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A SPECTRUM FOR NONCOMMUTATIVE RINGS

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by

RUSSELL RAINVILLE

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ABSTRACT

A SPECTRUM FOR NONCOMMUTATIVE RINGS

by

RUSSELL RAINVILLE

This paper constructs a generalization of Spec. If $A$ is a ring with identity (not necessarily commutative) let $\mathcal{J}(A)$ be a set of morphism, $x: A \rightarrow D_x$, such that the codomain of $x$, $D_x$, is a division ring and for all division rings $E$ the relation $\text{Im } x \subseteq E \subseteq D_x$ implies $E = D_x$; moreover, if $y: A \rightarrow D_y$ is any morphism with these properties then there is an $x \in \mathcal{J}(A)$ and an isomorphism $\psi: D_x \rightarrow D_y$ such that

```
  A
  |
  v
 D_x
  |
  v
  y
 x

commutes. If $s: A \rightarrow B$ is a morphism of rings with identity let $\tilde{\mathcal{J}}(s): \mathcal{J}(B) \rightarrow \mathcal{J}(A)$ such that $(\tilde{\mathcal{J}}(s))(y)$ is the one and only $x \in \mathcal{J}(A)$ for which there is a morphism $u: D_x \rightarrow D_y$ so that

```

iv
commutes.

This paper also gives a topology for $\mathfrak{J}(A)$ such that $\mathfrak{J}$ is a contravariant functor from rings with identity into topological spaces. We then show that $\mathfrak{J}$ is a generalization of Spec and that many of the theorems about Spec are extendable to $\mathfrak{J}$. 
INTRODUCTION

One of the more recent tools of Algebraic Geometry is the spectrum of a ring. If A is a commutative ring with identity then Spec A is a topological space whose underlying set is the set of prime ideals of A. For example consider F, K, I, V, A where K is a field extension of F with infinite transcendence degree, I is ideal of \( F[t_1, t_2, \ldots, t_n] \) the ring of polynomials in n indeterminants over \( F \), \( V = \{ x \in K^n \mid \text{for every } f \in I, f(x) = 0 \} \), the variety determined by I, and \( A = F[t_1, t_2, \ldots, t_n]/\sqrt{I} \) the coordinate ring of V. We may study V by examining Spec A since for each \( x \in V \) there is a function \( T_x: A \rightarrow K \) such that \( T_x(f) = f(x) \) and \( \ker T_x \in \text{Spec } A \). In fact the map \( x \mapsto \ker T_x \) is a continuous map of V onto Spec A which "identifies" points which are generic specializations of each other.

To see the possibilities for generalizing to the noncommutative case consider the morphism

\[ x: A \rightarrow K. \]

In the commutative case we chose K to be a field. In the noncommutative case we could choose K to be a simple ring, a prime ring, a primitive ring or a division ring. Most authors have chosen prime rings, some have even suggested injective indecomposable modules [Br]. George Bergman has suggested division rings [B], and this thesis takes up that suggestion.

In the commutative case the field of fractions of the image of x is determined by \( A/\ker x \) but this is not true if A is
noncommutative [C]. So rather than look at the kernels of the morphism we consider the morphism themselves. Chapter I contains the definition of $\mathcal{S}$, the contravariant functor that is proposed as a generalization of Spec. We then give some examples of spectrums—in particular, that the spectrum of a product of division rings is the Stone-Čech compactification of the disjoint union of the spectrums of the factors. Chapter III examines the topological properties of $\mathcal{S}(A)$. We show that $\mathcal{S}(A)$ is $T_0$, that every closed subset of $\mathcal{S}(A)$ is the homeomorphic image of $\mathcal{S}M$ (where $M$ is some discrete space), and that $s: A \to B$ is onto implies that $\mathcal{S}(s)$ is a closed embedding. The second of these results allows us to give alternate proofs of the two results mentioned in Bergman's paper; namely, $\mathcal{S}(A)$ is compact and any closed irreducible subset of $\mathcal{S}(A)$ has a generic point. In the last chapter we begin the development of a presheaf of rings over $\mathcal{S}(A)$.
CHAPTER I
THE FUNCTOR $\mathcal{S}$

In this thesis we shall be a bit slovenly with our notation. When we mean $(A,+_A,\cdot_A,0_A,1_A)$ is a ring with identity we shall say $A$ is a ring with identity, and if $x = ((A,+_A,\cdot_A,0_A,1_A), (B,+_B,\cdot_B,0_B,1_B))$ is a morphism of rings with identity we shall confuse $x$ and the function $f$. At some times it will be important to know the codomain of a morphism $x$ and in those instances we will write $R^x$ for the codomain, or $D^x$ if we know the codomain is a division ring. Since the only rings we will consider are rings with identity we will use the word ring to mean ring with identity.

We wish to define a contravariant functor, the spectrum, from the category of rings with identity (a morphism $x$ in this category has the property $x(1) = 1$) into the category of topological spaces. We first need to associate with each ring, $A$, a set. So consider the collection of ring morphisms, $x$, whose domain is $A$ and whose codomain, $D^x$, is a division ring with the following property: if $E$ is a division ring such that $\text{Im} x \subseteq E \subseteq D^x$ then $E = D^x$ ($\text{Im} x$ means the image of $x$, $\{x(a) | a \in A\}$). Two morphisms $x$ and $y$ of this collection are equivalent if there exists an isomorphism $u$ of $D^x$ onto $D^y$ such that $u \circ x = y$. Then $\mathcal{S}(A)$, the spectrum of $A$, is a set of these morphisms which contains exactly one from each equivalence class.

1 DEFINITION: If $A$ is a ring then $\mathcal{S}(A)$ is as above.
The object now is to define a topology on the set $\mathcal{A}(A)$. The method will be to define a set of functions whose domains are subsets of $\mathcal{A}(A)$ and then to use the domains of these functions as a base for the topology.

2 DEFINITION: If $A$ is a ring and $f, g$ are functions whose domains are subsets of $\mathcal{A}(A)$ such that for each $x$ in their domains $f(x), g(x) \in D_x$ then let:

1) $f^{-1}: \{x| x \in \text{dom } f, f(x) \neq 0\} \rightarrow \bigcup_{x \in \mathcal{A}(A)} D_x$ such that $f^{-1}(x) = [f(x)]^{-1}$,

2) $(f-g): \text{dom } f \cap \text{dom } g \rightarrow \bigcup_{x \in \mathcal{A}(A)} D_x$ such that $(f-g)(x) = f(x) - g(x)$,

3) $fg: \text{dom } f \cap \text{dom } g \rightarrow \bigcup_{x \in \mathcal{A}(A)} D_x$ such that $fg(x) = f(x)g(x)$.

3 DEFINITION: If $A$ is a ring:

1) for each $a \in A$ let $a^*: \mathcal{A}(A) \rightarrow \bigcup_{x \in \mathcal{A}(A)} D_x$ such that $a^*(x) = x(a)$,

2) let $A^*_0 = \{a^*| a \in A\}$,

3) for $n \geq 0$ let

$A^*_n+1 = \{f^{-1}| f \in A^*_n\} \cup \{(f-g)| f, g \in A^*_n\} \cup \{fg| f, g \in A^*_n\}$,

4) let $\mathcal{F}(A) = \bigcup_{n \geq 0} A^*_n$.

This set of functions, $\mathcal{F}(A)$, is the set we use to get our topology.
4 DEFINITION: If A is a ring let
\[ \mathcal{J}(A) = \{ U \mid U \subseteq \mathcal{J}(A) \text{ such that for each } x \in U \text{ there exists } f \in \mathcal{J}(A) \]
with \( x \in \text{dom} \ f \subseteq U \} .

Since \( f \) and \( g \) are elements of \( \mathcal{J}(A) \) implies \( \text{dom} f \cap \text{dom} g = \text{dom} fg \), we have the following theorem.

5 THEOREM: If A is a ring then \( \mathcal{J}(A) \) is a topology for \( \mathcal{J}(A) \).

There are some very close relationships between the closed subsets of \( \mathcal{J}(A) \) and certain subsets of \( \mathcal{J}(A) \). To investigate these we make the following definitions.

6 DEFINITION: (We denote the power set of a set \( X \) by \( \mathcal{P}(X) \).) If \( A \) is a ring:

1) let \( V_A : \mathcal{P}(\mathcal{J}(A)) \to \mathcal{P}(\mathcal{J}(A)) \) by \( V_A(G) = \mathcal{J}(A) \cap \bigcup_{g \in G} \text{dom} g^{-1} \),

2) let \( J_A : \mathcal{P}(\mathcal{J}(A)) \to \mathcal{P}(\mathcal{J}(A)) \) by \( J_A(X) = \{ f \mid f \in \mathcal{J}(A), X \cap \text{dom} f^{-1} = \emptyset \} \).

When it is clear which ring \( A \) we are discussing we will drop the subscript and write just \( V \) and \( J \).
7 THEOREM: If $A$ is a ring:

1) $G \subseteq \mathcal{F}(A)$ implies $V(G) \in \mathcal{F}(A)$-closed,

2) $V(\{0^*\}) = \mathcal{J}(A)$ ,

3) $V(\{1^*\}) = \emptyset$ ,

4) $G \subseteq F \subseteq \mathcal{F}(A)$ implies $V(F) \subseteq V(G)$ ,

5) if $\{G_i\}_{i \in I}$ is a family of subsets of $\mathcal{F}(A)$ then 
   $V(\bigcup_{i \in I} G_i) = \bigcap_{i \in I} V(G_i)$ ,

6) if $F, G \subseteq \mathcal{F}(A)$ and $FG = \{fg | f \in F, g \in G\}$ then 
   $V(FG) = V(F) \cup V(G)$ .

PROOF: 1) If $G \subseteq \mathcal{F}(A)$ and $g \in G$ then $\text{dom } g^{-1} \in \mathcal{F}(A)$ . So 

$V(G) = \mathcal{J}(A) \cup \bigcup_{g \in G} \text{dom } g^{-1} \in \mathcal{F}(A)$-closed .

2) $V(\{0^*\}) = \mathcal{J}(A) \cup \text{dom } 0^{-1} = \mathcal{J}(A) \cup \emptyset = \mathcal{J}(A)$ .

3) $V(\{1^*\}) = \mathcal{J}(A) \cup \text{dom } 1^{-1} = \mathcal{J}(A) \cup \text{dom } 1^* = \mathcal{J}(A) \cup \mathcal{J}(A) = \emptyset$ .

4) $G \subseteq F \subseteq \mathcal{F}(A)$ implies $\bigcup_{f \in F} \text{dom } f^{-1} \subseteq \mathcal{J}(A) \cup \bigcup_{g \in G} \text{dom } g^{-1} = V(G)$ .

5) If $\{G_j\}_{j \in J}$ is a family of subsets of $\mathcal{F}(A)$ then for each 
   $j \in J , \ G_j \subseteq \mathcal{G}_i \subseteq \mathcal{F}(A)$ . Thus by 4) above for each $j \in J$ , $V(\bigcup_{i \in I} G_i) \subseteq V(G_j)$ ,
   and so $V(\bigcup_{i \in I} G_i) \subseteq \bigcap_{i \in I} V(G_i)$ . To see the other inclusion suppose 
   $x \in \bigcap_{i \in I} V(G_i)$ . Then if $f \in \bigcup_{i \in I} G_i$ then there is a $j \in J$ such that $f \in G_j$ . Since $x \in V(G_j)$ we know $x \notin \text{dom } f^{-1}$ . Thus for all 
   $f \in \bigcup_{i \in I} G_i , \ x \in \text{dom } f^{-1}$ and so $x \in V(\bigcup_{i \in I} G_i)$ .

6) First we claim that if $f, g \in \mathcal{F}(A)$ then $(fg)^{-1} = g^{-1}f^{-1}$ .

If $x$ is in the domains of both $g^{-1}$ and $f^{-1}$ then it is clear that 

$\text{(fg)}^{-1}(x) = [fg(x)]^{-1} = [f(x)g(x)]^{-1} = [g(x)]^{-1}[f(x)]^{-1} = [g^{-1}(x)][f^{-1}(x)]$

$= (g^{-1}f^{-1})(x)$ . Thus the only question is whether $\text{dom}(fg)^{-1} = \text{dom } f^{-1}g^{-1}$ .
But \( \text{dom } (fg)^{-1} = \{ x \mid x \in \text{dom } fg, fg(x) \neq 0 \} = \{ x \mid x \in \text{dom } f \cap \text{dom } g, f(x)g(x) \neq 0 \} = \{ x \mid x \in \text{dom } f, x \in \text{dom } g, f(x) \neq 0, g(x) \neq 0 \} \). The reason for the last equality is that \( f(x), g(x) \in D_x \) which is a division ring. This in turn is equal to \( \{ x \mid x \in \text{dom } g, g(x) \neq 0 \} \cap \{ x \mid x \in \text{dom } f, f(x) \neq 0 \} = \text{dom } g^{-1} \cap \text{dom } f^{-1} = \text{dom } (fg)^{-1} \). So to prove part 6) of the theorem:

\[
V(G) \cup V(F) = \bigcup_{g \in G} \text{dom } g^{-1} \cup \bigcup_{f \in F} \text{dom } f^{-1}
\]

\[
= \mathcal{J}(A) \cup \left( \bigcup_{g \in G} \text{dom } g^{-1} \right) \cup \left( \bigcup_{f \in F} \text{dom } f^{-1} \right)
\]

\[
= \mathcal{J}(\text{dom } g^{-1} \cap \text{dom } f^{-1})
\]

\[
= \mathcal{J}(\text{dom } (fg)^{-1})
\]

\[
= V(FG).
\]

8 THEOREM: If \( A \) is a ring:

1) \( J(\emptyset) = \mathcal{J}(A) \),

2) \( J(\mathcal{J}(A)) = \{ f \mid f \in \mathcal{J}(A) \land f^{-1} = \emptyset \} \),

3) \( X \subseteq Y \subseteq \mathcal{J}(A) \) implies \( J(Y) \subseteq J(X) \),

4) if \( \{ X_i \}_{i \in I} \) is a family of subsets of \( \mathcal{J}(A) \) then

\[
J(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} J(X_i)
\]

5) \( X \subseteq \mathcal{J}(A) \) implies \( V(J(X)) = \overline{X} \) (the \( \mathcal{J}(A) \)-closure of \( X \)),

6) \( X \subseteq \mathcal{J}(A) \) implies \( J(X) = J(X) \).

PROOF: 1) \( J(\emptyset) = \{ f \mid f \in \mathcal{J}(A) \land \emptyset \cap \text{dom } f^{-1} = \emptyset \} = \mathcal{J}(A) \).

2) \( J(\mathcal{J}(A)) = \{ f \mid f \in \mathcal{J}(A) \land \mathcal{J}(A) \cap \text{dom } f^{-1} = \emptyset \}
\]

\[
= \{ f \mid f \in \mathcal{J}(A) \land \text{dom } f^{-1} = \emptyset \}
\]

\[
= \{ f \mid f \in \mathcal{J}(A) \land f^{-1} = \emptyset \}.
\]
3) If \( X \subseteq Y \) then for all \( f \in J(Y) \), \( X \cap \operatorname{dom} f^{-1} \subseteq Y \cap \operatorname{dom} f^{-1} = \emptyset \).

4) If \( \{X_i\}_{i \in I} \) is a family of subsets of \( \mathcal{A}(A) \) then for each \( j \in I \), \( X_j \subseteq \bigcup X_i \), and so by 3) \( J(\bigcup X_i) \subseteq J(X_j) \). Hence \( J(\bigcup X_i) \leq \bigcap J(X_i) \). On the other hand for every \( f \in \bigcap J(X_i) \) we have \( \operatorname{dom} f^{-1} \cap (\bigcup X_i) = \bigcup (\operatorname{dom} f^{-1} \cap X_i) = \bigcup \emptyset = \emptyset \). So for every \( f \in \bigcap J(X_i) \) we have \( f \in J(\bigcup X_i) \).

5) If \( x \in X \) then for all \( f \in J(X) \) we get \( x \notin \operatorname{dom} f^{-1} \).

