty(\Omega,\omega)$ with the norm α is dense in $L^\alpha(\Omega,\omega)$, they have the same dual spaces. We prove in the next lemma that the normed dual $(L^\alpha(\Omega,\omega),\alpha)^\# = (L^\infty(\Omega,\omega),\alpha)^\#$ can be viewed as a vector subspace of $L^1(\Omega,\omega)$. Suppose $w \in L^1(\Omega,\omega)$, we define the functional $\varphi_w : L^\infty(\Omega,\omega) \to \mathbb{C}$ by

$$\varphi_w(f) = \int_\Omega fw d\omega.$$

The following Lemma and two Theorems from the chapter 2.

Lemma 63. *Suppose (Ω,Σ,ω) is a probability space and α is a continuous normalized gauge norm on $L^\infty(\Omega,\omega)$. Then*

(1) if $\varphi : L^\infty(\Omega,\omega) \to \mathbb{C}$ is an α-continuous linear functional, then there is a $w \in L^1(\Omega,\omega)$ such

that $\varphi = \varphi_w$,

(2) if φ_w is α-continuous on $L^\infty(\Omega, \omega)$, then

(a) $\|w\|_{1,\omega} \leq \|\varphi_w\| = \left\|\varphi_{|w|}\right\|$,

(b) given φ in the dual of $L^\alpha(\Omega, \omega_2)$, i.e., $\varphi \in (L^\alpha(\Omega, \omega_2))^\#$, there exists a $w \in L^1(\Omega, \omega_2)$, such that

$$\forall f \in L^\infty(\Omega, \omega_2), \varphi(f) = \int_\Omega f w d\omega_2 \text{ and } wL^\alpha(\Omega, \omega_2) \subseteq L^1(\Omega, \omega_2).$$

Theorem 64. *Suppose $(\Omega, \Sigma, \omega_1)$ is a probability space and α is a continuous normalized gauge norm on $L^\infty(\Omega, \Sigma, \omega_1)$ and $\varepsilon > 0$. Then there exists a constant c with $1 - \varepsilon < c \leq 1$ and a probability measure ω_2 on Σ that is mutually absolutely continuous with respect to ω_1 such that α is $c\|\cdot\|_{1,\omega_2}$-dominating.*

If we take $\Omega = \Gamma$, Theorem 64 holds for the probability space $(\Omega, \omega) = (\Gamma, \omega)$. We can check the L^p-version of the Helson-Lowdenslager theorem also holds, in a sense, on Γ when ω_1 is replaced with a mutually absolutely continuous probability measure ω_2. Here the role of $H^p(\Gamma, \omega_2)$ is replaced with $\left(1/g^{\frac{1}{p}}\right) H^p(\Gamma, \omega_1)$. This result is well-known as the following corollary in [9].

Corollary 65. *Suppose ω_2 is a probability measure on Γ and $\omega_1 << \omega_2$ and $\omega_2 << \omega_1$. Let $g = d\omega_2/d\omega_1$ and suppose $1 \leq p < \infty$. Suppose W is a closed subspace of $L^p(\Gamma, \omega_2)$, and $zW \subset W$. Then $g^{\frac{1}{p}}W = \chi_E L^1(\Gamma, \omega_1)$ for some Borel subset E of Γ or $g^{\frac{1}{p}}W = \varphi H^p(\Gamma, \omega_1)$ for some unimodular function φ.*

Theorem 66. *Suppose ω_1 and ω_2 are probability measures on Σ and they are mutually absolutely continuous. i.e, $\omega_1 << \omega_2$ and $\omega_2 << \omega_1$. Let $g = d\omega_2/d\omega_1$ and suppose $1 \leq p < \infty$. Suppose W is a closed subspace of $L^p(\Omega, \Sigma, \omega_1)$ and that is invariant under M_ψ for every $\psi \in H^\infty(\Gamma)$. Then either*

(1) $g^{\frac{1}{p}}W = \chi_E L^p(\Gamma, \omega_1)$ for some Borel subset E of Γ or

(2) $g^{\frac{1}{p}}W = \varphi H^p(\Gamma, \omega_1)$ for some unimodular function $\varphi \in H^\infty(\Gamma)$ such that $|\varphi|$ is constant on each of the components of Γ.

Suppose α is a continuous normalized gauge norm on $L^{\infty}(\Gamma,\omega_1)$. suppose that $c>0$ and ω_2 is a probability measure on Γ such that $\omega_1 << \omega_2$ and $\omega_2 << \omega_1$ and α is $c\|\cdot\|_{1,\omega_2}$-dominating. We let $g = d\omega_2/d\omega_1$ and $g>0$. We consider two cases

(1) $\int |\log g|\, d\omega_1 < \infty$,

(2) $\int |\log g|\, d\omega_1 = \infty$.

We define $L^p(\Gamma,\omega_2)$ to be the $\|\cdot\|_{p,\omega_2}$-closure of $L^{\infty}(\Gamma,\omega_2)$ and define $H^p(\Gamma,\omega_2)$ to be $\|\cdot\|_{p,\omega_2}$-closure of the polynomials for $1 \le p < \infty$. Denote $L^{\infty}(\Gamma,\omega_1) = L^{\infty}(\omega_1)$, $L^p(\Gamma,\omega_1) = L^p(\omega_1)$ and $H^p(\Gamma,\omega_1) = H^p(\omega_1)$.

Lemma 67. *The following are true:*

(1) $\int |\log g|\, d\omega_1 < \infty \Leftrightarrow$ *there is an outer function* $h \in H^1(\omega_1)$ *with* $|h| = g$*,*

(2) $\int |\log g|\, d\omega_1 = \infty \Leftrightarrow H^1(\omega_2) = L^1(\omega_2)$.

Proof. Clearly $H^1(\omega_2)$ is a closed z-invariant subspace of $L^1(\omega_2)$. Thus, by corollary 65, either $gH^1(\omega_2) = \varphi H^1(\omega_1)$ for some unimodular φ or $gH^1(\omega_2) = \chi_E L^1(\omega_1)$ for some Borel set $E \subset \mathbb{T}$.

For (1), if $gH^1(\omega_2) = \varphi H^1(\omega_1)$ for some unimodular φ, and $0 < g \in gH^1(\omega_2)$, then $0 \neq \overline{\varphi} g \in H^1(\omega_1)$ which implies $\log g = \log|\overline{\varphi} g| \in L^1(\omega_1)$. It is a standard fact that if $g>0$ and $\log g$ are in $L^1(\omega_1)$, then there exists an outer function $h \in H^1(\omega_1)$ with the same modulus as g,(i.e., $|h| = g$). Therefore, (1) is proved by Lemma 3.2 in [10].

For (2), Since $gH^1(\omega_2) = \varphi H^1(\omega_1)$ if and only if $\int |\log g|\, d\omega_1 < \infty$. Suppose $\int |\log g|\, d\omega_1 = \infty$. Then $gH^1(\omega_2) = \chi_E L^1(\omega_1)$. We have $g = \chi_E f$ for some $f \in L^1(\omega_1)$, which implies $\chi_E = 1$ since $g>0$. Thus $gH^1(\omega_2) = L^1(\omega_1) = gL^1(\omega_1)$, which implies $H^1(\omega_2) = L^1(\omega_2)$. Conversely, if $H^1(\omega_2) = L^1(\omega_2)$, then $gH^1(\omega_2) = gL^1(\omega_2) = L^1(\omega_1) = \chi_{\mathbb{T}} L^1(\omega_1)$, which means $gH^1(\omega_2) \neq \varphi H^1(\omega_1)$, i.e., $\int |\log g|\, d\omega_1 = \infty$. □

There is an important characterization of outer functions in $H^1(\omega_1)$.

Lemma 68. *A function* f *is an outer function in* $H^1(\omega_1)$ *if and only there is a real harmonic function* u *with harmonic conjugate* $\overline{u}$ *such that*

(1) $u \in L^1(\omega_1)$,

(2) $f = e^{u+i\overline{u}}$,

(3) $f \in L^1(\omega_1)$.

Through the remainder of following sections we assume

1. α is a continuous normalized gauge norm on $L^\infty(\omega_1)$.

2. and that $c > 0$ and ω_2 is a probability measure on $\mathbb{T}$ such that $\omega_2 << \omega_1$ and $\omega_1 << \omega_2$ and such that α is $c\|\cdot\|_{1,\omega_2}$-dominating.

3. $h \in H^1(\omega_1)$ is an outer function, η is unimodular and $\bar{\eta}h = g = d\omega_2/d\omega_1$.

Since ω_2 and ω_1 are mutually absolutely continuous we have $L^\infty(\omega_1) = L^\infty(\omega_2), L^\alpha(\omega_1) = L^\alpha(\omega_2)$ and $H^\alpha(\omega_1) = H^\alpha(\omega_2)$, we will use L^∞ to denote $L^\infty(\omega_1)$ and $L^\infty(\omega_2)$, use L^α to denote $L^\alpha(\omega_1)$ and $L^\alpha(\omega_2)$, use H^α to denote $H^\alpha(\omega_1)$ and $H^\alpha(\omega_2)$. It follows that $L^\alpha, L^\infty, H^\alpha$ do not depend on ω_2 or ω_1. However, this notation slightly conflicts with the classical notation for $L^1(\omega_1) = L^{\|\cdot\|_{1,\omega_1}}$ or $H^1(\omega_1) = H^{\|\cdot\|_{1,\omega_1}}$, so we will add the measure to the notation when we are talking about L^p or H^p.

Theorem 69. *Let $g = d\omega_2/d\omega_1$ and $g > 0$, there exists a $h \in H^1(\Gamma, \omega_1)$ is an outer function, η is unimodular and $\bar{\eta}h = g = d\omega_2/d\omega_1$. We have $hL^1(\Gamma, \omega_2) = L^1(\Gamma, \omega_1)$ and $hH^1(\Gamma, \omega_2) = H^1(\Gamma, \omega_1)$.*

Proof. We know from our assumption (3) that $hL^1(\omega_2) = g\eta L^1(\omega_2) = gL^1(\omega_2) = L^1(\omega_1)$. By Lemma 67(1), we have $gH^1(\omega_2) = \eta H^1(\omega_1)$, so

$$hH^1(\omega_2) = \eta gH^1(\omega_2) = \eta\eta H^1(\omega_1) = H^1(\omega_1).$$

□

Corollary 70. *$gH^1(\omega_2) = \gamma H^1(\omega_1)$ for some unimodular $\gamma \Leftrightarrow \int_{\mathbb{T}} |\log g|\, d\omega_1 < \infty$.*

Proof. Assume $gH^1(\omega_2) = \gamma H^1(\omega_1)$, Since $1 \in H^1(\omega_2), g \in gH^1(\omega_2)$, $\exists \phi \in H^1(\omega_1)$ such that $g = \gamma\phi$. Since $\phi \in H^1(\omega_1), \phi = \psi h$, where ψ is an inner function and h is an outer function. Thus, $\int_{\mathbb{T}} |\log g|\, d\omega_1 = \int_{\mathbb{T}} \log |g| d\omega_1 = \int_{\mathbb{T}} \log |h| d\omega_1 < \infty$, since h is an outer function.

Assume $\int_{\mathbb{T}} |\log g|\, d\omega_1 < \infty, g$ and $\log g \in L^1(\omega_1), g > 0$. Thus there exists an outer function $h \in H^1(\omega_1)$, such that $|h| = |g| = g, |h| = \phi h, |\phi| = 1, g = \eta h$, Define $V : L^1(\omega_2) \longrightarrow L^1(\omega_1)$ by $Vf = hf$, as in Theorem 69, we have $hH^1(\omega_2) = H^1(\omega_1)$, so $gH^1(\omega_2) = \eta h H^1(\omega_2) = \eta H^1(\omega_1)$. Let $\gamma = \eta$, then $gH^1(\omega_2) = \gamma H^1(\omega_1)$. $\square$

5.3 Beurling-Helson-Lowdenslager Theorem for $L^\alpha(\Gamma, \omega)$

In this section, we let $g = d\omega_2/d\omega$ with $g > 0$ and we consider the first case $\int |\log g|\, d\omega < \infty$. Let α be a continuous, normalized gauge norm on $L^\infty(\Gamma, \omega)$, i.e. $\alpha \in \mathfrak{N}$. To generalize Beurling-Helson-Lowdenslager Theorem to the spaces $L^\alpha(\Gamma, \omega)$, we use the same technique as that of the first author in [10] with a few modifications necessitated by the multiple connectedness of the domain of the members of $H^\alpha(\Omega)$. In that paper invariant subspaces of the single operator multiplication by z on $L^\alpha(\mathbb{T}, m)$ were considered, in which case invariance under that operator is enough to imply invariance under multiplication by all $H^\infty(\mathbb{T})$ functions. In the multiply connected case, the invariant subspaces of the operator multiplication by z are more complicated (see [42, 1, 2]), and so we assume invariance under multiplication by all $H^\infty(\Gamma)$ functions. A basic idea in [10] was also devised earlier by Gamelin [26, 27] to study invariant subspaces in certain generalized H^p spaces.

Lemma 71. *Suppose f is analytic and has no zeros on Ω and a_j is a point in the jth hole of Ω. Then there exists $k_j \in \mathcal{Z}, 1 \leq j \leq n$ and $g \in H(\Omega)$ such that $\varphi(z) = (z - a_1)(z - a_2)\cdots(z - a_n)e^{g(z)}$.*

Lemma 72. *If u is a real-valuled harmonic function on Ω, then there exists a harmonic unit u_a such that $u + u_a$ is the real part of an analytic function on Ω.*

Lemma 73. $H^\alpha(\Gamma) = H^1(\Gamma) \cap L^\alpha(\Gamma, \omega)$.

