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Secondary Preservice, In-Service, and Student Teachers’ Noticing of Mathematical Work and Thinking in Trigonometry

May Chaar

University of New Hampshire, Durham

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SECONDARY PRESERVICE, IN-SERVICE, AND STUDENT TEACHERS’ NOTICING OF MATHEMATICAL WORK AND THINKING IN TRIGONOMETRY

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DISSERTATION

Submitted to the University of New Hampshire
In Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy

In

Mathematics Education

September, 2015
This dissertation has been examined and approved in partial fulfillment of the requirements for the degree of in Ph.D. in Mathematics Education by:

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On June 16, 2015

Original approval signatures are on file with the University of New Hampshire Graduate School.
DEDICATION

This dissertation is dedicated to each of the mathematics teachers who participated in this study. This dissertation was only made possible because you volunteered your time and your ideas. I will forever be grateful for your generosity and for all I learned from you.

This dissertation is also dedicated to my family, especially my parents and my husband, because while it is likely that you will not read the final product, it is, in large part, a result of your hard work, encouragement, love, and patience.
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Alex's Work on Task Two: “Since 20° and 70° are complementary, we know that sin 20° + sin 70° = sin 90° and so sin 20° ≈ 1 - 0.9397”

Work That Challenged but Did Not Further Their Thinking

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ABSTRACT

SECONDARY PRESERVICE, IN-SERVICE, AND STUDENT TEACHERS’ NOTICING OF MATHEMATICAL WORK AND THINKING IN TRIGONOMETRY

by

May Chaar

University of New Hampshire, September, 2015

Recognizing and responding to students’ work and thinking are central to reform-minded mathematics teaching; in particular, recent educational reforms advocate for instruction that builds on students’ thinking, requiring teachers’ continual assessment of students’ verbal and written strategies. Despite its significance however, little is known about how secondary mathematics teachers analyze and respond to students’ work and thinking. This study aimed to help explain how teachers carry out this work. In particular, it sought to explain what types of knowledge and other resources enable or inhibit teachers’ in-depth analysis of students’ work and thinking while more generally describing the ways in which preservice, in-service, and student teachers attend to, interpret, and respond to students’ work and thinking in trigonometry. These findings serve to inform efforts to improve these skills in future teacher preparation and professional development programs.

This study also provides insight into secondary mathematics teachers’ understandings of various concepts in trigonometry and characterizes how and to what extent teachers’ previously held mathematical conceptions were challenged as they attempted to make sense of solutions, contributing to a budding discussion of how the study of student solutions stimulates teachers’ engagement in mathematics.
CHAPTER I

INTRODUCTION

The classroom environment is complex; teachers are, whether consciously or not, constantly making choices about what to attend to and what to ignore, making sense of those events they do attend to, and deciding how to respond to these events. Unpacking this in-the-moment decision-making is imperative to understanding and improving current teaching practices. Some researchers have begun the unpacking process using the *Mathematics Teacher Noticing* framework (Jacobs, Lamb, & Philipp, 2010; Sherin, Jacobs, & Philipp, 2011). Although mathematics educators have yet to agree on one precise definition of *teacher noticing*, a few ideas are consistently present: (1) teacher noticing is not another category of teacher knowledge, (2) in a broader sense, teacher noticing encompasses the process through which the teacher interacts with the classroom world, managing the “blooming buzzing confusion of sensory data” they are presented with while teaching (Jacobs, Lamb, & Philipp, 2010, p. 35), and (3) teacher noticing involves two interrelated and cyclical processes, (i) attending to classroom events and (ii) interpreting these events. One reason for the lack of one precise definition for noticing is that due to the complexity of classrooms, different researchers have looked at teachers’ noticing of a variety of different aspects of the classroom. By identifying a focus for noticing, researchers are able to attend less to the variety of what is noticed and more on how, as well as the extent to which, teachers notice particular aspects of the classroom (Jacobs, et al., 2010). The particular interest and focus of this dissertation is teachers’ noticing of students’ work and thinking in trigonometry.
Recent educational reforms advocate for instructional styles and actions such as student-centered teaching, adaptive and responsive instruction, diagnostic teaching, formative assessment, and learning from one’s own teaching. Central to these teaching styles and actions is the idea that instruction should build on students’ thinking, requiring teachers’ continual assessment of students’ verbal or written strategies and ideas. The National Council of Teachers of Mathematics (NCTM, 2000) states, “Effective teaching involves observing students, listening carefully to their ideas and explanations, having mathematical goals, and using the information to make instructional decisions” (p. 19). Attending to and interpreting student thinking are central tasks of teaching that provide an invaluable resource for teachers in making informed decisions and improving their instruction (Ball, Lubienski, & Mewborn, 2001; Crespo, 2000; NCTM, 2000). For example, in summarizing the results of several research projects, Chamberlin (2005) documented specific positive effects of attending to students’ mathematical thinking: “the ability on the part of teachers to construct or select worthwhile mathematical tasks including “higher levels of conceptual understandings by students” and “more positive attitudes held by teachers and students toward mathematics” (p. 141-142).

My interest in the way teachers attend to student work and thinking was initiated through my work with the Learning Mathematics Through Teaching (LMTT) research team in an ongoing study (funded by NSF) investigating preservice secondary teachers’ (PSTs’) mathematical knowledge for teaching within the context of a capstone course in mathematics and coordinated practicum experience. As a part of our research, we conducted a series of interviews with more than half of the course enrollees. During
each interview the PSTs were asked to solve a mathematics problem, immediately following were given two or three hypothetical students' work on this problem along with some descriptions of their rationale (created by the LMTT research team), and were then asked to talk about each student’s thinking and understanding. Among other things, these interviews allowed us to begin addressing the question, how do PSTs analyze student work and thinking? My experience conducting these interviews as well as the findings that arose from the data stirred several new questions. The research presented in this dissertation aims to both continue the discussion that the LMTT team started and extend this work in new ways by addressing the following research questions:

0. What content knowledge of radian, unit circle, and the sine and cosine functions do secondary mathematics preservice, in-service, and student teachers possess?

1. How do secondary mathematics preservice, in-service, and student teachers attend to, interpret, and respond to hypothetical students’ written work and thinking in trigonometry?

2. In what ways does the hypothetical students’ work on trigonometry problems challenge and/or further the preservice, in-service, and student teachers’ mathematical thinking?

In the following, I describe my motivation and defend each of these questions as both significant and under-researched in the field of mathematics education.

**Question 1: How do secondary mathematics preservice, in-service, and student teachers attend to, interpret, and respond to hypothetical students’ written work and thinking in trigonometry?**

Although teachers’ noticing of students’ mathematical thinking is appreciated by many as a central task of teaching, research has shown that teachers, at least those in the U.S., do not adequately attend to student work and thinking in their classrooms (Ball, Lubienski, & Mewborn, 2001; Bray, 2011; Santaga, 2005). In addition, when
teachers do attend to thinking there is a tendency to attribute conceptual understanding to work that only exhibits what only exhibits *procedural* understanding (Bartell, Webel, Bowen, & Dyson, 2012). Of additional concern is teachers’ lack of attention to errors and incorrect thinking. Whereas in high-achieving countries such as China and Japan teachers use students’ errors to stimulate further inquiry, teachers in the U.S. tend to avoid errors or respond to them by simply highlighting correct strategies (Bray, 2011; Santaga, 2005; Schleppenbach, Flevares, Sims, & Perry, 2007; Stevenson & Stigler, 1992; Stigler & Hiebert, 1999). Before we can improve teachers’ attention to and use of students’ ideas, we need to better understand the current situation. That is, we need to know how teachers typically attend to, interpret, and respond to students’ work and thinking in order to improve noticing. Yet little research exists documenting how secondary mathematics teachers attend to and respond to students’ strategies (Doerr, 2006).

As noted above, using hypothetical student work in task based interviews, the LMTT research team began highlighting specific ways that preservice teachers attended to student work and thinking as well as the resources they used in doing so. My dissertation expands on this work in several ways. In my research, I investigated both in-service and student teachers (or teaching interns) in addition to preservice teachers. LMTT found that experience working with students, as well as the knowledge of students that is acquired through classroom experience, are potential resources that can be utilized during one’s analysis of student work. For example, one of the LMTT PSTs often referenced common errors that students made during her tutoring sessions; unfortunately however, in general, the PSTs’ experience working with students was
limited. The goal of my work was to give a fuller picture of how resources influence teachers’ analysis. One way I did this was by including in-service teacher participants who have more experience working with students in order to better explore the role of one’s knowledge of students and/or teaching in one’s analysis of student work and thinking. The LMTT team also found that mathematical content knowledge was a significant resource for recognizing and responding to mathematical thinking; however many of our participants had weak mathematical knowledge/fluency, especially in trigonometry. In my own study, I was better able to investigate the participants’ mathematical content knowledge as a resource for noticing mathematical work and thinking because several of my participants, especially three teachers with precalculus teaching experience, had strong precalculus knowledge. In looking at the role of mathematical content knowledge in noticing, I am also expanding on the work of Bartell, Webel, Bowen, and Dyson (2013), who found that elementary teachers needed conceptual understanding in order to accurately recognize whether or not students’ work included evidence of conceptual understanding.

A second way in which I expand on the work of the LMTT research team is by looking not just at teachers’ analyses of student work, but also at how they decide to respond to this work and thinking. Doerr (2006) argues, “the link between the examination of students’ work and the teachers’ actions in the classroom is largely unexamined, particularly at the secondary level” (p.3). That is, research has not sufficiently explored how teachers’ examination of student work informs their instructional decision-making. Even outside of the classroom, preservice teachers’ responses to student errors have received limited attention in the research literature
(Son & Sinclair, 2010). I address this issue by investigating teachers’ instructional responses to hypothetical students’ work in addition to the resources they draw on to respond.