Hence \( x \in \mathcal{A}(A) \cap \bigcup_{f \in J(X)} \operatorname{dom} f^{-1} = V(J(X)) \), which gives us \( X \subseteq V(J(X)) \).

But by 7 part 1) \( V(J(X)) \) is closed, so \( \overline{X} \subseteq V(J(X)) \). However if \( x \notin \overline{X} \) then there exists \( f \in \mathcal{F}(A) \) such that \( x \notin \operatorname{dom} f \leq \mathcal{A}(A) \cap \overline{X} \).

Two cases arise, \( f(x) \neq 0 \) and \( f(x) = 0 \). The second may be reduced to the first by noticing that \( \operatorname{dom}(f^{-1}) = \operatorname{dom} f \cap J(A) = \operatorname{dom} f \) and \( (f^{-1} f(x)) = f(x) - 1 \). So we will assume \( f(x) \neq 0 \). Then one has \( x \in \operatorname{dom} f^{-1} \leq \operatorname{dom} f \leq \mathcal{A}(A) \cap \overline{X} \). So \( x \in \operatorname{dom} f^{-1} \) and \( \operatorname{dom} f^{-1} \cap X = \emptyset \).

Thus \( f \in J(X) \) and \( x \notin \mathcal{A}(A) \cap \bigcup_{g \in J(X)} \operatorname{dom} g^{-1} = V(J(X)) \).

6) Since \( X \subseteq \overline{X} \) by part 3) we have \( J(X) \subseteq J(X) \). For the other inclusion if \( f \in J(X) \) then \( \operatorname{dom} f^{-1} \cap X = \emptyset \), but since \( \operatorname{dom} f^{-1} \) is open this implies \( \operatorname{dom} f^{-1} \cap \overline{X} = \emptyset \) and so \( f \in J(\overline{X}) \).

It is clear that \( V_A \) and \( J_A \) establish a correspondence between the closed subsets of \( \mathcal{A}(A) \) and certain subsets of \( \mathcal{F}(A) \). We would like to know more about these subsets of \( \mathcal{F}(A) \).
9 DEFINITION: If $A$ is a ring,
1) for each $G \subseteq \mathfrak{J}(A)$ let $r_A(G) = J_A(V_A(G))$ ,
2) let $\text{rad}(A) = \{F | F \subseteq \mathfrak{J}(A)$ and there exists a $G \subseteq \mathfrak{J}(A)$ such that $F = r_A(G)\}$. 

10 THEOREM: If $A$ is a ring then $V|_{\text{rad}(A)}$ is a one-to-one correspondence between \text{rad}(A) and $\mathfrak{J}(A)$-closed.

PROOF: First we note that if $X \subseteq \mathfrak{J}(A)$ then $J(X) \in \text{rad}(A)$, because $r_A(J(X)) = J(V(J(X))) = J(X) = J(X)$.

By 7 part 1) we know that $V$ maps into the closed subsets of $\mathfrak{J}(A)$; to see that it maps $\text{rad}(A)$ onto the closed sets, note that for any closed set $X$ that $J(X) \in \text{rad}(A)$ and $V(J(X)) = X = X$.

The one-to-one property of $V|_{\text{rad}(A)}$ is proven in the usual way. For if $F, G \in \text{rad}(A)$ then there exist $H, I \subseteq \mathfrak{J}(A)$ such that $F = r_A(H)$ and $G = r_A(I)$. Thus if $V(F) = V(G)$ we have $V(J(V(H))) = V(J(V(I)))$. By 8 part 5) and 7 part 1) we have $V(H) = V(I)$ and so $F = J(V(H)) = J(V(I)) = G$. Therefore $V|_{\text{rad}(A)}$ is a one-to-one correspondence.

11 LEMMA: If $A$ is a ring then
1) $X \subseteq \mathfrak{J}(A)$ implies $J(X) \in \text{rad}(A)$ , and
2) $F \in \text{rad}(A)$ implies $r_A(F) = F$ .

PROOF: Part 1) was proven in the theorem. To prove part 2) let $G \subseteq \mathfrak{J}(A)$ such that $F = r_A(G)$. Then $J(V(F)) = J(V(J(V(G)))) = J(V(G)) = J(V(G)) = F$. 

The terms $r_A(F)$ and $\text{rad}(A)$ are not accidental but were chosen because of the similarities and relationships between $r_A(F)$ and the radical of an ideal (for a discussion of $\sqrt{I}$, the radical of the ideal $I$ see [Z + S, p. 147-149]).

12 THEOREM: If $A$ is a ring, $F \in \text{rad}(A)$, $f, g \in F$, and $h \in F(A)$ then $f - g \in F$, $f^{-1} \in F$, $fh \in F$, and $hf \in F$. Furthermore if for some $n \in \mathbb{Z}$, $h^n \in F$ then $h \in F$ (where $h^0$ means $hh^{-1}$).

PROOF: If $f, g \in F = r_A(F) = J(V(F))$ then $\text{dom } f^{-1} \cap V(F) = \phi$ and $\text{dom } g^{-1} \cap V(F) = \phi$. So if $x \in V(F) \cap \text{dom}(f-g) = V(F) \cap \text{dom } f \cap \text{dom } g$ then $f(x) = g(x) = 0$. Hence $V(F) \cap \text{dom}(f-g)^{-1} = \phi$ and $(f-g) \in J(V(F)) = F$.

Since $\text{dom}(f^ {-1}) = \text{dom } f^{-1}$ and $V(F) \cap \text{dom } f^{-1} = \phi$, we have $f^{-1} \in J(V(F)) = F$.

Since $\text{dom}(hf)^{-1} = \text{dom } f^{-1}h^{-1} = \text{dom } f^{-1} \cap \text{dom } h^{-1}$, we have $V(F) \cap \text{dom}(hf)^{-1} = V(F) \cap \text{dom } f^{-1} \cap \text{dom } h^{-1} = \phi \cap \text{dom } h^{-1} = \phi$. The proof that $fh \in F$ is similar.

Lastly suppose $h^n \in F$ then three cases arise: $n > 0$, $n < 0$, and $n = 0$. If $n > 0$ then $\text{dom}(h^n)^{-1} = \text{dom } h^{-1} = \bigcap_{i=1}^{n} \text{dom } h^{-1} = \text{dom } h^{-1}$. If $n < 0$ then $\text{dom}(h^n)^{-1} = \text{dom } h^n = \text{dom } h^{-1}n = \text{dom } h^{-1}$. If $n = 0$ then $\text{dom}(h^n)^{-1} = \text{dom } (hh^{-1})^{-1} = \text{dom } (h^{-1})^{-1}h^{-1} = \text{dom } h^{-1} \cdot h^{-1} = \text{dom } h^{-1}$. Thus $\text{dom}(h^n)^{-1} = \text{dom } h^{-1}$ and $V(F) \cap \text{dom}(h^n)^{-1} = \phi$ implies $V(F) \cap \text{dom } h^{-1} = \phi$ and $h \in J(V(F)) = F$.

13 COROLLARY: If $A$ is a ring, $F \in \text{rad}(A)$, and $I = \{a | a \in A, a^* \in F\}$ then $I$ is a two sided ideal of $A$ and $\sqrt{I} = I$. 
There is a theorem that an ideal which is equal to its radical is the intersection of the prime ideals which contain it. In our case the sets $J\{x\}$ play the part of the prime ideals.

**14 Theorem:** If $A$ is a ring, $x \in \mathcal{J}(A)$, and $f, g \in \mathcal{J}(A)$ with $fg \in J\{x\}$ then either $f \in J\{x\}$ or $g \in J\{x\}$.

**Proof:** Suppose $g \notin J\{x\}$, then since $\text{dom } g^{-1} \cap \{x\} \neq \emptyset$ we have $\text{dom } g^{-1} \cap \{x\} = \{x\}$. Now $\phi = \text{dom}(fg)^{-1} \cap \{x\} = \text{dom } g^{-1}x^{-1} \cap \{x\} = \text{dom } f^{-1} \cap \text{dom } g^{-1} \cap \{x\} = \text{dom } f^{-1} \cap \{x\}$. Thus $f \in J\{x\}$.

**15 Theorem:** If $A$ is a ring and $F \in \text{rad}(A)$ then

$$F = \bigcap_{x \in V(F)} J\{x\}.$$  

**Proof:** $F = r_A(F) = J(V(F)) = J\left( \bigcup_{x \in V(F)} \{x\} \right) = \bigcap_{x \in V(F)} J\{x\}.$

We will later show that the $J\{x\}$'s are the only members of $	ext{rad}(A)$ which have the property given in 14. But now we will return to complete our definition of the functor $\mathcal{J}$. We want $\mathcal{J}$ to be a contravariant functor from the category of rings with identity into the category of topological spaces. So with each morphism of rings with identity we must associate a continuous function.

If $A$ and $B$ are rings, $s: A \to B$ is a morphism (recall that in addition to the usual property of a ring morphism that $s(1) = 1$), and $x \in \mathcal{J}(B)$ then $x \circ s: A \to D_x$. If $D_{s,x} = \bigcap \{E | E \text{ is a division ring, Im}(xos) \subseteq E \subseteq D_x \}$ and $t: A \to D_{s,x}$ by $t(a) = (xos)(a)$ then there is
exactly one element, say \( y \), of \( \mathcal{J}(A) \) which is equivalent to \( t \). Thus we have the following diagram; where \( u \) is the equivalence isomorphism and \( i \) is the natural injection:

![Diagram]

16 **DEFINITION:** If \( A \) and \( B \) are rings and \( s: A \to B \) is a morphism then let \( \mathcal{J}(s): \mathcal{J}(B) \to \mathcal{J}(A) \) such that if \( x \in \mathcal{J}(B) \) then \( (\mathcal{J}(s))(x) \) is the \( y \) described above. Furthermore let \( s_x = i \circ u \) where \( i \) and \( u \) are as above.

17 **LEMMA:** If \( A \) and \( B \) are rings, \( s: A \to B \), \( x \in \mathcal{J}(B) \), \( z \in \mathcal{J}(A) \), and \( v: D_z \to D_x \) such that

![Diagram]

commutes then \( z = (\mathcal{J}(s))(x) \) and \( v = s_x \).

**PROOF:** First we note that since \( v(1) = 1 \) and \( D_z \) is a division ring, \( v \) is a monomorphism. Secondly to simplify the writing let \( y = (\mathcal{J}(s))(x) \).
Then from the definition of $\mathcal{J}$

$$
\begin{array}{c}
A \\
Y
\end{array}
\xymatrix{
& B \\
Y \\
D}
$$

commutes and what we must show is that there exists an isomorphism $w: D_y \to D_y$ such that $wz = y$.

Let $E = s_x^+[\text{Im } v]$. Since $\text{Im } v \cap \text{Im } s_x$ is a division ring and $s_x$ is a monomorphism, $E$ is a division ring. Now if $a \in A$ then $s_x(y(a)) = x(s(a)) = v(z(a))$ and $y(a) \in E$. Thus $\text{Im } y \subseteq E \subseteq D_y$, and so, by the definition of $\mathcal{J}(A)$, we have $E = D_y$. This implies $\text{Im } s_x \subseteq \text{Im } v$. By a similar argument the reverse inclusion is also true.

So $w: D_z \to D_y$ given by $w(d) = s_x^+(v(d))$ is an isomorphism and $wz = y$. This proves that $z$ and $y$ are equivalent, but $\mathcal{J}(A)$ contains exactly one member from each equivalence class and so $z = y = (s_x)(x)$.

To prove that $v = s_x$ let $E = \{d \mid d \in D_y \text{ and } w(d) = d\}$. Then $E$ is a division ring. If $a \in A$ then $w(y(a)) = s_x^+(v(y(a))) = s_x^+(v(z(a))) = s_x^+(x(s(a))) = y(a)$. Hence $\text{Im } y \subseteq E \subseteq D_y$ and by the definition of $\mathcal{J}(A)$ this means $E = D_y$, that is $w = \text{id}_{D_y}$. So $s_x = s_x \circ \text{id}_{D_y} = s_x \circ w = v$.

18 THEOREM: If $A$, $B$, and $C$ are rings and $s: A \to B$ and $t: B \to C$ are morphisms then $\mathcal{J}(tos) = \mathcal{J}(s) \circ \mathcal{J}(t)$. 
PROOF: Let \( x \in \mathcal{L}(A) \), \( y \in \mathcal{L}(B) \), \( z \in \mathcal{L}(C) \) such that \( y = (\mathcal{L}(t))(z) \) and \( x = (\mathcal{L}(s))(y) \). Then the two squares in the following diagram commute.

\[
\begin{array}{ccc}
A & \xrightarrow{s} & B & \xrightarrow{t} & C \\
\downarrow{x} & & \downarrow{y} & & \downarrow{z} \\
D & \xrightarrow{s_y} & B & \xrightarrow{t_z} & D \\
\end{array}
\]

Thus the rectangle commutes and by 17 \( (\mathcal{L}(t))(z) = x = (\mathcal{L}(s))(y) = (\mathcal{L}(s))( (\mathcal{L}(t))(z)) = (\mathcal{L}(s) \circ \mathcal{L}(t))(z) \).

19 COROLLARY: Under the hypothesis of the theorem and with \( y \) and \( z \) as in the proof then \( (\mathcal{L}(t))(z) = t \circ s \).

For \( \mathcal{L} \) to be a functor we must show that for each morphism \( s: A \to B \) that \( \mathcal{L}(s) \) is a morphism, that is a continuous function.

Since the topologies for \( \mathcal{L}(A) \) and \( \mathcal{L}(B) \) are defined in terms of the sets of functions \( \mathcal{F}(A) \) and \( \mathcal{F}(B) \) it is natural that our proof will require a knowledge of how these sets interact with \( s \).

20 DEFINITION: If \( A \) and \( B \) are rings and \( s: A \to B \) is a morphism let \( \mathcal{F}(s): \mathcal{F}(A) \to \mathcal{F}(B) \) such that

1) if \( f \in \mathcal{F}(A) \) then \( \text{dom}(\mathcal{F}(s))(f) = \mathcal{L}(s) \uparrow [\text{dom } f] \),

2) if \( x \in \text{dom}(\mathcal{F}(s))(f) \) then \( (\mathcal{F}(s)(f))(x) = s(\mathcal{F}(f)((\mathcal{L}(s))(x))) \).
From this definition it is clear that \( \text{dom}(\mathcal{F}(s))(f) \) is a subset of \( \mathcal{B}(B) \) and that for each \( x \in \text{dom}(\mathcal{F}(s))(f) \) we have \( (\mathcal{F}(s))(f)(x) \in D_x \), but it is not obvious that \( \mathcal{F}(s)(f) \in \mathcal{B}(B) \). Thus we need the following lemma.

21 LEMMA: If \( A \) and \( B \) are rings and \( s: A \rightarrow B \) is a morphism then for each \( f \in \mathcal{F}(A) \) it follows that \( (\mathcal{F}(s))(f) \in \mathcal{F}(B) \).

PROOF: Our proof involves the use of facts which are of sufficient interest to be mentioned by themselves.

22 LEMMA: If \( A \) and \( B \) are rings, \( s: A \rightarrow B \) is a morphism, and \( a \in A \) then \( (\mathcal{F}(s))(a^*) = (s(a))^* \).

PROOF: Clearly \( \text{dom}(s(a))^* = \mathcal{B}(B) = \mathcal{B}(s)^{t^\perp}[\mathcal{A}(A)] = \mathcal{B}(s)^{t^\perp}[\text{dom } a^*] \)

\( = \text{dom}(\mathcal{F}(s))(a^*) \). If \( x \in \text{dom}(s(a))^* \) and \( y = (\mathcal{B}(s))(x) \) then

\[
\begin{array}{c}
\text{A} \xrightarrow{s} \text{B} \\
\text{Y} \downarrow \quad \uparrow \text{x} \\
\text{D} \xrightarrow{s} \text{X}
\end{array}
\]

commutes and so \( (s(a))^*(x) = x(s(a)) = s_x(y(a)) = s_x(a^*(y)) = s_x(a^*((\mathcal{B}(s))(x))) \)

\( = ((\mathcal{F}(s))(a^*))(x) \).