Proof. Suppose $\varphi \in L^\alpha(\Gamma,\omega)^\#$ is in the annihilator of $H^\alpha(\Gamma)$. By Proposition 3, there exists $F \in \mathcal{L}^{\alpha'}(\Gamma,\omega)$ such that for all $f \in L^\alpha(\Gamma,\omega)$, $fF \in L^1(\Gamma,\omega)$ and $\varphi(f) = \int_\Gamma fF\,\mathrm{d}\omega$.

Because φ is in the annihilator of $H^\alpha(\Gamma)$ we have $\int_\Gamma fF\,\mathrm{d}\omega = 0$ for all $f \in H^\infty(\Gamma)$, and it follows from Theorem 4.8 of [24] that $PF \in H^1(\Gamma)$, where P is the polynomial whose zeros are the critical points of the Green's function of Ω with pole at $\widehat{w}$. Further, because $P \cdot (H^1(\Gamma) + N(\Gamma)) = H^1(\Gamma)$, it follows that $F \in H^1(\Gamma)+N(\Gamma)$, and thus $F = F_1+F_N$, where $F_1 \in H^1(\Gamma)$ and $F_N \in N(\Gamma)$. Since $1 \in H^\infty(\Gamma)$ and $\int_\Gamma f\,\mathrm{d}\omega = 0$ for all $f \in N(\Gamma)$, $\int_\Gamma 1F_1\,\mathrm{d}\omega = \int_\Gamma 1F\,\mathrm{d}\omega = 0$, and thus $F_1 \in H_0^1(\Gamma)$.

Suppose $g \in H^1(\Gamma) \cap L^\alpha(\Gamma,\omega)$. Then $gF \in L^1(\Gamma,\omega)$, and because F_N is bounded, $gF_1 \in L^1(\Gamma,\omega)$, and consequently $C_\tau(gF_1) \in L^1(\mathbb{T},m)$. Also, from $g \in H^1(\Gamma)$ and $F_1 \in H_0^1(\Gamma)$, it follows that $C_\tau(g) \in H^1(\mathbb{T})$ and $C_\tau(F_1) \in H_0^1(\mathbb{T})$. Thus the product of $C_\tau(g)$ and $C_\tau(F_1)$ is in $H_0^1(\mathbb{T})$, which implies $\int_\Gamma gF_1\,\mathrm{d}\omega = \int_\mathbb{T}(gF_1)\circ\tau\,\mathrm{d}m = 0$. Since $H^2(\Gamma)^*$ and $N(\Gamma)$ are orthogonal in $L^2(\Gamma,\omega)$, and since $H^2(\Gamma)$ is dense in $H^1(\Gamma)$, it follows that $\int_\Gamma gF_N\,\mathrm{d}\omega = 0$. Consequently $\varphi(g) = 0$, and the Hahn-Banach theorem now implies that $g \in H^\alpha(\Gamma)$, thereby giving us the required opposite inclusion. □

Lemma 74. *If $b \in L^\infty(\Gamma,\omega)$ and $1/b \in L^\alpha(\Gamma,\omega)$, then there exists a function ψ having ω-a.e. constant modulus on each connected component of Γ, and there exists an outer function $h \in H^\infty(\Gamma)$ such that $b = \psi h$ and $1/h \in H^\alpha(\Gamma)$.*

Proof. If b satisfies the hypothesis, then, since $L^\alpha(\Gamma,\omega) \subset L^1(\Gamma,\omega)$, it follows that $\log|b|$ is integrable, and hence there exists a harmonic function u on Ω with $\log|b|$ as its boundary function. Thus, there exists a harmonic unit u_0 such that $u - u_0$ has a harmonic conjugate function v on Ω. Put $h = \exp(u - u_0 + iv)$ to get an outer function on Ω such that $|h|$ has a boundary function $|b|e^{-u_0}$, and thus $h \in H^\infty(\Gamma)$. If $\psi = b/h$, then $|\psi| = e^{u_0}$ which is constant on each of the sets Γ_j. Finally, $1/h$ is in both $H^1(\Gamma)$ and $L^\alpha(\Gamma,\omega)$, and thus Lemma 73 implies $1/h \in H^\alpha(\Gamma)$. □

Proposition 4. *Let M be a weak* closed subspace of $L^\infty(\Gamma,\omega)$ that is invariant under multiplication by all members of $H^\infty(\Gamma)$. If $\mathcal{M}$ is the closure of M in $L^\alpha(\Gamma,\omega)$, then $\mathcal{M}$ is also invariant under multiplication by members of $H^\infty(\Gamma)$ and $M = \mathcal{M} \cap L^\infty(\Gamma,\omega)$.*

Proposition 5. *Let $\mathcal{M}$ be a closed subspace of $L^\alpha(\Gamma,\omega)$ that is invariant under multiplication by all members of $H^\infty(\Gamma)$. If $M = \mathcal{M} \cap L^\infty(\Gamma,\omega)$, then M is weak* closed and invariant and $\mathcal{M} = M^{-\alpha}$.*

Proof. That M is weak* closed follows from the Krein-Šmulian theorem and the method as in [10]. Invariance is immediate.

It is clear that $M^{-\alpha} \subset \mathcal{M}$. Consider $f \in \mathcal{M}$, and apply Lemma 74 to $b = 1/(|f|+1)$, thereby producing a function ψ with ω-a.e. constant modulus on each component of Γ and an outer function $h \in H^\infty(\Gamma)$ with $1/h \in H^\alpha(\Gamma)$ such that $b = \psi h$. There exists a sequence (h_ω) in $H^\infty(\Gamma)$ such that $\alpha(1/h - h_\omega) \to 0$ as $\omega \to \infty$. Since $|hf| = |\psi||f|/(|f|+1)$, it follows that $hf \in \mathcal{M}$ and hf is bounded, and hence $hf \in M$. The same is true of each $h_\omega hf$, and $\alpha(f - h_\omega hf) \leq \alpha(1/h - h_\omega)\|hf\|_\infty \to 0$ as $\omega \to \infty$. Therefore $f \in M^{-\alpha}$, and the proof is complete □

With Propositions 4 and 5 in hand we can now prove the principal result, the Beurling, Helson-Lowdenslager theorem for a space with a continuous, normalized, gauge norm on a multiply connected domain. As mentioned in the introduction, the last statement contains Royden's version of Beurling's theorem in [51, Theorem 1].

Theorem 75. *Suppose ω is harmonic measure on Γ and α is a continuous normalized gauge norm on $L^\infty(\Gamma,\omega)$. Suppose also that $c > 0$ and ω_2 is a harmonic measur that is mutually absolutely continuous with respect to ω such that α is $c\,\|\|_{1,\omega_2}$-dominating and $\log|d\omega_2/d\omega| \in L^1(\Gamma,\omega)$. Let $\mathcal{M}$ be a closed subspace of $L^\alpha(\Gamma,\omega)$ that is invariant under M_ψ for every $\psi \in H^\infty(\Gamma)$. Then either*

(i) $\mathcal{M} = \chi_{\mathbb{E}} L^\alpha(\Gamma,\omega)$ for some measurable subset $\mathbb{E}$ of Γ, or

(ii) $\mathcal{M} = \varphi H^\alpha(\Gamma)$ for some $\varphi \in L^\infty(\Gamma,\omega)$ such that $|\varphi|$ is constant on each of the components of Γ.

The result is also true in the case where α is the essential supremum norm when $\mathcal{M}$ is weak closed. When $\mathcal{M} \subset H^\alpha(\Gamma)$ case (ii) holds and the function φ is a Royden inner function.*

Proof. Suppose M is a weak* closed subspace of $L^\infty(\Gamma, \omega)$ that is invariant under M_ψ for every $\psi \in H^\infty(\Gamma)$, and let $\mathcal{M}$ be the closure of M in $L^2(\Gamma, \omega)$. The preceding case then applies to $\mathcal{M}$, and Proposition 4 implies that M is obtained by intersecting $\mathcal{M}$ with $L^\infty(\Gamma, \omega)$. Since the intersection of $\chi_{\mathbb{E}} L^2(\Gamma, \omega)$ with $L^\infty(\Gamma, \omega)$ is $\chi_{\mathbb{E}} L^\infty(\Gamma, \omega)$ and the intersection of $\varphi H^2(\Gamma, \omega)$ with $L^\infty(\Gamma, \omega)$ is $\varphi H^\infty(\Gamma, \omega)$, this case is proved.

Next let $\mathcal{M}$ be a closed subspace of $L^\alpha(\Gamma, \omega)$ for $\alpha \in \mathfrak{N}$. By Proposition 5, if $M = \mathcal{M} \cap L^\infty(\Gamma, \omega)$, then M is weak* closed and invariant under each M_ψ with $\psi \in H^\infty(\Gamma)$. The preceding case now implies that either $M = \chi_{\mathbb{E}} L^\infty(\Gamma, \omega)$ or $M = \varphi H^\infty(\Gamma)$. The closure of M in the α topology is $\mathcal{M}$, by Proposition 5, the α closure of $\chi_{\mathbb{E}} L^\infty(\Gamma, \omega)$ is $\chi_{\mathbb{E}} L^\alpha(\Gamma, \omega)$, and the α closure of $\varphi H^\infty(\Gamma)$ is $\varphi H^\alpha(\Gamma)$. Thus the proof is completed. $\square$

CHAPTER 6

APPROXIMATE EQUIVALENCE IN VON NEUMANN ALGEBRAS

Suppose $\mathcal{A}$ is a separable unital ASH C*-algebra, $\mathcal{R}$ is a sigma-finite II$_\infty$ factor von Neumann algebra, and $\pi, \rho : \mathcal{A} \to \mathcal{R}$ are unital $*$-homomorphisms such that, for every $a \in \mathcal{A}$, the range projections of $\pi(a)$ and $\rho(a)$ are Murray von Neuman equivalent in $\mathcal{R}$.In the chapter, we prove that π and ρ are approximately unitarily equivalent modulo $\mathcal{K}_{\mathcal{R}}$, where $\mathcal{K}_{\mathcal{R}}$ is the norm closed ideal generated by the finite projections in $\mathcal{R}$. We also prove a very general result concerning approximate equivalence in arbitrary finite von Neumann algebras.

6.1 Introduction

In 1977 D. Voiculescu [66] proved a remarkable theorem concerning approximate (unitary) equivalence of representations of a separable unital C*-algebra on a separable Hilbert space. The beauty of the theorem is that the characterization was in terms of purely algebraic terms. This was made explicit in the reformulation of Voiculescu's theorem in [29] in terms of rank.

Theorem 76. *[66]Suppose $B(H)$ is the set of operators on a separable Hilbert space H and $\mathcal{K}(H)$ is the ideal of compact operators. Suppose $\mathcal{A}$ is a separable unital C*-algebra, and $\pi, \rho : \mathcal{A} \to B(H)$ are unital $*$-homomorphisms. The following are equivalent:*

1. *There is a sequence $\{U_n\}$ of unitary operators in $B(H)$ such that*

   (a) *$U_n \pi(a) U_n^* - \rho(a) \in \mathcal{K}(H)$ for every $n \in \mathbb{N}$ and every $a \in \mathcal{A}$.*

   (b) *$\|U_n \pi(a) U_n^* - \rho(a)\| \to 0$ for every $a \in \mathcal{A}$.*

2. *There is a sequence* $\{U_n\}$ *of unitary operators in* $B(H)$ *such that, for every* $a \in \mathcal{A}$,

$$\|U_n \pi(a) U_n^* - \rho(a)\| \to 0.$$

3. *For every* $a \in \mathcal{A}$,

$$\textit{rank}(\pi(a)) = \textit{rank}(\rho(a)) \ .$$

If $\pi : \mathcal{A} \to B(H)$ is a unital $*$-homomorphisms. We will write $\pi \sim_a \rho$ in $B(H)$ to mean that statement (2) in the preceding theorem holds and we will write $\pi \sim_a \rho$ ($\mathcal{K}(H)$) in $B(H)$ to indicate statement (1) holds. When the C*-algebra $\mathcal{A}$ is not separable, $\pi \sim_a \rho$ means that there is a net of unitaries $\{U_\lambda\}$ such that, for every $a \in \mathcal{A}$, $\|U_\lambda \pi(a) U_\lambda^* - \rho(a)\| \to 0$. It was shown in [29] that $\pi \sim_a \rho$ if and only if rank$(\pi(a))$ =rank$(\rho(a))$ always holds even when $\mathcal{A}$ or H is not separable, where, for $T \in B(H)$, rank(T) is the Hilbert-space dimension of the projection $\mathfrak{R}(T)$ onto the closure of the range of T.

Later H. Ding and the first author extended the notion of rank to operators in a von Neumann algebra $\mathcal{M}$, i.e., if $T \in \mathcal{M}$, then $\mathcal{M}$-rank(T) is the Murray von Neumann equivalence class of the projection $\mathfrak{R}(T)$. If p and q are projections in a C*-algebra $\mathcal{W}$, we write $p \sim q$ in $\mathcal{W}$ to mean there is a partial isometry $v \in \mathcal{W}$ such that $v^*v = p$ and $vv^* = q$. Thus $\mathcal{M}$-rank$(T) = \mathcal{M}$-rank(S) if and only if $\mathfrak{R}(S) \sim \mathfrak{R}(T)$. In [17] they extended Voiculescu's theorem for representations of a separable AH C*-algebra into a von Neumann algebra on a separable Hilbert space. When the algebra $\mathcal{A}$ is ASH, their characterization works when the von Neumann algebra is a II_1 factor [17] (See Theorem 79.)

