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Brauer-Picard groups and pointed braided tensor categories

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BRAUER-PICARD GROUPS AND POINTED BRAIDED TENSOR CATEGORIES

BY

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ABSTRACT

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Tensor categories are ubiquitous in areas of mathematics involving algebraic structures. They appear, also, in other fields, such as mathematical physics (conformal field theory) and theoretical computer science (quantum computation). The study of tensor categories is, thus, a useful undertaking.

Two classes of tensor categories arise naturally in this study. One consists of group-graded extensions and another of pointed tensor categories. Understanding the former involves knowledge of the Brauer-Picard group of a tensor category, while results about pointed Hopf algebras provide insights into the structure of the latter.

This work consists of two main parts. In the first one we compute the Brauer-Picard group of a class of symmetric non-semisimple finite tensor categories by studying a canonical action on a vector space. In the second one we use results from the theory of Hopf algebras to prove an equivalence between the groupoid of pointed braided finite tensor categories admitting a fiber functor and a groupoid of metric quadruples.

CHAPTER 1

INTRODUCTION

The theory of tensor categories has become an important field of mathematics. It has connections with other mathematical fields (representation theory, Hopf algebras, subfactor theory, low-dimensional topology), as well as with other areas of science, namely, mathematical physics (conformal field theory, quantum statistics) and theoretical computer science (quantum computation).

It is desirable, then, to have a good understanding of tensor categories. More precisely, we want to have classification results for these objects based on their properties.

A class of tensor categories that arises naturally is formed by those that are group-graded. If $\mathcal{C}$ is such a category, with grading $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$, and if $\mathcal{C}_e$ is the component associated to the identity element e of G, we say that $\mathcal{C}$ is a G-extension of $\mathcal{C}_e$. Such categories were studied in [ENO10]. It was shown there that the G-extensions of $\mathcal{D}$ are classified by group homomorphisms from G into the Brauer-Picard group of $\mathcal{D}$, $\mathsf{BrPic}(\mathcal{D})$, and additional cohomological data.

Thus, in order to study group extensions of tensor categories, we need to understand Brauer-Picard groups. This turns out not to be a trivial task. No general descriptions of Brauer-Picard groups are known, except in a very few cases. For example, if G is a finite abelian group then it was shown in [ENO10] that $\mathsf{BrPic}(\mathsf{Rep}\, G) \cong O(G \oplus \widehat{G})$, the group of orthogonal automorphisms of $G \oplus \widehat{G}$. The first computations of $\mathsf{BrPic}(\mathsf{Rep}\, G)$, for non-abelian G, were obtained in [NR14] and, furthermore, in [MN16].

Every tensor category has a pointed subcategory, i.e. one whose simple objects are invertible with respect to the tensor product. Pointed braided tensor categories are fairly well-understood. One of the earliest classification results in the theory of tensor categories was given by A. Joyal and R. Street in [JS93], where it was shown that the category of pointed braided fusion categories is equivalent to the category of pre-metric groups. Every pointed braided finite tensor category admitting a fiber functor is equivalent to the category of co-representations of a finite dimensional pointed Hopf algebras with abelian coradical, and the latter are almost completely known.

This thesis contributes to the study of tensor categories by describing the Brauer-Picard group of a class of symmetric non-semisimple finite tensor categories and by giving a classification result for pointed braided finite tensor categories admitting a fiber functor.

The tools used come from the theory of Hopf algebras. These objects are intimately connected with tensor categories. The representation category $\mathsf{Rep}\, H$ of a Hopf algebra H is a tensor category, and, by Tannaka-Krein reconstruction theory, every tensor category admitting a fiber functor is the representation category of a Hopf algebra.

Hopf algebras appear in a variety of mathematical contexts. Due to their ubiquity they are the subject of intense study by researchers. One of the main research efforts is concentrated on the classification of finite dimensional Hopf algebras. The most impressive and effective program was developed by N. Andruskiewitsch and H.-J. Schneider for classifying finite dimensional pointed Hopf algebras [AS02]. Their method, for example, was used to classify all the liftings of a quantum linear space $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$, a certain type of Hopf algebra obtained from an abelian group Γ and a Γ-graded Γ-module V [AS98].

The first part of this thesis deals with the computation of the Brauer-Picard group of the representation categories of Nichols Hopf algebras, $E(n)$, $n = 1, 2, ...$ [BN15]. These algebras are quantum linear spaces obtained from the group of order two and the exterior algebra of a vector space. We show in Theorem 3.8.6 that

$$\mathsf{BrPic}(\mathsf{Rep}\, E(n)) \cong \mathsf{PSp}_n(\mathbf{k}) \times \mathbb{Z}/2\mathbb{Z},$$

where $\mathsf{PSp}_n(\mathbf{k})$ is the projective symplectic group of degree $2n$.

This result gives the first description of the Brauer-Picard group in the case of a non-semisimple tensor category and also solves a long-standing problem in the theory of Hopf algebras. It turns out that the Brauer-Picard group of $\mathsf{Rep}\, H$ coincides with the full Brauer group of H, $\mathsf{BQ}(\mathbf{k}, H)$, as defined in [CvOZ97]. The group $\mathsf{BQ}(\mathbf{k}, H_4)$, where H_4 is Sweedler's Hopf algebra, the smallest non-commutative, non-cocommutative Hopf algebra, was studied in [vOZ01] and [CC11], but its description eluded experts. Since $H_4 = E(1)$, it follows from our work that

$$\mathsf{BQ}(\mathbf{k}, H_4) \cong \mathsf{SL}_2(\mathbf{k}) \times \mathbb{Z}/2\mathbb{Z}.$$

The second part of the thesis deals with a classification result regarding pointed braided finite tensor categories admitting a fiber functor [BN17]. By Tannaka-Krein reconstruction theory these categories can be described as co-representation categories of finite dimensional pointed co-quasitriangular Hopf algebras. Using results from the theory of Hopf algebras, we show that these algebras are deformations of quantum linear spaces $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. This allows us to describe, in Theorem 4.7.3, the aforementioned categories in terms of metric quadruples (Γ, q, V, r), consisting of a finite abelian group Γ, a quadratic form q on Γ, a Γ-module V and an alternating bilinear map $r : V \times V \to \mathbf{k}$ satisfying certain conditions.

My work can be extended in several ways:

1. One can try to generalize the computation of the Brauer-Picard group of $\mathsf{Rep}\, E(n)$ to the case of quantum linear spaces $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. For this, one should use the identification of the Brauer-Picard group of $\mathsf{Rep}\, H$ with the group of braided autoequivalences of the center of $\mathsf{Rep}\, H$ and study the action of the latter on a categorical Lagrangian Grassmanian or as symmetries of the maximal pointed subcategory of the center. For the latter approach, the description obtained in [BN17], is likely to prove useful.

2. One can try to extend the result of [BN17] to the case of pointed braided finite tensor categories not necessarily having a fiber functor. Since such categories are equivalent to representation categories of quasi-Hopf algebras, one approach could start by investigating how much of the theory of pointed Hopf algebras extends to the 'quasi' case.

3. Another direction of research is to use the description of the Brauer-Picard group of $\mathsf{Rep}\, E(n)$ to study various G-extensions of $\mathsf{Rep}\, E(n)$. These extensions are described by group homomorphisms from $G \to \mathsf{BrPic}(\mathsf{Rep}\, E(n))$ and by certain cohomological data [ENO10]. The easiest case to consider would be $G = \mathbb{Z}/2\mathbb{Z}$.

CHAPTER 2

PRELIMINARIES

As the title says, this chapter consists of preliminary material. None of this material is new and it can be found in the literature. We mention only those concepts and results which are needed for understanding the rest of the work.

A major theme that runs through the chapter is the use of concepts from Hopf algebra theory to illustrate notions from the theory of tensor categories. This is not coincidental as the tensor categories that will appear in this work are representation or corepresentation categories of Hopf algebras.

We point out that both theories are much richer than they might seem from this short exposition. It is where they intersect that fruitful interaction happens, as the next two chapters will show.

2.1 Bialgebras and Hopf algebras

In this section we recall basic notions and results from Hopf algebra theory. For more details the reader can consult [Mont93], [DNR01] or [Rad12].

Throughout, $\mathbf{k}$ is an arbitrary field and unadorned $\otimes$ means $\otimes_{\mathbf{k}}$.

We begin by recasting the definition of a $\mathbf{k}$-algebra and morphism between algebras in a diagrammatic way. The advantages of this will be made clear shortly.

Definition 2.1.1. (1) A $\mathbf{k}$-*algebra* is a triple (A, m, u), consisting of a $\mathbf{k}$-vector space A and two linear maps $m : A \otimes A \to A$ and $u : \mathbf{k} \to A$, called *multiplication* and *unit*, respectively, such that the following diagrams are commutative:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes \mathrm{id}_A"'] & & A \otimes (A \otimes A) \arrow[d, "\mathrm{id}_A \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes A \arrow[dd, "m"] & \\
\mathbf{k} \otimes A \arrow[ur, "u \otimes \mathrm{id}_A"] \arrow[dr, "l_A"'] & & A \otimes \mathbf{k} \arrow[ul, "\mathrm{id}_A \otimes u"'] \arrow[dl, "r_A"] \\
& A &
\end{tikzcd}$$

(1) (2)

where $a_{A,A,A}$, l_A and r_A are the obvious maps.

(2) An *algebra map* (or a *morphism*) between two $\mathbf{k}$-algebras (A, m_A, u_A) and (B, m_B, u_B) is a linear map $f : A \to B$ making the following two diagrams commutative:

$$\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "m_A"'] & B \otimes B \arrow[d, "m_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
& \mathbf{k} \arrow[ul, "u_A"] \arrow[ur, "u_B"'] &
\end{tikzcd}$$

Remark 2.1.2. Commutativity of diagram (1) is called the *associativity condition*, and commutativity of diagram (2) is called the *unit condition*.

Remark 2.1.3. The advantage of this definition is twofold. First, it suggests that we can make this definition in any category for which there exists a notion of a "tensor product", a "unit object", and "associativity" and "left and right unit maps". We will make this precise in Section 2.8. Secondly, we can "dualize" it, that is, consider maps and diagrams with reversed arrows. What we obtain, when we do this, are the notions of "coalgebra" and "morphism" between coalgebras.

Definition 2.1.4. (1) A $\mathbf{k}$-*coalgebra* is a triple (C, Δ, ε), consisting of a $\mathbf{k}$-vector space C and two linear maps, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbf{k}$, called *comultiplication* and *counit*, respectively, such that the following diagrams are commutative:

$$\begin{tikzcd}[column sep=small]
(C \otimes C) \otimes C & & C \otimes (C \otimes C) \arrow[ll, "a^{-1}_{C,C,C}"'] \\
C \otimes C \arrow[u, "\Delta \otimes \mathrm{id}_C"] & & C \otimes C \arrow[u, "\mathrm{id}_C \otimes \Delta"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& C \otimes C \arrow[dl, "\varepsilon \otimes \mathrm{id}_C"'] \arrow[dr, "\mathrm{id} \otimes \varepsilon"] & \\
\mathbf{k} \otimes C & & C \otimes \mathbf{k} \\
& C \arrow[ul, "l_C^{-1}"] \arrow[ur, "r_C^{-1}"'] \arrow[uu, "\Delta"] &
\end{tikzcd}$$

(3) (4)

(2) A *coalgebra map* (or a *morphism*) between two $\mathbf{k}$-coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ is a linear map $f : C \to D$ making the following two diagrams commutative:

$$\begin{tikzcd}
C \otimes C \arrow[r, "f \otimes f"] & D \otimes D \\
C \arrow[u, "\Delta_C"] \arrow[r, "f"'] & D \arrow[u, "\Delta_D"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\
& \mathbf{k} &
\end{tikzcd}$$

Remark 2.1.5. Commutativity of diagram (3) is called the *coassociativity condition*, and commutativity of diagram (4), the *counit condition*.

Let us give some examples of these objects.

Example 2.1.6. Let G be a finite group and let $\mathbf{k}[G]$ be the $\mathbf{k}$-vector space with basis $\{g\}_{g \in G}$ consisting of the elements of G. Then $\mathbf{k}[G]$ is a $\mathbf{k}$-algebra with multiplication

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) = \sum_{g,h \in G} a_g b_h gh = \sum_{l \in G} \left(\sum_{gh=l} a_g b_h \right) l$$

and unit element 1_G. This is called the *group algebra* of G over the field $\mathbf{k}$.

There is also a $\mathbf{k}$-coalgebra structure on $\mathbf{k}[G]$. It is given by:

$$\begin{aligned}
\Delta &: \mathbf{k}[G] \to \mathbf{k}[G] \otimes \mathbf{k}[G], & \Delta(g) &= g \otimes g, \\
\varepsilon &: \mathbf{k}[G] \to \mathbf{k}, & \varepsilon(g) &= 1,
\end{aligned}$$

for all $g \in G$. With this structure, $\mathbf{k}[G]$ is called the *group coalgebra* of G over $\mathbf{k}$.

Example 2.1.7. Another example associated to a group G is the following. Let $\mathbf{k}^G$ be the $\mathbf{k}$-vector space of $\mathbf{k}$-valued functions defined on G. Then $\mathbf{k}^G$ is an algebra with pointwise multiplication:

$$(pq)(g) = p(g)q(g), \quad p, q \in \mathbf{k}^G, g \in G.$$

The unit element is the function that sends all elements of G to $1_{\mathbf{k}}$. This algebra is called the *algebra of functions on* G.

If G is finite then $\mathbf{k}^G$ admits, also, a $\mathbf{k}$-coalgebra structure. The comultiplication and counit on the basis elements p_g, $g \in G$, where $p_g(h) = \delta_{g,h}$, for all $g, h \in G$, are given by:

$$\begin{aligned} \Delta : \mathbf{k}^G \to \mathbf{k}^G \otimes \mathbf{k}^G, \qquad & \Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \\ \varepsilon : \mathbf{k}^G \to \mathbf{k}, \qquad & \varepsilon(p_g) = \delta_{g,1_G}, \end{aligned}$$

where $\delta_{g,h}$ is Kronecker's symbol ($\delta_{g,h} = 1$ if $g = h$, and 0, otherwise). With this structure, $\mathbf{k}^G$ is called the *coalgebra of functions on* G.

Example 2.1.8. Let $n \geq 1$ be a positive integer and let $M_n(\mathbf{k})$ be the $\mathbf{k}$-vector space of $n \times n$ matrices with entries in $\mathbf{k}$. This is an algebra with usual matrix multiplication.

There is a $\mathbf{k}$-coalgebra structure on $M_n(\mathbf{k})$, as well. The comultiplication and counit on the basis elements e_{ij}, $1 \leq i, j \leq n$, where e_{ij} is the unit matrix with 1 in the (i, j)-th entry and 0 elsewhere, are given by

$$\begin{aligned} \Delta : M_n(\mathbf{k}) \to M_n(\mathbf{k}) \otimes M_n(\mathbf{k}), \qquad & \Delta(e_{ij}) = \sum_{l=1}^{n} e_{il} \otimes e_{lj}, \\ \varepsilon : M_n(\mathbf{k}) \to \mathbf{k}, \qquad & \varepsilon(e_{ij}) = \delta_{ij}, \end{aligned}$$

where δ_{ij} is Kronecker's symbol. With this structure, $M_n(\mathbf{k})$ is called the $n \times n$ *matrix coalgebra*.

There is an important notation associated with coalgebras, which simplifies computations greatly. It is called *Sweedler's notation* or *the sigma notation* and is defined as follows. If (C, Δ, ε) is a coalgebra and $c \in C$, then we write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)},$$

where the subscripts (1) and (2) are symbolic.

Coassociativity of Δ allows us to extend this notation to the case when there are more than two tensorands. We have $(\Delta \otimes \mathsf{id}_C)\Delta(c) = (\mathsf{id}_C \otimes \Delta)\Delta(c)$, so

$$\sum (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}.$$

We denote this element by $\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$.

In general, if we define Δ_n to be the composition of the following n maps:

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\Delta \otimes \mathsf{id}_C} C \otimes C \otimes C \to \cdots \to C^{\otimes(n)} \xrightarrow{\Delta \otimes \mathsf{id}_{C^{\otimes(n-1)}}} C^{\otimes(n+1)},$$

where $C^{\otimes i}$ is the tensor product of C with itself, i times, then one can show that

$$\Delta_n = (\mathsf{id}_{C^{\otimes i}} \otimes \Delta \otimes \mathsf{id}_{C^{\otimes(n-1-i)}}) \circ \Delta_{n-1},$$

for every $i = 0, \ldots, n-1$. This is called the *generalized coassociativity law*. The element $\Delta_n(c)$ is denoted by $\sum c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n+1)}$.

We saw that some vector spaces have both an algebra structure and a coalgebra structure. In general, these are independent of each other. When the two structures are compatible, in the sense that the comultiplication and the counit maps are algebra maps, we say that the object in question is a bialgebra.

Definition 2.1.9. A *bialgebra* over a field $\mathbf{k}$ is a $\mathbf{k}$-vector space B, together with an algebra structure (B, m, u) and a coalgebra structure (B, Δ, ε), such that Δ and ε are algebra maps.

Remark 2.1.10. It seems that in the definition m and u are ignored and it would not be unreasonable to require that m and u be coalgebra maps, as well. It turns out that there is no need for this, as it can be shown that Δ and ε are algebra maps if and only if m and u are coalgebra maps [DNR01, Proposition 4.1.1].

Example 2.1.11. The group algebra and coalgebra $\mathbf{k}[G]$ is a bialgebra. Likewise, the algebra and coalgebra of functions $\mathbf{k}^G$, on a finite group G, is a bialgebra. The algebra of matrices $M_n(\mathbf{k})$, with the matrix coalgebra structure, is a bialgebra if and only if $n = 1$. To see this, note that a non-zero linear map $M_n(\mathbf{k}) \to \mathbf{k}$ must be injective, and this happens only when $n = 1$.

As with algebras and coalgebras, there is a notion of a morphism between two bialgebras. The requirements are the ones that one would expect.

Definition 2.1.12. Let B and B' be $\mathbf{k}$-bialgebras. A *bialgebra map* (or a *morphism of bialgebras*) from B to B', is a linear map $f : B \to B'$ which is both an algebra map and a coalgebra map.

Hopf algebras are bialgebras admitting a linear version of the "inverse" function from group theory: $G \to G$, $g \mapsto g^{-1}$. More precisely, we have the following:

Definition 2.1.13. A *Hopf algebra* is a bialgebra H for which there exists a linear map $S : H \to H$, called an *antipode*, such that

$$S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \varepsilon(h)1_H,$$

for all $h \in H$.

Remark 2.1.14. It can be shown that, if an antipode exists, then it is unique. Moreover, the antipode is an antimorphism of algebras and an antimorphism of coalgebras, that is, if $S : H \to H$ is an antipode of H, then the following relations hold:

$$\begin{aligned}
S(gh) &= S(h)S(g), \\
S(1) &= 1, \\
\Delta\big(S(h)\big) &= S(h_{(2)}) \otimes S(h_{(1)}), \\
\varepsilon S(h) &= \varepsilon(h),
\end{aligned}$$

for all $g, h \in H$.

Remark 2.1.15. It can also be shown that if H is a finite dimensional Hopf algebra then the antipode S is invertible. For a proof see [DNR01, Theorem 7.1.7] or [Rad12, Theorem 7.1.14].

Example 2.1.16. The group bialgebra $\mathbf{k}[G]$ is a Hopf algebra. Its antipode is the map

$$S : \mathbf{k}[G] \to \mathbf{k}[G], \quad S(g) = g^{-1}, \quad g \in G.$$

We call $\mathbf{k}[G]$ the *group Hopf algebra of* G.

Example 2.1.17. Let G be a finite group. The bialgebra of functions $\mathbf{k}^G$ is a Hopf algebra. The antipode is

$$S : \mathbf{k}^G \to \mathbf{k}^G, \quad S(p_g) = p_{g^{-1}}, \quad g \in G.$$

We call $\mathbf{k}^G$ the *Hopf algebra of functions on* G.

Remark 2.1.18. If $f : K \to H$ is a bialgebra map between two Hopf algebras K and H, then it is not hard to check that $S_H f = f S_K$. Bialgebra maps between Hopf algebras are called *Hopf algebra maps*, or *morphisms of Hopf algebras*.

We present next some common constructions with algebras, coalgebras, bialgebras and Hopf algebras, and provide more examples of Hopf algebras.

Definition 2.1.19. Let $\mathbf{k}$ be a field and, for any two $\mathbf{k}$-vector spaces U and V, let $\tau_{U,V} : U \otimes V \to V \otimes U$ be the *transposition* (or the *flip*) map: $\tau_{U,V}(u \otimes v) = v \otimes u$, for all $u \in U$ and $v \in V$.

(1) The *opposite algebra* of an algebra (A, m, u) is $A^{\text{op}} = (A, m^{\text{op}} = m\tau_{A,A}, u)$. If $A = A^{\text{op}}$ then A is said to be *commutative*.

(2) The *co-opposite coalgebra* of a coalgebra (C, Δ, ε) is $C^{\text{cop}} = (C, \Delta^{\text{cop}} = \tau_{C,C}\Delta, \varepsilon)$. If $C = C^{\text{cop}}$, then C is said to be *co-commutative*.

(3) The *opposite bialgebra* of a bialgebra $(B, m, u, \Delta, \varepsilon)$ is $B^{\text{op}} = (B, m^{\text{op}}, u, \Delta, \varepsilon)$, and the *co-opposite bialgebra* of $(B, m, u, \Delta, \varepsilon)$ is $B^{\text{cop}} = (B, m, u, \Delta^{\text{cop}}, \varepsilon)$. If $B = B^{\text{op}}$ then B is said to be *commutative*, and, if $B = B^{\text{cop}}$ then B is said to be *co-commutative*.

Remark 2.1.20. If H is a Hopf algebra then H^{op} is a Hopf algebra if and only if the antipode of H is invertible. The same is true for H^{cop}. If the antipode of H is invertible then $S_{H^{\text{op}}} = S_{H^{\text{cop}}} = S_H^{-1}$.

Definition 2.1.21. Let H be a Hopf algebra with invertible antipode. Then H^{op} is called the *opposite Hopf algebra* of H, and H^{cop} is called the *co-opposite Hopf algebra* of H. If $H = H^{\text{op}}$ then H is said to be *commutative*, and, if $H = H^{\text{cop}}$ then H is said to be *co-commutative*.

Example 2.1.22. The group Hopf algebra $\mathbf{k}[G]$ is co-commutative. It is commutative if and only if G is an abelian group. The Hopf algebra of functions $\mathbf{k}^G$ on a finite group G is commutative. It is co-commutative if and only if G is an abelian group.

Example 2.1.23. The smallest non-commutative, non-co-commutative Hopf algebra is *Sweedler's Hopf algebra* H_4. It was introduced by M. Sweedler and it is described as follows. Let $\mathbf{k}$ be a field of characteristic $\neq 2$. As an algebra, H_4 is the quotient of the free algebra $\mathbf{k}\{g, x\}$ on two generators, g and x, by the two-sided ideal generated by $g^2 - 1$, x^2 and $gx + xg$. Thus,

$$H_4 = \mathbf{k}\{g, x\}/(g^2 - 1, x^2, gx + xg).$$

The comultiplication, the counit and the antipode of H_4 are:

$$\begin{aligned} \Delta(g) &= g \otimes g, & \varepsilon(g) &= 1, & S(g) &= g^{-1}, \\ \Delta(x) &= 1 \otimes x + x \otimes g, & \varepsilon(x) &= 0, & S(x) &= g^{-1}x. \end{aligned}$$

A $\mathbf{k}$-basis of H_4 is $\{1, g, x, gx\}$. The antipode of H_4 has order 4. It can be shown that, up to isomorphism, H_4 is the unique non-commutative, non-co-commutative Hopf algebra of dimension 4. Thus, $H_4^{\text{op}} \cong H_4^{\text{cop}} \cong H_4$.

Example 2.1.24. In [T71] E.J. Taft introduced a family of non-commutative, non-co-commutative Hopf algebras, containing Sweedler's Hopf algebra as a particular case. The family is defined as follows. Let $N \geq 2$ and $m \geq 1$ be two integers. Let $\mathbf{k}$ be a field containing a primitive N-th root of unity ξ. Then

$$T_{N,m} = \mathbf{k}\{g_1, \ldots, g_m, x\}/(g_i^N - 1, x^N, g_i g_j - g_j g_i, x g_i - \xi g_i x)$$

is a Hopf algebra with comultiplication, counit and antipode:

$$\begin{aligned}
\Delta(g_i) &= g_i \otimes g_i, & \varepsilon(g_i) &= 1, & S(g_i) &= g_i^{-1}, \\
\Delta(x) &= 1 \otimes x + x \otimes g_1, & \varepsilon(x) &= 0, & S(x) &= -\xi^{-1} g_1^{-1} x,
\end{aligned}$$

for all $i = 1, \ldots, n$. $H_{N,m}$ is a non-commutative, non-co-commutative Hopf algebra of dimension N^{m+1}, a $\mathbf{k}$-basis being $\{g_1^{i_1} \cdots g_m^{i_m} x^i \mid 0 \leq i_1, \ldots, i_m, i \leq N-1\}$. The antipode of $T_{N,m}$ has order $2N$.

$T_{N,m}$ is called the *general Taft algebra*, while $T_{N,1}$ is called the *two-generator Taft algebra*. Note that $T_{2,1}$ is Sweedler's Hopf algebra.

Example 2.1.25. Another family of non-commutative, non-co-commutative Hopf algebras, containing Sweedler's Hopf algebra as a particular case, is formed by *Nichols Hopf algebras*. These were introduced by W.D. Nichols in [Nich78] and they are defined as follows. Let $n \geq 1$ be a positive integer. Then

$$E(n) = \mathbf{k}\{c, x_1, \ldots, x_n\}/(c^2 - 1, x_i^2, c x_i + x_i c, x_i x_j + x_j x_i)$$

is a Hopf algebra with comultiplication, counit and antipode:

$$\begin{aligned}
\Delta(c) &= c \otimes c, & \varepsilon(c) &= 1, & S(c) &= c^{-1}, \\
\Delta(x_i) &= 1 \otimes x_i + x_i \otimes c, & \varepsilon(x_i) &= 0, & S(x_i) &= c^{-1} x_i,
\end{aligned}$$

for all $i = 1, \ldots, n$. $E(n)$ is a non-commutative, non-co-commutative Hopf algebra of dimension 2^{n+1}. A $\mathbf{k}$-basis is $\{c^i x_P \mid i = 0, 1,\ P \subseteq \{1, \ldots, n\}\}$, where, for a subset $P = \{i_1, i_2, \ldots, i_s\} \subseteq$

$\{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_s$, we denote $x_P = x_{i_1}x_{i_2}\cdots x_{i_s}$, and $x_\emptyset = 1$. We note that $E(1)$ is Sweedler's Hopf algebra.

Note that, if U and V are two $\mathbf{k}$-vector spaces and $U^* = \mathsf{Hom}(U, \mathbf{k})$ and $V^* = \mathsf{Hom}(V, \mathbf{k})$ are their duals, then the map

$$\rho_{U,V} : U^* \otimes V^* \to (U \otimes V)^*, \quad (p \otimes q)(u \otimes v) = p(u)q(v),$$

for all $u \in U$, $v \in V$, $p \in U^*$ and $q \in V^*$, is injective. If U or V is finite dimensional, then $\rho_{U,V}$ is an isomorphism.

Recall, also, that, if $f : U \to V$ is a linear map, then its *transpose* is $f^* : V^* \to U^*$, $f^*(p) = pf$, for all $p \in V^*$.

Using the map ρ, we can transpose the comultiplication and counit of a coalgebra C to a multiplication and unit on C^*. Similarly, we can transpose the multiplication and unit of an algebra A to a comultiplication and counit on A^*. For the latter, we need that A be finite dimensional, so that ρ is invertible.

Definition 2.1.26. Let $\mathbf{k}$ be a field.

(1) The *dual algebra* of a coalgebra (C, Δ, ε) is $(C^*, \Delta^*\rho_{C,C}, \varepsilon^*)$. Explicitly, the multiplication of C^*, called the *convolution product*, is

$$(p * q)(c) = \sum p(c_{(1)})q(c_{(2)}), \qquad c \in C,\ p, q \in C^*.$$

The unit element of C^* is ε.

(2) The *dual coalgebra* of a finite dimensional algebra (A, m, u) is $(A^*, \rho_{A,A}^{-1}m^*, u^*)$. Explicitly, the comultiplication of A^* is

$$\Delta(p)(a \otimes b) = \sum p_{(1)}(a)p_{(2)}(b), \qquad a, b \in A,\ p \in A^*.$$

The counit of A^* is given by $\varepsilon(p) = p(1)$, for all $p \in A^*$.

Remark 2.1.27. If B is a finite dimensional bialgebra, then the dual algebra and coalgebra B^* is a bialgebra. It is called the *dual bialgebra* of B.

Remark 2.1.28. If H is a finite dimensional Hopf algebra, then the dual bialgebra H^* is a Hopf algebra. Its antipode is $S_{H^*} = S_H^*$. H^* is called the *dual Hopf algebra* of H. If $H^* \cong H$, then H is said to be *self-dual*.

Example 2.1.29. If G is a finite group then $(\mathbf{k}[G])^* \cong \mathbf{k}^G$ and $(\mathbf{k}^G)^* \cong \mathbf{k}[G]$.

Example 2.1.30. The Nichols Hopf algebra $E(n)$ is self-dual. An isomorphism $E(n) \to E(n)^*$ maps

$$c \mapsto 1^* - c^* \quad \text{and} \quad x_i \mapsto x_i^* + (cx_i)^*,$$

for all $i = 1, \dots, n$, where $\{(c^i x_P)^*\}_{i,P}$ is the dual basis of $\{c^i x_P\}_{i,P}$.

2.2 The language of categories

In this section and the next we recall basic definitions and results from category theory. For more details see [P70] or [MacL98].

Definition 2.2.1. A category $\mathcal{C}$ consists of

1. A class $\mathsf{Ob}(\mathcal{C})$ of *objects*, which we usually denote by capital letters: $X, Y, Z, ... \in \mathsf{Ob}(\mathcal{C})$.
2. For each ordered pair of objects (X, Y), a set $\mathsf{Hom}_{\mathcal{C}}(X, Y)$, whose elements are called *morphisms* with *domain* X and *codomain* Y. These sets are called *hom-sets* and morphisms are usually denoted by small letters: $f, g, h, ... \in \mathsf{Hom}_{\mathcal{C}}(X, Y)$.
3. For each ordered triple (X, Y, Z), a map

$$\mathsf{Hom}_{\mathcal{C}}(X, Y) \times \mathsf{Hom}_{\mathcal{C}}(Y, Z) \to \mathsf{Hom}_{\mathcal{C}}(X, Z), \quad (f, g) \mapsto gf$$

called *composition*.

These are required to satisfy the following conditions:

(C1) Disjointedness of hom-sets: If $(X,Y) \neq (Z,W)$ then $\mathsf{Hom}_{\mathcal{C}}(X,Y)$ and $\mathsf{Hom}_{\mathcal{C}}(Z,W)$ are disjoint sets.

(C2) *Associativity* of composition: If $f \in \mathsf{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathsf{Hom}_{\mathcal{C}}(Y,Z)$, and $h \in \mathsf{Hom}_{\mathcal{C}}(Z,W)$, then $(hg)f = h(gf)$.

(C3) Existence of *identity elements*: For every object X there is an element $\mathsf{id}_X \in \mathsf{Hom}_{\mathcal{C}}(X,X)$, called the *identity morphism* of X, such that $f\,\mathsf{id}_X = f$, for every $f \in \mathsf{Hom}_{\mathcal{C}}(X,Y)$, and $\mathsf{id}_X\, g = g$, for every $g \in \mathsf{Hom}_{\mathcal{C}}(Y,X)$.

Example 2.2.2. The category of $\mathbf{k}$-vector spaces, $\mathbf{k}$-**Vec**, is the category whose objects are vector spaces over $\mathbf{k}$ and whose morphisms are $\mathbf{k}$-linear maps.

Example 2.2.3. Let G be a group. Then the category $\mathbf{k}$-$\mathbf{Vec}_G$ of *G-graded vector spaces* over $\mathbf{k}$ is the category with:

- Objects: Pairs $(V, \{V_g\}_{g\in G})$, where V is a $\mathbf{k}$-vector space and V_g, $g \in G$, are subspaces of V such that $V = \bigoplus_{g\in G} V_g$. We usually write V instead of $(V, \{V_g\}_{g\in G})$.

- Morphisms: $f : (U, \{U_g\}_{g\in G}) \to (V, \{V_g\}_{g\in G})$ is a *morphism of G-graded vector spaces* if $f : U \to V$ is a linear map such that $f(U_g) \subseteq V_g$, for all $g \in G$.

The fact that G is a group is irrelevant for this example. Later on, we will see that the group structure of G turns $\mathbf{k}$-$\mathbf{Vec}_G$ into a "rigid monoidal" category.

Example 2.2.4. If (A, m, u) is a $\mathbf{k}$-algebra then the category $\mathsf{Rep}\, A$ of finite dimensional *representations of A* (or (*left*) *A-modules*), is the category with

- Objects: Pairs (V, ρ), where V is a finite-dimensional $\mathbf{k}$-vector space and $\rho : A \otimes V \to V$ is a linear map, called the *action of A on V*, such that the following two diagrams are commutative:

$$
\begin{tikzcd}
(A \otimes A) \otimes V \arrow[rr, "a_{A,A,V}"] \arrow[d, "m \otimes \mathrm{id}_V"'] & & A \otimes (A \otimes V) \arrow[d, "\mathrm{id}_A \otimes \rho"] \\
A \otimes V \arrow[dr, "\rho"'] & & A \otimes V \arrow[dl, "\rho"] \\
& V &
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[r, "u \otimes \mathrm{id}_V"] & A \otimes V \arrow[d, "\rho"] \\
V \arrow[u, "l_v^{-1}"] \arrow[r, "\mathrm{id}_V"] & V
\end{tikzcd}
$$

where $a_{A,A,V}$ and l_V are the obvious maps. We usually write V instead of (V, ρ).

- Morphisms: $f : (U, \rho_U) \to (V, \rho_V)$ is a *morphism of A-modules*, or an *A-linear map*, if $f : U \to V$ is a linear map such that the following diagram is commutative:

$$
\begin{tikzcd}
A \otimes U \arrow[r, "\mathrm{id}_A \otimes f"] \arrow[d, "\rho_U"'] & A \otimes V \arrow[d, "\rho_V"] \\
U \arrow[r, "f"'] & V
\end{tikzcd}
$$

For a group algebra $\mathbf{k}[G]$ we denote $\mathsf{Rep}\, \mathbf{k}[G]$ by $\mathsf{Rep}\, G$. Note that $\mathsf{Rep}\, \mathbf{k} = \mathbf{k}$-Vec.

Example 2.2.5. Let (C, Δ, ε) be a $\mathbf{k}$-coalgebra. The category $\mathsf{Corep}\, C$ of finite-dimensional *corepresentations of C* (or (*right*) *C-comodules*) is the category with

- Objects: Pairs (V, δ), where V is a vector space and $\delta : V \to V \otimes C$ is a linear map, called the *coaction of C on V*, such that diagrams

$$
\begin{tikzcd}
& V \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\
V \otimes C \arrow[d, "\delta \otimes \mathrm{id}_C"'] & & V \otimes C \arrow[d, "\mathrm{id}_V \otimes \Delta"] \\
(V \otimes C) \otimes C \arrow[rr, "a_{V,C,C}"'] & & V \otimes (C \otimes C)
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\mathrm{id}_V"] \arrow[d, "\delta"'] & V \\
V \otimes C \arrow[r, "\mathrm{id}_V \otimes \varepsilon"] & V \otimes k \arrow[u, "r_V"']
\end{tikzcd}
$$

are commutative. We usually denote (V, δ) by V.

- Morphisms: $f : (U, \delta_U) \to (V, \delta_V)$ is a *morphism of C-comodules*, or a *C-colinear map*, if $f : U \to V$ is a linear map such that the following diagram is commutative:

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\delta_U"'] & V \arrow[d, "\delta_V"] \\
U \otimes C \arrow[r, "f\otimes \mathsf{id}_C"'] & V \otimes C
\end{tikzcd}$$

For a group coalgebra $\mathbf{k}[G]$ we denote $\mathsf{Corep}\,\mathbf{k}[G]$ by $\mathsf{Corep}\,G$. Note that $\mathsf{Corep}\,\mathbf{k} = \mathbf{k}$-Vec.

Remark 2.2.6. Sweedler's notation extends to comodules. If V is a C-comodule, with coaction δ, and $v \in V$ then $\delta(v)$ is denoted by $\sum v_{(0)} \otimes v_{(1)}$. The common element $(\mathsf{id}_V \otimes \Delta)\delta(v) = a_{V,C,C}(\delta \otimes \mathsf{id}_C)\delta(v)$ is denoted by $\sum v_{(0)} \otimes v_{(1)} \otimes v_{(2)}$, and so forth.

Definition 2.2.7. A category $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if $\mathsf{Ob}(\mathcal{D})$ is a subclass of $\mathsf{Ob}(\mathcal{C})$, $\mathsf{Hom}_{\mathcal{D}}(X,Y)$ is a subset of $\mathsf{Hom}_{\mathcal{C}}(X,Y)$, for any X, $Y \in \mathsf{Ob}(\mathcal{D})$, and the composition and identity morphisms of $\mathcal{D}$ coincide with those of $\mathcal{C}$. If $\mathsf{Hom}_{\mathcal{D}}(X,Y) = \mathsf{Hom}_{\mathcal{C}}(X,Y)$, for all X, $Y \in \mathsf{Ob}(\mathcal{D})$, then $\mathcal{D}$ is said to be a *full* subcategory of $\mathcal{C}$.

Example 2.2.8. The category $\mathbf{k}$-Vec of finite dimensional $\mathbf{k}$-vector spaces is a full subcategory of $\mathbf{k}$-**Vec**. Similary, the category $\mathbf{k}$-Vec$_G$ of finite dimensional G-graded $\mathbf{k}$-vector spaces is a full subcategory of $\mathbf{k}$-**Vec**$_G$.

Definition 2.2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$, denoted by $F : \mathcal{C} \to \mathcal{D}$, consists of:

1. An assignment $\mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D})$, $X \mapsto F(X)$.

2. For each pair of objects (X,Y), a function

$$\mathsf{Hom}_{\mathcal{C}}(X,Y) \to \mathsf{Hom}_{\mathcal{D}}\big(F(X), F(Y)\big), \quad f \mapsto F(f),$$

satisfying the following:

(F1) $F(fg) = F(f)F(g)$, for all morphisms f and g.

(F2) $F(\mathsf{id}_X) = \mathsf{id}_{F(X)}$, for every $X \in \mathsf{Ob}(\mathcal{C})$.

Example 2.2.10. An algebra map $f : B \to A$ induces a functor $Res_f : \mathsf{Rep}\,A \to \mathsf{Rep}\,B$, called the *restriction of scalars functor* and defined in the following way:

- If $V \in \mathsf{Rep}\,A$ then $Res_f(V) = V$, with B-action: $b \cdot v = f(b)v$, for all $b \in B$ and $v \in V$.

- If f is a morphism in $\mathsf{Rep}\,A$ then $F(f) = f$ is a morphism in $\mathsf{Rep}\,B$.

The unit map $u : \mathbf{k} \to A$ induces the *forgetful functor* $F_A = Res_u : \mathsf{Rep}\,A \to \mathbf{k}\text{-Vec}$. It is called forgetful because it takes an A-module (V, ρ) to its underlying vector space V, "forgetting" the action ρ.

Example 2.2.11. Similarly, a coalgebra map $f : C \to D$ induces a functor $Ext_f : \mathsf{Corep}\,C \to \mathsf{Corep}\,D$, called the *extension of scalars functor* and defined in the following way:

- If $V \in \mathsf{Corep}\,C$, with C-coaction δ, then $Ext_f(V) = V$, with D-action: $(\mathsf{id}_V \otimes f)\delta$.

- If f is a morphism in $\mathsf{Corep}\,C$ then $F(f) = f$ is a morphism in $\mathsf{Corep}\,D$.

The counit map $\varepsilon : C \to \mathbf{k}$ induces the *forgetful functor* $F_C = Ext_\varepsilon : \mathsf{Corep}\,C \to \mathbf{k}\text{-Vec}$. It takes a C-comodule (V, δ) to its underlying vector space V.

Remark 2.2.12. If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ then $GF : \mathcal{C} \to \mathcal{E}$ is the functor: $GF(X) = G(F(X))$, for all $X \in \mathsf{Ob}(\mathcal{C})$, and $GF(f) = G(F(f))$, for all morphisms f in $\mathcal{C}$. GF is called the *composition of* G *with* F. For example, if $f : C \to B$ and $g : B \to A$ are algebra maps then $Res_f\, Res_g = Res_{gf}$. If $f : C \to D$ and $g : D \to E$ are coalgebra maps then $Ext_g\, Ext_f = Ext_{gf}$.

Functors between two categories form the objects of a category. Morphisms in this category are called natural transformations.

Definition 2.2.13. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* μ from F to G, denoted by $\mu : F \to G$, is a family of morphisms $\mu = \{\mu_X : F(X) \to G(X)\}_{X \in \mathsf{Ob}(\mathcal{C})}$ such that, for every morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram is commutative:

$$\begin{tikzcd} F(X) \arrow[r, "\mu_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\mu_Y"'] & G(Y) \end{tikzcd}$$

Remark 2.2.14. If $\mathcal{C}$ and $\mathcal{D}$ are two categories, then the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations between functors is denoted by $\mathsf{Fun}(\mathcal{C},\mathcal{D})$. The composition of two natural transformations μ and ν is $\mu\nu = \{\mu_X\nu_X\}_{X\in\mathsf{Ob}\,\mathcal{C}}$. The identity morphism of a functor F is the identity natural transformation of F: $\mathsf{id}_F = \{\mathsf{id}_{F(X)}\}_{X\in\mathsf{Ob}\,\mathcal{C}}$.

The category $\mathsf{End}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C},\mathcal{C})$ is called the *category of endofunctors* of $\mathcal{C}$.

Definition 2.2.15. (1) A morphism $f : X \to Y$ in a category $\mathcal{C}$ is said to be an *isomorphism* if there exists a morphism $g : Y \to X$ such that $gf = \mathsf{id}_X$ and $fg = \mathsf{id}_Y$. Two objects, X and Y, of a category $\mathcal{C}$ are *isomorphic*, and we write this $X \cong Y$, if there exists an isomorphism $f : X \to Y$.

(2) A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF = \mathsf{id}_\mathcal{C}$ and $FG = \mathsf{id}_\mathcal{D}$. Two categories, $\mathcal{C}$ and $\mathcal{D}$, are *isomorphic*, and we write this $\mathcal{C} \cong \mathcal{D}$, if there exists an isomorphism $F : \mathcal{C} \to \mathcal{D}$.

(3) A natural transformation $\mu : F \to G$ is said to be a *natural isomorphism* if there exists a natural transformation $\nu : G \to F$ such that $\nu\mu = \mathsf{id}_F$ and $\mu\nu = \mathsf{id}_G$. Two functors, F and G, are said to be *isomorphic*, and we write this $F \cong G$, if there exists a natural isomorphism $\mu : F \to G$.

Example 2.2.16. If G is a group then $\mathsf{Corep}\,G$ and $\mathbf{k}$-Vec_G are isomorphic. To see this, note that, if (V,δ) is a corepresentation of $\mathbf{k}[G]$, then $V = \bigoplus_{g\in G} V_g$, where $V_g = \{v \in V \mid \delta(v) = v \otimes g\}$. Indeed, if $v \in V$ then there exist $g_1,\dots,g_n \in G$ and $v_1,\dots,v_n \in V$ such that $\delta(v) = \sum_i v_i \otimes g_i$. Since

$$\textstyle\sum_i (v_i \otimes g_i) \otimes g_i = a^{-1}_{V,C,C}(\mathsf{id}_V \otimes \Delta)\delta(v) = (\delta \otimes \mathsf{id}_{\mathbf{k}[G]})\delta(v) = \sum_i \delta(v_i) \otimes g_i$$

it follows that $\delta(v_i) = v_i \otimes g_i$. Moreover,

$$v = r_V(\mathsf{id}_V \otimes \varepsilon)\delta(v) = \sum_i v_i\varepsilon(g_i) = \sum_i v_i$$

so, $V = \sum_{g\in G} V_g$. To see that the sum is direct, suppose $g_1, \ldots, g_n$ are distinct elements of G and $v_i \in V_{g_i}$, $i = 1, \ldots, n$, satisfy $v_1 + \cdots + v_n = 0$. Applying δ to this relation, we obtain $\sum_i v_i \otimes g_i = 0$. It follows from this that $v_i = 0$, for all $i = 1, \ldots, n$. Thus, $V = \oplus_{g\in G} V_g$.

The association $(V, \delta) \mapsto (V, \{V_g\}_{g\in G})$ is an isomorphism from $\mathsf{Corep}\, G$ to $\mathbf{k}\text{-Vec}_G$. The inverse functor sends a G-graded vector space $V = \bigoplus_{g\in G} V_g$ to (V, δ), where $\delta(v) = v \otimes g$, for all $v \in V_g$ and $g \in G$. Both functors leave morphisms unchanged.

Example 2.2.17. If A is a finite dimensional algebra and A^* is the dual coalgebra of A, then $\mathsf{Rep}\, A$ and $\mathsf{Corep}\, A^*$ are isomorphic. An isomorphism $F : \mathsf{Rep}\, A \to \mathsf{Corep}\, A^*$ is given as follows.

Let $\{e_i\}$ be a basis of A and let $\{e_i^*\}$ be its dual basis. If $V \in \mathsf{Rep}\, A$ then $F(V) = V$, with A^*-coaction given by

$$v \mapsto \sum_i e_i v \otimes e_i^*, \qquad v \in V.$$

The inverse functor takes an A^*-comodule V and sends it to V, with A-action:

$$av = \sum v_{(1)}(a)v_{(0)}, \qquad a \in A, v \in V,$$

for all $a \in A$ and $v \in V$. Both functors leave morphisms unchanged.

Remark 2.2.18. In a sense, the condition for two categories to be isomorphic is too strong. It requires, in particular, that there be a one-one correspondence between the objects of the two categories. Since, in most cases, we are more interested in isomorphism classes of objects, a more useful notion is that of equivalence of categories. The condition for two categories to be equivalent ensures that there is a one-one correspondence between the isomorphism classes of objects of the two categories.

Definition 2.2.19. A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathsf{id}_{\mathcal{C}}$ and $FG \cong \mathsf{id}_{\mathcal{D}}$. Two categories, $\mathcal{C}$ and $\mathcal{D}$, are said to be *equivalent*, and we write this $\mathcal{C} \simeq \mathcal{D}$, if there exists an equivalence $F : \mathcal{C} \to \mathcal{D}$.

There is a useful criterion for checking that a given functor is an equivalence. To state it, we need the following definitions.

Definition 2.2.20. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. F is said to be *faithful* (respectively, *full*) if for all X and $Y \in \mathsf{Ob}(\mathcal{C})$, the maps

$$\mathsf{Hom}_{\mathcal{C}}(X, Y) \to \mathsf{Hom}_{\mathcal{D}}\big(F(X), F(Y)\big)$$

   are injective (respectively, surjective).

2. F is said to be *fully faithful* if F is both faithful and full.

3. F is said to be *essentially surjective* if for any $Y \in \mathcal{D}$ there exists $X \in \mathcal{C}$ such that $Y \cong F(X)$.

Theorem 2.2.21. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is essentially surjective and fully faithful.*

Proof. See [P70, Proposition 2.1.3] or [MacL98, Theorem IV.4.1]. □

Definition 2.2.22. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is defined by: $\mathsf{Ob}(\mathcal{C} \times \mathcal{D}) = \mathsf{Ob}(\mathcal{C}) \times \mathsf{Ob}(\mathcal{D})$ and

$$\mathsf{Hom}_{\mathcal{C}\times\mathcal{D}}((X, Y), (X', Y')) = \mathsf{Hom}_{\mathcal{C}}(X, X') \times \mathsf{Hom}_{\mathcal{D}}(Y, Y'),$$

for all X, $X' \in \mathcal{C}$ and Y, $Y' \in \mathcal{D}$, with composition induced from $\mathcal{C}$ and $\mathcal{D}$.

Definition 2.2.23. A functor from a product category of two categories into another category is called a *bifunctor*.

2.3 Finite $\mathbf{k}$-linear abelian categories

Categories of finite dimensional representations of finite dimensional $\mathbf{k}$-algebras can be described as those categories that are finite, $\mathbf{k}$-linear, and abelian. Since these requirements are met by the objects of our study, finite tensor categories, it is useful to review their definition and see their properties.

Definition 2.3.1. An *additive category* is a category $\mathcal{C}$ satisfying the following axioms:

(1) Every hom-set $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ is equipped with an abelian group structure, written additively, such that composition of morphisms is biadditive with respect to addition.