Finally, I am also interpreting the data through the lens of mathematics teacher noticing. In particular, I focus on the participants’ professional noticing of students’ written work and thinking which involves three inter-related skills: (1) attending to the students’ written work and thinking (2) interpreting students’ thinking and ways of knowing, and (3) deciding how to respond. I explain this construct in more detail in the next chapter. I was most interested in how and to what extent the participants interpreted thinking and how they responded to the students and what informed their responses. Thus my first research question is: How do preservice, in-service, and student teachers attend to, interpret and respond to written work and thinking in trigonometry? The following are the corresponding sub-questions:

a. What do teachers do when asked to notice? To what extent is their attention on students’ thinking?

b. What resources do teachers draw upon to interpret students’ thinking based on their written work?

c. What resources do teachers draw upon to respond to students based on their written work?

d. What are the goals of teachers proposed instructional responses, and how are they informed by their interpretations of students’ thinking?

**Question 2: In what ways does the hypothetical students’ work on trigonometry problems presented challenge and/or further the preservice, in-service, and student teachers’ mathematical thinking?**

“The first thing you had to do was solve it yourself, and at first [you] like I didn't really understand what I was doing, um, and or like I just like answered the question but I didn't understand like, what it meant. Um, and then we would start going through and looking at all the answers and
seeing what they had and correcting small mathematical errors or very large conceptual errors and it makes you realize what is happening with the problem more, so you're able to understand it better by seeing all their answers, and you're also able to understand the common misconceptions that go into students doing those problems.”

Catherine, an LMTT participant, shared this during one of her interviews.

Through my experience conducting interviews I began to, like Catherine, see the task of analyzing student work as a means to push and extend the PSTs’ mathematical understanding. In other words, in addition to being a method of gathering useful information about the participants and their analyses of student work, I began to also perceive our interviews as a means through which to engage them in mathematical thinking. For example, some PSTs didn’t know how to begin a problem and the student work acted as a springboard for them. For others, even if they had solved a problem correctly, the hypothetical students’ ideas caused the PSTs to question standard procedure. This was not our goal, nor was it something I expected. Supporting this hypothesis, the LMTT team found an unanticipated significant difference in pre/post improvement on a precalculus exam between those who participated in the interviews and those who did not. Of course, this might be a reflection of the subgroup of students who agreed to be interviewed rather than the effect of the interviews themselves; however, several researchers have found more conclusive results that analyzing student work generates engagement with mathematics (Crespo, 2000; Francisco & Maher, 2011; Herbel-Eisenmann & Phillips, 2005).

I have become increasingly curious about the idea of incorporating analysis of student work and thinking in the preparation of teachers. This curiosity was stirred not only by my perception that the PSTs’ mathematical understanding was extended
through their engagement in these tasks, but also because of positive comments they made regarding these tasks. Based on their responses in interviews, the PSTs rarely perceived the work they did in and for their capstone course, such as engaging in mathematical modeling, as helpful or meaningful towards their future teaching; however they did respond, almost unanimously, that they appreciated and valued analyzing students’ work.

The idea of using student work as a catalyst for teacher change is not original. Little (1999) states, “one of the most powerful and least costly occasions of teacher learning is the systematic, sustained study of student work” (p. 235). This strategy is part of a bigger movement to situate teacher learning in the authentic tasks of teaching (Ball & Cohen, 1999; Little, 1999). Ball & Cohen (1999) explain that, “Centering professional education in practice is not a statement about either a physical locale or some stereotypical professional work. Rather it is a statement about a terrain of action and analysis that is defined first by identifying the central activities of teaching practice and, second, by selecting or creating materials that usefully depict that work and could be selected, represented, or otherwise modified to create opportunities for novice and experienced practitioners to learn” (p. 13). In this study, I use hypothetical student work to explore how it might be used as a means toward improving teachers’ mathematical understanding. Although some evidence existed that attending to student thinking can improve teachers’ understanding of mathematics content knowledge (Crespo, 2000; Francisco & Maher, 2011; Herbel-Eisenmann & Philips, 2005), there was need for further investigation of this “reversed relationship” by looking beyond whether or not growth occurs to looking at how it occurs (Crespo, 2000). I begin addressing this how
piece by investigating what mathematics the teachers engage in and with as a result of analyzing the student work. In addition, I form possible hypotheses as to how various characteristics of the problems, the student work, or teachers’ experiences may have influenced the extent to which the teachers engaged in additional mathematical inquiry. More specifically I address the following research question and corresponding sub-questions: *In what ways does the hypothetical students’ work on trigonometry problems challenge and/or further the preservice, in-service, and student teachers’ mathematical thinking?*

a. What mathematics do teachers engage in as a result of examining the hypothetical students’ work that they hadn’t engaged in by simply working on the problem?

b. Which possible characteristics of the problems, student work, or interview prompts, might be related to the teachers’ mathematical engagement?

**Question 0: What content knowledge of radian, unit circle, and the sine and cosine functions do secondary mathematics preservice, in-service, and student teachers possess?**

In order to (1) assess which resources, especially mathematical knowledge, teachers draw upon in the work of evaluating and noticing students’ work and thinking (addressing my first research question) and to (2) understand how engaging with student work encourages teachers to think about mathematical topics (addressing my second research question), it was important that I have insight into the mathematical knowledge of my participants. Thus, it was important that I (i) focus the teachers’ noticing on a small selection of targeted mathematical concepts and (ii) assess their understanding of these particular concepts. The area of mathematics I targeted is
trigonometry. More specifically, I targeted the concepts of radian, unit circle, and the sine and cosine functions, central ideas within the study of trigonometry.

Thus, as a consequence of necessary data collection toward addressing my main two research questions, I was provided an opportunity to address the following preliminary research question, *What content knowledge of radian, unit circle, and the sine and cosine functions do secondary mathematics preservice, in-service, and student teachers possess?*
Summary of Research Questions

0. What content knowledge of radian, unit circle, and the sine and cosine functions do secondary mathematics preservice, in-service, and student teachers possess?

1. How do secondary mathematics preservice, in-service, and student teachers attend to, interpret, and respond to hypothetical students' work and thinking in trigonometry?
   a. What do teachers do when asked to notice? To what extent is their focus on students' thinking?
   b. What resources do teachers draw upon to interpret students' thinking based on the students' written work?
   c. What resources do teachers draw upon to respond to students based on the students' written work?
   d. What are the goals of teachers' proposed instructional responses, and how are they informed by their interpretations of students' thinking?

2. In what ways does the hypothetical students' work on trigonometry problems challenge and/or further the preservice, in-service, and student teachers' mathematical thinking?
   a. What mathematics do teachers engage in as a result of examining the hypothetical students' work that they hadn't engaged in by simply working on the problem?
   b. Which possible characteristics of the problems, student work, or interview prompts, might be related to the teachers' mathematical engagement?
CHAPTER II

BACKGROUND LITERATURE

Theoretical Framework: A Situative Perspective

Guiding my work, including research questions, data collection, and analysis, is the belief that all learning and knowing is situated. That is, I hold a situative perspective through which I assume learning, knowledge, and beliefs to be inseparable from context and situation in which they are enacted. Thus, in the case of teachers, knowledge is important only to the extent that it is appropriately situated within the context of the work of teaching (Borko, Peressini, Romagnano, Knuth, Yorker, Wooley, Hoevrmill, & Masarik, 2000). Whereas from a cognitive perspective, knowledge is acquired in one setting and transferred to another, from a situative perspective, the consideration of transfer is more complicated (Perisinni, Borko, Romagnano, Knuth, & Willis, 2004). This distinction has implications for my research.

From a situative perspective, researchers would not ask whether or how knowledge transfers across situations, but rather, “ask questions about the consistency of patterns of participation across situations, conditions under which successful participation in an activity in one type of situation facilitates successful participation in other types of situations, and the process of recontextualizing resources and discourses in new situations” (Perissini, Borko, Romagnano, Knuth, & Willis, 2004, p.70). Thus, rather than attempting to correlate certain proxies for teacher knowledge (such as mathematics test scores) with quality of teaching, within the situative tradition one might investigate which resources teachers draw upon in the actual work of teaching. In the case of my own work this means looking at how the resources acquired by preservice
and in-service teachers through various experiences such as college mathematics courses and working with students, are recontextualized in analyzing and responding to student work and thinking. For example, if I gather information about a teacher’s knowledge of a specific procedure, I would be interested in how the teacher uses this knowledge in deciding whether or not the student is correct, how serious the student’s errors are, and how to engage the student to improve his or her understanding.

Teachers often complain that their preparation programs were too removed from the work of teaching (Putnam & Borko, 2000). From, a situative perspective, learning is situated and thus it seems essential to situate teachers’ preparation in the authentic tasks of teaching like analyzing student work and thinking. Putnam and Borko (2000) argue that while some may suggest that a situative perspective implies that all learning experiences for teachers should take place in actual classrooms, “for some purposes, in fact, situating learning experiences for teachers outside of the classroom may be important- indeed essential- for powerful learning” and that the situative perspective simply “focuses researchers' attention on how various settings for teachers' learning give rise to different kinds of knowing” (p.6). For example, in addressing the second research question of this study, I investigate how analyzing hypothetical students’ work outside of an actual classroom may give rise to different kinds of mathematical knowledge and thinking.

Professional Mathematics Teacher Noticing

Defining Noticing

The idea of professional noticing is not a new idea, nor is it unique to mathematics teaching. In 1994, Goodwin termed professional vision to explain
perceptual frameworks developed within particular professions. Similarly, in a general sense, “professional noticing is an ability to recognize and act on key indicators significant to one’s profession” (p. 380, Schack, Fisher, Thomas, Eisenhardt, Tassell, & Yoder, 2013). Mason (2002) discussed intentional noticing, or noticing of a particular profession, and noted that with respect to teaching in particular, “Every act of teaching depends on noticing: noticing what children are doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be or done next” (p. 7).