When $\mathcal{M}$ is a type II_∞ factor with a faithful normal tracial weight τ, there are analogues $\mathcal{F}_\mathcal{M}$ and $\mathcal{K}_\mathcal{M}$ of the finite rank operators and compact operators; namely $\mathcal{F}_\mathcal{M}$ is the ideal generated by the projections $P \in \mathcal{M}$ with $\tau(P) < \infty$, and $\mathcal{K}_\mathcal{M}$ is the norm closure of $\mathcal{F}_\mathcal{M}$.

In [33] Rui Shi and the first author proved that if $\mathcal{M}$ is a sigma finite type II_∞ factor, $\mathcal{A}$ is a separable unital commutative C*-algebra and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms, then

$\pi \sim_a \rho(\mathcal{M})$ if and only if the sequence $\{U_n\}$ of unitary operators in $\mathcal{M}$ for which part (2) in Theorem 76 can be chosen so that, for every $n \in \mathbb{N}$ and every $a \in \mathcal{A}$,

$$U_n \pi(a) U_n^* - \rho(a) \in \mathcal{K}_{\mathcal{M}} .$$

More recently this result has been extended [23] to the case when $\mathcal{A}$ is an AF C*-algebra, and in [60] to the case when $\mathcal{A}$ is AH. In this chapter we extend the results to the case when $\mathcal{A}$ is a ASH C*-algebra. We also extend some of the results in [17] for arbitrary finite von Neumann algebras.

6.2 Finite von Neumann Algebras

A separable C*-algebra is AF if it is a direct limit of finite-dimensional C*-algebras. A separable C*-algebra is *homogeneous* if it is a finite direct sum of algebras of the form $\mathbb{M}_n(C(X))$, where X is a compact metric space. Such algebras are characterized by the fact that every irreducible representation is on an n-dimensional Hilbert space. A unital C*-algebra is *subhomogeneous* if there is an $n \in \mathbb{N}$, such that every representation is on a Hilbert space of dimension at most n; equivalently, if $x^n = 0$ for every nilpotent $x \in \mathcal{A}$. Every subhomogeneous algebra is a subalgebra of a homogeneous one. Every subhomogeneous von Neumann algebra is homogeneous; in particular, if $\mathcal{A}$ is subhomogeneous, then $\mathcal{A}^{\#\#}$ is homogeneous.

There has been a lot of work determining which separable C*-algebras are AF-embeddable. A (possibly nonseparable) C*-algebra $\mathcal{B}$ is LF if, for every finite subset $F \subset \mathcal{B}$ and every $\varepsilon > 0$ there is a finite-dimensional C*-algebra $\mathcal{D}$ of $\mathcal{B}$ such that, for every $b \in F$, dist$(b, \mathcal{D}) < \varepsilon$. Every separable unital C*-subalgebra of a LF C*-algebra is contained in a separable AF subalgebra.

We are interested in a more general property. We say that a unital C*-algebra $\mathcal{A}$ is *strongly LF-embeddable* if there is an LF C*-algebra $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{A}^{\#\#}$. It is easily shown that a ASH algebra is strongly LF-embeddable.

Lemma 77. *Suppose $\mathcal{B}$ is a unital LF C*-algebra and $\mathcal{D} = \mathbb{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{nk}(\mathbb{C})$ and $\mathcal{W}$ is a unital C*-algebra.*

1. *If $\pi, \rho : \mathcal{D} \to \mathcal{W}$ are unital $*$-homomorphisms and $\pi(e_{11,s}) \sim \rho(e_{11,s})$ for $1 \le s \le k$, where $\{e_{ij,s}\}$ is the system of matrix units for $\mathbb{M}_{ns}(\mathbb{C})$, then π and ρ are unitarily equivalent in $\mathcal{W}$.*

2. *If $\pi, \rho : \mathcal{B} \to \mathcal{W}$ are unital $*$-homomorphisms such that $\pi(p) \sim \rho(p)$ in $\mathcal{W}$ for every projection $p \in \mathcal{B}$, $\pi \sim_a \rho$ in $\mathcal{W}$.*

Proof. (1) Since $e_{ii,s} \sim e_{11,s}$ in $\mathcal{D}$ for $1 \le i \le n_s$ and $1 \le s \le k$, we see that $\pi(e_{ii,s}) \sim \rho(e_{ii,s})$ in $\mathcal{W}$ for $1 \le i \le n_s$ and $1 \le s \le k$. It follows from [17, Theorem 2] that π and ρ are unitarily equivalent in $\mathcal{W}$.

(2) Suppose Λ is the set of all pairs $\lambda = (F_\lambda, \varepsilon_\lambda)$ with F_λ a finite subset of $\mathcal{B}$ and $\varepsilon_\lambda > 0$. Clearly Λ is directed by $(\subset, \ge)$. For $\lambda \in \Lambda$, we can choose a finite-dimensional algebra $\mathcal{D}_\lambda \subset \mathcal{B}$ such that, for every $x \in F_\lambda$, dist$(x, \mathcal{D}_\lambda) < \varepsilon_\lambda$. It follows from part (1) that there is a unitary operator $U_\lambda \in \mathcal{W}$ such that, for every $x \in F_\lambda$, $U\pi(x)U^* = \rho(x)$. For each $a \in F_{\lambda,}we$ can choose $x_a \in \mathcal{D}_\lambda$ such that $\|a - x_a\| < \varepsilon_{\lambda..}$ Hence, for every $a \in F_\lambda$

$$\|U_\lambda \pi(a) U_\lambda^* - \rho(a)\| = \|U_\lambda \pi(a - x_\lambda) U_\lambda^* - \rho(a - x_\lambda)\| < 2\varepsilon_\lambda.$$

It follows that, for every $a \in \mathcal{A}$,

$$\lim_\lambda \|U_\lambda \pi(a) U^* - \rho(a)\| = 0.$$

□

It was shown in [17] that $(1) \Rightarrow (2) \Leftrightarrow (3)$ in Theorem holds for every C*-algebra $\mathcal{A}$ when $\mathcal{M}$ acts on a separable Hilbert space. The key property of a finite von Neumann algebra $\mathcal{M}$, is that there is a faithful normal tracial conditional expectation Φ from $\mathcal{M}$ to its center $\mathcal{Z}(\mathcal{M})$, and that for projections p and q in $\mathcal{M}$, we have p and q are Murray-von Neumann equivalent if and only if $\Phi(p) = \Phi(q)$.

Lemma 78. *Suppose $\mathcal{A}$ is a (possibly nonunital) C*-algebra, $\mathcal{M}$ is a finite von Neumann algebra with center-valued trace $\Phi : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$. If $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are $*$-homomorphisms such that, for every $a \in \mathcal{A}$,*

$$\mathcal{M}\text{-}rank\,(\pi(a)) = \mathcal{M}\text{-}rank\,(\rho(a)),$$

then

$$\Phi \circ \pi = \Phi \circ \rho .$$

Proof. We can extend π and ρ to weak*-weak* continuous *-homomorphisms $\hat{\pi}, \hat{\rho} : \mathcal{A}^{\#\#} \to \mathcal{M}$. Suppose $x \in \mathcal{A}$ and $0 \leq x \leq 1$. Suppose $0 < \alpha < 1$ and define $f_\alpha : [0,1] \to [0,1]$ by

$$f(t) = dist\,(t, [0, \alpha]) .$$

Since $f(0) = 0$, we see that $f(x) \in \mathcal{A}$, and $\chi_{(\alpha,1]}(x) = \text{weak*-}\lim_{n\to\infty} f(x)^{1/n} \in \mathcal{A}^{\#\#}$, so

$$\mathfrak{R}(f(x)) = \chi_{(\alpha,1]}(x) \ .$$

It follows that

$$\hat{\pi}\left(\chi_{(\alpha,1]}(x)\right) = \mathfrak{R}(\pi(f_\alpha(x))) = \chi_{(\alpha,1]}(\pi(x))$$

and

$$\hat{\rho}\left(\chi_{(\alpha,1]}(x)\right) = \mathfrak{R}(\rho(f_\alpha(x))) = \chi_{(\alpha,1]}(\rho(x)) .$$

Hence

$$\Phi\left(\hat{\pi}\left(\chi_{(\alpha,1]}(x)\right)\right) = \Phi\left(\hat{\rho}\left(\chi_{(\alpha,1]}(x)\right)\right) .$$

Thus, Suppose $0 < \alpha < \beta < 1$. Since $\chi_{(\alpha,\beta]} = \chi_{(\alpha,1]} - \chi_{(\beta,1]}$, we see that

$$\Phi\left(\hat{\pi}\left(\chi_{(\alpha,\beta]}(x)\right)\right) = \Phi\left(\hat{\rho}\left(\chi_{(\alpha,\beta]}(x)\right)\right) .$$

Thus, for all $n \in \mathbb{N}$,

$$\Phi\left(\hat{\pi}\left(\sum_{k-1}^{n-1} \frac{k}{n} \chi_{\left(\frac{k}{n}, \frac{k+1}{n}\right]}(x)\right)\right) = \Phi\left(\hat{\rho}\left(\sum_{k-1}^{n-1} \frac{k}{n} \chi_{\left(\frac{k}{n}, \frac{k+1}{n}\right]}(x)\right)\right) .$$

Since, for every $n \in \mathbb{N}$,

$$\left\| x - \sum_{k-1}^{n-1} \frac{k}{n} \chi_{\left(\frac{k}{n}, \frac{k+1}{n}\right]}(x) \right\| \leq 1/n,$$

it follows that

$$\Phi(\pi(x)) = \Phi(\hat{\pi}(x)) = \Phi(\hat{\rho}(x)) = \Phi(\rho(x)).$$

Since $\mathcal{A}$ is the linear span of its positive contractions, $\Phi \circ \pi = \Phi \circ \rho$. □

Theorem 79. *Suppose $\mathcal{A}$ is a strongly LF-embeddable, $\mathcal{M}$ is a finite von Neumann algebra with center-valued trace $\Phi : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$. If $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms, the following are equivalent:*

1. $\pi \sim_a \rho(\mathcal{M})$.

2. *$\mathcal{M}$-rank$(\pi(a)) = \mathcal{M}$-rank$(\rho(a))$ for every $a \in \mathcal{A}$.*

3. $\Phi \circ \pi = \Phi \circ \rho$.

Proof. $(3) \Rightarrow (1)$. We can extend π and ρ to weak*-weak* continuous $*$-homomorphisms $\hat{\pi}, \hat{\rho}$: $\mathcal{A}^{\#\#} \to \mathcal{M}$. Since Φ is weak*-weak* continuous, it follows that $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$. Since $\mathcal{A}$ is strongly LF-embeddable, there is an LF algebra $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{A}^{\#\#}$. For every projection $p \in \mathcal{B}$ we have

$$\Phi(\hat{\pi}(p)) = \Phi(\hat{\rho}(p)),$$

which implies that $\hat{\pi}(p) \sim \hat{\rho}(p)$. Hence,by Lemma 77, $\hat{\pi}|_{\mathcal{B}} \sim_a \hat{\rho}|_{\mathcal{B}}$. in $\mathcal{M}$. Thus $\pi \sim_a \rho(\mathcal{M})$.

$(1) \Rightarrow (3)$. Suppose $\{U_\lambda\}$ is a net of unitaries in $\mathcal{M}$ such that, for every $a \in \mathcal{A}$,

$$\|U_\lambda \pi(a) U_\lambda^* - \rho(a)\| \to 0.$$

Thus, since Φ is tracial and continuous,

$$\Phi(\rho(a)) = \lim_\lambda \Phi(U_\lambda \pi(a) U_\lambda^*) = \Phi(\pi(a)).$$

$(3) \Rightarrow (2)$. Assume (3). Then, for any $a \in \mathcal{A}$,

$$\Phi(\Re(\pi(a))) = \lim_{n\to\infty} \Phi\left(\pi\left((aa^*)^{1/n}\right)\right) = \lim_{n\to\infty} \Phi\left(\rho\left((aa^*)^{1/n}\right)\right) = \Phi(\Re(\pi(a))).$$

Hence $\Re(\pi(a)) \sim \Re(\rho(a))$. Thus $\mathcal{M}$-rank$(\pi(a)) = \mathcal{M}$-rank$(\rho(a))$.

$(2) \Rightarrow (3)$. This is Lemma 78. □

Remark 1. *It is important to note that the proof of* $(2) \Rightarrow (3)$ *holds even when* $\mathcal{A}$ *is not unital.*

In [29] it was shown that if $\mathcal{A}$ is a separable unital C*-algebra and π and ρ are representations on separable Hilbert spaces such that, for every $x \in \mathcal{A}$

$$\text{rank}\pi(x) \leq \text{rank}\rho(x),$$

then there is a representation σ such that

$$\pi \oplus \sigma \sim_a \rho.$$

In [33], Rui Shi and the first author proved an analogue for representations of separable abelian C*-algebras into II_1 factor von Neumann algebras. This result was extended by Rui Shi and Junsheng Fang [23] to separable AF C*-algebras. We extend this result further, including separable ASH C*-algebras.