23 LEMMA: If \( A \) and \( B \) are rings and \( s: A \rightarrow B \) is a morphism then
for all \( f \) and \( g \in \mathcal{F}(A) \)

1) \( (\mathcal{F}(s))(f^{-1}) = ((\mathcal{F}(s))(f))^{-1} \),

2) \( (\mathcal{F}(s))(f-g) = (\mathcal{F}(s))(f) - (\mathcal{F}(s))(g) \),

3) \( (\mathcal{F}(s))(fg) = ((\mathcal{F}(s))(f))((\mathcal{F}(s))(g)) \).

**PROOF:** All three parts are similar and we will content ourselves with the proof of part 1). We have

\[
\text{dom}[(\mathcal{F}(s))(f)]^{-1} = \text{dom}(\mathcal{F}(s))(f) \cap \{x | (\mathcal{F}(s))(f)(x) = 0\}
\]

\[
= \mathcal{J}(s)^+[\text{dom } f] \cap \{x | s_x(f((\mathcal{J}(s))(x))) = 0\}
\]

(since for each \( x \in \mathcal{J}(B) \), \( s_x \) is a monomorphism)

\[
= \mathcal{J}(s)^+[\text{dom } f] \cap \{y | f(y) = 0\}
\]

\[
= \mathcal{J}(s)^+[\text{dom } f^{-1}]
\]

\[
= \text{dom}(\mathcal{F}(s))(f)^{-1}.
\]

If \( x \in \text{dom}(\mathcal{F}(s))(f)^{-1} \) then we have

\[
[(\mathcal{F}(s))(f)^{-1}(x) = [((\mathcal{F}(s))(f))(x)]^{-1}
\]

\[
= [s_x(f((\mathcal{J}(s))(x)))^{-1}
\]

\[
= s_x((f((\mathcal{J}(s))(x)))^{-1})
\]

\[
= s_x(f^{-1}((\mathcal{J}(s))(x)))
\]

\[
= ((\mathcal{F}(s))(f^{-1}))(x).
\]

With these two lemmas we can complete the proof of 21. Recall that \( \mathcal{F}(A) \) was defined as \( \bigcup_{n \geq 0} A_n^* \) and that the \( A_n^* \)'s were defined recursively, hence our proof will be by induction on \( n \) such that \( f \in A_n^* \).

If \( f \in A_0^* \) then there exists an \( a \in A \) such that \( f = a^* \) and so \( (\mathcal{F}(s))(f) = (\mathcal{F}(s))(a^*) = (s(a))^* \in \mathcal{F}(B) \). Now suppose that
17

(\mathcal{F}(s))[A^n_+] \subseteq \mathcal{F}(B) and that \ f \in A^{n+1}_+. Then there are three cases:

1) there exists \ g \in A^n_+ such that \ f = g^{-1},

2) there exist \ g, h \in A^n_+ such that \ f = g - h,

3) there exist \ g, h \in A^n_+ such that \ f = g \cdot h.

If there is a \ g \in A^n_+ such that \ f = g^{-1} then by the induction hypothesis

(\mathcal{F}(s))(g) \in \mathcal{F}(B). Thus by 23 \ (\mathcal{F}(s))(f) = (\mathcal{F}(s))(g)^{-1} = ([\mathcal{F}(s)](g))^{-1} \in \mathcal{F}(B).

Cases 2) and 3) give the same result in a similar way.

24 LEMMA: If A, B, and C are rings and \ s: A \rightarrow B, t: B \rightarrow C

are morphisms then \ \mathcal{F}(tos) = \mathcal{F}(t) \circ \mathcal{F}(s).

PROOF: If \ f \in \mathcal{F}(A) then, by 18

\text{dom}(\mathcal{F}(tos))(f) = \mathcal{J}(tos)^+[\text{dom } f]
= (\mathcal{J}(s) \circ \mathcal{J}(t))^+[\text{dom } f]
= (\mathcal{J}(t)^+ \circ \mathcal{J}(s)^+)[\text{dom } f]
= \mathcal{J}(t)^+[\mathcal{J}(s)^+[\text{dom } f]]
= \mathcal{J}(t)^+[\text{dom}(\mathcal{F}(s))(f)]
= \text{dom}(\mathcal{F}(t))((\mathcal{F}(s))(f))
= \text{dom}(\mathcal{F}(t) \circ \mathcal{F}(s))(f).

If \ x \in \text{dom}(\mathcal{F}(tos))(f) and \ y \in \mathcal{J}(B) such that \ y = (\mathcal{J}(t))(x) then

((\mathcal{F}(tos))(f))(x) = (tos)_x((\mathcal{J}(tos))(x))
= (t \circ os_y)((\mathcal{J}(s) \circ \mathcal{J}(t))(x))
= (t \circ os_y)((\mathcal{J}(s))(y))
= t_x(((\mathcal{F}(s))(f))(y))
= t_x(((\mathcal{F}(s))(f))((\mathcal{J}(t))(x)))
= ((\mathcal{F}(t))((\mathcal{F}(s))(f)))(x)
= ((\mathcal{F}(t) \circ \mathcal{F}(s))(f))(x).
Thus we have the desired result, namely $\tilde{g}(t) \circ \tilde{f}(s) = \tilde{f}(tos)$.

25 DEFINITION: If $A$ and $B$ are rings with identity and $s: A \to B$ is a morphism then let $\ker \tilde{f}(s) = \{f | f \in \mathcal{J}(A) \text{ and } ((\tilde{f}(s))(f))^{-1} = \phi \}$.

26 LEMMA: If $A$ and $B$ are rings and $s: A \to B$ is a morphism then $\ker \tilde{f}(s) = J_A(\text{Im } \mathcal{J}(s))$.

PROOF: From 23 we have: if $f \in \mathcal{J}(A)$ then $((\tilde{f}(s))(f))^{-1} = (\tilde{f}(s))(f^{-1})$.

Hence $f \in \ker \tilde{f}(s)$ iff $(\tilde{f}(s))(f^{-1}) = \phi$ iff $\text{dom}(\tilde{f}(s))(f^{-1}) = \phi$ iff $\mathcal{J}(s)^+[\text{dom } f^{-1}] = \phi$ iff $\text{dom } f^{-1} \cap \text{Im } \mathcal{J}(s) = \phi$ iff $f \in J_A(\text{Im } \mathcal{J}(s))$.

27 LEMMA: If $s: A \to B$ is a morphism of rings then $V_A(\ker \tilde{f}(s)) = \overline{\text{Im } \mathcal{J}(s)}$.

PROOF: We use 26 and part 5) of 8. $V_A(\ker \tilde{f}(s)) = V_A(J_A(\text{Im } \mathcal{J}(s))) = \overline{\text{Im } \mathcal{J}(s)}$.

28 LEMMA: If $s: A \to B$ is a morphism of rings and $E \subseteq \mathcal{J}(A)$ then $(\mathcal{J}(s)^+[V_A(E)]) = V_B(\tilde{f}(s)[E])$.

PROOF: Suppose $x \in \mathcal{J}(s)^+[V_A(E)]$. Then for any $f \in E$ it follows that $(\mathcal{J}(s))(x) \notin \text{dom } f^{-1}$. Thus for any $f \in E$, $x \notin \text{dom}(\tilde{f}(s))(f^{-1})$ and $\text{dom}((\tilde{f}(s))(f))^{-1}$, which is to say that $x \in V_B(\tilde{f}(s)[E])$. 
On the other hand, suppose \( y \in V_B(\mathcal{F}(s)[E]) \). Then for any \( f \in E \) we may infer that \( y \notin \text{dom}((\mathcal{F}(s)(f))^{-1}) = \text{dom}(\mathcal{F}(s)(f^{-1})) \). Hence for all \( f \in E \), \( (\mathcal{J}(s))(y) \notin \text{dom} f^{-1} \), that is \( (\mathcal{J}(s))(y) \in V_A(E) \). Therefore we have the desired result: namely that \( y \in \mathcal{J}(s) \uparrow [V_A(E)] \).

With the aid of this last lemma the continuity of \( \mathcal{J}(s) \) follows easily.

**29 Theorem:** If \( s: A \to B \) is a morphism of rings then \( \mathcal{J}(s) \) is \( \mathcal{J}(B) - \mathcal{J}(A) \) -continuous.

**Proof:** From 10 we have, \( V_A \) is a one-to-one correspondence between the elements of \( \text{rad}(A) \) and the closed subsets of \( \mathcal{J}(A) \). Thus every closed subset of \( \mathcal{J}(A) \) is of the form \( V_A(E) \) for some \( E \) in \( \text{rad}(A) \).

By the last lemma, however, \( \mathcal{J}(s) \uparrow [V_A(E)] = V_B(\mathcal{F}(s)[E]) \) which is closed. Hence \( \mathcal{J}(s) \) is continuous.

**30 Theorem:** \( \mathcal{J} \) is a contravariant functor from rings with identity into topological spaces.

**Proof:** This theorem is a compilation of 5, 29, and 18.
CHAPTER II
EXAMPLES

It is time that we gave some examples of the spectrum of a ring. In this chapter we will show that for commutative rings, \( \mathcal{S} \) is naturally equivalent to Spec, the usual spectrum for commutative rings. Then we shall construct the spectrums of a division ring, the product of division rings, the ring of \( n \times n \) matrices over a division ring, and \( \mathbb{Z}[i,j,k] \), the ring of quaternions with integer coefficients.

One may find a discussion of Spec in [M]. We give here only the definitions and results which are necessary to our purpose.

3.1 DEFINITION: If \( s: A \to B \) is a morphism of commutative rings with identity let

1) \( \text{Spec } A = \{ P | P \text{ is a prime ideal of } A \} \),

2) \( v_A(E) = \{ P | P \in \text{Spec } A, E \subseteq P \} \) for all \( E \subseteq A \),

3) \( (\text{Spec } s)(P) = s^+(P) \) for all \( P \in \text{Spec } B \).

With this definition, \( \{ \text{Spec } A \sim v\{a\} | a \in A \} \) is a base for a topology on \( \text{Spec } A \) and \( v_A \) is a one-to-one correspondence between the radical ideals of \( A \) and the closed subsets of \( \text{Spec } A \). (If \( I \) is an ideal of \( A \) then \( \sqrt{I} = \{ a | a \in A \text{ such that for some } n > 0, a^n \in I \} \). \( I \) is a radical ideal if \( I = \sqrt{I} \), see for example, [Z + S, p. 147].)

Spec is a contravariant functor from the category of commutative rings with identity into topological spaces. For \( \mathcal{S} \) to be naturally equivalent to Spec there must exist for each ring \( A \) a homeomorphism.
$H_A: \mathcal{J}(A) \to \text{Spec } A$ such that for each morphism $s: A \to B$ the diagram

\[
\begin{array}{ccc}
\mathcal{J}(A) & \xrightarrow{H_A} & \text{Spec } A \\
\downarrow \mathcal{J}(s) & & \downarrow \text{Spec } s \\
\mathcal{J}(B) & \xrightarrow{H_B} & \text{Spec } B
\end{array}
\]

commutes.

**32 DEFINITION:** If $A$ is a commutative ring let $H_A: \mathcal{J}(A) \to \text{Spec } A$ be $H_A(x) = \ker x$.

Every closed set in $\text{Spec } A$ is of the form $v_A(I)$ where $I$ is an ideal of $A$ such that $\sqrt{I} = I$. Now

\[
H_A[v_A(I)] = \{x | x \in \mathcal{J}(A), I \subseteq \ker x\}
\]

\[
= \{x | x \in \mathcal{J}(A) \text{ such that for all } a \in I, x(a) = 0\}
\]

\[
= \{x | x \in \mathcal{J}(A) \text{ such that for all } a \in I, a^*(x) = 0\}
\]

\[
= \{x | x \in \mathcal{J}(A) \text{ such that for all } a \in I, x \notin \text{dom } a^{*-1}\}
\]

\[
= \mathcal{J}(A) \cap \bigcup_{a \in I} \text{dom } a^{*-1}
\]

\[
= v_A(\{a^* | a \in I\}) \text{ which is closed.}
\]

So $H_A$ is continuous.

If $P$ is a prime ideal of a commutative ring $A$ then $A/P$ is an integral domain and has a field of quotients $K_p$. If $p$ is the projection of $A$ onto $A/P$, $i$ the injection of $A/P$ into $K_p$, and $s = iop$ then there exists an $x \in \mathcal{J}(A)$ which is equivalent to $s$; that is there is an
isomorphism \( u \) which makes

\[
\begin{array}{c}
A \\
\downarrow x \\
D_x \\
\downarrow u \\
K_p \\
\end{array}
\quad \begin{array}{c}
A \\
\downarrow s \\
\downarrow u \\
\end{array}
\quad \begin{array}{c}
A \\
\downarrow u \\
K_p \\
\end{array}
\]

commute. Then \( H_A(x) = \ker x = \{a \mid a \in A, x(a) = 0\} = \{a \mid a \in A, u(x(a)) = 0\} = \{a \mid a \in A, s(a) = 0\} = \ker s = P \). Hence \( H_A \) is onto.

On the other hand if \( x \in \mathcal{J}(A) \) and \( \ker x = P \) then there is a monomorphism \( t : A/P \to D_x \) such that

\[
\begin{array}{c}
A \\
\downarrow x \\
D_x \\
\downarrow t \\
A/P \\
\end{array}
\]

commutes. Thus \( D_x \) is isomorphic to \( K_p \) and \( x \) is equivalent to \( i \circ p \) where \( i \) is the injection of \( A/P \) into its field of quotients. Therefore if \( x, y \in \mathcal{J}(A) \) and \( H_A(x) = H_A(y) \) it follows that \( x \) is equivalent to \( i \circ p \) which is equivalent to \( y \), and, since no two members of the spectrum are equivalent, we may infer \( x = y \). Thus \( H_A \) is one to one.

To show that \( H_A \) is a homeomorphism it remains to prove that \( H_A \) is a closed map. To do this we need the following two lemmas.
33 LEMMA: If $\mathbf{A}$ is a commutative ring then $F = \{a*b^{-1} \mid a, b \in \mathbf{A}\} = \mathcal{J}(\mathbf{A})$.

PROOF: Clearly $F \subseteq \mathcal{J}(\mathbf{A})$. To show the other inclusion we induct on $n$, showing $\mathbf{A}_n^* \subseteq F$ for all $n$.

If $f \in \mathbf{A}_n^*$ then there exists $a \in \mathbf{A}$ such that $f = a^* = a^*1^*1^{-1} \in F$.

Suppose that $\mathbf{A}_n^* \subseteq F$ and that $f \in \mathbf{A}_{n+1}^*$ then either

1) there exists $g \in \mathbf{A}_n^*$ such that $f = g^{-1}$,

2) there exists $g, h \in \mathbf{A}_n^*$ such that $f = g - h$,

3) there exists $g, h \in \mathbf{A}_n^*$ such that $f = gh$.

If 1) is true then, since $\mathbf{A}_n^* \subseteq F$, there exist $a, b \in \mathbf{A}$ and $g = a^*b^*1^{-1}$.

We claim then that $f = (b^2)^*(ab)^*1^{-1}$. For

$$\text{dom } f = \text{dom } g \cup \{x \mid x \in \mathcal{J}(\mathbf{A}), g(x) = 0\}$$

$$= \mathcal{J}(\mathbf{A}) \cup \{x \mid x \in \mathcal{J}(\mathbf{A}), x(b) = 0\} \cup \{x \mid x \in \mathcal{J}(\mathbf{A}), x(a) = 0\}$$

$$= \mathcal{J}(\mathbf{A}) \cup \{x \mid x \in \mathcal{J}(\mathbf{A}), x(a) = 0\}$$

$$= \text{dom}(b^2)^*(ab)^*1^{-1}.$$

If $x \in \text{dom } f$ we have

$$f(x) = (g(x))^{-1} = (a^*b^{-1}(x))^{-1} = (x(a)(x(b))^{-1})^{-1} = x(b)(x(a))^{-1}$$

$$= x(b^2)(x(ab))^{-1} = ((b^2)^*(ab)^*1^{-1})(x).$$

Therefore $f = (b^2)^*(ab)^*1^{-1} \in F$.