(2) $\mathcal{C}$ has a *zero object*, that is, an object 0 such that all hom-sets, $\mathsf{Hom}_{\mathcal{C}}(0, X)$ and $\mathsf{Hom}_{\mathcal{C}}(X, 0)$, $X \in \mathsf{Ob}(\mathcal{C})$, are singletons.

(3) For any objects X and Y of $\mathcal{C}$, there exists an object B and morphisms $i_X : X \to B$, $i_Y : Y \to B$, $p_X : B \to X$ and $p_Y : B \to Y$ such that $p_X i_X = \mathsf{id}_X$, $p_Y i_Y = \mathsf{id}_Y$ and $i_X p_X + i_Y p_Y = \mathsf{id}_B$.

Remark 2.3.2. A zero object is unique, up to a unique isomorphism. If 0 is a zero object of $\mathcal{C}$ then every hom-set $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ has a distinguished element, namely the composition $X \to 0 \to Y$. This morphism is called the *zero morphism* and it is denoted by $0_{X,Y}$ or, if it is clear from the context what the domain and co-domain are, by 0. If $\mathcal{C}$ is an additive category then $0_{X,Y}$ is the zero element of the additive group $\mathsf{Hom}_{\mathcal{C}}(X, Y)$.

Remark 2.3.3. The object B of condition (3) is unique, up to a unique isomorphism, with the stated properties. It is called the *direct sum* of X and Y and it is denoted by $X \oplus Y$.

Definition 2.3.4. Let $\mathbf{k}$ be a field. A $\mathbf{k}$-*linear category* is an additive category whose hom-sets are $\mathbf{k}$-vector spaces such that composition of morphisms is $\mathbf{k}$-bilinear.

Definition 2.3.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $\mathcal{C}$ and $\mathcal{D}$ are additive (respectively, $\mathbf{k}$-linear), then F is *additive* (respectively, $\mathbf{k}$-*linear*) if the associated maps

$$\mathsf{Hom}_{\mathcal{C}}(X, Y) \to \mathsf{Hom}_{\mathcal{D}}\big(F(X), F(Y)\big)$$

are morphisms of groups (respectively, $\mathbf{k}$-linear maps).

Next, we recall the definition of abelian categories. These are categories where the notion of an exact sequence can be defined. We need the concepts of a kernel and cokernel of a morphism.

Definition 2.3.6. Let $\mathcal{C}$ be a category with a zero object and let $f : X \to Y$ be a morphism in $\mathcal{C}$.

1. A *kernel* of f is pair (K, i), where $i : K \to X$ is a morphism such that $fi = 0$, and, if $i' : K' \to X$ satisfies $fi' = 0$, then there is a unique $k : K' \to K$ such that $ik = i'$.

2. A *cokernel* of f is a pair (C, p), where $p : Y \to C$ is a morphism such that $pf = 0$ and, if $p' : Y \to C'$ satisfies $p'f = 0$, then there is a unique $c : C \to C'$ such that $cp = p'$.

Remark 2.3.7. If they exist, the kernel and cokernel are unique, up to a unique isomorphism. The kernel of f is denoted by $\mathsf{Ker}(f)$ and the cokernel by $\mathsf{Coker}(f)$.

Definition 2.3.8. An *abelian category* is an additive category $\mathcal{C}$ with the property that, for every morphism $f : X \to Y$, there exists a sequence

$$K \xrightarrow{i} X \xrightarrow{u} I \xrightarrow{v} Y \xrightarrow{p} C$$

such that $f = vu$, (K, i) is a kernel of f, (I, u) is a cokernel of i, (I, v) is a kernel of p, and (C, p) is a cokernel of f. The object I is called the *image of f* and is denoted by $\mathsf{Im}(f)$.

Example 2.3.9. If A is a $\mathbf{k}$-algebra then $\mathsf{Rep}\, A$ is a $\mathbf{k}$-linear abelian category. Similarly, if C is a $\mathbf{k}$-coalgebra then $\mathsf{Corep}\, C$ is a $\mathbf{k}$-linear abelian category.

Next, we recall the notion of a simple object in an abelian category. For this, we need the concept of subobject.

Definition 2.3.10. Let $\mathcal{C}$ be an abelian category. A morphism $f : X \to Y$ is said to be a *monomorphism* if $\mathsf{Ker}(f) = 0$, and an *epimorphism* if $\mathsf{Coker}(f) = 0$.

Definition 2.3.11. Let $\mathcal{C}$ be an abelian category and Y an object of $\mathcal{C}$. A *subobject* of Y is a pair (X, i), where $i : X \to Y$ is a monomorphism. A *quotient object* of Y is a pair (Z, p), where $p : Y \to Z$ is an epimorphism.

Remark 2.3.12. It is common to omit mentioning the monomorphism i when saying that (X, i) is a subobject of Y. Thus, we say X is a subobject of Y and we write $X \subseteq Y$. The cokernel of the monomorphism $i : X \to Y$ is denoted by Y/X.

Definition 2.3.13. Let $\mathcal{C}$ be an abelian category.

(1) A nonzero object X in $\mathcal{C}$ is *simple* if 0 and X are its only subobjects.

(2) An object X in $\mathcal{C}$ is *semisimple* if it is a direct sum of simple objects.

(3) $\mathcal{C}$ is *semisimple* is every object of $\mathcal{C}$ is semisimple.

The Jordan-Hölder theorem holds in abelian categories.

Definition 2.3.14. Let $\mathcal{C}$ be an abelian category. An object X of $\mathcal{C}$ is said to have *finite length* if X admits a *composition series* (or a *Jordan-Hölder series*), that is, a filtration

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X$$

such that X_i/X_{i-1} is simple, for all $i = 1, \ldots, n$. The objects X_i/X_{i-1} are called the *factors* of the composition series, and n is called the *length* of the composition series.

Theorem 2.3.15 (Jordan-Hölder)**.** *Let $\mathcal{C}$ be an abelian category. If $X \in \mathsf{Ob}(\mathcal{C})$ has finite length, then all composition series of X have the same length and isomorphic factors, up to order.*

It follows from Jordan-Hölder theorem that the number of factors in a composition series of an object X is independent of the composition series. This number is called the *length* of X.

As we previously said, abelian categories allow definition of the notion of exact sequence.

Definition 2.3.16. Let $\mathcal{C}$ be an abelian category. A sequence of morphisms in $\mathcal{C}$

$$\cdots \to X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \to \cdots$$

is *exact in degree* i if $\mathsf{Im}(f_{i-1}) = \mathsf{Ker}(f_i)$. The sequence is *exact* if it is exact in every degree.

An exact sequence $0 \to X \to Y \to Z \to 0$ is called a *short exact sequence*.

Definition 2.3.17. Let $\mathcal{C}$ be an abelian category and let X and Y be objects of $\mathcal{C}$.

(1) An *extension* of Y by X is a short exact sequence $S : 0 \to X \xrightarrow{i} E \xrightarrow{p} Y \to 0$.

(2) A *morphism* from $S : 0 \to X \xrightarrow{i} E \xrightarrow{p} Y \to 0$ to $S' : 0 \to X \xrightarrow{i'} E' \xrightarrow{p'} Y \to 0$ is a morphisms $f : E \to E'$ making the following diagram commutative:

$$\begin{tikzcd}
 & & E \arrow[dd, "f"] \arrow[dr, "p"] & & \\
0 \arrow[r] & X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[r] & 0 \\
 & & E' \arrow[ur, "p'"'] & &
\end{tikzcd}$$

Remark 2.3.18. The set of isomorphism classes of extensions of Y by X is denoted by $\mathsf{Ext}^1(Y, X)$. It is an abelian group with the following operation. Let $S : 0 \to X \xrightarrow{i} E \xrightarrow{p} Y \to 0$ and $S' : 0 \to X \xrightarrow{i'} E' \xrightarrow{p'} Y \to 0$ be two extensions of Y by X. The image N of $(i, -i') : X \to E \oplus E'$ is a subobject of the kernel M of $pp_E - p'p_{E'} : E \oplus E' \to Y$, where p_E and $p_{E'}$ are the projections of $E \oplus E'$ to its summands. The sum of the isomorphism classes of S and S' is the isomorphism class of the exact sequence

$$0 \to X \xrightarrow{(i,0)} M/N \xrightarrow{pp_E} Y \to 0$$

This operation is called the *Baer sum*. The zero element of $\mathsf{Ext}^1(Y, X)$ is the class of the split extension $0 \to X \to X \oplus Y \to Y \to 0$.

If $\mathcal{C}$ is a $\mathbf{k}$-linear abelian category, then $\mathsf{Ext}^1(Y, X)$ is a $\mathbf{k}$-vector space. The multiple of the class of $0 \to X \xrightarrow{i} E \xrightarrow{p} Y \to 0$ by the non-zero scalar $\lambda \in \mathbf{k}^\times$ is the class of

$$0 \to X \xrightarrow{\lambda^{-1} i} E \xrightarrow{p} Y \to 0$$

One can check that if λ, $\mu \in \mathbf{k}^\times$, then $0 \to X \xrightarrow{\lambda i} E \xrightarrow{\mu p} Y \to 0$ and $0 \to X \xrightarrow{\lambda \mu i} E \xrightarrow{p} Y \to 0$ are isomorphic.

Definition 2.3.19. Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* (respectively, *right exact*) if given any short exact sequence in $\mathcal{C}$, $0 \to X \to Y \to Z \to 0$, the sequence

$$0 \to F(X) \to F(Y) \to F(Z) \quad \text{(respectively, } F(X) \to F(Y) \to F(Z) \to 0)$$

is exact in $\mathcal{D}$. A left exact and right exact functor is said to be *exact*.

Example 2.3.20. If X is an object of an abelian category $\mathcal{C}$, then the functor $\mathsf{Hom}_{\mathcal{C}}(X, -)$ from $\mathcal{C}$ to the category of abelian groups is left exact.

Example 2.3.21. If A and B are finite dimensional $\mathbf{k}$-algebras and P is a (B, A)-bimodule, then $P \otimes_A (-) : \mathsf{Rep}\, A \to \mathsf{Rep}\, B$ is right exact.

Definition 2.3.22. Let $\mathcal{C}$ be an abelian category.

(1) An object P of $\mathcal{C}$ is *projective* if the functor $\mathsf{Hom}_{\mathcal{C}}(P, -)$ is exact.

(2) A *projective cover* of $X \in \mathsf{Ob}(\mathcal{C})$ is a projective object $P(X)$ in $\mathcal{C}$, together with an epimorphism $p : P(X) \to X$ such that, if $g : P \to X$ is an epimorphism from a projective object P to X, then there exists an epimorphism $h : P \to P(X)$ such that $ph = g$.

We are now ready to state an intrinsic characterization of finite $\mathbf{k}$-linear abelian categories.

Definition 2.3.23. Let $\mathbf{k}$ be a field. A $\mathbf{k}$-linear abelian category $\mathcal{C}$ is *finite* if:

(1) $\mathcal{C}$ has finite dimensional hom-sets.

(2) All objects of $\mathcal{C}$ have finite length.

(3) There are finitely many isomorphism classes of simple objects.

(4) Every simple object has a projective cover.

Theorem 2.3.24. *If $\mathcal{C}$ is a finite, $\mathbf{k}$-linear, abelian category then there exists a finite dimensional $\mathbf{k}$-algebra A such that $\mathcal{C}$ is equivalent to* $\mathsf{Rep}\, A$*.*

Proof. See [E11, Theorem 9.6.4]. □

Since functors between representation categories of finite dimensional algebras will play an important role in our work, we close this section by making a few observations concerning them.

First, if $F : \mathcal{C} \to \mathcal{D}$ is a functor between two $\mathbf{k}$-linear categories, then $\mathsf{End}(F)$, the set of functorial endomorphisms of F, is a $\mathbf{k}$-algebra with addition, scalar multiplication and product given by:

$$(\mu + \nu)_X = \mu_X + \nu_X, \quad (\lambda\mu)_X = \lambda\mu_X, \quad (\mu\nu)_X = \mu_X \nu_X$$

for all $X \in \mathsf{Ob}(\mathcal{C})$, $\lambda \in \mathbf{k}$, and $\mu, \nu \in \mathsf{End}(F)$.

Functorial endomorphisms of forgetful functors have a nice description.

Lemma 2.3.25. *Let A be a finite dimensional $\mathbf{k}$-algebra and $F_A : \mathsf{Rep}\, A \to$ Vec the forgetful functor. Then*

$$\varphi : \mathsf{End}(F_A) \to A, \quad \varphi(\mu) = \mu_A(1), \quad \mu \in \mathsf{End}(F_A)$$

is an isomorphism of $\mathbf{k}$-algebras, with inverse

$$\varphi^{-1} : A \to \mathsf{End}(F_A), \quad \varphi^{-1}(a)_V(v) = av, \quad v \in V \in \mathsf{Rep}(A),\, a \in A$$

Proof. Straightforward. □

Remark 2.3.26. Let $\varphi : \mathsf{End}(F_A) \to A$ be the isomorphism of the previous Lemma and let $Res_\varphi : \mathsf{Rep}\, A \to \mathsf{Rep}\, \mathsf{End}(F_A)$ be the restriction of scalars functor associated to φ. If $V \in \mathsf{Rep}\, A$ then $Res_\varphi(V) = V$ with $\mathsf{End}(F_A)$-action given by $\mu \cdot v = \mu_V(v)$, for all $v \in V$ and $\mu \in \mathsf{End}(F_A)$. In other words, $\mu_V(v) = \varphi(\mu)v = \mu_A(1)v$, for all v, V and μ. Indeed, consider for $v \in V$ the A-linear map $f_v : A \to V$, $f_v(a) = av$, for all $a \in A$. From the naturality of μ we have $\mu_V f_v = f_v \mu_A$. Thus, $\mu_V(v) = \mu_v f_v(1) = f_v \mu_A(1) = \mu_A(1)v$.

Restriction of scalars functors preserve forgetful functors. The converse is also true, as we next show.

Proposition 2.3.27. *Let A and B be two finite dimensional $\mathbf{k}$-algebras and let $F_A : \mathsf{Rep}\, A \to$ Vec and $F_B : \mathsf{Rep}\, B \to$ Vec be the forgetful functors. If $F : \mathsf{Rep}\, B \to \mathsf{Rep}\, A$ is a $\mathbf{k}$-linear functor such that $F_A F = F_B$ then there exists an algebra map $f : A \to B$ such that $F = Res_f$.*

Proof. Let $\varphi_A : \mathsf{End}(F_A) \to A$ and $\varphi_B : \mathsf{End}(F_B) \to B$ be the algebra isomorphisms from Lemma 2.3.25. Define

$$g : \mathsf{End}(F_A) \to \mathsf{End}(F_B) = \mathsf{End}(F_A F), \quad g(\mu)_V = \mu_{F(V)}$$

for all $V \in \mathsf{Rep}\, B$ and $\mu \in \mathsf{End}(F_A)$. It is easy to check that g is a well defined algebra homomorphism.

Let $f : A \to B$ be the algebra map $f = \varphi_B g \varphi_A^{-1}$. We claim that $F = Res_f$. Since $Res_f = Res_{\varphi_A^{-1}} Res_g Res_{\varphi_B}$, it suffices to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{Rep}\, B \arrow[r, "F"] \arrow[d, "Res_{\varphi_B}"'] & \mathsf{Rep}\, A \arrow[d, "Res_{\varphi_A}"] \\
\mathsf{Rep}\left(\mathsf{End}(F_B)\right) \arrow[r, "Res_g"'] & \mathsf{Rep}\left(\mathsf{End}(F_A)\right)
\end{tikzcd}$$

Let $V \in \mathsf{Rep}\, B$. Taking into account Remark 2.3.26, we have that $Res_{\varphi_A} F(V)$ is V with $\mathsf{End}(F_A)$-action given by $\mu \cdot v = \mu_{F(V)}(v)$, while $Res_g\, Res_{\varphi_B} V$ is V with $\mathsf{End}(F_A)$-action $\mu \cdot v = g(\mu)_V(v) = \mu_{F(V)}(v)$. Thus, $F = Res_f$. $\square$

Functors that are natural isomorphic to restriction of scalars functors can be described as those functors that preserve dimensions.

Proposition 2.3.28. *Let A and B be two finite dimensional $\mathbf{k}$-algebras. A functor $F : \mathsf{Rep}\, B \to \mathsf{Rep}\, A$ is isomorphic to $Res\ f$, for some algebra map $f : A \to B$, if and only if $\dim F(V) = \dim V$, for all $V \in \mathsf{Rep}\, B$.*

Proof. Suppose $\dim F(V) = \dim V$, for all $V \in \mathsf{Rep}\, B$. For each such V choose an isomorphism of $\mathbf{k}$-vector spaces $\mu_V : F(V) \to V$. Then on V there is a unique A-module structure such that μ_V is A-linear. Denote this A-module by $G(V)$. For a B-linear map $f : U \to V$ define $G(f) = \mu_V F(f)\mu_u^{-1}$. It is easy to check that, with these choices, G is a functor from $\mathsf{Rep}\, B$ to $\mathsf{Rep}\, A$ and $\mu = \{\mu_V\}$ is an isomorphism from F to G. Since G preserves the forgetful functors, it follows from Proposition 2.3.27 that $G = Res_f$, for some algebra map $f : A \to B$. The converse is trivial. $\square$

2.4 Rigid monoidal categories

In this section we present the categorical counterpart of the monoid from group theory, namely, the monoidal category. We also talk about rigidity in monoidal categories and, at the end, introduce the main objects of our study, finite tensor categories.

Definition 2.4.1. A *monoidal category* is a sextuple $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ consisting of a category $\mathcal{C}$, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called *tensor product*, an object $\mathbf{1} \in \mathcal{C}$, called a *unit object*, and natural isomorphisms:

$$\begin{aligned} a &= \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathsf{Ob}(\mathcal{C})}, \\ l &= \{l_X : \mathbf{1} \otimes X \to X\}_{X \in \mathsf{Ob}(\mathcal{C})}, \\ r &= \{r_X : X \otimes \mathbf{1} \to X\}_{X \in \mathsf{Ob}(\mathcal{C})} \end{aligned}$$

called, respectively, the *associativity*, *left unit* and *right unit constraints*, satisfying the *pentagon axiom*, i.e. diagram

$$\begin{tikzcd} & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "a_{W,X,Y \otimes Z}"] & \\ ((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "a_{W \otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y} \otimes \mathsf{id}_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\ (W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X \otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathsf{id}_W \otimes a_{X,Y,Z}"'] \end{tikzcd}$$

commutes for all $W, X, Y, Z \in \mathsf{Ob}(\mathcal{C})$, and the *triangle axiom*, i.e. diagram

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "a_{X,\mathbf{1},Y}"] \arrow[dr, "r_X \otimes \mathsf{id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathsf{id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

commutes for all $X, Y \in \mathsf{Ob}(\mathcal{C})$.

Remark 2.4.2. If $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is a monoidal category then we say that $(\otimes, \mathbf{1}, a, l, r)$ constitutes a *monoidal structure* on $\mathcal{C}$. If this structure is clear from the context then we write $\mathcal{C}$ instead of $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$. This will be the case with the next examples.

Example 2.4.3. The category **k**-**Vec** is a monoidal category. The tensor product is the usual tensor product of vector spaces, $\otimes = \otimes_{\mathbf{k}}$, the unit element is **k**, and a, l and r are the obvious maps:

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w),$$
$$l_V : \mathbf{k} \otimes V \to V, \quad \lambda \otimes v \mapsto \lambda v,$$
$$r_V : V \otimes \mathbf{k} \to V, \quad v \otimes \lambda \mapsto \lambda v,$$

for all $u \in U$, $v \in V$, $w \in W$, $\lambda \in \mathbf{k}$, and $U, V, W \in$ **k**- **Vec**. The same data defines a monoidal structure on **k**-Vec, the category of finite dimensional **k**-vector spaces.

Example 2.4.4. If B is a bialgebra then the comultiplication and the counit of B make $\mathsf{Rep}\ B$ a monoidal category. Namely, if $U, V \in \mathsf{Rep}\, B$ then $U \otimes V = U \otimes_{\mathbf{k}} V$, with action of B given by

$$b \cdot (u \otimes v) = \sum b_{(1)} u \otimes b_{(2)} v, \qquad b \in B,\ u \in U,\ v \in V.$$

The unit object is **k**, on which B acts via ε: $b \cdot \lambda = \varepsilon(b)\lambda$, for all $b \in B$ and $\lambda \in \mathbf{k}$. The associativity and the left and right unit constraints are the same as for **k**-Vec.

Example 2.4.5. Again, let B be a bialgebra. Then the multiplication and the unit of B give $\mathsf{Corep}\ B$ a monoidal structure. If $U, V \in \mathsf{Corep}\ H$ then $U \otimes V = U \otimes_{\mathbf{k}} V$, with co-action of B given by

$$u \otimes v \mapsto \sum u_{(0)} \otimes v_{(0)} \otimes u_{(1)} v_{(1)}, \qquad u \in U,\ v \in V.$$

The unit object is $\mathbf{k}$, with the trivial coaction: $\lambda \mapsto \lambda \otimes 1_H$, for all $\lambda \in \mathbf{k}$. The associativity and the left and right unit constraints are the same as for $\mathbf{k}$-Vec.

Definition 2.4.6. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. The monoidal category *opposite to* $\mathcal{C}$ is $\mathcal{C}^{\mathrm{op}} = (\mathcal{C}, \otimes^{\mathrm{op}}, \mathbf{1}, a^{\mathrm{op}}, \ldots)$, where $X \otimes^{\mathrm{op}} Y = Y \otimes X$ and $a^{\mathrm{op}}_{X,Y,Z} = a^{-1}_{Z,Y,X}$.

Example 2.4.7. Given a bialgebra B, $(\mathsf{Rep}\, B)^{\mathrm{op}} \cong \mathsf{Rep}\, B^{\mathrm{cop}}$.

Definition 2.4.8. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', l', r')$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a triple (F, J, φ), where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, $\varphi : \mathbf{1}' \to F(\mathbf{1})$ is an isomorphism, and

$$J_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y)$$

is a natural isomorphism, such that the following diagrams are commutative

$$\begin{tikzcd}
\big(F(X) \otimes' F(Y)\big) \otimes' F(Z) \arrow[rr, "a'_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes' \mathsf{id}_{F(Z)}"'] & & F(X) \otimes' \big(F(Y) \otimes' F(Z)\big) \arrow[d, "\mathsf{id}_{F(X)} \otimes' J_{Y,Z}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\
F\big((X \otimes Y) \otimes Z\big) \arrow[rr, "F(a_{X,Y,Z})"'] & & F\big(X \otimes (Y \otimes Z)\big)
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
\mathbf{1}' \otimes' F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\varphi \otimes \mathsf{id}_{F(X)}"'] & F(X) & F(X) \otimes' \mathbf{1}' \arrow[r, "r'_{F(X)}"] \arrow[d, "\mathsf{id}_{F(X)} \otimes \varphi"'] & F(X) \\
F(\mathbf{1}) \otimes' F(X) \arrow[r, "J_{\mathbf{1},X}"'] & F(\mathbf{1} \otimes X) \arrow[u, "F(l_X)"'] & F(X) \otimes' F(\mathbf{1}) \arrow[r, "J_{X,\mathbf{1}}"'] & F(X \otimes \mathbf{1}) \arrow[u, "F(r_X)"']
\end{tikzcd} \tag{2.2}$$

The pair (J, φ) is called a *monoidal structure* on F.

Remark 2.4.9. Monoidal functors can be composed. If (F, J, φ) is a monoidal functor from $(\mathcal{C}, \otimes, \mathbf{1})$ to $(\mathcal{C}', \otimes', \mathbf{1}')$, and (F', J', φ') is a monoidal functor from $(\mathcal{C}', \otimes', \mathbf{1}')$ to $(\mathcal{C}'', \otimes'', \mathbf{1}'')$ then the composition of (F', J', φ') with (F, J, φ) is $(F'F, J'F'(J), F'(\varphi)\varphi')$, where $J'F'(J)$ is the natural transformation given by the composition:

$$F'F(X)\otimes'' F'F(Y) \xrightarrow{J'_{F(X),F(Y)}} F'\big(F(X)\otimes' F(Y)\big) \xrightarrow{F'(J_{X,Y})} F'F(X\otimes Y),$$

for all $X, Y \in \mathcal{C}$.

The following results provide some examples of monoidal functors.

Lemma 2.4.10. *Let K and H be two bialgebras and $f : K \to H$ an algebra map. The set of monoidal structures on the restriction of scalars functor $Res_f : \mathsf{Rep}\, H \to \mathsf{Rep}\, K$, with $\varphi = \mathsf{id}_{\mathbf{k}}$, is in one-to-one correspondence with the set of invertible elements $T \in H \otimes H$ satisfying the following conditions:*

$$(f\otimes f)\Delta(x) \;=\; T^{-1}\Delta\big(f(x)\big)T, \text{ for all } x \in K, \tag{2.3}$$

$$(\Delta\otimes \mathsf{id}_H)(T)(T\otimes 1) \;=\; (\mathsf{id}_H \otimes\Delta)(T)(1\otimes T), \tag{2.4}$$

$$(\varepsilon\otimes \mathsf{id}_H)(T) \;=\; (\mathsf{id}_H \otimes\varepsilon)(T) = 1. \tag{2.5}$$

The monoidal structure corresponding to T is $(J^T, \mathsf{id}_{\boldsymbol{k}})$, where

$$J^T_{U,V} : U\otimes V \to U\otimes V, \quad u\otimes v \mapsto T(u\otimes v).$$

Proof. Straightforward. □

Remark 2.4.11. We will denote the monoidal functor $(Res_f, T, \mathsf{id}_{\mathbf{1}})$ by (f, T). It is easy to check that the composition of (f', T') with (f, T) is $\big(ff', T(f\otimes f)(T')\big)$.

Remark 2.4.12. The tensor functors (f, T) have been studied by A. Davydov in [Dav10]. We will say more about them in Section 2.10.

Corollary 2.4.13. *Let H be a bialgebra. The set of monoidal structures on the forgetful functor $F_H : \mathsf{Rep}\, H \to$ Vec, with $\varphi = \mathsf{id}_{\mathbf{k}}$, is in bijection with the set of invertible elements $T \in H \otimes H$ satisfying conditions* (2.4) *and* (2.5).

Definition 2.4.14. Let H be a bialgebra. An invertible element $T \in H \otimes H$ satisfying conditions (2.4) and (2.5) is called a (*right*) *twist on H*.

Corollary 2.4.15. *Let H be a bialgebra. The set of monoidal structures on the identity functor $\mathsf{id}_{\mathsf{Rep}\,H} : \mathsf{Rep}\,H \to \mathsf{Rep}\,H$, with $\varphi = \mathsf{id}_{\mathbf{k}}$, is in bijection with the set of twists on H satisfying $T\Delta(x) = \Delta(x)T$, for all $x \in H$.*

Definition 2.4.16. A twist T on a bialgebra H is *invariant* if $T\Delta(x) = \Delta(x)T$, for all $x \in H$.

Lemma 2.4.17. *Let K and H be two bialgebras and $f : H \to K$ a coalgebra map. The set of monoidal structures on the extension of scalars functor $Ext_f : \mathsf{Corep}\,H \to \mathsf{Corep}\,K$, with $\varphi = \mathsf{id}_{\mathbf{k}}$, is in one-to-one correspondence with the set of convolution invertible elements $\sigma : H \otimes H \to \mathbf{k}$ satisfying the following conditions:*

$$f(x)f(y) = \sigma^{-1}(x_{(1)}, y_{(1)})f(x_{(2)}y_{(2)})\sigma(x_{(3)}, y_{(3)}), \tag{2.6}$$

$$\sigma(x_{(1)}y_{(1)}, z)\sigma(x_{(2)}, y_{(2)}) = \sigma(x, y_{(1)}z_{(1)})\sigma(y_{(2)}, z_{(2)}), \tag{2.7}$$

$$\sigma(x, 1) = \sigma(1, x) = \varepsilon(x), \tag{2.8}$$

for all x, y, $z \in H$. The monoidal structure corresponding to σ is $(J^\sigma, \mathsf{id}_{\mathbf{k}})$, where

$$J^\sigma_{U,V} : U \otimes V \to U \otimes V, \quad u \otimes v \mapsto \sigma(u_{(1)}, v_{(1)})u_{(0)} \otimes v_{(0)}.$$

Proof. Straightforward. □

Remark 2.4.18. We will denote the monoidal functor $(\mathsf{Ext}_f, J^\sigma, \mathsf{id}_{\mathbf{1}})$ by (f, σ). The composition of (f', σ') with (f, σ) is $(f'f,\ \sigma * (\sigma' \circ (f \otimes f)))$.

Corollary 2.4.19. *Let H be a bialgebra. The set of monoidal structures on the forgetful functor $F_H : \mathsf{Corep}\,H \to Vec$ is in bijection with the set of convolution invertible elements $\sigma : H \otimes H \to k$ satisfying* (2.7) *and* (2.8).

Definition 2.4.20. Let H be a bialgebra. A convolution invertible element $\sigma : H \otimes H \to k$ satisfying (2.7) and (2.8) is called a (*right*) 2-*cocycle on* H.

Corollary 2.4.21. *Let H be a bialgebra. The set of monoidal structures on the identity functor $\mathsf{id}_{\mathsf{Corep}\,H} : \mathsf{Corep}\,H \to \mathsf{Corep}\,H$ is in bijection with the set of 2-cocycles on H satisfying $\sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = x_{(1)}y_{(1)}\sigma(x_{(2)}, y_{(2)})$.*

Definition 2.4.22. An *invariant* 2-cocycle on a bialgebra H is a 2-cocycle σ satisfying

$$\sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = x_{(1)}y_{(1)}\sigma(x_{(2)}, y_{(2)}),$$

for all $x, y \in H$.

Definition 2.4.23. Let (F, J, φ) and (F', J', φ') be two monoidal functors from $(\mathcal{C}, \otimes, \mathbf{1})$ to $(\mathcal{C}', \otimes', \mathbf{1}')$. A *natural monoidal transformation* from (F, J, φ) to (F', J', φ') is a natural transformation $\mu : F \to F'$ such that the following diagrams commute:

$$\begin{tikzcd} F(X) \otimes' F(Y) \arrow[r, "J_{X,Y}"] \arrow[d, "\mu_X \otimes' \mu_Y"'] & F(X \otimes Y) \arrow[d, "\mu_{X \otimes Y}"] \\ F'(X) \otimes' F'(Y) \arrow[r, "J'_{X,Y}"'] & F'(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} F(\mathbf{1}) \arrow[rr, "\mu_{\mathbf{1}}"] & & F'(\mathbf{1}) \\ & \mathbf{1}' \arrow[ul, "\varphi"] \arrow[ur, "\varphi'"'] & \end{tikzcd} \tag{2.9}$$

A *natural monoidal isomorphism* is a natural monoidal transformation that is a natural isomorphism.

Remark 2.4.24. Suppose (F, J, φ) is a monoidal functor from $(\mathcal{C}, \otimes, \mathbf{1})$ to $(\mathcal{C}', \otimes', \mathbf{1}')$. If $F' : \mathcal{C} \to \mathcal{C}'$ is a functor, and $\mu : F \to F'$ is a natural isomorphism, then there is a unique monoidal structure on F' such that μ becomes a natural monoidal isomorphism. Namely, this structure is (J', φ'), where J' and φ' are defined by diagrams (2.9).

Lemma 2.4.25. *Let H and K be two bialgebras, and (f, T) and (f', T') two monoidal functors from $\mathsf{Rep}\, H$ to $\mathsf{Rep}\, K$. The set of natural monoidal transformations from (f, T) to (f', T') is in bijection with the set of elements $u \in H$ satisfying the following conditions:*

$$uf(x) = f'(x)u, \quad \text{for all } x \in K, \tag{2.10}$$

$$T'(u \otimes u) = \Delta(u)T, \tag{2.11}$$

$$\varepsilon(u) = 1. \tag{2.12}$$

The natural monoidal transformation corresponding to u is μ^u, where

$$\mu_V^u : V \to V, \quad v \mapsto uv, \quad v \in V \in \mathsf{Rep}\, H.$$

The transformation μ^u is a natural monoidal isomorphism if and only if u is invertible.

Proof. Straightforward. $\square$

Corollary 2.4.26. *Let T and T' be two twists of a bialgebra H and let $F_H : \mathsf{Rep}\, H \to \mathbf{k}$-Vec be the forgetful functor. Then (F_H, T) and (F_H, T') are monoidal natural isomorphic if and only if there exists an invertible element $u \in H$ such that $\varepsilon(h) = 1$ and*

$$T' = \Delta(u)T(u^{-1} \otimes u^{-1}).$$

Definition 2.4.27. Two twists, T and T', of a bialgebra H are *gauge equivalent* if (F_H, T) and (F_H, T') are monoidal natural isomorphic.

Lemma 2.4.28. *Let H and K be two bialgebras and (f, σ) and (f', σ') be two monoidal functors from $\mathsf{Corep}\, H$ to $\mathsf{Corep}\, K$. Then the set of natural monoidal transformations from (f, σ) to (f', σ') is in bijection with the set of linear maps $\alpha : H \to k$ satisfying the following conditions:*

$$f'(x_{(1)})\alpha(x_{(2)}) = \alpha(x_{(1)})f(x_{(2)}), \quad \text{for all } x \in H, \tag{2.13}$$

$$\alpha(x_{(1)}y_{(1)})\sigma(x_{(2)}, y_{(2)}) = \sigma'(x_{(1)}, y_{(1)})\alpha(x_{(2)})\alpha(y_{(2)}), \tag{2.14}$$

$$\alpha(1) = 1. \tag{2.15}$$

The natural monoidal transformation corresponding to α is μ^α, where

$$\mu_V^\alpha : V \to V, \quad v \mapsto \sum \alpha(v_{(1)})v_{(0)}, \quad v \in V \in \mathsf{Corep}\, H.$$

μ^α is a natural monoidal isomorphism if and only if α is convolution invertible.

Corollary 2.4.29. *Let σ and σ' be two 2-cocycles on the bialgebra H and let $F_H : \mathsf{Corep}\, H \to \mathbf{k}$-Vec be the forgetful functor. Then (F_H, σ) is natural monoidal isomorphic to (F_H, σ') if and only if there exists a convolution invertible linear map $\alpha : H \to \mathbf{k}$ such that $\alpha(1_H) = 1_{\mathbf{k}}$ and*

$$\sigma'(x, y) = \alpha(x_{(1)}y_{(1)})\sigma(x_{(2)}, y_{(2)})\alpha^{-1}(x_{(3)})\alpha^{-1}(y_{(3)}).$$

Definition 2.4.30. Two 2-cocycles, σ and σ', on a bialgebra H are *gauge equivalent* if (F_H, σ) and (F_H, σ') are natural monoidal isomorphic.

Next, we discuss rigidity in monoidal categories.

Definition 2.4.31. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category and X an object of $\mathcal{C}$. In what follows we suppress the unit constraints l and r.

(1) An object X^* of $\mathcal{C}$ is a *left dual* of X if there exist morphisms

$$\mathsf{ev}_X : X^* \otimes X \to \mathbf{1} \quad \text{and} \quad \mathsf{coev}_X : \mathbf{1} \to X \otimes X^*$$

called *evaluation* and *coevaluation* morphisms, such that the compositions

$$X \xrightarrow{\mathsf{coev}_X \otimes \mathsf{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathsf{id}_X \otimes \mathsf{ev}_X} X,$$
$$X^* \xrightarrow{\mathsf{id}_{X^*} \otimes \mathsf{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathsf{ev}_X \otimes \mathsf{id}_{X^*}} X^*$$

are the identity morphisms.

(2) An object ${}^*X$ of $\mathcal{C}$ is a *right dual* of X if there exist morphisms

$$\mathsf{ev}'_X : X \otimes {}^*X \to \mathbf{1} \quad \text{and} \quad \mathsf{coev}'_X : \mathbf{1} \to {}^*X \otimes X$$

also called *evaluation* and *coevaluation* morphisms, such that the compositions

$$X \xrightarrow{\mathsf{id}_X \otimes \mathsf{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X) \xrightarrow{\mathsf{ev}'_X \otimes \mathsf{id}_X} X,$$
$${}^*X \xrightarrow{\mathsf{coev}'_X \otimes \mathsf{id}_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathsf{id}_{{}^*X} \otimes \mathsf{ev}'_X} {}^*X$$

are the identity morphisms.

Definition 2.4.32. An object of a monoidal category is *rigid* if it has both left and right duals. A monoidal category is called *rigid* if all of its objects are rigid.

Example 2.4.33. The category $\mathbf{k}$-Vec of finite dimensional $\mathbf{k}$-vector spaces is rigid. The left and right dual of a finite dimensional vector space V are $V^* = \mathsf{Hom}(V, \mathbf{k})$. If $\{e_i\}$ is a basis of V and $\{e_i^*\}$ is its dual basis, then the evaluation and coevaluation morphisms are given by:

$$\begin{aligned}
\mathsf{ev}_V &: V^* \otimes V \to \mathbf{k}, \quad (p, v) \mapsto p(v), \\
\mathsf{coev}_V &: \mathbf{k} \to V \otimes V^*, \quad 1 \mapsto \sum_i e_i \otimes e_i^*, \\
\mathsf{ev}'_V &: V \otimes V^* \to \mathbf{k}, \quad (v, p) \mapsto p(v), \\
\mathsf{coev}'_V &: \mathbf{1} \to V^* \otimes V, \quad 1 \mapsto \sum_i e_i^* \otimes e_i.
\end{aligned}$$

The category **k-Vec**, of all $\mathbf{k}$-vector spaces, is not rigid, as can easily be verified.

Example 2.4.34. Let H be a Hopf algebra and $\mathsf{Rep}\, H$ the category of finite dimensional representations of H. If V is an H-module then V^*, with H-action given by

$$(h \cdot f)(v) = f\big(S(h)v\big), \quad v \in V, f \in V^*, h \in H,$$

is a left dual of V. If the antipode of H is bijective, then V^*, with H-action given by

$$(h \cdot f)(v) = f\big(S^{-1}(h)v\big), \quad v \in V, f \in V^*, h \in H,$$

is a right dual of V. The evaluation and coevaluation morphisms are the same as for finite dimensional vector spaces.

Example 2.4.35. Let H be a Hopf algebra and $\mathsf{Corep}\, H$ the category of finite dimensional corepresentations of H. If $V \in \mathsf{Corep}\, H$ and $\{e_i\}$ is a basis of V, with dual basis $\{e_i^*\}$, then V^*, with H-coaction given by

$$V^* \to V^* \otimes H, \quad f \mapsto \textstyle\sum_i f\big((e_i)_{(0)}\big) e_i^* \otimes S\big((e_i)_{(1)}\big),$$

is a left dual of V. If S is invertible, then V^*, with H-coaction:

$$V^* \to V^* \otimes H, \quad f \mapsto \textstyle\sum_i f\big((e_i)_{(0)}\big) e_i^* \otimes S^{-1}\big((e_i)_{(1)}\big),$$

is a right dual of V. The evaluation and coevaluation morphisms are the same as for finite dimensional vector spaces.

2.5 Pointed finite tensor categories and pointed Hopf algebras

We introduce in this section the main objects of our study, namely, pointed finite tensor categories. Since pointed Hopf algebras provide examples of such categories, we introduce more concepts from Hopf algebra theory that pertain to their study and which will be of use in this dissertation.

Definition 2.5.1. (1) A *finite tensor category* is a rigid, monoidal, finite, $\mathbf{k}$-linear, abelian category $\mathcal{C}$ such that the tensor bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms and $\mathsf{End}_{\mathcal{C}}(\mathbf{1}) \cong \mathbf{k}$.

(2) A *fusion category* is a finite tensor category which is semisimple.

(3) A *tensor functor* between two finite tensor categories, $\mathcal{C}$ and $\mathcal{D}$, is a monoidal functor $(F, J) : \mathcal{C} \to \mathcal{D}$ such that F is exact, faithful and $\mathbf{k}$-linear.

Example 2.5.2. If H is a finite dimensional Hopf algebra then $\mathsf{Rep}\, H$ and $\mathsf{Corep}\, H$ are finite tensor categories. If, moreover H is semisimple then $\mathsf{Rep}\, H$ and $\mathsf{Corep}\, H$ are fusion categories.

The (co)-representation categories of finite dimensional Hopf algebras are those finite tensor categories that admit a fiber functor.

Definition 2.5.3. A *fiber functor on* $\mathcal{C}$ is tensor functor from $\mathcal{C}$ to $\mathbf{k}$-Vec.

Theorem 2.5.4. *If $\mathcal{C}$ is a finite tensor category and $F : \mathcal{C} \to \mathbf{k}$-Vec is a fiber functor then $H = \mathsf{End}(F)$ is a finite dimensional Hopf algebra and $\mathcal{C} \simeq \mathsf{Rep}(H)$.*

Proof. See [Ulb90] and the references therein. ☐

Remark 2.5.5. The procedure of obtaining the Hopf algebra $\mathsf{End}(F)$ from a fiber functor F is called *Tannaka-Krein reconstruction*. This is because, in the case of a Hopf algebra H we recover H from the forgetful functor $F : \mathsf{Rep}\, H \to \mathbf{k}$-Vec: $\mathsf{End}(F) \cong H$.

We will be interested in those finite tensor categories that are pointed, i.e. those tensor categories whose simple objects are invertible.

Definition 2.5.6. Let $\mathcal{C}$ be a finite tensor category.

(1) An object X of $\mathcal{C}$ is *invertible* if $\mathsf{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathsf{coev}_X : \mathbf{1} \to X \otimes X^*$ are isomorphisms.

(2) $\mathcal{C}$ is *pointed* if every simple object of $\mathcal{C}$ is invertible.

Example 2.5.7. Let H be a finite dimensional Hopf algebra. Then the invertible objects of $\mathsf{Rep}(H)$ are the 1-dimensional representations of H. Similarly, the invertible objects of $\mathsf{Corep}\, H$ are the 1-dimensional co-represention of H.

Definition 2.5.8. Let H be a finite dimensional Hopf algebra. H is said to be *basic* if $\mathsf{Rep}\, H$ is pointed. H is said to be *pointed* if $\mathsf{Corep}\, H$ is pointed.

Example 2.5.9. Let G be a finite group. Then $\mathbf{k}[G]$ is a pointed Hopf algebra. To see this, recall that the objects of $\mathsf{Corep}\, G$ can be equivalently described as G-graded vector spaces: if $V \in \mathsf{Corep}\, G$ then $V = \oplus_{g \in G} V_g$, where $V_g = \{v \in V \mid \delta(v) = v \otimes g\}$. Thus, the simple objects of $\mathsf{Corep}\, G$ are the 1-dimensional G-graded vector spaces δ_g, $g \in G$, with the grading of δ_g given by

$$(\delta_g)_h = \begin{cases} \mathbf{k} & \text{if} \quad h = g \\ 0 & \text{if} \quad h \neq g \end{cases}$$

If G is abelian then $\mathbf{k}[G]$ is also basic.

There is an equivalent way of describing pointed Hopf algebras. For this, we have to introduce additional notions from Hopf algebra theory.

Definition 2.5.10. Let (C, Δ, ε) be a coalgebra.

(1) A *subcoalgebra* of C is a subspace D of C such that $\Delta(D) \subseteq D \otimes D$.

(2) C is said to be *simple* if 0 and C are its only subcoalgebras.

(3) The *coradical* of C is the sum of all simple subcoalgebras of C. It is denoted by C_0.

Remark 2.5.11. If D is a 1-dimensional subcoalgebra of C then there exists a non-zero element $g \in C$ such that $D = \mathbf{k}g$ and $\Delta(g) = g \otimes g$. Conversely, if g is a non-zero element of C and $\Delta(g) = g \otimes g$ then $\mathbf{k}g$ is a 1-dimensional subcoalgebra of C.

Elements such as g play an important role in the theory of Hopf algebras. Because their comultiplication resembles the comultiplication of the group elements of a group coalgebra, they are said to be group-like.

Definition 2.5.12. A non-zero element g of a coalgebra C is *group-like* if $\Delta(g) = g \otimes g$. The set of group-like elements of C is denoted by $G(C)$.

Remark 2.5.13. If g is a group-like element then $g = \varepsilon(g)g$, so $\varepsilon(g) = 1$.

Remark 2.5.14. It is easy to see that $G(C)$ is a linearly independent subset of C. Moreover, the subspace $\mathbf{k}[G(C)]$ generated by $G(C)$ is contained in the coradical C_0 of C.

Remark 2.5.15. If H is a Hopf algebra then $G(H)$ is a group. Indeed, since Δ is an algebra map, we have $\Delta(1_H) = 1_H \otimes 1_H$, and, if g, $h \in G(H)$, then $\Delta(gh) = \Delta(g)\Delta(h) = gh \otimes gh$. The inverse of $g \in G(H)$ is $g^{-1} = S(g)$, since, by the defining property of the antipode, we have $gS(g) = S(g)g = \varepsilon(g)1_H = 1_H$.

Theorem 2.5.16. *Let H be a finite dimensional Hopf algebra. The following are equivalent:*

(1) H is pointed, i.e. every simple corepresentation of H is 1-dimensional.

(2) Every simple subcoalgebra of H is 1-dimensional.

(3) $H_0 = \mathbf{k}[G(H)]$.

Remark 2.5.17. The classification of finite-dimensional Hopf algebras is the subject of ongoing research. The effort is focused on two cases: 1) $H = H_0$, which is equivalent to H being semisimple, and 2) $H_0 = \mathbf{k}[G(H)]$, i.e. H is pointed. Of the two cases, the second one is the better investigated thanks to the program initiated by N. Andruskiewitsch and H.-J. Schneider in [AS98]. Their method has led, for example, to the classification of all finite dimensional pointed Hopf algebras H such that $G(H)$ is abelian and the only prime numbers that divide $|G(H)|$ are greater than 7 [AS10].

We end this section by mentioning a few more concepts that arise in the theory of pointed Hopf algebras. These will be important when we will study homomorphisms between such objects.

If C is a coalgebra then the coradical C_0 is the bottom piece of a filtration of C:

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \tag{2.16}$$

called the *coradical filtration*. The $(j+1)$-th piece is

$$C_{j+1} = \{c \in C \mid \Delta(c) \in C_j \otimes C + C \otimes C_0\}.$$

The *graded coalgebra* associated to this filtration is

$$\mathsf{gr}(C) = \bigoplus_{n \geq 0} C_n / C_{n-1},$$

where $C_{-1} = 0$.

Remark 2.5.18. If H is a pointed Hopf algebra (more generally, if H_0 is a Hopf subalgebra of H) then (2.16) is, also, an algebra filtration and gr H is a graded Hopf algebra. A Hopf algebra L isomorphic to gr H is called a *lifting of* H.

Definition 2.5.19. Let g and h be group-like elements of a coalgebra C. An element $x \in C$ is called (g, h)-*primitive* if $\Delta(x) = g \otimes x + x \otimes h$. When $(g, h) = (1, 1)$ we say that x is a *primitive element*, otherwise, x is a *skew-primitive element*.

Remark 2.5.20. The set of all (g, h)-primitive elements of C is denoted by $P_{g,h}(C)$. It is a subspace of C. Notice that $\mathbf{k}(g-h) \subseteq P_{g,h}(C)$.

Remark 2.5.21. If $f : C \to D$ is a morphism of coalgebras and $g \in C$ is a group-like element then $f(c)$ is a group-like element. Moreover, if $x \in C$ is a (g, h)-primitive element then $f(x)$ is an $(f(g), f(h))$-primitive element.

Remark 2.5.22. It can be shown that any Hopf algebra generated by group-like elements and skew-primitive elements is pointed. The converse is a conjecture posed by N. Andruskiewitsch and H.-J. Schneider in [AS00].

Conjecture 2.5.23. *Any finite dimensional pointed Hopf algebra over an algebraically closed field of characteristic zero is generated by group-like elements and skew-primitive elements.*

Remark 2.5.24. The conjecture holds in all known examples and was verified in various cases. The most general one is due to I. Angiono who showed in [Ang13] that the conjecture is true when the group of group-like elements is abelian.

A categorical way of thinking about group-like elements and skew-primitive elements is giving by the following.

Proposition 2.5.25. *Let H be a finite dimensional Hopf algebra. Then the group-like elements of H^* can be identified with the 1-dimensional representations of H. If γ, $\eta \in G(H^*)$ then*

$$P_{\gamma,\eta}(H^*)/\mathbf{k}(\gamma - \eta) \cong \mathsf{Ext}^1_{\mathsf{Rep}\, H}(\eta, \gamma).$$

Proof. It is easy to check that $\gamma \in G(H^*)$ if and only if $\gamma : H \to \mathbf{k}$ is an algebra map, hence the first assertion.

For the second one, let $0 \to \gamma \xrightarrow{i} E \xrightarrow{p} \eta \to 0$ be an extension of η by γ and let $e_1, e_2 \in E$ be such that $e_1 = i(1)$ and $p(e_2) = 1$. Then $\{e_1, e_2\}$ is a $\mathbf{k}$-basis of E such that $h \cdot e_1 = \gamma(h)e_1$ and $h \cdot e_2 = \xi(h)e_1 + \eta(h)e_2$, for all $h \in H$ and some $\xi \in H^*$. In fact, $\xi \in P_{\gamma,\eta}(H^*)$, since

$$(hl) \cdot e_2 = h \cdot (\xi(l)e_1 + \eta(l)e_2) = (\gamma(h)\xi(l) + \xi(h)\eta(l))e_1 + \eta(hl)e_2$$

implies that $\xi(hl) = \gamma(h)\xi(l) + \xi(h)\eta(l)$, for all $h, l \in H$, whence $\Delta(\xi) = \gamma \otimes \xi + \xi \otimes \eta$.