Theorem 80. *Suppose $\mathcal{A}$ is a separable strongly LF-embeddable C*-algebra and $\mathcal{M}$ is a II_1 factor von Neumann algebra with a faithful normal tracial state τ. Suppose P is a projection in $\mathcal{M}$ and $\pi : \mathcal{A} \to P\mathcal{M}P$ and $\rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that, for every $a \in \mathcal{A}$,*

$$\mathcal{M}\text{-rank}\left(\pi\left(a\right)\right) \leq \mathcal{M}\text{-rank}\left(\rho\left(a\right)\right).$$

Then there is a unital $*$-homomorphism $\sigma : \mathcal{A} \to P^{\perp}\mathcal{M}P^{\perp}$ such that

$$\pi \oplus \sigma \sim_a \rho \ (\mathcal{M}) \ .$$

Proof. As in the proof of Theorem 79 choose a separable AF C*-algebra $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{A}^{\#\#}$, and extend π and ρ to unital weak*-weak* continuous $*$-homomorphisms $\hat{\pi}$ and $\hat{\rho}$ with domain $\mathcal{A}^{\#\#}$. It was shown in [17] that the condition on π and ρ is equivalent to: for every $a \in \mathcal{M}$ with $0 \leq a$, $\tau\left(\pi\left(a\right)\right) \leq \tau\left(\rho\left(a\right)\right)$. It follows from weak* continuity that, for every $a \in \mathcal{A}^{\#\#}$ with $0 \leq a$, $\tau\left(\hat{\pi}\left(a\right)\right) \leq \tau\left(\hat{\rho}\left(a\right)\right)$. In particular this holds for $0 \leq a \in \mathcal{B}$. However, since $\mathcal{B}$ is AF, it follows from [23] that there is a unital $*$-homomorphism $\gamma : \mathcal{B} \to P^{\perp}\mathcal{A}P^{\perp}$ such that

$$(\hat{\pi}|_{\mathcal{B}}) \oplus \gamma \sim_a \hat{\rho}|_{\mathcal{B}} \ (\mathcal{M}).$$

If we let $\sigma = \gamma|_{\mathcal{A}}$, we see $\pi \oplus \sigma \sim_a \rho\left(\mathcal{M}\right)$. □

6.3 Representations of ASH algebras relative to ideals

We prove a version of Voiculescu's theorem for representations of a separable ASH C*-algebras into sigma-finite type II_∞ factor von Neumann algebras. We first prove a more general result. We begin with a probably well-known lemma.

Lemma 81. *Suppose $\mathcal{J}$ is a norm closed two-sided ideal in a von Neumann algebra $\mathcal{M}$ and $\mathcal{J}_0$ is the ideal in $\mathcal{M}$ generated by the projections in $\mathcal{J}$. Suppose also that $\mathcal{A}$ is a C*-algebra and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphism. Then*

1. *$\mathcal{J}$ is the norm closed linear span of the set of projections in $\mathcal{J}$, so $\mathcal{J}_0^{-\|\|} = \mathcal{J}$,*

2. *$\mathcal{J}_0 = \{T \in \mathcal{M} : T = PTP$ for some projection $P \in \mathcal{J}\}$,*

3. *$T \in \mathcal{J}_0$ if and only if $\chi_{(0,\infty)}(|T|) = \mathfrak{R}(T) \in \mathcal{J}_0$,*

4. *If P and Q are projections in $\mathcal{J}_0$ then $P \vee Q = \mathfrak{R}(P+Q) \in \mathcal{J}_0$,*

5. *$\pi^{-1}(\mathcal{J}_0)^{-\|\|} = \pi^{-1}(\mathcal{J})$,*

6. *If $\{\mathcal{A}_i : i \in I\}$ is an increasingly directed family of unital C*-subalgebras of $\mathcal{A}$ and $\mathcal{A} = [\cup_{i\in I}\mathcal{A}_i]^{-\|\|}$, then*
$$\left[\cup_{i\in I}\mathcal{A}_i \cap \pi^{-1}(\mathcal{J}_0)\right]^{-\|\|} = \pi^{-1}(\mathcal{J}) .$$

Proof. $(1), (2), (3)$ can be found in [44].

(4). Suppose $a \in \pi^{-1}(\mathcal{J})$. Then $\pi(a) \in \mathcal{J}$, so

$$\pi(g_\varepsilon(|a|)) = g_\varepsilon(|\pi(a)|)\,\chi_{(\varepsilon/2,\infty)}(|\pi(a)|) \in \mathcal{J}_0,$$

and

$$\|a - ag_\varepsilon(|a|)\| \leq \varepsilon.$$

(5). Let $\eta : \mathcal{M} \to \mathcal{M}/\mathcal{J}$ be the quotient map. Suppose $a \in \pi^{-1}(\mathcal{J})$ and $\varepsilon > 0$. Then there is an $i \in I$ and a $b \in \mathcal{A}_i$ such that $\|a - b\| < \varepsilon$. Thus

$$\|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| = \|(\eta \circ \pi)(b)\| = \|(\eta \circ \pi)(b-a)\| \leq \varepsilon,$$

so there is a $w \in \mathcal{A}_i$ so that

$$\|w\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(w)\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| \leq \varepsilon.$$

$z = b - w \in \ker(\eta \circ (\pi|_{\mathcal{A}_i})) = \pi^{-1}(\mathcal{J}) \cap \mathcal{A}_i$, and $\|b - z\| = \|w\| < \varepsilon$. It follows from part (2) that there is a $v \in \pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_i$ such that $\|z - v\| \leq \varepsilon$. Hence $\|a - v\| \leq \|a - b\| + \|b - z\| + \|z - v\| \leq 3\varepsilon$.

(6) Let $\eta : \mathcal{M} \to \mathcal{M}/\mathcal{J}$ be the quotient map. Suppose $a \in \pi^{-1}(\mathcal{J})$ and $\varepsilon > 0$. Then there is an $i \in I$ and a $b \in \mathcal{A}_i$ such that $\|a - b\| < \varepsilon$. Thus

$$\|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| = \|(\eta \circ \pi)(b)\| = \|(\eta \circ \pi)(b - a)\| \leq \varepsilon,$$

so there is a $w \in \mathcal{A}_i$ so that

$$\|w\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(w)\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| \leq \varepsilon.$$

$z = b - w \in \ker(\eta \circ (\pi|_{\mathcal{A}_i})) = \pi^{-1}(\mathcal{J}) \cap \mathcal{A}_i$, and $\|b - z\| = \|w\| < \varepsilon$. It follows from part (5) that there is a $v \in \pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_i$ such that $\|z - v\| \leq \varepsilon$. Hence $\|a - v\| \leq \|a - b\| + \|b - z\| + \|z - v\| \leq 3\varepsilon$. □

Suppose $\mathcal{A}$ is a unital C*-algebra, $\mathcal{M} \subset B(H)$ is a von Neumann algebra with a norm-closed ideal $\mathcal{J}$ and $\pi : \mathcal{A} \to \mathcal{M}$ is a unital $*$-homomorphism. We define

$$H_{\pi,\mathcal{J}} = \text{sp}^{-\|\|}\left(\cup \{\text{ran}\pi(a) : a \in \mathcal{A} \text{ and } \pi(a) \in \mathcal{J}\}\right).$$

It is clear that $H_{\pi,\mathcal{J}}$ is a reducing subspace for π and we call the summand $\pi(\cdot)|_{H_{\pi,\mathcal{J}}} = \pi_{\mathcal{J}}$.

In Voiculescu's theorem, where $\pi, \rho : \mathcal{A} \to B(H)$ and $\mathcal{A}$ and H are separable, we write

$$\pi = \pi_{\mathcal{K}(H)} \oplus \pi_1, \ \rho = \rho_{\mathcal{K}(H)} \oplus \rho_1.$$

The proof of Voiculescu's theorem involves showing

$$\pi \sim_a \pi \oplus \rho_1 = \pi_{\mathcal{K}(H)} \oplus \pi_1 \oplus \rho_1,$$

and

$$\rho \sim_a \rho \oplus \pi_1 \backsimeq \rho_{\mathcal{K}(H)} \oplus \pi_1 \oplus \rho_1,$$

which was the hard part. Using descriptions of C*-algebras of compact operators and their representations, it is not too hard to show that the equality of rank conditions imply that $\pi_{\mathcal{K}(H)}$ and $\rho_{\mathcal{K}(H)}$ are unitarily equivalent. When $B(H)$ is replaced with a sigma-finite type II_∞ factor von Neumann algebra $\mathcal{M}$ and $\mathcal{K}(H)$ is replaced with the closed ideal $\mathcal{K}_\mathcal{M}$ generated by the finite projections, the hard part is harder (and unsolved) and the easy part is not true.

In a deep and beautiful paper [46] of Qihui Li, Junhao Shen, and Rui Shi proved the best-to-date attack of the hard part.

Theorem 82. *Suppose $\mathcal{A}$ is a separable nuclear C*-algebra, $\mathcal{M}$ is a sigma-finite type II_∞ factor von Neumann algebra and $\pi, \sigma : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that*

$$\sigma|_{\pi^{-1}(\mathcal{K}_\mathcal{M})} = 0.$$

Then

$$\pi \sim_a \pi \oplus \sigma \ (\mathcal{K}_\mathcal{M}) \ .$$

The following is a fairly general version of the analogue of the "easier part" of the proof of Voiculescu's theorem when the C*-algebra is ASH. In particular, there is no assumption that the von Neumann algebra $\mathcal{M}$ is a factor or acts on a separable Hilbert space.

Theorem 83. *Suppose $\mathcal{A}$ is a separable ASH C*-algebra, $\mathcal{M} \subset B(H)$ is a von Neumann algebra with a norm closed two-sided ideal $\mathcal{J}$. Suppose $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that*

1. *Every projection in $\mathcal{J}$ is finite,*
2. *$\mathcal{M}$-rank$(\pi(a)) = \mathcal{M}$-rank$(\rho(a))$ for every $a \in \mathcal{A}$.*

Then there is a sequence $\{W_n\}$ of partial isometries in $\mathcal{M}$ such that

3. *$W_n^* W_n$ is the projection onto $H_{\pi,\mathcal{J}}$ and $W_n W_n^*$ is the projection onto $H_{\rho,\mathcal{J}}$,*

4. $W_n \pi_{\mathcal{J}}(a) W_n^* - \rho_{\mathcal{J}}(a) \in \mathcal{J}$ *for every* $n \in \mathbb{N}$ *and every* $a \in \mathcal{A}$,

5. $\lim_{n\to\infty} \| W_n \pi_{\mathcal{J}}(a) W_n^* - \rho_{\mathcal{J}}(a) \| = 0$ *for every* $a \in \mathcal{A}$.

Proof. First, suppose $x \in \mathcal{A}$ and $x = x^*$. It follows from [59] that there is a sequence $\{U_n\}$ such that

$$\| U_n \pi(x) U_n^* - \rho(x) \| \to 0.$$

It follows that $\pi(x) \in \mathcal{J}$ if and only if $\rho(x) \in \mathcal{J}$ when $x = x^*$. However, for any $a \in \mathcal{A}$, we get $\pi(a) \in \mathcal{J}$ if and only if $\pi(|a|) \in \mathcal{J}$. Hence $\pi^{-1}(\mathcal{J}) = \rho^{-1}(\mathcal{J})$. Also, $\pi(a) \in \mathcal{J}_0$ if and only if $\mathfrak{R}(\pi(a)) \in \mathcal{J}_0$. Since $\mathfrak{R}(\pi(a))$ and $\mathfrak{R}(\rho(a))$ are Murray von Neumann equivalent (from (2)), we see that $\pi(a) \in \mathcal{J}_0$ if and only if $\rho(a) \in \mathcal{J}_0$. It follows that $\pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n = \rho^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n$ for each $n \in \mathbb{N}$, and, from Lemma 81,

$$\left[\bigcup_{n=1}^{\infty} \pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n \right]^{-\|\|} = \left[\bigcup_{n=1}^{\infty} \rho^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n \right]^{-\|\|} = \pi^{-1}(\mathcal{J}) = \rho^{-1}(\mathcal{J}).$$

Since $\mathcal{A}$ is a ASH algebra, we can assume that there is a sequence

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$$

of subalgebras of $\mathcal{A}$ such that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is norm dense in $\mathcal{A}$ such that, for each $n \in \mathbb{N}$,

$$\mathcal{A}_n^{\#\#} = \mathcal{M}_{k(n,1)} \left(C(X_{n,1}) \oplus \cdots \oplus \mathcal{M}_{k(n,s_n)} (X_{n,s_n}) \right)$$

with $X_{n,1}, \ldots, X_{n,s_n}$ compact Hausdorff spaces.

Suppose $T = (f_{ij}) \in \mathbb{M}_k(C(X))$ is a $k \times k$ matrix of functions. We define $T^{\maltese} = \text{diag}(f, f, \ldots, f)$ where $f = \sum_{i,j=1}^{k} |f_{ij}|^2$. It is clear that if $T \geq 0$, then $\mathfrak{R}(T) \leq \mathfrak{R}\left(T^{\maltese}\right)$. Since $f_{ij} e_{\mathfrak{ss}} = e_{si} T e_{js}$, we have

$$|f_{ij}|^2 e_{\mathfrak{ss}} = (e_{si} T e_{js})^* (e_{si} T e_{js}) = e_{sj} T^* e_{is} e_{si} T e_{js} = e_{js}^* T^* e_{ii} T e_{js}.$$

Thus

$$T^{\maltese}=\sum_{s=1}^{g}\sum_{i,j=1}^{k}|f_{ij}|^{2}e_{\mathfrak{ss}}=\sum_{s=1}^{g}\sum_{i,j=1}^{k}e_{js}^{*}T^{*}e_{ii}Te_{js}.$$

Suppose $A=A_1\oplus\cdots\oplus A_{s_n}\in\mathcal{A}_n^{\#\#}$. We define

$$\Delta_n(A)=A_1^{\maltese}\oplus\cdots\oplus A_{s_n}^{\maltese}.$$

Thus if $A\in\mathcal{A}_n^{\#\#}$, then $\Delta_n(A)$ has the form

$$\Delta_n(A)=\sum_{k=1}^{m}B_kAC_k,$$

with $B_1,C_1,\ldots,B_m,C_m\in\mathcal{A}_n^{\#\#}$.

It is clear that

a. $\Delta_n\left(\mathcal{A}_n^{\#\#}\right)$ is contained in the center $\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ of $\mathcal{A}_n^{\#\#}$, and

b. If $A\geq 0$, then $\mathfrak{R}(A)\leq\mathfrak{R}(\Delta_n(A))\in\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$.