In case 2) there exist $a, b, c, d \in \mathbf{A}$ such that $f = g - h$.

$$= a^*b^*1^{-1} = (ad-cb)^*(bd)^*1^{-1} \in F.$$

In case 3) there exist $a, b, c, d \in \mathbf{A}$ such that $f = gh$.

$$= (a^*b^*1^{-1})(c^*d^*1^{-1}) = (ac)^*(bd)^*1^{-1} \in F.$$
34 LEMMA: If \( A \) is a commutative ring and \( T: \text{rad}(A) \to \text{ideal} A \) by \( T(G) = \{a|a \in A \text{ and } a^* \in G\} \) then \( T \) is a one-to-one correspondence between \( \text{rad}(A) \) and \( \{I|I \in \text{ideal} A \text{ and } \sqrt{I} = I\} \).

PROOF: In 13 we proved \( \text{Im} T \subseteq \{I|I \in \text{ideal} A \text{ and } \sqrt{I} = I\} \). To show the opposite inclusion suppose \( I \) is an ideal of \( A \) such that \( I = \sqrt{I} \) and let \( I^* = \{a^*|a \in I\} \). Then \( F = J(V(I^*)) \) is an element of \( \text{rad}(A) \) such that \( I \subseteq T(F) \). We also have,

\[
V(I^*) = \mathcal{J}(A) \cup \bigcup_{b^* \in I^*} \text{dom } b^{*\text{-}1}
\]

\[
= \mathcal{J}(A) \cup \bigcup_{b \in I} \{x|x \in \mathcal{J}(A), x(b) \neq 0\}
\]

\[
= \mathcal{J}(A) \cup \bigcap_{b \in I} \{x|x \in \mathcal{J}(A), x(b) = 0\}
\]

\[
= \{x|x \in \mathcal{J}(A), I \subseteq \ker x\}
\]

\[
= \{x|x \in \mathcal{J}(A), I \subseteq H_A(x)\}.
\]

Thus \( T(F) = T(J(V(I^*))) = T(J(\{x|x \in \mathcal{J}(A), I \subseteq H_A(x)\})) \)

\[
= \{a|a \in A, \{x|x \in \mathcal{J}(A), I \subseteq H_A(x)\} \cap \text{dom } a^{*\text{-}1} = \emptyset\}
\]

\[
= \{a|a \in A \text{ and } I \subseteq H_A(x) \text{ implies } x(a) = 0\}
\]

\[
= \bigcap_{I \subseteq H_A(x)} \mathcal{H}_A(x) \quad \text{(since } H_A \text{ maps onto the prime ideal of } A) \text{ maps onto the prime ideal of } A)
\]

\[
= \bigcap_{P \text{ prime}} P \quad \text{(where } P \text{ is prime)}
\]

\[
= \sqrt{I} = I \quad [\text{Z} + S, \text{p. 151, Note II}].
\]

Thus all that remains is to show that \( T \) is one to one. To this end, let \( F, G \in \text{rad}(A) \) such that \( T(F) = T(G) \). Now, if \( f \in \mathcal{V}(A) \), by
the previous lemma, there exists \( a, b \in A \) such that \( f = a^*b^{*-1} \) and \( \text{dom } f^{-1} = \text{dom } f \sim \{ x \mid f(x) = 0 \} \)
\[
= (\mathfrak{J}(A) \sim \{ x \mid x(b) = 0 \}) \sim \{ x \mid x(a) = 0 \}
= \mathfrak{J}(A) \sim \{ x \mid x(b) = 0 \text{ or } x(a) = 0 \}
= \mathfrak{J}(A) \sim \{ x \mid x(ab) = 0 \}
= \text{dom}(ab)^{-1}.
\]
Hence \( f \in F = J(V(F)) \) iff \( V(F) \cap \text{dom } f^{-1} = \phi \)
\[
\text{iff } V(F) \cap \text{dom}(ab)^{-1} = \phi
\text{iff } (ab)^* \in J(V(F)) = F
\text{iff } ab \in T(F) = T(G)
\text{iff } (ab)^* \in G = J(V(G))
\text{iff } V(G) \cap \text{dom}(ab)^{-1} = \phi
\text{iff } V(G) \cap \text{dom } f^{-1} = \phi
\text{iff } f \in J(V(G)) = G.
\]
Therefore \( T \) is a one-to-one correspondence between \( \text{rad}(A) \) and \( \{ \mathfrak{I} | \mathfrak{I} \text{ is ideal } A \text{ and } \sqrt{\mathfrak{I}} = \mathfrak{I} \} \).

Returning to our consideration of \( H_A \) we remark that since \( V_A \)
is a one-to-one correspondence between \( \text{rad}(A) \) and the closed subsets of \( \mathfrak{J}(A) \), and \( T \) is a one-to-one correspondence between \( \text{rad}(A) \) and the radical ideals of \( A \) that every closed set in \( \mathfrak{J}(A) \) is of the form \( V(T^+(\mathfrak{I})) = V(J(V(T^+(\mathfrak{I})))) = V(I^*) = V(I^+) = \{ x \mid x \in \mathfrak{J}(A), I \subseteq H_A(x) \} \).
Therefore \( H_A[V(T^+(\mathfrak{I}))] = \{ H_A(x) | x \in \mathfrak{J}(A), I \subseteq H_A(x) \} = \{ P | P \text{ a prime ideal of } A, I \subseteq P \} = V(I) \). Which is closed in \( \text{Spec } A \).
Hence we have the following lemma.
35 LEMMA: If $A$ is a commutative ring then $H_A$ is a homeomorphism of $\mathcal{J}(A)$ onto $\text{Spec } A$.

36 THEOREM: For commutative rings $\mathcal{J}$ is naturally equivalent to $\text{Spec}$.

PROOF: Suppose $s: A \to B$ is a morphism of commutative rings. In view of 35, all that we need show is that

$$
\begin{array}{ccc}
\mathcal{J}(A) & \xrightarrow{H_A} & \text{Spec } A \\
\downarrow \mathcal{J}(s) & & \uparrow \text{Spec } s \\
\mathcal{J}(B) & \xrightarrow{H_B} & \text{Spec } B
\end{array}
$$

commutes.

To this end suppose $y \in \mathcal{J}(B)$ and $y' = (\mathcal{J}(s))(y)$. Then

$$
\begin{array}{ccc}
A & \xrightarrow{s} & B \\
\downarrow y' & & \uparrow y \\
D_{y'} & \xrightarrow{s_y} & D_y
\end{array}
$$

commutes and $H_A((\mathcal{J}(s))(y)) = H_A(y') = \ker y' = \ker(s_y \circ y') = \ker(y \circ s) = s^\dagger(\ker y) = \text{Spec } s(\ker y) = \text{Spec } s(H_B(y))$. Thus $H_A \circ \mathcal{J}(s) = \text{Spec } s \circ H_B$, and $\mathcal{J}$ is naturally equivalent to $\text{Spec}$.

We now consider a division ring, $D$, and its spectrum $\mathcal{J}(D)$.

If $x, y \in \mathcal{J}(D)$ then since $x(1) = 1$ and $y(1) = 1$ we may conclude that $x$ and $y$ are isomorphisms and therefore that $w = y \circ x^\dagger: D_x \to D_y$.
is an isomorphism such that $y = w \circ x$. Thus $x$ is equivalent to $y$
and since $x, y \in \mathcal{S}(D)$ we have $x = y$. This shows that $\mathcal{S}(D)$ is
a singleton and we shall assume that $\mathcal{S}(D) = \{\text{id}_D\}$.

37 THEOREM: If $D$ is a division ring then $\mathcal{S}(D)$ is a singleton.

The next class of rings whose spectrums we shall find are the
products of division rings. What we shall show is that if $M$ is a non
null set, $\{E(m)\}_{m \in M}$ is a family of division rings, and $A = \bigcap_{m \in M}$ then
$\mathcal{S}(A)$ is homeomorphic to the Stone-Cech compactification of the disjoint
union of the spectrums of its factors, $\bigcup_{m \in M} \mathcal{S}(E(m))$.

We may simplify our notation a bit by noting that, since for each
$m \in M$, $\mathcal{S}(E(m))$ is a singleton, $\bigcup_{m \in M} \mathcal{S}(E(m))$ is a discrete space with
cardinality equal to the cardinality of $M$. Thus what we will do is show
that $\mathcal{S}(A)$ is homeomorphic to $\beta M$, the Stone-Cech compactification of the
discrete space $M$.

The Stone-Cech compactification may be characterized in different
ways depending on our knowledge of the original space. Kelley [K, p. 167, R],
provides that for normal spaces we may use the Wallman compactification.
Since a discrete space is normal we may, and do, use this construction of
$\beta M$.

38 DEFINITION: If $M$ is a non null set, $U \subseteq \mathcal{P}(M)$ is a filter of $M$ if:

1) $U \neq \emptyset$,
2) $\emptyset \notin U$,
3) $K, L \in U$ implies $K \cap L \in U$,
4) $K \in U$ and $K \subseteq L \subseteq M$ implies $L \in U$. 
A filter $U$ is an ultrafilter of $M$ if it is maximal, with respect to inclusion, among the filters of $M$.

39 LEMMA: If $M$ is a non null set and $U$ is a filter of $M$ then $U$ is an ultrafilter of $M$ if and only if for every $K \subseteq M$ either $K \in U$ or $(M \smallsetminus K) \in U$.

A proof of 39 and some uses of ultrafilters in ring theory may be found in [H], pages 179-186.

40 DEFINITION: If $M$ is a non null set with the discrete topology:

1) $\beta M = \{U|U$ is an ultrafilter of $M\}$ ,

2) for each $K \subseteq M$, $v'(K) = \{U|U \in \beta M, K \in U\}$ ,

3) $\{v'(K)|K \subseteq M\}$ is a base for the closed sets of the topology of $\beta M$; that is $X \subseteq \beta M$ is closed if for every $U \in \beta M \sim X$ there exists $K \subseteq M$ such that $U \notin v'(K)$ and $X \subseteq v'(K)$.

With this definition every closed set in $\beta M$ is of the form $\bigcap_{K \in X} v'(K)$ for some set $X \subseteq \mathcal{P}(M)$.

In [H] on page 180 the following is shown. If $M$ is a non null set, $\{R_m\}_{m \in M}$ is a family of rings, and $U$ is an ultrafilter of $M$ then there is an equivalence relation on $\Pi R_m$ given by $a \sim b$ if $\{m|m \in M, a(m) = b(m)\} \in U$. The set $I = \{a|a \in \Pi R_m, a \sim 0\}$ is then an ideal of $\Pi R_m$ and we write $\Pi R_m/U$ for the ring $\Pi R_m/I$ and call it an ultraproduct of the $R_m$'s.
41 DEFINITION: If $M$ is a non null set, $\{R_m\}_{m \in M}$ is a family of rings, and $U$ is an ultrafilter of $M$ then the ultraproduct, $\prod R(m)/U$ is as above.

42 LEMMA: An ultraproduct of division rings is a division ring.

PROOF: Let $M$ be a non null set, $\{E(m)\}_{m \in M}$ be a family of division rings, and $U$ be an ultrafilter of $M$. We will show that $I_U = \{a | a \in \Pi E(m), a \sim 0\}$ is a maximal ideal.

Suppose $a \in \Pi E(m) \sim I_U$. Then, since $a \notin I_U$, $\{m | m \in M, a(m) = 0\} \notin U$. $U$ is an ultrafilter implies $K = \{m | m \in M, a(m) \neq 0\} = M \sim \{m | m \in M, a(m) = 0\} \in U$. Let

$$b \in \Pi E(m) \text{ given by } b(m) = \begin{cases} 0 & \text{if } m \in K \\ 1 & \text{otherwise} \end{cases}$$

$$c \in \Pi E(m) \text{ given by } c(m) = \begin{cases} [a(m)]^{-1} & \text{if } m \in K \\ 1 & \text{otherwise} \end{cases}.$$ 

Note that $K \in U$ implies $b \in I_U$.

$$((a+b)c)(m) = \begin{cases} (a(m)+0)(a(m))^{-1} = 1 & \text{if } m \in K \\ (0+1)(1) = 1 & \text{otherwise} \end{cases}.$$ 

So $1 = (a+b)c$ is an element of the ideal generated by $I_U$ and $a$.

Thus either $I_U = \Pi E(m)$ or $I_U$ is maximal. Now $I_U \neq \Pi E(m)$ since $\{m | m \in M, l(m) = 0\} = \phi \notin U$ implies $1 \notin I_U$.

43 LEMMA: If $M$ is a non null set, $\{E(m)\}_{m \in M}$ be a family of division rings, $A = \Pi E(m)$, $U$ an ultrafilter of $M$, and $p_U: A \to \Pi E(m)/U$ is the natural projection, then there exists an $x \in \mathcal{S}(A)$ which is equivalent to $p_U$. 
PROOF: By 41, \( p_u \) is a morphism of \( A \) onto a division ring, and so
by the definition of \( \mathcal{J}(A) \) there is an \( x \in \mathcal{J}(A) \) which is equivalent
to \( p_u \).

44 DEFINITION: If \( M \) is a non null set, \( \{E(m)\}_{m \in M} \) be a family of
division rings, and \( A = \Pi E(m) \) let
\[
q: \mathcal{J}(A) \to \beta M \text{ by } q(x) = \{\{m|a(m) = 0\} | a \in \ker x\}.
\]
We will eventually show that \( q \) is a homeomorphism.

45 LEMMA: With \( M, E, \) and \( A \) as in 43, \( q \) is a one-to-one correspondence
between \( \mathcal{J}(A) \) and \( \beta M \).

PROOF: First we show that the image of \( q \) is a subset of \( \beta M \). That
is, if \( x \in \mathcal{J}(A) \) then \( q(x) \) is an ultrafilter of \( M \).

If \( a \in A \) and \( \{m|a(m) = 0\} = \emptyset \) then let \( b \in A \) given by
\( b(m) = (a(m))^{-1} \). Thus since \( ab = 1 \) and \( x(1) = 1 \) it follows that
\( a \notin \ker x \). Therefore \( \emptyset \notin q(x) \).

If \( K, L \in q(x) \) let \( a, b \in \ker x \) such that \( K = \{m|a(m) = 0\} \)
and \( L = \{m|b(m) = 0\} \). Also let
\[
a' \in A \text{ given by } a'(m) = \begin{cases} 0 & \text{if } m \in K \\ (a(m))^{-1} & \text{if } m \in M \setminus K \end{cases},
\]
\[
b' \in A \text{ given by } b'(m) = \begin{cases} 0 & \text{if } m \in L \\ (b(m))^{-1} & \text{if } m \in M \setminus L \end{cases}.
\]
Then \( c = aa' \) and \( d = bb' \) are elements of \( \ker x \) and
\[
c(m) = \begin{cases} 0 & \text{if } m \in K \\ 1 & \text{otherwise} \end{cases},
\]
\[
d(m) = \begin{cases} 0 & \text{if } m \in L \\ 1 & \text{otherwise} \end{cases}.
\]
Let \( e \in A \) given by 
\[
e(m) = \begin{cases} 
0 & \text{if } c(m) = 1 \text{ and } \text{Char } E(m) = 2 \\
1 & \text{otherwise}
\end{cases}
\]

Then \( c + de \in \ker x \) and
\[
(c+de)(m) = \begin{cases} 
0 & \text{if } c(m) = d(m) = 0 \\
2 & \text{if } c(m) = d(m) = 1 \text{ and } \text{Char } E(m) \neq 2 \\
1 & \text{otherwise}
\end{cases}
\]

So \( K \cap L = \{m|c(m) = 0\} \cap \{m|d(m) = 0\} \)
\[
= \{m|c(m) = 0 \text{ and } d(m) = 0\}
= \{m|(c+de)(m) = 0\} \in q(x) .
\]

If \( K \in q(x) \) and \( K \subseteq L \subseteq M \) let \( a \in \ker x \) such that 
\[
K = \{m|a(m) = 0\} \quad \text{and let} \quad b \in A \quad \text{given by}
\]
\[
b(m) = \begin{cases} 
0 & \text{if } m \in L \\
1 & \text{otherwise}
\end{cases}
\]

Then \( ab \in \ker x \) and \( \{m|(ab)(m) = 0\} = \{m|a(m) = 0 \text{ or } b(m) = 0\} \)
\[
= \{m|a(m) = 0\} \cup \{m|b(m) = 0\} = K \cup L = L . \quad \text{Hence } L \in q(x) . \quad \text{We have shown that } q(x) \text{ is a filter of } M .
\]

To see that \( q(x) \) is an ultrafilter suppose \( K \subseteq M \) and let \( a, b \in A \) such that 
\[
a(m) = \begin{cases} 
0 & \text{if } m \in K \\
1 & \text{otherwise}
\end{cases} , \quad b(m) = \begin{cases} 
1 & \text{if } m \in K \\
0 & \text{otherwise}
\end{cases}
\]

Then, since \( ab = 0 \in \ker x \), either \( a \in \ker x \) or \( b \in \ker x \). Hence 
\( K \in q(x) \) or \( M \supseteq K \in q(x) \), and so \( q(x) \) is an ultrafilter.