If e_2' and ξ' are such that $p(e_2') = 1$ and $h \cdot e_2' = \xi'(h)e_1 + \eta(h)e_2'$ then $\xi' - \xi \in \mathbf{k}(\gamma - \eta)$. Indeed, there exists $a \in \mathbf{k}$ such that $e_2 - e_2' = ae_1$, and action of $h \in H$ on this relation yields $\xi - \xi' = a(\gamma - \eta)$.

Similarly, the equivalence class of ξ modulo $\mathbf{k}(\gamma - \eta)$ remains the same if we pass to an extension equivalent to E. Thus, the map $\mathsf{Ext}^1_{\mathsf{Rep}(H)}(\eta, \gamma) \ni [E] \mapsto \widehat{\xi} \in P_{\gamma,\eta}(H^*)/k(\gamma - \eta)$, where $\widehat{\xi}$ denotes the class of ξ, is well defined and is easily seen to be a $\mathbf{k}$-vector space isomorphism. $\square$

2.6 Braided monoidal categories

If tensor categories should be thought of as the categorical counterparts of rings, then braided tensor categories should be thought of as the counterparts of commutative rings. The identification of the two ways of tensoring two objects is realized by a braiding.

Definition 2.6.1. Let $\mathcal{C}$ be a monoidal category. A *braiding*, or a *commutativity constraint*, on $\mathcal{C}$ is a natural isomorphism $c = \{c_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{C}}$, such that the two hexagonal diagrams

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] \arrow[dl, "a_{X,Y,Z}"'] & Z \otimes (X \otimes Y) & \\
X \otimes (Y \otimes Z) \arrow[dr, "\mathsf{id}_X \otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \arrow[ul, "a_{Z,X,Y}"'] \\
 & X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes \mathsf{id}_Y"'] &
\end{tikzcd} \tag{2.17}$$

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] \arrow[dl, "a^{-1}_{X,Y,Z}"'] & (Y \otimes Z) \otimes X & \\
(X \otimes Y) \otimes Z \arrow[dr, "c_{X,Y} \otimes \mathsf{id}_Z"'] & & & Y \otimes (Z \otimes X) \arrow[ul, "a^{-1}_{Y,Z,X}"'] \\
 & (Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[ur, "\mathsf{id}_Y \otimes c_{X,Z}"'] &
\end{tikzcd} \tag{2.18}$$

are commutative for all X, Y and Z in $\mathcal{C}$.

Definition 2.6.2. A *braided* monoidal category is a pair $(\mathcal{C}, c)$ consisting of a monoidal category $\mathcal{C}$ and a braiding c on $\mathcal{C}$.

Remark 2.6.3. When talking about a braided monoidal category we usually write $\mathcal{C}$, instead of $(\mathcal{C}, c)$, with the understanding that there is a fixed braiding on $\mathcal{C}$.

Remark 2.6.4. If c is a braiding on $\mathcal{C}$ then

$$c^{\mathsf{rev}} = \{c_{Y,X}^{-1} : X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{C}} \tag{2.19}$$

is also a braiding on $\mathcal{C}$, called the *reverse* braiding of c. The braided monoidal category $(\mathcal{C}, c^{\mathsf{rev}})$ is called the *reversed* category of $(\mathcal{C}, c)$ and is denoted by $\mathcal{C}^{\mathsf{rev}}$.

If $c^{\mathsf{rev}} = c$, that is, if $c_{Y,X} \circ c_{X,Y} = \mathsf{id}_{X \otimes Y}$, for all X, $Y \in \mathcal{C}$, then c is said to be *symmetric*. In this case, $\mathcal{C}^{\mathsf{rev}} = \mathcal{C}$ and $\mathcal{C}$ is said to be a *symmetric* (braided) monoidal category.

The following results provide examples of braidings and braided monoidal categories.

Lemma 2.6.5. *Let H be a Hopf algebra. The set of braidings on* $\mathsf{Rep}\, H$ *is in bijection with the set of elements $R = \sum R^1 \otimes R^2 \in H \otimes H$, satisfying the following conditions:*

$$\begin{aligned}
(\Delta \otimes \mathsf{id}_H)(R) &= R^{13}R^{23}, & (2.20)\\
\sum \varepsilon(R^1)R^2 &= 1, & (2.21)\\
(\mathsf{id}_H \otimes \Delta)(R) &= R^{13}R^{12}, & (2.22)\\
\sum R^1\varepsilon(R^2) &= 1, & (2.23)\\
\Delta^{cop}(h)R &= R\Delta(h), & (2.24)
\end{aligned}$$

for all $h \in H$. Above, by R^{12}, R^{13} and R^{23}, we mean $\sum R^1 \otimes R^2 \otimes 1$, $\sum R^1 \otimes 1 \otimes R^2$ and $\sum 1 \otimes R^1 \otimes R^2$, respectively. The braiding corresponding to R is:

$$c_{U,V} : U \otimes V \to V \otimes U, \quad u \otimes v \mapsto \sum R^2 v \otimes R^1 u.$$

We denote by $\mathsf{Rep}(H, R)$ *the braided monoidal category* $\mathsf{Rep}\, H$ *with braiding afforded by* R*.*

Proof. See [Mont93, Theorem 10.4.2] or [Maj95, Theorem 9.2.4]. □

Definition 2.6.6. (1) Let H be a Hopf algebra. An element $R \in H \otimes H$, satisfying conditions (2.20)-(2.24), is called a *(universal) R-matrix*, or a *quasitriangular structure*, of H.

(2) A quasitriangular Hopf algebra is a pair (H, R), formed with a Hopf algebra H and a quasitriangular structure R of H.

Remark 2.6.7. If R is an R-matrix of H then R is invertible and

$$R^{-1} = (S \otimes \mathsf{id}_H)(R).$$

Moreover, $\tau(R^{-1})$ is an R-matrix of H, where τ denotes the transposition map.

If c is the braiding of $\mathsf{Rep}\, H$ afforded by R, then c^{rev} is the braiding of $\mathsf{Rep}\, H$ afforded by $\tau(R^{-1})$. We have $c^{\mathsf{rev}} = c$ if and only if $R^{-1} = \tau(R)$. In this case we say that R is a *triangular structure*.

In finding quasitriangular structures on a Hopf algebra H, the following observation is useful.

Remark 2.6.8. If V is a vector space then the map

$$\phi_V : V \otimes V \to \mathsf{Hom}(V^*, V), \quad \phi_V(R)(p) = (p \otimes \mathsf{id})(R),$$

for all $R \in V \otimes V$ and $p \in V^*$, is injective. When V is finite dimensional, ϕ_V is an isomorphism, with inverse

$$\phi_V^{-1} : \mathsf{Hom}(V^*, V) \to V \otimes V, \quad \phi_V^{-1}(f) = \sum_i e_i \otimes f(e_i^*),$$

where $\{e_i\}$ is a basis of V and $\{e_i^*\}$ is its dual basis.

Now let H be a Hopf algebra and $R \in H \otimes H$. Then R satisfies (2.20) and (2.21) if and only if $\phi_H(R)$ is an algebra map, and, when H is finite dimensional, R satisfies (2.22) and (2.23) if and only if $\phi_H(R)$ is a coalgebra anti-homomorphism. Thus, in the finite-dimensional setting, the set of elements of $H \otimes H$ satisfying (2.20)-(2.23) is in bijection with the set of bialgebra maps $H^{*\mathrm{cop}} \to H$.

Example 2.6.9. Let G be a finite group. It was shown by A. Davydov in [Dav97] that quasitriangular structures on $\mathbf{k}[G]$ are parameterized by triples (A, B, β), where A and B are two isomorphic normal abelian subgroups of G and $\beta : \widehat{A} \times \widehat{B} \to k^\times$ is a non-degenerate, bi-multiplicative, G-invariant form. The R-matrix corresponding to (A, B, β) is

$$R_{(A,B,\beta)} = \frac{1}{|A|^2} \sum_{\substack{a \in A \\ b \in B}} \sum_{\substack{\chi \in \widehat{A} \\ \xi \in \widehat{B}}} \beta(\chi, \xi)\chi(a)\xi(b)\, a \otimes b.$$

Example 2.6.10. The R-matrices of Sweedler's Hopf algebra H_4 were described by D. Radford in [Rad93]. They are parameterized by the base field $\mathbf{k}$. The R-matrix corresponding to $\lambda \in \mathbf{k}$ is

$$R_\lambda = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) + \frac{\lambda}{2}(x \otimes x + x \otimes gx + gx \otimes gx - gx \otimes x).$$

Example 2.6.11. The R-matrices of Nichols Hopf algebra $E(n)$ were described by F. Panaite and F. van Oystaeyen in [PvO99]. They are parameterized by the set of $n \times n$ matrices with coefficients in the base field $\mathbf{k}$. For a matrix $A = (a_{ij}) \in M_n(\mathbf{k})$ and two subsets $P = \{i_1, i_2, \ldots, i_r\}$ and $F = \{j_1, j_2, \ldots, j_r\}$ of $\{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_r$, let $[A]_{P,F}$ denote the $r \times r$ minor of A found at the intersection of rows $i_1, \ldots, i_r$ with columns $j_1, \ldots, j_r$. Then the R-matrix of $E(n)$, corresponding to A, was described in [PvO99, Remark 2] as:

$$\begin{aligned} R_A = &\frac{1}{2}(1 \otimes 1 + 1 \otimes c + c \otimes 1 - c \otimes c) + \frac{1}{2} \sum_{|P|=|F|} (-1)^{\frac{|P|(|P|-1)}{2}} [A]_{P,F} \times \\ &\times (x_P \otimes c^{|P|} x_F + c x_P \otimes c^{|P|} x_F + x_P \otimes c^{|P|+1} x_F - c x_P \otimes c^{|P|+1} x_F), \end{aligned}$$

where the sum is over all non-empty subsets P and F of $\{1, \dots, n\}$. We will use the following equivalent expression for R_A:

$$R_A = \frac{1}{2}\sum_{i=0}^{n}(-1)^{\frac{i(i-1)}{2}} \sum_{|P|=|F|=i} [A]_{P,F}(x_P \otimes x_F + x_P \otimes cx_F + \\ + (-1)^i cx_P \otimes x_F + (-1)^{i+1} cx_P \otimes cx_F)$$

where the sum is over all subsets P and F of $\{1, \dots, n\}$ and the convention is that $[A]_{\emptyset,\emptyset} = 1$. Notice that for $n = 1$ we recover the R-matrices of $H_4 = E(1)$ from Example 2.6.10.

G. Carnovale and J. Cuadra have showed in [CC04b] that the quasitriangular structure R_A is triangular if and only if A is symmetric.

The notion dual to R-matrix is that of r-form. It corresponds to a braiding on $\mathsf{Corep}\, H$.

Lemma 2.6.12. *Let H be a Hopf algebra. There is a bijective correspondence between the set of braidings on $\mathsf{Corep}\, H$ and the set of linear maps $r : H \otimes H \to k$, satisfying the following conditions:*

$$\begin{aligned} r(xy, z) &= r(x, z_{(1)})r(y, z_{(2)}), && (2.25)\\ r(1, x) &= \varepsilon(x), && (2.26)\\ r(x, yz) &= r(x_{(1)}, z)r(x_{(2)}, y), && (2.27)\\ r(x, 1) &= \varepsilon(x), && (2.28)\\ x_{(1)}y_{(1)}r(y_{(2)}, x_{(2)}) &= r(y_{(1)}, x_{(1)})y_{(2)}x_{(2)}, && (2.29)\end{aligned}$$

for all x, y, $z \in H$. The braiding corresponding to r is

$$c_{U,V} : U \otimes V \to V \otimes U, \quad u \otimes v \mapsto \sum r(u_{(1)}, v_{(1)})v_{(0)} \otimes u_{(0)}.$$

We denote by $\mathsf{Corep}(H, r)$ the braided monoidal category $\mathsf{Corep}\, H$ with braiding given by r.

Definition 2.6.13. (1) Let H be a Hopf algebra. A *coquasitriangular structure*, or an *r-form*, on H is a linear map $r : H \otimes H \to \mathbf{k}$ satisfying conditions (2.25)-(2.29).

(2) A *co-quasitriangular Hopf algebra* is a pair (H, r), formed with a Hopf algebra H and an r-form r on H.

Remark 2.6.14. Let H be a finite dimensional Hopf algebra. Let $\{e_i\}$ be a basis of H and $\{e_i^*\}$ its dual basis. Then the map

$$\psi : (H \otimes H)^* \to H^* \otimes H^*, \quad \psi(r) = \textstyle\sum_{i,j} r(e_i, e_j)\, e_i^* \otimes e_j^*$$

is invertible, with inverse given by $\psi^{-1}(\alpha \otimes \beta)(x \otimes y) = \alpha(x)\beta(y)$, for all α, $\beta \in H^*$ and x, $y \in H$. It is straightforward to check that r is an r-form on H if and only if $\psi(r)$ is an R-matrix of H^*.

The following two remarks mirror Remark 2.6.7 and Remark 2.6.8.

Remark 2.6.15. If r is an r-form on H then r is convolution invertible and

$$r^{-1} = r \circ (S \otimes \mathsf{id}_H).$$

Moreover, $r^{-1}\tau$ is an r-form on H, where τ is the transposition map.

If c is the braiding of $\mathsf{Corep}\, H$ corresponding to r, then c^{rev} is the braiding of $\mathsf{Corep}\, H$ corresponding to $r^{-1}\tau$. We have $c^{\mathsf{rev}} = c$ if and only if $r^{-1} = r\tau$. In this case we say that r is a *triangular structure*.

Remark 2.6.16. If V is a vector space then the map

$$\varphi_V : (V \otimes V)^* \to \mathsf{Hom}(V, V^*), \quad \varphi(r)(v) = r(v, -)$$

for all $v \in V$, is invertible, with inverse

$$\varphi^{-1} : \mathsf{Hom}(V, V^*) \to (V \otimes V)^*, \quad \varphi^{-1}(f)(v \otimes w) = f(v)(w)$$

for all v and $w \in V$.

If H is a Hopf algebra and $r : H \otimes H \to \mathbf{k}$ is a linear map, then r satisfies (2.25) and (2.26) if and only if $\varphi_H(r)$ is an algebra homomorphism, and r satisfies (2.27) and (2.28) if and only if $\varphi_H(r)$ is a coalgebra antihomomorphism. Thus, the set of elements $r : H \otimes H \to k$ satisfying (2.25)-(2.28) is in bijection with the set of bialgebra maps $H \to H^{*\,\mathrm{cop}}$.

Example 2.6.17. Let G be a finite group. Then $\mathbf{k}[G]$ admits a co-quasitriangular structure if and only if G is abelian. In this case, restriction to $G \times G$ establishes a one-one correspondence between r-forms on $\mathbf{k}[G]$ and bicharacters of G.

Definition 2.6.18. Let $(\mathcal{C}, c)$ and $(\mathcal{D}, d)$ be two braided monoidal categories. A monoidal functor $(F, J) : \mathcal{C} \to \mathcal{D}$ is said to be *braided* if the diagram

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "d_{F(X),F(Y)}"] \arrow[d, "J_{X,Y}"'] & F(Y) \otimes F(X) \arrow[d, "J_{Y,X}"] \\ F(X \otimes Y) \arrow[r, "F(c_{X,Y})"'] & F(Y \otimes X) \end{tikzcd} \tag{2.30}$$

commutes for all $X, Y \in \mathcal{C}$.

Definition 2.6.19. A *braided monoidal equivalence* of braided monoidal categories is a braided monoidal functor which is also an equivalence of categories.

Example 2.6.20. Let (H, R_H) and (K, R_K) be two quasitriangular Hopf algebras. Consider the monoidal functor (f, T) from Lemma 2.4.10. Then (f, T) is a braided monoidal functor from $\mathsf{Rep}(H, R_H)$ to $\mathsf{Rep}(K, H_K)$ if and only if $(f \otimes f)(R_K) = \tau(T^{-1}) R_H T$.

Example 2.6.21. Let (H, r_H) and (K, r_K) be two co-quasitriangular Hopf algebras. Then the monoidal functor (f, σ) from Lemma 2.4.17 is a braided monoidal functor from $\mathsf{Corep}(H, r_H)$ to $\mathsf{Corep}(K, r_K)$ if and only if $r_K(f \otimes f)(x, y) = \sigma^{-1}(y_{(1)}, x_{(1)}) r(x_{(2)}, y_{(2)}) \sigma(x_{(3)}, y_{(3)})$, for all x, $y \in H$.

Definition 2.6.22. Let $\mathcal{C}$ be a braided tensor category with braiding c.

(1) The *centralizer* $\mathcal{D}'$ of a tensor subcategory $\mathcal{D} \subseteq \mathcal{C}$ is the full tensor subcategory of $\mathcal{C}$ consisting of all objects Y such that $c_{Y,X} \circ c_{X,Y} = \mathsf{id}_{X \otimes Y}$, for all $X \in \mathcal{C}$.

(2) The *symmetric center of* $\mathcal{C}$ is $\mathcal{Z}_{sym}(\mathcal{C}) = \mathcal{C}'$.

Example 2.6.23. If (H, R) is a quasitriangular Hopf algebra then $\mathcal{Z}_{sym}(\mathsf{Rep}(H, R)) = \mathsf{Rep}\, H_{sym}$, where H_{sym} is the following quotient Hopf algebra of H. Let $R_{21} = \tau(R)$, where τ is the transposition map, and let

$$\Phi_R : H^* \to H, \qquad \Phi_R(p) = (p \otimes \mathsf{id}_H)(R_{21}R), \qquad p \in H^*.$$

Let $K = \Phi_R(H^*)$ and $K^+ = K \cap \mathsf{Ker}(\varepsilon)$. Then $HK^+ = \{hk \mid h \in H, k \in K^+\}$ is a Hopf ideal of H, and $H_{sym} = H/HK^+$.

The fact that $\mathcal{Z}_{sym}(\mathsf{Rep}(H, R)) = \mathsf{Rep}\, H_{sym}$ was shown by S. Natale in [Nat06, Theorem 4.4].

Example 2.6.24. This example is the dual version of the previous one. The symmetric center of $\mathsf{Corep}(H, r)$, where (H, r) is a coquasitriangular Hopf algebra, is $\mathcal{Z}(\mathsf{Corep}(H, r)) = \mathsf{Corep}\, H_{sym}$, where H_{sym} is the following Hopf subalgebra of H. Let

$$\Phi_r : H \to H^*, \qquad \Phi_r(x)(y) = r(y_{(1)}, x_{(1)})r(x_{(2)}, y_{(2)}), \qquad x, y \in H.$$

Then $H_{sym} = (H^*\Phi_r(H)^+)^\perp$, where, for a subset I of H^* we denote by $I^\perp$ its annihilator, i.e. $I^\perp = \{x \in H \mid f(x) = 0 \text{ for all } f \in I\}$. Explicitly,

$$H_{sym} = \{x \in H \mid x_{(1)}r(x_{(2)}, y_{(1)})r(y_{(2)}, x_{(3)}) = \varepsilon(y)x, \text{ for all } y \in H\}, \tag{2.31}$$

Equivalently, H_{sym} consists of all $x \in H$ such that the squared braiding $c^2_{H,H} : H \otimes H \to H \otimes H$ fixes $x \otimes y$, for all $y \in H$.

Example 2.6.25. This is a particular case of the previous example. If Γ is a finite abelian group and $r : \Gamma \times \Gamma \to \mathbf{k}^\times$ is a bicharacter, then

$$\mathcal{Z}_{sym}(\mathsf{Corep}(\Gamma, r)) = \mathsf{Corep}(\Gamma^\perp, r|_{\Gamma^\perp}),$$

where $\Gamma^\perp$ is the radical of Γ relative to the bicharacter $b : \Gamma \times \Gamma \to \mathbf{k}^\times$, $b(g,h) = r(g,h)r(h,g)$, $g, h \in \Gamma$.

Remark 2.6.26. A braided tensor category is symmetric if and only if $\mathcal{Z}_{sym}(\mathcal{C}) = \mathcal{C}$. Symmetric tensor categories have been described by P. Deligne. It is shown in [Del02] that any symmetric finite tensor category is equivalent to the representation category of a finite supergroup. As explained in [AEG01], any such category can be realized as the representation category of a modified supergroup Hopf algebra $\wedge V \rtimes \mathbf{k}G$, where G is a finite group with a fixed central element u such that $u^2 = 1$ and V is a finite dimensional representation of G on which u acts by -1. The coalgebra structure of $\wedge V \rtimes \mathbf{k}G$ is determined by

$$\Delta(g) = g \otimes g,\ \varepsilon(g) = 1,\ g \in G, \qquad \Delta(v) = 1 \otimes v + v \otimes u,\ \varepsilon(v) = 0,\ v \in V,$$

and the antipode is given by $S(g) = g^{-1}$, $S(v) = -v$. This category is semisimple if and only if $V = 0$.

Any symmetric finite tensor category has a unique, up to isomorphism, super-fiber functor (i.e., a braided tensor functor to the category sVec of super vector spaces). This functor is identified with the forgetful tensor functor

$$\mathsf{Rep}(\wedge V \rtimes \mathbf{k}G) \to \mathsf{sVec}.$$

To the opposite extreme of symmetric categories are the factorizable categories.

Definition 2.6.27. A braided tensor category is $\mathcal{C}$ is *factorizable* if $\mathcal{Z}_{sym}(\mathcal{C}) \simeq \mathrm{Vec}$.

We close this section by talking about ribbon structures. These are used to construct knot invariants and 3-manifold invariants.

Definition 2.6.28. Let $\mathcal{C}$ be a braided tensor category with braiding c. A *ribbon structure on* $\mathcal{C}$ is a natural isomorphism $\theta = \{\theta_X : X \to X\}_{X \in \mathsf{Ob}(\mathcal{C})}$ such that

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}, \tag{2.32}$$

$$(\theta_X)^* = \theta_{X^*}, \tag{2.33}$$

for all $X, Y \in \mathcal{C}$.

A *ribbon* tensor category is a braided tensor category together with a *ribbon structure* on it.

Lemma 2.6.29. *Let (H, r) be a coquasitriangular Hopf algebra. The set of ribbon structures on $\mathsf{Corep}(H, r)$ is in bijection with the set of ribbon elements of (H, r), i.e. convolution invertible central elements $\alpha \in H^*$ such that $\alpha \circ S = \alpha$ and*

$$\alpha(xy) = \alpha(x_{(1)})\alpha(y_{(1)})(r_{21} * r)(x_{(2)}, y_{(2)}),$$

for all x, $y \in H$. The ribbon structure associated to the ribbon element α is

$$\theta_V : V \to V, \quad v \mapsto \sum \alpha(v_{(1)})v_{(0)}.$$

Proof. Straightforward. □

Remark 2.6.30. If (H, r) is a triangular Hopf algebra, i.e. if $r^{-1} = r_{21}$, then the set of ribbon elements of (H, r) is the set of involutive central group-like elements of H^*: $\{\alpha \in G(H^*) \cap Z(H^*) \mid \alpha^2 = \varepsilon\}$.

Example 2.6.31. Let Γ be a finite abelian group and $r : \Gamma \times \Gamma \to \mathbf{k}^\times$ a bicharacter on Γ. Then a ribbon element of $(\mathbf{k}[\Gamma], r)$ is the same thing as a function $\alpha : \Gamma \to \mathbf{k}^\times$ satisfying:

- $\alpha(g^{-1}) = \alpha(g)$, for all $g \in \Gamma$,
- $\dfrac{\alpha(g+h)}{\alpha(g)\alpha(h)} = r(h, g)r(g, h)$, for all g, $h \in \Gamma$.

Such a function is an example of a *quadratic form on* Γ (see Definition 4.1.2). The ribbon structure on $\mathsf{Corep}(\Gamma, r)$ associated to α is

$$\theta_V : V \to V, \quad v \mapsto \alpha(g)v, \quad v \in V_g.$$

If $(\mathbf{k}[\Gamma], r)$ is triangular, then the set of ribbon elements of $(\mathbf{k}[\Gamma], r)$ is the set of involutive characters of Γ: $\{\chi \in \widehat{\Gamma} \mid \chi^2 = 1\}$.

Remark 2.6.32. The ribbon elements of (H, r) can be determined in the following way (see [Rad94, Proposition 2] where the result appears in dual form). Let η be the *Drinfeld element* of (H, r), i.e. $\eta : H \to \mathbf{k}$, $\eta(h) = r(h_{(2)}, S(h_{(1)}))$, for all $h \in H$. Then η is convolution invertible, with inverse $\eta^{-1}(h) = r(S^2(h_{(2)}), h_{(1)})$, for all $h \in H$, and the element $(\eta \circ S) * \eta^{-1}$ is a group-like element of H^*. The map

$$\gamma \mapsto \gamma * \eta$$

establishes a one-to-one correspondence between the set of group-like elements $\gamma \in H^*$, satisfying $\gamma^2 = (\eta \circ S) * \eta^{-1}$ and $S^2_{H^*}(p) = \gamma^{-1} * p * \gamma$, for all $p \in H^*$, and the set of ribbon elements of (H, r).

Remark 2.6.33. Any symmetric fusion category $\mathcal{C}$ has a canonical ribbon structure θ [EGNO15, Section 9.5]. It differs from the trivial ribbon structure by a tensor automorphism of the identity endofunctor of $\mathcal{C}$. This give rise to a (possibly trivial) $\mathbb{Z}/2\mathbb{Z}$-grading:

$$\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-, \tag{2.34}$$

where $\mathcal{C}_+$ is the maximal Tannakian subcategory of $\mathcal{C}$ [Del02]. In terms of the canonical ribbon structure θ, one has $\theta_X = \pm\, \mathsf{id}_X$ when $X \in \mathcal{C}_\pm$.

Example 2.6.34. Let Γ be a finite abelian group and $r : \Gamma \times \Gamma \to \mathbf{k}^\times$ a bicharacter on Γ. If $\mathsf{Corep}(\Gamma, r)$ is symmetric then the canonical ribbon structure of $\mathsf{Corep}(\Gamma, r)$ is

$$\theta_V : V \to V, \quad v \mapsto r(g, g)v, \quad v \in V_g.$$

In this case, the objects of $\mathsf{Corep}(\Gamma, r)_+$ (respectively, of $\mathsf{Corep}(\Gamma, r)_-$) are those Γ-graded vector spaces with support contained in $\{g \in \Gamma \mid r(g, g) = 1\}$ (respectively, $\{g \in \Gamma \mid r(g, g) = -1\}$).

2.7 The center construction

An important construction in the theory of tensor categories is the center construction. It associates to a tensor category $\mathcal{C}$ a braided factorizable category $\mathcal{Z}(\mathcal{C})$. It is defined as follows.

Definition 2.7.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. The *center* of $\mathcal{C}$ is the category $\mathcal{Z}(\mathcal{C})$ constructed as follows:

- An object of $\mathcal{Z}(\mathcal{C})$ is a pair $(Z, \gamma_{-,Z})$, where Z is an object of $\mathcal{C}$ and $\gamma_{-,Z} = \{\gamma_{X,Z} : X \otimes Z \to Z \otimes X\}_{X \in \mathcal{C}}$ is a natural isomorphism, making the following diagram commutative:

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "\gamma_{X\otimes Y,Z}"] \arrow[dl, "a_{X,Y,Z}"'] & Z \otimes (X \otimes Y) & \\
X \otimes (Y \otimes Z) \arrow[dr, "\mathrm{id}_X \otimes \gamma_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \arrow[ul, "a_{Z,X,Y}"'] \\
 & X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "\gamma_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

  for all $X, Y \in \mathcal{C}$.

- A morphism $f : (Z, \gamma_{-,Z}) \to (Z', \gamma_{-,Z'})$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f : Z \to Z'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
X \otimes Z \arrow[r, "\gamma_{X,Z}"] \arrow[d, "\mathrm{id}_X \otimes f"'] & Z \otimes X \arrow[d, "f \otimes \mathrm{id}_X"] \\
X \otimes Z' \arrow[r, "\gamma_{X,Z'}"'] & Z' \otimes X
\end{tikzcd}$$

  is commutative for all $X \in \mathcal{C}$.

Remark 2.7.2. $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category. The tensor product of two objects $(Y, \gamma_{-,Y})$ and $(Z, \gamma_{-,Z})$ is

$$(Y, \gamma_{-,Y}) \otimes (Z, \gamma_{-,Z}) = (Y \otimes Z, \gamma_{-,Y\otimes Z}),$$

where $\gamma_{X,Y\otimes Z}$, for $X \in \mathcal{C}$, is the morphism defined by the following commutative diagram:

$$
\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "\gamma_{X,Y\otimes Z}"] \arrow[dl, "a^{-1}_{X,Y,Z}"'] & (Y \otimes Z) \otimes X & \\
(X \otimes Y) \otimes Z \arrow[dr, "\gamma_{X,Y}\otimes \mathsf{id}_Z"'] & & & Y \otimes (Z \otimes X) \arrow[ul, "a^{-1}_{Y,Z,X}"'] \\
& (Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[ur, "\mathsf{id}_Y \otimes \gamma_{X,Z}"'] &
\end{tikzcd}
$$

The braiding of $\mathcal{Z}(\mathcal{C})$ is given by

$$\gamma_{X,Y} : (X, \gamma_{-,X}) \otimes (Y, \gamma_{-,Y}) \to (Y, \gamma_{-,Y}) \otimes (X, \gamma_{-,X}).$$

Remark 2.7.3. There is an obvious forgetful monoidal functor

$$F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}, \quad (Z, \gamma_{-,Z}) \mapsto Z.$$

If $\mathcal{C}$ is a finite tensor category then $\mathcal{Z}(\mathcal{C})$ is a finite tensor category. In particular, given a finite dimensional Hopf algebra H, $\mathcal{Z}(\mathsf{Rep}\, H)$ is a finite tensor category. Moreover, there is a fiber functor $\tilde{F} : \mathcal{Z}(\mathsf{Rep}\, H) \to$ Vec, namely, the composition of the forgetful functors $\mathcal{Z}(\mathsf{Rep}\, H) \to \mathsf{Rep}\, H \to$ Vec. By Tannaka-Krein reconstruction, $D(H) = \mathsf{End}(\tilde{F})$ is a Hopf algebra and $\mathcal{Z}(\mathsf{Rep}\, H) \simeq \mathsf{Rep}\, D(H)$. This Hopf algebra is called the Drinfeld double of H. It can be constructed explicitly as follows.

Let H^* be the dual of H. Then H^* is an H-bimodule with the following actions:

$$h \rightharpoonup p = p_{(2)}(h)p_{(1)} \quad \text{and} \quad p \leftharpoonup h = p_{(1)}(h)p_{(2)},$$

for all $h \in H$ and $p \in H^*$.

Definition 2.7.4. The *Drinfeld double* of a finite dimensional Hopf algebra H is the Hopf algebra $D(H)$ with underlying vector space $H^* \otimes H$, product

$$(p \otimes h)(p' \otimes h') = p\big(h_{(1)} \rightharpoonup p' \leftharpoonup S^{-1}(h_{(3)})\big) \otimes h_{(2)}h' \tag{2.35}$$

and the tensor product coalgebra structure of $H^{*\,\mathrm{cop}} \otimes H$.

Remark 2.7.5. The canonical braiding of $\mathcal{Z}(\mathsf{Rep}\, H)$ corresponds to the braiding of $\mathsf{Rep}\, D(H)$ associated to the R-matrix

$$\mathcal{R} = \sum (1 \otimes e_i) \otimes (e_i^* \otimes 1),$$

where $\{e_i\}$ is a basis of H and $\{e_i^*\}$ is the dual basis.

It is well known that a representation of a Drinfeld double $D(H)$ is the same thing as a (left-right) Yetter-Drinfeld H-module. We recall this equivalent definition and the connection between the two notions.

Definition 2.7.6. Let H be a finite dimensional Hopf algebra. The category ${}_H\mathcal{YD}^H$ of finite-dimensional *(left-right) Yetter-Drinfeld H-modules* (or *crossed H-bimodules*) is the category with

- Objects: finite-dimensional vector spaces V endowed with a left H-module structure $'\cdot'$ and a right H-comodule structure δ such that:

  $$\delta(h \cdot v) = \sum h_{(2)} \cdot v_{(0)} \otimes h_{(3)} v_{(1)} S^{-1}(h_{(1)}),$$

  for all $h \in H$ and $v \in V$.

- Morphisms: linear maps that are H-linear and H-colinear.

Remark 2.7.7. The category of Yetter-Drinfeld modules is braided and monoidal. The tensor product of two objects, $V, W \in {}_H\mathcal{YD}^H$, is the vector space $V \otimes W$ with H-action and H-coaction given, respectively, by

$$\begin{aligned} h \cdot (v \otimes w) &= \sum h_{(1)} \cdot v \otimes h_{(2)} \cdot w, \\ \delta(v \otimes w) &= \sum v_{(0)} \otimes w_{(0)} \otimes w_{(1)} v_{(1)}. \end{aligned}$$

The unit object is $\mathbf{k}$ with trivial H-action and trivial H-coaction. The associativity and the left and right unit constraints are the same as for the category of vector spaces. The braiding of ${}_H\mathcal{YD}^H$ is:

$$c_{U,V} : U \otimes V \to V \otimes U, \quad u \otimes v \mapsto \sum v_{(0)} \otimes v_{(1)} \cdot u, \tag{2.36}$$

for all $u \in U$, $v \in V$ and $U, V \in {}_H\mathcal{YD}^H$.

Proposition 2.7.8. *Given a finite dimensional Hopf algebra H, there is a braided isomorphism*

$$\mathsf{Rep}(D(H), \mathcal{R}) \cong {}_H\mathcal{YD}^H.$$

Proof. Let $(F, J) : \mathsf{Rep}(D(H), \mathcal{R}) \to {}_H\mathcal{YD}^H$ be the monoidal functor defined as follows:

- If $V \in \mathsf{Rep}\, D(H)$, then $F(V) = V$, with H-action and H-coaction given, respectively, by

$$\begin{aligned} h \cdot v &= (1 \bowtie h)v, && (2.37)\\ v &\mapsto \sum_i (e_i^* \bowtie 1)v \otimes e_i, && (2.38) \end{aligned}$$

  for all $h \in H$ and $v \in V$, where $\{e_i\}$ is a basis of H and $\{e_i^*\}$ is the dual basis.

- If f is a morphism in $\mathsf{Rep}\, D(H)$ then $F(f) = f$ is a morphism in ${}_H\mathcal{YD}^H$.

- $J_{U,V} : F(U) \otimes F(V) \to F(U \otimes V), \quad u \otimes v \mapsto u \otimes v, \quad u \in U,\ v \in V.$

It is easy to check that (F, J) is well defined and is an isomorphism of braided categories. For more details see [Mont93, Proposition 10.6.16] or [Kas95, Theorem IX.5.2]. $\square$

Remark 2.7.9. There is also a notion of a *left-left Yetter-Drinfeld H-module*. Namely, such an object is a vector space V together with a left H-action $'\cdot'$ and left H-coaction δ, such that

$$\delta(h \cdot v) = \sum h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)},$$

for all $h \in H$ and $v \in V$.

Finite dimensional left-left Yetter-Drinfeld modules form a braided monoidal category denoted ${}^H_H\mathcal{YD}$. The tensor product of two objects $V, W \in {}^H_H\mathcal{YD}$ is the vector space $V \otimes W$ with H-action and H-coaction given, respectively, by

$$\begin{aligned} h \cdot (v \otimes w) &= \sum h_{(1)} \cdot v \otimes h_{(2)} \cdot w,\\ \delta(v \otimes w) &= \sum v_{(-1)} w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}. \end{aligned}$$

The braiding is given by

$$c_{U,V} : U \otimes V \to V \otimes U, \quad u \otimes v \mapsto \sum u_{(-1)} \cdot v \otimes u_{(0)},$$

for all $u \in U$, $v \in V$ and $U, V \in {}^H_H\mathcal{YD}$.

Remark 2.7.10. If G is a finite group then a left-left (or a left-right) Yetter-Drinfeld $\mathbf{k}G$-module is the same thing as a G-graded vector space $V = \oplus_{g \in G} V_g$, together with a G-module structure, such that $h \cdot V_g \subseteq V_{ghg^{-1}}$, for all $g, h \in G$. We denote ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ by ${}^G_G\mathcal{YD}$.

If G is abelian then the simple objects of ${}^G_G\mathcal{YD}$ are $\{\delta_{(g,\chi)}\}_{g \in G, \chi \in \widehat{G}}$, where $\delta_{(g,\chi)} = \mathbf{k}$ with G-action $h \cdot 1 = \chi(h)$, $h \in G$, and G-coaction $1 \mapsto g \otimes 1$. If $V \in {}^G_G\mathcal{YD}$ then

$$V = \bigoplus_{g \in G} V_g = \bigoplus_{g \in G} \bigoplus_{\chi \in \widehat{G}} V_g^{\chi} \cong \bigoplus_{g \in G, \chi \in \widehat{G}} \dim(V_g^{\chi}) \delta_{(g,\chi)},$$

where $V_g^{\chi} = \{v \in V_g \mid h \cdot v = \chi(h) v, \text{ for all } h \in G\}$.

Remark 2.7.11. A group homomorphism $\alpha : G \to G'$ induces a functor $\mathsf{ind}_{\alpha} : {}^G_G\mathcal{YD} \to {}^{G'}_{G'}\mathcal{YD}$. If $V \in {}^G_G\mathcal{YD}$, then $\mathsf{ind}_{\alpha}(V) = V$ with action $\cdot'$ and coaction δ' given by

$$h' \cdot' v = \alpha^{-1}(h') \cdot v \qquad \text{and} \qquad \delta'(v) = (\alpha \otimes \mathsf{id}) \delta(v),$$

for all $v \in V$ and $g' \in G'$.

Remark 2.7.12. If $\mathcal{C}$ is a braided monoidal category, with braiding c, then there exist canonical braided embeddings

$$\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c_{-,X}) \quad \text{and} \quad \mathcal{C}^{\mathsf{rev}} \hookrightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c^{-1}_{X,-}). \tag{2.39}$$

The intersection of the images of $\mathcal{C}$ and $\mathcal{C}^{\mathsf{rev}}$ in $\mathcal{Z}(\mathcal{C})$ is equivalent to $\mathcal{Z}_{sym}(\mathcal{C})$, the symmetric center of $\mathcal{C}$.

Example 2.7.13. Consider a finite abelian group Γ and a bicharacter r on Γ. We have $\mathcal{Z}(\mathsf{Corep}\,\Gamma) \cong {}^\Gamma_\Gamma\mathcal{YD}$. Using the description of the objects of ${}^\Gamma_\Gamma\mathcal{YD}$ given in Remark 2.7.10, the image of $\mathsf{Corep}(\Gamma, r)$ in ${}^\Gamma_\Gamma\mathcal{YD}$ consists of those Γ-graded vector spaces $V = \oplus_{g \in \Gamma} V_g$, with Γ-action given by: $h \cdot v = r(h, g) v$, for all $v \in V_g$ and $g, h \in \Gamma$.

Now $(\mathsf{Corep}(\Gamma, r))^{\mathsf{rev}} = \mathsf{Corep}(\Gamma, r^{-1} \circ \tau)$, so the image of $(\mathsf{Corep}(\Gamma, r))^{\mathsf{rev}}$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ consists of those Γ-graded vector spaces $V = \oplus_{g \in \Gamma} V_g$, with Γ-action given by: $h \cdot v = r^{-1}(g, h)v$, for all $v \in V_g$ and $g, h \in \Gamma$.

Thus, the intersection of $\mathsf{Corep}(\Gamma, r)$ and $(\mathsf{Corep}(\Gamma, r))^{\mathsf{rev}}$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ consists of those Γ-graded vector spaces $V = \oplus_{g \in \Gamma} V_g$ with Γ-action given by

$$h \cdot v = r(h, g)v = r^{-1}(g, h)v,$$

for all $v \in V_g$ and $g, h \in \Gamma$. In particular, if $V_g \neq 0$ then $r(g, h)r(h, g) = 1$, for all $h \in \Gamma$. Thus, $V = \oplus_{g \in \Gamma^{\perp}} V_g \in \mathsf{Corep}\,\Gamma^{\perp} \cong \mathcal{Z}_{sym}(\mathsf{Corep}(\Gamma, r))$ (see Example 2.6.25).

Remark 2.7.14. The braided monoidal category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ has a canonical ribbon structure θ, given by

$$\theta_V : V \to V, \qquad \theta_V(v) = \chi(g)v, \qquad v \in V_g^{\chi}.$$

2.8 Module categories

Just as modules are useful in studying algebras, so too module categories are useful in studying tensor categories.

Definition 2.8.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a finite tensor category. A (*left*) $\mathcal{C}$-*module category* is a quadruple $(\mathcal{M}, \otimes, m, l)$, consisting of a finite, $\mathbf{k}$-linear, abelian category $\mathcal{M}$, a bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, called the *action* (or *module product*) *bifunctor*, and two natural isomorphisms

$$\begin{aligned} m &= \{m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)\}_{X,Y \in \mathsf{Ob}\,\mathcal{C}, M \in \mathsf{Ob}(\mathcal{M})}, \\ l &= \{l_M : \mathbf{1} \otimes M \to M\}_{M \in \mathsf{Ob}(\mathcal{M})}, \end{aligned}$$

called the *module associativity* and *module unit constraints*, respectively, satisfying the *pentagon* and the *triangle axioms*:

$$\begin{tikzcd}
 & (X\otimes Y)\otimes(Z\otimes M) \arrow[dr, "m_{X,Y,Z\otimes M}"] & \\
((X\otimes Y)\otimes Z)\otimes M \arrow[ur, "m_{X\otimes Y,Z,M}"] \arrow[d, "a_{X,Y,Z}\otimes \mathsf{id}_M"'] & & X\otimes(Y\otimes(Z\otimes M)) \\
(X\otimes(Y\otimes Z))\otimes M \arrow[rr, "m_{X,Y\otimes Z,M}"'] & & X\otimes((Y\otimes Z)\otimes M) \arrow[u, "\mathsf{id}_X\otimes m_{Y,Z,M}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(X\otimes \mathbf{1})\otimes M \arrow[rr, "m_{X,\mathbf{1},M}"] \arrow[dr, "r_X\otimes \mathsf{id}_M"'] & & X\otimes(\mathbf{1}\otimes M) \arrow[dl, "\mathsf{id}_X\otimes l_M"] \\
 & X\otimes M &
\end{tikzcd}$$

Remark 2.8.2. There is also a notion of a *right module category*, defined in a similar way. Equivalently, a right $\mathcal{C}$-module category is the same thing as a left $\mathcal{C}^{\mathrm{op}}$-module category.

Example 2.8.3. Let G be a finite group. If H is a subgroup of G and $\psi \in \mathsf{Z}^2(H, \mathbf{k}^\times)$ then the category $\mathsf{Rep}_\psi H$, of projective representations of H with Schur multiplier ψ, is a left $\mathsf{Rep}\, G$-module category. The module product is

$$W \otimes V := Res_H^G(W) \otimes V, \qquad W \in \mathsf{Rep}\, G,\ V \in \mathsf{Rep}_\psi H,$$

where $Res_H^G : \mathsf{Rep}\, G \to \mathsf{Rep}\, H$ is the restriction functor.

Definition 2.8.4. Let $\mathcal{C}$ and $\mathcal{D}$ be finite tensor categories. A $(\mathcal{C}, \mathcal{D})$-*bimodule category* is a (locally) finite $\mathbf{k}$-linear abelian category $\mathcal{M}$ together with a left $\mathcal{C}$-module category structure $(\otimes, m, l)$, a right $\mathcal{D}$-module category structure $(\otimes, n, r)$, and a natural isomorphism

$$b = \{b_{X,M,Y} : (X \otimes M) \otimes Y \to X \otimes (M \otimes Y)\}_{X\in\mathcal{C}, M\in\mathcal{M}, Y\in\mathcal{D}}$$

called the *middle associativity constraint*, such that the diagrams

$$\begin{tikzcd}
& (X \otimes Y) \otimes (M \otimes Z) \arrow[dr, "m_{X,Y,M\otimes Z}"] & \\
((X \otimes Y) \otimes M) \otimes Z \arrow[ur, "b_{X\otimes Y,M,Z}"] \arrow[d, "m_{X,Y,M}\otimes \mathrm{id}_Z"'] & & X \otimes (Y \otimes (M \otimes Z)) \\
(X \otimes (Y \otimes M)) \otimes Z \arrow[rr, "b_{X,Y\otimes M,Z}"'] & & X \otimes ((Y \otimes M) \otimes Z) \arrow[u, "\mathrm{id}_X \otimes b_{Y,M,Z}"']
\end{tikzcd}$$

$$\begin{tikzcd}
& (X \otimes M) \otimes (W \otimes Z) \arrow[dr, "n_{X\otimes M,W,Z}"] & \\
X \otimes (M \otimes (W \otimes Z)) \arrow[ur, "b^{-1}_{X,M,W\otimes Z}"] \arrow[d, "\mathrm{id}_X \otimes n_{M,W,Z}"'] & & ((X \otimes M) \otimes W) \otimes Z \\
X \otimes ((M \otimes W) \otimes Z) \arrow[rr, "b^{-1}_{X,M\otimes W,Z}"'] & & (X \otimes (M \otimes W)) \otimes Z \arrow[u, "b^{-1}_{X,M,W}\otimes \mathrm{id}_Z"']
\end{tikzcd}$$

commute for all $X, Y \in \mathcal{C}$, $Z, W \in \mathcal{D}$ and $M \in \mathcal{M}$.

Remark 2.8.5. It can be shown that a $(\mathcal{C}, \mathcal{D})$-bimodule category is the same thing as a left $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$-module category.

Definition 2.8.6. Let $(\mathcal{M}, \otimes, m, l)$ and $(\mathcal{M}', \otimes', m', l')$ be two module categories over a finite tensor category $\mathcal{C}$. A *$\mathcal{C}$-module functor* from $\mathcal{M}$ to $\mathcal{M}'$ is a pair (F, s), where $F : \mathcal{M} \to \mathcal{M}'$ is a functor and

$$s_{X,M} : F(X \otimes M) \to X \otimes' F(M)$$

is a natural isomorphism, such that the following diagrams are commutative:

$$\begin{tikzcd}
& (X \otimes Y) \otimes' F(M) \arrow[dr, "m'_{X,Y,F(M)}"] & \\
F((X \otimes Y) \otimes M) \arrow[ur, "s_{X\otimes Y,M}"] \arrow[d, "F(m_{X,Y,M})"'] & & X \otimes' (Y \otimes' F(M)) \\
F(X \otimes (Y \otimes M)) \arrow[rr, "s_{X,Y\otimes M}"'] & & X \otimes' F(Y \otimes M) \arrow[u, "\mathrm{id}_{X'} \otimes s_{Y,M}"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(\mathbf{1} \otimes M) \arrow[rr, "s_{\mathbf{1},M}"] \arrow[dr, "F(l_M)"'] & & \mathbf{1} \otimes' F(M) \arrow[dl, "l'_{F(M)}"] \\
& F(M) &
\end{tikzcd}$$

for all $X, Y \in \mathsf{Ob}(\mathcal{C})$ and all $M \in \mathcal{M}$.

Definition 2.8.7. Let (F, s), $(G, t) : \mathcal{M} \to \mathcal{M}'$ be two $\mathcal{C}$-module functors. A *morphism of $\mathcal{C}$-module functors* from (F, s) to (G, t) is a natural transformation $\mu : F \to G$ such that the following diagram commutes for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$:

$$\begin{tikzcd}
F(X \otimes M) \arrow[r, "s_{X,M}"] \arrow[d, "\mu_{X\otimes M}"] & X \otimes' F(M) \arrow[d, "\mathsf{id}_X \otimes \mu_M"] \\
G(X \otimes M) \arrow[r, "t_{X,M}"] & X \otimes' G(M)
\end{tikzcd}$$

Remark 2.8.8. $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{M}'$ and morphisms of $\mathcal{C}$-module functors form a category, denoted by $\mathsf{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}')$.

Module categories over a finite tensor category $\mathcal{C}$ can be studied by studying algebras in $\mathcal{C}$. It can be shown (see Proposition 2.8.16) that every $\mathcal{C}$-module category is equivalent to the category of modules in $\mathcal{C}$ over an algebra A in $\mathcal{C}$.

In Remark 2.1.3 we pointed out that the notion of an algebra can be defined in any category equipped with a tensor product, a unit element and associativity and left and right unit constraints. We make that precise now.