We call a projection $Q\in\mathcal{A}_n^{\#\#}$ **good** if

c. $\hat{\pi}(Q),\hat{\rho}(Q)\in\mathcal{J}_0$

d. $Q\in\left[\mathcal{A}_n\cap\pi^{-1}(\mathcal{J}_0)\right]^{-\text{weak}*}$

e. For all $T\in Q\mathcal{A}^{\#\#}Q$, $\mathcal{M}$-rank$(\hat{\pi}(T))=\mathcal{M}$-rank$(\hat{\rho}(T))$.

Our proof is based on two claims.

Claim 1: If $Q_1,Q_2\in\mathcal{A}_n^{\#\#}$ are good projections, then there is a good projection $P\in\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ such that $Q_1,Q_2\leq P$.

Proof: Suppose $Q\in\mathcal{A}_n^{\#\#}$ is a good projection. Then there are $B_1,C_1,\ldots,B_k,C_k\in\mathcal{A}_n^{\#\#}$ such that

$$E=\sum_{k=1}^{m}B_kQC_k=\Delta_n(Q).$$

Since $\hat{\pi}(Q)$, $\hat{\rho}(Q)\in\mathcal{J}_0$, we see that $\hat{\pi}(E)$ and $\hat{\rho}(E)\in\mathcal{J}_0$, which, in turn, implies $\hat{\pi}(\mathfrak{R}(E))$ and $\hat{\rho}(\mathfrak{R}(E))\in\mathcal{J}_0$. Let $F=\hat{\pi}(\mathfrak{R}(E))\vee\hat{\rho}(\mathfrak{R}(E))\in\mathcal{J}_0$ is a finite projection. Thus $F\mathcal{M}F$ is a finite von Neumann algebra. Let Φ_F be the center-valued trace on $F\mathcal{M}F$. Since Q is a good projection and in $E\mathcal{A}^{\#\#}E$, we know from Lemma 78, for every $A \in \mathcal{A}^{\#\#}$,

$$\Phi_F\left(\hat{\pi}\left(QAQ\right)\right) = \Phi_F\left(\hat{\rho}\left(QAQ\right)\right).$$

Now $\hat{\pi}, \hat{\rho} : E\mathcal{A}^{\#\#}E \to F\mathcal{M}F$ is a $*$-homomorphism, and, for $A \in \mathcal{A}^{\#\#}$,

$$\Phi_F\left(\hat{\pi}\left(EAE\right)\right) = \sum_{j,k=1}^{m} \Phi_F\left(\left[\hat{\pi}\left(B_k\right)\hat{\pi}\left(Q\right)\right]\left[\hat{\pi}\left(Q\right)\hat{\pi}\left(C_k\right)\hat{\pi}\left(A\right)\hat{\pi}\left(B_j\right)\hat{\pi}\left(Q\right)\hat{\pi}\left(C_j\right)\right]\right) =$$

$$= \sum_{j,k=1}^{m} \Phi_F\left(\left[\hat{\pi}\left(Q\right)\hat{\pi}\left(C_k\right)\hat{\pi}\left(A\right)\hat{\pi}\left(B_j\right)\hat{\pi}\left(Q\right)\hat{\pi}\left(C_j\right)\right]\left[\hat{\pi}\left(B_k\right)\hat{\pi}\left(Q\right)\right]\right)$$
$$= \sum_{j,k=1}^{m} \Phi_F\left(\left[\hat{\rho}\left(Q\right)\hat{\rho}\left(C_k\right)\hat{\rho}\left(A\right)\hat{\rho}\left(B_j\right)\hat{\rho}\left(Q\right)\hat{\rho}\left(C_j\right)\right]\left[\hat{\rho}\left(B_k\right)\hat{\rho}\left(Q\right)\right]\right)$$
$$= \sum_{k=1}^{m} \Phi_F\left(\hat{\rho}\left(B_k\right)\hat{\rho}\left(Q\right)\hat{\rho}\left(C_k\right)\right) = \Phi_F\left(\hat{\rho}\left(EAE\right)\right).$$

Thus $\Phi_F \circ \hat{\pi} = \Phi_F \circ \hat{\rho}$ on $E\mathcal{A}^{\#\#}E$, and since $\hat{\pi}, \hat{\rho}$, and Φ_F are weak* continuous, we have $\Phi_F \circ \hat{\pi} = \Phi_F \circ \hat{\rho}$ on $\left(E\mathcal{A}^{\#\#}E\right)^{-\text{weak*}} = \mathfrak{R}\left(E\right)\mathcal{A}^{\#\#}\mathfrak{R}\left(E\right)$.

Finally, since $\left[\mathcal{A}_n \cap \pi^{-1}\left(\mathcal{J}_0\right)\right]^{-\text{weak*}}$ is a weak* closed $*$-algebra, and an ideal for $\mathcal{A}_n^{\#\#}$, we see that

$$E = \Delta_n\left(Q\right) = \sum_{k=1}^{m} B_k Q C_k \in \left[\mathcal{A}_n \cap \pi^{-1}\left(\mathcal{J}_0\right)\right]^{-\text{weak*}},$$

so $P = \mathfrak{R}\left(E\right) \in \left[\mathcal{A}_n \cap \pi^{-1}\left(\mathcal{J}_0\right)\right]^{-\text{weak*}}$. Thus $\mathfrak{R}\left(E\right) \in \mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ is a good projection and $Q \leq \mathfrak{R}\left(E\right)$.

Now if $Q_1, Q_2 \in \mathcal{A}_n^{\#\#}$ are good projections, the $P_k = \mathfrak{R}\left(\Delta_n\left(Q_k\right)\right)$ is a good projection in $\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ and $Q_k \leq P_k$ for $k = 1, 2$. However, $P_1P_2 = P_2P_1$, and it easily follows that

$$P = P_1 \vee P_2 = P_1\left(1 - P_2\right) + P_1P_2 + P_2\left(1 - P_1\right)$$

is the direct sum of three good projections and is therefore good. Thus Claim 1 is proved.

Claim 2: If $x \in \mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0)$, then $\mathfrak{R}(x) \in \mathcal{A}_n^{\#\#}$ is good.

Proof: We know that $\hat{\pi}(\mathfrak{R}(x))$ and $\hat{\rho}(\mathfrak{R}(x))$ are Murray von Neumann equivalent and, since $\mathcal{M}$-rank$(\pi(x))$ and $\mathcal{M}$-rank$(\rho(x))$ are equal. Since $\pi(x) \in \mathcal{J}_0$, we know $\hat{\pi}(\mathfrak{R}(x))$,$\hat{\rho}(\mathfrak{R}(x)) \in \mathcal{J}_0$. Arguing as in the proof of Claim 1, we let $F = \hat{\pi}(\mathfrak{R}(x)) \vee \hat{\rho}(\mathfrak{R}(x)) \in \mathcal{J}_0$ and get

$$\hat{\pi}, \hat{\rho} : [x\mathcal{A}x]^{-\|\|} \to F\mathcal{M}F$$

satisfy $\Phi_F \circ \hat{\pi} = \Phi_F \circ \hat{\rho}$. Thus $\Phi_F \circ \hat{\pi} = \Phi_F \circ \hat{\rho}$ on $[x\mathcal{A}x]^{-\text{weak*}} = \mathfrak{R}(x)\mathcal{A}^{\#\#}\mathfrak{R}(x)$. Thus $\mathfrak{R}(x)$ is a good projection.

We can choose a countable dense set $\{b_1, b_2, \ldots\}$ of $\cup_{n=1}^{\infty}(\mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0))$ whose closure is $\pi^{-1}(\mathcal{J})$.

We now want to define a sequence $0 = P_0 \leq P_1 \leq P_2 \leq \cdots$ of good projections such that

1. $P_n \in \mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ for all $n \in \mathbb{N}$,

2. If $1 \leq k \leq n$ and $b_k \in \mathcal{A}_n$, then $\mathfrak{R}(b_k) \leq P_n$, i.e.

$$b_k = b_k P_n$$

Define $P_0 = 0$. Suppose $n \in \mathbb{N}$ and P_k has been define for $0 \leq k < n$. We let $x_n = \sum_{k \leq n, b_k \in \mathcal{A}_n} b_k b_k^* \in \mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0)$. Thus, by Claim 2, P_{n-1} and $\mathfrak{R}(x_n)$ are good projections in $\mathcal{A}_n^{\#\#}$. By Claim 1, there is a good projection $P_n \in \mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ such that $P_{n-1} \leq P_n$ and $\mathfrak{R}(x_n) \leq P_n$. Clearly, if $1 \leq k \leq n$ and $b_k \in \mathcal{A}_n$, we have $\mathfrak{R}(b_k) \leq \mathfrak{R}(x_n) \leq P_n$.

Since P_n is a good projection, $P_n \in [\mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0)]^{-\text{weak*}}$. Thus

$$P_n \leq \sup\left\{\mathfrak{R}(x) : x \in \mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0)\right\} \in \mathcal{A}_n^{\#\#}.$$

Thus $\hat{\pi}(P_n) \leq P_{\pi,\mathcal{J}}$ (the projection onto $H_{\pi,\mathcal{J}}$) and $\hat{\rho}(P_n) \leq P_{\rho,\mathcal{J}}$ (the projection onto $H_{\rho,\mathcal{J}}$). Let $P_e = \lim_{n\to\infty} P_n$ (weak*). Thus $\hat{\pi}(P_e) \leq P_{\pi,\mathcal{J}}$ and $\hat{\rho}(P_e) \leq P_{\rho,\mathcal{J}}$. On the other hand, for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \|b_k - b_k P_n\| = 0.$$

This implies

$$bP_e = b \text{ for every } b \in \left[\pi^{-1}(\mathcal{J})\right]^{-\|\|} .$$

Thus $\hat{\pi}(P_e) = P_{\pi,\mathcal{J}}$ and $\hat{\rho}(P_e) = P_{\rho,\mathcal{J}}$. Thus $P_{\pi,\mathcal{J}}$ and $P_{\rho,\mathcal{J}}$ are Murray von Neumann equivalent.

Since $P_n \in \mathcal{A}_n'$ for each $n \in \mathbb{N}$, we have of every $A \in \cup_{k=1}^{\infty}\mathcal{A}_k$,

$$\lim_{n\to\infty} \|AP_n - P_n A\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|AP_n - P_n A\| = 0$$

holds for every $A \in \mathcal{A}$.

Choose a dense subset $\{A_1, A_2, \ldots\}$ of $\mathcal{A}$. Suppose $\varepsilon > 0$ and $m \in \mathbb{N}$. It follows that we can choose a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ such that, for all $1 \leq n \leq m$,

$$\sum_{k=1}^{\infty} \|A_n P_k - P_k A_n\| < \infty$$

and

$$\sum_{k=1}^{\infty} \|A_n P_k - P_k A_n\| < \varepsilon/37.$$

Define $E_k = P_{n_k} - P_{n_{k-1}}$ ($P_{n_0} = 0$) and define $\varphi : \mathcal{A} \to \sum_{1\leq k<\infty}^{\oplus} E_k \mathcal{A} E_k$ by

$$\varphi(T) = \sum_{k=1}^{\infty} E_k T E_k.$$

It follows from [37, page 903] that the above conditions on $\|A_n P_{n_k} - P_{n_k} A_n\|$ that, for all $k \in \mathbb{N}$,

$$A_k - \varphi(A_k) \in \hat{\pi}^{-1}(\mathcal{J}) \cap \hat{\rho}^{-1}(\mathcal{J})$$

and

$$\|A_n P_e - \varphi(A_n)\| < \varepsilon/4$$

for $1 \leq n \leq m$.

Suppose $k \in \mathbb{N}$. For each $n \geq n_k$, $E_k \mathcal{A}_n E_k \subset \mathcal{A}_n^{\#\#}$, which is homogeneous. Hence $C^*(E_k \mathcal{A}_n E_k)$ is subhomogeneous. Thus $C^*(E_k \mathcal{A} E_k)$ is ASH. If we let $E_k = \hat{\pi}(e_k) = \hat{\rho}(e_k)$ for each $k \in \mathbb{N}$, we have

$$\hat{\pi}, \hat{\rho} : C^*(e_k \mathcal{A} e_k) \to E_k \mathcal{R} E_k,$$

and since

$$\tau \circ \left(\hat{\pi}|_{C^*(e_k \mathcal{A} e_k)}\right) = \tau \circ \left(\hat{\rho}|_{C^*(e_k \mathcal{A} e_k)}\right),$$

and $C^*(e_k \mathcal{A} e_k)$ is ASH, it follows from [17], that

$$\hat{\pi}|_{C^*(e_k \mathcal{A} e_k)} \sim_a \hat{\rho}|_{C^*(e_k \mathcal{A} e_k)} \ (E_k \mathcal{R} E_k).$$

We can therefore choose a unitary $U_k \in E_k \mathcal{R} E_k$ such that

$$\|U_k E_k \pi(a_n) E_k U_k^* - E_k \rho(a_n) E_k\| < \frac{1}{37(4^k)m}$$

when $1 \leq n \leq k+m$. Let $W_m = \sum_{k=1}^{\infty} U_k \oplus (1-p_e)$ is a unitary operator in $\mathcal{R}$.

Also, for each $n \in \mathbb{N}$,

$$W_m \hat{\pi}(\varphi(a_n)) W_m^* - \hat{\rho}(\varphi(a_n)) \in \mathcal{K}_{\mathcal{R}},$$

and

$$\|W_m \hat{\pi}(\varphi(a_n)) W_m^* - \hat{\rho}(\varphi(a_n))\| < \frac{1}{37m}$$

for $1 \leq n \leq m$.

It follows that

$$W_m \hat{\pi}\left(\varphi\left(a\right)\right) W_m^* - \hat{\rho}\left(\varphi\left(a\right)\right) \in \mathcal{K}_{\mathcal{R}}$$

for every $a \in \mathcal{A}$.