Thus \( q \) maps \( \mathcal{J}(A) \) into \( \mathcal{B}M \). On the other hand if \( U \in \mathcal{B}M \) then by 43 there exists \( x \in \mathcal{J}(A) \) such that \( x \) is equivalent to \( p_U \) and 
\[
q(x) = \{\{m|a(m) = 0\}|a \in \ker x\}
= \{\{m|a(m) = 0\}|a \in \ker p_U\}
= \{\{m|a(m) = 0\}|\{m|a(m) = 0\} \in U\}
= U .
\]
Therefore \( q \) is onto.

If \( x, y \in \mathcal{J}(A) \) and \( q(x) = q(y) = U \) then by 43 and what is above \( x \) is equivalent to \( p_U \) and \( y \) is equivalent to \( p_U \). Thus, since \( x, y \in \mathcal{J}(A) \) and \( x \) is equivalent to \( y \), we conclude that \( x = y \).

Hence \( q \) is a one-to-one correspondence between \( \mathcal{J}(A) \) and \( \beta M \).

Because of 43 and 44, if \( M \) is a non null set, \( \{E(m)\}_{m \in M} \) is a family of division rings, and \( A = \Pi E(m) \) we will assume \( \mathcal{J}(A) = \{p_U \mid U \in \beta M\} \).

46 DEFINITION: If \( M \) is a non null set and \( U \) is an ultrafilter of \( M \) then \( U \) is free if \( \cap_{K \in U} K = \emptyset \) and \( U \) is fixed if \( \cap_{K \in U} K \neq \emptyset \).

47 LEMMA: If \( M \) is a non null set and \( U \) is an ultrafilter of \( M \) then \( U \) is fixed if and only if there exists an \( m \in M \) such that \( U = \{K \mid K \subseteq M, m \in K\} \).

PROOF: If \( U \) is fixed then \( \cap_{K \in U} K \neq \emptyset \), so let \( m \in \cap_{K \in U} K \). Then since \( m \in \cap_{K \in U} K \), \( U \) is an ultrafilter either \( \{m\} \in U \) or \( M \setminus \{m\} \in U \). If \( M \setminus \{m\} \in U \) then \( m \notin \cap_{K \in U} K \subseteq M \setminus \{m\} \). So \( \{m\} \in U \) and hence \( \{K \mid K \subseteq M, m \in K\} \subseteq U \).

They are equal since if \( m \notin L \subseteq M \) then \( L \) cannot be in \( U \) since \( \{m\} \cap L = \emptyset \). The other implication is obvious.

48 LEMMA: If \( M \) is a non null set, \( \{E(m)\}_{m \in M} \) is a family of division rings, \( A = \Pi E(m) \), and \( U \) is a fixed ultrafilter of \( M \) then there exists an \( m \in M \) such that \( p_U \) is equivalent to \( \pi_m : A \to E(m) \) the natural projection.
PROOF: Since \( U \) is fixed there exists \( m \in M \) and \( U = \{ K | K \in M, m \in K \} \).

Now \( \ker p_U = \{ a | a \in A, \{ n | a(n) = 0 \} \in U \} = \{ a | a \in A, a(m) = 0 \} = \ker p_m \). Thus, since both \( p_U \) and \( p_m \) are epimorphisms, \( \tilde{w}_m = p_U \circ \tilde{p} \) is an isomorphism and \( p_U = \tilde{w}_m \circ \tilde{p}_m \).

49 DEFINITION: With \( M, E, A, U, \) and \( m \) as in Lemma 48 let \( \tilde{w}_m = p_U \circ \tilde{p}_m \).

50 LEMMA: If \( M \) is a non null set, \( \{ E(m) \}_{m \in M} \) is a family of division rings, and \( A = \prod E(m) \) then \( \mathcal{A}(A) = \{ a \cdot b^{-1} | a, b \in A \} \), in fact if \( f \in \mathcal{A}(A) \) and \( a, b \in A \) such that

\[
\begin{align*}
\tilde{w}_m & = \begin{cases} 
\tilde{p}_m & \text{if } (\tilde{w}_m \circ \tilde{p}_m) \in \text{dom } f \\
0 & \text{otherwise}
\end{cases} \\
a(m) & = \begin{cases} 
w_m(f(w \circ p_m)) & \text{if } (w_m \circ p_m) \in \text{dom } f \\
0 & \text{otherwise}
\end{cases} \\
b(m) & = \begin{cases} 
1 & \text{if } (w_m \circ p_m) \in \text{dom } f \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

then \( f = a \cdot b^{-1} \).

PROOF: If \( g \in \mathcal{A}(A) \) let \( K_g = \{ m | w_m \circ p_m \in \text{dom } g \} \)

\[
\begin{align*}
a_g(m) & = \begin{cases} 
w_m(g(w_m \circ p_m)) & \text{if } m \in K_g \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad b_g(m) = \begin{cases} 
1 & \text{if } m \in K_g \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Suppose \( f \in \mathcal{A}(A) \); our proof will be by induction on \( n \) such that \( f \in A_n^* \).

If \( f \in A_0^* \) then there exists \( c \in A \) such that \( f = c^* \).

This implies \( \text{dom } f = \mathcal{A}(A) \) and so \( K_f = M \) and \( b_f = 1 \). Also for
all \( m \in M \), since if \( p_U = w_m \circ \pi_m \), \( c(m) = \pi_m(c) = \overrightarrow{w_m(p_U(c))} \)

\( = \overrightarrow{w_m(c^*(p_U))} = \overrightarrow{w_m(f(p_U))} = a_f(m) \). Thus \( c = a_f \) and

\( f = c^* = a_f^{**-1} = a_f b_f^{**-1} \).

If \( f \in \mathbb{A}_{n+1}^* \) and for all \( g \in \mathbb{A}_n^* \), \( g = a_b^{**-1} \) then there are three cases,

1) there exists \( g \in \mathbb{A}_n^* \), \( f = g^{-1} \),
2) there exists \( g, h \in \mathbb{A}_n^* \), \( f = g - h \),
3) there exists \( g, h \in \mathbb{A}_n^* \), \( f = gh \).

In case 1), \( p_U \in \text{dom } f \) iff \( p_U \in \text{dom } g \) and \( g(p_U) \neq 0 \)

iff \( p_U(b_g) \neq 0 \) and \( p_U(a_g) \neq 0 \)

iff \( \{ m | b_g(m) = 0 \} \notin U \) and \( \{ m | a_g(m) = 0 \} \notin U \)

iff \( \{ m | b_g(m) \neq 0 \} \in U \) and \( \{ m | a_g(m) \neq 0 \} \in U \)

iff \( \{ m | a_g(m) \neq 0 \} \in U \) (since

\( \{ m | a_g(m) \neq 0 \} \subset \{ m | b_g(m) \neq 0 \} \))

iff \( \{ m | w_m \circ \pi_m \in \text{dom } g, g(w_m \circ \pi_m) \neq 0 \} \in U \)

iff \( \{ m | w_m \circ \pi_m \in \text{dom } f \} \in U \)

iff \( K_f \in U \).

Thus we have \( p_U \in \text{dom } f \) iff \( K_f \in U \), but

\( K_f \in U \) iff \( \{ m | b_f(m) \neq 0 \} \notin U \)

iff \( p_U(b_f) \neq 0 \)

iff \( p_U \in \text{dom } b_f^{**-1} \)

iff \( p_U \in \text{dom } a_f b_f^{**-1} \).

So \( \text{dom } f = \text{dom } a_f b_f^{**-1} \), and if \( p_U \in \text{dom } f \) then \( K_f \in U \) and for all \( m \in K_f \),

\[
(1 - a_f a_f)(m) = 1 - [w_m^+(f(w_m \circ \pi_m))] [w_m^+(g(w_m \circ \pi_m))] \\
= 1 - w_m^+(f(w_m \circ \pi_m) g(w_m \circ \pi_m)) \\
= 1 - w_m^+(1) \\
= 0 .
\]
Thus \( 0 = p_U(1-a_f a_g) = 1 - p_U(a_f)p_U(a_g) \) and so

\[
f(p_U) = (p_U(a_g))^{-1} = p_U(a_f) = (a_f b_f^{* -1})(p_U) .
\]

Cases 2) and 3) are similar and we will not do them here.

51 LEMMA: If \( M \) is a non null set, \( \{E(m)\}_{m \in M} \) is a family of division rings, \( A = \Pi E(m), F \in \text{rad}(A) \), and \( I = \{a|a \in A, a^* \in F\} \) then

\[
\forall (F) = \bigcup_{a \in I} \text{dom } a^{*-1} .
\]

PROOF: If \( f \in F \) then by 50 there exists \( a, b \in A \) such that

\[
f = a^* b^{* -1} . \text{ Let } c = ab , \text{ then } \text{dom } f^{-1} = \text{dom}(a^*b^{*-1})^{-1}
\]

\[
= \{p_U|p_U(a) \neq 0, p_U(b) \neq 0\} \cap \{p_U|p_U(a) \neq 0\} = \text{dom } b^{*-1} \cap \text{dom } a^{*-1} = \text{dom } (a^*b^*)^{-1} = \text{dom } (ab)^{-1} = \text{dom } c^{*-1} .
\]

Since \( \text{dom } c^{*-1} = \text{dom } f^{-1} \subseteq \bigcup \text{dom } g^{-1} \) we have \( \forall (F) \cap \text{dom } c^{*-1} = \phi \). Hence \( c^* \in F \) and \( c \in I \).

Therefore if \( f \in F \), \( \text{dom } f^{-1} \subseteq \bigcup \text{dom } a^{*-1} \). Hence \( \bigcup \text{dom } f^{-1} \subseteq \bigcup \text{dom } a^{*-1} \) and the reverse inclusion is obvious.

52 THEOREM: If \( M \) is a non null set, \( \{E(m)\}_{m \in M} \) is a family of division rings, and \( A = \Pi E(m) \) then \( q \) is a homeomorphism of \( \hat{\hat{\mathcal{A}}} \) onto \( \mathcal{B}M \).

PROOF: By 44 \( q \) is one to one and onto. By 10 every closed set in \( \hat{\hat{\mathcal{A}}} \) is of the form \( \forall (F) \) for some \( F \in \text{rad}(A) \). So, let
Consider $q[V(F)]$. By 51 we have,

$$q[V(F)] = q[\mathcal{J}(A) \cup \text{dom } a^{-1}]$$

$$= q[\bigcap_{a \in I} \{p_\mathcal{U} | p_\mathcal{U}(a) = 0\}]$$

$$= q[\bigcap_{a \in I} \{m | a(m) = 0 \in U\}]$$

$$= \bigcap_{a \in I} q[\{m | a(m) = 0 \in U\}]$$

$$= \bigcap_{a \in I} \{u | \{m | a(m) = 0 \in U\} \in \mathcal{U}\}$$

$$= \bigcap_{a \in I} v'((m | a(m) = 0)) \text{ is closed in } \mathcal{J}(A).$$

Now $\{v'(K) | K \subseteq M\}$ is a bases for the closed subsets of $\mathcal{J}(M)$, so if $K \subseteq M$ let $a \in A$ given by

$$a(m) = \begin{cases} 0 & \text{if } m \in K \\ 1 & \text{otherwise} \end{cases}$$

Then $q^{*}[v'(K)] = q^{*}[\{U | U \in \mathcal{J}(M), K \in U\}]$

$$= \{p_\mathcal{U} | p_\mathcal{U} \in \mathcal{J}(A), K \in U\}$$

$$= \{p_\mathcal{U} | p_\mathcal{U} \in \mathcal{J}(A), p_\mathcal{U}(a) = 0\}$$

$$= \{p_\mathcal{U} | p_\mathcal{U} \in \mathcal{J}(A), a^{*}(p_\mathcal{U}) = 0\}$$

$$= \mathcal{J}(A) \cap \text{dom } a^{*-1} \text{ which is closed in } \mathcal{J}(A).$$

Therefore $q$ is a homeomorphism.

We turn now to our fourth example, the ring of $n \times n$ matrices, $D_n$, over the division ring $D$. If $x$ is a morphism of $D_n$ into a division ring $E$ such that $x(1) = 1$ then, since $D_n$ is simple, $x$ is a monomorphism. But if $n > 1$ this is absurd since $D_n$ has zero divisors. Thus there can be no morphisms of $D_n$ into a division ring and we have the following.
53 THEOREM: If $D$ is a division ring and $n > 1$ then $\mathcal{S}(D_n) = \emptyset$.

Our last example is the spectrum of $\mathbb{Z}[i,j,k]$, the ring of quaternions with integer coefficients. Because of the confusion in the literature we find it convenient to make the following notational remarks: $\mathbb{Z}_p$ will mean the integers localized at the prime $(p)$; and, $\mathbb{Z}/p$ will mean the integers mod $(p)$.

54 DEFINITION: Let $\tau: \mathbb{Z}[i,j,k] \to \mathbb{Z}/2$ by $\tau(a+bi+cj+dk) = (a+b+c+d) + (2)$, and $\iota: \mathbb{Z}[i,j,k] \to \mathbb{Q}[i,j,k]$ by $\iota(a+bi+cj+dk) = a + bi + cj + dk$.

55 LEMMA: There exist $x$ and $y$ in $\mathcal{S}(\mathbb{Z}[i,j,k])$ such that $x$ is equivalent to $\iota$ and $y$ is equivalent to $\tau$.

PROOF: We will consider $\iota$ first. It is clear that $\iota$ is a morphism of rings with identity. Suppose that $D$ is a division ring and $\text{Im } \iota \subseteq D \subseteq D_\iota = \mathbb{Q}[i,j,k]$. If $a + bi + cj + dk \in D_\iota$ let $m$ be a common multiple of the denominators of $a, b, c, \text{ and } d$, that is

$$a + bi + cj + dk = \left(\frac{1}{m}\right)[am + bmi + cmj + dmk] \text{ where } m, am, bm, cm, dm \in \mathbb{Z}.$$ 

So $m$ and $am + bmi + cmj + dmk \in \text{Im } \iota \subseteq D$. Since $D$ is a division ring $\frac{1}{m} \in D$ and hence $a + bi + cj + dk \in D$. Thus $D = D_\iota = \mathbb{Q}[i,j,k]$ and by the definition of $\mathcal{S}$ there exist an $x \in \mathcal{S}(\mathbb{Z}[i,j,k])$ such that $x$ is equivalent to $\iota$.

We now consider $\tau$. Since it is clear that $\tau$ is onto and $\tau(1) = 1$ all we need show is that $\tau$ is a ring morphism. Let
a, b ∈ Z[i,j,k], a = a_1 + a_2 + a_3 + a_4, and b = b_1 + b_2 + b_3 + b_4.