Definition 2.8.9. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. An *algebra in $\mathcal{C}$* is a triple (A, m, u), consisting of an object A in $\mathcal{C}$ and two morphisms $m : A \otimes A \to A$ and $u : \mathbf{1} \to A$, called *multiplication* and *unit*, respectively, such that the following diagrams are commutative:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes\mathsf{id}_A"'] & & A \otimes (A \otimes A) \arrow[d, "\mathsf{id}_A \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes A \arrow[dd, "m"] & \\
\mathbf{1} \otimes A \arrow[ur, "u\otimes\mathsf{id}_A"] \arrow[dr, "l_A"'] & & A \otimes \mathbf{1} \arrow[ul, "\mathsf{id}_A \otimes u"'] \arrow[dl, "r_A"] \\
& A &
\end{tikzcd}$$

where $a_{A,A,A}$, l_A and r_A are the obvious maps.

Example 2.8.10. An algebra in **k-Vec** is a usual **k**-algebra.

Example 2.8.11. Let B be a bialgebra. Then an algebra in $\mathsf{Rep}\, B$ is a finite dimensional $\mathbf{k}$-algebra (A, m, u) together with a B-action, such that m and u are B-linear, that is

$$\begin{aligned} b \cdot (xy) &= \sum (b_{(1)} \cdot x)(b_{(2)} \cdot y), \\ b \cdot 1_A &= \varepsilon(b) 1_A, \end{aligned}$$

for all $b \in B$ and $x, y \in A$. Algebras in $\mathsf{Rep}\, B$ are called (*left*) B-*module algebras*.

Example 2.8.12. Let B be a bialgebra. Then an algebra in $\mathsf{Corep}\, B$ is a finite dimensional $\mathbf{k}$-algebra (A, m, u) together with a B-coaction δ, such that m and u are B-colinear, that is

$$\begin{aligned} \delta(xy) &= \sum x_{(0)} y_{(0)} \otimes x_{(1)} y_{(1)}, \\ \delta(1_A) &= 1_A \otimes 1_B, \end{aligned}$$

for all $x, y \in A$. Algebras in $\mathsf{Corep}\, B$ are called (*right*) B-*comodule algebras*.

Example 2.8.13. If $\mathcal{C}$ is a monoidal category and $X \in \mathcal{C}$ has a left dual X^*, then $X \otimes X^*$ is an algebra in $\mathcal{C}$ with multiplication $m = \mathsf{id}_X \otimes \mathsf{ev}_X \otimes \mathsf{id}_{X^*}$ and unit $u = \mathsf{coev}_X$ (notice that we have suppressed the associativity and the left and right unit constraints). Similarly, if X has a right dual ${}^*X$ then ${}^*X \otimes X$ is an algebra in $\mathcal{C}$ with multiplication $m = \mathsf{id}_{{}^*X} \otimes \mathsf{ev}'_X \otimes \mathsf{id}_X$ and unit $u = \mathsf{coev}'_X$.

Accompanying the notion of an algebra A in $\mathcal{C}$ is the notion of A-module in $\mathcal{C}$.

Definition 2.8.14. Let (A, m, u) be an algebra in a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$. Then the category $\mathcal{C}_A$, of (*right*) A-*modules in* $\mathcal{C}$, is the category with:

(1) Objects: Pairs (M, ρ), where M is an object of $\mathcal{C}$ and $\rho : M \otimes A \to M$ is a morphism in $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd}[column sep=small]
M \otimes (A \otimes A) \arrow[rr, "a^{-1}_{M,A,A}"] \arrow[d, "\mathsf{id}_M \otimes m"'] & & (M \otimes A) \otimes A \arrow[d, "\rho \otimes \mathsf{id}_A"] \\
M \otimes A \arrow[dr, "\rho"'] & & M \otimes A \arrow[dl, "\rho"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes \mathbf{1} \arrow[r, "\mathsf{id}_M \otimes u"] & M \otimes A \arrow[d, "\rho"] \\
M \arrow[u, "r^{-1}_M"] \arrow[r, "\mathsf{id}_M"] & M
\end{tikzcd}$$

(2) Morphisms: $f : (M, \rho_M) \to (N, \rho_N)$ is a *morphism of A-modules in $\mathcal{C}$* if $f : M \to N$ is a morphism in $\mathcal{C}$ such that the following diagram is commutative:

$$\begin{tikzcd}
M \otimes A \arrow[r, "f \otimes \mathsf{id}_A"] \arrow[d, "\rho_M"'] & N \otimes A \arrow[d, "\rho_N"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

Proposition 2.8.15. *Let A be an algebra in a monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$. Then the tensor product, the associativity constraint and the left unit constraint induce a $\mathcal{C}$-module structure on $\mathcal{C}_A$. More precisely, the module product bifunctor is*

$$\mathcal{C} \times \mathcal{C}_A \to \mathcal{C}_A, \quad (X, (M, \rho)) \mapsto (X \otimes M, (\mathsf{id}_X \otimes \rho) a_{X,M,A})$$

the module associativity is $\{a_{X,Y,M}\}_{X,Y \in \mathcal{C}, M \in \mathcal{C}_A}$, and the module unit constraint is $\{l_M\}_{M \in \mathcal{C}_A}$.

Proof. Straightforward. □

Proposition 2.8.16. *Let $\mathcal{C}$ be a finite tensor category. If A is an algebra in $\mathcal{C}$ then $\mathcal{C}_A$ is a finite $\mathbf{k}$-linear abelian $\mathcal{C}$-module category. Conversely, if $\mathcal{M}$ is a finite $\mathbf{k}$-linear abelian $\mathcal{C}$-module category then there exists an algebra A in $\mathcal{C}$ such that $\mathcal{M} \cong \mathcal{C}_A$ as $\mathcal{C}$-module categories.*

Proof. See [Ost03, Theorem 3.1] and [EO04, Theorem 3.17]. □

Definition 2.8.17. Two algebras, A and B, in a finite tensor category $\mathcal{C}$ are *Morita equivalent* if and only if $\mathcal{C}_A$ and $\mathcal{C}_B$ are isomorphic as left $\mathcal{C}$-module categories.

Remark 2.8.18. When $\mathcal{C} = \mathbf{k}$-Vec we recover the classical notion of Morita equivalence for $\mathbf{k}$-algebras.

The class of module categories most useful to study is that formed by exact module categories. These should be thought as the counterparts of projective modules from ring theory.

Definition 2.8.19. Let $\mathcal{C}$ be a finite tensor category.

1. A $\mathcal{C}$-module category $\mathcal{M}$ is *exact* if, for any projective object $P \in \mathcal{C}$ and any object $M \in \mathcal{M}$, the object $P \otimes M$ is projective in $\mathcal{M}$.
2. An algebra A in $\mathcal{C}$ is *exact* if $\mathcal{C}_A$ is exact.

Example 2.8.20. A $\mathbf{k}$-algebra A is exact, as an algebra in $\mathbf{k}$-Vec, if and only if A is semisimple. Indeed, since $\mathbf{k}$ is a projective object of $\mathbf{k}$-Vec, $(\mathbf{k}\text{-Vec})_A = \mathsf{Rep}\, A^{\mathrm{op}}$ is exact if and only if every A^{op}-module $M = \mathbf{k} \otimes M$ is projective.

2.9 2-cocycles and Galois objects

We will see in the next section that tensor functors between representation, respectively corepresentation, categories of Hopf algebras are completely determined, up to monoidal natural isomorphism, by morphisms and twists, respectively 2-cocycles, of Hopf algebras. Describing the latter turns out to be an involved task. One way to achieve this is with the help of Galois objects. We present in this section left 2-cocycles and discuss their connection to Galois objects.

We begin by recalling the definition of right 2-cocycles and introducing their left counterpart.

Definition 2.9.1. Let H be a Hopf algebra.

(1) A *right* 2*-cocycle* on H is a convolution invertible map $\sigma : H \otimes H \to k$ such that $\sigma(1, x) = \sigma(x, 1) = \varepsilon(x)$, for all $x \in H$, and

$$\sigma(x_{(1)}y_{(1)}, z)\,\sigma(x_{(2)}, y_{(2)}) = \sigma(x, y_{(1)}z_{(1)})\,\sigma(y_{(2)}, z_{(2)})$$

for all $x, y, z \in H$.

(1) A *left* 2-*cocycle* on H is a convolution invertible linear map $\sigma : H \otimes H \to k$ such that $\sigma(1, x) = \sigma(x, 1) = \varepsilon(x)$, for all $x \in H$, and

$$\sigma(x_{(1)}, y_{(1)})\, \sigma(x_{(2)} y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\, \sigma(x, y_{(2)} z_{(2)})$$

for all x, y, $z \in H$.

Remark 2.9.2. The set of right 2-cocycles on H is denoted by $\mathsf{Z}^2_r(H)$, while the set of left 2-cocycles on H, by $\mathsf{Z}^2_l(H)$. If $\sigma : H \otimes H \to \mathbf{k}$ is a convolution invertible map then $\sigma \in \mathsf{Z}^2_r(H)$ if and only if $\sigma^{-1} \in \mathsf{Z}^2_l(H)$. Thus, it suffices to know, and work with, only one type of 2-cocycles.

Gauge equivalence for left 2-cocycles is as follows (compare with Definition 2.4.30):

Definition 2.9.3. Two left 2-cocycles σ and σ' are *gauge equivalent* if and only if there exists a convolution invertible map $\alpha : H \to \mathbf{k}$ such that $\alpha(1_H) = 1_{\mathbf{k}}$ and

$$\sigma'(x, y) = \alpha^{-1}(x_{(1)}) \alpha^{-1}(y_{(1)}) \sigma(x_{(2)}, y_{(2)}) \alpha(x_{(3)} y_{(3)})$$

for all x, $y \in H$.

The condition for a left 2-cocycle to be invariant is the same as for right 2-cocycles (see Definition 2.4.22).

Definition 2.9.4. A left 2-cocycle σ on H is *invariant* if and only if

$$\sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} = x_{(1)} y_{(1)} \sigma(x_{(2)}, y_{(2)}) \tag{2.40}$$

for all x, $y \in H$.

Remark 2.9.5. An invariant left 2-cocycle is an invariant right 2-cocycle, and vice versa. The set of invariant 2-cocycles of H is denoted by $\mathsf{Z}^2_{\mathsf{inv}}(H)$. It is a group with convolution product.

Remark 2.9.6. If H is a cocommutative Hopf algebra then $\mathsf{Z}^2_l(H) = \mathsf{Z}^2_r(H) = \mathsf{Z}^2_{\mathsf{inv}}(H)$. For example, if G is a group then the notions of right and left 2-cocycles on $\mathbf{k}[G]$ coincide. In this case, the restriction to $G \times G$ of a 2-cocycle on $\mathbf{k}[G]$ is a usual 2-cocycle on G. Thus, $\mathsf{Z}^2_{\mathsf{inv}}(\mathbf{k}[G]) \cong \mathsf{Z}^2(G, \mathbf{k}^\times)$.

Remark 2.9.7. If $\alpha : H \to \mathbf{k}$ is a convolution invertible map such that $\alpha(1_H) = 1_{\mathbf{k}}$ and $x_{(1)}\alpha(x_{(2)}) = \alpha(x_{(1)})x_2$, for all $x \in H$, then $\partial(\alpha) : H \otimes H \to \mathbf{k}$ defined by

$$\partial(\alpha)(x) = \alpha^{-1}(x_{(1)})\alpha^{-1}(y_{(1)})\alpha(x_{(2)}y_{(2)}), \quad x, y \in H$$

is an invariant 2-cocycle on H. Such 2-cocycles are called 2-*coboundaries*. The set of 2-coboundaries is a central subgroup of $\mathsf{Z}^2_{\mathsf{inv}}(H)$, denoted by $\mathsf{B}^2_{\mathsf{inv}}(H)$.

Remark 2.9.8. Two invariant 2-cocycles σ and τ are gauge equivalent if and only if there exists a 2-coboundary $\partial(\alpha)$ such that $\sigma = \partial(\alpha)\tau$. We also say, in this case, that σ and τ are *cohomologous*.

Definition 2.9.9. The *second invariant cohomology group of* H is the quotient group

$$\mathsf{H}^2_{\mathsf{inv}}(H) = \mathsf{Z}^2_{\mathsf{inv}}(H) / \mathsf{B}^2_{\mathsf{inv}}(H)$$

Remark 2.9.10. The second invariant cohomology group was introduced by P. Schauenburg in [Sch02] in his generalization of Kac's exact sequence [Kac68]. J. Bichon and G. Carnovale have given a comprehensive study of the group in [BC06]. Their motivation came, primarily, from the study of the biGalois group $\mathsf{BiGal}(H)$ of a Hopf algebra H, but also from the interplay of invariant cohomology with Brauer groups and projective representations. If the elements of $\mathsf{BiGal}(H)$ can be thought of as isomorphism classes of $\mathbf{k}$-linear monoidal autoequivalences of $\mathsf{Corep}\, H$, then Bichon and Carnovale showed in [BC06, Theorem 3.8] that the elements of $\mathsf{H}^2_{\mathsf{inv}}(H)$ can be identified with those classes of autoequivalences that are isomorphic, as functors, with the identity functor.

As we said at the beginning, 2-cocycles can be studied by studying Galois objects.

Definition 2.9.11. Let H be a Hopf algebra. A *right* H*-Galois object* is a non-zero right H-comodule algebra A such that $\{a \in A \mid \delta(a) = a \otimes 1_H\} = \mathbf{k}1_A$, and the following composition is bijective:

$$A \otimes A \xrightarrow{\mathsf{id}_A \otimes \delta} A \otimes A \otimes H \xrightarrow{m_A \otimes \mathsf{id}_H} A \otimes H$$

A *morphism of right* H*-Galois objects* is an algebra map which is right H-colinear.

Remark 2.9.12. The set of isomorphism classes of right H-Galois objects is denoted by $\mathsf{Gal}(H)$.

Every 2-cocycle gives rise to a Galois object in the following way. If σ is a left 2-cocycle on a Hopf algebra H, then the vector space H, with the product

$$x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}, \qquad x, y \in H,$$

and right coaction given by the comultiplication Δ of H, is a right H-Galois object and it is denoted by ${}_\sigma H$. If σ' is another left 2-cocycle on H then ${}_{\sigma'}H$ is isomorphic to ${}_\sigma H$ if and only if σ' and σ are gauge equivalent.

The H-Galois objects of the type ${}_\sigma H$ are the ones with the *normal basis property*, i.e. those that are isomorphic to H as H-comodules. If A is a right H-Galois object with the normal basis property and if $\psi : H \to A$ is a right H-colinear isomorphism with $\psi(1) = 1$, then $A \cong {}_\sigma H$, where

$$\sigma(x, y) = \varepsilon\Big(\psi^{-1}\big(\psi(x)\psi(y)\big)\Big), \qquad x, y \in H$$

It is known that all H-Galois objects are cleft if either H is finite dimensional [KC76] or H is pointed [G99]. In these cases, we see, from this discussion, that the set of gauge equivalence classes of 2-cocycles on H is in bijection with $\mathsf{Gal}(H)$.

Example 2.9.13. The set $\mathsf{Gal}(E(n))$ was described in [PvO00]. Using this description, invariant 2-cocycles on $E(n)$ were studied in [CC04b] and [BC06]. It was shown in [BC06] that $\mathsf{H}^2_{\mathsf{inv}}(E(n)) \cong \mathsf{Sym}_n(\mathbf{k})$, the additive group of $n \times n$ symmetric matrices with entries in $\mathbf{k}$. The cohomology class corresponding to $M = (m_{ij}) \in \mathsf{Sym}_n(\mathbf{k})$ is represented by the invariant 2-cocycle $\sigma_M : E(n) \otimes E(n) \to k$ defined by:

$$\sigma_M(c \otimes c) = 1, \quad \sigma_M(x_i \otimes x_j) = m_{ij}, \quad i, j = 1, \dots, n,$$

$$\sigma_M(x_P \otimes x_Q) = \sigma_M(cx_P \otimes x_Q) = (-1)^{|P|}\sigma_M(x_P \otimes cx_Q) = (-1)^{|P|}\sigma_M(cx_P \otimes cx_Q),$$

for all P, $Q \subseteq \{1, \dots, n\}$, $\sigma_M(x_P \otimes x_Q) = 0$ if $|P| \neq |Q|$, and some recurrence formula allowing to compute $\sigma_M(x_P, x_Q)$ when $|P| = |Q|$.

Given a left 2-cocycle σ on a Hopf algebra H, we can deform the multiplication of H to obtain a new Hopf algebra H^σ. This procedure was introduced by Y. Doi [Doi93].

Definition 2.9.14. Let σ be a left 2-cocycle on a Hopf algebra H. The *σ-deformation of H* is the Hopf algebra H^σ with the coalgebra structure of H and the product

$$x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} \sigma^{-1}(x_{(3)}, y_{(3)}), \qquad x, y \in H.$$

Remark 2.9.15. A left 2-cocycle σ on H is invariant if and only if the σ-deformation does not change the multiplication of H; in other words, if $H^\sigma = H$ as Hopf algebras.

Remark 2.9.16. Let σ be a left 2-cocycle on H. If r is an r-form on H, then $r^\sigma = (\sigma\tau) * r * \sigma^{-1}$, is an r-form on H^σ. Explicitly,

$$r^\sigma(x, y) = \sigma(y_{(1)}, x_{(1)}) r(x_{(2)}, y_{(2)}) \sigma^{-1}(x_{(3)}, y_{(3)}), \qquad x, y \in H. \tag{2.41}$$

We close this section by considering the dual version of 2-cocycles, namely twists. We mention only those results which will be used later.

Definition 2.9.17. Let H be a Hopf algebra.

(1) A *right twist on H* is an invertible element $T \in H \otimes H$ such that $(\varepsilon \otimes \mathsf{id})(T) = (\mathsf{id} \otimes \varepsilon)(T) = 1$ and

$$(\Delta \otimes \mathsf{id})(T)(T \otimes 1) = (\mathsf{id} \otimes \Delta)(T)(1 \otimes T).$$

(2) A *left twist on H* is an invertible element $T \in H \otimes H$ such that $(\varepsilon \otimes \mathsf{id})(T) = (\mathsf{id} \otimes \varepsilon)(T) = 1$ and

$$(T \otimes 1)(\Delta \otimes \mathsf{id})(T) = (1 \otimes T)(\mathsf{id} \otimes \Delta)(T).$$

(3) A left, or right, twist T is *invariant* if $T\Delta(x) = \Delta(x)T$, for all $x \in H$.

Remark 2.9.18. Let H be a finite dimensional Hopf algebra. Consider the invertible map:

$$\psi : (H \otimes H)^* \to H^* \otimes H^*, \quad \psi(\sigma) = \sum_{i,j} \sigma(e_i, e_j)\, e_i^* \otimes e_j^*,$$

where $\{e_i\}$ be a basis of H and $\{e_i^*\}$ is the dual basis. It is easy to check that σ is a left (right) 2-cocycle on H if and only if $\psi(\sigma)$ is a left (right) twist on H^*.

Remark 2.9.19. An invariant left twist is the same thing as an invariant right twist. We call these, simply, invariant twists. Just as with cocycles, we can define a second invariant cohomology group of H by considering cohomology classes of invariant twists. By the previous remark, this group is $\mathsf{H}^2_{\mathsf{inv}}(H^*)$.

The following is the twist analogue of the σ-deformation.

Definition 2.9.20. Let T be a left twist on a Hopf algebra H. The *twist deformation of H by T* is the Hopf algebra H^T with the algebra structure of H and the comultiplication

$$\Delta^T(x) = T\Delta(x)T^{-1}, \qquad x \in H.$$

Remark 2.9.21. If T is a left twist on H and R is an R-matrix on H then $R^T = \tau(T)RT^{-1}$ is an R-matrix on H^T.

2.10 Autoequivalences of finite tensor categories

To better understand tensor categories it is important to know their symmetries. In the case of the (co)-representation categories of Hopf algebras, these symmetries are determined by morphisms and twists (2-cocycles) of Hopf algebras. We make this precise in the present section and provide examples of tensor auto-equivalences of tensor categories. The results presented here are based on the exposition of A. Davydov in [Dav10].

Given a finite tensor category $\mathcal{C}$ we denote by $\mathsf{Aut}^{\otimes}(\mathcal{C})$ the group of monoidal isomorphism classes of tensor auto-equivalences of $\mathcal{C}$.

Recall from Lemma 2.4.10 that, if $f : K \to H$ is an algebra map between two Hopf algebras and $T \in H \otimes H$ is a right twist of H satisfying

$$(f \otimes f)\Delta(x) = T^{-1}\Delta\big(f(x)\big)T, \tag{2.42}$$

for all $x \in K$, then $(f, T) : \mathsf{Rep}\, H \to \mathsf{Rep}\, K$ is a tensor functor.

We can rephrase this in a slightly different form. Notice that T^{-1} is a left twist of H and $\Delta^{T^{-1}}(x) = T^{-1}\Delta\big(x\big)T$, $x \in H$, is the comultiplication of the twist deformation $H^{T^{-1}}$. Condition (2.42) is then equivalent to requiring $f : K \to H^{T^{-1}}$ to be a coalgebra map.

The following concept was introduced and studied by A. Davydov in [Dav10].

Definition 2.10.1. Let H and K be Hopf algebras. A *twisted homomorphism* from K to H is a pair (T, f), where T is a right twist of H and $f : K \to H^{T^{-1}}$ is a Hopf algebra map.

We see from the above discussion that every twisted homomorphism $(T, f) : H \to K$ gives rise to a tensor functor $(f, T) : \mathsf{Rep}\, H \to \mathsf{Rep}\, K$.

Any tensor functor $\mathsf{Rep}\, H \to \mathsf{Rep}\, K$ is monoidal isomorphic to a functor (f, T). To see this, let us assume for now that a tensor functor $(F, J) : \mathsf{Rep}\, H \to \mathsf{Rep}\, K$ preserves dimensions. Then, it follows from Proposition 2.3.28, that F is isomorphic to Res_f, for some algebra map $f : K \to H$. Using Remark 2.4.24, there exists a monoidal structure J' on Res_f such that (F, J) is natural monoidal isomorphic to (Res_f, J'). According to Lemma 2.4.10, $J' = J^T$, for some right twist T on H, satisfying $(f \otimes f)\Delta(x) = T^{-1}\Delta(f(x))T$, for all $x \in K$. Thus, $(F, J) \cong (f, T)$.

We have proved the following:

Proposition 2.10.2. *Let H and K be two Hopf algebras. If $(F, J) : \mathsf{Rep}\, H \to \mathsf{Rep}\, K$ is a tensor functor then there exists a twisted homomorphism $(T, f) : K \to H$ such that $(F, J) \cong (f, T)$.*

Remark 2.10.3. The fact that a tensor functor $F : \mathsf{Rep}\, H \to \mathsf{Rep}\, K$ preserves dimensions follows from a more general fact in the theory of tensor categories. Namely, any object X in a finite tensor category $\mathcal{C}$ has a Frobenius-Peron dimension. It is defined as the largest non-negative real eigenvalue of the matrix $N = (N_{ij})_{1\le i,j\le n}$, where N_{ij}, $j = 1, \dots, n$, are the multiplicities with which the simple objects of $\mathcal{C}$, $X_1, \dots, X_n$, appear in the decomposition of $X \otimes X_i$: $X \otimes X_i \cong \sum_j N_{ij} X_j$. If V is a finite-dimensional representation of a finite dimensional Hopf algebra H then its Frobenius-Perron dimension equals its dimension as a vector space. It can be shown that tensor functors between finite tensor categories preserve Frobenius-Perron dimensions (see [EGNO15, Proposition 4.5.7]).

Since we are interested in equivalences of tensor categories, we can ask when is (f, T) an equivalence. The following should come as no surprise.

Proposition 2.10.4. *Let H and K be Hopf algebras and $(f, T) : \mathsf{Rep}\, H \to \mathsf{Rep}\, K$ a tensor functor. Then (f, T) is an equivalence if and only if $f : K \to H^{T^{-1}}$ is an isomorphism.*

Proof. Straightforward. $\square$

Corollary 2.10.5. *If $\mathsf{Rep}\, H$ and $\mathsf{Rep}\, K$ are tensor equivalent then $K \cong H^{T^{-1}}$, for some right twist T on H.*

Let $\mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H)$ be the set of *twisted automorphisms* of H, i.e. the set of those twisted homomorphisms $(T, f) : H \to H$, with f an invertible homomorphism from H to $H^{T^{-1}}$. There is a group operation on $\mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H)$, namely

$$(T', f')(T, f) = (T'(f' \otimes f')(T), f'f), \qquad (T, f), (T', f') \in \mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H).$$

The inverse of (T, f) is $((f^{-1} \otimes f^{-1})(T^{-1}), f^{-1})$ and the identity element is $(1 \otimes 1, \mathsf{id}_H)$.

It follows from Proposition 2.10.2 and Proposition 2.10.4 that there is a surjective group homomorphism

$$\varphi : \mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H) \to \mathsf{Aut}^{\otimes}(\mathsf{Rep}\, H), \qquad (T, f) \mapsto \widehat{(f, T)}^{-1},$$

where $\widehat{(f, T)}$ denotes the isomorphism class of (f, T).

The kernel of φ is given as follows. Let $H^{\times}_{\varepsilon}$ be the set of invertible elements $u \in H$ with $\varepsilon(u) = 1$. Then there is an injective group homomorphism

$$\partial : H_\varepsilon^\times \to \mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H), \qquad \partial(u) = (\Delta(u)u^{-1} \otimes u^{-1},\, {}^u(-)),$$

where ${}^u(-) : H \to H$ is the conjugation automorphism: ${}^u(x) = uxu^{-1}$, $x \in H$. The kernel of φ coincides with the image of ∂.

Thus, we have the following description of the group of symmetries of the tensor category $\mathsf{Rep}\, H$:

Proposition 2.10.6. *The sequence*

$$1 \to H_\varepsilon^\times \xrightarrow{\partial} \mathsf{Aut}^{tw}_{Hopf}(H) \xrightarrow{\varphi} \mathsf{Aut}^{\otimes}(\mathsf{Rep}\, H) \to 1$$

is a short exact sequence of groups.

Corollary 2.10.7. *If H is a commutative finite dimensional Hopf algebra then*

$$\mathsf{Aut}^{\otimes}(\mathsf{Rep}\, H) \cong \mathsf{H}^2_{\mathsf{inv}}(H^*) \rtimes \mathsf{Aut}_{Hopf}(H).$$

Proof. By Proposition 2.10.6, $\mathsf{Aut}^{\otimes}(\mathsf{Rep}\, H)$ is isomorphic to the quotient group $\mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H)/\partial(H_\varepsilon^\times)$. Since H is commutative, the elements of $\partial(H_\varepsilon^\times)$ have the form $(\Delta(u)(u^{-1} \otimes u^{-1}), \mathsf{id}_H)$, with $u \in H_\varepsilon^\times$. Moreover, any twist of H is an invariant twist, and, if $(T, f) \in \mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H)$ then

$$\partial(H_\varepsilon^\times)(T, f) = \{(\Delta(u)(u^{-1} \otimes u^{-1})T, f) \mid u \in H_\varepsilon^\times\}.$$

Thus, the map

$$\mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H)/\partial(H_\varepsilon^\times) \to \mathsf{H}^2_{\mathsf{inv}}(H^*) \times \mathsf{Aut}_{\text{Hopf}}(H), \qquad \partial(H_\varepsilon^\times)(T, f) \mapsto (\widehat{T}, f),$$

is well defined and is a bijection. The same map is easily seen to be a group homomorphism between $\mathsf{Aut}^{\text{tw}}_{\text{Hopf}}(H)/\partial(H_\varepsilon^\times)$ and the semidirect product $\mathsf{H}^2_{\mathsf{inv}}(H^*) \rtimes \mathsf{Aut}_{\text{Hopf}}(H)$ with respect to the left action of $\mathsf{Aut}_{\text{Hopf}}(H)$ on $\mathsf{H}^2_{\mathsf{inv}}(H^*)$: $f \triangleright \widehat{T} = \widehat{(f \otimes f)(T)}$. $\square$

Remark 2.10.8. There are two group homomorphisms:

$$\begin{aligned} \mathsf{Aut}_{\mathrm{Hopf}}(H) &\to \mathsf{Aut}^{\mathrm{tw}}_{\mathrm{Hopf}}(H), \quad f \mapsto (1 \otimes 1, f), \\ \mathsf{H}^2_{\mathsf{inv}}(H^*) &\to \mathsf{Aut}^{\mathrm{tw}}_{\mathrm{Hopf}}(H), \quad \widehat{T} \mapsto (T, \mathsf{id}_H). \end{aligned}$$

Composing these with the map $\varphi : \mathsf{Aut}^{\mathrm{tw}}_{\mathrm{Hopf}}(H) \to \mathsf{Aut}^{\otimes}(\mathsf{Rep}\, H)$, we obtain the following group homomorphisms:

$$\begin{aligned} \mathsf{Aut}_{\mathrm{Hopf}}(H) &\to \mathsf{Aut}^{\otimes}(\mathsf{Rep}\, H), \quad f \mapsto (\widehat{f^{-1}, 1 \otimes} 1), \\ \mathsf{H}^2_{\mathsf{inv}}(H^*) &\to \mathsf{Aut}^{\otimes}(\mathsf{Rep}\, H), \quad \widehat{T} \mapsto (\mathsf{id}\widehat{_H, T^{-}}^1). \end{aligned}$$

If (H, R_H) and (K, R_K) are quasitriangular Hopf algebras, a natural question to ask is when $(f, T) : \mathsf{Rep}(H, R_H) \to \mathsf{Rep}(K, R_K)$ is a braided tensor functor. The answer is given by the following:

Proposition 2.10.9. *The tensor functor $(f, T) : \mathsf{Rep}(H, R_H) \to \mathsf{Rep}(K, R_K)$ is braided if and only if $f : (K, R_K) \to (H^{T^{-1}}, R_H^{T^{-1}})$ is a morphism of quasitriangular Hopf algebras.*

Proof. It follows from Example 2.6.20. □

For future reference, we end this section by providing the dual versions of some of the previous results.

Proposition 2.10.10. *Let H and K be two Hopf algebras. If $(F, J) : \mathsf{Corep}\, H \to \mathsf{Corep}\, K$ is a tensor functor then:*

(1) There exist a left 2-cocycle σ on H and a Hopf algebra map $f : H^\sigma \to K$ such that $(F, J) \cong (f, \sigma^{-1})$.

(2) F is an equivalence if and only if f is an isomorphism.

(3) If r_H and r_K are r-forms on H and K, respectively, then (F, J) is a braided functor from $\mathsf{Corep}(H, r_H)$ to $\mathsf{Corep}(K, r_K)$ if and only if $f : (H^\sigma, r_H^\sigma) \to (K, r_K)$ is a morphism of coquasitriangular Hopf algebras.

Corollary 2.10.11. *If $\mathsf{Corep}\, H$ and $\mathsf{Corep}\, K$ are tensor equivalent then $K \cong H^\sigma$, for some left 2-cocycle σ on H.*

Corollary 2.10.12. *If G is a finite group then $\mathsf{Aut}^{\otimes}(\mathsf{Corep}\, G) \cong \mathsf{H}^2(G, \mathbf{k}^\times) \rtimes \mathsf{Aut}(G)$.*

Remark 2.10.13. The previous two results were proved for the first time by P. Schauenburg in [Sch96], using the concept of BiGalois object.

CHAPTER 3

THE BRAUER-PICARD GROUP OF A FINITE SYMMETRIC TENSOR CATEGORY

In this chapter we compute the Brauer-Picard group of the representation category of Nichols Hopf algebra $E(n)$. If $\mathcal{C}_n = \mathsf{Rep}(E_n)$, we prove in Theorem 3.8.6 that

$$\mathsf{BrPic}(\mathcal{C}_n) \cong \mathsf{PSp}_n(\mathbf{k}) \times \mathbb{Z}/2\mathbb{Z},$$

where $\mathsf{PSp}_n(\mathbf{k})$ is the projective symplectic group of degree $2n$.

Our method relies on a canonical representation of the group of braided autoequivalences of $\mathcal{Z}(\mathcal{C}_n)$ on the space of extensions of two simple objects, as well as on a description of the group of braided autoequivalences of $\mathcal{Z}(\mathcal{C}_n)$ trivializable on $\mathcal{C}_n$ with a symmetric braiding, obtained in [CC04b].

The material is organized as follows.

In Section 3.1 we define the Brauer group of a braided finite tensor category based on the more general construction of [vOZ98]. At the end we give a short review of the literature on the computation of Brauer groups of Hopf algebras.

In Section 3.2 we define the Brauer-Picard group of a finite tensor category and the Picard group of a braided finite tensor category. The latter is a subgroup of the Brauer-Picard group and it coincides with the Brauer group defined in Section 3.1.

In Section 3.3 we review the properties of Nichols Hopf algebra $E(n)$. We recall the description of the quasitriangular structures, the invariant 2-cocycles and invariant twist from Chapter 2, and describe its Drinfeld double.

In Section 3.4 we show that $\mathcal{Z}(\mathcal{C}_n)$ has precisely two invertible objects, ε and χ, and the group of braided tensor autoequivalences of $\mathcal{Z}(\mathcal{C}_n)$ acts projectively on the space $\mathsf{Ext}^1(\chi, \varepsilon)$ of extensions of χ by ε. We prove that this space is $2n$-dimensional.

In Section 3.5 we show that the action of Section 3.4 preserves a symplectic form on $\mathbf{k}^{2n}$. This allows us to view the Brauer-Picard group of $\mathcal{C}_n$ as a symplectic group.

In Section 3.6 we show that the sets of subcategories of $\mathcal{Z}(\mathcal{C}_n)$ which are tensor, respectively braided, equivalent to $\mathcal{C}_n$, are parametrized by the sets of n-dimensional subspaces and Lagrangian subspaces, respectively, of $\mathbf{k}^{2n}$. We show that there is a one-to-one correspondence with the n-dimensional subspaces of $\mathsf{Ext}^1(\chi, \varepsilon)$.

In Section 3.7 we describe two ways of constructing elements of the Brauer-Picard group. One induces braided autoequivalences of $\mathcal{Z}(\mathcal{C}_n)$ from tensor autoequivalences of $\mathcal{C}_n$. The other one induces invertible $\mathcal{Z}(\mathcal{C}_n)$-module categories from invertible $\mathcal{D}$-module categories, where $\mathcal{D}$ is a tensor subcategory of $\mathcal{Z}(\mathcal{C}_n)$. In this way We construct various group homomorphisms into $\mathsf{BrPic}(\mathcal{C}_n)$.

In the final section we put together the information from the previous sections to compute $\mathsf{BrPic}(\mathcal{C}_n)$.

The results of this chapter appeared in [BN15].

3.1 The Brauer group of a braided finite tensor category

We recall here the classical definition of the Brauer group of a field and show how it generalizes to the case of a braided finite tensor category.

Definition 3.1.1. Let $\mathbf{k}$ be a field.

(1) A *central simple* $\mathbf{k}$-algebra is a $\mathbf{k}$-algebra A which is *simple*, i.e. it has no proper non-zero ideals, and *central*, i.e. $Z(A) = \mathbf{k}$.

(2) An *Azumaya* $\mathbf{k}$-algebra is a finite-dimensional central simple algebra.

Remark 3.1.2. We collect here some facts about central simple algebras:

(1) The algebra $M_n(\mathbf{k})$, of $n \times n$ matrices with entries in a field $\mathbf{k}$, is an Azumaya $\mathbf{k}$-algebra.

(2) If A is a central simple algebra then A^{op} is a central simple algebra.

(3) If A and B are central simple algebras then $A \otimes B$ is a central simple algebra.

(4) A is an Azumaya $\mathbf{k}$-algebra if and only if A is finite-dimensional and the map

$$A \otimes A^{\text{op}} \to \mathsf{End}(A), \quad (a \otimes b)(x) = axb$$

is an isomorphism of algebras.

There is an equivalence relation on the set of Azumaya algebras. It was introduced by R. Brauer in 1929 in his study of division rings.

Definition 3.1.3. Two Azumaya $\mathbf{k}$-algebras, A and B, are *equivalent*, and we write this $A \sim B$, if there exists an isomorphism of $\mathbf{k}$-algebras:

$$A \otimes M_m(\mathbf{k}) \cong B \otimes M_n(\mathbf{k}),$$

for some integers m and n. The equivalence class of an Azumaya $\mathbf{k}$-algebra A is denoted by $[A]$.

Remark 3.1.4. The above equivalence is the Morita equivalence: A and B are equivalent if and only if $\mathsf{Rep}\, A \simeq \mathsf{Rep}\, B$.

Definition 3.1.5. The set of equivalence classes of Azumaya $\mathbf{k}$-algebras, with respect to '$\sim$', is a group with the following operation:

$$[A] \cdot [B] = [A \otimes B]$$

This group is denoted by $\mathsf{Br}(\mathbf{k})$ and is called the *Brauer group* of $\mathbf{k}$.

Remark 3.1.6. $\mathsf{Br}(\mathbf{k})$ is a commutative group. The inverse of $[A]$ is $[A^{\text{op}}]$, and the unit element is $[\mathbf{k}]$.

Example 3.1.7. If $\mathbf{k}$ is an algebraically closed field then $\mathsf{Br}(\mathbf{k}) = \{0\}$. Thus, $\mathsf{Br}(\mathbb{C}) = 0$.

Example 3.1.8. $\mathsf{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, where the non-zero element is represented by the algebra $\mathbb{H}$ of quaternions.

The construction of the Brauer group of a field was extended, over time, to various contexts. It was defined for commutative rings, for commutative rings graded by abelian groups, for schemes, for commutative and cocommutative Hopf algebras over a commutative ring, etc. An important step in this sequence of extensions was made by B. Pareigis who, in 1975, defined the Brauer group of a symmetric monoidal category [P75]. After A. Joyal and R. Street introduced, in 1993, the concept of a braided monoidal category [JS93], F. Van Oystaeyen and Y. Zhang extended, in 1998, Pareigis' definition of the Brauer group [vOZ98] to such categories. This is the most general definition of the Brauer group, encompassing all known instances of the construction.

Since the definition of the Brauer group in its greatest generality exceeds the purpose of this exposition, we will restrict our attention to the case of finite braided tensor categories. The generalization to this context will be more clearly understood if we recast the definition of Azumaya algebras and its accompanying equivalence relation in terms only of finite dimensional vector spaces. This means that we work in the category $\mathbf{k}$-Vec.

Notice first that, if V is a $\mathbf{k}$-vector space of dimension n, then there exist isomorphisms of algebras:

$$M_n(\mathbf{k}) \cong \mathsf{End}(V) \cong V \otimes V^*,$$

where V^* is the dual vector space of V. Here, the product of $V \otimes V^*$ is $(v \otimes p)(w \otimes q) = p(w)v \otimes q$, for all v, $w \in V$ and p, $q \in V^*$. The isomorphism $\mathsf{End}(V) \to V \otimes V^*$ maps f to $\sum_i f(e_i) \otimes e_i^*$, where $\{e_i\}$ is a basis of V and $\{e_i^*\}$ is its dual basis.

An Azumaya $\mathbf{k}$-algebra is then an algebra (A, m, u) in $\mathbf{k}$-Vec such that, if $\{e_i\}$ is a basis of A, then the map

$$F : A \otimes A^{\mathrm{op}} \to A \otimes A^*, \quad F(a \otimes b) = \textstyle\sum_i a e_i b \otimes e_i^*,$$

which makes the following diagram commutative,

$$\begin{tikzcd}
A \otimes A^{\mathrm{op}} \arrow[rrr, "F"] \arrow[d, "\cong"'] & & & A \otimes A^* \\
A \otimes A^{\mathrm{op}} \otimes \mathbf{k} \arrow[r, "\mathsf{id} \otimes \mathsf{coev}_A"'] & A \otimes A^{\mathrm{op}} \otimes A \otimes A^* \arrow[r, "\mathsf{id} \otimes c_{A,A} \otimes \mathsf{id}"'] & A \otimes A \otimes A^{\mathrm{op}} \otimes A^* \arrow[r, "m_A \otimes \mathsf{id}"'] & A \otimes A^{\mathrm{op}} \otimes A^* \arrow[u, "m_A \otimes \mathsf{id}"']
\end{tikzcd}$$

is an isomorphism of algebras.

Two Azumaya algebras, A and B, are equivalent if and only if there exist finite dimensional vector spaces, U and V, such that

$$A \otimes U \otimes U^* \cong B \otimes V \otimes V^*.$$

We see from the above that the Brauer group can be defined for a category which has a tensor product, a braided structure, and whose objects admit duals. This is the case with braided finite tensor categories.

Recall that, given a braided finite tensor category $(\mathcal{C}, \otimes, \mathbf{1}, c)$, the opposite algebra A^{op} of an algebra A in $\mathcal{C}$ is A with multiplication $m_A c_{A,A}$. If A and B are algebras in $\mathcal{C}$ then $A \otimes_{\mathcal{C}} B$ is the algebra with underlying object $A \otimes B$ and multiplication $(m_A \otimes m_B)(\mathsf{id}_A \otimes c_{B,A} \otimes \mathsf{id}_B)$. Also, if X is an object of $\mathcal{C}$ with left and right duals, X^* and ${}^*X$, respectively, then $X \otimes X^*$ and ${}^*X \otimes X$ are algebras with multiplication $\mathsf{id}_X \otimes \mathsf{ev}_X \otimes \mathsf{id}_{X^*}$ and $\mathsf{id}_{{}^*X} \otimes \mathsf{ev}'_X \otimes \mathsf{id}_X$, respectively.

Definition 3.1.9. Let $(\mathcal{C}, \otimes, \mathbf{1}, c)$ be a braided finite tensor category. A *$\mathcal{C}$-Azumaya algebra* is an algebra A in $\mathcal{C}$ such that morphisms

$$F : A \otimes_{\mathcal{C}} A^{\text{op}} \to A \otimes A^* \quad \text{and} \quad G : A^{\text{op}} \otimes_{\mathcal{C}} A \to {}^*A \otimes A$$

making the following diagrams commutative,

$$\begin{tikzcd}
A \otimes A^{\text{op}} \arrow[rrr, "F"] \arrow[d, "r^{-1}_{A\otimes A^{\text{op}}}"'] & & & A \otimes A^* \\
A \otimes A^{\text{op}} \otimes \mathbf{k} \arrow[r, "\mathsf{id} \otimes \mathsf{coev}_A"'] & A \otimes A^{\text{op}} \otimes A \otimes A^* \arrow[r, "\mathsf{id} \otimes c_{A^{\text{op}},A} \otimes \mathsf{id}"'] & A \otimes A \otimes A^{\text{op}} \otimes A^* \arrow[r, "m_A \otimes \mathsf{id}"'] & A \otimes A^{\text{op}} \otimes A^* \arrow[u, "m_A \otimes \mathsf{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A^{\text{op}} \otimes A \arrow[rrr, "G"] \arrow[d, "l^{-1}_{A^{\text{op}}\otimes A}"'] & & & {}^*A \otimes A \\
\mathbf{k} \otimes A^{\text{op}} \otimes A \arrow[r, "\mathsf{coev}'_A \otimes \mathsf{id}"'] & {}^*A \otimes A \otimes A^{\text{op}} \otimes A \arrow[r, "\mathsf{id} \otimes c_{A,A^{\text{op}}} \otimes \mathsf{id}"'] & {}^*A \otimes A^{\text{op}} \otimes A \otimes A \arrow[r, "\mathsf{id} \otimes m_A"'] & {}^*A \otimes A^{\text{op}} \otimes A \arrow[u, "\mathsf{id} \otimes m_A"']
\end{tikzcd}$$

are isomorphisms of algebras in $\mathcal{C}$.

Remark 3.1.10. The reason we consider both morphisms F and G is that, in general, $A \otimes_{\mathcal{C}} A^{\text{op}}$ and $A^{\text{op}} \otimes_{\mathcal{C}} A$ are not isomorphic as algebras in $\mathcal{C}$.

Remark 3.1.11. It can be shown that an algebra A in $\mathcal{C}$ is $\mathcal{C}$-Azumaya if and only if the following two functors are equivalence functors (see [vOZ98, Theorem 3.1]):

$$\begin{aligned} \mathcal{C} &\to {}_{A\otimes_{\mathcal{C}} A^{\mathrm{op}}}\mathcal{C}, \qquad X \mapsto A \otimes X, \\ \mathcal{C} &\to \mathcal{C}_{A^{\mathrm{op}}\otimes_{\mathcal{C}} A}, \qquad X \mapsto X \otimes X. \end{aligned}$$

Remark 3.1.12. The following hold true in a braided finite tensor category $\mathcal{C}$:

(1) $X \otimes X^*$ is a $\mathcal{C}$-Azumaya algebra for every $X \in \mathcal{C}$.

(2) If A is a $\mathcal{C}$-Azumaya algebra then A^{op} is a $\mathcal{C}$-Azumaya algebra.

(3) If A and B are $\mathcal{C}$-Azumaya algebras then $A \otimes_{\mathcal{C}} B$ is a $\mathcal{C}$-Azumaya algebra.

(4) Any $\mathcal{C}$-Azumaya algebra is exact.

Definition 3.1.13. Let $\mathcal{C}$ be a braided finite tensor category.

(1) Two $\mathcal{C}$-Azumaya algebras A and B are *equivalent*, and we write this $A \sim B$, if there exist two objects, X and Y, in $\mathcal{C}$ and an isomorphism of $\mathcal{C}$-algebras

$$A \otimes_{\mathcal{C}} (X \otimes X^*) \cong B \otimes_{\mathcal{C}} (Y \otimes Y^*).$$

The equivalence class of A is denoted by $[A]$.

(2) The set of equivalence classes of $\mathcal{C}$-Azumaya algebras is a group with the operation

$$[A] \cdot [B] = [A \otimes_{\mathcal{C}} B].$$

This group is denoted by $\mathsf{Br}(\mathcal{C})$ and called the *Brauer group of* $\mathcal{C}$. The inverse of $[A]$ is $[A^{\mathrm{op}}]$, and the identity element is $[\mathbf{1}]$.

Remark 3.1.14. It can be shown that the equivalence relation on the set of $\mathcal{C}$-Azumaya algebras is the Morita equivalence, that is $A \sim B$ if and only if $\mathcal{C}_A \simeq \mathcal{C}_B$ as $\mathcal{C}$-module categories.

Example 3.1.15. It is obvious that $\mathsf{Br}(\mathbf{k}\text{-}\mathsf{Vec}) = \mathsf{Br}(\mathbf{k})$.

Example 3.1.16. $\mathsf{Br}(\mathbf{k}\text{-}\,\mathsf{sVec})$ is called *the Brauer-Wall group of* $\mathbf{k}$ and is denoted by $BW(\mathbf{k})$. This group was introduced by C.T.C. Wall and it classifies equivalence classes of finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded central simple algebras [Wall64]. We have $BW(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$ and $BW(\mathbb{R}) = \mathbb{Z}/8\mathbb{Z}$.

Example 3.1.17. For a quasitriangular Hopf algebra (H, R) the group $\mathsf{Br}\left(\mathsf{Rep}(H, R)\right)$ is denoted by $\mathsf{BM}(\mathbf{k}, H, R)$. If (H, r) is coquasitriangular, then $\mathsf{Br}\left(\mathsf{Corep}(H, r)\right)$ is denoted by $\mathsf{BC}(\mathbf{k}, H, r)$. Since $\mathsf{Rep}(H, R)$ is braided equivalent to $\mathsf{Corep}(H^*, R^*)$, where R^* is the r-form dual to R, we have $\mathsf{BM}(\mathbf{k}, H, R) \cong \mathsf{BC}(\mathbf{k}, H^*, R^*)$.

These groups were studied by a number of Hopf algebraists. The first explicit computation was performed by F. van Oystaeyen and Y. Zhang in [vOZ01], who showed that

$$\mathsf{BM}(\mathbf{k}, H_4, R_0) \cong (\mathbf{k}, +) \times BW(\mathbf{k})$$

In [C01] G. Carnovale showed that the groups $\mathsf{BM}(\mathbf{k}, H_4, R_\lambda)$, where R_λ, $\lambda \in \mathbf{k}$, are the R-matrices from Example 2.6.10, are all isomorphic. She used, in fact, the self-duality of H_4, and worked with the groups $\mathsf{BC}(\mathbf{k}, H_4, r_\lambda) \cong \mathsf{BM}(\mathbf{k}, H_4, R_\lambda)$. Other examples were considered in [CC03] and [CC04a]. For a symmetric $n \times n$ matrix A, it was shown in [CC04b] that

$$\mathsf{BM}(\mathbf{k}, E(n), R_A) \cong \mathsf{Sym}_n(\mathbf{k}) \times BW(\mathbf{k}), \tag{3.1}$$

where $\mathsf{Sym}_n(\mathbf{k})$ is the additive group of symmetric $n \times n$ matrices with entries in $\mathbf{k}$. If A is not symmetric, then the description of $\mathsf{BM}(\mathbf{k}, E(n), R_A)$ is still possible, but is a bit more involved. Finally, BM was computed for a large class of triangular Hopf algebras, called *modified supergroup algebras*, in [C06]. Taking into account [EG01] this yields computation of BM for all finite dimensional triangular Hopf algebras.