Thus, for every $a \in \mathcal{A}$

$$W_m \pi\left(a\right) W_m^* - \rho\left(a\right) = W_m \hat{\pi}\left(a - \varphi\left(a\right)\right) W_m^*$$

$$+ W_m \left[\hat{\pi}\left(\varphi\left(a\right)\right) - \hat{\rho}\left(\varphi\left(a\right)\right)\right] W_m^* + W_m \hat{\pi}\left(\varphi\left(a\right) - a\right) W_m^*,$$

so

$$W_m \pi\left(a\right) W_m^* - \rho\left(a\right) \in \mathcal{K}_{\mathcal{R}} \ .$$

Moreover, the same computation show that

$$\left\| W_m \pi\left(a_n\right) W_m^* - \rho\left(a_n\right) \right\| < \frac{1}{m}.$$

for $1 \leq n \leq m$.

Hence, for every $n \in \mathbb{N}$,

$$\left\| W_m \pi\left(a_n\right) W_m^* - \rho\left(a_n\right) \right\| \to 0 \ .$$

Thus, for every $a \in \mathcal{A}$,

$$\left\| W_m \pi\left(a\right) W_m^* - \rho\left(a\right) \right\| \to 0.$$

Thus $\pi \sim_a \rho\left(\mathcal{K}_{\mathcal{R}}\right)$. □

Remark 2. *In two cases, namely, when $H_{\pi,\mathcal{J}} = H_{\rho,\mathcal{J}} = H$, or when $\pi\left(\cdot\right)|_{H^{\perp}_{\pi,\mathcal{J}}}$ and $\rho\left(\cdot\right)|_{H^{\perp}_{\rho,\mathcal{J}}}$ are unitarily equivalent, the conclusion in Theorem 83 becomes*

$$\pi \sim_a \rho \ \left(\mathcal{J}\right) \ .$$

When $\mathcal{A}$ is a separable ASH C*-algebra and $\mathcal{M}$ is a sigma-finite II_∞ factor von Neumann algebra, we can use Theorems 83 and 82 to have both parts of Voiculescu's theorem, including an extension of results in [17].

Corollary 84. *Suppose $\mathcal{A}$ is a separable ASH C*-algebra, $\mathcal{M}$ is a sigma-finite type II_∞ factor von Neumann algebra on a Hilbert space H, and τ is a faithful normal tracial weight on $\mathcal{M}$. Suppose $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that, for every $a \in \mathcal{A}$*

$$\mathcal{M}\text{-}rank\left(\pi\left(a\right)\right) = \mathcal{M}\text{-}rank\left(\rho\left(a\right)\right) .$$

Then $\pi \sim_a \rho \left(\mathcal{K}_{\mathcal{M}}\right)$.

Theorem 85. *Suppose $\mathcal{M} \subset B\left(H\right)$ is a semifinite von Neumann algebra with no finite summands, H is separable, and $\mathcal{A}$ is a separable unital ASH C*-algebra. Also suppose $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that, for every $a \in \mathcal{A}$*

$$\mathcal{M}\text{-}rank\left(\pi\left(a\right)\right) = \mathcal{M}\text{-}rank\left(\rho\left(a\right)\right) .$$

Then $\pi \sim_a \rho \left(\mathcal{K}_{\mathcal{M}}\right)$.

Proof. We can write $\pi = \pi_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1$ and $\rho = \rho_{\mathcal{K}_{\mathcal{M}}} \oplus \rho_1$. It follows from Theorem 82 that

$$\pi \sim_a \pi_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1 \oplus \rho_1 \;\left(\mathcal{K}_{\mathcal{M}}\right) \text{ and } \rho \sim_a \rho_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1 \oplus \rho_1 \;\left(\mathcal{K}_{\mathcal{M}}\right).$$

It follows from Theorem 83 that $\pi \sim_a \rho \left(\mathcal{K}_{\mathcal{M}}\right)$. □

CHAPTER 7

A CHARACTERIZATION OF TRACIALLY NUCLEAR C*-ALGEBRAS

In the chapter, we give two characterizations of tracially nuclear C*-algebras. The first is that the finite summand of the second dual is hyperfinite. The second is in terms of a variant of the weak* uniqueness property. The necessary condition holds for all tracially nuclear C*-algebras. When the algebra is separable, we prove the sufficiency.

7.1 Introduction

Suppose $\mathcal{A}$ is a unital C*-algebra, $\mathcal{M}$ is a von Neumann algebra and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms. We say that π and ρ are *weak* approximately unitarily equivalent in* $\mathcal{M}$ if and only if, there are nets $\{U_\lambda\}$ and $\{V_\lambda\}$ of unitary operators in $\mathcal{M}$, such that, for every $a \in \mathcal{A}$,

$$U_\lambda \pi(a) U_\lambda^* \to \rho(a) \text{ and } V_\lambda \rho(a) V_\lambda^* \to \pi(a)$$

in the weak*-topology. In [17] H. Ding and D. Hadwin defined the $\mathcal{M}$-rank(T) of an operator T in $\mathcal{M}$ as the Murray von Neumann equivalence class of the projection onto the closure of the range of T.

In [13] A. Ciuperca, T. Giordano, P. W. Ng, Z. Niu proved that if $\mathcal{A}$ is a separable C*-algebra, then the following are equivalent:

1. For every separably acting von Neumann algebra $\mathcal{M}$ and all representations $\pi, \rho : \mathcal{A} \to \mathcal{M}$,

$$\pi \text{ is weak* approximately unitarily equivalent to } \rho \;\Leftrightarrow\; (\mathcal{M}\text{-rank}) \circ \pi = (\mathcal{M}\text{-rank}) \circ \rho.$$

2. $\mathcal{A}$ is nuclear.

In this chapter we address the question of how does the Ciuperca-Giordano-Ng-Niu theorem change if in statement (1) we restrict $\mathcal{M}$ to be a finite von Neumann algebra. The answer turns out to be the condition that $\mathcal{A}$ is *tracially nuclear*, a condition defined in [31].

It is known [62] that a C*-algebra is *nuclear* if and only if, for every Hilbert space H and every unital $*$-homomorphism $\pi : \mathcal{A} \to B(H)$, the von Neumann algebra $\pi(\mathcal{A})''$ generated by $\pi(\mathcal{A})$ is hyperfinite. In [31] a unital C*-algebra $\mathcal{A}$ was defined to be *tracially nuclear* if, for every tracial state τ on $\mathcal{A}$, if π_τ is the GNS representation for τ, then $\pi_\tau(\mathcal{A})''$ is hyperfinite. Tracially nuclear algebras also played a key role in the theory of tracially stable C*-algebras [35].

In this chapter we give two new characterizations of tracially nuclear C*-algebras, the first (Theorem 87) in terms of the second dual of the algebra, and the second (Theorem 93) in terms of weak* approximate equivalence of representations into finite von Neumann algebras. In one direction, we show (Theorem 90) that if $\mathcal{A}$ is any tracially nuclear C*-algebra, and $\mathcal{M}$ is any finite von Neumann algebra, then the rank condition in [17] on two representations $\pi, \rho : \mathcal{A} \to \mathcal{M}$ implies a strong version of weak* approximate equivalence of π and ρ. When $\mathcal{A}$ is separable we prove the converse (Theorem 93). Thus the second characterization is an analogue of the characterization if nuclearity given in [13].

When $\mathcal{A}$ is separable, we only need to check $\pi_\tau(\mathcal{A})''$ is hyperfinite when τ is an *infinite-dimensional factor state*, i.e., $\pi_\tau(\mathcal{A})''$ is a II_1 factor von Neumann algebra.

Lemma 86. *Suppose $\mathcal{A}$ is a separable unital C*-algebra. Then $\mathcal{A}$ is tracially nuclear if and only if, for every infinite-dimensional factor tracial state τ on $\mathcal{A}$, $\pi_\tau(\mathcal{A})''$ is hyperfinite.*

Proof. We let $N = \pi_\tau(\mathcal{A})''$. Since $\mathcal{A}$ is separable, N acts on a separable Hilbert space. Using the central decomposition we can write $N = \int_\Omega^\oplus N_\omega d\mu(\omega)$ where each N_ω is a factor von Neumann algebra, and we can write $\pi_\tau = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$ and $\tau = \int_\Omega^\oplus \tau_\omega d\mu(\omega)$ with each τ_ω a factor state, each $\pi_\omega = \pi_{\tau_\omega}$, and each $\pi_{\tau_\omega}(\mathcal{A})'' = N_\omega$. Since N is hyperfinite if and only if almost every N_ω is hyperfinite, and since every finite-dimensional factor is hyperfinite, the lemma is proved. □

7.2 The second dual $\mathcal{A}^{\#\#}$

If $\mathcal{R} \subset B(H)$ is a finite von Neumann algebra, then we can write $H = \sum_{\gamma\in\Gamma}^{\oplus} H_\gamma$ and $\mathcal{R} = \sum_{\gamma\in\Gamma}^{\oplus} \mathcal{R}_\gamma$, where each $\mathcal{R}_\gamma \subset B(H_\gamma)$ has a faithful normal tracial state τ_γ. We can extend each τ_γ to a tracial state on $\mathcal{R}$ by saying if $T = \sum_{\lambda\in\Gamma}^{\oplus} T_\lambda$, then $\tau_\gamma(T) = \tau_\gamma(T_\gamma)$. Each τ_γ gives a seminorm $\|T\|_{2,\gamma} = \tau_\gamma(T^*T)^{1/2}$. It is a simple fact that on bounded subsets of $\mathcal{R}$, the strong (SOT) and $*$-strong ($*$-SOT) operator topologies coincide and are generated by the family $\left\{\|\cdot\|_{2,\gamma} : \gamma \in \Gamma\right\}$. Thus a bounded net $\{T_n\}$ in $\mathcal{R}$ converges in SOT or $*$-SOT to $T \in \mathcal{R}$ if and only if, for every $\gamma \in \Gamma$,

$$\|T_n - T\|_{2,\gamma} \to 0.$$

Also every von Neumann algebra $\mathcal{R}$ can uniquely be decomposed into a direct sum $\mathcal{R} = \mathcal{R}_f \oplus \mathcal{R}_i$, where $\mathcal{R}_f$ is a finite von Neumann algebra and $\mathcal{R}_i$ has no finite direct summands. Equivalently, $\mathcal{R}_i$ has no normal tracial states. Relative to this decomposition, we write $Q_{f,\mathcal{R}} = 1 \oplus 0$.

If $\mathcal{A}$ is a unital C*-algebra, then $\mathcal{A}^{\#\#}$ is a von Neumann algebra, and, using the universal representation, we can assume $\mathcal{A} \subset \mathcal{A}^{\#\#} \subset B(\mathcal{H})$ where the weak* topology on $\mathcal{A}^{\#\#}$ coincides with the weak operator topology, so that $\mathcal{A}'' = \mathcal{A}^{\#\#}$. Moreover, for every von Neumann algebra $\mathcal{R}$ and every unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{R}$, there is a weak*-weak* continuous unital $*$-homomorphism $\hat{\pi} : \mathcal{A}^{\#\#} \to \mathcal{R}$ such that $\hat{\pi}|_{\mathcal{A}} = \pi$. Moreover, $\ker\hat{\pi}$ being a weak* closed two-sided ideal in $\mathcal{A}^{\#\#}$ has the form

$$\ker\hat{\pi} = (1 - P_\pi)\,\mathcal{A}^{\#\#}, \text{ with } P_\pi = P_\pi^2 = P_\pi^* \in \mathcal{Z}\left(\mathcal{A}^{\#\#}\right),$$

where $\mathcal{Z}(\mathcal{M})$ denotes the center of a von Neumann algebra $\mathcal{M}$. Thus

$$\mathcal{A}^{\#\#} = P_\pi \mathcal{A}^{\#\#} \oplus \ker\hat{\pi}\,.$$

The following theorem contains our first characterization of tracially nuclear C*-algebras.

Theorem 87. *If $\mathcal{A}$ is a unital C*-algebra, then*

1. *For every unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{M}$ with $\mathcal{M}$ a finite von Neumann algebra, $P_\pi \leq Q_{f,\mathcal{A}^{\#\#}}$.*

2. *$\mathcal{A}$ is tracially nuclear if and only if $\left(\mathcal{A}^{\#\#}\right)_f$ is a hyperfinite von Neumann algebra.*

Proof. (1). Assume, via contradiction, $\hat{\pi}\left(1 - Q_{f,\mathcal{A}^{\#\#}}\right) \neq 0$. Since $\mathcal{M}$ is finite, there is a normal tracial state τ on $\mathcal{M}$ such that

$$s = \tau\left(\hat{\pi}\left(1 - Q_{f,\mathcal{A}^{\#\#}}\right)\right) \neq 0.$$

Hence the map $\gamma : \left(\mathcal{A}^{\#\#}\right)_i \to \mathbb{C}$ defined by

$$\gamma(T) = \frac{1}{s}\hat{\pi}(0 \oplus T)$$

is a faithful normal tracial state on $\left(\mathcal{A}^{\#\#}\right)_i$, which is a contradiction. Thus

$$\hat{\pi}\left(1 - Q_{f,\mathcal{A}^{\#\#}}\right) = 0,$$

which means that $P_\pi \leq Q_{f,\mathcal{A}^{\#\#}}$.