By applying Goodwin’s (1994) idea of professional vision to mathematics education, Sherin and van Es have carried out significant work, bringing attention to mathematics teacher noticing (Sherin & van Es, 2005, 2009; van Es & Sherin, 2002, 2006, 2008). In general mathematics educators define teacher noticing to include two main processes: (i) attending to particular events in an instructional setting and (ii) making sense of events in an instructional setting (Sherin, Jacobs, & Philipp, 2011). When attending to particular events teachers choose, either consciously or subconsciously, to focus on certain aspects of the classroom (and choose for how long to focus) and also which aspects not to pay attention to. These choices have significant impact on the trajectory of any given class. In contrast to attending, making sense of events involves interpretation, “relating observed events to abstract categories and characterizing what they see in terms of familiar instructional episodes” (Sherin, Jacobs, & Philipp, 2011, p.5). More recently, some researchers (Jacobs, Lamb, & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011; Kazemi et al., 2011) have argued that making sense of classroom events involves not only interpretation but also, deciding
*how to respond*, a skill that “reflects intended responding, not the actual execution of the response” (Jacobs, Lamb, Philipp, & Schappelle, 2011, p.99). The argument for inclusion in the noticing framework is that responding is conceptually linked to both attending and interpreting, and thus, it is valuable to investigate these skills in conjunction with each other.

There are additional variations in how researchers conceptualize and utilize the framework of noticing resulting from differences in: (i) focus, or object, of noticing and (ii) specific content. In this particular study, I focused on the content of trigonometry, namely radian, unit circle, and the sine and cosine functions. With respect to the focus of noticing, I sought to investigate teachers’ interactions with students’ work and thinking. Building on previous more general research on noticing (Sherin & van Es, 2005, 2009; van Es & Sherin, 2002, 2006, 2008), Jacobs, Lamb, and Philipp (2010) introduced the construct of *professional noticing of children’s mathematical thinking* as “a set of three interrelated skills: attending to children’s strategies, interpreting children’s understanding, and deciding how to respond on the basis of children’s understandings” (p.172). By *attending to children’s strategies*, the authors referred to teachers’ attention to the mathematical details in the students’ strategies. I defined attending (as well as interpreting) slightly differently. In the following paragraphs I will explain and describe these choices further.

In my own study, some participants made interpretive claims about students not on the basis of the students’ strategies, but on the students’ result or particular diagrams the students’ drew. Further, in my study, I presented the teachers’ with hypothetical students’ work which didn’t always include extensive detail of the students’
strategies, and thus in order to pay attention to students’ strategies or what the students did, the teachers often needed to pay attention to students’ thinking. Thus, instead of considering specific attention to strategy, I considered attention more generally than Jacobs, Lamb and Philipp (2010). In particular, I considered the participants’ attention to both the mathematical details of the written work as well as their attention to thinking, or any aspects of a student’s strategies or understandings that was not explicitly written on the paper. I was particularly interested in the extent to which the participants’ attention or focus was on students’ thinking.

Jacobs, Lamb, and Philipp (2010) included “interpreting children’s understanding” as the second skill of noticing, but they did not provide a definition of “understanding.” I found it more helpful to include any interpretive claims (i.e. claims that were not immediately obvious based on the students’ work) within the category of interpreting. In particular, I used interpreting to encapsulate both interpreting students’ thinking about the problem as well as interpreting the students’ more general ways of knowing.

As I discussed, while I find their work helpful and informative to my study, I do not adopt the exact construct used by Jacobs, Lamb, and Philipp’s (2010). I consider professional noticing of students’ written mathematical work and thinking to involve: (1) attending to students’ written work and thinking, (2) interpreting students’ thinking and ways of knowing, and (3) deciding to respond. Note that my modified construct involves the same skills as Jacobs, Lamb and Philipp’s (2010) however the three skills are slightly restructured to minimize ambiguity and maximize its utility in making sense of my own data. This construct guided my first research question, data collection and analysis; in particular, I looked specifically at these three skills in an attempt to (i)
describe these skills as exhibited by my participants, (ii) draw connections between
these skills, and (iii) extrapolate which resources teachers draw upon in carrying out
these skills. Jacobs and colleagues investigated teachers’ professional noticing of
children’s mathematical thinking using data from a cross-sectional study, which I
discuss in the following section dedicated to research related to professional noticing of
students’ mathematical thinking.

Research Related to Professional Noticing of Students’ Mathematical Thinking

In this section I provide background as to what research has been done with
respect to teachers’ noticing of students’ thinking, as well as how this research
influences my own study. In reviewing the literature I found four main categories of such
projects: (i) studies examining the relationship between noticing students’ thinking and
quality of teaching, (ii) studies aimed at developing or improving noticing of students’
thinking, (iii) studies aimed at describing and evaluating teachers’ noticing students’
thinking, and (iv) studies aimed at examining the relationship between teacher
knowledge and noticing. My own study falls in the latter two categories, and thus, these
are discussed in more detail. In the following, I briefly address the first two categories
and then elaborate further on the third. The fourth category of research is summarized
in a later section within the section on the relationship between knowledge and noticing.

Studies examining the relationship between noticing students’ thinking and
quality of teaching. Several research projects have found that attending to students’
mathematical thinking has a positive effect in the classroom (Chamberlin, 2005). In
particular, Cognitively Guided Instruction (CGI) research illustrated that in classrooms in
which teachers’ attended to students’ thinking, (i) the teachers were transformed into
learners (ii) students could solve a wider variety of problems and use a wider range of strategies to solve those problems, (iii) there was a shift in emphasis from skills to concepts and problem solving, and (iv) students had higher levels of conceptual understandings (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Franke & Kazemi, 2001). Additional research has also shown that attention to children's thinking can have a positive affect on student learning (Kersting, Givvin, Sotelo, & Stigler, 2010).

Sherin, Jacobs, & Philipp (2011) highlighted three additional teaching and research practices advocated for by the mathematics education community that indicate the importance of professional noticing: (i) adaptive and responsive teaching, (ii) learning from teaching, and (iii) decomposing practice. In addition, research and policy emphasizing formative assessment in mathematics teaching also highlights the relevance of noticing. While noticing is recognized as an important teaching skill, it proves to be difficult for teachers (Bray, 2011; Santaga, 2005; Schleppenbach, Flevares, Sims, & Perry, 2007; Stevenson & Stigler, 1992; Stigler & Hiebert, 1999). Thus, many mathematics educators have attempted to improve preservice and inservice teachers’ noticing through professional development, methods courses, and research interventions. Several of these attempts are discussed in the following section.

**Studies aimed at developing or improving noticing of students’ thinking.** In reviewing the literature related to noticing, I found that most studies were designed to improve teachers’ noticing. Further, most of these studies aimed to do so by increasing participants’ engagement with student thinking. Although my study does not aim to develop teachers’ noticing, but rather describe their current noticing to inform future
development, I find it necessary to include some background on this form of research due to its prevalence, in order to further situate my own work within the overall work done with respect to professional noticing. In addition, while it is not the focus of my research, this literature suggests that the participants in my own study may become better at noticing through their prompted noticing in the interview settings.

Current research on developing teachers’ noticing of student thinking advocates most strongly for interaction with classroom artifacts. More specifically, it is well established that professional development and teacher preparation programs should involve investigating students’ work (Ball, 1997; Chamberlin, 2005; Crespo, 2000; Driscoll & Moyer, 2001). Ball argued that having teachers discuss artifacts of teaching and learning, including student work, held “promise for equipping teachers with the intellectual resources likely to be helpful in navigating the uncertainties of interpreting student thinking” (Ball, 1997, p.808). In the following, I discuss three particular studies that provide evidence of the benefit of using students’ work. I take care to acknowledge the precise ways in which each project utilized artifacts as well as other factors that may have contributed to the development of teachers’ noticing.

In the Linked Learning in Mathematics Project (LLMP), middle school teachers gained a deeper understanding of students’ algebraic thinking after examining students’ work (Driscoll & Moyer, 2001). Their improvement in making sense of students’ algebraic thinking cannot simply be attributed to the students’ work; two other significant aspects of the program included: (i) discussion of the students’ work in small groups, and (ii) guidelines for analyzing algebraic thinking which were given to the teachers, helping them to focus on particular algebraic habits of mind (Driscoll & Moyer, 2001).
Within the research and development project, Cognitively Guided Instruction (CGI), Franke and Kazemi (2001) also observed benefits using focused teacher group discussions centered around students’ work. In particular, they placed twelve K-5 teachers in four work groups, and each month the teachers would pose a mathematics problem to their own students, and then bring their students’ work to a work group discussion meeting facilitated by the researchers. Crespo (2000) also used authentic student work in her attempt to improve preservice elementary teachers’ ability to make sense of students thinking via a mathematical letter exchange program. In this program, the preservice teachers exchanged letters with fourth grade students who attempted to make sense of a particular mathematics problem. Crespo (2000) noted changes in focus from correctness (evaluation) to meaning and changes in responses from quick and conclusive to thoughtful and tentative.

Other programs designed to develop teachers’ noticing utilized video as opposed to students’ written work. For example, McDuffie and colleagues attempted to support prospective K-8 teachers’ noticing by engaging them in video analysis of excerpts of mathematics lessons (Roth McDuffie, Foote, Bolson, Turner, Aguirre, Bartell, Drake, & Land, 2014). The mathematics teacher educators structured the video analyses by providing background reading and focusing the teachers’ discussion on four aspects of a mathematics lesson: teaching, learning, task, as well as power and participation. They observed increased depth of noticing and a move from attending primarily to what they saw, or teacher moves, especially teachers’ interactions with students, to interpreting the importance and effects of the moves or teacher-student interactions (Roth McDuffie et al., 2013). In another recent example, Schack and colleagues reported successfully
developing the professional noticing abilities of preservice elementary teachers by having them engage with video excerpts of interviews with children doing mathematics (Schack et al., 2013). In addition to observing video, the preservice elementary teachers also engaged in group discussions and were provided instruction regarding professional noticing.