Example 3.1.18. Let ${}_H\mathcal{YD}^H$ be the category of Yetter-Drinfeld modules over a finite dimensional Hopf algebra H. Then $\mathsf{Br}({}_H\mathcal{YD}^H)$ is denoted by $\mathsf{BQ}(\mathbf{k}, H)$ and is called the *full Brauer group of* H. This was introduced in [CvOZ97]. The justification for the name comes from the fact that

$\mathsf{BQ}(\mathbf{k}, H)$ contains $\mathsf{BM}(\mathbf{k}, H, R)$ and $\mathsf{BC}(\mathbf{k}, H, r)$ as a subgroups, when (H, R) is quasitriangular and/or (H, r) is co-quasitriangular.

Notice that, since ${}_H\mathcal{YD}^H$ is braided equivalent to $\mathsf{Rep}\big(D(H), \mathcal{R}\big)$, we have $\mathsf{BQ}(\mathbf{k}, H) \cong \mathsf{BM}(\mathbf{k}, D(H), \mathcal{R})$.

Before the systematic study of tensor categories was initiated, the full Brauer group was known only for some special classes of group Hopf algebras of abelian groups, e.g. $\mathbf{k}\mathbb{Z}/n\mathbb{Z}$, with n square-free and $\mathbf{k}$ algebraically closed with of characteristic $\nmid n$ [Lon74], or n a power of an odd prime number and some mild assumptions on $\mathbf{k}$ [BC89].

A major step in advancing our knowledge on the subject was made when it was realized that the Brauer-Picard group $\mathsf{BrPic}(\mathcal{C})$ of a finite tensor category $\mathcal{C}$ coincides with the Brauer group $\mathsf{Br}(\mathcal{Z}(\mathcal{C}))$ of the center of $\mathcal{C}$ [DN13]. Thus, if H is a finite dimensional Hopf algebra and $\mathbf{k}$ is an algebraically closed field of characteristic 0, $\mathsf{BrPic}(\mathsf{Rep}\, H) \cong \mathsf{BQ}(\mathbf{k}, H)$. In particular, $\mathsf{BrPic}(\mathsf{Rep}\,\mathbf{k}[G]) \cong \mathsf{BQ}(\mathbf{k}, \mathbf{k}[G])$, for any finite group G. This allowed for the use of methods from the theory of tensor categories to compute the full Brauer group. For example, it follows easily from this theory that, if A is a finite abelian group, then $\mathsf{BrPic}(\mathsf{Rep}\, A) \cong O(A \oplus \widehat{A})$, the orthogonal group of $A \oplus \widehat{A}$ with respect to the canonical quadratic form on $q : A \oplus \widehat{A} \to \mathbf{k}$, $q(a, \chi) = \chi(a)$, for all $a \in A$ and $\chi \in \widehat{A}$. The first descriptions of $\mathsf{BrPic}(\mathsf{Rep}\, G)$, when G is not abelian, were obtained by B. Riepel and D. Nikshych in [NR14], and furthermore by I. Marshall and D. Nikshych in [MN16].

An interesting problem that remained unsolved was to describe $\mathsf{BQ}(\mathbf{k}, H_4)$ where H_4 is Sweedler's Hopf algebra, the smallest non-semisimple Hopf algebra. Athough attempts were made (see [vOZ01] and [CC11]), the answer seemed to be out of reach. In this chapter, I will present my contribution to the problem of computing Brauer groups by computing $\mathsf{BrPic}(\mathsf{Rep}\, E(n)) = \mathsf{BQ}(\mathbf{k}, E(n))$, where $\{E(n)\}$ are the Nichols Hopf algebras. Since $H_4 = E(1)$, this solves, in particular, the problem of describing $\mathsf{BQ}(\mathbf{k}, H_4)$.

3.2 The Brauer-Picard group of a tensor category

In this section we give the definition of the Brauer-Picard group of a finite tensor category $\mathcal{C}$, mention its equivalent description as the group of braided autoequivalences of the center of $\mathcal{C}$, and, when $\mathcal{C}$ is braided, discuss an important subgroup, the Picard group of $\mathcal{C}$. We show that the Picard group of $\mathcal{C}$ coincides with the Brauer group of $\mathcal{C}$, as defined in the previous section. This creates a bridge between the theory of tensor categories and the theory of Brauer groups, and allows for the application of results from one field to the other. Throughout, the base field $\mathbf{k}$ is assumed to be algebraically closed of characteristic 0.

Recall the definition of the Picard group of an algebra A. An A-bimodule P is *invertible* if there exists an A-bimodule Q such that $P \otimes_A Q$ and $Q \otimes_A P$ are isomorphic to A as A-bimodules. The *Picard group* of A, denoted by $\mathsf{Pic}(A)$, is the set of isomorphism classes of invertible A-bimodules, with the group operation induced by the tensor product over A:

$$[P] \cdot [Q] = [P \otimes_A Q],$$

where $[P]$ and $[Q]$ denote the isomorphism classes of the invertible A-bimodules P and Q, respectively.

The counterpart of the Picard group of an algebra in the theory of tensor categories is the Brauer-Picard group of a finite tensor category. To define it, we need to introduce the tensor product of two module categories.

Definition 3.2.1. Let $\mathcal{M}$ be a right module category and $\mathcal{N}$ a left module category over a finite tensor category $\mathcal{C}$. Let $\mathcal{A}$ be a $\mathbf{k}$-linear abelian category. A bifunctor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$, additive and $\mathbf{k}$-linear in each argument, is said to be *$\mathcal{C}$-balanced* if there is a natural isomorphism

$$b = \{b_{M,X,N} : F(M \otimes X, N) \to F(M, X \otimes N)\}_{M \in \mathcal{M}, X \in \mathcal{C}, N \in \mathcal{N}},$$

making the following diagram commutative

$$\begin{tikzcd}
F(M \otimes (X \otimes Y), N) \arrow[rr, "b_{M,X\otimes Y,N}"] \arrow[d, "m_{M,X,Y}"'] & & F(M, (X \otimes Y) \otimes N) \\
F((M \otimes X) \otimes Y, N) \arrow[dr, "b_{M\otimes X,Y,N}"'] & & F(M, X \otimes (Y \otimes N)) \arrow[u, "n^{-1}_{X,Y,N}"'] \\
& F(M \otimes X, Y \otimes N) \arrow[ur, "b_{M,X,Y\otimes N}"'] &
\end{tikzcd}$$

for every $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $X, Y \in \mathcal{C}$.

Definition 3.2.2. Let $\mathcal{C}$ be a finite tensor category. A *tensor product* of a right $\mathcal{C}$-module category $\mathcal{M}$ and a left $\mathcal{C}$-module category $\mathcal{N}$ is a $\mathbf{k}$-linear abelian category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$, together with a $\mathcal{C}$-balanced bifunctor

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N},$$

which is right exact in each variable and which induces, for every $\mathbf{k}$-linear abelian category $\mathcal{A}$, an equivalence between the category of $\mathcal{C}$-balanced, right exact in each variable, bifunctors from $\mathcal{M} \times \mathcal{N} \to \mathcal{A}$ and the category of right exact functors from $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ to $\mathcal{A}$:

$$\mathsf{Fun}_{bal,re}(\mathcal{M} \times \mathcal{N}, \mathcal{A}) \simeq \mathsf{Fun}_{re}(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \mathcal{A}).$$

Remark 3.2.3. The subscripts bal and re indicate that the functors in question are balanced, respectively, right exact.

Remark 3.2.4. It was shown in [ENO10, Section 3.2] that the tensor product of a right $\mathcal{C}$-module category $\mathcal{M}$ and a left $\mathcal{C}$-module category $\mathcal{N}$ exists, and that there is an equivalence of abelian categories:

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \simeq \mathsf{Fun}_{\mathcal{C},re}(\mathcal{M}^{\mathrm{op}}, \mathcal{N}).$$

Remark 3.2.5. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{C}$-bimodule categories, then the left $\mathcal{C}$-module structure of $\mathcal{M}$ and the right $\mathcal{C}$-module structure of $\mathcal{N}$ induce a $\mathcal{C}$-bimodule structure on $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$.

Remark 3.2.6. It was shown in [DN13, Proposition 2.10] that for two exact $\mathcal{C}$-bimodule categories, $\mathcal{M}$ and $\mathcal{N}$, the tensor product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is an exact $\mathcal{C}$-bimodule category.

Definition 3.2.7. Let $\mathcal{C}$ be a finite tensor category.

(1) An exact $\mathcal{C}$-bimodule category $\mathcal{M}$ is *invertible* if there exists an exact $\mathcal{C}$-bimodule category $\mathcal{N}$ such $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ and $\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M}$ are equivalent to $\mathcal{C}$ as $\mathcal{C}$-bimodule categories.

(2) The *Brauer-Picard group of* $\mathcal{C}$, denoted by $\mathsf{BrPic}(\mathcal{C})$, is the set of equivalence classes of invertible, exact, $\mathcal{C}$-bimodule categories, with the group operation induced by the tensor product over $\mathcal{C}$:

$$[\mathcal{M}] \cdot [\mathcal{N}] = [\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}],$$

where $[\mathcal{M}]$ and$[\mathcal{N}]$ denote the equivalence classes of the invertible, exact, $\mathcal{C}$-bimodule categories $\mathcal{M}$ and $\mathcal{N}$.

Remark 3.2.8. The definition of the Brauer-Picard group justifies the inclusion of Picard's name. Brauer's name is included because $\mathsf{BrPic}(\mathbf{k}\text{-}\mathsf{Vec}) \cong \mathsf{Br}(\mathbf{k})$.

Given an algebra A, there is a homomorphism

$$\phi : \mathsf{Pic}(A) \to \mathsf{Aut}(Z(A)),$$

where $Z(A)$ is the center of A, defined as follows. For an invertible A-bimodule M and $z \in Z(A)$, $\phi(M)(z)$ is that element of $Z(A)$ such that the endomorphism of M given by left multiplication with $\phi(M)(z)$ is the same as the endomorphism given by right multiplication with z.

There is an analogue of homomorphism ϕ at the categorical level. Consider a finite tensor category $\mathcal{C}$ and an invertible $\mathcal{C}$-bimodule category $\mathcal{M}$. Then $\mathcal{Z}(\mathcal{C})$ can be identified with the category of $\mathcal{C}$-bimodule endofunctors of $\mathcal{M}$ in two ways: via the functors $Z \mapsto Z \otimes -$ and $Z \mapsto - \otimes Z$. We can define a braided autoequivalence $\Phi_{\mathcal{M}}$ of $\mathcal{Z}(\mathcal{C})$ in such a way that there is an isomorphism of $\mathcal{C}$-bimodule functors

$$\Phi_{\mathcal{M}}(Z) \otimes - \cong - \otimes Z,$$

for all $Z \in \mathcal{Z}(\mathcal{C})$.

The next result was established for fusion categories in [ENO10] and extended to tensor categories in [DN13].

Theorem 3.2.9. *Let $\mathcal{C}$ be a finite tensor category. Then*

$$\Phi : \mathsf{BrPic}(\mathcal{C}) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C})), \quad \mathcal{M} \mapsto \Phi_{\mathcal{M}}, \tag{3.2}$$

is a group isomorphism.

Remark 3.2.10. Much of the progress on computing Brauer-Picard groups relies on Theorem 3.2.9. This is because, in practice, it is much easier to work with functors than with module categories. Moreover, $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}))$ can be viewed as a classical orthogonal group, which allows for important geometric insights. For example, the computations of the Brauer-Picard groups in [NR14] and [MN16], were achieved by studying the action of $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}))$ on categorical analogues of Grassmannians.

Example 3.2.11. Let A be a finite abelian group. It follows from Theorem 3.2.9 that

$$\mathsf{BrPic}(\mathsf{Rep}\, A) \cong O(A \oplus \widehat{A}, q),$$

where $O(A \oplus \widehat{A}, q)$ is the group of automorphisms of $A \oplus \widehat{A}$ preserving the canonical quadratic form $q : A \oplus \widehat{A} \to \mathbf{k}^{\times}$, $q(a, \chi) = \chi(a)$, for all $a \in A$, and $\chi \in \widehat{A}$.

If $\mathcal{C}$ is a braided tensor category then $\mathsf{BrPic}(\mathcal{C})$ contains the Brauer group $\mathsf{Br}(\mathcal{C})$ as a subgroup. We close this section by discussing this relationship.

Suppose $c = \{c_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{C}}$ is a braiding on $\mathcal{C}$. Then, just as modules over a commutative ring become bimodules over that ring, so too, modules categories over $\mathcal{C}$ become $\mathcal{C}$-bimodules. If $(\mathcal{M}, \otimes, m, l)$ is a left $\mathcal{C}$-module category then $\mathcal{M}$ is a $\mathcal{C}$-bimodule category with

- Right action: $M \otimes X := X \otimes M$, for all $M \in \mathcal{M}$ and $X \in \mathcal{C}$.

- Right module associativity constraint given by

$$\begin{tikzcd}
(M \otimes X) \otimes Y \arrow[rr, "n_{M,X,Y}"] \arrow[d, equal] & & M \otimes (X \otimes Y) . \arrow[d, equal] \\
Y \otimes (X \otimes M) \arrow[r, "m_{Y,X,M}"'] & (Y \otimes X) \otimes M \arrow[r, "c_{Y,X}"'] & (X \otimes Y) \otimes M
\end{tikzcd}$$

- Right unit constraint $r = l$.

- Middle associativity constraint given by

$$\begin{tikzcd}
X \otimes (M \otimes Y) \arrow[rrr, "b_{X,M,Y}"] \arrow[d, equal] & & & (X \otimes M) \otimes Y . \arrow[d, equal] \\
X \otimes (Y \otimes M) \arrow[r, "m_{X,Y,M}"'] & (X \otimes Y) \otimes M \arrow[r, "c_{X,Y}"'] & (Y \otimes X) \otimes M \arrow[r, "m^{-1}_{Y,X,M}"'] & Y \otimes (X \otimes M)
\end{tikzcd}$$

Definition 3.2.12. Let $\mathcal{C}$ be a braided finite tensor category.

(1) A $\mathcal{C}$-bimodule category is said to be *one-sided* if it is equivalent to a $\mathcal{C}$-bimodule category constructed, in the manner presented earlier, from a left $\mathcal{C}$-module category.

(2) The *Picard group of* $\mathcal{C}$, denoted by $\mathsf{Pic}(\mathcal{C})$, is the subgroup of $\mathsf{BrPic}(\mathcal{C})$ consisting of equivalence classes of invertible, one-sided, exact, $\mathcal{C}$-bimodule categories.

Let $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}); \mathcal{C})$ be the subgroup of $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}))$ consisting of braided autoequivalences of $\mathcal{Z}(\mathcal{C})$ trivializable on $\mathcal{C}$. The following result was proved in [DN13].

Theorem 3.2.13. *The image of* $\mathsf{Pic}(\mathcal{C})$*, under the isomorphism* (3.2.9)*, is* $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C})$.

Remark 3.2.14. The Picard group of a braided finite tensor category $\mathcal{C}$ is nothing but the Brauer group of $\mathcal{C}$. To see this, recall that every exact left $\mathcal{C}$-module category is equivalent to a category $\mathcal{C}_A$, of right A-modules in $\mathcal{C}$, for some exact algebra A in $\mathcal{C}$. It can be shown that $\mathcal{C}_A$ is invertible if and only if the two functors from Remark 3.1.11 are equivalences, i.e. A is a $\mathcal{C}$-Azumaya algebra. Recall also that the equivalence relation on the set of $\mathcal{C}$-Azumaya algebras is the Morita equivalence: $A \sim B$ if and only if $\mathcal{C}_A \simeq \mathcal{C}_B$.

Thus, the map

$$\mathsf{Pic}(\mathcal{C}) \to \mathsf{Br}(\mathcal{C}), \quad [\mathcal{C}_A] \mapsto [A]$$

where $[\mathcal{C}_A]$ and $[A]$ denote equivalence classes, is well defined and is a bijection. Moreover, it is an isomorphism, since, for two exact algebras, A and B, in $\mathcal{C}$ there is an equivalence of $\mathcal{C}$-module categories $\mathcal{C}_A \boxtimes_{\mathcal{C}} \mathcal{C}_B \simeq \mathcal{C}_{A\otimes B}$ (see [DN13, Proposition 3.4]).

Remark 3.2.15. It follows from [ENO10] that

$$\mathsf{BrPic}(\mathcal{C}) \cong \mathsf{Pic}(\mathcal{Z}(\mathcal{C})).$$

Thus, if H is a finite dimensional Hopf algebra, we have

$$\mathsf{BrPic}(\mathsf{Rep}\, H) \cong \mathsf{Pic}(\mathcal{Z}(\mathsf{Rep}\, H)) \cong \mathsf{Br}({}_H\mathcal{YD}^H) = \mathsf{BQ}(\mathbf{k}, H).$$

Example 3.2.16. Let $\mathbf{k}$ be an algebraically closed field of characteristic $\neq 0$. Let R_0 be the R-matrix of $E(n)$ associated to the zero $n \times n$ matrix. Taking into account (3.1), we have

$$\mathsf{Pic}(\mathsf{Rep}(E(n), R_0)) \cong \mathsf{BM}(\mathbf{k}, E(n), R_0) \cong \mathsf{Sym}_n(\mathbf{k}) \times \mathbb{Z}/2\mathbb{Z}. \tag{3.3}$$

3.3 A finite symmetric tensor category

In this section we recall the definition and properties of the Nichols Hopf algebra $E(n)$. We review the quasitriangular structures, the invariant 2-cocycles and invariant twists. At the end we describe the Drinfeld double $D(E(n))$.

Recall Example 2.1.25. The Nichols Hopf algebra $E(n)$ is :

$$E(n) = \mathbf{k}\{c, x_1, \ldots, x_n\}/(c^2 - 1, x_i^2, cx_i + x_ic, x_ix_j + x_jx_i).$$

with comultiplication, counit and antipode given by:

$$\begin{aligned}
\Delta(c) &= c \otimes c, & \varepsilon(c) &= 1, & S(c) &= c^{-1},\\
\Delta(x_i) &= 1 \otimes x_i + x_i \otimes c, & \varepsilon(x_i) &= 0, & S(x_i) &= c^{-1}x_i,
\end{aligned}$$

for all $i = 1, \ldots, n$.

$E(n)$ is a pointed Hopf algebra with coradical $\mathbf{k}[C_2]$, where $C_2 = \langle c \rangle$. A $\mathbf{k}$-basis of $E(n)$ is $\{c^i x_P \mid i = 0, 1,\ P \subseteq \{1, \dots, n\}\}$. The comultiplication of a basis element is given as follows. For a subset $F = \{i_{j_1}, \dots, i_{j_r}\}$ of $P = \{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, n\}$, let

$$S(F,P) = \begin{cases} (j_1 + \cdots + j_r) - r(r+1)/2 & \text{if} \quad F \neq \emptyset \\ 0 & \text{if} \quad F = \emptyset. \end{cases}$$

If we denote the number of elements of F by $|F|$, then

$$\Delta(x_P) = \textstyle\sum_{F \subseteq P} (-1)^{S(F,P)} x_F \otimes c^{|F|} x_{P \setminus F}.$$

The group of Hopf automorphisms of $E(n)$ was computed in [PvO99]. We have

$$\mathsf{Aut}_{\text{Hopf}}(E(n)) \simeq \mathsf{GL}_n(k), \tag{3.4}$$

with the automorphism corresponding to $T = (t_{ij}) \in \mathsf{GL}_n(k)$, being given by $c \mapsto c$ and $x_i \mapsto \sum_j t_{ji} x_j$, for all $i = 1, \dots, n$. The inner automorphisms of $E(n)$ correspond to $T = \pm I_n$.

The quasitriangular structures of $E(n)$ were described in Example 2.6.11. The set of R-matrices of $E(n)$ is parameterized by $M_n(\mathbf{k})$. The R-matrix corresponding to $A \in M_n(k)$ is

$$R_A = \frac{1}{2} \sum_{i=0}^{n} (-1)^{\frac{i(i-1)}{2}} \sum_{|P|=|F|=i} [A]_{P,F} (x_P \otimes x_F + x_P \otimes c x_F + \\ + (-1)^i c x_P \otimes x_F + (-1)^{i+1} c x_P \otimes c x_F),$$

and R_A is triangular if and only if A is symmetric.

Remark 3.3.1. The category $\mathsf{Rep}\, E(n)$ with symmetric braiding is equivalent to the representation category of a finite supergroup $\wedge \mathbf{k}^n \rtimes \mathbb{Z}/2\mathbb{Z}$. It is the most general example of a non-semisimple symmetric tensor category without non-trivial Tannakian subcategories. We will denote $\mathsf{Rep}\, E(n)$ with a symmetric braiding by $\mathcal{C}_n$.

Proposition 3.3.2. $\mathsf{Aut}^{br}(\mathcal{C}_n) \cong \mathsf{GL}_n(\mathbf{k})/\{\pm I_n\}$.

Proof. By [Del02] the symmetric category $\mathcal{C}_n$ has a unique, up to isomorphism, braided tensor functor to sVec. Let F denote the composition of this functor with the forgetful functor $\mathsf{sVec} \to \mathsf{Vec}$. Then $E(n) \cong \mathsf{End}(F)$. Since every braided tensor autoequivalence of $\mathcal{C}_n = \mathsf{Rep}(E(n))$ preserves F it must come from a Hopf automorphism of $E(n)$. By (3.4) we have $\mathsf{Aut}_{\mathrm{Hopf}}(E(n)) = \mathsf{GL}_n(\mathbf{k})$. Tensor autoequivalences of $\mathcal{C}_n$ isomorphic to the identity functor come from inner automorphisms of $E(n)$. The statement follows from the observation that the group of inner Hopf automorphisms of $E(n)$ is generated by the conjugation by c and is isomorphic to $\{\pm I_n\}$. $\square$

Recall the description of the invariant 2-cocycles of $E(n)$ from Example 2.9.13. We have $\mathsf{H}^2_{\mathsf{inv}}(E(n)) \cong \mathsf{Sym}_n(\mathbf{k})$. A representative of the cohomology class corresponding to $M = (m_{ij}) \in \mathsf{Sym}_n(\mathbf{k})$ is the invariant 2-cocycle $\sigma_M : E(n) \otimes E(n) \to k$ defined by:

$$\sigma_M(c \otimes c) = 1, \quad \sigma_M(x_i \otimes x_j) = m_{ij}, \quad i, j = 1, \ldots, n,$$

$$\sigma_M(x_P \otimes x_Q) = \sigma_M(cx_P \otimes x_Q) = (-1)^{|P|}\sigma_M(x_P \otimes cx_Q) = (-1)^{|P|}\sigma_M(cx_P \otimes cx_Q),$$

for all P, $Q \subseteq \{1, \ldots, n\}$, $\sigma_M(x_P \otimes x_Q) = 0$ if $|P| \neq |Q|$, and a recurrence formula allowing to compute $\sigma_M(x_P, x_Q)$ when $|P| = |Q|$. In particular, we have $\sigma_M(c^i x_k \otimes c^j x_l) = (-1)^j m_{kl}$, for all $i, j = 0, 1$ and $k, l = 1, \ldots, n$.

Since $E(n)$ is self-dual, we have $\mathsf{H}^2_{\mathsf{inv}}(E(n)^*) \cong \mathsf{Sym}_n(\mathbf{k})$. A representative for the cohomology class corresponding to $M = (m_{ij}) \in \mathsf{Sym}_n(k)$ is the invariant twist

$$J_M = \tfrac{1}{4} \textstyle\sum_{i,j,P,Q} \sigma_M(c^i x_P \otimes c^j x_Q)(x_P + (-1)^i cx_P) \otimes (x_Q + (-1)^j cx_Q).$$

Finally, we will need a description of $D(E(n))$. Recall that $D(E(n))$ contains $E(n)$ and $E(n)^{*\mathrm{cop}}$ as Hopf subalgebras and multiplication is given by formula (2.35). Composing the two Hopf algebra isomorphisms $E(n) \to E(n)^*$, $c \mapsto 1^* - c^*$, $x_i \mapsto x_i^* + (cx_i)^*$, and $E(n)^{\mathrm{cop}} \to E(n)$, $c \mapsto c$, $x_i \mapsto cx_i$, we obtain the isomorphism $E(n) \to E(n)^{*\mathrm{cop}}$, $c \mapsto 1^* - c^*$, $x_i \mapsto x_i^* - (cx_i)^*$. Thus, the Drinfeld double, $D(E(n))$, is generated by two copies of $E(n)$.

Let $C = 1^* - c^*$ and $X_i = x_i^* - (cx_i)^*$, $i = 1, \ldots, n$. Then, viewing $E(n)$ and $E(n)^{*\mathrm{cop}}$ as Hopf subalgebras of $D(E(n))$ and taking into account formula (2.35), we see that $D(E(n))$ is generated by the grouplike elements c and C, the $(1, c)$-primitive elements $x_1, \ldots, x_n$ and the $(1, C)$-primitive elements $X_1, \ldots, X_n$, subject to the following relations:

$$c^2 = 1,\ x_i^2 = 0,\ x_i c + c x_i = 0,\ x_i x_j + x_j x_i = 0, \tag{3.5}$$

$$C^2 = 1,\ X_i^2 = 0,\ X_i C + C X_i = 0,\ X_i X_j + X_j X_i = 0, \tag{3.6}$$

$$cC = Cc,\ X_i c + c X_i = x_i C + C x_i = 0,\ x_i X_j + X_j x_i = \delta_{i,j}(1 - Cc), \tag{3.7}$$

for all $i, j = 1, \ldots, n$.

Lemma 3.3.3. *If P is a subset of $\{1, \ldots, n\}$ then*

$$\begin{aligned} x_P^* &= (-1)^{\frac{|P|(|P|-1)}{2}}(X_P + CX_P) \\ (cx_P)^* &= (-1)^{\frac{|P|(|P|+1)}{2}}(X_P - CX_P). \end{aligned}$$

Proof. For $P \subseteq \{1, \ldots, n\}$ define $Y_P = x_P^* + (cx_P)^*$. An easy argument using induction on $|P|$ shows that, if $P = \{i_1, \ldots, i_r\}$, with $i_1 < i_2 < \cdots < i_r$, then $Y_P = Y_{i_1} Y_{i_2} \cdots Y_{i_r}$. Moreover, since

$$(1^* - c^*)(x_P^* + (cx_P)^*)(c^i x_Q) = \begin{cases} 0 & \text{if } Q \neq P \\ (-1)^i & \text{if } Q = P \end{cases}$$

we have $CY_P = x_P^* - (cx_P)^*$. In particular, $CY_i = X_i$, for all $i = 1, \ldots, n$, and, because C is an element of order 2 that anti-commutes with X_i, we also have $Y_i C = -CY_i$, for all i. Consider now $i \in \{0, 1\}$ and $P = \{i_1, \ldots, i_r\}$, with $i_1 < i_2 < \cdots < i_r$. Then

$$
\begin{aligned}
C^i X_P &= C^i X_{i_1} X_{i_2} \cdots X_{i_r} \\
&= C^i (CY_{i_1})(CY_{i_2}) \cdots (CY_{i_r}) \\
&= (-1)^{\frac{r(r-1)}{2}} C^{r+i} Y_{i_1} Y_{i_2} \cdots Y_{i_r} \\
&= (-1)^{\frac{|P|(|P|-1)}{2}} C^{|P|+i} Y_P \\
&= (-1)^{\frac{|P|(|P|-1)}{2}} (x_P^* + (-1)^{|P|+i} (cx_P)^*).
\end{aligned}
$$

From this we easily obtain the formulas for x_P^* and $(cx_P)^*$. $\square$

Remark 3.3.4. For $i \in \{0,1\}$ and $P \subseteq \{1,\ldots,n\}$ we have

$$
(c^i x_P)^* = \frac{1}{2}(-1)^{\frac{|P|(|P|-1)}{2} + i|P|} (X_P + (-1)^i C X_P). \tag{3.8}
$$

3.4 A canonical representation of $\mathsf{BrPic}(\mathcal{C}_n)$

We show in this section that $\mathcal{Z}(\mathcal{C}_n)$ has precisely two invertible objects: ε and χ. Thus, there is a canonical action of $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n))$ on the space of extensions of χ by ε. We prove that this space is a $2n$-dimensional vector space.

We start with the following observation.

Proposition 3.4.1. *Let $\mathcal{C}$ be a tensor category and let U and V be two simple objects of $\mathcal{C}$ such that $\alpha(V) = V$ and $\alpha(U) = U$ for all tensor autoequivalences $\alpha : \mathcal{C} \to \mathcal{C}$. Then isomorphisms*

$$
\rho(\alpha) : \mathsf{Ext}^1(U, V) \xrightarrow{\sim} \mathsf{Ext}^1(\alpha(U), \alpha(V)) = \mathsf{Ext}^1(U, V), \tag{3.9}
$$

where the image of extension $0 \to V \xrightarrow{i} E \xrightarrow{p} U \to 0$ under $\rho(\alpha)$ is

$$
0 \to V \xrightarrow{\alpha(i)} \alpha(E) \xrightarrow{\alpha(p)} U \to 0
$$

give rise to a projective representation of $\mathsf{Aut}^{\otimes}(\mathcal{C})$ *on* $\mathsf{Ext}^1(U, V)$.

Proof. Let α, $\alpha' : \mathcal{C} \to \mathcal{C}$ be tensor autoequivalences and let $\phi : \alpha \to \alpha'$ be a tensor isomorphism between them (so that α and α' determine the same element of $\mathsf{Aut}^{\otimes}(\mathcal{C})$). We have an isomorphism of extensions:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\alpha(i)"] \arrow[d, "\phi_V"] & \alpha(E) \arrow[r, "\alpha(p)"] \arrow[d, "\phi_E"] & U \arrow[r] \arrow[d, "\phi_U"] & 0 \\ 0 \arrow[r] & V \arrow[r, "\alpha'(i)"] & \alpha'(E) \arrow[r, "\alpha'(p)"] & U \arrow[r] & 0, \end{tikzcd}$$

where ϕ_V, ϕ_U are non-zero scalars and ϕ_E is an isomorphism. Thus, the equivalence classes of $\rho(\alpha)$ and $\rho(\alpha')$ differ by the scalar $\phi_V \phi_U^{-1}$. Hence, the map

$$\rho : \mathsf{Aut}(\mathcal{C}) \to PGL(\mathsf{Ext}^1(U, V))$$

is well defined. It is clear that this map is a group homomorphism. $\square$

$\mathcal{Z}(\mathcal{C}_n) = \mathsf{Rep}\, D(E(n))$ has precisely two invertible objects: the trivial representation ε and a one-dimensional representation χ, as we next show.

Lemma 3.4.2. *The algebra* $D(E(n))$ *has a unique non-trivial one-dimensional representation,* $\chi : D(E(n)) \to k$, *defined by*

$$\chi(C) = \chi(c) = -1, \quad \chi(x_i) = \chi(X_i) = 0, \quad i = 1, \dots, n.$$

Proof. It follows from relations (3.5)-(3.7) that, for a one-dimensional representation $\chi : D(E(n)) \to k$, one has $\chi(X_i) = \chi(x_i) = 0$, for all $i = 1, \dots, n$, $\chi(c)^2 = \chi(C)^2 = 1$, and $\chi(cC) = 1$. This implies the claim. $\square$

Proposition 3.4.3. *The space* $\mathsf{Ext}^1(\chi, \varepsilon)$ *of equivalence classes of extensions of the one-dimensional representation* χ *by the trivial representation* ε *is isomorphic to* $\mathbf{k}^{2n}$. *The equivalence class corresponding to* $\mathbf{a} = (a_1, \dots, a_{2n}) \in \mathbf{k}^{2n}$ *is the one associated to the extension*

$$0 \to \varepsilon \xrightarrow{i} V_{\mathbf{a}} \xrightarrow{p} \chi \to 0$$

where $V_{\mathbf{a}} = \mathbf{k}^2$ is the 2-dimensional $D(E(n))$-module with basis $\{v_1 = (1,0), v_2 = (0,1)\}$, $D(E(n))$-action given in matrix form by

$$C, c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X_i \mapsto \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}, \ x_i \mapsto \begin{pmatrix} 0 & a_{n+i} \\ 0 & 0 \end{pmatrix}, \ i = 1, \dots, n \tag{3.10}$$

and the maps i and p are such that $i(1) = v_1$ and $p(v_2) = 1$.

Proof. Let V be an extension of χ by ε. Then V comes equipped with two maps i and p such that

$$0 \to \varepsilon \xrightarrow{i} V \xrightarrow{p} \chi \to 0$$

is an exact sequence. Let $v_1 = i(1)$ and choose $v_2 \in V$ such that $p(v_2) = 1$. Then $\{v_1, v_2\}$ is a $\mathbf{k}$-basis of V on which the elements of $D(E(n))$ act by $h \cdot v_1 = \varepsilon(h)v_1$ and $h \cdot v_2 = f(h)v_1 + \chi(h)v_2$, for all $h \in D(E(n))$, and for some linear map $f \in \mathsf{Hom}(D(E(n)), k)$.

Consider now another extension

$$0 \to \varepsilon \xrightarrow{i'} V' \xrightarrow{p'} \chi \to 0$$

of χ by ε and associate to V', as above, a basis $\{v_1', v_2'\}$ and a linear map $f' \in \mathsf{Hom}(D(E(n)), k)$. We claim that if there exists a homomorphism of extensions $\varphi : V \to V'$ then f and f' differ by a multiple of $\chi - \varepsilon$. Indeed, if φ is such a map then, then from $\varphi \circ i = i'$ and $p' \circ \varphi = p$ we readily deduce that $\varphi(v_1) = v_1'$ and $\varphi(v_2) = \lambda v_1' + v_2'$, for some $\lambda \in k$. Letting $h \in D(E(n))$ act on the latter relation and taking into account that φ commutes with the action of $D(E(n))$, we arrive at the equality $\big(f(h) + \lambda\chi(h)\big)v_1' + \chi(h)v_2' = \big(\lambda\varepsilon(h) + f'(h)\big)v_1' + \chi(h)v_2'$, which shows that $f' - f = \lambda(\chi - \varepsilon)$.

In particular, if we take $V' = V$ and $\varphi = \mathsf{id}_V$, we see that the $2n$-tuple $(f(X_1), \dots, f(X_n), f(x_1), \dots f(x_n))$ does not depend on the choice of v_2. Also, the above discussion shows that the same $2n$-tuple depends only on the equivalence class of V. We can, thus, define a map $\mathsf{Ext}^1(\chi, \varepsilon) \to \mathbf{k}^{2n}$ sending the equivalence class of V to $(f(X_1), \dots, f(X_n), f(x_1), \dots f(x_n))$. This map is easily seen to be one-to-one and onto, sending the equivalence class of the extension $V_{\mathbf{a}}$, in the statement, to $\mathbf{a} \in \mathbf{k}^{2n}$. $\square$

Using Proposition 3.4.1, we see that there is a group homomorphism

$$\rho : \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}_n)) \to \mathsf{PGL}(\mathsf{Ext}^1(\varepsilon,\, \chi)) = \mathsf{PGL}_{2n}(\mathbf{k}). \tag{3.11}$$

3.5 $\mathsf{BrPic}(\mathcal{C}_n)$ as a symplectic group

In this section we show that the image of (3.11) lies, actually, in the projective symplectic group $\mathsf{PSp}_{2n}(\mathbf{k})$. We do this by proving that the elements of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}_n))$ preserve a symplectic form on $\mathsf{Ext}^1(\varepsilon,\, \chi)$. Thus, $\mathsf{BrPic}(\mathcal{C}_n)$ can be viewed as a symplectic group.

Consider the sympletic bilinear form

$$\omega : \mathbf{k}^{2n} \times \mathbf{k}^{2n} \to \mathbf{k}, \qquad \omega(\mathbf{a},\, \mathbf{b}) = \textstyle\sum_{i=1}^{n}(a_i b_{n+i} - a_{n+i} b_i),$$

for all $\mathbf{a} = (a_1, \ldots, a_{2n})$, $\mathbf{b} = (b_1, \ldots, b_{2n}) \in \mathbf{k}^{2n}$. The *symplectic group* and the *projective symplectic group* of degree $2n$ are, respectively:

$$\begin{aligned} \mathsf{Sp}_{2n}(\mathbf{k}) &= \{T \in \mathsf{GL}_{2n}(\mathbf{k}) \mid \omega(T(\mathbf{a}),\, T(\mathbf{b})) = \omega(\mathbf{a},\, \mathbf{b})\}, \\ \mathsf{PSp}_{2n}(\mathbf{k}) &= \mathsf{Sp}_{2n}(\mathbf{k})/\{\pm I_{2n}\}. \end{aligned}$$

To see how the braided tensor category $\mathcal{Z}(\mathcal{C}_n)$ gives rise to a symplectic form, let us switch to the language of Yetter-Drinfeld modules. In particular, let us describe the extensions of χ by ε as Yetter-Drinfeld modules.

Lemma 3.5.1. *Let $V_{\mathbf{a}}$ be the extension of χ by ε associated to $\mathbf{a} = (a_1, \ldots, a_{2n}) \in \mathbf{k}^{2n}$. If $\{v_1, v_2\}$ is a basis of $V_{\mathbf{a}}$ on which $D(E(n))$ acts by* (3.10)*, then the Yetter-Drinfeld module structure of $V_{\mathbf{a}}$ is given by the $E(n)$-action*

$$c \cdot v_1 = v_1, \quad x_i \cdot v_1 = 0, \quad c \cdot v_2 = -v_2, \quad x_i \cdot v_2 = a_{n+i} v_1$$

for all $i = 1, \ldots, n$, *and the* $E(n)$*-coaction*

$$\delta(v_1) = v_1 \otimes 1 \quad \textit{and} \quad \delta(v_2) = \textstyle\sum_{j=1}^{n} a_j v_1 \otimes x_j + v_2 \otimes c$$

Proof. This follows from Proposition 2.7.8. To see that the $E(n)$-coaction δ is the one stated, we use that

$$\delta(v) = \textstyle\sum_{i,P} (c^i x_P)^* \cdot v \otimes c^i x_P$$

formulas (3.8) and the following relations, which are straightforward to check:

$$(C^i X_P) \cdot v_1 = \begin{cases} v_1 & \text{if} \quad P = \emptyset \\ 0 & \text{if} \quad P \neq \emptyset \end{cases} \quad \text{and} \quad (C^i X_P) \cdot v_2 = \begin{cases} (-1)^i v_2 & \text{if} \quad P = \emptyset \\ a_j v_1 & \text{if} \quad P = \{j\} \\ 0 & \text{if} \quad |P| \geq 2 \end{cases}$$

□

Proposition 3.5.2. *Let* $0 \to \varepsilon \xrightarrow{i} V_{\mathbf{a}} \xrightarrow{p} \chi \to 0$ *and* $0 \to \varepsilon \xrightarrow{j} V_{\mathbf{b}} \xrightarrow{q} \chi \to 0$ *be two extensions of* χ *by* ε *associated to* $\mathbf{a} = (a_1, \ldots, a_{2n})$ *and* $\mathbf{b} = (b_1, \ldots, b_{2n})$*. Then*

$$c_{V_{\mathbf{b}}, V_{\mathbf{a}}} \circ c_{V_{\mathbf{a}}, V_{\mathbf{b}}} = \mathsf{id}_{V_{\mathbf{a}} \otimes V_{\mathbf{b}}} + \omega(\mathbf{a}, \mathbf{b})\,(i \otimes j) \circ (p \otimes q)$$

Proof. Let $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$ be bases of $V_{\mathbf{a}}$ and $V_{\mathbf{b}}$, respectively, on which $D(E(n))$ acts as in (3.10). Then it follows from (2.36) and the Yetter-Drinfeld module structures of $V_{\mathbf{a}}$ and $V_{\mathbf{b}}$ that the matrix of $c_{V_{\mathbf{b}}, V_{\mathbf{a}}} \circ c_{V_{\mathbf{a}}, V_{\mathbf{b}}}$ in the basis $\{v_1 \otimes v'_1,\ v_1 \otimes v'_2,\ v_2 \otimes v'_1,\ v_2 \otimes v'_2\}$ is

$$\begin{pmatrix} 1 & 0 & 0 & \omega(\mathbf{b}, \mathbf{a}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For example,

$$\begin{aligned}
c_{V_{\mathbf{b}},V_{\mathbf{a}}} \circ c_{V_{\mathbf{a}},V_{\mathbf{b}}}(v_2 \otimes v_2') &= c_{V_{\mathbf{b}},V_{\mathbf{a}}}\left(\sum_{i=1}^{n} b_i v_1' \otimes x_i \cdot v_2 + v_2' \otimes c \cdot v_2\right) \\
&= c_{V_{\mathbf{b}},V_{\mathbf{a}}}\left(\sum_{i=1}^{n} b_i a_{n+i} v_1' \otimes v_1 - v_2' \otimes v_2\right) \\
&= \sum_{i=1}^{n} b_i a_{n+i} v_1 \otimes v_1' - \sum_{i=1}^{n} a_i v_1 \otimes x_i \cdot v_2' - v_2 \otimes c \cdot v_2' \\
&= \sum_{i=1}^{n} b_i a_{n+i} v_1 \otimes v_1' - \sum_{i=1}^{n} b_{n+i} a_i v_1 \otimes v_1' + v_2 \otimes v_2' \\
&= \omega(\mathbf{b},\mathbf{a}) v_1 \otimes v_1' + v_2 \otimes v_2'.
\end{aligned}$$

The same matrix is obtained for the map $\mathsf{id}_{V_{\mathbf{a}} \otimes V_{\mathbf{b}}} + \omega(\mathbf{a},\mathbf{b})\,(i \otimes j) \circ (p \otimes q)$, if $(i \otimes j) \circ (p \otimes q)$ is the appropriate composition

$$V_{\mathbf{a}} \otimes V_{\mathbf{b}} \xrightarrow{p \otimes q} \chi \otimes \chi \cong \varepsilon \cong \varepsilon \otimes \varepsilon \xrightarrow{i \otimes j} V_{\mathbf{a}} \otimes V_{\mathbf{b}}$$

□

Proposition 3.5.3. *Let* $0 \to \varepsilon \xrightarrow{i} V_{\mathbf{a}} \xrightarrow{p} \chi \to 0$ *and* $0 \to \varepsilon \xrightarrow{j} V_{\mathbf{b}} \xrightarrow{q} \chi \to 0$ *be two extensions of* χ *by* ε *associated to* $\mathbf{a}$ *and* $\mathbf{b} \in \mathbf{k}^{2n}$. *If* $\alpha \in \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}_n))$, $\alpha(V_{\mathbf{a}}) = V_{\alpha(\mathbf{a})}$ *and* $\alpha(V_{\mathbf{b}}) = V_{\alpha(\mathbf{b})}$ *then* $\omega(\alpha(\mathbf{a}), \alpha(\mathbf{b})) = \omega(\mathbf{a}, \mathbf{b})$.

Proof. Let J be the monoidal structure of α. Since α is a braided functor, the following diagram is commutative

$$\begin{tikzcd}
V_{\alpha(\mathbf{a})} \otimes V_{\alpha(\mathbf{b})} \arrow[r, "c_{V_{\alpha(\mathbf{a})},V_{\alpha(\mathbf{b})}}"] \arrow[d, "J_{V_{\mathbf{a}},V_{\mathbf{b}}}"'] & V_{\alpha(\mathbf{b})} \otimes V_{\alpha(\mathbf{a})} \arrow[r, "c_{V_{\alpha(\mathbf{b})},V_{\alpha(\mathbf{a})}}"] & V_{\alpha(\mathbf{a})} \otimes V_{\alpha(\mathbf{b})} \arrow[d, "J_{V_{\mathbf{a}},V_{\mathbf{b}}}"] \\
\alpha(V_{\mathbf{a}} \otimes V_{\mathbf{b}}) \arrow[r, "\alpha(c_{V_{\mathbf{a}},V_{\mathbf{b}}})"'] & \alpha(V_{\mathbf{b}} \otimes V_{\mathbf{a}}) \arrow[r, "\alpha(c_{V_{\mathbf{b}},V_{\mathbf{a}}})"'] & \alpha(V_{\mathbf{a}} \otimes V_{\mathbf{b}})
\end{tikzcd}$$

Using Proposition 3.5.2, we have

$$\begin{aligned}
c_{V_{\alpha(\mathbf{b})},V_{\alpha(\mathbf{a})}} \circ c_{V_{\alpha(\mathbf{a})},V_{\alpha(\mathbf{b})}} &= J^{-1}_{V_{\mathbf{a}},V_{\mathbf{b}}} \alpha(c_{V_{\mathbf{b}},V_{\mathbf{a}}} \circ c_{V_{\mathbf{a}},V_{\mathbf{b}}}) J_{V_{\mathbf{a}},V_{\mathbf{b}}} \\
&= J^{-1}_{V_{\mathbf{a}},V_{\mathbf{b}}} \Big(\alpha(\mathsf{id}_{V_{\mathbf{a}}\otimes V_{\mathbf{b}}}) + \omega(\mathbf{a},\mathbf{b})\, \alpha\big((i\otimes j)\circ(p\otimes q)\big) \Big) J_{V_{\mathbf{a}},V_{\mathbf{b}}} \\
&= \mathsf{id}_{V_{\alpha(\mathbf{a})}\otimes V_{\alpha(\mathbf{b})}} + \omega(\mathbf{a},\mathbf{b})\, J^{-1}_{V_{\mathbf{a}},V_{\mathbf{b}}} \alpha\big((i\otimes j)\circ(p\otimes q)\big)\Big) J_{V_{\mathbf{a}},V_{\mathbf{b}}} \\
&= \mathsf{id}_{V_{\alpha(\mathbf{a})}\otimes V_{\alpha(\mathbf{b})}} + \omega(\mathbf{a},\mathbf{b}) \Big(\big(\alpha(i)\otimes\alpha(j)\big) \circ \big(\alpha(p)\otimes\alpha(q)\big) \Big)
\end{aligned}$$

where the last equality follows from the commutativity of diagram

$$\begin{tikzcd}
V_{\alpha(\mathbf{a})} \otimes V_{\alpha(\mathbf{b})} \arrow[r, "\alpha(p)\otimes\alpha(q)"] \arrow[d, "J_{V_{\mathbf{a}},V_{\mathbf{b}}}"'] & \chi\otimes\chi \cong \varepsilon \cong \varepsilon\otimes\varepsilon \arrow[r, "\alpha(i)\otimes\alpha(j)"] & V_{\alpha(\mathbf{a})} \otimes V_{\alpha(\mathbf{b})} \arrow[d, "J_{V_{\mathbf{a}},V_{\mathbf{b}}}"] \\
\alpha(V_{\mathbf{a}} \otimes V_{\mathbf{b}}) \arrow[r, "\alpha(p\otimes q)"'] & \alpha(\chi\otimes\chi) \cong \alpha(\varepsilon) \cong \alpha(\varepsilon\otimes\varepsilon) \arrow[r, "\alpha(i\otimes j)"'] & \alpha(V_{\mathbf{a}} \otimes V_{\mathbf{b}})
\end{tikzcd}$$

From Proposition 3.5.2 it follows that $\omega(\alpha(\mathbf{a}),\alpha(\mathbf{b})) = \omega(\mathbf{a},\mathbf{b})$. $\square$

Corollary 3.5.4. *The image of the group homomorphism* (3.11) *belongs to* $\mathsf{PSp}_{2n}(\mathbf{k})$.

3.6 Subcategories of $\mathcal{Z}(\mathcal{C}_n)$

In this section we describe the sets $\mathbb{L}(\mathcal{C}_n)$ and $\mathbb{L}_0(\mathcal{C}_n)$ of subcategories of $\mathcal{Z}(\mathcal{C}_n)$ which are tensor, respectively braided, equivalent to $\mathcal{C}_n$. We show that $\mathbb{L}(\mathcal{C}_n)$ can be identified with the set $\mathsf{Gr}(n,2n)$ of n-dimensional subspaces of $\mathbf{k}^{2n}$, while $\mathbb{L}_0(\mathcal{C}_n)$ can be identified with the subset $\mathsf{Lag}(n,2n)$ of $\mathsf{Gr}(n,2n)$ consisting of Lagrangian subspaces of $\mathbf{k}^{2n}$, i.e. n-dimensional subspaces of $\mathbf{k}^{2n}$ on which the symplectic form ω vanishes. We then relate $\mathbb{L}(\mathcal{C}_n)$ and $\mathbb{L}_0(\mathcal{C}_n)$ with $\mathsf{Ext}^1(\chi,\varepsilon) \cong \mathbf{k}^{2n}$. We show that there is a one-to-one correspondence between $\mathbb{L}(\mathcal{C}_n)$ and the set of n-dimensional subspaces of $\mathsf{Ext}^1(\chi,\varepsilon)$. The correspondence preserves the action of the Brauer-Picard group, a fact which will be useful later, when we determine the kernel of the action of $\mathsf{BrPic}(\mathcal{C}_n)$ on $\mathsf{Ext}^1(\chi,\varepsilon)$.

By a *tensor subcategory* $\mathcal{D}$ of a tensor category $\mathcal{C}$ we mean the image of a fully faithful tensor functor (i.e., embedding) $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$. Tensor subcategories of $\mathsf{Rep}\, H$, for a Hopf algebra H, can be described as follows.