Then \( \tau(a-b) = \tau(a_1 - b_1 + (a_2 - b_2)i + (a_3 - b_3)j + (a_4 - b_4)k) \)

\[
\begin{align*}
&= \left( \sum_{n=1}^{4} a_n \right) + (2) - \left( \sum_{n=1}^{4} b_n \right) + (2) \\
&= \tau(a) - \tau(b),
\end{align*}
\]

and \( \tau(ab) = \tau((a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4) + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)i + \\
(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)j + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1)k) \)

\[
\begin{align*}
&= \left( \sum_{n=1}^{4} a_n \right) + (2) + \left( \sum_{n=1}^{4} b_n \right) + (2) \\
&= \tau(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4) + (a_1b_2 + a_2b_1) + (a_3b_4 + a_4b_3) + \\
(a_1b_3 + a_2b_4 + a_3b_1 + a_4b_2) + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1) + (2) \\
&= \tau((a_1 + a_2 + a_3 + a_4) + (2)) \left( (b_1 + b_2 + b_3 + b_4) + (2) \right) \\
&= \tau(a) \tau(b).
\end{align*}
\]

Therefore \( \tau \) is a ring morphism and there exists \( y ∈ \mathcal{J}(Z[i,j,k]) \)
such that \( y \) is equivalent to \( \tau \).

56 LEMMA: If \( x ∈ \mathcal{J}(Z[i,j,k]) \) then either \( x \) is equivalent to \( \tau \) or \( x \) is equivalent to \( 1 \).

PROOF: We first claim that if \( x ∈ \mathcal{J}(Z[i,j,k]) \) and \( x(i), x(j), \) or \( x(k) \) is in the center of \( \text{Im} \ x \) then \( \text{Char} \ D_x = 2 \).

To prove this claim suppose \( x ∈ \mathcal{J}(Z[i,j,k]) \) and that \( x(i) \)
is in the center of \( \text{Im} \ x \). Then \( x(ji-i-j) = x(j)x(i) - x(i)x(j) = 0 \) and
so \( 0 = x(k) = x(ji-i-j)x(k) = x((-j-i-j)k) = x((-k-k)k) = x(-k^2-k^2) \)
\( = x(l+l) = x(l) + x(l) = 1 + 1 \). Thus \( \text{Char} \ D_x = 2 \). Either of the
assumptions, \( x(j) \) is in the center or \( x(k) \) is in the center, leads to the same result in an analogous way.

Now suppose \( x \in \mathcal{M}(Z[i,j,k]) \) and \( \text{Char} \, D_x = p \neq 0 \). Then \( \text{Im} \, x \) is a finite subring of \( D_x \) (card (\( \text{Im} \, x \)) \( \leq p^4 \)) so by Wedderburn's theorem \( \text{Im} \, x \) is a field. Thus by our claim \( \text{Char} \, D_x = 2 \) and \( \text{Im} \, x \) is a finite, and hence algebraic, extension of \( Z/2 \). But \( x(i), x(j), \) and \( x(k) \) are roots of \( y^2 + 1 \) which splits over \( Z/2 \). Hence \( x(i) = x(j) = x(k) = 1 \) and \( \text{Im} \, x \cong Z/2 \). Since \( \text{Im} \, x \) is a division ring and \( \text{Im} \, x \subseteq \text{Im} \, x \subseteq D_x \) we have \( \text{Im} \, x = D_x \). If \( u \) is the isomorphism of \( D_x \) onto \( Z/2 \) it is clear that

\[
\begin{array}{c}
\text{Z[i,j,k]} \\
\downarrow \tau \\
x \\
\downarrow \\
\text{Z/2} \\
\leftarrow u \\
\rightarrow D_x
\end{array}
\]

commutes and therefore \( x \) is equivalent to \( \tau \).

Before we consider the case, \( \text{Char} \, D_x = 0 \), we note some elementary facts about quaternions. If \( a = a_1 + a_2i + a_3j + a_4k \) then we call \( \overline{a} \) the conjugate of \( a \) if \( \overline{a} = a_1 - a_2i - a_3j - a_4k \) and observe that \( \overline{a}a = a\overline{a} = a_1^2 + a_2^2 + a_3^2 + a_4^2 \). Thus, if \( x(\overline{a}) \) is invertible in some ring \( A \) where \( \text{Im} \, x \subseteq A \subseteq D_x \) then \( x(a) \) is also invertible in \( A \):

\[ ([x(\overline{a})]^{-1}x(a))x(a) = [x(\overline{a})]^{-1}x(\overline{a}) = 1. \]

Suppose that \( \text{Char} \, D_x = 0 \). Then we claim that \( x \) is one to one. \( a, b \in Z[i,j,k] \) and \( x(a) = x(b) \) implies \( x(a-b) = 0 \)

imply \( x((a-b)(\overline{a-b})) = 0 \)

imply \( \sum_{n=1}^{4} (x(a-nb)) = 0 \)
(since $\text{Char } D_x = 0$) implies for each $n$, $x(a - b) = 0$

(since $x|_Z$ is one to one) implies for each $n$, $a_n = b_n$

implies $a = b$.

From (†) and the fact that $x$ is one to one we conclude that

$D_x = \{(x(a))^{-1}(x(b))| a \in Z, a > 0, \text{ and } b \in Z[i,j,k]\}$ and that

$u: Q[i,j,k] \rightarrow D_x$ by $u(a^{-1}b) = (x(a))^{-1}(x(b))$ is an isomorphism such that

commutes. Hence $x$ is equivalent to $\tau$. In view of this lemma we
will assume that $\mathcal{A}(Z[i,j,k]) = \{\tau, \tau\}$.

We would like to describe the topology on $\{\tau, \tau\}$. We note that

since $\tau(2) = 0$ we have $\{i\} = \text{dom } 2^{-1}$, which is open. Therefore the

only question that remains is whether $\{\tau\}$ is open.

57 THEOREM: 1) $\mathcal{F}(Z[i,j,k]) = \{a^{-1}b^*| a \in Z \text{ and } b \in Z[i,j,k]\}$

2) $\mathcal{F}(Z[i,j,k]) = \{\emptyset, \{i\}, \{\tau, \tau\}\}$.

PROOF: We prove 1) by induction on $n$ such that $f \in Z[i,j,k]^*_n$. If

$f \in Z[i,j,k]^*_0$ then there exists an $a \in Z[i,j,k]$ such that $f = a^* = 1^{*-1}a^*$.

Suppose for all $g \in Z[i,j,k]^*_n$ there exists $a \in Z$ and $b \in Z[i,j,k]$ such that $g = a^*-b^*$. So if $f \in Z[i,j,k]^*_n$ then either,

a) there exists $g \in Z[i,j,k]^*_n$ and $f = g^{-1}$,

b) there exist $g, h \in Z[i,j,k]^*_n$ and $f = g - h$, or

c) there exist $g, h \in Z[i,j,k]^*_n$ and $f = gh$.
All the proofs are easy, we give a hint for a) and leave the calculation to the reader. In a) if \( g = a^{*-1}b^* \) show that \( f = (abb)^{*-1}(a^2b)^* \).

To show part 2) of the theorem observe that if \( f = a^{*-1}b^* \) and \( 1 \notin \text{dom } f \) then \( i(a^*) = a^*(i) = 0 \). So \( a \in \ker i = \{0\} \) and hence \( f = 0^{*-1}b^* = \phi b^* = \phi \). Thus \( \text{dom } f = \phi \) and \( 1 \notin \text{dom } f \).
CHAPTER III

TOPOLOGICAL PROPERTIES OF $\mathcal{S}$

In this chapter we develop some topological properties of $\mathcal{S}(A)$.

In particular it is shown that $\mathcal{S}(A)$ is $T_0$ and compact. Theorems 61, $\mathcal{S}(A)$ is compact, and 64, every closed irreducible subset of $\mathcal{S}(A)$ has a generic point, are given by Bergman in [B]; the proofs presented, however, are different and the author believes more natural.

We also relate algebraic properties of a morphism $s$ to topological properties of $\mathcal{S}(s)$. For example, it is shown that if $s$ is an epimorphism then $\mathcal{S}(s)$ is a closed embedding.

58 LEMMA: If $A$ is a ring and $x \in \mathcal{S}(A)$ then

$$D_x = \{ f(x) | f \in \mathcal{S}(A) \text{ and } x \in \text{dom } f \}.$$ 

PROOF: Clearly $\text{Im } x = \{ a^*(x) | a \in A \} \subseteq \{ f(x) | f \in \mathcal{S}(A) \text{ and } x \in \text{dom } f \} \subseteq D_x$.

Because of the definition of $\mathcal{S}(A)$ we will be finished if we can show that $E = \{ f(x) | f \in \mathcal{S}(A) \text{ and } x \in \text{dom } f \}$ is a division ring.

If $d, e \in E$ then there exist $f, g \in \mathcal{S}(A)$ such that $x \in \text{dom } f$, $x \in \text{dom } g$, $f(x) = d$ and $g(x) = e$. Hence $x \in \text{dom } f \cap \text{dom } g$ $= \text{dom } f - g = \text{dom } fg$ and $(f-g)(x) = d - e$ while $(fg)(x) = de$. Therefore $E$ is a ring. Furthermore if $f(x) = d \neq 0$ then $x \in \text{dom } f^{-1}$ and $f^{-1}(x) = d^{-1}$. Thus $E$ is a division ring and the lemma is proven.

59 THEOREM: If $A$ is a ring then $\mathcal{S}(A)$ is a $T_0$-space.
PROOF: Suppose $x, y \in \mathcal{S}(A)$ and for all $f \in \mathcal{F}(A)$, $x \in \text{dom } f$ iff $y \in \text{dom } f$. Then for all $f \in \mathcal{F}(A)$ we have $x \in \text{dom } f$ and $f(x) = 0$ iff $y \in \text{dom } f$ and $f(y) = 0$, for otherwise either $x \in \text{dom } f^{-1}$ and $y \notin \text{dom } f^{-1}$ or vice versa.

Let $u: D_x \to D_y$ such that for all $f \in \mathcal{F}(A)$ with $x \in \text{dom } f$, $u(f(x)) = f(y)$. The preceding lemma shows that $\text{dom } u = D_x$ and if $d \in D_y$, there is an $f \in \mathcal{F}(A)$ such that $f(y) = d$. Since $y \in \text{dom } f$ iff $x \in \text{dom } f$ we have $u(f(x)) = f(y) = d$, that is, $u$ is onto.

The following shows that if $u$ is a function then $u$ is a morphism:

- $u(f(x) - g(x)) = u((f - g)(x)) = (f - g)(y) = f(y) - g(y) = u(f(x)) - u(g(x))$
- $u(f(x)g(x)) = u(fg(x)) = fg(y) = f(y)g(y) = (u(f(x))(u(g(x)))$
- $u(1) = u(1)(1)) = u(1^*(x)) = 1^*(y) = y(1) = 1$

Thus if we show $u$ is a function we will know that $u$ is an isomorphism. To see this suppose $f, g \in \mathcal{F}(A)$, $x \in \text{dom } f \cap \text{dom } g$, and $f(x) = g(x)$ then $(f - g)(x) = 0$. As we showed at the beginning of this proof $y \in \text{dom } f \cap \text{dom } g$ and $(f - g)(y) = 0$. Thus $f(x) = g(x)$ implies $u(f(x)) = f(y) = g(y) = u(g(x))$, and we have shown that $u$ is an isomorphism.

Now if $a \in A$ then $u(x(a)) = u(a^*(x)) = a^*(y) = y(a)$ and the following diagram commutes:

```
\begin{array}{c}
A \\
\downarrow x \\
D_x \\
\frac{u}{y} \\
\downarrow D_y
\end{array}
```

But that means that $x$ and $y$ are equivalent members of $\mathcal{S}(A)$, and so $x = y$. 

Therefore if \( x \) and \( y \) are distinct members of \( \mathcal{J}(A) \) then either there exists \( f \in \mathcal{F}(A) \) such that \( x \in \text{dom } f \) and \( y \notin \text{dom } f \) or there exists \( f \in \mathcal{F}(A) \) with \( x \notin \text{dom } f \) and \( y \in \text{dom } f \). That is, \( \mathcal{J}(A) \) is a \( T_0 \) space.

60 LEMMA: If \( A \) is a ring, \( X \) a closed subset of \( \mathcal{J}(A) \), \( B = \prod_{x \in X} D_x \), and \( s: A \to B \) such that for each \( a \in A \) and each \( x \in X \), \( (s(a))(x) = x(a) \) then \( \text{Im } \mathcal{J}(s) = X \).

PROOF: If \( x \in X \) let \( U \) be the fixed filter of \( X \) determined by \( x \), that is \( U = \{ K \mid x \in K \subset X \} \). Then \( p_U \in \mathcal{J}(B) \) and the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{s} & B \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & B/U \\
\downarrow & & \downarrow \\
D & \xrightarrow{\nu} & B/U
\end{array}
\]

Where \( \pi_x \) is the projection and \( \nu_x = p_U \circ \pi_x^+ \). It was shown, in 47, that \( \nu_x \) is an isomorphism. Since the diagram commutes we have \( x = (\mathcal{J}(s))(p_U) \). Thus \( X \subseteq \text{Im } \mathcal{J}(s) \).

Now consider \( \mathcal{J}(s)^+[X] \). Since \( X \) is closed and \( \{ p_U \mid U \text{ is a fixed ultrafilter of } X \} \subseteq \mathcal{J}(s)^+[X] \) we know that \( \mathcal{J}(B) \subseteq \mathcal{J}(s)^+[X] \). The reason for this last statement is that \( \mathcal{J}(B) \) is the Stone-Cech compactification of the set of fixed ultrafilters and so this set is dense in \( \mathcal{J}(B) \), [K, page 151].

This completes the proof since \( \mathcal{J}(B) \subseteq \mathcal{J}(s)^+[X] \) implies \( \text{Im } \mathcal{J}(s) \subseteq X \).
61 THEOREM: If $A$ is a ring then $\mathcal{J}(A)$ is compact.

PROOF: Let $B = \prod D_x$ and $s: A \rightarrow B$ such that for each $a \in A$ and each $x \in \mathcal{J}(A)$, $(s(a))(x) = x(a)$. Then, by the previous lemma we have $\text{Im} \mathcal{J}(s) = \mathcal{J}(A)$. Thus $\mathcal{J}(A)$ is the continuous image of a compact set and hence is compact [K, page 141, 8th].

62 DEFINITION: If $Y$ is a topological space, $X \subseteq Y$ then $x$ is a generic point of $X$ if $X = \{x\}$ (the closure of singleton $x$).

63 DEFINITION: If $Y$ is a topological space and $X \subseteq Y$ then $X$ is irreducible if for every pair of open sets $O_1$ and $O_2$, $O_1 \cap X \neq \emptyset$ and $O_2 \cap X \neq \emptyset$ implies $O_1 \cap O_2 \cap X \neq \emptyset$.

64 THEOREM: If $A$ is a ring and $X$ is a non-null, closed, irreducible subset of $\mathcal{J}(A)$ then $X$ has a generic point.

PROOF: Let $B = \prod D_x$ and $s: A \rightarrow B$ such that for each $a \in A$ and each $x \in X$, $(s(a))(x) = x(a)$. Then by Lemma 60, $\text{Im} \mathcal{J}(s) = X$. Let $F = \{K|K \subseteq X \text{ and such that there exists } f \in \mathcal{V}(A) \text{ with } \text{dom } f \cap X \neq \emptyset \text{ and } (\text{dom } f \cap X) \subseteq K\}$.

We claim that $F$ is a filter of $X$.

1) Clearly $X \neq \emptyset$ implies $F \neq \emptyset$.

2) It is also obvious that $\emptyset \nsubseteq F$.

3) If $K, L \in F$ then there exists $f, g \in \mathcal{V}(A)$, $\emptyset \neq (\text{dom } f \cap X) \subseteq K$ and $\emptyset \neq (\text{dom } g \cap X) \subseteq L$. Then $fg \in \mathcal{V}(A)$ and,
since $X$ is irreducible $\text{dom } fg \cap X = \text{dom } f \cap \text{dom } g \cap X \neq \phi$ and
\[ \text{dom } fg \cap X = (\text{dom } f \cap X) \cap (\text{dom } g \cap X) \subseteq K \cap L. \] Hence $K \cap L \in F$.

4) If $K \in F$ and $K \subseteq L \subseteq X$ then there exists $f \in \mathcal{F}(A)$ such that $\phi \neq (\text{dom } f \cap X) \subseteq K \subseteq L$. Hence $L \in F$.