There were significantly more studies aimed at improving teachers’ noticing than there were studies aimed at understanding how teachers’ make sense of students’ work and thinking, and how they decide how to respond to students. I argue that programs designed to improve teachers’ noticing might be better informed by having a better understanding, or picture, of how teachers typically attend to, interpret, and respond to students’ work and thinking and then building on their pre-existing noticing. Studies that attempt to contribute to such a picture are discussed in the following section (Shack et al., 2013).

**Studies aimed at describing teachers’ noticing of students’ thinking.** Since their *construct of professional noticing of children’s mathematical thinking* as well as their research methodologies and findings were influential on my own study, I begin this section by discussing, in detail, the work of Jacobs and colleagues (Jacobs, Lamb, & Philipp, 2010; Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011).

Jacobs and colleagues investigated teachers’ noticing of children’s thinking using data from a cross-sectional study, “Studying Teachers’ Evolving Perspectives” (STEP) of 131 preservice and in-service K-3 teachers (Jacobs, Lamb, & Philipp, 2010; Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011). Jacobs, Lamb & Philipp (2010) used two assessments to capture participants’ noticing. In one they were asked
to watch a video clip of a classroom, and in the other they were given a set of written
student work. After observing either the video or student work, the participants were
asked to write responses to various prompts, namely: (i) Please describe in detail what
you think each child did in response to the problem; (ii) Please explain what you learned
about these children’s understandings; and (iii) Pretend that you are the teacher of
these children. What problem or problems might you pose next? (Jacobs, Lamb, &
Philipp, 2010). These three prompts were designed to correspond with each of the three
skills of noticing the authors had outlined – attending, interpreting, and responding –
respectively. Jacobs and colleagues also report using a video of a student named Rex
solving subtraction and addition word problems (Jacobs & Philipp, 2010; Jacobs, Lamb,
Philipp, & Schappelle, 2011). Similarly, participants were asked to respond in writing to
two prompts: (i) please describe in detail what Rex said and did in response to this
tadpole problem; and (ii) describe some ways you might respond to Rex, and explain
why you chose those responses (Jacobs, Lamb, Philipp, & Schappelle, 2011). These
prompts only corresponded to two skills – attending and responding- respectively. As
will be explained in Chapter III, my own interview prompts were adapted from the
prompts in both studies. One important choice I made was to ask participants to
describe how they would respond to students (as in the second study) instead of more
specifically asking them to explain what problems they would pose (as in the first study).
I believed the latter would influence and restrict participants’ responses, since some
teachers might not decide to pose a problem, but instead would ask the student a
question, share information with the student, etc. I also believed that asking participants
to first “describe what each student did” could potentially restrict and influence what and
how they noticed, and thus, my first prompt to the participants was simply, “Please take
a look at the work and tell me what you notice.” This prompt allowed me to focus on
what the participants would typically do when looking at students’ work and allowed me
to focus on what they attended to on their own, especially the extent to which they
attended to thinking.

In both studies (Jacobs, Lamb, & Philipp, 2010; Jacobs, Lamb, Philipp, &
Schappelle, 2011), Jacobs and colleagues assigned scores to the responses in each of
the categories. Responses to attending prompts were identified as containing evidence
(1) or lack of evidence (0) of the participants’ attention to the student’s strategies
(Jacobs, Lamb, & Philipp, 2010, p.179). Responses to interpreting prompts were
identified as containing robust evidence (2), limited evidence (1), or lack of evidence (0)
of the participants interpreting the children’s’ understandings on the basis of the student
work. Finally, responses to deciding-how-to-respond prompts were also identified as
containing robust evidence (2), limited evidence (1), or lack of evidence (0) of the
participants deciding how to respond on the basis of the children’s understandings.
Overall means and the means of subgroups (defined by the participants level of
experience teaching and involvement in professional development), and in both studies
(Jacobs, Lamb, & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011), they found
that prospective teachers had little engagement with the children’s thinking in all three
skills, but that “emerging teacher leaders,” or individuals with the most experience with
children’s thinking, showed significantly more engagement. This finding is consistent
with additional, less well-substantiated claims that the ability to notice characterizes
expert teachers (Ainley & Luntley, 2007; Dreher & Kuntze, 2014; Mason, 2002).
Jacobs and colleagues looked at a large set of data, assigned scores to the data and analyzed quantitatively, allowing them to come to conclusions about differences between subgroups of participants. In contrast, I aimed to capture a more nuanced picture of how teachers (with varying levels of experience) attend to, interpret and respond, to students’ work and thinking. Thus, although I adopt a similar construct to these authors as well as similar data collection techniques and prompts, I have collected data from a smaller number of participants, making use of interviews and qualitative analysis to provide more in-depth descriptions of how teachers engage and react to student work in trigonometry, including what they do when they notice and what resources enable or inhibit their noticing. Also, in contrast to Jacobs and colleagues, I looked at secondary, rather than elementary, preservice, in-service, and student teachers’ noticing.

With respect to one particular skill, deciding to respond, Jacobs and colleagues did begin to provide a more nuanced picture of their participants beyond simply assigning and comparing scores (Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011). In particular, they articulated that when teachers did not decide respond to students on the basis of the students’ thinking, their responses tended to fall into three other categories (Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011). First, some responses were general responses with very little specifics as to the reasoning or underlying steps; this included responses such like, “I would ask questions along the way as to guide students in the process of how to start and where to go next” (Jacobs, Lamb, Philipp, & Schappelle, 2011, p.107). Second, some responses focused on the teacher’s own thinking rather than the thinking of the
Participant responses in this category are specific to the problem but not the student, corresponding instead to the way the participant might solve or think about the problem. For example, a participant might decide to explain or lead a student to solve the problem using a strategy that is different from the one the student had attempted. The third category classified responses that focused on the student’s affect (Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011). These responses were geared toward raising the students’ confidence. I investigated three of these four categories of reasoning (student’s mathematical thinking, the student’s affect, and general teaching moves) to decide to respond in analyzing my own data.

As I mentioned earlier, one distinguishing factor between the work of Jacobs and colleagues and this study is that the focus was on elementary teachers rather than secondary mathematics teachers. In fact, the majority of studies investigating teachers’ noticing of students’ thinking, regardless of whether they used the teacher noticing terminology or not, have been conducted with elementary school teachers with a focus on content such as number sense and rational numbers; whereas similar research at the secondary level is considerably more limited (Doerr, 2006; Haltiwanger & Simpson, 2014). Further, the research studies conducted on teacher noticing have addressed a narrow range of mathematical topics: place value, division, natural numbers, fractions, and geometry (almost exclusively area and perimeter), mostly all of which are elementary level content areas (Mewborn, 2001; Son & Sinclaire, 2010). I address these gaps not only by investigating at the secondary level, but also by looking at
noticing of student work in trigonometry, an area that has received little attention in mathematics education.

I did identify one study, conducted recently, in which the researchers did examine noticing of students’ thinking at the secondary level using the framework laid out by Jacobs and colleagues. Using this construct, and borrowing from their methodology, Haltiwanger and Simpson (2014) investigated how secondary mathematics preservice teachers attended to, interpreted, and responded to high school students’ mathematical thinking about extending a pattern. They asked 30 preservice teachers to respond to three samples of student work and respond to three prompts, the precise prompts used by Jacobs, Lamb, & Philipp (2010), including the prompt, Pretend that you are the teacher of this student. What problem or problems might you pose next and why?, For analysis Haltiwanger and Simpson (2014) used the scale outlined by Jacobs and colleagues, and they also investigated the participants’ responses based on the four categories previously mentioned: general move, student’s affect, teacher’s mathematical thinking, or student’s mathematical thinking. Haltiwanger and Simpson (2014) reported several findings (i) senior level participants attended to students’ thinking more so than their sophomore or junior peers, (ii) participants had an overall tendency to focus on what students did understand rather than what they did not, (iii) no participants responded in a way that would support students’ affect (i.e. support their confidence, emotional well-being, etc.), (iv) 20% of participants provided responses that focused on their own mathematical thinking, (v) most provided a general response, missing specificity, and (vi) only 20% of responses focused on students’ thinking.
Although they did not use the noticing framework, Son and Crespo (2009) investigated preservice elementary and secondary teachers reasoning and instructional responses to a student’s non-traditional strategy for dividing fractions. Among their findings, Son and Crespo concluded that those who unpacked the mathematics of the student’s strategy tended to use teacher-focused responses (i.e. telling, explaining, or showing), whereas those who put less work into their analyses, tended to use student-focused responses, such as providing opportunities for students to explain/justify strategies or engage in sense making. Similarly, Son and Sinclair (2010) revealed two different forms of address: (i) “show” and “tell” and (ii) “give” or “ask.” Responses in the “show” and “tell” category involved showing, telling, or talking about properties of reflection. Responses in the “give” or “ask” category involved having the student do something with a manipulative, image, or question, requiring the student to deliver verbal or nonverbal information. Son and Sinclair (2010) found that 32 out of 43 responses fell under the category “show” and “tell”, evidence that preservice teachers prefer to deliver information.

During my own analysis, I attempted to make sense of and situate the categories of responses outlined by (i) Jacobs and colleagues (Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011) (ii) Son and Crespo (2009), and (iii) Son and Sinclair’ (2010). While I found teacher-focused and student-focused to be too similar to the categories of Jacobs and colleagues, coding data using Son and Sinclair’s (2010) distinction between “show” and “tell” and “give” or “ask” ended up being a helpful addition to Jacobs’ and colleagues categories; in particular it helped me make sense of and distinguish between the participants’ responses within the four main categories.
I find it important to mention one final study, which I believe highlights the focus of the following section. In their study, Asquith, Stephens, Knuth, and Alibali (2007) asked middle school teachers to make predictions of students’ responses to written assessment items focused on the equal sign and variable. The teachers’ predictions of students’ understanding of variable was well aligned with students’ actual responses, whereas the teachers’ predictions of students’ understanding of the equal sign was not well aligned with actual responses. These results are significant in that they suggest the contextual nature of noticing students’ mathematical thinking. This made it necessary to focus on particular concepts, discussed in the trigonometry section. My own results confirm and explore the contextual nature of noticing by describing how content knowledge enabled and/or inhibited the participants’ noticing.