(2). Suppose $\mathcal{A}$ is tracially nuclear. $(\mathcal{A}^{\#\#})_f = \Sigma^{\oplus}_{\lambda\in\Lambda}(\mathcal{R}_\lambda, \tau_\lambda)$, where τ_λ is a faithful normal tracial state on $\mathcal{R}_\lambda$. Then $\mathcal{A}^{\#\#} = (\mathcal{A}^{\#\#})_f \oplus \left(\mathcal{A}^{\#\#}\right)_i = \Sigma^{\oplus}_{\lambda\in\Lambda}\mathcal{R}_\lambda \oplus \left(\mathcal{A}^{\#\#}\right)_i$ relative to $\mathcal{H} = \sum^{\oplus}_{\lambda\in\Lambda}\mathcal{H}_\lambda \oplus \mathcal{H}_i$. Viewing $\mathcal{A} \subset \mathcal{A}^{\#\#}$, we let $\pi_\lambda : \mathcal{A} \to \mathcal{R}_\lambda$ be defined by $\pi_\lambda(A) = A|_{\mathcal{H}_\lambda}$. Then $\psi_\lambda = \tau_\lambda \circ \pi_\lambda$ is a tracial state on $\mathcal{A}$ and, since $\mathcal{A}$ is weak*-dense in $\mathcal{A}^{\#\#}$, $\pi_{\psi_\lambda}(\mathcal{A})^{-\text{weak*}} = \mathcal{R}_\lambda$. Since $\mathcal{A}$ is tracially nuclear, $\mathcal{R}_\lambda$ must be hyperfinite. Hence, $(\mathcal{A}^{\sharp\sharp})_f = \Sigma^{\oplus}_{\lambda\in\Lambda}\mathcal{R}_\lambda$ is hyperfinite.

Conversely, suppose $\left(\mathcal{A}^{\#\#}\right)_f$ is hyperfinite, and suppose τ is a tracial state on $\mathcal{A}$. Since $\pi_\tau(\mathcal{A})''$ has a faithful normal tracial state, it must be finite. Thus $P_{\pi_\tau} \leq Q_{f,\mathcal{A}^{\#\#}}$. This means that $P_{\pi_\tau}\mathcal{A}^{\#\#}$ is a direct summand of $\left(\mathcal{A}^{\#\#}\right)_f$, and is therefore hyperfinite. But this summand is isomorphic to $\pi_\tau(\mathcal{A})''$. Thus $\mathcal{A}$ is tracially nuclear. □

7.3 Weak* approximate equivalence in finite von Neumann algebras

Suppose $\mathcal{A}$ is a unital C*-algebra, $\mathcal{R}$ is a von Neumann algebra and $\pi, \rho : \mathcal{A} \to \mathcal{R}$ are unital $*$-homomorphisms. Following [13], π and ρ are *weak* approximately equivalent* if there are nets $\{U_\lambda\}$ and $\{V_\lambda\}$ of unitary operators in $\mathcal{R}$ such that, for every $A \in \mathcal{A}$,

$$U_\lambda^* \pi\left(A\right) U_\lambda \overset{\text{weak*}}{\to} \rho\left(A\right) \text{ and } V_\lambda^* \rho\left(A\right) V_\lambda \overset{\text{weak*}}{\to} \pi\left(A\right).$$

It was observed in [13] that it follows that the convergence above actually occurs in the $*$-strong operator topology ($*$-SOT).

Suppose $\mathcal{M}$ is a von Neumann algebra and $T \in \mathcal{M}$. Following [17], $\mathcal{M}$-rank(T) is defined to be the Murray von Neumann equivalence class in $\mathcal{M}$ of the projection onto the closure of the range of T. In [13] it was shown that if $\mathcal{A}$ is a separable nuclear C*-algebra and $\mathcal{M}$ is a von Neumann algebra acting on a separable Hilbert space, then two unital $*$-homomorphisms $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are weak* approximately equivalent if and only if, $(\mathcal{M}\text{-rank}) \circ \pi = (\mathcal{M}\text{-rank}) \circ \rho$. They also proved that this property for $\mathcal{A}$ is equivalent to nuclearity.

The following result is from [32].For completeness we include a short proof.

Lemma 88. *[32]Suppose $a = a^*$ in $\mathcal{B}(H)$, $0 \leq a \leq 1$ and $\mathcal{C}_0^*(a)$ is the norm-closure of $\{p(a), p \in \mathbb{C}\left[z\right], p(0) = 0\}$. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a center-valued trace $\Phi : \mathcal{M} \to \mathcal{Z}(M)$, and $\pi, \rho : \mathcal{C}_0^*(a) \to \mathcal{M}$ are *-homomorphisms. Then the following are equivalent:*

(1). $\forall x \in \mathcal{C}_0^*(a)$, $\mathcal{M}$-$rank\ \pi(x) = \mathcal{M}$-$rank\ \rho(x)$,

(2). $\Phi \circ \pi = \Phi \circ \rho$.

Proof. $(1) \Rightarrow (2)$. We can extend π and ρ to weak*-weak* continuous *-homomorphisms $\hat{\pi}, \hat{\rho} : \mathcal{C}_0^*(a)^{\#\#} \to \mathcal{M}$. Suppose $x \in \mathcal{C}_0^*(a)$ and $0 \leq x \leq 1$. Suppose $0 < \alpha < 1$ and define $f_\alpha : [0,1] \to [0,1]$ by

$$f\left(t\right) = dist\left(t, [0, \alpha]\right).$$

Since $f(0)=0$, we see that $f(x)\in\mathcal{A}$, and $\chi_{(\alpha,1]}(x) = $ weak*-$\lim_{n\to\infty} f(x)^{1/n} \in \mathcal{A}^{\#\#}$, so

$$\hat{\pi}\left(\chi_{(\alpha,1]}(x)\right), \text{ and } \hat{\rho}\left(\chi_{(\alpha,1]}(x)\right)$$

are the range projections for $\pi(f(x))$ and $\rho(f(x))$, respectively. Since

$$\mathcal{M}-rank\ \pi(f(x)) = \mathcal{M}-rank\ \rho(f(x)),$$

we see that $\hat{\rho}\left(\chi_{(\alpha,1]}(x)\right)$ and $\hat{\pi}\left(\chi_{(\alpha,1]}(x)\right)$ are Murray von Neumann equivalent. Hence

$$\Phi\left(\hat{\pi}\left(\chi_{(\alpha,1]}(x)\right)\right) = \Phi\left(\hat{\rho}\left(\chi_{(\alpha,1]}(x)\right)\right).$$

Thus, Suppose $0<\alpha<\beta<1$. Since $\chi_{(\alpha,\beta]} = \chi_{(\alpha,1]} - \chi_{(\beta,1]}$, we see that

$$\Phi\left(\hat{\pi}\left(\chi_{(\alpha,\beta]}(x)\right)\right) = \Phi\left(\hat{\rho}\left(\chi_{(\alpha,\beta]}(x)\right)\right).$$

Thus, for all $n\in\mathbb{N}$,

$$\Phi\left(\hat{\pi}\left(\sum_{k-1}^{n-1}\frac{k}{n}\chi_{(\frac{k}{n},\frac{k+1}{n}]}(x)\right)\right) = \Phi\left(\hat{\rho}\left(\sum_{k-1}^{n-1}\frac{k}{n}\chi_{(\frac{k}{n},\frac{k+1}{n}]}(x)\right)\right).$$

Since, for every $n\in\mathbb{N}$,

$$\left\|x-\sum_{k-1}^{n-1}\frac{k}{n}\chi_{(\frac{k}{n},\frac{k+1}{n}]}(x)\right\| \leq 1/n,$$

it follows that

$$\Phi(\pi(x)) = \Phi(\hat{\pi}(x)) = \Phi(\hat{\rho}(x)) = \Phi(\rho(x)).$$

Since $\mathcal{A}$ is the linear span of its positive contractions, $\Phi\circ\pi = \Phi\circ\rho$.

$(2) \Rightarrow (1)$. Since Φ, $\hat{\pi}$ and $\hat{\rho}$ are weak*-weak* continuous, it follows that $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$, so we see, for any $x \in \mathcal{C}_0^*(a)$ that

$$\Phi\left(\hat{\pi}\left(\chi_{(0,\infty)}\left(|x|\right)\right)\right) = \Phi\left(\hat{\rho}\left(\chi_{(0,\infty)}\left(|x|\right)\right)\right),$$

which implies that $\chi_{(0,\infty)}\left(|\pi\left(x\right)|\right)$ and $\chi_{(0,\infty)}\left(|\rho\left(x\right)|\right)$. Thus $\mathcal{M}$-$rank\ \pi(x) = \mathcal{M}$-$rank\ \rho(x)$. □

The following lemma is from [17].

Lemma 89. *[17]Suppose $\mathcal{B} = \Sigma_{m=1}^{t}\mathcal{M}_{k_m}(\mathbb{C})$ with matrix units $e_{i,j,m}$, $\mathcal{D}$ is a unital C*-algebra, and $\pi, \rho : \mathcal{B} \to \mathcal{D}$ are unital *-homomorphisms such that $\pi(e_{i,i,m}) \sim \rho(e_{i,i,m})$. Then, there exists a unitary $w \in \mathcal{D}$ such that $\pi\left(\cdot\right) = w^*\rho\left(\cdot\right)w$.*

Theorem 90. *Suppose $\mathcal{A}$ is a unital tracially nuclear C*-algebra, $\mathcal{M}$ is a finite von Neumann algebra with center-valued trace Φ, and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital *-homomorphisms. The following are equivalent:*

1. *For every $a \in \mathcal{A}$, $\mathcal{M}$-rank $\pi(a) = \mathcal{M}$-rank $\rho(a)$.*
2. *$\Phi \circ \pi = \Phi \circ \rho$.*
3. *The representations π and ρ are weak* approximately equivalent.*
4. *There is a net $\{U_n\}$ of unitary operators in $\mathcal{M}$ such that, for every $a \in \mathcal{A}^{\#\#}$,*
   - *(a) $U_n\pi(a)U_n^* \to \rho(a)$ in the $*$strong operator topology, and*
   - *(b) $U_n^*\rho(a)U_n \to \pi(a)$ in the $*$strong operator topology.*

Proof. Clearly, $(4) \Rightarrow (3) \Rightarrow (2)$.

$(1) \Leftrightarrow (2)$. This is proved in 88.

$(2) \Rightarrow (4)$. Let $\hat{\pi}, \hat{\rho} : \mathcal{A}^{\sharp\sharp} \to \mathcal{M}$ be the weak*-weak* continuous extensions of π and ρ, respectively. Since Φ is weak*-weak* continuous, we see that $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$. Since $\mathcal{M}$ is finite,

$\mathcal{M}$ can be written as $\mathcal{M} = \Sigma^{\oplus}_{\gamma\in\Gamma}(\mathcal{M}_\gamma, \beta_\gamma)$, where β_γ is a faithful normal tracial state of $\mathcal{M}_\gamma$. Similarly, we can write $(\mathcal{A}^{\sharp\sharp})_f = \Sigma^{\oplus}_{\lambda\in\Lambda}(\mathcal{R}_\lambda, \tau_\lambda)$ where τ_λ is a faithful normal tracial state on $\mathcal{R}_\lambda$ for each $\lambda \in \Lambda$. Thus $\mathcal{A}^{\#\#} = \Sigma^{\oplus}_{\lambda\in\Lambda}(\mathcal{R}_\lambda, \tau_\lambda) \oplus \left(A^{\#\#}\right)_i$. If $S \in \mathcal{M}$ and $T \in \mathcal{A}^{\#\#}$, we write

$$S = \sum_{\gamma\in\Gamma} S(\gamma) \text{ and } T = \sum_{\lambda\in\Lambda} T(\lambda) \oplus T(i).$$

Since $\mathcal{M}$ is finite, we know from 87 that $\hat{\pi}\left(Q_{f,\mathcal{A}^{\#\#}}\right) = \hat{\rho}\left(Q_{f,\mathcal{A}^{\#\#}}\right) = 1$. We also know that $\hat{\pi}$ and $\hat{\rho}$ are continuous in the strong operator topology. Thus if $\{T_j\}$ is a norm-bounded net in $\mathcal{A}^{\#\#}$ and $T \in \mathcal{A}^{\#\#}$, and $T_j Q_{f,\mathcal{A}^{\#\#}} \to T Q_{f,\mathcal{A}^{\#\#}}$ in the strong operator topology, then $\hat{\pi}(T_j) = \hat{\pi}\left(T_j Q_{f,\mathcal{A}^{\#\#}}\right) \to \hat{\pi}(T)$ and $\hat{\rho}(T_j) \to \hat{\rho}(T)$ in the strong operator topology. This means that, if, for every $\lambda \in \Lambda$, we have $\|T_j(\lambda) - T(\lambda)\|_{2,\tau_\lambda} \to 0$, then, for every $\gamma \in \Gamma$, we have

$$\|\hat{\pi}(T_j)(\gamma) - \hat{\pi}(T)(\gamma)\|_{2,\beta_\gamma} \to 0 \text{ and } \|\hat{\rho}(T_j)(\gamma) - \hat{\rho}(T)(\gamma)\|_{2,\beta_\gamma} \to 0\,.$$

Suppose $A \subset \text{ball}\left(\mathcal{A}^{\#\#}\right)$ is finite, $L \subset \Lambda$ is finite and $\varepsilon > 0$. Then there is a $\delta > 0$ and a finite subset $G \subset \Gamma$ such that, if $T \in A$, $S \in 2\text{ball}\left(\mathcal{A}^{\#\#}\right)$ and, for every $\lambda \in L$, we have $\|T(\lambda) - S(\lambda)\|_{2,\tau_\lambda} < \delta$, then

$$\sum_{\gamma\in G}\left[\|\hat{\pi}(S)(\gamma) - \hat{\pi}(T)(\gamma)\|_{2,\beta_\gamma} + \|\hat{\rho}(S)(\gamma) - \hat{\rho}(T)(\gamma)\|_{2,\beta_\gamma}\right] < \varepsilon/37\,.$$