Now since $F$ is a filter of $X$ there is an ultrafilter, $U$, of $X$ such that $F \subseteq U$. We claim that if $y = (\mathcal{A}(s))(p_U)$ then $y$ is a generic point of $X$.

Suppose $x \in X$ and $f \in \mathcal{F}(A)$ with $x \in \text{dom } f$ and consider $((\mathcal{F}(s))(f)) \in \mathcal{F}(B)$. The domain of $((\mathcal{F}(s))(f))$, as we showed in the proof of 50, is the set of all $p_U$, such that
\[ \{z \mid w_z \circ \pi_z \in \text{dom } ((\mathcal{F}(s))(f))\} \in U'. \] But
\[ \{z \mid w_z \circ \pi_z \in \text{dom } ((\mathcal{F}(s))(f))\} = \{z \mid (\mathcal{A}(s))(w_z \circ \pi_z) \in \text{dom } f\} = \{z \mid z \in \text{dom } f \cap X\} = \text{dom } f \cap X. \]

Thus $p_U \in \text{dom } ((\mathcal{F}(s))(f))$ iff $(\text{dom } f \cap X) \in U'$. If we look now at $U$ we discover that, since $\phi \neq \text{dom } f \cap X$, $\text{dom } f \cap X \in F \subseteq U$. Thus $p_U \in \text{dom } ((\mathcal{F}(s))(f))$ and $y = (\mathcal{A}(s))(p_U) \in \text{dom } f$. Therefore, since we have shown that for any $x \in X$ every basic neighborhood of $x$ contains $y$, we have $X \subseteq \{y\} \subseteq \text{Im } \mathcal{A}(s) = \overline{X} = X$.

In Chapter I we promised to show that $J({x})'$s were the only members of $\text{rad}(A)$, with the exception of $\mathcal{F}(A)$, which had the property: for all $f, g \in \mathcal{F}(A)$, $fg \in G$ implies $f \in G$ or $g \in G$. So, we have the following theorem.
THEOREM: If $A$ is a ring, $G \in \text{rad}(A)$, $V(G) \neq \phi$, and for all $f, g \in \mathcal{F}(A)$, $fg \in G$ implies $f \in G$ or $g \in G$ then there exists $x \in \mathcal{J}(A)$ and $G = J(\{x\})$.

PROOF: Let $X = V(G)$ then $X$ is a non-null closed subset of $\mathcal{J}(A)$, and we would like to prove that it is irreducible.

We first choose a convenient base for the topology on $\mathcal{J}(A)$. Our choice is $\{\text{dom } f^{-1} | f \in \mathcal{F}(A)\}$; we already know $\{\text{dom } f | f \in \mathcal{F}(A)\}$ is a base so to see that the smaller set is also, it is sufficient to note that for each $f \in \mathcal{F}(A)$, $\text{dom } f = (\text{dom } f^{-1}) \cup (\text{dom}(l^* - f)^{-1})$.

Assume $f, g \in \mathcal{F}(A)$, $\text{dom } f^{-1} \cap X \neq \phi$ and $\text{dom } g^{-1} \cap X \neq \phi$. Then since $G = J(X)$ we have $f, g \notin G$ and by hypothesis therefore $fg \notin G$. Hence $\text{dom}(fg)^{-1} \cap X \neq \phi$. But $\text{dom}(fg)^{-1} = \text{dom } g^{-1} f^{-1} = \text{dom } g^{-1} \cap \text{dom } f^{-1}$. Thus $\text{dom } f^{-1} \cap \text{dom } g^{-1} \cap X \neq \phi$ and hence $X$ is irreducible. So $X$ has a generic point, say $x$, and we have $G = J(V(G)) = J(X) = J(\{x\}) = J(\{x\})$.

This fulfills our promise since if $V(G) = \phi$ we have

$G = J(V(G)) = J(\phi) = \mathcal{F}(A)$.

We now look at some topological properties of the continuous function, $\mathcal{J}(s)$, and the relationships between the spectrums of the domain and codomain of $s$.

LEMMA: If $s: A \to B$ is a ring morphism, $\mathcal{F}(s)$ is onto, and $y \in \mathcal{J}(B)$ then $s_y$ is an isomorphism.
PROOF: If \( b \in B \) then, since \( \mathcal{F}(s) \) is onto, there exists \( f \in \mathcal{F}(A) \) such that \( (\mathcal{F}(s))(f) = b^* \). Thus we have \( y(b) = b^*(y) \)

\[
= ((\mathcal{F}(s))(f))(y) = s_y(f((\mathcal{A}(s))(y)))
\]

Hence \( \text{Im } y \subseteq \text{Im } s_y \subseteq D_y \) and \( \text{Im } s_y \) is a division ring. So, by the definition of \( \mathcal{A}(B) \), \( \text{Im } s_y = D_y \).

Since \( s_y \) is always a monomorphism, the proof is done.

67 LEMMA: If \( s: A \to B \) is a ring morphism and \( \mathcal{A}(s) \) is onto then \( \mathcal{A}(s) \) is one to one.

PROOF: Suppose \( x, y \in \mathcal{A}(B) \) and \( (\mathcal{A}(s))(x) = (\mathcal{A}(s))(y) = z \).

Then, by 66, \( s_y \circ s_x^- \) is an isomorphism of \( D_x \) onto \( D_y \). Moreover, if \( b \in B \) then there exists \( f \in \mathcal{F}(A) \) such that \( (\mathcal{F}(s))(f) = b^* \).

Thus we have:

\[
y(b) = b^*(y)
\]

\[
= ((\mathcal{F}(s))(f))(y)
\]

\[
= s_y(f((\mathcal{A}(s))(y)))
\]

\[
= s_y(f((\mathcal{A}(s))(x)))
\]

\[
= s_y(s_x^+((\mathcal{F}(s))(f))(x)))
\]

\[
= s_y(s_x^+(b^*(x)))
\]

\[
= s_y(s_x^+(x(b)))
\]

\[
= (s_y \circ s_x^+)(x(b))
\]

Therefore \( y = (s_y \circ s_x^+) \circ x \) and \( y \) is equivalent to \( x \). But since \( x, y \in \mathcal{A}(B) \) this means \( x = y \).
68 LEMMA: If \( s: A \to B \) is a ring morphism and \( \mathcal{F}(s) \) is onto then \( \mathcal{J}(s) \) is a homeomorphism of \( \mathcal{J}(B) \) onto \( \text{Im} \mathcal{J}(s) \).

PROOF: Since \( \mathcal{J}(s) \) is continuous and by 67 it is one to one, the only thing that must be proven is that \( \mathcal{J}(s) \) is an open map. It suffices to show that the \( \text{Im} \) of a basic open set is open. Suppose, therefore, that \( f \in \mathcal{F}(B) \). Then, since \( \mathcal{F}(s) \) is onto, there exists \( g \in \mathcal{F}(A) \) such that \( f = (\mathcal{F}(s))(g) \). But this means \( \text{dom} f = \mathcal{J}(s)^{-1} \text{dom} g \). Thus \( \mathcal{J}(s)[\text{dom} f] = \text{dom} g \cap \text{Im} \mathcal{J}(s) \) which is open in the relative topology on \( \text{Im} \mathcal{J}(s) \).

In order to use the last three lemmas we need some criterion for deciding when \( \mathcal{F}(s) \) is onto. The following lemma provides this. The proof of the lemma is by induction, the interested reader may readily do it.

69 LEMMA: If \( s: A \to B \) is a ring morphism and for each \( b \in B \) there exists an \( f \in \mathcal{F}(A) \) such that \( (\mathcal{F}(s))(f) = b^* \) then \( \mathcal{F}(s) \) is onto.

70 THEOREM: If \( s: A \to B \) is an epimorphism of rings then \( \mathcal{J}(s) \) is a homeomorphism of \( \mathcal{J}(B) \) onto \( \text{Im} \mathcal{J}(s) \).

PROOF: In view of 68 and 69 it is enough to show that for all \( b \in B \) there is \( f \in \mathcal{F}(A) \) such that \( (\mathcal{F}(s))(f) = b^* \). But since \( s \) is onto, there is an \( a \in A \) such that \( s(a) = b \), and this implies \( (\mathcal{F}(s))(a^*) = (s(a))^* = b^* \).
We would like to improve this result by giving a characterization of \( \text{Im} \, \mathcal{S}(s) \). We have previously shown that \( \text{Im} \, \mathcal{S}(s) = V_A(\ker \mathcal{S}(s)) \) and for commutative rings that \( V_A(\ker \mathcal{S}(s)) = \{x | x \in \mathcal{S}(A), \ker s \subseteq \ker x\} \).

The best general, non-topological, result that we can get about these three sets is:

**71 Lemma:** If \( s: A \to B \) is a ring morphism then \( \text{Im} \, \mathcal{S}(s) \subseteq V_A(\ker \mathcal{S}(s)) \subseteq \{x | x \in \mathcal{S}(A), \ker s \subseteq \ker x\} \).

**Proof:** The first inclusion was proven in 27. To prove the second suppose \( y \in V_A(\ker \mathcal{S}(s)) \) and \( a \in \ker s \). Then, since \( \{(\mathcal{S}(s))(a^*)\}^{-1} = [(s(a))^*]^{-1} = 0^{-1} = \phi \), we have \( a^* \in \ker \mathcal{S}(s) \). Therefore, since \( V_A(\ker \mathcal{S}(s)) = \mathcal{S}(A) \cap \bigcup_{f \in \ker \mathcal{S}(s)} \text{dom } f^{-1} \), \( y(a) = a^*(y) = 0 \).

To see that this is the best result we can get, observe first that if \( D \) is a division ring and \( s: D \to D_2 \) by \( s(d) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \) then since \( \mathcal{S}(D_2) = \phi \) we have \( \phi = \text{Im} \, \mathcal{S}(s) = V_A(\ker \mathcal{S}(s)) \not\subset \{x | x \in \mathcal{S}(D), \ker s \subseteq \ker x\} = \{id_D\} = \mathcal{S}(D) \). Secondly note that for \( \imath: Z[i,j,k] \to Q[i,j,k] \), one gets \( \{\imath\} = \text{Im} \, \mathcal{S}(\imath) \not\subset \{x | x \in \mathcal{S}(Z[i,j,k]) \), \( \ker \imath \subseteq \ker x\} = \{\imath, \imath\} \).

However, if we add the hypothesis that \( s \) is onto, then both inclusions can be replaced by equality and we have,

**72 Theorem:** If \( s: A \to B \) is a ring epimorphism then \( \mathcal{S}(B) \) is homeomorphic to \( \text{Im} \, \mathcal{S}(s) = V_A(\ker \mathcal{S}(s)) = \{x | x \in \mathcal{S}(A), \ker s \subseteq \ker x\} \) and \( \mathcal{S}(s) \) is a closed embedding.
PROOF: All that remains to prove is that \( \{ x \mid x \in \mathcal{J}(A), \ker s \subseteq \ker x \} \subseteq \text{Im} \ \mathcal{J}(s) \). Suppose, therefore, that \( x \in \mathcal{J}(A) \) and \( \ker s \subseteq \ker x \) and let \( y: B \rightarrow D_x \) such that for each \( a \in A \), \( y(s(a)) = x(a) \). Since \( \ker s \subseteq \ker x \), \( y \) is a function and, since \( s \) is onto, the domain of \( y \) is \( B \). It is clear then that \( y \) is a ring morphism of \( B \) into a division ring. Now, if \( E \) is a division ring, and \( \text{Im} \ y \subseteq E \subseteq D_x \) then, since \( \text{Im} \ y = \text{Im} \ x \) and \( x \in \mathcal{J}(A) \), we have \( E = D_x \). Therefore the definition of \( \mathcal{J}(B) \) gives us a \( z \in \mathcal{J}(B) \) which is equivalent to \( y \). That is, there is a isomorphism of \( D_x \) onto \( D_z \) such that the lower triangle in the following diagram commutes.

Thus if \( a \in A \), \( z(s(a)) = u(y(s(a)) = u(x(a)) \). Therefore the square commutes and by 17, \( x = (\mathcal{J}(s))(z) \in \text{Im} \ \mathcal{J}(s) \).

This completes the proof since the words "closed embedding" mean a homeomorphism whose image is closed and in this instance \( \text{Im} \mathcal{J}(s) = V_A(\ker \mathcal{J}(s)) \) is closed.

72 THEOREM: If \( A \) is a ring, \( \mathcal{J}(A) \) is not null, and \( I = \bigcap_{x \in \mathcal{J}(A)} \ker x \) then \( \mathcal{J}(A) \) is homeomorphic to \( \mathcal{J}(A/I) \).

PROOF: Let \( p: A \rightarrow A/I \) be the natural projection of \( A \) onto \( A/I \).

Then by the previous theorem \( \mathcal{J}(A/I) \) is homeomorphic to
\( \{ x \mid x \in \mathcal{J}(A), I \leq \ker x \} \). But, since \( I = \bigcap_{x \in \mathcal{J}(A)} \ker x \), this is all of \( \mathcal{J}(A) \).

To this point all the theorems in this chapter have been generalizations of theorems from the commutative case. However, in the commutative case there is the nice result, "if \( s: A \to B \) is a monomorphism then \( \text{Spec } s \) is dominant, that is \( \text{Im Spec } s = \text{Spec } A \)."

This result does not carry over to the noncommutative case, for example, if \( D \) is a division ring and \( s: D \to D_2 \) by \( s(d) = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \) then \( s \) is a monomorphism but \( \phi = \text{Im } \mathcal{J}(s) \neq \mathcal{J}(D) = \{ \text{id}_D \} \).
CHAPTER IV
A PRESHFEEF ON \( A \)

In this chapter we develop a presheaf of rings on the topological space \( A \).

73 DEFINITION: If \( A \) is a ring and \( U \in \mathcal{U}(A) \) then let
\[ K_A(U) = \{ f \mid f \in \mathcal{F}(A), U \subseteq \text{dom} f \} \]
and if \( f, g \in K_A(U) \) say \( f \sim_U g \)
whenever \( f - g \in J_A(U) \).

74 LEMMA: If \( A \) is a ring and \( U \in \mathcal{U}(A) \) then \( \sim_U \) is an equivalence relation on \( K_A(U) \).

PROOF: If \( f \in K(U) \) then \( (f-f)^{-1} = \phi \in J(U) \). If \( f, g \in K(U) \) and
\( (f-g) \in J(U) \) then, since \( J(U) \subseteq \text{rad}(A) \), by 12 we have
\( g - f = 0^* - (f-g) \in J(U) \). Finally, if \( f, g, h \in K_A(U) \) and \( f - g \)
and \( g - h \in J(U) \) then for all \( x \in U \), \( f(x) = g(x) = h(x) \). Hence
\( f - h \in J(U) \).

75 DEFINITION: If \( A \) is a ring, \( U \in \mathcal{U}(A) \), and \( f \in K_A(U) \) let
\[ [f]_U = \{ g \mid g \in K_A, f - g \in J(U) \} \]
and \( \Gamma(A,U) = \{ [f]_U \mid f \in K_A \} \).

We also define an addition and a multiplication on this set of equivalence classes
\[ [f]_U + [g]_U = [f - (0^* - g)]_U \]
\[ [f]_U \cdot [g]_U = [fg]_U \].
LEMMA: If $A$ is a ring and $U \in \mathcal{J}(A)$ then

1) $U = \emptyset$ implies $\Gamma(A, U)$ is a zero ring,

2) $U \neq \emptyset$ implies $\Gamma(A, U)$ is a ring with identity.

PROOF: If $U = \emptyset$ then $K(U) = \mathcal{J}(A)$ and $J(U) = \mathcal{J}(A)$. Thus for all $f, g \in \mathcal{J}(A)$, $f \cap U g$.

If $U \neq \emptyset$ then $0^*, 1^* \in K(U)$ and $1^* - 0^* = 1^* \downarrow J(U)$.

It is clear that if our operations are "well defined" that $\Gamma(A, U)$ is a ring with identity. We will present the proof for the addition and leave the remaining proof for the disbelieving reader.