**The Relationship between Teachers’ Knowledge and Noticing**

Many research studies and policy documents have acknowledged a connection between teachers’ knowledge and their teaching practices (Ernest, 1989; Escudero & Sanchez, 2007; Fennema and Franke, 1992; National Council of Teachers of Mathematics, 2000; Shulman, 1986). In my own work, I am interested in the bidirectional relationship between teachers’ knowledge and the particular task of noticing students’ work and thinking. In doing so, I rely on two frameworks to establish this connection and categorize teacher knowledge that may inform and be informed by noticing: (1) Mathematical Knowledge for Teaching (MKT) and (2) the Mathematics Teaching Cycle. In this section, I summarize these two frameworks and describe how they inform my work. I then summarize literature and research that establishes or
hypothesizes connections between teacher knowledge and mathematical noticing of students’ work and thinking in particular.

**Mathematical Knowledge for Teaching (MKT): Resources for Noticing**

As with most discussions of teacher knowledge, I begin with a brief review of Shulman’s groundbreaking work in categorizing teacher-specific knowledge. In 1986, Shulman suggested three categories of content-specific knowledge to address what he argued as “a blind spot with respect to content that characterizes most research on teaching, and as a consequence, most of our state-level programs of teacher evaluation and teacher certification (pp. 7-8). The three categories were (i) subject matter content knowledge, (ii) pedagogical content knowledge, and (iii) curricular knowledge (Shulman, 1986). Within curricular knowledge, Shulman (1986) proposed that teachers should have knowledge not only of his or her own curricula, but also of alternative curricula, and knowledge of the curriculum materials that one’s students are learning under in other subjects. Subject matter knowledge is rather self-explanatory; however pedagogical content knowledge “goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching” (p.9, Shulman, 1986). Shulman (1986) suggested that pedagogical knowledge included knowledge of powerful representations of content for teaching, knowledge of which content-specific topics are difficult for students to learn and why, and knowledge of instructional strategies that help students overcome commonly held misconceptions. Through the work of Ball and colleagues (Ball & Bass, 2003, 2009; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008) these domains of knowledge have been
mathematized and built upon (as well as slightly reorganized) as a practice-based model of Mathematical Knowledge for Teaching (MKT).

Ball and colleagues conducted extensive qualitative analysis of elementary school teaching practice to investigate (i) the mathematical work of teaching and (ii) the mathematical knowledge, skills, and sensibilities entailed by this work (Ball & Bass, 2003, 2009; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). Ball and colleagues found that teachers need “to know more, and different, mathematics—not less” (Ball, Thames, & Phelps, 2008, p. 397). The result of this analysis was a continually developing theory of the Mathematical Knowledge for Teaching (MKT). As a theory, MKT not only hypothesizes what knowledge is necessary, but also proposes an organization and classification of the different types of knowledge including an initial division of MKT into two main subgroups: Subject Matter Knowledge and Pedagogical Content Knowledge (PCK). Both of these are further distinguished into three subgroups. A summary of the division is provided in Figure 1.1 (Ball & Bass, 2009). I elaborate on these distinctions and their implications within my research in the remainder of this section.
Figure 1.1. Model of Teacher's Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008)

**Subject Matter Knowledge.** Subject Matter Knowledge consists of three subgroups: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Knowledge at the mathematical horizon. Common Content Knowledge (CCK) refers to mathematical “knowledge and skill used in settings other than teaching” (Ball, Thames, & Phelps, 2008, p.400). For example, a teacher’s ability to solve certain content-specific problems, knowledge of meanings of mathematical terms and definitions, or knowledge of correct notation all could impact his or her teaching and subsequent student learning; however, they are not uniquely necessary to teaching. In contrast, Specialized Content Knowledge (SCK) is “the mathematical knowledge and
skill unique to teaching” (Ball, Thames, & Phelps, 2008, p.401). This is not to say that a mathematician who does not teach would not hold this knowledge; rather, it means that this knowledge is not typically needed to be a successful mathematician, nor is it typically needed in every day life, yet is typically needed in the work of teaching. Ball, Thames, and Phelps (2008) provide the example, “Accountants have to calculate and reconcile numbers and engineers have to mathematically model properties of materials, but neither group need to explain why, when you multiply by 10, you ‘add a zero’” (p. 402). The third category of MKT, knowledge at the mathematical horizon, is defined as “awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball, Thames, & Phelps, 2008, p.404). Ball and Bass (2009) further elaborate on four elements of this horizon knowledge: (i) a sense of the mathematical environment surrounding the current “location” in instruction, (ii) major disciplinary ideas and structures, (iii) key mathematical practices, and (iv) core mathematical values and sensibilities.

All three of these types of knowledge serve as potential resources when analyzing student work and thinking. For example, identifying an error involves CCK whereas “sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK)” (Ball, Thames, & Phelps, 2008, p.401). Interpreting students’ understandings one must also have “sufficient understanding of the mathematical landscape to connect how those strategies reflect understanding of mathematical concepts”, or knowledge at the mathematical horizon (Jacobs, Lamb, & Philipp, 2010, p. 195). While these
categories within subject matter knowledge informed my interpretation of the data, I experienced difficulty making clear distinctions between the three within my interview data, and thus I currently do not make theses distinctions explicit in my analysis or reporting of my findings. Instead, I refer to instances of subject matter knowledge as simply, “mathematical knowledge.” In contrast, I did code data (and will report results) with respect to the three different categories of PCK, which I discuss in the remainder of this section.

Pedagogical Content Knowledge (PCK). Pedagogical Content Knowledge (PCK) consists of: (i) Knowledge of Content and Curriculum, (ii) Knowledge of Content and Teaching (KCT), and (iii) Knowledge of Content and Students (KCS). Knowledge of Content and Curriculum is included in “the egg” in an attempt to show the correspondence between Shulman’s (1986) categories and their own (Ball, Thames, & Phelps), and thus in analyzing my data and reporting results, I use Shulman’s definition of curricular knowledge, as was discussed previously. In this definition, “curriculum is represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances” (Shulman, 1986, p.10).

Knowledge of Content and Teaching (KCT) combines knowledge about teaching and one’s knowledge about mathematics. This sort of knowledge includes knowing the advantages and disadvantages to sequencing instruction of specific content in a given way or the advantages and disadvantages of using certain representations of a specific
idea. One can easily envision that KCT would manifest itself in one’s ability to respond (within the framework of noticing), and I was curious how it manifested itself. I found that participants’ drew on KCT, not only when deciding to respond to students but also to interpret the students’ work thinking. In particular, in some cases, participants explained how certain teaching strategies were likely responsible for students’ errors. This finding is elaborated on in the results. During analysis, two useful distinctions within KCT also emerged from my data: knowledge of content and common teaching and knowledge of content and ideal teaching.

Knowledge of Content and Students (KCS) refers to “content knowledge intertwined with knowledge of how students think about, know, or learn this particular content” (Hill, Ball, & Schilling, 2008, p.375) including knowledge about students’ mathematical thinking, errors, and misconceptions (Hill, Ball, & Schilling, 2008). This category of PCK is arguably the most directly related to one’s professional noticing of students’ thinking. For example one’s knowledge of common errors or misconceptions might influence what he or she attends to and how he or she interprets what is noticed. Similarly, one’s instructional response might be informed by knowledge of common errors and misconceptions. Ball, Thames, and Phelps (2008) provide the following example: “In teaching students to add fractions, a teacher might be aware that students, who often have difficulty with the multiplicative nature of fractions, are likely to add the numerators and denominators of two fractions. Such knowledge might help her design instruction to address this likely issue” (p.125). Similarly, Hiebert, Morris, Berk, and Jansen (2007) argued that moving from analysis of student work to instructional practice
“requires a set of competencies or skills that draw directly on subject matter knowledge, combined with knowledge of student thinking” (p.52).

**The Mathematics Teaching Cycle**

Through the analysis of three teaching episodes, Simon (1995) developed a schematic model, the Mathematics Teaching Cycle, to help explain the “cyclical interrelationship of aspects of teacher knowledge, thinking, decision making, and activity” (p.135). In this model the teacher’s knowledge of mathematics, knowledge of mathematical activities and representations, hypothesis of student’s knowledge, theories about mathematics learning and teaching, and knowledge of student learning of particular content, guides the teacher’s hypothetical learning trajectory (HLT), or the teacher’s prediction as to the path of student learning that will occur. For my own purposes, I am not concerned with the model as a whole, but rather a subgroup of the relationships that exist within the model. In the remainder of this section, I will discuss these relationships within the model and within the context of my own research.

With regards to professional noticing, Simon’s framework elaborates that once actual classroom interaction and activities have taken place, the teacher’s assessment of students’ knowledge is filtered through his or her own teacher knowledge. This assessment in turn affects the teacher’s own knowledge (Simon, 1995). In particular with respect to mathematical knowledge, Simon (1995) notes, “As the teacher my perception of students’ mathematical understandings is structured by my understandings of the mathematics in question. Conversely, what I observe in the students’ mathematical thinking affects my understandings of the mathematical ideas involved and their interconnection” (p. 135). In my own work, in addition to categorizing
participants’ noticing, I also investigated this bidirectional relationship between teacher knowledge and their noticing. In particular, I sought to understand which aspects of MKT (including PCK) informed teachers’ noticing, and conversely, how their experience noticing affected their mathematical knowledge. The affects of noticing on one’s PCK lie outside the scope of this study.