Proposition 3.6.1. *Let H be a finite-dimensional Hopf algebra. The set of tensor subcategories of $\mathsf{Rep}(H)$ is in bijection with the set of equivalence classes of surjective Hopf algebra homomorphisms $p : H \to K$, under the following equivalence relation: two surjective homomorphisms $p : H \to K$ and $p' : H \to K'$ are equivalent if there is a Hopf algebra isomorphism $f : K \xrightarrow{\sim} K'$ such that $f \circ p = p'$.*

Proof. If $p : H \to K$ is a surjective Hopf algebra map then the restriction of scalars functor $Res_p : \mathsf{Rep}(K) \to \mathsf{Rep}(H)$ is an embedding. The image of Res_p consists of isomorphism classes of representations of H that factor through p, so, it does not change when p is composed with an isomorphism.

Conversely, let $\iota : \mathcal{D} \to \mathsf{Rep}(H)$ be a tensor embedding. Let $F : \mathsf{Rep}(H) \to \mathbf{k}$-Vec be the forgetful tensor functor. By Tannakian formalism, $K = \mathsf{End}(F \circ \iota)$ is a Hopf algebra such that $\mathcal{D}$ is canonically equivalent to $\mathsf{Rep}(K)$. Also, $H \cong \mathsf{End}(F)$. The natural map $p : H = \mathsf{End}(F) \to \mathsf{End}(F \circ \iota) = K$ is a surjective homomorphism of Hopf algebras. It is clear that any embedding $\iota' : \mathcal{D}' \to \mathsf{Rep}(H)$ with $\iota(\mathcal{D}) = \iota(\mathcal{D}')$ results in a homomorphism $p' : H \to K'$ equivalent to p. □

Lemma 3.6.2. *The set of surjective Hopf algebra maps $D(E(n)) \to E(n)$ is in bijection with the set of $n \times 2n$ matrices of rank n. The homomorphism f corresponding to $(A|B) \in M_{n\times 2n}(k)$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices, is given by*

$$f(C) = f(c) = c, \quad f(X_i) = \sum_{j=1}^{n} a_{ji} x_j, \quad f(x_i) = \sum_{j=1}^{n} b_{ji} x_j, \quad i = 1, \ldots, n \tag{3.12}$$

Proof. Let $f : D(E(n)) \to E(n)$ be a Hopf algebra map. Since C and c are group-like elements of $D(E(n))$, we have $f(C), f(c) \in G(E(n)) = \{1, c\}$.

If $(f(C), f(c)) = (1, 1)$ then $f(X_i)$ and $f(x_i)$, $i = 1, \ldots, n$, are primitive elements of $E(n)$, so $f(X_i) = f(x_i) = 0$, for all $i = 1, \ldots, n$. Thus, f is the trivial homomorphism, $f(h) = \varepsilon(h)1$, which is not surjective.

If $(f(C), f(c)) = (1, c)$ then $f(X_i) = 0$, for all $i = 1, \ldots, n$. Applying f to the relation $x_iX_i + X_ix_i = 1 - Cc$ we obtain $0 = 1 - c$, which is not possible. Similarly, if $(f(C), f(c)) = (c, 1)$.

If $(f(C), f(c)) = (c, c)$ then $f(X_i)$ and $f(x_i)$ are $(1, c)$-primitive elements of $E(n)$, for all $i = 1, \ldots, n$. Since the space of $(1,\ c)$-primitive elements of $E(n)$ is $k(1 - c) \oplus kx_1 \oplus \cdots \oplus kx_n$ it follows that there exist a, b, a_{ij}, $b_{ij} \in k$, $i, j = 1, \ldots, n$, such that $f(X_i) = a(1 - c) + \sum_j a_{ji}x_j$ and $f(x_i) = b(1 - c) + \sum_j b_{ji}x_j$, for all $i = 1, \ldots, n$. Using the relations $x_ic + cx_i = 0$ and $X_iC + CX_i = 0$, we readily deduce that $a = b = 0$. Since the remaining relations impose no other restrictions on the scalars a_{ij} and b_{ij} we are left to see under what conditions the homomorphism associated to these scalars is surjective.

We claim that f is surjective if and only if f maps $U = \text{span}\{X_1, \ldots, X_n, x_1, \ldots, x_n\}$ onto $\text{span}\{x_1, \ldots, x_n\}$. For this, it suffices to prove that if x_i is in the image of f then it is in the image of the restriction of f to U.

Suppose $x_i = f(h)$, for some $h \in D(E(n))$. Since $B = \{C^jX_Pc^lx_Q \mid j, l \in \{0, 1\}, P, Q \subseteq \{1, \ldots, n\}\}$ is a basis of $D(E(n))$ there exist $u \in U$, $v \in V = \text{span}\{CX_jc, Ccx_j \mid j = 1, \ldots, n\}$ and $w \in W = \text{span}\, B \setminus \{X_j, x_j, CX_jc, Ccx_j \mid j = 1, \ldots, n\}$ such that $h = u + v + w$. Now $f(u), f(v) \in \text{span}\{x_1, \ldots, x_n\}$ and $f(w) \in \text{span}\{c^jx_P\} \setminus \{x_1, \ldots, x_n\}$, so, from $x_i = f(u) + f(v) + f(w)$ we deduce that $f(w) = 0$. Taking into account that $f(CX_jc) = -f(X_j)$ and $f(Ccx_j) = f(x_j)$, for all $j = 1, \ldots, n$, we see that $f(v) \in f(U)$, hence $x_i \in f(U)$.

Thus, f is surjective if and only if f maps U onto $\text{span}\{x_1, \ldots, x_n\}$. In terms of the scalars a_{ij} and b_{ij} this is equivalent to saying that the rank of the $n \times 2n$ matrix $(A|B)$, where $A = (a_{ij})$ and $B = (b_{ij})$, is n. The lemma is proved. □

Proposition 3.6.3. *The set $\mathbb{L}(\mathcal{C}_n)$ of subcategories of $\mathcal{Z}(\mathcal{C}_n)$ tensor equivalent to $\mathcal{C}_n$ is identified with* Gr $(n, 2n)$*, the Grassmannian of n-dimensional subspaces of a $2n$-dimensional vector space.*

Proof. Taking into account Proposition 3.6.1, the description (3.4) of the automorphisms of $E(n)$, and Proposition 3.6.2, $\mathbb{L}(\mathcal{C}_n)$ is identified with the set of equivalence classes of $n \times 2n$ matrices of rank n, under the equivalence relation induced by left multiplication with invertible $n \times n$ matrices.

The latter set is $\mathsf{Gr}(n, 2n)$. Indeed, if $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times 2n$ matrices of rank n then the rows of A, $r_1(A), \ldots, r_n(A)$, and the rows of B, $r_1(B), \ldots, r_n(B)$, generate the same subspace of $\mathbf{k}^{2n}$ if and only if there exists $T = (t_{ij}) \in \mathsf{GL}_n(k)$ such that $r_i(B) = \sum_j t_{ij} r_j(A) = r_i(TA)$, for all $i = 1, \ldots, n$, that is, if and only if $B = TA$. $\square$

We now prove that $\mathbb{L}_0(\mathcal{C}_n)$ can be identified with the set of Lagrangian subspaces of the symplectic space $(\mathbf{k}^{2n}, \omega)$, where

$$\omega : \mathbf{k}^{2n} \times \mathbf{k}^{2n} \to \mathbf{k}, \quad \omega(\mathbf{a}, \mathbf{b}) = \textstyle\sum_{i=1}^{n} (a_i b_{n+i} - a_{n+i} b_i)$$

for all $\mathbf{a} = (a_1, \ldots, a_{2n})$ and $\mathbf{b} = (b_1, \ldots, b_{2n}) \in \mathbf{k}^{2n}$.

Recall that a subspace V of $\mathbf{k}^{2n}$ is called *isotropic* if $\omega(\mathbf{a}, \mathbf{b}) = 0$, for all $\mathbf{a}, \mathbf{b} \in V$. An isotropic subspace V is called *Lagrangian* if $\dim_{\mathbf{k}}(V) = n$ (which is the maximal possible dimension of an isotropic subspace).

We need the following result.

Lemma 3.6.4. *If $f : D(E(n)) \to E(n)$ is given by* (3.12) *and $P = \{i_1, \ldots, i_r\}$ is a subset of $\{1, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_r$, then*

$$f(X_P) = \sum_{|F|=|P|} [A]_{F,P} x_F \qquad \textit{and} \qquad f(x_P) = \sum_{|F|=|P|} [B]_{F,P} x_F \tag{3.13}$$

Proof. We have

$$
\begin{aligned}
f(X_P) &= f(X_{i_1})\cdots f(X_{i_r}) \\
&= \sum_{j_1,\dots,j_r} a_{j_1,i_1}\cdots a_{j_r,i_r} x_{j_1}\cdots x_{j_r} \\
&= \sum_{\substack{j_1<\cdots<j_r \\ \sigma\in S_r}} a_{\sigma(j_1),i_1}\cdots a_{\sigma(j_r),i_r} x_{\sigma(j_1)}\cdots x_{\sigma(j_r)} \\
&= \sum_{j_1<\cdots<j_r} \left(\sum_{\sigma\in S_r} \operatorname{sgn}(\sigma) a_{\sigma(j_1),i_1}\cdots a_{\sigma(j_r),i_r}\right) x_{j_1}\cdots x_{j_r} \\
&= \sum_{|F|=|P|} [A]_{F,P} x_F
\end{aligned}
$$

and similarly for $f(x_P)$. □

Proposition 3.6.5. $\mathbb{L}_0(\mathcal{C}_n) = \mathrm{Lag}(n, 2n)$*, the Grassmannian of Lagrangian subspaces of the symplectic space* $(\mathbf{k}^{2n}, \omega)$.

Proof. Under the identification of Proposition 3.6.1 $\mathbb{L}_0(\mathcal{C}_n)$ corresponds to the set of equivalence classes of surjective Hopf algebra maps $D(E(n)) \to E(n)$ that take the canonical R-matrix of $D(E(n))$ to a triangular structure on $E(n)$.

Let A, $B \in M_n(k)$ be such that the two block matrix $M = (A|B)$ has rank n and let $f : D(E(n)) \to E(n)$ be the map given by (3.12). Let $R = \sum_{i,P} c^i x_P \otimes (c^i x_P)^*$ be the canonical R-matrix of $D(E(n))$. Then, taking into account (3.13) and using (3.8), we have:

$$
\begin{aligned}
(f\otimes f)(R) &= \frac{1}{2}\sum_{i,P}(-1)^{\frac{|P|(|P|-1)}{2}+i|P|}f(c^i x_P)\otimes f(X_P+(-1)^i CX_P)\\
&= \frac{1}{2}\sum_{i,|E|=|F|=|P|}(-1)^{\frac{|P|(|P|-1)}{2}+i|P|}[A]_{E,P}[B]_{F,P}c^i x_E\otimes (x_F+(-1)^i cx_F)\\
&= \frac{1}{2}\sum_{|E|=|F|=|P|}(-1)^{\frac{|P|(|P|-1)}{2}}[A]_{E,P}[B]_{F,P}\Big(x_E\otimes x_F+x_E\otimes cx_F+\\
&\quad +(-1)^{|P|}cx_E\otimes x_F+(-1)^{|P|+1}cx_E\otimes cx_F\Big)\\
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{\frac{j(j-1)}{2}}\sum_{|E|=|F|=|P|=j}[A]_{E,P}[B]_{F,P}\Big(x_E\otimes x_F+x_E\otimes cx_F+\\
&\quad +(-1)^{j}cx_E\otimes x_F+(-1)^{j+1}cx_E\otimes cx_F\Big)\\
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{\frac{j(j-1)}{2}}\sum_{|E|=|F|=j}\left(\sum_{|P|=j}[A]_{E,P}[B]_{F,P}\right)\Big(x_E\otimes x_F+\\
&\quad +x_E\otimes cx_F+(-1)^{j}cx_E\otimes x_F+(-1)^{j+1}cx_E\otimes cx_F\Big)\\
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{\frac{j(j-1)}{2}}\sum_{|E|=|F|=j}[AB^t]_{E,F}\Big(x_E\otimes x_F+x_E\otimes cx_F+\\
&\quad +(-1)^{j}cx_E\otimes x_F+(-1)^{j+1}cx_E\otimes cx_F\Big)\\
&= R_{AB^t}
\end{aligned}
$$

where B^t denotes the transpose matrix of B and where we used the well known formula for the minor of a product of two matrices, $[AB]_{E,F}=\sum_{|P|=|E|}[A]_{E,P}[B]_{P,F}$.

Thus, f takes the canonical R-matrix of $D(E(n))$ to the R-matrix corresponding to AB^t. Recall that the latter is a triangular structure if and only if AB^t is symmetric. This is equivalent to $AB^t=BA^t$, or, what is the same, to $\sum_{l=1}^n a_{il}b_{jl}=\sum_{l=1}^n b_{il}a_{jl}$, for all i, $j=1,\ldots,n$. Subtracting the right hand term in the previous equality from the other, we obtain $\sum_{l=1}^n(a_{il}b_{jl}-b_{il}a_{jl})=0$, for all i, $j=1,\ldots,n$. If $r_1(M),\ldots,r_n(M)$ denote the rows of M, then the last condition is equivalent to $\omega(r_i(M),r_j(M))=0$, for all i, $j=1,\ldots,n$.

Thus, the surjective Hopf algebra maps $D(E(n))\to E(n)$ which take the canonical quasitriangular structure of $D(E(n))$ to a triangular structure of $E(n)$ correspond to $n\times 2n$ matrices, of rank

n, with entries from $\mathbf{k}$, such that the symplectic form ω on $\mathbf{k}^{2n}$ vanishes on the subspace generated by their rows. Equivalence classes of such maps have, as their correspondent in $\mathsf{Gr}(n, 2n)$, those subspaces on which the symplectic form vanishes, whence the assertion in the statement. $\square$

Remark 3.6.6. If $\mathcal{C}$ is a braided tensor category then $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}))$ acts on $\mathbb{L}_0(\mathcal{C})$ by permutation of categories:

$$\alpha \cdot \mathcal{L} = \alpha(\mathcal{L}), \quad \alpha \in \mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C})), \quad \mathcal{L} \in \mathbb{L}_0(\mathcal{C}).$$

The stabilizer of $\iota_{\mathcal{C}}(\mathcal{C})$ in $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}))$, where $\iota_{\mathcal{C}} : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$, is the embedding $Z \mapsto (Z, c_{-,Z})$, is

$$\mathsf{St}(\iota_{\mathcal{C}}(\mathcal{C})) = \mathsf{Pic}(\mathcal{C}) \rtimes \mathsf{Aut}^{\mathrm{br}}(\mathcal{C}).$$

The proof is the same as of [NR14, Proposition 6.8].

We now relate $\mathbb{L}(\mathcal{C}_n)$ and $\mathbb{L}_0(\mathcal{C}_n)$ with $\mathsf{Ext}^1(\chi, \varepsilon)$. For a tensor subcategory $\mathcal{C}$ of $\mathcal{Z}(\mathcal{C}_n)$ let $\mathcal{C} \cap \mathsf{Ext}^1(\chi, \varepsilon)$ denote the subspace of $\mathsf{Ext}^1(\chi, \varepsilon)$ consisting of equivalence classes of extensions $0 \to \varepsilon \to V \to \chi \to 0$ such that V belongs to $\mathcal{C}$.

Proposition 3.6.7. *The assignment* $\mathcal{C} \to \mathcal{C} \cap \mathsf{Ext}^1(\chi, \varepsilon)$ *induces bijections* $\mathbb{L}(\mathcal{C}_n) \to \mathsf{Gr}(n, 2n)$ *and* $\mathbb{L}_0(\mathcal{C}_n) \to \mathsf{Lag}(n, 2n)$.

Proof. We saw in Proposition 3.6.3 that $\mathbb{L}(\mathcal{C}_n) = \mathsf{Gr}(n, 2n)$. The subcategory of $\mathcal{Z}(\mathcal{C}_n)$, tensor equivalent to $\mathcal{C}_n$, corresponding to $U \in \mathsf{Gr}(n, 2n)$ is $\mathcal{C}_U$, described as follows. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices such that the rows $r_1(M), \dots, r_n(M)$ of $M = (A|B) \in M_{n\times 2n}(\mathbf{k})$, form a basis of U. Then $\mathcal{C}_U$ is the image of the restriction functor associated to $f : D(E(n)) \to E(n)$, $f(C) = f(c) = c$, $f(X_i) = \sum_{j=1}^{n} a_{ji} x_j$, $f(x_i) = \sum_{j=1}^{n} b_{ji} x_j$, $i = 1, \dots, n$. We will show that, under the isomorphism of Proposition 3.4.3, $\mathcal{C}_U \cap \mathsf{Ext}^1(\chi, \varepsilon) = U$, which will prove the claim.

Let $0 \to \varepsilon \xrightarrow{i} V \xrightarrow{p} \chi \to 0$ be an element of $\mathcal{C}_U \cap \mathsf{Ext}(\chi, \varepsilon)$ and let $\{v_1, v_2\}$ be a basis of V such that $v_1 = i(1)$ and $p(v_2) = 1$. If $\{v_1^*, v_2^*\}$ is the dual basis of $\{v_1, v_2\}$ then the element of $\mathbf{k}^{2n}$ corresponding to V, under the isomorphism of Proposition 3.4.3, is

$$\mathbf{a}_V = \big(v_1^*(X_1 \cdot v_2), \dots, v_1^*(X_n \cdot v_2), v_1^*(x_1 \cdot v_2), \dots, v_1^*(X_1 \cdot v_2)\big)$$

Since $v_1^*(X_i \cdot v_2) = v_1^*\big(f(X_i)v_2\big) = v_1^*(\sum_j a_{ji}x_jv_2) = \sum_j a_{ji}v_1^*(x_jv_2)$ and $v_1^*(x_i \cdot v_2) = v_1^*\big(f(x_i)v_2\big) = v_1^*(\sum_j b_{ji}x_jv_2) = \sum_j b_{ji}v_1^*(x_jv_2)$, for all $i = 1, \dots, n$, we deduce that

$$\mathbf{a}_V = \textstyle\sum_{j=1}^n v_1^*(x_jv_2)(a_{j1}, \dots, a_{jn}, b_{j1}, \dots, b_{jn}) = \sum_{j=1}^n v_1^*(x_jv_2)r_j(M)$$

Thus, $\mathbf{a}_V \in U$, for all $V \in \mathcal{C}_U \cap \mathsf{Ext}^1(\chi, \varepsilon)$.

To complete the proof we need only show that $V_{r_i(M)} \in \mathcal{C}_U$, for all $i = 1, \dots, n$. A quick check shows that the representation $V_{r_i(M)}$ is obtained from the following matrix representation of $E(n)$:

$$E(n) \to M_2(\mathbf{k}), \quad c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_j \mapsto \begin{pmatrix} 0 & \delta_{i,j} \\ 0 & 0 \end{pmatrix}, \quad j = 1, \dots, n$$

by restriction of scalars via f. □

Remark 3.6.8. It is easy to see that, if $\alpha \in \mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n))$ and $\mathcal{C}_U$ is the tensor subcategory of $\mathcal{Z}(\mathcal{C}_n)$ corresponding to $U \in \mathsf{Gr}(n, 2n)$, then $\alpha(\mathcal{C}_U) = \mathcal{C}_{\alpha(U)}$.

3.7 Induction homomorphisms

In this section we discuss two ways of constructing elements of the Brauer-Picard group. One is to induce braided autoequivalences of $\mathcal{Z}(\mathcal{C})$ from tensor autoequivalences of $\mathcal{C}$. The other one is to induce invertible $\mathcal{Z}(\mathcal{C})$-module categories from invertible $\mathcal{D}$-module categories, where $\mathcal{D}$ is a tensor subcategory of $\mathcal{Z}(\mathcal{C})$.

For any tensor category $\mathcal{C}$ there is an induction homomorphism

$$\Gamma : \mathsf{Aut}^{\otimes}(\mathcal{C}) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C})) : \alpha \mapsto \Gamma_\alpha, \tag{3.14}$$

where $\Gamma_\alpha(Z, \gamma) = (\alpha(Z), \gamma^\alpha)$ and γ^α is defined by the following commutative diagram

$$
\begin{tikzcd}
X \otimes \alpha(Z) \arrow[rr, "\gamma_X^\alpha"] \arrow[d] & & \alpha(Z) \otimes X \arrow[d] \\
\alpha(\alpha^{-1}(X)) \otimes \alpha(Z) \arrow[d, "J_{\alpha^{-1}(X),Z}"'] & & \alpha(Z) \otimes \alpha(\alpha^{-1}(X)) \arrow[d, "J_{Z,\alpha^{-1}(X)}"] \\
\alpha(\alpha^{-1}(X) \otimes Z) \arrow[rr, "\alpha(\gamma_{\alpha^{-1}(X)})"] & & \alpha(Z \otimes \alpha^{-1}(X)).
\end{tikzcd}
$$

Here α^{-1} is a quasi-inverse of α and $J_{X,Y} : \alpha(X) \otimes \alpha(Z) \xrightarrow{\sim} \alpha(X \otimes Z)$ is the tensor functor structure of α.

Let us describe the braided autoequivalences of $\mathcal{Z}(\mathsf{Rep}\, H)$ induced from the tensor autoequivalences of $\mathsf{Rep}(H)$ of Remark 2.10.8. We will identify $\mathcal{Z}(\mathsf{Rep}\, H)$ with ${}_H\mathcal{YD}^H$, the category of Yetter-Drinfeld modules over H (see Proposition 2.7.8).

Example 3.7.1. Let $(\alpha, 1 \otimes 1)$ be the tensor autoequivalence of $\mathsf{Rep}\, H$, corresponding to a Hopf algebra automorphism α of H. Then $\Gamma_{(\alpha,1\otimes 1)}(V) = V$ as a vector space, with the H-action and H-coaction given by

$$
h \cdot v = \alpha(h)v, \qquad \delta_\alpha(v) = v_{(0)} \otimes \alpha(v_{(1)}) \qquad h \in H,\, v \in V. \tag{3.15}
$$

Example 3.7.2. Let (id, T) be the tensor autoequivalence of $\mathsf{Rep}\, H$ corresponding to an invariant twist T on H. Then $\Gamma_{(\mathsf{id},T)}(V) = V$ as an H-module, with H-coaction given by

$$
\delta_T(v) = (T^{-1})^2 \cdot (T^1 \cdot v)_{(0)} \otimes (T^{-1})^1 (T^1 \cdot v)_{(1)} T^2, \qquad v \in V. \tag{3.16}
$$

Here $T^1 \otimes T^2$ stands for T, and $T^{-1} \otimes T^{-2}$ for the inverse of T. Formula (3.16) appeared, also, in [CZ07].

Example 3.7.3. Let $\sigma \in (H \otimes H)^*$ be an invariant 2-cocycle on H. The dual map σ^* can be seen as an invariant twist on H^* and, as such, it gives rise to an autoequivalence (id, σ^*) of $\mathsf{Rep}(H^*)$. By Example 3.7.2, this induces an autoequivalence $\Gamma_{(\mathsf{id},\sigma^*)}$ of ${}_{H^*}\mathcal{YD}^{H^*}$. Since ${}_H\mathcal{YD}^H$ and ${}_{H^*}\mathcal{YD}^{H^*}$ are tensor equivalent, via the functor that dualizes module and comodule structures, we obtain an autoequivalence of ${}_H\mathcal{YD}^H$, which we continue to denote by $\Gamma_{(\mathsf{id},\sigma^*)}$. If $V \in {}_H\mathcal{YD}^H$ then $\Gamma_{(\mathsf{id},\sigma^*)}(V) = V$ as an H-comodule, with the H-action given by

$$h \cdot v = \sigma^{-1}\Big(\big(h_{(2)} \cdot v_{(0)}\big)_{(1)} \otimes h_{(1)}\Big)\sigma(h_{(3)} \otimes v_{(1)})\big(h_{(2)} \cdot v_{(0)}\big)_{(0)}, \quad h \in H, v \in V. \tag{3.17}$$

Remark 3.7.4. Autoequivalences described in Examples 3.7.1, 3.7.2, 3.7.3 give rise to group homomorphisms

$$\begin{aligned}
\iota_1 &: \mathsf{Aut}_{\mathrm{Hopf}}(H) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Rep}\, H)),\\
\iota_2 &: \mathsf{H}^2_{\mathsf{inv}}(H) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Rep}\, H)),\\
\iota_3 &: \mathsf{H}^2_{\mathsf{inv}}(H^*) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Rep}\, H)).
\end{aligned}$$

Now let $\mathcal{C}$ be a braided tensor category and $\mathcal{D} \subseteq \mathcal{C}$ a tensor subcategory. If $\mathcal{M}$ is an invertible $\mathcal{D}$-module category then $\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{M}$ is an invertible $\mathcal{C}$-module category. The assignment

$$\mathsf{Ind}_{\mathcal{D}}^{\mathcal{C}} : \mathsf{Pic}(\mathcal{D}) \to \mathsf{Pic}(\mathcal{C}), \qquad \mathcal{M} \mapsto \mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{M} \tag{3.18}$$

is a group homomorphism.

Remark 3.7.5. If A is an algebra in $\mathcal{D}$ such that $\mathcal{M}$ is equivalent to the category of A-modules in $\mathcal{D}$ then $\mathsf{Ind}_{\mathcal{D}}^{\mathcal{C}}(\mathcal{M})$ is the category of A-modules in $\mathcal{C}$.

The homomorphism (3.18) is not injective in general. Its kernel can be described as follows. Let $\mathcal{C} = \bigoplus_{\alpha \in \Sigma} \mathcal{C}_\alpha$ be the decomposition of $\mathcal{C}$ into a direct sum of $\mathcal{D}$-module subcategories (this decomposition exists and is unique by [EO04]) and let

$$\Sigma_0 = \{\alpha \in \Sigma \mid \mathcal{C}_\alpha \text{ is an invertible } \mathcal{D}\text{-module category}\}.$$

Proposition 3.7.6. *The kernel of homomorphism* (3.18) *is precisely the set of the equivalence classes of categories $\mathcal{C}_\alpha$, $\alpha \in \Sigma_0$.*

Proof. Suppose that $\mathcal{M}$ is an invertible $\mathcal{D}$-module category such that

$$\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{M} \cong \mathcal{C} \tag{3.19}$$

as a $\mathcal{C}$-module category. From the $\mathcal{D}$-module decomposition of both sides of (3.19) we obtain

$$\bigoplus_{\alpha \in \Sigma} (\mathcal{C}_\alpha \boxtimes_{\mathcal{D}} \mathcal{M}) \cong \bigoplus_{\beta \in \Sigma} \mathcal{C}_\beta.$$

Taking α such that $\mathcal{C}_\alpha = \mathcal{D}$ we conclude that $\mathcal{M} \cong \mathcal{C}_\beta$ for some $\beta \in \Sigma_0$.

Conversely, suppose that $\mathcal{M} \cong \mathcal{C}_\beta$ for some $\beta \in \Sigma_0$. Note that

$$\mathcal{E} = \bigoplus_{\alpha \in \Sigma_0} \mathcal{C}_\alpha$$

is a tensor subcategory of $\mathcal{C}$. It is group graded with the trivial component $\mathcal{D}$. Thus, $\mathsf{Ind}_{\mathcal{D}}^{\mathcal{E}} \mathcal{M} \cong \mathcal{E}$ and, hence, $\mathsf{Ind}_{\mathcal{D}}^{\mathcal{C}} \mathcal{M} = \mathsf{Ind}_{\mathcal{E}}^{\mathcal{C}}(\mathsf{Ind}_{\mathcal{D}}^{\mathcal{E}} \mathcal{M}) \cong \mathcal{C}$. □

Proposition 3.7.7. *Let $\mathcal{C}$ be a braided tensor category and let $\mathcal{D} \subseteq \mathcal{C}$ be a tensor subcategory. The image of the composition*

$$\mathsf{Pic}(\mathcal{D}) \xrightarrow{\mathsf{Ind}_{\mathcal{D}}^{\mathcal{C}}} \mathsf{Pic}(\mathcal{C}) \to \mathsf{Aut}^{br}(\mathcal{C})$$

is contained in $\mathsf{Aut}^{br}(\mathcal{C};\, \mathcal{D}')$. Here $\mathcal{D}'$ denotes the centralizer of $\mathcal{D}$ in $\mathcal{C}$.

Proof. Let $\mathcal{M} \in \mathsf{Pic}(\mathcal{D})$ be an invertible $\mathcal{D}$-module category and let $A \in \mathcal{D}$ be an algebra such that $\mathcal{M}$ is identified with the category of A-modules in $\mathcal{D}$. When we view A as an algebra in $\mathcal{C}$ the corresponding element $\partial_{\mathcal{M}}$ of $\mathsf{Aut}^{\mathrm{br}}(\mathcal{C})$ is determined by the existence of a natural tensor isomorphism

$$A \otimes X \cong \partial_{\mathcal{M}}(X) \otimes A, \qquad X \in \mathcal{C},$$

of A-modules [DN13]. When X centralizes $\mathcal{D}$ (and, hence, centralizes A) we get a natural tensor isomorphism $\partial_{\mathcal{M}}(X) \cong X$, i.e., $\partial_{\mathcal{M}}|_{\mathcal{D}'} \cong \mathsf{id}_{\mathcal{D}'}$. □

Example 3.7.8. Consider the triangular structure $R_0 = \frac{1}{2}(1\otimes 1 + 1\otimes c + c\otimes 1 - c\otimes c)$ of $E(n)$. Let $C_2 = \langle c\rangle \cong \mathbb{Z}/2\mathbb{Z}$. We have following sequence of surjective quasitriangular homomorphisms:

$$(D(E(n)), \mathcal{R}) \xrightarrow{f} (E(n), R_0) \xrightarrow{g} (\mathbf{k}[C_2], R_0), \tag{3.20}$$

where $f(p \bowtie x) = (p \otimes \mathsf{id})(R_0)x$, for all $p \in E(n)^*$ and $x \in E(n)$, and $g(c) = c$ and $g(x_i) = 0$, for all $i = 1, \ldots, n$. Sequence (3.20) induces a sequence of tensor embeddings

$$\mathsf{sVec} \xrightarrow{Res_g} (\mathcal{C}_n, c_0) \xrightarrow{Res_f} \mathcal{Z}(\mathcal{C}_n) \tag{3.21}$$

where $\mathsf{sVec} = \mathsf{Rep}(\mathbf{k}[C_2], R_0)$ and $(\mathcal{C}_n, c_0) = \mathsf{Rep}(E(n), R_0)$. This induces, in turn, a sequence of group homomorphisms

$$\mathsf{Pic}(\mathsf{sVec}) \xrightarrow{\mathsf{Ind}_{\mathsf{sVec}}^{\mathcal{C}_n}} \mathsf{Pic}(\mathcal{C}_n) \xrightarrow{\mathsf{Ind}_{\mathcal{C}_n}^{\mathcal{Z}(\mathcal{C}_n)}} \mathsf{Pic}(\mathcal{Z}(\mathcal{C}_n)) = \mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n)) \tag{3.22}$$

Since the image of sVec in $\mathcal{Z}(\mathcal{C}_n)$, under composition (3.21), is the tensor subcategory generated by the invertible object χ, we have that the image of $\mathsf{Pic}(\mathsf{sVec})$ in $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n))$ consists of those braided autoequivalences of $\mathcal{Z}(\mathcal{C}_n)$ that are trivializable on the centralizer of χ.

3.8 Computing $\mathsf{BrPic}(\mathcal{C}_n)$

We begin by describing the images of the compositions of the map $\rho : \mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n)) \to \mathsf{PGL}_n(\mathbf{k})$ of (3.11), with the homomorphisms of Remark 3.7.4, in the case $H = E(n)$.

Proposition 3.8.1. *We have*

(i) $\rho \circ \iota_1\big(\mathsf{Aut}_{\mathrm{Hopf}}(E(n))\big) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in \mathsf{GL}_{2n}(\mathbf{k}) \right\},$

(ii) $\rho \circ \iota_2\big(\mathsf{H}^2_{\mathsf{inv}}(E(n))\big) = \left\{ \begin{pmatrix} I_n & 0 \\ B & I_n \end{pmatrix} \mid B = B^t \right\},$

(iii) $\rho \circ \iota_3\big(\mathsf{H}^2_{\mathsf{inv}}(E(n)^*)\big) = \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \mid B = B^t \right\}.$

Here each matrix denotes the class in $\mathsf{PSp}_{2n}(\mathbf{k})$.

Proof. (i) This is clear in view of Proposition 3.3.2.

(ii) Let $\sigma_M \in (E(n) \otimes E(n))^*$ be the invariant 2-cocycle associated to $M = (m_{ij}) \in \mathsf{Sym}_n(k)$, and let $\Gamma_{(\mathsf{id}, \sigma^*_M)}$ be the autoequivalence of $\mathcal{Z}(\mathcal{C}_n)$ induced by σ_M.

Let $V_{\mathbf{a}} \in \mathsf{Ext}^1(\chi, \varepsilon)$. According to Example 3.7.3, $\Gamma_{(\mathsf{id}, \sigma^*_M)}(V_{\mathbf{a}}) = V_{\mathbf{a}}$ as an $E(n)$-comodule, with the $E(n)$-module structure given by

$$h \cdot v = \sigma_M^{-1}\Big(\big(h_{(2)} \cdot z_{(0)}\big)_{(1)} \otimes h_{(1)}\Big)\sigma_M(h_{(3)} \otimes z_{(1)})\big(h_{(2)} \cdot z_{(0)}\big)_{(0)}, \quad h \in E(n),\, v \in V_{\mathbf{a}}.$$

Consider a basis $\{v_1, v_2\}$ of $V_{\mathbf{a}}$ such that the $D(E(n))$-module structure of $V_{\mathbf{a}}$ is given by (3.10). Then, a straightforward computation shows that $h \cdot v_1 = \varepsilon(h)v_1$, for all $h \in E(n)$, $c \cdot v_2 = -v_2$ and

$$x_i \cdot v_2 = a_{n+j} + \Big(\textstyle\sum_{i=1}^n (m_{ij} + m_{ji})a_i\Big)v_1,$$

for all $i = 1, \dots, n$. Thus, $\Gamma_{(\mathsf{id}, \sigma^*_M)}(V_{\mathbf{a}}) = V_{\mathbf{a}'}$, where

$$\mathbf{a}'^t = \begin{pmatrix} I_n & 0 \\ M + M^t & I_n \end{pmatrix} \mathbf{a}^t$$

and the result follows.

(iii) Let

$$T_M = \tfrac{1}{4}\textstyle\sum_{i,j,P,Q} \sigma_M(c^i x_P \otimes c^j x_Q)(x_P + (-1)^i c x_P) \otimes (x_Q + (-1)^j c x_Q)$$

be the invariant twist associated to $M = (m_{ij}) \in \mathsf{Sym}_n(\mathbf{k})$. Observe that

$$T_M = 1 \otimes 1 + \textstyle\sum_{j,l=1}^n m_{jl} x_j \otimes c x_l + L$$

where L is a linear combination of $c^i x_P \otimes c^j x_Q$, with $i, j \in \{0, 1\}$ and P, $Q \subseteq \{1, \dots, n\}$ such that $|P| \geq 2$ or $|Q| \geq 2$. Let $\Gamma_{(\mathsf{id}, T_M)}$ be the autoequivalence of $\mathcal{Z}(\mathcal{C}_n)$ induced by T_M.

If $V_{\mathbf{a}} \in \mathsf{Ext}^1(\chi, \varepsilon)$ then $\Gamma_{(\mathsf{id}, T_M)}(V_{\mathbf{a}})$ is $V_{\mathbf{a}}$ as an $E(n)$-module, with the comodule structure given by

$$\delta_{T_M}(v) = (T_M^{-1})^2 \cdot (T_M^1 \cdot v)_{(0)} \otimes (T_M^{-1})^1 (T_M^1 \cdot v)_{(1)} T_M^2,$$

for all $v \in V_{\mathbf{a}}$.

Consider $\{v_1, v_2\}$ a basis for $V_{\mathbf{a}}$ such that the action of $D(E(n))$ on $V_{\mathbf{a}}$ is given by (3.10). Then one can easily check, using Lemma 3.5.1, that $\delta_{T_M}(v_1) = v_1 \otimes 1$ and

$$\delta_{T_M}(v_2) = \textstyle\sum_{i=1}^n a_i v_1 \otimes x_i + \sum_{i=1}^n \left(\sum_{j=1}^n a_{n+j}(m_{ij} + m_{ji}) \right) v_1 \otimes c x_i + v_2 \otimes c.$$

Taking into account that the induced $E(n)^*$-module structure of $\Gamma_{(\mathsf{id}, T_M)}(V_{\mathbf{a}})$ is $f \cdot v = \sum f(v_{(1)}) v_{(0)}$, for all $f \in E(n)^*$ and $v \in \Gamma_{(\mathsf{id}, T_M)}(V_{\mathbf{a}})$, we readily deduce the $D(E(n))$-module structure of $\Gamma_{(\mathsf{id}, T_M)}(V_{\mathbf{a}})$. We have $C \cdot v_1 = v_1$, $C \cdot v_2 = -v_2$, $X_i \cdot v_1 = 0$ and

$$X_i \cdot v_2 = (x_i^* - (c x_i)^*) \cdot v_2 = \left(a_i - \textstyle\sum_{j=1}^n a_{n+j}(m_{ij} + m_{ji}) \right) v_1.$$

Thus, $\Gamma_{(\mathsf{id}, T_M)}(V_{\mathbf{a}}) = V_{\mathbf{a}'}$, where

$$\mathbf{a}'^t = \begin{pmatrix} I_n & -(M + M^t) \\ 0 & I_n \end{pmatrix} \mathbf{a}^t$$

which concludes the proof. □

Corollary 3.8.2. *The image of homomorphism* (3.11) *is* $\mathsf{PSp}_{2n}(\mathbf{k})$.

Proof. The three subgroups from Proposition 3.8.1 generate $\mathsf{PSp}_{2n}(\mathbf{k})$, so the statement follows from Proposition 3.5.3. □

We now determine the kernel of ρ. We need the following result.

Lemma 3.8.3. *Let $V_{\mathbf{a}}$ be the underlying object of an extension of χ by ε. Then $V_{\mathbf{a}}$ centralizes χ, i.e., the squared braiding $c_{V_{\mathbf{a}}, \chi} \circ c_{\chi, V_{\mathbf{a}}}$ is the identity morphism.*

Proof. As a Yetter-Drinfeld module over $E(n)$, χ has the module structure given by $c \cdot 1 = -1$, $x_i \cdot 1 = 0$, for all $i = 1, \dots, n$, and the comodule structure $1 \mapsto 1 \otimes c$.

Let $\{v_1, v_2\}$ be a basis of $V_{\mathbf{a}}$ such that the action of $D(E(n))$ on $V_{\mathbf{a}}$ is given by (3.10). Then, from Lemma 3.5.1 and (2.36), we deduce that

$$c_{V_{\mathbf{a}},\chi} \circ c_{\chi,V_{\mathbf{a}}}(1 \otimes v_1) = c_{V_{\mathbf{a}},\chi}(v_1 \otimes 1) = 1 \otimes c \cdot v_1 = 1 \otimes v_1$$

and

$$\begin{aligned} c_{V_{\mathbf{a}},\chi} \circ c_{\chi,V_{\mathbf{a}}}(1 \otimes v_2) &= c_{V_{\mathbf{a}},\chi}\left(\sum_{j=1}^{n} a_j v_1 \otimes x_j \cdot 1 + v_2 \otimes c \cdot 1\right) \\ &= -c_{V_{\mathbf{a}},\chi}(v_2 \otimes 1) \\ &= -1 \otimes c \cdot v_2 \\ &= 1 \otimes v_2 \end{aligned}$$

whence the claim. $\square$

Proposition 3.8.4. *The kernel of homomorphism* (3.11) *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$*.*

Proof. Taking into account Remark 3.6.6, Proposition 3.6.7, Proposition 3.3.2 and Theorem 3.2.13 the kernel of ρ is a subgroup of $\mathsf{Pic}(\mathcal{C}_n) = \mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n); \mathcal{C}_n)$. From Example 3.2.16, we know that $\mathsf{Pic}(\mathcal{C}_n) \cong \mathsf{Sym}_n(\mathbf{k}) \times \mathbb{Z}/2\mathbb{Z}$.

The subgroups of $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n); \mathcal{C}_n)$ corresponding to $\mathbb{Z}/2\mathbb{Z}$ and $\mathsf{Sym}_n(\mathbf{k})$ are the following. $\mathbb{Z}/2\mathbb{Z}$ corresponds to the image of $\mathsf{Pic}(\mathsf{sVec})$ in $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n))$ under composition (3.22), and $\mathsf{Sym}_n(\mathbf{k})$ corresponds to the image of ι_2.

According to Example 3.7.8 the image of $\mathsf{Pic}(\mathsf{sVec})$ in $\mathsf{Aut}^{\mathrm{br}}(\mathcal{Z}(\mathcal{C}_n))$ consists of braided autoequivalences that are trivializable on the centralizer of χ. Since $V_{\mathbf{a}}$ centralizes χ, for every $\mathbf{a} \in \mathbf{k}^{2n}$, we have $\mathbb{Z}/2\mathbb{Z} \subseteq \mathsf{Ker}(\rho)$. Now $\mathsf{Ker}(\rho) \cap \mathsf{Im}(\iota_2) = 1$, so $\mathsf{Ker}(\rho) = \mathbb{Z}/2\mathbb{Z}$. $\square$

Proposition 3.8.5. *The restriction of homomorphism* (3.11) *on the subgroup of* $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}_n))$ *generated by the images of* $\mathsf{Aut}_{\mathrm{Hopf}}(E(n))$*,* $\mathsf{H}^2_{\mathsf{inv}}(E(n))$*,* $\mathsf{H}^2_{\mathsf{inv}}(E(n)^*)$ *is injective.*

Proof. Every matrix $M \in \mathsf{Sp}_{2n}(\mathbf{k})$ can be uniquely written as $M = XYZ$, where X, Y, Z are matrices from parts (i), (ii), and (iii) of Proposition 3.8.1, respectively, so the claim holds. $\square$

Theorem 3.8.6. *We have*

$$\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}_n)) \cong \mathsf{PSp}_{2n}(\mathbf{k}) \times \mathbb{Z}/2\mathbb{Z}. \tag{3.23}$$

The action of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}_n))$ *on* $\mathbb{L}_0(\mathcal{C}_n)$ *corresponds to the action of* $\mathsf{PSp}_{2n}(\mathbf{k})$ *on the Lagrangian Grassmannian* $\mathsf{Lag}(n, 2n)$.

Proof. According to Corollary 3.8.2 and Proposition 3.8.4, we have a central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathsf{Aut}^{\mathsf{br}}(\mathcal{Z}(\mathcal{C}_n)) \to \mathsf{PSp}_{2n}(\mathbf{k}) \to 1.$$

This extension splits by Proposition 3.8.5. The Lagrangian equivariance follows from Remark 3.6.8. $\square$

Corollary 3.8.7. $\mathsf{BrPic}(\mathcal{C}_n) \cong \mathsf{PSp}_{2n}(\mathbf{k}) \times \mathbb{Z}/2\mathbb{Z}$.

CHAPTER 4

POINTED BRAIDED FINITE TENSOR CATEGORIES

One of the earliest results on the classification of tensor categories was given by A. Joyal and R. Street in [JS93], who showed that the category with objects pointed braided fusion categories and whose morphisms are natural isomorphisms of braided tensor functors is equivalent to the category of pre-metric groups. The objects in the latter category are pairs (Γ, q), where Γ is a finite abelian group and $q : \Gamma \to \mathbf{k}^\times$ is a quadratic form.

In this chapter we prove a weak generalization of this result. We show in Theorem 4.7.3 that the category with objects pointed braided tensor categories admitting a fiber functor and morphisms natural isomorphisms of braided equivalences is equivalent to a category of metric quadruples. The objects of the latter category are quadruples (Γ, q, V, r) consisting of a finite abelian group Γ, a diagonalizable quadratic form $q : \Gamma \to \mathbf{k}^\times$, a Γ-graded vector space V whose support is contained in $\{g \in \Gamma \mid \frac{q(g+h)}{q(h)} = -1, \text{ for all } h \in \Gamma\}$ (equivalently, $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, q))_-$) and an anti-symmetric Γ-graded map $r : V \otimes V \to \mathbf{k}$. We achieve this by using results from the classification theory of finite dimensional pointed Hopf algebras.

The material in this chapter is organized as follows.

In Section 4.1 we prove the result of A. Joyal and R. Street in the case of pointed braided fusion categories admitting a fiber functor. This will serve as a prototype for the proof of Theorem 4.7.3.

In Section 4.2 we describe the objects that will serve as tools in our study of pointed braided tensor categories. These arose from the classification program of N. Andruskiewitsch and H.-J. Schneider and are called quantum linear spaces. They are certain type of pointed Hopf algebras associated to a finite abelian group Γ and a Γ-graded Γ-module V.

In Sections 4.3 and 4.4 we describe the 2-cocycles and the co-quasitriangular structures, respectively, of quantum linear spaces. For the former we use M. Mombelli's classification of module categories over the representation category of a quantum linear space in [Momb11]. The latter objects were described in a different form by A. Nenciu in [Nen04]. Our approach is categorical in nature and sheds more light on the structure of r-forms on quantum linear spaces.

At the end of Section 4.4 we associate to a quadruple (Γ, r_0, V, r_1) formed with a finite abelian group Γ, a bicharacter r_0 on Γ, a certain Γ-module V and a morphism $r_1 : V \otimes V \to \mathbf{k}$, a pointed braided finite tensor category admitting a fiber functor, denoted by $\mathcal{C}(\Gamma, r_0, V, r_1)$. We describe in Sections 4.5 and 4.6 the symmetric center and the ribbon structures of $\mathcal{C}(\Gamma, r_0, V, r_1)$.

In Section 4.7 we prove Theorem 4.7.3. As an application of our work, we describe in Section 4.8 the metric quadruple associated to the adjoint subcategory of the center of $\mathcal{C}(\Gamma, q, V, r)$, for a carefully chosen V.

The results of this chapter are based on [BN17].

4.1 Pointed braided fusion categories

In this section we prove a weak form of Joyal and Street's result. Namely, we show that the category whose objects are pointed braided fusion categories admitting a fiber functor, and whose morphisms are natural isomorphisms of braided tensor functors is equivalent to the category of pre-metric groups with a diagonalizable quadratic form. This will serve as a prototype for the more general result which we will prove in Section 4.7.

We start by recalling a result from group cohomology, that we will use later.

Let Γ be a finite abelian group and let $\mathsf{Alt}^2(\Gamma)$ be the abelian group of alternating bicharacters on Γ, i.e. those bicharacters $b : \Gamma \times \Gamma \to \mathbf{k}^\times$ such that $b(g, g) = 1$, for all $g \in \Gamma$.

Proposition 4.1.1. *The map* $\mathrm{alt} : \mathsf{Z}^2(\Gamma, \mathbf{k}^\times) \to \mathsf{Alt}^2(\Gamma)$*, defined by*

$$\mathrm{alt}(\sigma)(g, h) = \frac{\sigma(g, h)}{\sigma(h, g)}, \quad g, h \in \Gamma, \tag{4.1}$$

induces an isomorphism

$$\mathsf{H}^2(\Gamma, \mathbf{k}^\times) \cong \mathsf{Alt}^2(\Gamma).$$

Proof. See [Karp87, Theorem 2.6.7] where the result appears in an equivalent form. □

Definition 4.1.2. Let Γ be an abelian group. A *quadratic form* on Γ (with values in $\mathbf{k}$) is a map $q : \Gamma \to \mathbf{k}^\times$ such that:

- $q(g) = q(g^{-1})$, for all $g \in \Gamma$.
- The function $b : \Gamma \times \Gamma \to \mathbf{k}^\times$, $b(g, h) = \dfrac{q(gh)}{q(g)q(h)}$, for all g, $h \in \Gamma$, is a bicharacter.

The set of quadratic forms on Γ is denoted by $\mathsf{Quad}(\Gamma)$. A quadratic form q is *non-degenerate* if the associated bicharacter b is non-degenerate.

Example 4.1.3. Every bicharacter $r : \Gamma \times \Gamma \to \mathbf{k}^\times$ gives rise to a quadratic form on Γ, namely

$$q : \Gamma \to \mathbf{k}^\times, \qquad q(g) = r(g, g), \qquad g \in \Gamma.$$

Quadratic forms obtained in this way are said to be *diagonalizable*. The set of diagonalizable quadratic forms on Γ is denoted by $\mathsf{Quad}_d(\Gamma)$.

Definition 4.1.4. (1) A *pre-metric group* (over $\mathbf{k}$) is a pair (Γ, q), formed with a finite abelian group Γ and a quadratic form $q : \Gamma \to \mathbf{k}^\times$.

(2) An *orthogonal homomorphism* of pre-metric groups from (Γ, q) to (Γ', q') is a group homomorphism $\alpha : \Gamma \to \Gamma'$ such that $q' \circ \alpha = q$.

Let $\mathcal{G}$ be the category whose objects are pre-metric groups (Γ, q), with $q \in \mathsf{Quad}_d(\Gamma)$, and whose morphisms are orthogonal homomorphisms of pre-metric groups. Let $\mathcal{F}$ be the category whose objects are pointed braided fusion categories admitting a fiber functor and whose morphisms are isomorphism classes of braided tensor functors.