If $[f]_U = [g]_U$ then $[f]_U + [h]_U = [f - (0^*-h)]_U$ and $[g]_U + [h]_U = [g - (0^*-h)]_U$, but for all $x \in U$ we have

$((g - (0^*-h)) - (f - (0^*-h)))(x) = g(x) + h(x) - f(x) - h(x) = g(x) - f(x) = 0$.

Therefore $[f - (0^*-h)]_U = [g - (0^*-h)]_U$.

So with each open set, $U$, of $\mathcal{J}(A)$ we have associated a ring $\Gamma(A, U)$ . In order to have a presheaf whenever $W$ and $U$ are open sets such that $U \subseteq W$ we must have a morphism $\rho_{U,W} : \Gamma(A, W) \to \Gamma(A, U)$, moreover if $T \subseteq U$ is also open then $\rho_{T,W} = \rho_{T,U} \circ \rho_{U,W}$ . The following definition provides these morphisms.

DEFINITION: If $A$ is a ring and $U, W \in \mathcal{J}(A)$ such that $U \subseteq W$ let $\rho_{U,W} : \Gamma(A, W) \to \Gamma(A, U)$ by $\rho_{U,W}([f]_W) = [f]_U$.

LEMMA: If $A$ is a ring and $T, U, W \in \mathcal{J}(A)$ such that $T \neq U \subseteq W$ then

1) $\rho_{U,W}$ is a morphism,
2) \( \rho_{T,W} = \rho_{T,U} \circ \rho_{U,W} \)

3) \( \rho_{U,U} \) is the identity on \( \Gamma(A,U) \).

**Proof:** In part 1) we will prove that \( \rho_{U,W} \) is a function, believing the rest to be obvious. If \( [f]_W = [g]_W \) then \( f - g \in J(W) \), but by \( \theta \), \( U \subseteq W \) implies \( J(W) \subseteq J(U) \). So \( f - g \in J(U) \) and hence

\[ \rho_{U,W}([f]_W) = [f]_U = [g]_U = \rho_{U,W}([g]_W) \]

To prove part 2): \( (\rho_{T,U} \circ \rho_{U,W})([f]_W) = \rho_{T,U}([f]_U) = [f]_T = \rho_{T,W}([f]_W) \).

Part 3) is obvious.

In view of this last lemma we have,

**79 Theorem:** If \( A \) is a ring then \( (\Gamma(A,\cdot), \rho) \) is a presheaf of rings over \( \mathcal{J}(A) \).

We would like to know how this presheaf breaks up the spectrum of \( A \). That is, how is \( \mathcal{J}(\Gamma(A,U)) \) related to \( \mathcal{J}(A) \) and \( U \). We can use the information from the last chapter if we can get a natural morphism of \( A \) into \( \Gamma(A,U) \).

**80 Definition:** If \( A \) is a ring and \( U \in \mathcal{J}(A) \) let \( \gamma_U: A \rightarrow \Gamma(A,U) \) by \( \gamma_U(a) = [a^*]_U \).

**81 Theorem:** If \( A \) is a ring and \( \phi \neq U \in \mathcal{J}(A) \) then \( \gamma_U \) is a morphism.
PROOF: If $a, b \in A$ we have $(a+b)^* = a^* + b^*$ and $(ab)^* = a^*b^*$. Thus
\[
\gamma_u(a+b) = [(a+b)^*]_U = [a^*+b^*]_U = [a^*]_U + [b^*]_U = \gamma_u(a) + \gamma_u(b),
\]
\[
\gamma_u(ab) = [(ab)^*]_U = [a^*b^*]_U = [a^*]_U[b^*]_U = \gamma_u(a)\gamma_u(b), \text{ and } \gamma_u(1) = [1^*]_U
\]
is the identity in $\Gamma(A, U)$.

We use this theorem and the following lemma to find the structure of $\mathcal{J}(\Gamma(A, U))$.

**82 Lemma:** If $A$ is a ring, $\phi \neq U \in \mathcal{J}(A)$, and $f \in K_A(U)$ then
\[
\mathcal{J}(\gamma_U)(f) = [f]_U^*.
\]

**Proof:** We proceed by induction on $n$ such that $f \in A_n^*$.

If $f \in A_n^*$ then there exists $a \in A$ such that $f = a^*$. So by 22 we have
\[
\mathcal{J}(\gamma_U)(f) = \mathcal{J}(\gamma_U)(a^*) = (\gamma_U(a))^* = [a^*]_U^* = [f]_U^*.
\]

Now suppose that for each $g \in A_n^* \cap K_A(U)$ we have
\[
\mathcal{J}(\gamma_U)(g) = [g]_U^*, \text{ and suppose } f \in A_{n+1}^* \cap K_A(U) .
\]

There are then three cases:

1) there exists $g \in A_n^*$ such that $f = g^{-1}$,
2) there exists $g, h \in A_n^*$ such that $f = g - h$,
3) there exists $g, h \in A_n^*$ such that $f = gh$.

First consider case 1). Then $U \subseteq \text{dom } f \subseteq \text{dom } g$ and so
\[
g \in K_A(U) .
\]

Therefore, by the induction hypothesis and 23,
\[
\mathcal{J}(\gamma_U)(f) = \mathcal{J}(\gamma_U)(g^{-1}) = (\mathcal{J}(\gamma_U)(g))^{-1} = ([g]_U^*)^{-1} .
\]

However, for all $x \in U$ we have $f(x) = (g(x))^{-1}$; hence, $[f]_U = [g]_U^{-1}$ and $[f]_U^* = ([g]_U^*)^{-1} = \mathcal{J}(\gamma_U)(f)$.

Cases 2) and 3) are similar.
This lemma, 69, and 68 give us the following theorem,

83 THEOREM: If $A$ is a ring and $\phi \neq U \in \mathcal{J}(A)$ then $\mathcal{J}(\Gamma(A,U))$ is homeomorphic to $\text{Im } \mathcal{J}(\gamma_U)$.

84 LEMMA: If $A$ is a ring and $\phi \neq U \in \mathcal{J}(A)$ then $U \subseteq \text{Im } \mathcal{J}(\gamma_U)$.

PROOF: Suppose $x \in U$ and let $y: \Gamma(A,U) \to D_\infty$ by $y([f]_U) = f(x)$. We claim $y$ is a morphism. If $[f]_U = [g]_U$ then for all $z \in U$, $f(z) = g(z)$, in particular $y([f]_U) = f(x) = g(x) = y([g]_U)$ and so $y$ is a function. The calculations that $y$ is a morphism are then easy.

Now if $a \in A$ then $y(\gamma_U(a)) = y([a]_U) = a*(x) = x(a)$, so $y \circ \gamma_U = x$ and $\text{Im } x \subseteq \text{Im } y$. Therefore if $D'$ is a division ring and $\text{Im } y \subseteq D' \subseteq D_\infty$ then $\text{Im } x \subseteq D' \subseteq D_\infty$. Since $x \in \mathcal{J}(A)$ this implies $D' = D_\infty$. Because $y$ has this property there exists $z \in \mathcal{J}(\Gamma(A,U))$ and an isomorphism $w: D_\infty \to D_z$ such that $w \circ y = z$.

Since both triangles in the following diagram commute

```
\begin{tikzcd}
A \arrow{r}{Y} \arrow{d}{x} & \Gamma(A,U) \arrow{d}{y} \\
D_x \arrow{r}{w} \arrow{ur}{z} & D_z
\end{tikzcd}
```

we conclude that the square commutes. Thus by 17, $x = \mathcal{J}(\gamma_U)(z) \in \text{Im } \mathcal{J}(\gamma_U)$. 

For the basic open sets of the topology on \( \mathcal{J}(A) \) we can improve on this lemma.

**85 Theorem:** If \( A \) is a ring, \( f \in \mathfrak{F}(A) \) and \( \phi \neq \text{dom } f = U \) then \( \mathfrak{J}(\Gamma(A,U)) \) is homeomorphic to \( U \).

**Proof:** In view of 83 and 84 all that remains to be proven is that \( \text{Im } \mathfrak{J}(\gamma_U) \subseteq U \). Since \( \text{dom } f = U \), we have \( f \in K_A(U) \) and so, by 82, \( \mathfrak{F}(\gamma_U)(f) = \mathfrak{J}(\gamma_U)^*[\text{dom } f] = \mathfrak{J}(\gamma_U)^+[U] \). Therefore \( \text{Im } \mathfrak{J}(\gamma_U) \subseteq U \).

This theorem tells us that any topological property that we prove about the spectrum is also true of the basic open sets of the spectrum. For example:

**86 Theorem:** If \( A \) is a ring and \( f \in \mathfrak{F}(A) \) then \( \text{dom } f \) is compact.

If \( A \) is a ring and \( x \in \mathfrak{J}(A) \) then the open sets which contain \( x \) form a collection which is directed by inclusion, and with each pair of open sets \( U, W \) such that \( x \in U \subseteq W \) there is a morphism \( \rho_{U,W}: \Gamma(A,W) \rightarrow \Gamma(A,U) \). Thus we can take the direct limit of the \( \Gamma(A,U) \) such that \( x \in U \).

**87 Definition:** If \( A \) is a ring and \( x \in \mathfrak{J}(A) \) then let \( (\Gamma\text{-stalk-} x, \sigma^X) \) be the direct limit of the \( \Gamma(A,U), \rho_{U,W} \) where \( x \in U \subseteq W \).
To say that \((\Gamma\text{-stalk-}x, \sigma^x_U)\) is the direct limit means that each open set, \(x \in U \in \mathcal{U}(A)\), \(\sigma^x_U\) is a ring morphism of \(\Gamma(A,U)\) into the ring \(\Gamma\text{-stalk-}x\) such that for any pair of open sets \(U, W\) we have \(U \subseteq W\) implies \(\sigma^x_U \circ \rho^x_{U,W} = \sigma^x_W\). Moreover, if \((R, \theta)\) is a pair with the same properties then there exists a morphism \(\psi: \Gamma\text{-stalk-}x \to R\) such that for each \(U \in \mathcal{U}(A)\) with \(x \in U\) then \(\theta_U = \psi \circ \sigma^x_U\).

It can be shown that if \(a \in \Gamma\text{-stalk-}x\) then there exists a \(U\) and an \(f, x \in U \in \mathcal{U}(A), f \in K_A(U)\) such that \(a = \sigma^x_U([f])\).

88 DEFINITION: If \(A\) is a ring and \(x \in \mathcal{J}(A)\) let \(\delta^x = \sigma^x_{\mathcal{J}(A)} \circ \gamma_{\mathcal{J}(A)}\).

89 LEMMA: If \(A\) is a ring and \(x \in \mathcal{J}(A)\) then

1) for all \(U, W \in \mathcal{J}(A), U \subseteq W\) implies \(\gamma_U = \rho^x_{U,W} \circ \gamma_W\),

2) for all \(U \in \mathcal{J}(A), x \in U\) implies \(\delta^x = \sigma^x_U \circ \gamma_U\).

PROOF: If \(a \in A\) then \((\rho^x_{U,W} \circ \gamma_W)(a) = \rho^x_{U,W}([a^*]_W) = [a^*]_U = \gamma_U(a)\).

Therefore, we also have, for any \(U \in \mathcal{J}(A)\) such that \(x \in U\),
\[
\sigma^x_U \circ \gamma_U = \sigma^x_U \circ (\rho^x_{U, \mathcal{J}(A)} \circ \gamma_{\mathcal{J}(A)}) = (\sigma^x_U \circ \rho^x_{U, \mathcal{J}(A)}) \circ \gamma_{\mathcal{J}(A)} = \sigma^x_{\mathcal{J}(A)} \circ \gamma_{\mathcal{J}(A)} = \delta^x.
\]

90 LEMMA: If \(A\) is a ring and \(x \in \mathcal{J}(A)\) then \(\text{Im } \mathcal{J}(\delta^x)\)
\[
= \bigcap_{x \in U \in \mathcal{J}(A), x \in \{y\}} U = \{y|y \in \mathcal{J}(A), x \in \{y\}\}.
\]

PROOF: Since \(\mathcal{J}\) is a contravariant functor we have, for each \(U \in \mathcal{J}(A)\) such that \(x \in U\), \(\mathcal{J}(\delta^x) = \mathcal{J}(\gamma_U) \circ \mathcal{J}(\sigma^x_U)\). Hence \(\text{Im } \mathcal{J}(\delta^x) \subseteq \text{Im } \mathcal{J}(\gamma_U)\).

Now, since \(\{\text{dom } f|f \in \mathcal{J}(A)\}\) is a base for \(\mathcal{J}(A)\), \(\bigcap_{x \in U \in \mathcal{J}(A)} U\)
\[
= \bigcap_{x \in \text{dom } f} \bigcap_{f \in \mathcal{J}(A)} \text{Im } \mathcal{J}(\gamma_{\text{dom } f}). \text{ Therefore } \text{Im } \mathcal{J}(\delta^x) \subseteq \bigcap_{x \in U \in \mathcal{J}(A)} U.
\]
On the other hand if $y \in \bigcap_{x \in U} \mathcal{Y}(A)$ with $x \in W$ let $\theta_W: \Gamma(A,W) \to D_y$ by $\theta_W([f]_W) = f(y)$. This makes sense because $f \in K_A(W)$ implies $y \in W \subseteq \text{dom } f$. Now if $U, W \in \mathcal{Y}(A)$ with $x \in U \subseteq W$ then for each $f \in K_A(W)$ we have $(\theta_U \circ \rho_{U,W})([f]_W) = \theta_U([f]_U) = f(y) = \theta_W([f]_W)$. Because $\Gamma$-stalk-$x$ is the direct limit there exists $\zeta: \Gamma$-stalk-$x \to D_y$ such that for all $U$, $\theta_U = \zeta \circ \sigma^x_U$. Now if $a \in A$ then $y(a) = a^*(y) = \theta_x \mathcal{J}(A)([a^*] \mathcal{J}(A)) = \zeta(\sigma^x \mathcal{J}(A)([a^*] \mathcal{J}(A))) \in \text{Im } \zeta$ and hence $\text{Im } y \subseteq \text{Im } \zeta$. So if $D'$ is a division ring such that $\text{Im } \zeta \subseteq D' \subseteq D_y$ then since $y \in \mathcal{J}(A)$ we have $D' = D_y$. Therefore there is a $z \in \mathcal{J}(\Gamma$-stalk-$x$) which is equivalent to $\zeta$; that is, there is an isomorphism $\psi$ of $D_y$ onto $D_z$ such that $\psi \circ \zeta = z$. We claim $y = \mathcal{J}(\delta^x)(z)$.

For any $a \in A$ we have $\psi(y(a)) = \psi(\zeta(\sigma^x \mathcal{J}(A)([a^*] \mathcal{J}(A))))$

$= z(\sigma^x \mathcal{J}(A)(y \mathcal{J}(A)(a))) = z(\delta^x(a))$. So the square commutes and by 17, $y = \mathcal{J}(\delta^x)(z)$.

91 THEOREM: If $A$ is a ring and $x \in \mathcal{J}(A)$ then $\mathcal{J}(\Gamma$-stalk-$x$) is homeomorphic to $\bigcap_{x \in U} \mathcal{Y}(A)$. 
PROOF: Since \( \bigcap_{x \in U} f(A) \) is the \( \text{Im} \, \delta^X \) lemmas 69 and 68 imply that it is enough to show that if \( a \in \Gamma \)-stalk-\( x \) then there is an \( f \in f(A) \) such that \( \mathcal{O} (\delta^X) (f) = a^* \).

Suppose \( a \in \Gamma \)-stalk-\( x \) then there is a \( U \in f(A) \) and an \( f \in K_A(U) \) such that \( a = c^X_U([f]_U) \). Now for this \( f \) we have

\[
\mathcal{O} (\delta^X) (f) = \mathcal{O} (\sigma^X_U \circ \gamma_U) (f) = (\mathcal{O} (\sigma^X_U) \circ \mathcal{O} (\gamma_U)) (f) = \mathcal{O} (\sigma^X_U) ([f]_U) = (\sigma^X_U([f]_U))^* = a^* .
\]
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