Thus far, I have introduced the Mathematics Teaching Cycle, and used it to conceptualize the relationship between MKT and attending and interpreting. The question remains, how does Simon’s (1995) model relate one’s MKT and the act of responding? In Simon’s model (1995), teachers respond to students in particular ways based on hypothetical learning trajectories (HLTs), which they develop for their students. These trajectories consist of three components: (i) learning goal, (ii) learning activities, and (iii) a prediction of how the students’ thinking will evolve (Simon, 1995; Simon & Tzur, 2004). In my own study, the prompt I gave participants only explicitly asked for their hypothetical response; thus, I only attempt to access the participants “learning activities.” The idea is that the learning goal and prediction of the students’ thinking is implicit, guiding the individual’s choice of learning activities; however as I will discuss in the results section, several participants made their learning goals and predictions explicit.

In Simon’s (1995) model, both teacher knowledge and one’s assessment of student knowledge inform the HLT which results in one’s instructional decisions. In analyzing my own data, I investigated the MKT that informed participants’ responses as well as to what extent their responses were based in their assessments, and in particular, their interpretations of students’ thinking. Simon (1995) also noted that,
“further generation of hypotheses of student conceptual development depends on the nature of anticipated activities” (p. 136). This is one way in which I evaluated my participants’ hypothetical responses: did their responses aim to (or allow for) further access to students’ thinking and understanding?

Figure 1.2 is a summary of the Mathematics Teaching Cycle. As I mentioned earlier, it is important to note that Simon (1995) acknowledges that teacher knowledge informs assessment of student knowledge directly; however this model does not include these arrows in order to “simplify the diagram” (p.137).

Figure 1.2. The Mathematics Teaching Cycle (Simon, 1995)
Research and Literature Regarding the Relationship between Teacher Knowledge and Professional Noticing

Many researchers have acknowledged the relationship between teachers’ knowledge and teachers’ noticing in particular (Dreher & Kuntze, 2014; Schifter, 2001, 2011; van Es, 2011). Dating back to the early 1900s, Bartlett (1932, Cited in Jacobs, Lamb, & Philipp, 2010) showed that one’s knowledge influences how he or she observes the world and events in it. Ball (1993) explained that mathematics teachers in particular must have a bifocal perspective, “perceiving mathematics through the mind of the learner while perceiving the mind of the learner through the mathematics (p.159).” Similarly, Schoenfeld (2011) argued, “Noticing is essential, but it does not suffice by itself. It takes place within the context of teachers’ knowledge and orientations; and the decisions that teachers make regarding whether and how to follow up on what they notice are shaped by the teachers’ knowledge (more broadly resources) and orientations (p.233). Steffe (1990) also contended, “Using their own mathematical knowledge, mathematics teachers must interpret the language and actions of their students and then make decisions about possible mathematical knowledge their students might learn” (p.395).

Haltiwanger and Simpson (2014) found that of their 30 preservice teacher participants, seniors and MAT participants attended to students’ thinking more often than sophomores and juniors. In fact they found a statistically significant difference, and hypothesized that this difference could be attributed to their mathematical coursework. Dreher and Kuntze (2014) proposed, “So teacher’s noticing on the one hand and their knowledge and views on the other hand are somehow connected, but how is this
relationship constituted?” (p.94). They found a weak to medium relationship between noticing and any given particular aspect of professional knowledge; however, they suggest that the relationship between professional knowledge and noticing is more complicated (Dreher & Kuntze, 2014). In particular, they noted that, “drawing on a variety of different components of professional knowledge and views can result in successful theme-specific noticing” (Dreher & Kuntze, 2014, p. 106). They also found that (i) lack of knowledge, (ii) inappropriate views, and (iii) selective use of knowledge were factors that hindered one’s ability to notice (Dreher & Kuntze, 2014). Dreher and Kuntze (2014) use their research to make an argument for more research on the ways in which teachers’ noticing and their professional knowledge are related, especially in other domains of mathematics. This dissertation attempts to address this call.

Several studies also investigated the “reversed relationship,” the effects of attending to students’ work and thinking on teachers’ mathematical knowledge, thinking, and engagement. Herbel-Eisenmann and Phillips (2005) found that through the analysis of eighth grade students’ work on problems in algebra, teachers became more aware of the mathematics embedded in the problems and deepened their algebraic understandings. More specifically, the authors conjectured that the teachers’ understandings were deepened because this sort of analysis encouraged them to think about algebra more broadly as well as make new connections (Herbel-Eisenmann & Phillips, 2005). Francisco & Maher (2011) observed an instance in which teachers gained new mathematical knowledge in response to one student’s understanding of fractions. In a study of PSTs’ correspondence with students via letters, Crespo (2000)
observed instances where teachers’ analyses of student work generated additional study of the mathematics at hand.

Although evidence exists that attending to student thinking can improve teachers’ understanding of some mathematics, Crespo (2000) argues that there is need for further investigation of this “reversed relationship” by looking beyond whether or not growth occurs to how it occurs. Similarly, Kazemi and Franke (2004) argue that it is necessary to investigate the best ways to center teacher inquiry around student work. Accordingly, my study was not a pre-/post- study in which I investigated their content knowledge or understanding before and after providing opportunities to investigate solutions; instead, my study examined how different aspects of the mathematical solutions and prompts encouraged or challenged the teachers’ mathematical thinking in the moment.

**Trigonometry**

Trigonometry has traditionally been a significant focus in the secondary precalculus curriculum and is one of the first topics students encounter that link algebraic, geometric, and graphical reasoning (Weber, 2005). In addition, an understanding of trigonometry is a pre-requisite for understanding topics in physics, engineering, applied mathematics, and periodic phenomenon (Moore, 2013; Weber, 2005). Despite the topic’s significance however, research on students’ and secondary teachers’ understanding of trigonometry is sparse (Akkoc, 2008; Moore, 2009; Topcu, Kertil, Akkoc, Yilmaz, & Onder, 2006; Weber, 2005). Findings that have been reported suggest that both students’ and teachers’ understanding of trigonometry is often shallow.
For this study, it was important to narrow my focus on trigonometry to a few topics so that I could examine closely the reciprocal relationship between mathematical knowledge and noticing of mathematical thinking. Two main criteria were important in choosing the more focused trigonometry topics: (1) the topics needed to be central ideas in that they exist throughout most trigonometry curricula and are prerequisites for other trigonometric topics, and (2) enough research needed to have been done on students’ understanding of these topics so that the hypothetical student work could be designed and informed by the findings of this research. The three topics that I felt best fit these criteria were: radian, unit circle, and trigonometric functions. In addition, all three of these concepts are highly emphasized in the Common Core State Standards for Mathematics (CCSSI, 2010). In the remainder of this sub-section, I focus on findings with respect to students’ and teachers’ understanding of these three particular areas of trigonometry, and then I discuss how this prior research informed my methodology.

**Research on Students’ and Teachers’ Knowledge of Radian**

Several researchers have observed that students and teachers are often more comfortable with degree angle measurements than radian angle measurements. Fi (2003) found that although most preservice teachers could convert between radian and degree measurements, none of his fourteen participants offered a correct definition of radian. Akkoc (2008), investigating 42 PSTs’ concept images of radian, found similar results. In addition to noting that concept images of degree dominated the concept images of radian, he also reported that many participants claimed that radian measures
are only given in terms of $\pi$ (Akkoc, 2008). These insufficient understandings of radian
are problematic as angle measure serves as a common foundation between various
trigonometric contexts (Moore, 2009).

**Research on Students’ and Teachers’ Knowledge of the Unit Circle**

Less research seems to have been conducted with regards to students and
teachers’ understanding of the unit circle. Fi (2003) found that although most of them
could identify unit circle as a circle with radius of one unit, the preservice teachers in his
study were less able to discuss the functionality of the unit circle. In particular, they did
not discuss the importance of the unit circle with respect to evaluating trigonometric
functions at angles greater than 90°, considering clockwise rotations, the periodicity of
trigonometric functions, considering coterminal angles, or proving trigonometric
identities and formulas. In addition, Moore, LaForest, and Kim (2012) found that prior to
their intervention, two preservice secondary mathematics teachers described the unit
circle as having a radius of one, but “one” did not represent “one radius length.” One of
the participants even found it necessary to attach units (i.e. inches) when discussing or
drawing the unit circle (Moore, LaForest, & Kim, 2012). These preservice teachers had
difficulty using the unit circle in novel contexts such as determining arc length given a
radius and angle measure as well as determining angle measure when given the radius
and subtended arc length. (Moore, LaForest, & Kim, 2012).

**Research on Students’ and Teachers’ Knowledge of Trigonometric Functions**

Several researchers have found that students and teachers have weak
understandings of trigonometric functions (Brown, 2006; Fi, 2003; Weber, 2005). In
particular, Brown (2006) found that students had an incomplete understanding of the
three major ways of knowing sine and cosine: as coordinates of a point on the unit circle, as horizontal and vertical distances on the unit circle, and as ratios of sides of a reference triangle. Weber (2005) found that many undergraduate students could not approximate output values of trigonometric functions when given various input values, nor could they discuss various properties of trigonometric functions. He argued that their inability to do so was evidence of their lack of understanding of the sine function. Fi (2003) similarly found that many preservice teachers showed weakness in their knowledge of sinusoids and in inverse trigonometric functions.

How this Research on Trigonometry Informed my Methodology

As I’ve reported, research indicates that students and teachers have significant difficulty with trigonometry. I have purposefully chosen to focus my research on challenging concepts in order to address my second research question, investigating how and in what ways the mathematical work challenges or furthers the teachers’ thinking in this area. As I will discuss later, I have designed hypothetical student work that reflects the misconceptions highlighted in the research. I hoped that via noticing of this work, the participants would benefit from this experience by reflecting on their own conceptions.