We can define a functor $F : \mathcal{G} \to \mathcal{F}$ in the following way. Fix, for every object (Γ, q) in $\mathcal{G}$, a bicharacter $r : \Gamma \times \Gamma \to \mathbf{k}^\times$ such that $q(g) = r(g, g)$, for all $g \in \Gamma$. Define F on objects by

$$F(\Gamma, q) = \mathsf{Corep}(\Gamma, r).$$

Consider now an orthogonal homomorphism $\alpha : (\Gamma, q) \to (\Gamma', q')$. Then $r' \circ (\alpha \times \alpha)/r$ is an alternating bicharacter on Γ, so, by Proposition 4.1.1, there exists a 2-cocycle $\sigma \in \mathsf{Z}^2(\Gamma, \mathbf{k}^\times)$ such that $r' \circ (\alpha \times \alpha)/r = \text{alt}(\sigma)$, i.e. $r'(\alpha(g), \alpha(h)) = \sigma^{-1}(h, g)r(g, h)\sigma(g, h)$, for all $g, h \in \Gamma$. According to Example 2.6.21, $(\alpha, \sigma) : \mathsf{Corep}(\Gamma, r) \to \mathsf{Corep}(\Gamma', r')$ is a braided tensor functor.

Define F on morphisms by $F(\alpha) = \widetilde{(\alpha, \sigma)}$, where $\widetilde{(\alpha, \sigma)}$ is the isomorphism class of (α, σ). Note that F is well defined, since the 2-cocycle σ is determined up to cohomology.

Proposition 4.1.5. *The functor $F : \mathcal{G} \to \mathcal{F}$, defined above, is an equivalence of categories.*

Proof. Taking into account Theorem 2.2.21, we have to show that F is essentially surjective and fully faithful. The fact that F is fully faithful follows from Corollary 2.10.12. Let us show that F is essentially surjective.

Consider a pointed braided fusion category $\mathcal{C}$ admitting a fiber functor. By Tannaka-Krein reconstruction, $\mathcal{C} \simeq \mathsf{Corep}\, H$, for some finite dimensional Hopf algebra H. Since $\mathcal{C}$ is pointed and semisimple, it follows that H is pointed and co-semisimple. Thus, $H = \mathbf{k}[\Gamma]$, where Γ is the group of group-like elements of H. Since $\mathcal{C}$ is braided, we have that Γ is abelian, and there exists a bicharacter r on Γ such that $\mathcal{C}$ is braided equivalent to $\mathsf{Corep}(\Gamma, r)$. It is not hard to see that $\mathsf{Corep}(\Gamma, r) \simeq F(\Gamma, q)$, where $q : \Gamma \to \mathbf{k}^\times$, $q(g) = r(g, g)$, for all $g \in \Gamma$. Thus, F is essentially surjective and the proof is complete. $\square$

Remark 4.1.6. Given a finite abelian group Γ and a bicharacter r on Γ, we will use the following notation:

$$\mathcal{C}(\Gamma, r) := \mathsf{Corep}(\Gamma, r).$$

Corollary 4.1.7. (1) *$\mathcal{C}(\Gamma, r)$ and $\mathcal{C}(\Gamma', r')$ are braided equivalent if and only if there is an orthogonal isomorphism $(\Gamma, q) \to (\Gamma', q')$, where $q(g) = r(g, g)$ and $q'(g') = r'(g', g')$, for all $g \in \Gamma$, $g' \in \Gamma'$.*

(2) We have

$$\mathsf{Aut}^{br}(\mathcal{C}(\Gamma, r)) \cong O(\Gamma, q),$$

where $q : \Gamma \to \mathbf{k}^\times$, $q(g) = r(g, g)$, *for all* $g \in \Gamma$, *and* $O(\Gamma, q)$ *is the group of orthogonal automorphisms of* (Γ, q).

Remark 4.1.8. If $\mathcal{C}$ is a pointed braided fusion category then the isomorphism classes of simple objects of $\mathcal{C}$ form a finite abelian group Γ. The braiding of $\mathcal{C}$ determines a function $c : \Gamma \times \Gamma \to \mathbf{k}^\times$ and the function $q : \Gamma \to \mathbf{k}^\times$, $q(g) = c(g, g)$, $g \in \Gamma$, is a quadratic form on Γ. It was shown in [JS93] (see also [DGNO10, Appendix D]) that the assignment

$$\mathcal{C} \mapsto (\Gamma, q)$$

is an equivalence between the category with objects pointed braided fusion categories and morphisms natural isomorphisms of braided tensor functors and the category of pre-metric groups. We will denote a pointed braided fusion category associated to (Γ, q) by $\mathcal{C}(\Gamma, q)$.

Remark 4.1.9. If r is a bicharacter on Γ and $q(g) = r(g, g)$, for all $g \in \Gamma$, then

$$\mathcal{C}(\Gamma, r) \simeq \mathcal{C}(\Gamma, q).$$

4.2 Quantum linear spaces

In this section we describe a class of Hopf algebras, called quantum linear spaces, that were introduced by N. Andruskiewitsch and H.-J. Schneider in [AS98], as part of a classification program for finite dimensional pointed Hopf algebras. Remarkable progress has been made since then and the classification problem is approaching completion [A14]. It turns out that quantum linear spaces are all that we need in order to study pointed braided finite tensor categories admitting a fiber functor, so a good understanding of their properties is essential. We will continue our discussion in the next two sections, where we will give a description of their 2-cocycles and co-quasitriangular structures.

Let Γ be a finite abelian group. Let $g_1, \dots, g_n$ be elements of Γ and $\chi_1, \dots, \chi_n$ be elements of the character group $\widehat{\Gamma}$ such that

$$\chi_i(g_i) \neq 1 \quad \text{and} \quad \chi_j(g_i)\chi_i(g_j) = 1,$$

for all $i, j = 1, \dots, n$, $i \neq j$.

Definition 4.2.1. A *quantum linear space* associated to the above datum $(g_1, \dots, g_n, \chi_1, \dots, \chi_n)$ is a Yetter-Drinfeld module

$$V = \bigoplus_{i=1}^{n} \mathbf{k} x_i \in {}^{\Gamma}_{\Gamma}\mathcal{YD}, \tag{4.2}$$

with $h \cdot x_i = \chi_i(h) x_i$, for all $h \in \Gamma$, and $\delta(x_i) = g_i \otimes x_i$, for all i.

Remark 4.2.2. With the notation of Remark 2.7.10, we have $x_i \in V_{g_i}^{\chi_i}$, for all $i = 1, \dots, n$.

Remark 4.2.3. The braiding of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ on $V \otimes V$ is given, on the basic elements $x_i \otimes x_j$, $i, j = 1 \dots, n$, by

$$c_{V,V}(x_i \otimes x_j) = \chi_j(g_i)\, x_j \otimes x_i. \tag{4.3}$$

Definition 4.2.4. We say that a quantum linear space V is of *symmetric type* if $\chi_i(g_i) = -1$, for all $i = 1, \dots, n$.

Remark 4.2.5. In categorical terms, a quantum linear space of symmetric type is an object $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that $c_{V,V}^2 = \mathrm{id}_{V \otimes V}$ and $\theta_V = -\,\mathrm{id}_V$, where θ is the canonical ribbon structure of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ (see Remark 2.7.14). The advantage of this definition is that it does not depend on the choice of "basis" g_i, χ_i, $i = 1, \dots, n$.

Remark 4.2.6. Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$. A linear map $\beta : V \otimes V \to \mathbf{k}$ is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ if and only if

$$\beta(v \otimes v')(1 - gg') = 0 \qquad \text{and} \qquad \beta(v \otimes v')(1 - \chi\chi') = 0,$$

for all $v \in V_g^{\chi}$ and $v' \in V_{g'}^{\chi'}$.

Note, also, that the transposition map

$$\tau_{U,V} : U \otimes V \to V \otimes U, \qquad u \otimes v \mapsto v \otimes u, \qquad u \in U, v \in V,$$

is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

Lemma 4.2.7. *Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be a quantum linear space of symmetric type and let $\beta : V \otimes V \to \mathbf{k}$ be a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Then $\beta \circ c_{V,V} = -\beta \circ \tau_{V,V}$.*

Proof. Let $\{x_i\}$ be a basis of V. Taking into account Remarks 4.2.3 and 4.2.6, we have

$$\beta \circ c_{V,V}(x_i \otimes x_j) = \chi_j(g_i)\beta(x_j \otimes x_i) = \chi_i^{-1}(g_i)\beta(x_j \otimes x_i) = -\beta(x_j \otimes x_i),$$

for all i and j. $\square$

We denote by $\mathsf{Sym}^2_{{}^{\Gamma}_{\Gamma}\mathcal{YD}}(V^*)$ (respectively, $\mathsf{Alt}^2_{{}^{\Gamma}_{\Gamma}\mathcal{YD}}(V^*)$) the space of morphisms $\beta : V \otimes V \to \mathbf{k}$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, that are symmetric (respectively, anti-symmetric) in the usual sense, i.e., $\beta \circ \tau_{V,V} = \beta$ (respectively, $\beta \circ \tau_{V,V} = -\beta$). Thus,

$$\mathsf{Sym}^2_{{}^{\Gamma}_{\Gamma}\mathcal{YD}}(V^*) \subset S^2(V^*), \quad \mathsf{Alt}^2_{{}^{\Gamma}_{\Gamma}\mathcal{YD}}(V^*) \subset \wedge^2(V^*),$$

where $S^2(V^*)$ and $\wedge^2(V^*)$ are the usual spaces of symmetric, respectively, alternating bilinear forms on the vector space V.

Remark 4.2.8. Every linear map $\beta : V \otimes V \to \mathbf{k}$ has a canonical decomposition into a sum of symmetric and alternating parts:

$$\beta = \beta_{sym} + \beta_{alt}, \tag{4.4}$$

where $\beta_{sym} = \frac{1}{2}(\beta + \beta \circ \tau_{V,V})$ and $\beta_{alt} = \frac{1}{2}(\beta - \beta \circ \tau_{V,V})$.

Given a quantum linear space V we associate to it a Hopf algebra $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$, called the *bosonization of the Nichols algebra* $\mathfrak{B}(V)$ *by* $\mathbf{k}[\Gamma]$ [1]. If Γ is generated by $F \subseteq \Gamma$, with relations R, then $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ is generated by the group-like elements $g \in F$ and the $(g_i, 1)$-skew primitive elements x_i (i.e., such that $\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1$), $i = 1, \dots, n$, satisfying the following relations:

$$R, \qquad hx_i = \chi_i(h)x_ih, \qquad x_i^{r_i} = 0, \qquad h \in \Gamma,\ i = 1, \dots, n,$$

[1]We will abuse the terminology and will also refer to $\mathfrak{B}(V)\#k[\Gamma]$ as a quantum linear space.

$$x_i x_j = \chi_j(g_i) x_j x_i, \qquad i,\, j = 1, \dots, n,$$

where r_i is the order of the root of unity $\chi_i(g_i)$. The set

$$\{g x_1^{i_1} \cdots x_n^{i_n} \mid g \in \Gamma,\, 0 \le i_j < r_j,\ j = 1, \dots, n\}$$

is a basis of $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$.

Remark 4.2.9. $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ is a pointed Hopf algebra and its coradical filtration is given by

$$H_m = \langle g x_1^{i_1} \dots x_n^{i_n} \mid g \in \Gamma,\, 0 \le i_j < r_j,\, \forall j,\, \textstyle\sum_j i_j \le m \rangle.$$

In particular,

$$\begin{aligned} P_{g_i,1}(H) &= \mathbf{k}(1 - g_i) \oplus \Big(\bigoplus_{j \,:\, g_j = g_i} \mathbf{k} x_j \Big), \quad 1 \le j \le n, \\ P_{g,1}(H) &= \mathbf{k}(1 - g), \quad \text{ if } g \notin \{g_i\}. \end{aligned}$$

Many classical examples of pointed Hopf algebras are quantum linear spaces.

Example 4.2.10. If $\Gamma = C_2 = \langle c \rangle$ is the cyclic group of order 2 and $\widehat{\Gamma} = \langle \chi \rangle$, then $(g_1, \dots, g_n, \chi_1, \dots, \chi_n) = (c, \dots, c, \chi, \dots, \chi)$ is a datum for a quantum linear space V and

$$\mathfrak{B}(V)\#\mathbf{k}[\Gamma] \cong E(n)^{\mathrm{cop}} \cong E(n).$$

Example 4.2.11. If $\Gamma = C_N = \langle g \rangle$ is the cyclic group of order N, ξ is a primitive N-th root of unity, and $\chi \in \widehat{\Gamma}$ is defined by $\chi(g) = \xi$, then $(g_1, \chi_1) = (g, \chi)$ is a datum for a quantum linear space V and

$$\mathfrak{B}(V)\#\mathbf{k}[\Gamma] \cong T_{N,1}^{\mathrm{cop}} \cong T_{N,1}.$$

If V is a quantum linear space then the *liftings* of $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$, i.e., the pointed Hopf algebras H for which there exists a Hopf algebra isomorphism

$$\mathrm{gr}\, H \simeq \mathfrak{B}(V)\#\mathbf{k}[\Gamma],$$

where gr H is the graded Hopf algebra associated to the coradical filtration of H, were classified in [AS98, Theorem 5.5]. Namely, for any such lifting H, there exist scalars $\mu_i \in \{0,1\}$ and $\lambda_{ij} \in \mathbf{k}$ $1 \leq i < j \leq n$, such that

- μ_i is arbitrary if $g_i^{r_i} \neq 1$ and $\chi^{r_i} = 1$, and $\mu_i = 0$ otherwise,
- λ_{ij} is arbitrary if $g_i g_j \neq 1$ and $\chi_i \chi_j = 1$, and $\lambda_{ij} = 0$ otherwise.

H is then generated by the group-like elements $g \in F$ and the $(g_i, 1)$-skew-primitive elements a_i, $i = 1, \ldots, n$, subject to the following relations:

$$R, \qquad g a_i = \chi_i(g) a_i g, \qquad a_i^{r_i} = \mu_i(1 - g_i^{r_i}), \qquad g \in F,\ i = 1, \ldots, n,$$

$$a_i a_j = \chi_j(g_i) a_j a_i + \lambda_{ij}(1 - g_i g_j), \qquad 1 \leq i < j \leq n.$$

It was shown in [Mas01] that these liftings are cocycle deformations of $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$.

Remark 4.2.12. Suppose that V is a quantum linear space of symmetric type. Then $x_i^2 = 0$ for all $i = 1, \ldots, n$. For a subset $P = \{i_1, i_2, \ldots, i_s\} \subseteq \{1, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_s$ we denote the element $x_{i_1} \cdots x_{i_s}$ by x_P and use the convention that $x_\emptyset = 1$. The set $\{g x_P \mid g \in \Gamma,\ P \subseteq \{1, \ldots, n\}\}$ is then a basis of $\mathfrak{B}(V)\#k[\Gamma]$.

Let $F \subseteq P$ be subsets of $\{1, \ldots, n\}$ and let $\psi(P, F)$ be the element of $\mathbf{k}$ such that $x_P = \psi(P, F) x_F x_{P \setminus F}$. Thus,

$$\psi(P, F) = \prod_{\substack{j \in F,\, i \in P \setminus F \\ i < j}} \chi_j(g_i). \tag{4.5}$$

It is easy to check that the comultiplication formula for x_P is given by

$$\Delta(x_P) = \sum_{F \subseteq P} \psi(P, F) g_F x_{P \setminus F} \otimes x_F, \tag{4.6}$$

where $g_F = \Pi_{i \in F}\, g_i$ and $g_\emptyset = 1$.

We end this section by introducing a construction which will appear in Section 4.8 when we discuss the adjoint subcategory of the center of a pointed braided finite tensor category.

Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be a quantum linear space of symmetric type associated to a datum $(g_1, \dots, g_n, \chi_1, \dots, \chi_n)$. Let Σ be the subgroup of $\Gamma \times \widehat{\Gamma}$ generated by (g_i, χ_i^{-1}), $i = 1, \dots, n$, and define characters $\varphi_i : \Sigma \to \mathbf{k}^{\times}$ by

$$\varphi_i(g, \chi) = \chi_i(g), \quad \text{for all } (g, \chi) \in \Sigma, \quad i = 1, \dots, n.$$

We have

$$\varphi_i(g_i, \chi_i^{-1}) = -1 \quad \text{and} \quad \varphi_j(g_i, \chi_i^{-1})\varphi_i(g_j, \chi_j^{-1}) = 1,$$

for all $i, j = 1, \dots, n$. Thus, we can consider the quantum linear space of symmetric type $W \in {}^{\Sigma}_{\Sigma}\mathcal{YD}$ associated to the datum $\big((g_1, \chi_1^{-1}), \dots, (g_n, \chi_n^{-1}), \varphi_1, \dots, \varphi_n\big)$.

Definition 4.2.13. We call the quantum linear space $D(V) := W \oplus W^* \in {}^{\Sigma}_{\Sigma}\mathcal{YD}$ the *Drinfeld double* of V.

Note that the quantum linear space $D(V)$ is of symmetric type.

Remark 4.2.14. There is a canonical bilinear form $r_{D(V)} : D(V) \otimes D(V) \to \mathbf{k}$, given by

$$r_{D(V)}((w, f), (w', f')) := \mathsf{ev}_W(f, w') + \mathsf{ev}_W\, c_{W,W^*}(w, f') \tag{4.7}$$

for all $w, w' \in W$, $f, f' \in W^*$, where $\mathsf{ev}_W : W^* \otimes W \to \mathbf{k}$ is the evaluation morphism and $c_{W,W^*} : W \otimes W^* \to W^* \otimes W$ is the braiding in ${}^{\Sigma}_{\Sigma}\mathcal{YD}$.

Note that $r_{D(V)}$ is a symplectic bilinear form on $W \oplus W^*$. Indeed, if $\{x_i\}$ is a basis of W such that $x_i \in W^{\varphi_i}_{(g_i, \chi_i^{-1})}$, $i = 1, \dots, n$, and $\{x_i^*\}$ is the dual basis of W^*, then the matrix of $r_{D(V)}$ with respect to the basis $(x_1, \dots, x_n, x_1^*, \dots, x_n^*)$ is

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix.

4.3 2-cocycles on quantum linear spaces

In this section we describe the gauge equivalence classes of 2-cocycles on the quantum linear space $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$, with V of symmetric type. We determine, also, the second invariant cohomology group of $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. For this, we exploit the relationship between 2-cocycles and Galois objects, illustrated in Section 2.9, and make use of a description of the latter objects in the work of M. Mombelli [Momb11].

Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ be a quantum linear space of symmetric type and let $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. In [Momb11] Mombelli classified equivalence classes of exact indecomposable $\mathsf{Rep}(H)$-module categories. In particular, he classified H-Galois objects and, hence, 2-cocycles on H. We recall here this classification, see [Momb11, Section 4] for details.

A typical H-Galois object is determined by a 2-cocycle $\psi \in \mathsf{Z}^2(\Gamma, \mathbf{k}^{\times})$ and two families of scalars $\xi = (\xi_i)_{i=1,\dots,n}$ and $\alpha = (\alpha_{ij})_{1\le i<j\le n}$, satisfying:

$$\xi_i = 0 \quad \text{if } \chi_i^2(g) \neq \frac{\psi(g,\, g_i^2)}{\psi(g_i^2,\, g)}, \tag{4.8}$$

$$\alpha_{ij} = 0 \quad \text{if } \chi_i\chi_j(g) \neq \frac{\psi(g,\, g_ig_j)}{\psi(g_ig_j,\, g)}, \tag{4.9}$$

for all $g \in \Gamma$. To this datum one assigns a left H-comodule algebra $\mathcal{A}(\psi, \xi, \alpha)$ generated as an algebra by $\{e_g\}_{g\in\Gamma}$ and $v_1, \dots, v_n$ subject to the relations:

$$e_f e_g = \psi(f,\, g)e_{fg}, \qquad f, g \in \Gamma,$$

$$e_f v_i = \chi_i(f) v_i e_f, \qquad f \in \Gamma,\ i = 1, \dots, n,$$

$$v_i v_j - \chi_j(g_i) v_j v_i = \alpha_{ij} e_{g_ig_j}, \qquad 1 \le i < j \le n,$$

$$v_i^2 = \xi_i e_{g_i^2}, \qquad i = 1, \dots, n.$$

The left H-comodule structure of $\mathcal{A}(\psi, \xi, \alpha)$ is $\lambda : \mathcal{A}(\psi, \xi, \alpha) \to H \otimes \mathcal{A}(\psi, \xi, \alpha)$,

$$\lambda(v_i) = g_i \otimes v_i + x_i \otimes 1 \qquad \text{and} \qquad \lambda(e_f) = f \otimes e_f,$$

for all $i = 1, \dots, n$ and $f \in \Gamma$. Two H-Galois objects $\mathcal{A}(\psi, \xi, \alpha)$ and $\mathcal{A}(\psi', \xi', \alpha')$ are isomorphic if and only ψ and ψ' are cohomologous, $\xi = \xi'$, and $\alpha = \alpha'$.

Remark 4.3.1. Suppose that $\psi = 1$. The 2-cocycle σ corresponding to the H-Galois object $\mathcal{A}(1, \xi, \alpha)$ satisfies

$$\sigma(x_i,\, x_j) - \chi_j(g_i)\sigma(x_j,\, x_i) = \alpha_{ij} \quad \text{and} \quad \sigma(x_i,\, x_i) = \xi_i, \qquad 1 \le i < j \le n.$$

In this case, conditions (4.8) and (4.9) are equivalent to $\sigma - \sigma \circ c_{V,V} : V \otimes V \to \mathbf{k}$ being a Γ-module map.

Proposition 4.3.2. *The association $\sigma \mapsto \sigma|_{V \otimes V} \circ (\mathsf{id}_{V \otimes V} - c_{V,V})$ establishes a bijection between the set of gauge equivalence classes of 2-cocycles on H, whose restriction on Γ is trivial, and the set of Γ-module maps $\beta : V \otimes V \to \mathbf{k}$ satisfying $\beta \circ c_{V,V} = -\beta$.*

Proof. The set of gauge equivalence classes of 2-cocycles on H, whose restriction on Γ is trivial, is in bijection with the set of isomorphism classes of Galois objects of the type $\mathcal{A}(1, \xi, \alpha)$. The latter are parameterized by pairs (ξ, α), where $\xi = (\xi_i)_{1 \le i \le n}$ and $\alpha = (\alpha_{ij})_{1 \le i < j \le n}$ are such that

$$\xi_i(1 - \chi_i^2),\ i = 1, \dots, n \qquad \text{and} \qquad \alpha_{ij}(1 - \chi_i\chi_j) = 0,\ 1 \le i < j \le n.$$

The pair (ξ, α) associated to the equivalence class of σ is, according to Remark 4.3.1, given by

$$\xi_i = \sigma(x_i, x_i),\, i = 1, \dots, n, \qquad \alpha_{ij} = \sigma(x_i, x_j) - \chi_j(g_i)\sigma(x_j, x_i),\, 1 \le i < j \le n,$$

It defines, uniquely, the map $\beta = \sigma|_{V \otimes V} \circ (\mathsf{id}_{V \otimes V} - c_{V,V})$, since $\beta(x_i, x_i) = 2\xi_i$ and $\beta(x_i, x_j) = \alpha_{ij}$.

Every Γ-linear map $\beta : V \otimes V \to \mathbf{k}$, such that $\beta \circ c_{V,V} = -\beta$, arises in this way. To see this, note that, if $\xi_i = \frac{1}{2}\beta(x_i, x_i)$, $i = 1, \dots, n$, and $\alpha_{ij} = \beta(x_i, x_j)$, $1 \le i < j \le n$, then $\mathcal{A}(1, \xi, \alpha)$ is an H-Galois object. This completes the proof. $\square$

Next, we describe invariant 2-cocycles on H.

Proposition 4.3.3. *Let σ be a 2-cocycle on H, whose restriction on Γ is trivial, and let $\beta = \sigma|_{V\otimes V} \circ (\mathrm{id}_{V\otimes V} - c_{V,V})$. Then σ is invariant if and only if $\beta(x_i, x_j)(1 - g_i g_j) = 0$, for all i and j, i.e., if and only if β is morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.*

Proof. Suppose that σ is invariant. Taking $x = x_i$, $y = g \in \Gamma$ in (2.40) we obtain $\sigma(x_i, g) = 0$. Similarly, taking $x = g$ and $y = x_i$ we get $\sigma(g, x_i) = 1$. Next, taking $x = x_i$, $y = x_j$ in (2.40) we obtain $\sigma(x_i, x_j)(1 - g_i g_j) = 0$. This holds for all $i, j = 1, \ldots, n$, so σ is a Γ-comodule map. Thus, β is a Γ-comodule map, and, combining this with Proposition 4.3.2, we see that β is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

Conversely, suppose that a 2-cocycle σ on H is such that $\sigma|_{\Gamma\times\Gamma} = 1$ and β is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Then the multiplication in the corresponding twisted Hopf algebra H^σ satisfies relations $g \cdot_\sigma x_i = \chi_i(g) x_i \cdot_\sigma g$ and

$$x_i \cdot_\sigma x_j - \chi_j(g_i) x_j \cdot_\sigma x_i = \alpha_{ij}(1 - g_i g_j), \quad i, j = 1, \ldots, n.$$

But the right hand side of the last equality is equal to 0, so $H^\sigma = H$, i.e., σ is invariant. □

Recall that $\mathsf{Sym}^2_{{}^{\Gamma}_{\Gamma}\mathcal{YD}}(V^*)$ denotes the set of morphisms $\beta : V \otimes V \to \mathbf{k}$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ that are symmetric: $\beta\tau_{V,V} = \beta$.

Corollary 4.3.4. *The map $\sigma \mapsto (\sigma|_{V\otimes V})_{sym}$ is an isomorphism between the set of gauge equivalence classes of invariant 2-cocycles on H, whose restriction to Γ is trivial, and the space $\mathsf{Sym}^2_{{}^{\Gamma}_{\Gamma}\mathcal{YD}}(V^*)$.*

Proof. According to Proposition 4.3.2 and Proposition 4.3.3, the association $\sigma \mapsto \beta = \sigma|_{V\otimes V} \circ (\mathrm{id}_{V\otimes V} - c_{V,V})$ is a bijection between the set of invariant 2-cocycles on H, with trivial restriction to Γ, and the set of morphisms $\beta : V \otimes V \to \mathbf{k}$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, such that $\beta \circ c_{V,V} = -\beta$. The latter condition on β is equivalent, according to Lemma 4.2.7, to $\beta \circ \tau_{V,V} = \beta$, i.e. $\beta \in \mathsf{Sym}^2_{{}^{\Gamma}_{\Gamma}\mathcal{YD}}(V^*)$.

We claim that $\beta = 2\,(\sigma|_{V\otimes V})_{sym}$. Indeed, $\mathrm{id}_{V\otimes V} - c_{V,V}$ is an invertible morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, with inverse $\mathrm{id}_{V\otimes V} + c_{V,V}$, so $\sigma_{V\otimes V}$ is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. According to Lemma 4.2.7, $\sigma_{V\otimes V} \circ c_{V,V} = -\sigma_{V\otimes V}\tau_{V,V}$. Thus,

$$\beta = \sigma|_{V\otimes V} \circ (\mathsf{id}_{V\otimes V} - c_{V,V}) = \sigma \circ (\mathsf{id}_{V\otimes V} + \tau_{V,V}) = 2\,(\sigma|_{V\otimes V})_{sym}.$$

$\square$

We now analyze the general situation when $\sigma|_{\Gamma\times\Gamma}$ is not necessarily trivial. Let Γ_0 be the subgroup of Γ generated by g_i, $i = 1, \dots, n$.

Proposition 4.3.5. *Let σ be an invariant 2-cocycle on H. There exists $\rho \in \mathsf{Z}^2(\Gamma/\Gamma_0,\, \mathbf{k}^\times)$ such that $\sigma|_{\Gamma\times\Gamma}$ is cohomologous to $\rho\circ(\pi_{\Gamma_0}\times\pi_{\Gamma_0})$, where $\pi_{\Gamma_0} : \Gamma \to \Gamma/\Gamma_0$ is the quotient homomomorphism.*

Proof. It suffices to check that the alternating bilinear form $\mathrm{alt}(\sigma) : \Gamma\times\Gamma \to \mathbf{k}^\times$ given by

$$\mathrm{alt}(\sigma)(g,\, h) = \frac{\sigma(g,\, h)}{\sigma(h,\, g)}, \qquad g, h \in \Gamma \tag{4.10}$$

vanishes on $\Gamma\times\Gamma_0$. But this follows from invariance of σ since we must have $\sigma(g_i,\, g) = \sigma(g,\, g_i) = 1$, for all $i = 1, \dots, n$ and $g \in \Gamma$. $\square$

Proposition 4.3.6. $\mathsf{H}^2_{\mathsf{inv}}(H) \cong \mathsf{H}^2(\Gamma/\Gamma_0,\, \mathbf{k}^\times) \times \mathsf{Sym}^2_{{}^\Gamma_\Gamma\mathcal{YD}}(V^*)$.

Proof. By Corollary 4.3.4 the group $\mathsf{Sym}^2_{{}^\Gamma_\Gamma\mathcal{YD}}(V^*)$ is identified with the normal subgroup of $\mathsf{H}^2_{\mathsf{inv}}(H)$ consisting of gauge equivalence classes of invariant 2-cocycles with trivial restriction on Γ.

Next, there is a surjective Hopf algebra homomorphism $p : H \to \mathbf{k}[\Gamma/\Gamma_0]$ obtained by composing the canonical projection $H \to \mathbf{k}[\Gamma]$ with $\pi_{\Gamma_0} : \Gamma \to \Gamma/\Gamma_0$. Thus, for any 2-cocycle $\rho \in \mathsf{Z}^2(\Gamma/\Gamma_0,\, \mathbf{k}^\times)$ its pullback $p^*(\rho)$ is a 2-cocycle on H.

Using the explicit formula (4.6) for the comultiplication on H we check that this 2-cocycle satisfies

$$p^*(\rho)(x_{(1)}, y_{(1)})x_{(2)} \otimes y_{(2)} = x_{(1)} \otimes y_{(1)} p^*(\rho)(x_{(2)}, y_{(2)}), \qquad x, y \in H.$$

Indeed, for $x = hx_P$, $y = fx_Q$, where $h,\, f \in \Gamma$, both sides of this equality are equal to $\rho(\pi_{\Gamma_0}(h),\, \pi_{\Gamma_0}(f))hx_P \otimes fx_Q$.

In particular, $p^*(\rho)$ is an invariant 2-cocycle on H and belongs to the center of $\mathsf{H}^2_{\mathsf{inv}}(H)$. Thus, there is a central embedding $\mathsf{H}^2(\Gamma/\Gamma_0,\, \mathbf{k}^\times) \subset \mathsf{H}^2_{\mathsf{inv}}(H)$. The statement follows from Proposition 4.3.5. $\square$

4.4 Co-quasitriangular structures on quantum linear spaces

In this section we prove that if $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ admits a co-quasitriangular structure then V is of symmetric type. In this case, r-forms on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ are parametrized by pairs (r_0, r_1) where $r_0 : \Gamma \times \Gamma$ is a bicharacter, satisfying a certain condition, and $r_1 : V \otimes V \to \mathbf{k}$ is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Quasitriangular structures on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ have been described by A. Nenciu in [Nen04] in terms of generators of Γ and a basis of V. By duality, one can deduce from that a classification of co-quasitriangular structures. Our approach is new and, by providing a basis free description, gives more insight into the structure of r-forms on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$.

We start with a couple of preliminary results.

Lemma 4.4.1. *Let $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ be a quantum linear space. If $r : H \otimes H \to k$ is a linear map satisfying conditions* (2.25)-(2.28) *then r is a co-quasitriangular structure on H if and only if condition* (2.29) *holds for all pairs $(x, y) \in H_1 \times H_1$, where H_1 is the second term in the coradical filtration of H.*

Proof. We need only prove sufficiency. By induction on m we show that condition (2.29) holds for all pairs (x, y) for which either x or y is in H_m, the m-th term of the coradical filtration.

Assume first that $x \in H_1$. Using induction on $k \geq 1$ we show that condition (2.29) holds for all pairs (x, y) and (y, x) with $y \in H_k$. If $k = 1$ there is nothing to prove. Assume that the claim is true for $k \geq 1$ and consider $z \in H_1$. Then, using the induction hypothesis and (2.25), we have

$$
\begin{aligned}
x_{(1)}(yz)_{(1)}r\big((yz)_{(2)}, x_{(2)}\big) &= x_{(1)}y_{(1)}z_{(1)}r(y_{(2)}z_{(2)}, x_{(2)}) \\
&= \underline{x_{(1)}y_{(1)}}z_{(1)}\underline{r(y_{(2)}, x_{(2)})}r(z_{(2)}, x_{(3)}) \\
&= y_{(2)}\underline{x_{(2)}z_{(1)}}r(y_{(1)}, x_{(1)})\underline{r(z_{(2)}, x_{(3)})} \\
&= y_{(2)}z_{(2)}x_{(3)}r(y_{(1)}, x_{(1)})r(z_{(1)}, x_{(2)}) \\
&= y_{(2)}z_{(2)}x_{(2)}r(y_{(1)}z_{(1)}, x_{(1)}) \\
&= (yz)_{(2)}x_{(2)}r\big((yz)_{(1)}, x_{(1)}\big)
\end{aligned}
$$

and using (2.27) we have

$$
\begin{aligned}
(yz)_{(1)}x_{(1)}r\big(x_{(2)}, (yz)_{(2)}\big) &= y_{(1)}z_{(1)}x_{(1)}r(x_{(2)}, y_{(2)}z_{(2)}) \\
&= y_{(1)}\underline{z_{(1)}x_{(1)}r(x_{(2)}, z_{(2)})}r(x_{(3)}, y_{(2)}) \\
&= \underline{y_{(1)}x_{(2)}}z_{(2)}r(x_{(1)}, z_{(1)})\underline{r(x_{(3)}, y_{(2)})} \\
&= x_{(3)}y_{(2)}z_{(2)}r(x_{(1)}, z_{(1)})r(x_{(2)}, y_{(1)}) \\
&= x_{(2)}y_{(2)}z_{(2)}r(x_{(1)}, y_{(1)}z_{(1)}) \\
&= x_{(2)}(yz)_{(2)}r\big(x_{(1)}, (yz)_{(1)}\big)
\end{aligned}
$$

Since $H_{k+1} = H_k H_1$ it follows that (2.29) holds for all pairs (x, y) and (y, x) with $y \in H_{k+1}$. Thus, (2.29) holds for all pairs (x, y) with either x or y in H_1.

Suppose now that (2.29) holds for all pairs (x, y) with either x or y in H_m. Then a similar argument as the previous one shows that (2.29) holds for all pairs (x, yz) and (yz, x) with $y \in H_m$, $z \in H_1$ and arbitrary x. Since $H_{m+1} = H_m H_1$, it follows that (2.29) is satisfied for all pairs (x, y) with either x or y in H_{m+1}. This proves the induction step and concludes the proof. $\square$

Lemma 4.4.2. *Let H be a Hopf algebra generated as an algebra by $h_1, \ldots, h_n$ and such that the vector space V spanned by $h_1, \ldots, h_n$ is a subcoalgebra. Then any co-quasitriangular structure on H is uniquely determined by its restriction to $V \otimes V$.*

Proof. Suppose r' and r'' are two co-quasitriangular structures on H such that $r'(h_i, h_j) = r''(h_i, h_j)$, for all $i, j \in \{1, \ldots, n\}$. If $i, i_1, \ldots, i_t \in \{1, \ldots, n\}$ then, using (2.25), we have

$$\begin{aligned} r'(h_{i_1} \cdots h_{i_t}, h_i) &= r'\big(h_{i_1}, (h_i)_{(1)}\big) \cdots r'\big(h_{i_t}, (h_i)_{(t)}\big) \\ &= r''\big(h_{i_1}, (h_i)_{(1)}\big) \cdots r''\big(h_{i_t}, (h_i)_{(t)}\big) \\ &= r''(h_{i_1} \cdots h_{i_t}, h_i) \end{aligned}$$

Thus, $r'(h, h_i) = r''(h, h_i)$, for all $h \in H$ and $i \in \{1, \ldots, n\}$. Let $h \in H$ and $i_1, \ldots, i_t \in \{1, \ldots, n\}$. Then, using (2.27), we have

$$\begin{aligned} r'(h, h_{i_1} \cdots h_{i_t}) &= r'\big(h_{(1)}, h_{i_t}\big) \cdots r'\big(h_{(t)}, h_{i_1}\big) \\ &= r''\big(h_{(1)}, h_{i_t}\big) \cdots r''\big(h_{(t)}, h_{i_1}\big) \\ &= r''(h, h_{i_1} \cdots h_{i_t}) \end{aligned}$$

Since $h_1, \ldots, h_n$ generate H as an algebra, we conclude that $r' = r''$. $\square$

We now prove the main result of this section.

Proposition 4.4.3. *Let $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ be a quantum linear space. If H admits a co-quasitriangular structure then V is of symmetric type. In this case, the set of r-forms on H is parameterized by the set of pairs (r_0, r_1), where $r_0 : \Gamma \times \Gamma \to \mathbf{k}^\times$ is a bicharacter such that $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0^{-1})) \subseteq {}^{\Gamma}_{\Gamma}\mathcal{YD}$, and $r_1 : V \otimes V \to \mathbf{k}$ is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. The pair (r_0, r_1) corresponding to a co-quasitriangular structure r is $(r_0, r_1) = (r|_{\Gamma\times\Gamma}, r|_{V\otimes V})$.*

Proof. Let r be an r-form on $\mathfrak{B}(V)\#k[\Gamma]$. Then the restriction of r to $\Gamma \times \Gamma$ is a bicharacter r_0. From condition (2.29) applied to the pair $(x, y) = (x_i, g)$ we obtain $r_0(g, g_i) = \chi_i^{-1}(g)$ and

$r(g, x_i) = 0$, and from the same condition applied to $(x, y) = (g, x_i)$ we get $r_0(g_i, g) = \chi_i(g)$ and $r(x_i, g) = 0$. Thus,

$$r_0(g_i, -) = \chi_i = r_0(-, g_i)^{-1},$$

for all $i = 1, \ldots, n$. This is equivalent to requiring $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0^{-1})) \subseteq {}^{\Gamma}_{\Gamma}\mathcal{YD}$. It also implies $\chi_i(g_i) = -1$, for all $i = 1, \ldots, n$, so V is a quantum linear space of symmetric type.

Let us check now that $r_1 = r|_{V\otimes V}$ is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Making use, again, of condition (2.29), this time for the pair $(x, y) = (x_i, x_j)$, we obtain that $r(x_i, x_j)(1 - g_i g_j) = 0$. Looking now at condition (2.27), for $(x, y) = (x_i, x_j g)$ and $(x, y) = (x_i, g x_j)$, we see that $r(x_i, x_j g) = \chi_i(g) r(x_i, x_j)$ and $r(x_i, g x_j) = r(x_i, x_j)$. Since $x_j g = \chi_j^{-1}(g) g x_j$, we have

$$\chi_i(g) r(x_i, x_j) = r(x_i, x_j g) = \chi_j^{-1}(g) r(x_i, g x_j) = \chi_j^{-1}(g) r(x_i, x_j)$$

Thus, $r(x_i, x_j)(1 - \chi_i \chi_j) = 0$. It follows from Remark 4.2.6 that $r|_{V\otimes V}$ is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

We have proved that if $\mathfrak{B}(V)\#k[\Gamma]$ admits a co-quasitriangular structure r then V is a quantum linear space of symmetric type and the pair $(r_0, r_1) = (r|_{\Gamma\times\Gamma}, r|_{V\otimes V})$ has the properties listed in the statement. Let us see that, when V is a quantum linear space of symmetric type and $H = \mathfrak{B}(V)\#k[\Gamma]$, any such pair comes from a unique r-form on H.

Suppose (r_0, r_1) is such a pair. Let $f : H \to H^{*\mathrm{cop}}$ be defined by

$$f(g) = \gamma_g \quad \text{and} \quad f(x_i) = \xi_i$$

where, for $g \in \Gamma$ and $i = 1, \ldots, n$, γ_g and ξ_i are given by

$$\gamma_g(h x_P) = \delta_{P,\emptyset} r_0(g, h) \quad \text{and} \quad \xi_i(h x_P) = \begin{cases} 0 & \text{if } |P| \neq 1, \\ r_1(x_i, x_j) & \text{if } P = \{j\}, \end{cases}$$

for all $h \in \Gamma$ and $P \subseteq \{1, \ldots, n\}$. It is not hard to see that γ_g is a group-like element and x_i is a $(\gamma_{g_i}, \varepsilon)$-skew primitive element of $H^{*\mathrm{cop}}$, for every $g \in \Gamma$ and $i = 1, \ldots, n$. Thus, f is a coalgebra map.

In addition, f is an algebra map. Indeed, since $r_1 : V \otimes V \to k$ is a morphism of Yetter-Drinfeld modules, we have $\xi_i(x_j)(1 - \chi_i \chi_j) = 0$ and $\xi_i(x_j)(1 - g_i g_j) = 0$, for all $i, j \in \{1, \ldots, n\}$. With these relations, it is straightforward to check that $\gamma_g \gamma_h = \gamma_{gh}$, $\gamma_g \xi_i = \chi_i(g) \xi_i \gamma_g$, $\xi_i \xi_j = \chi_j(g_i) \xi_j \xi_i$

and $\xi_i^2 = 0$, for all $g, h \in \Gamma$ and $i, j = 1, \dots, n$. For example, $\xi_i\xi_j(hx_P) = 0 = \chi_j(g_i)\xi_j\xi_i(hx_P)$ when $|P| \neq 2$. If $1 \leq u < v \leq n$, then

$$\begin{aligned}
\xi_i\xi_j(hx_ux_v) &= \xi_i(hg_ux_v)\xi_j(hx_u) + \chi_u^{-1}(g_v)\xi_i(hg_vx_u)\xi_j(hx_v) \\
&= \xi_i(x_v)\xi_j(x_u) + \chi_i(g_j)^{-1}\xi_i(x_u)\xi_j(x_v) \\
&= \chi_j(g_i)\big(\xi_i(x_u)\xi_j(x_v) + \chi_j(g_i)^{-1}\xi_i(x_v)\xi_j(x_u)\big) \\
&= \chi_j(g_i)\big(\xi_j(hg_ux_v)\xi_i(hx_u) + \chi_u^{-1}(g_v)\xi_j(hg_vx_u)\xi_i(hx_v)\big) \\
&= \chi_j(g_i)\xi_j\xi_i(hx_ux_v).
\end{aligned}$$

Thus, f is a bialgebra map from H to $H^{*\mathrm{cop}}$. It follows from Remark 2.6.16 that $r : H \otimes H \to k$, $r(x, y) = f(x)(y)$, for all $x, y \in H$, satisfies (2.25)-(2.28). To prove that r is a co-quasitriangular structure on H, it suffices, by virtue of Lemma 4.4.1, to check that (2.29) holds for every pair of elements in the second term of the coradical filtration of H. This is straightforward, as we next show for the pair (gx_i, hx_j):

$$\begin{aligned}
r\big((hx_j)_{(1)}, (gx_i)_{(1)}\big)(hx_j)_{(2)}(gx_i)_{(2)} &= r(hg_j, gg_i)hx_jgx_i + r(hx_j, gx_i)hg \\
&= r(h, g)\chi_i^{-1}(h)ghx_ix_j + r(hx_j, gx_i)gh \\
&= gx_ihx_jr(h, g) + gg_ihg_jr(hx_j, gx_i) \\
&= (gx_i)_{(1)}(hx_j)_{(1)}r\big((hx_j)_{(2)}, (gx_i)_{(2)}\big)
\end{aligned}$$

where, for the third equality, we use the fact that $r(hx_j, gx_i)(1 - g_ig_j) = 0$. Thus, r is a co-quasitriangular structure on H which restricts to r_0 on $\Gamma \times \Gamma$ and to r_1 on $V \otimes V$. The uniqueness of r follows from Lemma 4.4.2. □

Remark 4.4.4. Recall from Example 2.6.25 that $\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r)) = \mathcal{C}(\Gamma^{\perp}, r|_{\Gamma^{\perp}})$. Recall, also, from Example 2.6.34 that, if $\mathcal{C}(\Gamma, r)$ is symmetric then the objects of $\mathcal{C}(\Gamma, r)_-$ are those Γ-graded vector spaces with support contained in $\{g \in \Gamma \mid r(g, g) = -1\}$. Thus, for arbitrary (Γ, r), the objects of $\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r))_-$ are the Γ-graded vector spaces with support contained in

$$\{g \in \Gamma \mid r(g,h)r(h,g) = 1, \text{ for all } h \in \Gamma\} \cap \{g \in \Gamma \mid r(g,g) = -1\}.$$

Corollary 4.4.5. *Let Γ be a finite abelian group, $r_0 : \Gamma \times \Gamma \to \mathbf{k}^\times$ a bicharacter, $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0^{-1}))_-$, and $r_1 : V \otimes V \to \mathbf{k}$, a Γ-graded map. Then V is a quantum linear space of symmetric type and (r_0, r_1) defines a co-quasitriangular structure on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$.*

Proof. Recall from Example 2.7.13 that $\mathcal{C}(\Gamma, r_0^{-1})$ embeds into ${}_{\Gamma}^{\Gamma}\mathcal{YD}$. Thus, V is a Yetter-Drinfeld Γ-module with Γ-action given by:

$$h \cdot v = r_0^{-1}(h,g), \quad v \in V_g, \quad g, h \in \Gamma.$$

Let $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ and $x_i \in V$, $i = 1, \ldots, n$, be such that $V = \oplus_{i=1}^n \mathbf{k}x_i$ and $x_i \in V_{g_i}^{\chi_i}$. Taking into account Remark 4.4.4, we have $\chi_i = r_0(-, g_i)^{-1}$,

$$\chi_i(g_i) = -1 \quad \text{and} \quad \chi_i(g_j)\chi_j(g_i) = 1,$$

for all i, $j = 1, \ldots, n$. Thus, V is a quantum linear space of symmetric type.

To finish the proof we have to check that r_1 is Γ-linear. Since r_1 is Γ-co-linear, we have $r_1(x_i, x_j)(1 - g_i g_j) = 0$, for all i and j. Notice that, since $\chi_i = r_0(-, g_i)^{-1}$, we have that $g_i g_j \neq 1$ whenever $\chi_i \chi_j \neq 1$. Thus, $r_1(x_i, x_j)(1 - \chi_i \chi_j) = 0$, for all i and j, so r_1 is Γ-linear. □

Remark 4.4.6. Given Γ, r_0, V and r_1 as in Corollary 4.4.5, we use the following notation

$$\mathcal{C}(\Gamma, r_0, V, r_1) := \mathsf{Corep}(\mathfrak{B}(V)\#\mathbf{k}[\Gamma], (r_0, r_1)),$$

where (r_0, r_1) denotes the r-form on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ afforded by r_0 and r_1.

Example 4.4.7. Let $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and let $r_0 : \Gamma \times \Gamma \to \mathbf{k}^\times$ be the non-trivial bicharacter of Γ. Let V be a multiple of the non-identity simple object of $\mathsf{Corep}(\mathbb{Z}/2\mathbb{Z})$. Then V belongs to $\mathcal{Z}_{sym}(\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, r_0^{-1}))_- \subseteq {}_{\Gamma}^{\Gamma}\mathcal{YD}$, so the Nichols algebra

$$E(V) = \mathfrak{B}(V)\#\mathbf{k}[\mathbb{Z}/2\mathbb{Z}]$$

admits co-quasitriangular structures. According to Proposition 4.4.3 such structures are in bijection with bilinear forms $r_1 : V \otimes V \to \mathbf{k}$ or, equivalently, with $n \times n$ square matrices, where $n = \dim_{\mathbf{k}}(V)$. This agrees with the result of [PvO99] (see Example 2.6.11).

4.5 The symmetric center of $\mathcal{C}(\Gamma, r_0, V, r_1)$

In this section we describe the symmetric center of $\mathcal{C}(\Gamma, r_0, V, r_1)$. As a result, we obtain necessary and sufficient conditions for $\mathcal{C}(\Gamma, r_0, V, r_1)$ to be symmetric, semisimple and factorizable, respectively.

Let Γ, r_0, V, and r_1 be as in Remark 4.4.5. Let $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ and r the r-form on H corresponding to (r_0, r_1). We have

$$\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0, V, r_1)) \cong \mathsf{Corep}(H_{sym}, r|_{H_{sym}\otimes H_{sym}}),$$

where H_{sym} is the Hopf subalgebra of H defined by (2.31).

Let $b : \Gamma \times \Gamma \to \mathbf{k}^\times$ be the symmetric bicharacter given by

$$b(g, h) = r(g, h)r(h, g), \qquad g, h \in \Gamma.$$

Let $\Gamma^\perp$ and $V^\perp$ denote the radicals of b and $(r_1)_{alt}$, respectively, i.e.

$$\begin{aligned} \Gamma^\perp &= \{g \in \Gamma \mid b(g, h) = 1 \text{ for all } h \in \Gamma\}, \\ V^\perp &= \{v \in V \mid (r_1)_{alt}(v, w) = 0 \text{ for all } w \in V\}. \end{aligned}$$

Lemma 4.5.1. *H_{sym} is generated as an algebra by $\Gamma^\perp$ and $V^\perp$.*

Proof. Since H_{sym} is a Hopf subalgebra of a pointed Hopf algebra with abelian coradical, it is pointed with abelian coradical. By the result of Angiono [Ang13], H_{sym} is generated by its group-like and skew-primitive elements.