Since $\mathcal{A}$ is tracially nuclear, we know that, for every $\lambda \in \Lambda$, $\mathcal{R}_\lambda$ is hyperfinite. Thus, for each $\lambda \in L$, there is a finite-dimensional unital C*-subalgebra $\mathcal{B}_\lambda \subset \mathcal{R}_\lambda$ such that, for each $S \in A$, there is a $B_{\lambda,S} \in \mathcal{B}_\lambda$ such that $\|B_{\lambda,S}\| \le \|S(\lambda)\|$ and $\|S(\lambda) - B_{\lambda,S}\|_{2,\tau_\lambda} < \delta$. Then $\mathcal{B} = \sum^{\oplus}_{\lambda\in L}\mathcal{B}_\lambda$ is a finite-dimensional C*-subalgebra of $\mathcal{A}^{\#\#}$, and, for each $S \in A$, we define $B_s = \sum^{\oplus}_{\lambda\in L} B_{\lambda,S} \in \mathcal{B}$. It follows that

$$\sum_{S\in A}\sum_{\gamma\in G}\left[\|\hat{\pi}(S)(\gamma) - \hat{\pi}(B_S)(\gamma)\|_{2,\beta_\gamma} + \|\hat{\rho}(S)(\gamma) - \hat{\rho}(B_S)(\gamma)\|_{2,\beta_\gamma}\right] < \varepsilon/37\,.$$

We know from $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$ and Lemma 89 that there is a unitary operator $U = U_{(A,G,\varepsilon)} \in \mathcal{M}$ such that, for every $W \in \mathcal{B}$,

$$U\hat{\pi}(W)\,U^* = \hat{\rho}(W)\ .$$

We therefore have,

$$\sum_{S\in A}\sum_{\gamma\in G}\|U\hat{\pi}(S)\,U^*(\gamma)-\hat{\rho}(S)(\gamma)\|_{2,\beta_\gamma}$$

$$\leq \sum_{S\in A}\sum_{\gamma\in G}\left[\|U(\hat{\pi}(S)(\gamma)-\hat{\pi}(B_S)(\gamma))\,U^*+\|_{2,\beta_\gamma}+\|\hat{\rho}(B_S)(\gamma)-\hat{\rho}(S)(\gamma)\|_{2,\beta_\gamma}\right]$$

$$\leq \varepsilon/37+\varepsilon/37<\varepsilon.$$

Also

$$\sum_{S\in A}\sum_{\gamma\in G}\|\hat{\pi}(S)(\gamma)-U^*\hat{\rho}(S)\,U(\gamma)\|_{2,\beta_\gamma}$$

$$=\sum_{S\in A}\sum_{\gamma\in G}\|U\hat{\pi}(S)\,U^*(\gamma)-\hat{\rho}(S)(\gamma)\|_{2,\beta_\gamma}<\varepsilon\ .$$

If we order the triples (A,G,ε) by $(\subset,\subset,>)$, we have a net $\left\{U_{(A,G,\varepsilon)}\right\}$ of unitary operators in $\mathcal{M}$ such that, for every $T\in\mathcal{A}^{\#\#}$,

$$U_{(A,G,\varepsilon)}\hat{\pi}(T)U^*_{(A,G,\varepsilon)}\to\hat{\rho}(T)\text{ and }U^*_{(A,G,\varepsilon)}\hat{\rho}(T)U_{(A,G,\varepsilon)}\to\hat{\pi}(T)\ .$$

in the strong operator topology. □

7.4 FWU algebras: A converse

In this section we prove a converse of Theorem 90 when $\mathcal{A}$ is separable. We say that a unital C*-algebra $\mathcal{A}$ is an *FWU algebra*, or that $\mathcal{A}$ has the *finite weak*-uniqueness property*, if, for every finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ and every pair $\pi, \rho : \mathcal{A} \to \mathcal{M}$ of unital $*$-homomorphisms such that, for all $a \in \mathcal{A}$,

$$\mathcal{M}\text{-rank}\left(\pi\left(a\right)\right) = \mathcal{M}\text{-rank}\left(\rho\left(a\right)\right),$$

there is a net $\{U_i\}$ of unitary operators in $\mathcal{M}$, such that, for every $a \in \mathcal{A}$,

$$\left\|U_i \pi\left(a\right) U_i^* - \rho\left(a\right)\right\|_{2,\tau} \to 0.$$

Since every finite von Neumann algebra is a direct sum of algebras having a faithful normal tracial state [62], being an FWU algebra is equivalent to saying that for every finite von Neumann algebra and every pair $\pi, \rho : \mathcal{A} \to \mathcal{M}$ of unital $*$-homomorphisms such that, for all $a \in \mathcal{A}$,

$$\mathcal{M}\text{-rank}\left(\pi\left(a\right)\right) = \mathcal{M}\text{-rank}\left(\rho\left(a\right)\right),$$

we have that π and ρ are weak* approximately unitarily equivalent.

A key ingredient is a result of Alain Connes [14], who proved the following characterization of hyperfiniteness. If $\mathcal{N}$ is a von Neumann algebra, then the *flip automorphism* $\pi : \mathcal{N} \otimes \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ is the automorphism defined by $\pi\left(a \otimes b\right) = b \otimes a$.

Theorem 91. *[14]Suppose $\mathcal{N} \subset B\left(H\right)$ is a II_1 factor von Neumann algebra acting on a separable Hilbert space. The following are equivalent:*

1. *$\mathcal{N}$ is hyperfinite,*

2. *For every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{N}$, $y_1, \ldots, y_n \in \mathcal{N}'$,*

$$\left\|\sum_{k=1}^{n} x_k y_k\right\|_H = \left\|\sum_{k=1}^{n} x_k \otimes y_k\right\|_{H \otimes H},$$

3. *The flip automorphism π on $\mathcal{N} \otimes \mathcal{N}$ is weak* approximately unitarily equivalent in $\mathcal{N} \otimes \mathcal{N}$ to the identity representation.*

If $\mathcal{N}$ is a von Neumann algebra and the flip automorphism π is weak* approximately equivalent to the identity, it easily follows that the implementing net $\{U_\lambda\}$ of unitaries simultaneously makes the maps $\rho_1, \rho_2 : \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ by

$$\rho_1(a) = a \otimes 1,\ \rho_2(a) = 1 \otimes a \text{ for every } a \in \mathcal{N}.$$

weak* approximately equivalent. Using Connes' proof we obtain a stronger statement. This statement fills in the details of Remark 7 in [19], which says that if $\mathcal{M} = W^*(X_1, \ldots, X_m)$ is a separably acting factor von Neumann algebra with trace τ, then $\mathcal{M}$ is nuclear whenever the following condition holds:

For every $\varepsilon > 0$ there is a $\delta > 0$ and a positive integer N, such that, for every factor von Neumann algebra $\mathcal{N}$ with trace ρ, and for all $A_1, B_1 \ldots, A_n, B_n \in \mathcal{N}$, if

$$|\tau(m(X_1, \ldots, X_m)) - \rho(m(A_1, \ldots, A_m))| < \delta$$

and

$$|\tau(m(X_1, \ldots, X_m)) - \rho(m(B_1, \ldots, B_m))| < \delta$$

for all $*$-monomials m with degree$(m) \leq N$, there is a unitary operator $U \in \mathcal{N}$ such that

$$\sum_{k=1}^{m} \|UA_kU^* - B_k\|_{2,\rho}^2 < \varepsilon.$$

Theorem 92. *Suppose $\mathcal{N} \subset B(H)$ is a finite factor von Neumann algebra acting on a separable Hilbert space H. Define $\rho_1, \rho_2 : \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ by*

$$\rho_1(a) = a \otimes 1,\ \rho_2(a) = 1 \otimes a \text{ for every } a \in \mathcal{N}.$$

Suppose ρ_1 and ρ_2 are weak approximately equivalent in $\mathcal{N} \otimes \mathcal{N}$. Then $\mathcal{N}$ is hyperfinite.*

Proof. Let τ be the unique faithful normal tracial state on $\mathcal{N}$. Then $\tau \otimes \tau$ is a faithful normal tracial state on the factor $\mathcal{N} \otimes \mathcal{N} \subset B(H \otimes H)$. Suppose ρ_1 and ρ_2 are weak* approximately equivalent in $\mathcal{N} \otimes \mathcal{N}$. Thus we can choose a net $\{U_\lambda\}$ of unitary operators in $\mathcal{N} \otimes \mathcal{N}$ such that, for every $a \in N$,

$$\|U_\lambda^* (a \otimes 1) U_\lambda - (1 \otimes a)\|_{2,\tau\otimes\tau} \to 0.$$

Suppose $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{N}$, $y_1, \ldots, y_n \in \mathcal{N}'$. Since $U_\lambda \in \mathcal{N} \otimes \mathcal{N}$ and each $1 \otimes y_k \in (\mathcal{N} \otimes \mathcal{N})'$, we have

$$U_\lambda^* \left(\sum_{k=1}^n x_k \otimes y_k \right) U_\lambda = \sum_{k=1}^n U_\lambda^* (x_k \otimes 1)(1 \otimes y_k) U_\lambda =$$

$$\sum_{k=1}^n [U_\lambda^* (x_k \otimes 1) U_\lambda] (1 \otimes y_k) \overset{\text{weak*}}{\to} \sum_{k=1}^n (1 \otimes x_k)(1 \otimes y_k) = 1 \otimes \left(\sum_{k=1}^n x_k y_k \right).$$

Since, for every λ,

$$\left\| U_\lambda^* \left(\sum_{k=1}^n x_k \otimes y_k \right) U_\lambda \right\| = \left\| \sum_{k=1}^n x_k \otimes y_k \right\|,$$

it follows that

$$\left\| \sum_{k=1}^n x_k y_k \right\| \leq \left\| \sum_{k=1}^n x_k \otimes y_k \right\| .$$

It also follows that, for every $a \in \mathcal{N}$,

$$\|a \otimes 1 - U_\lambda (1 \otimes a) U_\lambda^*\|_{2,\tau\otimes\tau} = \|U_\lambda^* (a \otimes 1) U_\lambda - (1 \otimes a)\|_{2,\tau\otimes\tau} \to 0 .$$

Thus

$$U_\lambda \left[1 \otimes \left(\sum_{k=1}^n x_k y_k \right) \right] U_\lambda^* = U_\lambda \sum_{k=1}^n (1 \otimes x_k)(1 \otimes y_k) U_\lambda^*$$

$$= \sum_{k=1}^n U_\lambda (1 \otimes x_k) U_\lambda^* (1 \otimes y_k) \overset{\text{weak*}}{\to} \sum_{k=1}^n (x_k \otimes 1)(1 \otimes y_k) = \sum_{k=1}^n x_k \otimes y_k.$$

Thus

$$\left\| \sum_{k=1}^n x_k \otimes y_k \right\| \leq \left\| \sum_{k=1}^n x_k y_k \right\|.$$

Thus by Connes' theorem (Theorem 91), $\mathcal{N}$ is hyperfinite. □

We now prove our converse result.

Theorem 93. *A separable unital C*-algebra is an FWU algebra if and only if it is tracially nuclear.*

Proof. Suppose $\mathcal{A}$ is an FWU algebra. Suppose τ is a factor tracial state on $\mathcal{A}$. Let $\mathcal{N} = \pi_\tau(\mathcal{A})''$. Since $\mathcal{A}$ is separable and π_τ has a cyclic vector, $\mathcal{N}$ acts on a separable Hilbert space. If $\mathcal{N}$ is finite-dimensional, then $\mathcal{N}$ is hyperfinite. Thus we can assume that $\mathcal{N}$ is a II_1 factor. Then $\mathcal{N} \subset L^2(\mathcal{A}, \tau)$ and $\pi_\tau(\mathcal{A})$ is $\|\|_{2,\tau}$-dense in $\mathcal{N}$. Define $\rho_1, \rho_2 : \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$ by

$$\rho_1(b) = b \otimes 1, \ \rho_2(b) = 1 \otimes b \text{ for every } b \in \mathcal{N}.$$

For $k = 1, 2$, let $\sigma_k = \rho_k \circ \pi_\tau : \mathcal{A} \to \mathcal{N} \otimes \mathcal{N}$. Since $(\tau \otimes \tau) \circ \rho_1 = (\tau \otimes \tau) \circ \rho_2$, we see that $(\tau \otimes \tau) \circ \sigma_1 = (\tau \otimes \tau) \circ \sigma_2$. Since $\mathcal{A}$ is an FWU algebra, σ_1 and σ_2 are weak* approximately unitarily equivalent in $\mathcal{N} \otimes \mathcal{N}$. Thus there is a net $\{U_\lambda\}$ of unitary operators in $\mathcal{N} \otimes \mathcal{N}$ such that, for every $b \in \pi_\tau(\mathcal{A})$,

$$\|U_\lambda^* (b \otimes 1) U_\lambda - (1 \otimes b)\|_{2,\tau\otimes\tau} \to 0 .$$

Since, for each λ, the map

$$b \mapsto U_\lambda^* (b \otimes 1) U_\lambda - (1 \otimes b)$$

is $\|\|_{2,\tau\otimes\tau}$-continuous and linear on $\mathcal{N}$ and has norm at most 2, and since $\pi_\tau(\mathcal{A})$ is $\|\|_{2,\tau\otimes\tau}$ dense in $\mathcal{N}$, we see that, for every $b \in \mathcal{N}$,

$$\|U_\lambda^* (b \otimes 1) U_\lambda - (1 \otimes b)\|_{2,\tau\otimes\tau} \to 0 .$$

Thus, by Theorem 92, $\mathcal{N}$ is hyperfinite. It follows from Lemma 86 that $\mathcal{A}$ is tracially nuclear. The other direction is contained in Theorem 90. $\square$

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