While the difficult nature of trigonometry makes it an ideal choice for addressing the second research question, it poses challenges in addressing the first research question. In particular, based on prior research, I expected that if I recruited in-service, preservice, and student teachers without purpose, most participants will have weak understandings of trigonometric concepts, which would in turn make it difficult for them to say much about the students’ work and thinking. Thus, I incorporated a first phase of
research during which I conducted and analyzed a trigonometry assessment (informed by the research discussed here). The results helped me identify individuals with high levels of trigonometric fluency. I also included participants who did not perform as well on the assessment, but by including participants with greater knowledge of trigonometry, I was able to report a more complete picture of how mathematical knowledge is drawn on in professional noticing of students' thinking.

**Concept Image and Concept Definition**

Earlier I provided a brief summary of CCK and SCK; however these terms are defined not directly but rather by the tasks and activities through which they are required or called upon. Thus, a more detailed framework is necessary in order to discuss participants’ mathematical knowledge of trigonometry. To provide this detail, I rely on the constructs of *concept image* and *concept definition*. In this section, I provide a summary of these constructs which although older, “have weathered the years well” (Bingolbalu & Monaghan, 2008, p.19).

Tall and Vinner (1981) define concept definition as “a form of words used to specify that concept” (p.152). A concept definition can be provided by another person or resource and then memorized by the individual, or it can be constructed by one’s self. Accordingly, Tall and Vinner (1981) distinguish between *person concept definition* and *formal concept definition*. One might assume that engagement in cognitive tasks requires students to call upon mathematical definitions; however students are more likely to call upon examples, representations, or experiences they associate with the central concepts rather than any formal or informal definitions (Vinner, 1991; Vinner & Dreyfus, 1989). That is, the majority of the time, students call upon their *concept image*
Tall and Vinner (1981) define *concept image* vaguely as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). Vinner (1991) later articulates that the concept image “is something non-verbal associated in our mind with the concept name.” Although a concept-image is “something non-verbal,” the associations one holds with a specific concept name can be translated into verbal forms (Vinner, 1991). It is of course impossible to capture one’s entire concept image as certain stimuli only evoke certain aspects of the image (Tall & Vinner, 1981; Vinner & Dreyfus, 1989; Vinner, 1991; Bingolbali & Monaghan, 2008). That is, we as researchers can only gain insight into and discuss a person’s *evoked concept images*.

Over time one’s concept image is built up as he or she encounters different stimuli (Tall & Vinner, 1981). Often, change in one’s concept image or definition is the result of some sort of cognitive conflict. It is possible, and likely, that students will hold conflicting aspects within their image and/or definition of a given concept. Tall and Vinner (1981) refer to such aspects as *potential conflict factors*. When these potential conflict factors are evoked simultaneously, making them *cognitive conflict factors*, there is potential for change (Tall & Vinner, 1981). For example when part of one’s concept image conflicts with one’s concept definition and both are evoked simultaneously, one might distort or forget the definition or one’s concept image might change (Tall & Vinner, 1981; Vinner, 1991). It is also possible for no change to occur regardless of the conflict, and instead when asked for the definition the student provides the concept definition, but in all other situations she calls upon her concept image (Vinner, 1991). Tall and
Vinner (1981) further argue that, “cognitive conflict factors may be evoked subconsciously with the conflict only manifesting itself by a vague sense of unease. We suggest this is the underlying cause for such feelings in problem solving or research when the individual senses something wrong somewhere; it may be a considerable time later (if at all) that the reason for the conflict is consciously understood” (p. 154).

As mentioned previously, this study includes an investigation of the reciprocal relationship between a teacher’s mathematical understandings and the way he or she makes sense of mathematical solutions. In particular I investigated (i) how the participants’ concept definitions as well as aspects of their concept images of radian, unit circle, and trigonometric functions influenced, enabled, or inhibited their ability to make sense of students’ mathematical work and (ii) how the process of making sense of students’ mathematical work influenced their concept definitions or aspects of their concept images of radian, unit circle, and trigonometric functions. In an interview setting, the mathematical problems, students’ mathematical work, and interview prompts served as potential stimuli, causing their concept definition or certain aspects of the concept images (evoked concept image) to become apparent. As the participants were presented with varying mathematical strategies to the same problem, there was also potential for cognitive conflict. Because this is when there is most likely to be a change or growth in one’s understanding, I directed my focus to these instances, identifying whether or not such conflict occurred and whether or not the conflict was resolved.

Other Resources (beyond MKT) for Noticing
Recall that one of my research sub-questions was, *What resources (knowledge and experience) do teachers draw upon in their professional noticing of work and thinking?* Thus far, the only potential resource for noticing I have discussed is one’s knowledge, namely one’s MKT. While this is significant and arguably the most important resource for professional noticing of students' work and thinking, there is evidence that indicates other factors may result in differences in teachers’ ability to notice. In particular, potential influential factors, or resources for noticing, include one’s (i) beliefs or orientations as well as (ii) interpretive language. While my own research focuses mainly on how teachers draw on MKT, I did investigate and make some hypotheses regarding additional resources.

**Beliefs or Orientations**

Several researchers have established that teachers’ practices are shaped by their own beliefs and orientations (Aguirre & Speer, 2000; Handal, 2003; Philipp, 2007; Thompson, 1992). For example, Aguirre and Speer (2000) investigated two secondary mathematics teachers as they taught an algebra lesson, and found evidence that teachers’ beliefs interact in conjunction with their learning goals influence moment-to-moment actions of teaching. With respect to noticing in particular, Schoenfeld (2011) argued that “what you attend to- what you notice-is in large measure a function of your orientations.” One may not draw on one’s own orientation the way one would draw on one’s mathematical knowledge or KCS; however one’s orientations serve as a lens through which a teacher notices, and thus one’s orientations (measurable or not) have the potential to influence one’s noticing. Thus, I think of beliefs and orientations as a
potential resource for noticing, but in a different way than the way I think of MKT as a resource.

Son and Crespo (2009) found evidence that teachers’ beliefs may be influencing factors in their reasoning and responses to students’ work, found some evidence of this connection, noting the existence of “more consistencies than inconsistencies” between types of reasoning and responses and one’s beliefs. Similarly, the LMMT found that preservice teachers’ perseverance and persistence in making sense of the student’s ideas seemed to be a driving, or at least related, factor in determining how successfully they engaged with student thinking. We hypothesized that this difference was a result of differences in orientations towards mathematics and/or teaching. While beliefs were not a major focus of my dissertation research, I recognized their significance in my analysis, and coded any interview dialogue that indicated a belief about teaching or learning, e.g., beliefs about what good or bad teaching involves or beliefs about how students learn or what constitutes learning. Also, I included two items in the survey aimed at assessing one’s beliefs about teaching and learning. This helped me distinguish between participants, especially when choosing and composing the case studies.

**Interpretive Language.**

Nemirovsky, DiMattia, Ribiero, and Lara-Melou (2005) identified two types of discourse teachers use to discuss classroom episodes: *grounded narrative*, “whose aim is to articulate descriptions of classroom events accounting for the available evidence” (p.365) and *evaluative discourse*, centered on “the values, virtues and commitments at play in the case” (p.365). Van Es (2011) argues that learning a new discourse for talking about teaching is also “central to noticing,” one which is “more interpretive in nature, in
which the teachers’ goal is to make sense of student thinking” (p.135). Van Es contends that this discourse is similar to the type of productive discourse promoted by Borko, Jacobs, Eiteljorg, and Pitman (2008). This makes sense from a situative perspective, in that learning to become a legitimate participant in a community involves learning how to talk, or participate in discourse, of that particular community (Lave & Wenger, 1991).

I had considered language or discourse as a potential resource for noticing prior to collecting my data; however as I will mention in my results, I began to observe that one of the obstacles, especially for preservice teachers, to interpreting students’ thinking, was simply finding the language to talk about students’ thinking. Van Es’s (2011) idea of having versus not having an “interpretive discourse” was useful in distinguishing between these instances.
CHAPTER III

METHODOLOGY

Introduction

In order to investigate teachers’ mathematical noticing of students’ written strategies as well as the relationship between one’s noticing and one’s mathematical knowledge for teaching, I used a combination of (i) open-ended assessment items to access understanding of trigonometric concepts, (ii) survey items to access information regarding teaching experience and teaching style, (iii) task-based interviews during which the teachers engage in both problem solving and mathematical noticing, and finally (iv) semi-structured reflective writing prompts. All of these data sources were used to create case studies for the interview participants as well as cross-case comparisons across subgroups of participants and the participants as a whole.

Setting and Participants

The targeted population for this study included (i) secondary mathematics preservice teachers who were in their third or fourth year of their undergraduate program (prior to student teaching), (ii) secondary mathematics student teachers who had just completed their undergraduate program, as well as (iii) in-service secondary mathematics teachers (currently teaching mathematics at the high school level). Investigating across these groups, including participants with varying teaching experience ranging from zero to 19 years, allowed for a discussion of the role and importance of one’s teaching experience, and the knowledge gained from teaching, in determining one’s ability to recognize and respond to student thinking. All participants were recruited from the same university in Northeast; they were either currently enrolled
in the mathematics teaching program at the university or had previously graduated from the university. Thus many of the participants’ teacher preparation experiences are likely to be similar.

**Recruitment.** To recruit preservice teacher participants, I visited three different (content and methods) classes offered to students in the final years of their secondary mathematics education program at the university. At the beginning of the class, with the instructor outside of the room, I introduced myself, presented the goals, procedures, risks and benefits of the study, and handed out informed consent forms. Each student indicated on the form whether or not they would like to participate, and I collected the forms from all students. In one of the three classes, students were required to complete the trigonometry assessment items as a part of their class; thus choosing to participate, in this case, simply meant allowing me to use a copy of their completed assessment in this study. Each participant who agreed to participate in the assessment portion of the study was offered the opportunity to participate in interviews.