It is easy to see that an element $g \in \Gamma$ is in H_{sym} if and only if $r(g, h)r(h, g) = 1$, for all $h \in \Gamma$. Thus, the set of group-like elements of H_{sym} is $\Gamma^\perp$.

Let now $g \in \Gamma^\perp$ and let x be a $(g, 1)$-primitive element. If $g \notin \{g_i \mid i = 1, \dots, n\}$ then x is a scalar multiple of $1 - g$, so it is contained in H_{sym}. If $g = g_i$ for some $i \in \{1, \dots, n\}$ then

$x = a(1-g_i)+\sum_{j:g_j=g_i} a_jx_j$, for some $a, a_j \in k$. Let $y = \sum_{j:g_j=g_i} a_jx_j$. We claim that $x \in H_{sym}$ if and only if $y \in V^{\perp}$.

It is clear that $x \in H_{sym}$ if and only if $y \in H_{sym}$. Next,

$$y_{(1)}r(y_{(2)}, z_{(1)})r(z_{(2)}, y_{(3)}) = \sum_{j:g_j=g_i} a_j\Big(r(g_j, z_{(1)})r(z_{(2)}, x_j) + r(x_j, z)\Big)g_i + \varepsilon(z)y,$$

for all $z \in H$, so $y \in H_{sym}$ if and only if

$$\sum_{j:g_j=g_i} a_j\Big(r(g_j, z_{(1)})r(z_{(2)}, x_j) + r(x_j, z)\Big) = 0, \tag{4.11}$$

for all $z \in \{hx_l \mid h \in \Gamma, l = 1, \dots, n\}$. For $z = hx_l$, the left-hand side of (4.11) becomes

$$\begin{aligned}
\text{LHS(4.11)} &= \sum_{j:g_j=g_i} a_j\Big(r(g_j, hg_l)r(hx_l, x_j) + r(x_j, hx_l)\Big)\\
&= \sum_{j:g_j=g_i} a_j\Big(r(g_j, g_l)r(x_l, x_j) + r(x_j, x_l)\Big)\\
&= r(g_l, g_i)r(x_l, y) + r(y, x_l)\\
&= (r + r \circ c_{V,V})(y, x_l)\\
&= (r - r \circ \tau_{V,V})(y, x_l)\\
&= 2(r_1)_{alt}(y, x_l),
\end{aligned}$$

where we have used the fact that $r|_{V\otimes V}$ is a morphism in $\mathcal{YD}$ and Lemma 4.2.7.

Thus, $y \in H_{sym}$ if and only if $y \in V^{\perp}$. It follows that non-trivial skew-primitive elements of H_{sym} generate $V^{\perp}$, and the claim is proved. $\square$

Corollary 4.5.2. $H_{sym} = \mathfrak{B}(V^{\perp})\#\mathbf{k}[\Gamma^{\perp}]$ *and*

$$\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0, V, r_1)) \cong \mathcal{C}(\Gamma^{\perp}, r_0|_{\Gamma^{\perp}\times\Gamma^{\perp}}, V^{\perp}, r_1|_{V^{\perp}\otimes V^{\perp}}).$$

Corollary 4.5.3. *The following hold.*

(i) $\mathcal{C}(\Gamma, r_0, V, r_1)$ *is symmetric if and only if* $\mathcal{C}(\Gamma, r_0)$ *is symmetric and* r_1 *is symmetric.*

(ii) $\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0, V, r_1))$ *is semisimple if and only if* $(r_1)_{alt} : V \otimes V \to \mathbf{k}$ *is non-degenerate.*

(iii) $\mathcal{C}(\Gamma, r_0, V, r_1)$ *is factorizable if and only if* $\mathcal{C}(\Gamma, r_0)$ *is factorizable and* $V = 0$.

Example 4.5.4. Let $E(V)$ be the Hopf algebra from Example 4.4.7 and let $r_1 : V \otimes V \to k$ be a bilinear form. In this case $c_{V,V} = -\tau_{V,V}$, so Corollary 4.5.3(i) says that the co-quasitriangular structure determined by r_1 is symmetric if an only if r_1 is symmetric (in the usual linear algebra sense). This was proved in [CC04b].

4.6 Ribbon structures on $\mathcal{C}(\Gamma, r_0, V, r_1)$

In this section we classify ribbon structures of $\mathcal{C}(\Gamma, r_0, V, r_1)$.

We need the following result.

Lemma 4.6.1. *Let* A *be an abelian group and let* $a_1, a_2, \dots, a_n$ *be elements of* A. *There exists a group homomorphism* $\gamma : A \to \{\pm 1\}$ *such that* $\gamma(a_i) = -1$, *for all* $i = 1, \dots, n$ *if and only if there are no relations in* A *of the form* $a_{i_1} a_{i_2} \cdots a_{i_k} = x^2$, *with* k *odd.*

Proof. Suppose there are no relations in A of the form $a_{i_1} a_{i_2} \cdots a_{i_k} = x^2$, with k odd. Let $\pi : A \to A/A^2$ be the natural projection and let $\{\pi(a_{i_1}), \pi(a_{i_2}), \dots, \pi(a_{i_r})\}$ be a maximal linearly independent subset of $\{\pi(a_i) \mid i = 1, \dots, n\}$ in the elementary abelian 2-group A/A^2. Notice that, for any $i \in \{1, \dots, n\}$, $\pi(g_i)$ is a product of an odd number of elements of $\{\pi(a_{i_1}), \pi(a_{i_2}), \dots, \pi(a_{i_r})\}$. Thus, there is a homomorphism $f : A/A^2 \to \{\pm 1\}$ such that $f\big(\pi(a_i)\big) = -1$, for all $i = 1, \dots, n$. Composing f with π we obtain a homomorphism $A \to \{\pm 1\}$ sending each a_i to -1. The converse is trivial. $\square$

Proposition 4.6.2. *The set of ribbon structures on* $\mathcal{C}(\Gamma, r_0, V, r_1)$ *is non-empty and is in bijection with the set of group homomorphisms* $\gamma : \Gamma \to \{\pm 1\}$ *such that* $\gamma(g_i) = -1$, *for all* $i = 1, \dots, n$.

Proof. Let $H = \mathfrak{B}(V)\#k[\Gamma]$. As explained in Remark 2.6.32, ribbon structures on $\mathcal{C}(\Gamma, r_0, V, r_1)$ are in bijection with group-like elements $\gamma \in G(H^*)$ satisfying $\gamma^2 = (\eta \circ S) * \eta^{-1}$ and $S^2_{H^*}(p) = \gamma^{-1} * p * \gamma$, for all $p \in H^*$.

Let γ be such an element. We have

$$\gamma(g)^2 = \gamma^2(g) = (\eta \circ S) * \eta^{-1}(g) = r_0(g^{-1}, g)r_0(g, g) = 1$$

for all $g \in \Gamma$, so $\gamma(\Gamma) \subseteq \{\pm 1\}$. Now $S^2_{H^*}(p) = \gamma^{-1} * p * \gamma$, for all $p \in H^*$, if and only if $S^2_H = \gamma^{-1} * \mathsf{id}_H * \gamma$. Since both maps are algebra maps, they are equal if and only if they agree on algebra generators. We have

$$\begin{aligned} S^2_H(g) &= g, \qquad (\gamma^{-1} * \mathsf{id}_H * \gamma)(g) = g, \\ S^2_H(x_i) &= -x_i, \quad (\gamma^{-1} * \mathsf{id}_H * \gamma)(x_i) = \gamma^{-1}(g_i)x_i, \end{aligned}$$

for all $g \in \Gamma$ and $i = 1, \dots, n$. Thus, $S^2_H = \gamma^{-1} * \mathsf{id}_H * \gamma$ if and only if $\gamma(g_i) = -1$ for all i.

It remains to show that there always exists a homomorphism $\gamma : \Gamma \to \{\pm 1\}$ such that $\gamma(g_i) = -1$, for all $i = 1, \dots, n$. Since $\mathbf{k}^\times$ is an injective $\mathbb{Z}$-module, it is enough to show that there is such a homomorphism on the subgroup $\Gamma_0 = \langle g_1, \dots g_n \rangle \subset \Gamma$. Using Lemma 4.6.1, we have to show that there are no relations in Γ_0 of the form $g_{i_1} g_{i_2} \cdots g_{i_k} = x^2$ with k odd. If $x = g_{i_1}^{e_1} g_{i_2}^{e_2} \cdots g_{i_t}^{e_t}$ is an element of Γ_0 then

$$r_0(x,\, x) = \textstyle\prod_{r=1}^t r_0(g_{i_r}, g_{i_r})^{e_r^2} \prod_{1 \le r < s \le t} \big(r_0(g_{i_r}, g_{i_s}) r_0(g_{i_s}, g_{i_r})\big)^{e_r e_s} = (-1)^{\sum_{r=1}^t e_r^2}.$$

In particular, $r_0(x,\, x)^2 = 1$, for all $x \in \Gamma_0$. On the other hand, if $g_{i_1} g_{i_2} \cdots g_{i_k} = x^2$ with k odd, then $r_0(x,\, x)^4 = r_0(x^2, x^2) = (-1)^k = -1$, which is a contradiction. $\square$

4.7 Metric quadruples

In this section we generalize the result of Section 4.1 to include the non-semisimple case. We prove that the groupoid with objects pointed braided finite tensor categories admitting a fiber functor is equivalent to a groupoid of metric quadruples.

We start with the following result.

Theorem 4.7.1. *Let $\mathcal{C}$ be a pointed braided finite tensor category admitting a fiber functor. Then there exist a finite abelian group Γ, a bicharacter $r_0 : \Gamma\times\Gamma \to \mathbf{k}^\times$, an object $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0^{-1}))_-$ and a Γ-graded morphism $r_1 : V \otimes V \to \mathbf{k}$, such that*

$$\mathcal{C} \simeq \mathcal{C}(\Gamma, r_0, V, r_1).$$

Proof. From the reconstruction theory we know that there exists a finite dimensional pointed co-quasitriangular Hopf algebra (H, r) such that $\mathcal{C} \simeq \mathsf{Corep}(H, r)$.

Let Γ be the group of group-like elements of H. Since $\mathsf{Corep}\,\Gamma$ is a tensor subcategory of $\mathsf{Corep}\,H$, it is braided, hence Γ is abelian. Moreover, r restricts to a bicharacter r_0' on Γ.

For $g \in \Gamma$ let $V_g = P_{g,1}(H)$ be the set of $(g, 1)$-primitive elements of H. The group Γ acts on V_g by conjugation, so $V_g = \bigoplus_{\chi\in\widehat{\Gamma}} V_g^\chi$, where $V_g^\chi = \{x \in V_g \mid hxh^{-1} = \chi(h)x, \text{ for all } h \in \Gamma\}$. Let

$$V = \bigoplus_{\substack{g\in\Gamma\setminus\{1\}\\ \chi\in\widehat{\Gamma}\setminus\{1\}}} V_g^\chi = \bigoplus_{i=1}^{n} \mathbf{k}x_i, \quad x_i \in V_{g_i}^{\chi_i}.$$

Condition (2.29) for the pair $(x, y) = (x_i, g)$ yields $r_0'(g, g_i) = \chi_i^{-1}(g)$, and the same condition for the pair $(x, y) = (g, x_i)$ yields $r_0'(g_i, g) = \chi_i(g)$. Thus,

$$r_0'(g_i, -) = \chi_i = r_0'(-, g_i)^{-1},$$

for all $i = 1, \ldots, n$. In particular, $\chi_i(g_j)\chi_j(g_i) = 1$ for all $i, j = 1, \ldots, n$. Moreover, $\chi_i(g_i) = -1$, for all $i = 1, \ldots, n$. Indeed, if $\chi_i(g_i) = 1$, for some i, then the Hopf subalgebra of H generated by

g_i and x_i is non-semisimple commutative, and, hence, infinite dimensional. Thus, V is a quantum linear space of symmetric type.

From I. Angiono's result [Ang13] it follows that H is generated by Γ and V, so H is a lifting of $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. From the work of N. Andruskiewitsch and H.-J. Schneider [AS98], and A. Masuoka [Mas01], we have that H is a 2-cocycle deformation of $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. Taking into account Proposition 4.4.3 and the fact that cocycle deformation does not change the category of co-representation, we have

$$\mathcal{C} \simeq \mathsf{Corep}(H, r) \simeq \mathsf{Corep}(\mathfrak{B}(V)\#\mathbf{k}[\Gamma], (r_0, r_1)) = \mathcal{C}(\Gamma, r_0, V, r_1),$$

for a bicharacter r_0 of Γ such that $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0^{-1})_-$, and a Γ-graded map $r_1 : V \otimes V \to \mathbf{k}^\times$. $\square$

Recall that a *groupoid* is a category in which all morphisms are isomorphisms.

Let $\mathcal{P}$ be the groupoid with objects pointed braided finite tensor categories admitting a fiber functor and morphisms natural isomorphism classes of equivalences of braided tensor categories.

Denote by $\mathcal{Q}$ the groupoid whose objects are quadruples (Γ, q, V, r), where Γ is a finite abelian group, $q \in \mathsf{Quad}_d(\Gamma)$ is a diagonalizable quadratic form on Γ, V is an object in $\mathcal{Z}_{sym}(\mathcal{C}(\Gamma, q))_-$, and $r : V \otimes V \to \mathbf{k}$ is an alternating bilinear form in $\mathcal{C}(\Gamma, q)$. The set of morphisms in $\mathcal{Q}$ from (Γ, q, V, r) to (Γ', q', V', r') is the set of equivalence classes of pairs (α, f), where $\alpha : (\Gamma, q) \to (\Gamma', q')$ is an orthogonal group isomorphism and $f : \mathsf{ind}_\alpha(V) \to V'$ is an isomorphism in $\mathcal{C}(\Gamma', q')$ such that $r' \circ (f \otimes f) = \mathsf{ind}_\alpha(r)$. The equivalence relation identifies (α, f) and $(\alpha, -f)$.

Definition 4.7.2. We call $\mathcal{Q}$ the groupoid of *metric quadruples*.

Define a functor $F : \mathcal{Q} \to \mathcal{P}$ as follows. Choose, for every metric quadruple (Γ, q, V, r), a bicharacter $r_0 : \Gamma \times \Gamma \to \mathbf{k}^\times$ such that $q(g) = r_0(g, g)$, for all $g \in \Gamma$, and define F on objects by:

$$F(\Gamma, q, V, r) = \mathcal{C}(\Gamma, r_0, V, r). \tag{4.12}$$

Let now $(\alpha, f) : (\Gamma, q, V, r) \to (\Gamma', q', V', r')$ be a morphism in $\mathcal{Q}$. It gives rise to an isomorphism of Hopf algebras $\varphi_{(\alpha,f)} : \mathfrak{B}(V)\#\mathbf{k}[\Gamma] \to \mathfrak{B}(V')\#\mathbf{k}[\Gamma']$ given by

$$\varphi_{(\alpha,f)}(g) = \alpha(g), \quad \varphi_{(\alpha,f)}(x) = f(x), \tag{4.13}$$

for all $g \in \Gamma$ and $x \in V$. If $r'_0 : \Gamma' \times \Gamma' \to \mathbf{k}^\times$ is a bicharacter such that $q'(g) = r'_0(g, g)$, $g \in \Gamma'$, then $r'_0 \circ (\alpha \times \alpha)/r_0 = \mathrm{alt}(\mu)$, for some $\mu \in \mathsf{Z}^2(\Gamma/\Gamma_0, \mathbf{k}^\times)$, see (4.10). This means that the r-form $r'_0 \circ (\alpha \times \alpha)$ is a μ-deformation of r_0. But $\widehat{\mu} \in \mathsf{H}^2(\Gamma/\Gamma_0, \mathbf{k}^\times)$ defines an invariant 2-cocycle σ on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ by Proposition 4.3.6. Thus, $\varphi_{(\alpha,f)}$ gives rise to a well defined braided tensor equivalence $F(\alpha, f)$ between $\mathcal{C}(\Gamma, r_0, V, r)$ and $\mathcal{C}(\Gamma', r'_0, V', r')$, namely $(\varphi_{(\alpha,f)}, \sigma)$.

Our aim is to prove the following.

Theorem 4.7.3. *The functor*

$$F : \mathcal{Q} \to \mathcal{P} \tag{4.14}$$

is an equivalence of categories.

We start with a few observations.

Lemma 4.7.4. *Let (r_0, r_1) and (r'_0, r'_1) define co-quasitriangular structures on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ and $\mathfrak{B}(V')\#\mathbf{k}[\Gamma']$, respectively. Then the set of co-quasitriangular Hopf algebra isomorphisms*

$$(\mathfrak{B}(V)\#\mathbf{k}[\Gamma], (r_0, r_1)) \to (\mathfrak{B}(V')\#\mathbf{k}[\Gamma'], (r'_0, r'_1))$$

is in bijection with the set of pairs (α, φ), where $\alpha : \Gamma \to \Gamma'$ is a group isomorphism such that $r'_0 \circ (\alpha \times \alpha) = r_0$, and $\varphi : \mathsf{ind}_\alpha(V) \to V'$ is an isomorphism in $\mathcal{C}(\Gamma', r'_0)$ such that $r'_1 \circ (\varphi \otimes \varphi) = \mathsf{ind}_\alpha(r_1)$. The pair (α, φ) corresponding to an isomorphism f is $(\alpha, \varphi) = (f|_\Gamma, f|_V)$.

Proof. Let $f : (\mathfrak{B}(V)\#\mathbf{k}[\Gamma], (r_0, r_1)) \to (\mathfrak{B}(V')\#\mathbf{k}[\Gamma'], (r'_0, r'_1))$ be an isomorphism of coquasitriangular Hopf algebras. Since f takes group-like elements to group-like elements, it restricts to a group isomorphism $\alpha : \Gamma \to \Gamma'$. Let us show that f induces an isomorphism $\mathsf{ind}_\alpha(V) \to V'$.

Notice first that $\alpha(g_i) \in \{g'_j\}$. Indeed, if this is not the case, then $f(x_i) = a(1 - f(g_i))$, for some $a \in \mathbf{k}$. But $f(x_i)$ anti-commutes with $f(g_i)$, while $a(1 - f(g_i))$ commutes with $f(g_i)$, so $f(x_i) = 0$. This, however, contradicts the injectivity of f.

Thus, $\alpha(g_i) \in \{g'_j\}$, for all i. It follows that there exist scalars a_i, $b_{ij} \in \mathbf{k}$ such that $b_{ji}\big(g'_j - f(g_i)\big) = 0$ and

$$f(x_i) = a_i(1 - f(g_i)) + \textstyle\sum_j b_{ji} x'_j, \qquad i = 1, \dots, n.$$

We must have $f(g)f(x_i) = \chi_i(g) f(x_i) f(g)$, for all $g \in \Gamma$. Now

$$\begin{aligned} f(g)f(x_i) &= a_i(f(g) - f(gg_i)) + \sum_j b_{ji} f(g) x'_j \\ &= a_i(f(g) - f(gg_i)) + \sum_j b_{ji} \chi'_j\big(f(g)\big) x'_j f(g) \end{aligned}$$

and

$$\chi_i(g) f(x_i) f(g) = a_i \chi_i(g)(f(g) - f(gg_i)) + \textstyle\sum_j b_{ji} \chi_i(g) x'_j f(g).$$

Thus, $f(g)f(x_i) = \chi_i(g) f(x_i) f(g)$ if and only if $a_i = a_i \chi_i(g)$ and $\chi'_j\big(f(g)\big) b_{ji} = \chi_i(g) b_{ji}$, for all j and $g \in \Gamma$. Taking $g = g_i$ in the first condition, we obtain $a_i = 0$. The second condition is equivalent to $b_{ji}(\chi_i - \chi'_j \circ f) = 0$.

It follows that $f(x_i) = \sum_j b_{ji} x'_j$, where $b_{ji}\big(g'_j - \alpha(g_i)\big) = 0$ and $b_{ji}(\chi_i \alpha^{-1} - \chi'_j) = 0$. In particular, the restriction φ of f to V is a morphism in $\mathcal{C}(\Gamma', r'_0)$ from $\mathsf{ind}_\alpha(V)$ to V'. Since f is an isomorphism, φ is also an isomorphism.

We have proved that if $f : \mathfrak{B}(V)\#\mathbf{k}[\Gamma] \to \mathfrak{B}(V')\#\mathbf{k}[\Gamma']$ is an isomorphism of Hopf algebras then it induces, by restriction, isomorphisms $\alpha : \Gamma \to \Gamma'$ and $\varphi : \mathsf{ind}_\alpha(V) \to V'$. We have $(r'_0, r'_1) \circ (f \otimes f) = (r_0, r_1)$ if and only if $r'_0 \circ (\alpha \times \alpha) = r_0$ and $r'_1 \circ (\varphi \otimes \varphi) = \mathsf{ind}_\alpha(r_1)$.

It is easy to check that every data (α, φ) with the above properties comes from an isomorphism of co-quastriangular Hopf algebras $H \to H'$, so the theorem holds. $\square$

Lemma 4.7.5. *Let σ be an invariant 2-cocycle on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ such that $\sigma|_{\Gamma\times\Gamma} = 1$, and let r be an r-form on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$. Then the σ-deformation of r satisfies*

$$r^\sigma|_{V\otimes V} = r|_{V\otimes V} + 2(\sigma|_{V\otimes V})_{sym}. \tag{4.15}$$

Proof. Using formula (2.41), we compute

$$\begin{aligned} r^\sigma(x_i,\, x_j) &= r(g_i,\, g_j)\,\sigma^{-1}(x_i,\, x_j) + r(x_i,\, x_j) + \sigma(x_j,\, x_i) \\ &= r(x_i,\, x_j) + (\sigma(x_j,\, x_i) - \chi_i(g_j)\sigma(x_i,\, x_j)) \\ &= r(x_i,\, x_j) + (\sigma \circ \tau_{V,V} - \sigma \circ c_{V,V} \circ \tau_{V,V})(x_i,\, x_j) \end{aligned}$$

Since the restrictions of r and r^σ on $V\otimes V$ are morphisms in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ we conclude, using Lemma 4.2.7, that

$$\sigma|_{V\otimes V} \circ \tau_{V,V} - \sigma|_{V\otimes V} \circ c_{V,V} \circ \tau_{V,V} = 2(\sigma|_{V\otimes V})_{sym},$$

which implies the statement. ☐

We now proceed with the proof of Theorem 4.7.3.

Proof. We need to show that the functor F (4.14) is essentially surjective and fully faithful.

(1) *F is essentially surjective.* For this end it suffices to check that the co-quasitriangular structure on H defined using $r : V \otimes V \to \mathbf{k}$ is gauge equivalent, by means of an invariant 2-cocycle on H, to the one defined using an alternating morphism $V \otimes V \to \mathbf{k}$ in $\mathcal{C}(\Gamma, r_0)$. Let σ be an invariant 2-cocycle on H such that $\sigma|_{\Gamma\times\Gamma} = 1$ (such 2-cocycles are classified in Corollary 4.3.4) and let $r^\sigma = \sigma_{21} * r * \sigma^{-1}$. By Lemma 4.7.5 we have

$$r^\sigma|_{V\otimes V} = r|_{V\otimes V} + 2(\sigma|_{V\otimes V})_{sym}.$$

Thus, we can take σ such that $(\sigma|_{V\otimes V})_{sym} = -\frac{1}{2}(r \circ \tau)_{sym}$, so that $r^\sigma|_{V\otimes V} = (r|_{V\otimes V})_{alt}$, i.e., $r^\sigma|_{V\otimes V}$ is alternating.

Since $\mathcal{C}(\Gamma, r_0, V, r)$ and $\mathcal{C}(\Gamma, r_0, , r^\sigma)$ are equivalent braided tensor ategories, the surjectivity of F follows.

(2) F *is faithful.* We need to check that F is injective on morphisms. It is clear from definitions that $F(\alpha, f) = F(\alpha', f')$ implies $\alpha = \alpha'$. Therefore, it remains to check that for an automorphism (id_Γ, f) of $(\Gamma, q, V, r) \in \mathcal{Q}$ one has $F(\mathsf{id}_\Gamma, f) \cong \mathsf{id}_{\mathcal{C}(\Gamma,q,V,r)}$ as a tensor functor if and only if $f = \pm\, \mathsf{id}_V$. One implication is clear since $-\,\mathsf{id}_V$ preserves any bilinear form.

Note that the Hopf algebra automorphism $\varphi_{(\mathsf{id}_\Gamma, f)}$ of H defined in (4.13) gives rise to a trivial tensor autoequivalence of $\mathsf{Corep}(H)$ if and only if it is given by

$$h \mapsto \chi \rightharpoonup h \leftharpoonup \chi^{-1}, \quad h \in H$$

for some character $\chi \in H^*$. The condition that it preserves r is equivalent to $\chi(g_i)\chi(g_j) = 1$ for all $i, j = 1, \dots, n$, i.e., to χ being identically equal to 1 or -1 on the support of V. By the Remark **??** there exists χ such that this value is -1, so that $f = \pm\, \mathsf{id}_V$.

(3) F *is full.* We need to check that F is surjective on morphisms. We claim that any braided tensor equivalence Φ between $\mathsf{Corep}(\mathfrak{B}(V)\#k[\Gamma], r)$ and $\mathsf{Corep}(\mathfrak{B}(V')\#k[\Gamma'], r')$ is isomorphic to one coming from a coquasitriangular Hopf algebra isomorphism $\mathfrak{B}(V)\#\mathbf{k}[\Gamma] \to \mathfrak{B}(V')\#\mathbf{k}[\Gamma']$. By the result of A. Davydov [Dav10] Φ corresponds to a pair (f, σ), where σ is a 2-cocycle on $\mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ and $f : (\mathfrak{B}(V)\#k[\Gamma])^\sigma \to \mathfrak{B}(V')\#k[\Gamma']$ is Hopf algebra isomorphism such that

$$r \circ (f \otimes f) = r^\sigma.$$

The last condition corresponds to the braided property of the equivalence.

We must have $\sigma|_{\Gamma\times\Gamma} = 1$ since non-trivial twisting changes the braided equivalence class of $\mathsf{Corep}(k[\Gamma],\, r_0)$. By Lemma 4.7.5 we have

$$(r \circ (f \otimes f) - r)\,|_{V\otimes V} = 2(\sigma|_{V\otimes V})_{sym}.$$

The left hand side is of the above equality is alternating, while the right hand side is symmetric. Hence, both sides are equal to 0 and so σ is gauge equivalent to the trivial 2-cocyle by Proposition 4.3.2. This means that Φ is isomorphic to the equivalence induced by a co-quasitriangular Hopf algebra isomorphism, so the result follows from Lemma 4.7.4. $\square$

Remark 4.7.6. We can give a conceptual explanation of the reason why $(r_1)_{alt}$ is an invariant of the braided tensor category $\mathcal{C} := \mathcal{C}(\Gamma, r_0, V, r_1)$.

Let $g \in \Gamma$. We will also use g to denote the corresponding invertible H-comodule. Recall that $\mathsf{Ext}^1(g, 1) \cong P_{1,g}(H)/\mathbf{k}(1 - g)$, where $P_{1,g}(H)$ denotes the space of $(1, g)$-skew primitive elements of H. Explicitly, elements of $\mathsf{Ext}^1(g, 1)$ are in bijection with equivalence classes of short exact sequences

$$0 \to 1 \xrightarrow{\iota} V_x \xrightarrow{p} g \to 0,$$

where 1 denotes the trivial comodule $\mathbf{k}$. The 2-dimensional comodule V_x is a vector space with a basis v_0, v_1 and H-coaction given by

$$\rho(v_0) = v_0 \otimes 1, \qquad \rho(v_1) = v_0 \otimes x + v_1 \otimes g, \tag{4.16}$$

where $x \in P_{1,g}(H)$.

Let $x' \in P_{1,g'}(H)$, $g' \in \Gamma$, be another skew-primitive element of H, let

$$0 \to 1 \xrightarrow{\iota'} V_{x'} \xrightarrow{p'} g' \to 0$$

be the corresponding extension, and let v_0', v_1' be a basis of $V_{x'}$ defined analogously to (4.16).

Let $\beta_{x,x'} = c_{V_{x'},V_x} \circ c_{V_x,V_{x'}}$ denote the square of the braiding on $V_x \otimes V_{x'}$. Using formula (**??**) one computes

$$\begin{aligned}
\beta_{x,x'}(v_0 \otimes v_0') &= v_0 \otimes v_0', \\
\beta_{x,x'}(v_0 \otimes v_1') &= v_0 \otimes v_1', \\
\beta_{x,x'}(v_1 \otimes v_0') &= v_1 \otimes v_0', \\
\beta_{x,x'}(v_1 \otimes v_1') &= v_1 \otimes v_1' + (r_1(x,\, x') + r_0(g,\, g')r_1(x',\, x))v_0 \otimes v_0'.
\end{aligned}$$

Let $s := r_1 - r_1 \circ \tau$. Combining Lemma 4.2.7 with above computation we see that

$$\beta_{x,x'} = \mathsf{id}_{V_x \otimes V_{x'}} + s(x,\, x')(p \otimes p') \circ (\iota \otimes \iota') \tag{4.17}$$

for all $x, x' \in V = \mathsf{Ext}^1(\Gamma, 1)$ (note that $(p \otimes p') \circ (\iota \otimes \iota') \in \mathsf{End}_{\mathcal{C}}(V_x \otimes V_{x'})$ whenever $s(x,\, x') \neq 0$).

It follows from (4.17) that $s = (r_1)_{\text{alt}}$ is an invariant of the braided equivalence class of $\mathcal{C}$ (a computation establishing this fact is straightforward and can be found in the proof of [BN15, Proposition 6.7]).

Let $\mathcal{C} = \mathcal{C}(\Gamma, q, V, r)$. Theorem 4.7.3 allows to compute the group $\mathsf{Aut}^{\mathrm{br}}(\mathcal{C})$ of isomorphism classes of braided autoequivalences of $\mathcal{C}$. Namely, let

$$\begin{aligned}
\mathsf{Aut}(V, r) &:= \{f \in \mathsf{Aut}_{\mathcal{C}(\Gamma,q)}(V) \mid r \circ (f \otimes f) = r\}/\{\pm \mathsf{id}_V\}, \\
O(\Gamma, q, r) &:= \{\alpha \in \mathsf{Aut}(\Gamma) \mid q \circ \alpha = q \text{ and } \mathsf{ind}_\alpha(r),\, r \text{ are congruent in } \mathcal{C}(\Gamma, q)\}.
\end{aligned}$$

Corollary 4.7.7. *There is a short exact sequence*

$$1 \to \mathsf{Aut}(V, r) \to \mathsf{Aut}^{br}(\mathcal{C}) \to O(\Gamma, q, r) \to 1. \tag{4.18}$$

Example 4.7.8. Let us consider $\mathcal{C} = \mathsf{Corep}(E(V), r)$, where $E(V)$ is the Hopf algebra from Example 4.4.7 with the co-quasitriangular structure given by the zero bilinear form on V (this structure is symmetric). In this case $\Gamma = \mathbb{Z}/2\mathbb{Z}$, so $O(\Gamma, q, r) = 1$ and Corollary 4.7.7 implies that $\mathsf{Aut}^{\mathrm{br}}(\mathcal{C}) = GL_n(V)/\{\pm \mathsf{id}_V\}$, cf. [BN15].

4.8 The Drinfeld center of a pointed braided tensor category

It is well known that the Drinfeld center of a pointed braided fusion category is pointed (see, e.g., [DN13, Proposition 5.8]). This is no longer true in the non-semisimple case. Indeed, the Drinfeld center is always factorizable, cf. Corollary 4.5.3(iii). If (Γ, q, V, r) is a metric quadruple such that $V \in \mathcal{C}(\Gamma, q)$ is self-dual, then the adjoint subcategory of $\mathcal{Z}(\mathcal{C}(\Gamma, q, V, r))$ is pointed. We describe the metric quadruple associated to this subcategory.

Let H be a Hopf algebra. It was shown in [GN08] that, for any Hopf subalgebra K of H contained in the center of H, the tensor category $\mathsf{Rep}(H)$ is graded by $G(K^*)$. The trivial component of this grading is $\mathsf{Rep}(H/HK^+)$. The maximal central Hopf subalgebra of H provides the universal grading of $\mathsf{Rep}(H)$. In this case, the trivial component is $\mathsf{Rep}(H)_{\mathrm{ad}}$, the *adjoint subcategory* of $\mathsf{Rep}(H)$.

If H is quasitriangular then the maximal central Hopf subalgebra of H is the group algebra of $G(H) \cap Z(H)$, and the universal grading group of $\mathsf{Rep}(H)$ is the group of characters of $G(H) \cap Z(H)$.

Let $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ be a quantum linear space of symmetric type. We have

$$\mathcal{Z}(\mathsf{Corep}\, H) \cong \mathsf{Rep}\, D(H)^{\mathrm{cop}}.$$

Since $D(H)^{\mathrm{cop}}$ is quasitriangular, the universal grading group of $\mathcal{Z}(\mathsf{Corep}(H))$ is isomorphic to the group of characters of $G(D(H)) \cap Z(D(H))$.

The group of central group-like elements of $D(H)$ was described in [Rad93, Proposition 10]. We have

$$G\big(D(H)\big) \cap Z\big(D(H)\big) \cong G\big(D(H)^*\big),$$

and

$$G\big(D(H)^*\big) = \{g \otimes \gamma \mid (g,\gamma) \in G(H) \times G(H^*), \\ \textstyle\sum \gamma(h_{(1)})h_{(2)}g = \sum \gamma(h_{(2)})gh_{(1)},\ \forall h \in H\}$$

If $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ is a quantum linear space of symmetric type, it is straigtforward to check that

$$G\big(D(H)^*\big) = \{g \otimes \gamma \mid (g,\gamma) \in \Gamma \times \widehat{\Gamma},\ \gamma(g_i) = \chi_i(g), \text{for all } i = 1,\dots,n\}.$$

Equivalently, this can be described as follows. Let $b : (\Gamma \times \widehat{\Gamma}) \times (\Gamma \times \widehat{\Gamma}) \to \mathbf{k}^\times$ be the canonical non-degenerate bicharacter defined by

$$b\big((g,\chi),(g',\chi')\big) = \chi(g')\chi'(g), \qquad (g,\chi),(g',\chi') \in \Gamma \times \widehat{\Gamma}.$$

Consider the subgroup

$$\Sigma := \langle (g_i, \chi_i^{-1}) \mid i = 1,\dots,n \rangle \subseteq \Gamma \times \widehat{\Gamma}. \tag{4.19}$$

Then $G(D(H)^*)$ is isomorphic to $\Sigma^\perp$, the orthogonal complement of Σ with respect to the bicharacter b.

Thus, the universal grading group of $\mathcal{Z}(\mathsf{Corep}\, H)$ is $\widehat{\Sigma^\perp}$.

Remark 4.8.1. Σ is an isotropic subgroup of $\Gamma \times \widehat{\Gamma}$, i.e. $\Sigma \subset \Sigma^\perp$.

Let K be the group algebra of $G(D(H)) \cap Z(D(H)) \cong \Sigma^\perp$. Then

$$\mathcal{Z}(\mathsf{Corep}\, H)_{\text{ad}} \cong (\mathsf{Rep}\, D(H)^{\text{cop}})_{\text{ad}} \cong \mathsf{Rep}(D(H)/D(H)K^+)^{\text{cop}}. \tag{4.20}$$

Let $\pi : D(H) \to D(H)/D(H)K^+$ be the canonical projection. Then the transpose $\pi^* : \big(D(H)/D(H)K^+\big)^* \to D(H)^*$ is an injective Hopf algebra map. We describe next the image of this map.

Given a co-quasitriangular Hopf algebra (H, r), there is a Hopf algebra map

$$\iota_r : H \to D(H)^{*\,\text{op}}, \quad \iota_r(x) = x_{(1)} \otimes r(-, x_{(2)})$$

It corresponds to the embedding $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C})$ from Remark 2.7.12.

Recall from Proposition 4.4.3 that the r-forms on $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ are in bijection with pairs (r_0, r_1), where r_0 is a bicharacter of Γ such that $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))_-$ and $r_1 : V \otimes V \to \mathbf{k}$ is a morphism in $\mathcal{C}(\Gamma, r_0)$. Fix a bicharacter r_0 on Γ such that $V \in \mathcal{Z}_{sym}(\mathcal{C}(\Gamma, r_0))_-$, and let $(g_1, \dots, g_n, \chi_1, \dots, \chi_n)$ be the datum that defines V.

If r is the r-form on H corresponding to the pair $(r_0, 0)$, then $\iota_r(g_i) = g_i \otimes \chi_i^{-1}$ and $\iota_r(x_i) = x_i \otimes \varepsilon$. Thus, $D(H)^*$ contains the group-like elements $g_i \otimes \chi_i^{-1}$ and the $(g_i \otimes \chi_i^{-1}, 1)$-skew primitive elements $x_i \otimes \varepsilon$, $i = 1, \dots, n$.

Assume that V is a self-dual object of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Then the set $\{(g_i, \chi_i) \mid i = 1, \dots, n\}$ is closed under taking inverses and there exists a non-degenerate morphism $r_1 : V \otimes V \to \mathbf{k}$ in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Let r' be the r-form on H corresponding to the pair (r_0, r_1). Then $\iota_{r'}(g_i) = g_i \otimes \chi_i^{-1}$ and $\iota_{r'}(x_i) = g_i \otimes r'(-, x_i) + x_i \otimes \varepsilon$. Thus, $D(H)^*$ contains, also, the $(g_i \otimes \chi_i^{-1}, 1)$-skew primitive elements $g_i \otimes r'(-, x_i)$.

Let A be the Hopf subalgebra of $D(H)^*$ generated by group-like elements $g_i \otimes \chi_i^{-1}$ and skew-primitive elements $x_i \otimes \varepsilon$ and $g_i \otimes r'(-, x_i)$, $i = 1, \dots, n$.

Remark 4.8.2. By definition (4.19), the group of group-likes of A is Σ. The above skew-primitive elements $x_i \otimes \varepsilon$ and $g_i \otimes r'(-, x_i)$, $i = 1, \dots, n$ constructed above are linearly independent and form a $2n$-dimensional quantum linear space of symmetric type in ${}^{\Sigma}_{\Sigma}\mathcal{YD}$. Therefore,

$$\dim_k(A) = |\Sigma|\, 2^{2n}.$$

Proposition 4.8.3. *The image of the map* $\pi^* : \left(D(H)/D(H)K^+\right)^* \to D(H)^*$ *is* A*. Thus,*

$$D(H)/D(H)K^+ \cong A^*. \tag{4.21}$$

Proof. By definition, K is the group algebra of $\{\gamma \otimes g \mid (g, \gamma) \in \Sigma^\perp\}$. Let

$$e = \frac{1}{|\Sigma^\perp|} \sum_{(g,\gamma) \in \Sigma^\perp} \gamma \otimes g.$$

Then e is a central idempotent of $D(H)$, $ze = \varepsilon(z)e$, for all $z \in K$, and $K^+ = (1-e)K$. Thus, $D(H)K^+ = D(H)K(1-e) = (1-e)D(H)$. The image of π^* is

$$\mathrm{Im}(\pi^*) = \{f \in \big(D(H)\big)^* \mid f(z) = f(ez), \text{ for all } z \in D(H)\}.$$

It is easy to check that any $f \in \{g_i \otimes \chi_i^{-1}, x_i \otimes \varepsilon, g_i \otimes r'(-, x_i)\}$ satisfies $f((\gamma \otimes g)z) = f(z)$, for all $(g, \gamma) \in \Sigma^\perp$ and $z \in D(H)$. For example, if $(g, \gamma) \in \Sigma^\perp$, then

$$\begin{aligned}(x_i \otimes \varepsilon)\big((\gamma \otimes g)z\big) &= (g_i \otimes \chi_i^{-1})(\gamma \otimes g)\,(x_i \otimes \varepsilon)(z) + (x_i \otimes \varepsilon)(\gamma \otimes g)\,(1 \otimes \varepsilon)(z)\\ &= \gamma(g_i)\chi_i^{-1}(g)\,(x_i \otimes \varepsilon)(z)\\ &= (x_i \otimes \varepsilon)(z).\end{aligned}$$

It follows that the generators of A are contained in the image of π^*, so $A \subseteq \mathrm{Im}(\pi^*)$. Using Remark 4.8.2 we compute

$$\dim \mathrm{Im}(\pi^*) = \frac{\dim D(H)}{\dim K} = \frac{|\Gamma|^2 2^{2n}}{|\Sigma^\perp|} = |\Sigma|\, 2^{2n} = \dim A$$

and so $A = \mathrm{Im}(\pi^*)$. $\square$

We are now ready to describe the metric quadruple associated to $\mathcal{Z}\big(\mathcal{C}(\Gamma, q, V, r)\big)_{\mathrm{ad}}$. Recall first the notion of the Drinfeld double of V from Section 4.2. We have $D(V) = W \oplus W^* \in {}^{\Sigma}_{\Sigma}\mathcal{YD}$, where W is the quantum linear space associated to the datum $\big((g_1, \chi_1^{-1}), \ldots, (g_n, \chi_n^{-1}), \varphi_1, \ldots, \varphi_n\big)$, and $\varphi_i : \Sigma \to \mathbf{k}^\times$, $i = 1, \ldots, n$, are defined by $\varphi_i(g, \chi) = \chi_i(g)$, for all $(g, \chi) \in \Sigma$. Define a bicharacter $r_\Sigma : \Sigma \times \Sigma \to \mathbf{k}^\times$ by

$$r_\Sigma\left((g, \chi), (g', \chi')\right) = \chi'(g)\,.$$

The diagonal of this bicharacter is a quadratic form $q_\Sigma : \Sigma \to \mathbf{k}^\times$,

$$q_\Sigma(g, \chi) = \chi(g), \qquad (g, \chi) \in \Sigma.$$

Then $D(V) \in \mathcal{Z}_{sym}(\mathcal{C}(\Sigma, r_\Sigma))_-$.

Theorem 4.8.4. *Let (Γ, q, V, r) be a metric quadruple such that $V \in \mathcal{C}(\Gamma, q)$ is self-dual. There is an equivalence of braided tensor categories:*

$$\mathcal{Z}\big(\mathcal{C}(\Gamma, q, V, r)\big)_{\text{ad}} \cong \mathcal{C}(\Sigma, q_\Sigma, D(V), r_{D(V)}),$$

where $r_{D(V)}$ is the canonical symplectic form on $D(V)$ defined in (4.7).

Proof. Let $H = \mathfrak{B}(V)\#\mathbf{k}[\Gamma]$ and let A be the Hopf subalgebra of $D(H)^*$ generated by the group-like elements $g_i \otimes \chi_i^{-1}$ and by the skew-primitive elements $x_i \otimes \varepsilon$ and $g_i \otimes r'(-, x_i)$, $i = 1, \dots, n$, where r' is an r-form on H whose restriction to $V \otimes V$ is non-degenerate. Using Proposition 4.8.3, we have

$$\mathcal{Z}(\mathsf{Corep}\, H)_{\text{ad}} = \big(\mathsf{Rep}\, D(H)^{\text{cop}}\big)_{\text{ad}} = \mathsf{Rep}\,\big(A^{*\,\text{cop}}\big) = \big(\mathsf{Rep}\, A^*\big)^{\text{op}} \simeq \mathsf{Rep}\, A^* = \mathsf{Corep}\, A.$$

where the equivalence between $\mathsf{Rep}\, A^*$ and its opposite follows from the fact that $\mathsf{Rep}\, A^*$ is braided.

We claim that $A \cong \mathfrak{B}(D(V))\#k[\Sigma]$. Indeed, it is easy to check that for each $(g, \gamma) \in \Sigma$ we have

$$(g \otimes \gamma)(x_i \otimes \varepsilon) = \chi_i^{-1}(g)(x_i \otimes \varepsilon)(g \otimes \gamma),$$

$$(g \otimes \gamma)(g_i \otimes r(-, x_i)) = \chi_i^{-1}(g)(g_i \otimes r(-, x_i))(g \otimes \gamma),$$

$$(g_i \otimes r(-, x_i))(x_j \otimes \varepsilon) = \chi_j^{-1}(g_i)(x_j \otimes \varepsilon)(g_i \otimes r(-, x_i)),$$

for all $i,\ j = 1, \dots, n$. Note that W is self-dual because V is self-dual. Thus, there exists a basis $\{y_i\}_{i=1}^{2n}$ of $D(V)$ such that $y_i,\ y_{n+i} \in D(V)^{\varphi_i}_{(g_i, \chi_i^{-1})}$. It follows from the above, that the map $A \to \mathfrak{B}(D(V))\#k[\Sigma]$, given by

$$g \otimes \gamma \mapsto (g, \gamma), \quad x_i \otimes \varepsilon \mapsto y_i, \quad g_i \otimes r(-, x_i) \mapsto y_{n+i}$$

for all $(g, \gamma) \in \Sigma$ and $i = 1, \dots, n$, is a Hopf algebra isomorphism.

The braiding on $\mathcal{Z}\big(\mathcal{C}(\Gamma, q, V, r)\big)_{\text{ad}}$ is obtained by restriction of the braiding of $\mathcal{Z}\big(\mathcal{C}(\Gamma, q, V, r)\big)$. It corresponds to the braiding on $\mathsf{Corep}\, A$ coming from the restriction to A of the canonical r-form $r_{D(H)^*} : D(H)^* \otimes D(H)^* \to k$. on $D(H)^*$. The latter is given by

$$r_{D(H)^*}(\alpha,\beta) = \sum_{h\in\Gamma, P\subseteq\{1,\dots,n\}} \alpha(\varepsilon\otimes hx_P)\beta((hx_P)^*\otimes 1), \quad \alpha,\beta\in D(H)^*.$$

We have

$$\begin{aligned}
r_{D(H)^*}(g\otimes\gamma, g'\otimes\gamma') &= \sum_{h,P}\varepsilon(g)\gamma(hx_P)(hx_P)^*(g')\gamma'(1) = \gamma(g'),\\
r_{D(H)^*}(x_i\otimes\varepsilon, x_j\otimes\varepsilon) &= \sum_{h,P}\varepsilon(x_i)\varepsilon(hx_P)(hx_P)^*(x_j\varepsilon(1)) = 0,\\
r_{D(H)^*}\big(x_i\otimes\varepsilon, g_j\otimes r(-,x_j)\big) &= \sum_{h,P}\varepsilon(x_i)\varepsilon(hx_P)(hx_P)^*(g_j)r(1,x_j) = 0,\\
r_{D(H)^*}\big(g_j\otimes r(-,x_j), x_i\otimes\varepsilon\big) &= \sum_{h,P}\varepsilon(g_j)r(hx_P,x_j)(hx_P)^*(x_i)\varepsilon(1) = r(x_i,x_j),\\
r_{D(H)^*}\big(g_i\otimes r(-,x_i), g_j\otimes r(-,x_j)\big) &= \sum_{h,P}\varepsilon(g_i)r(hx_P,x_i)(hx_P)^*(g_j)r(1,x_j) = 0.
\end{aligned}$$

Thus, $\mathsf{Corep}(A, r_{D(H)^*}|_{A\otimes A}) \simeq \mathcal{C}(\Sigma, q_\Sigma, D(V), s)$, where the quadratic form $q_\Sigma : \Sigma\to\mathbf{k}^\times$ is given by $q_\Sigma(g,\gamma)=\gamma(g)$, for all $(g,\gamma)\in\Sigma$, and the matrix of $s : D(V)\otimes D(V)\to\mathbf{k}$ with respect to the basis $\{y_i\}$ is the block matrix

$$\begin{pmatrix} 0 & 0 \\ X^t & 0 \end{pmatrix}.$$

Here X^t is the transpose of the matrix $X = \big(s(x_i,x_j)\big)_{i,j}$. Changing s by a cocycle deformation s^σ will not change the braided equivalence class of $\mathcal{C}(\Sigma, q_\Sigma, D(V), s)$. As explained in Section 4.7, we can choose invariant σ such that s^σ is alternating. The matrix of s^σ with respect to the basis $\{y_i\}$ is then

$$\tfrac{1}{2}\begin{pmatrix} 0 & -X \\ X^t & 0 \end{pmatrix}.$$

This matrix is easily seen to be congruent to $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. So, after a change of basis, the matrices of s^σ and $r_{D(V)}$ coincide. Therefore, $\mathsf{Corep}(A, r_{D(H)^*}|_{A\otimes A}) \simeq \mathcal{C}(\Sigma, q_\Sigma, D(V), r_{D(V)})$.

□

Corollary 4.8.5. *$\mathcal{Z}(\mathcal{C}(\Gamma, q, V, r))$ is a $\widehat{\Sigma^{\perp}}$-extension of $\mathcal{C}(\Sigma, q_{\Sigma}, D(V), r_{D(V)})$.*

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