I also sent out 26 emails to high school mathematics teachers and student teachers. I obtained their emails from two faculty members at the University who had regularly worked in/with local schools and one of whom had supervised student teachers. In the email, I explained the goals, procedures, risks and benefits of the study. If an individual responded and indicated interest in participating in the study, I met with him or her at a location of their choosing, typically the high school at which he or she was currently working; I reviewed the study and provided time to read the consent form and ask questions before officially giving consent. Again, each participant who agreed to participate in the assessment portion of the study was offered the opportunity to
participate in interviews. All participants were made aware that (i) they could drop out of the study and/or pull any materials/data from the study at any time and (ii) they would not be compensated for their participation.

Participants. In my written proposal, I anticipated/hoped to have 15 to 30 total and that eight to twelve of them would participate in interviews. My participant numbers did fall within these ranges. Nineteen participants chose to participate in the first phase of data collection, completing a written assessment/survey, twelve of the nineteen chose to participate in interviews, and eight of the interview participants submitted written responses to the reflective prompts. A summary of the participants, their background, and their level of participation is provided in Table 3.1. In this table, “Test” refers to the phase one assessment/survey items, “I1” and “I2” refer to interviews one and two, respectively, and “R” refers to the reflective writing asked of participants at the completion of the study.
Table 3.1. 
Overview of Participants

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<th>Pseudonym</th>
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<th>I2</th>
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Phase One: Trigonometry Assessment and Survey of Teaching

Experience/Style/Beliefs

The first phase of data collection involved administering a written inventory that included (i) open-ended assessment items designed to capture the participants’ trigonometric knowledge - in particular their understandings of radian, unit circle, and
trigonometric functions - as well as (ii) a brief survey of teaching experience, style and beliefs. This data provided insight into which resources (mathematical knowledge and teaching experience) were available to the participants. Data about these resources informed my analysis of how they attended to, interpreted, and decided to respond to students' mathematical work. The results of this assessment were also used to (i) ensure participants had varying levels of mathematical knowledge and teaching experience and (ii) situate the case studies amongst a slightly larger group of participants.

**Trigonometry Items**

To ensure that the mathematical concepts being assessed were both appropriate for the participants and significant to the work of secondary mathematics teaching, I first consulted the current Common Core State Standards for Mathematics (CCSSM, 2010) in the creation of the assessment. In particular, within the assessment, I aimed to address the following trigonometry content standards:

- **CCSS.Math.Content.HSF-TF.A.1** Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
- **CCSS.Math.Content.HSF-TF.A.2** Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
- **CCSS.Math.Content.HSF-TF.A.4 (+)** Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

In addition, in the design of this assessment, I consulted and borrowed trigonometry items from assessment and interview items used in multiple studies.
(Akkoc, 2008; Fi, 2003; Moore, LaForest, & Kim, 2012; Weber, 2005; Weber, 2008). Within the discussion of my findings, I compare the results from this study to those of the previously listed researchers. Some of the results I have found confirm and thus strengthen the claims of these researchers. Alternatively, there were inconsistencies between their findings and my own which provide direction for further inquiry.

The six trigonometry items were taken directly or modified from various sources: Akkoc's (2008) questionnaire and interview protocol aimed at investigating PSTs' concept images of radian; Fi's (2003) test of trigonometric knowledge administered to PSTs; Moore, LaForest and Kim's (2012)'s teaching experiment investigation of secondary PSTs' understanding of unit circle; and Weber's (2005) five question assessment aimed at identifying college students' understanding of trigonometric functions. These items are summarized in Table 3.2. The trigonometry items, as they were given to participants, are included in Appendix A.

Table 3.2. Summary of Trigonometry Assessment Items

<table>
<thead>
<tr>
<th>Assessment Item</th>
<th>Source</th>
<th>Purpose</th>
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| 1. Convert the given angles in radian measure to degrees and vice versa: \(5\pi/2, 3\pi/5, 19, 50^\circ, 570^\circ\). Show your work. | • Original Item  
• Idea from Akkoc (2008) | • Assess whether or not the participants can accurately convert back and forth between radians and degrees |
| 2. (a) Define \textit{radian}. (b) Explain the relationship between radian and degree angle measures. | • Similar to Akkoc (2008)  
• Similar to Fi (2003) | • (a) Assess the participants’ understanding of radian as an angle measure.  
• (b) Assess once again whether or not the participants’ know how to convert back and forth between radians and degrees.  
• (b) Deter participants from responding to (a) by referring
3. Given that the following angle measurement $\theta$ is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius length of 2 inches, 2.4 inches, and 2.9 inches (figure not to scale). Please explain your work and thinking in solving for the length of one arc.

- Directly from Moore, LaForest and Kim (2012) with the exception of the last sentence.
- Assess participants’ ability to find arc length given an angle and radius length.
- Provides participants an opportunity to exhibit understanding of radian as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle.
- Provides participants an opportunity for participants to exhibit that they associate the unit circle to measuring in radii.

4. (a) What is unit circle? Explain.
(b) In your opinion, why is the unit circle taught in trigonometry? Can you name some specific ways that the unit circle is useful in the learning and teaching of trigonometry?

- Similar to Akkoc (2008)
- Similar to Fi (2003)
- (a) Assess participants’ understanding of the unit circle as a circle with a radius of one.
- (b) Assess participants’ understanding of the functionality of the unit circle in the study of trigonometry.

5. Approximate (a) $\sin 340^\circ$ and (b) $\cos 340^\circ$. Please explain how you reached these approximations. Could you find a closer approximation? If so, how? If not, why not?
(c) Could you provide a closer approximation if you were given $\sin 70^\circ$? Please show and explain.

- (a and b) Directly from Weber (2005) except for addition “Please explain how you reached these approximations. Could you find a closer approximation? If so, how? If
- (a & b) Assess the participants’ understanding of the sine and cosine functions by assessing whether or not they know how to evaluate the functions at any angle and anticipate the approximate result.
- (c) Assess whether or not the study participants understand sine and cosine as cofunctions.
6. (a) Identify the (i) domain, (ii) period and (iii) symmetry of the cosine function. (b) Use the unit circle to explain and defend each of your responses in part (a).

<table>
<thead>
<tr>
<th>Teaching Experience, Style, and Beliefs</th>
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<td>In phase one of the study, participants were also asked to provide background information with respect to their teaching experience, teaching style and beliefs. These items are included in Appendix B. The purpose of gathering this information was to (i) ensure that interview participants had a range of teaching experiences and thus, the interview participants would have varied resources to draw upon in the work of noticing, (ii) offer more background on the case studies to potentially explain differences in how they notice or approach mathematics problems. In addition, this background information will help readers determine the applicability and generalizability of the findings of this study.</td>
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<th>Phase Two: Task-Based Interviews</th>
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<td>Task-based interviews are interviews in which the interviewer and subject interact in relation to one or more questions, problems, or activities (Goldin, 2000). In mathematics education research, task-based interviews are used most often to investigate conceptual understanding and higher-level thinking that is not as easily assessed using written assessments. Certainly, the participants’ mathematical noticing</td>
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</table>
of students' work and thinking qualifies as such higher-level thinking which could be better assessed in "conversation" as opposed to written responses. In this study, over the course of two interviews, participants were asked to respond to prompts corresponding to five trigonometry problems (some involving multiple parts). After working through each mathematics problem to the best of their ability, participants were then presented and asked to analyze students' work on that same problem-solving task, rank the students and then identify how they would respond to each student. Thus, in each interview, there were four major types of tasks: (i) doing the mathematics, (ii) attending and interpreting students' work, (iii) ranking students' understanding, and (iv) deciding on instructional responses. I refer to each set as a “task sequence.” The first portion of the task sequences are unstructured in that they involve free problem solving; however, I built in more structured prompts within the analysis, ranking, and response tasks.

In designing these interviews, I adhered to the principles of task-based interviews outlined by Goldin (1997). First, the mathematical ideas encompassed by the tasks were all appropriate in that they were all ideas that should be addressed at the secondary level. Second, each interview included a series of questions posed in the context of one or more tasks. Third, I ordered the questions in the analysis and response sections of the interviews such that hints, prompts, and new questions were posed only after the participants were allowed the maximum opportunity for free response and problem solving. Fourth, in the design stages, I identified major contingencies, or ways in which the participants might struggle with the mathematics based on prior studies; I also identified ideal analyses of the students' work. Finally, I
provided paper, pencils, and non-scientific calculators, so that the participants were able to construct external representations.

As I have discussed, twelve participants (four preservice teachers, three student teachers, and five in-service teachers) participated in phase two of this study, these task-based interviews. I used these interviews to assess (1) additional information about their understanding of three targeted concepts: radian, unit circle, and the sine and cosine functions, (2) what they did and paid attention to when presented with students' work, (3) to what extent they made claims about student thinking and what informed their interpretation, (4) how they drew on their interpretations of student thinking in their proposed instructional responses, (5) the goals of their proposed instructional responses, and (6) how various resources, especially their MKT, influenced their professional noticing of work and thinking during the interviews.

In addition, I also wanted to learn what mathematics the participants engaged in as a result of the student work, and especially, what ways the ideas presented in the work challenged the participants' own thinking. In order to achieve this, more than one interview was needed, so that (1) data corresponding to each concept was available from multiple sources and (2) it was possible to look at change in the participants' understanding of these concepts as a result of the interviews themselves. Thus, I conducted two video-recorded interviews with each of the participants in this second phase. Below I discuss the task sequences I included in these interviews. The actual interview protocols are provided in Appendices H and I.

Task Sequences
As I have discussed, the interviews consisted of different task sequences, which involve participants working through some mathematics and responding to prompts about different students’ work. Prior to describing the task sequences in more detail, it seems a justification is warranted of two choices: (i) the choice to use written work as opposed to video of students working on these problems and (ii) the choice to use hypothetical, self-designed work as opposed to the work of actual students. I also made the choice to use written work instead of video. In support of this decision, Goldsmith and Seago (2011) argue that, “working with video artifacts reduces the burden of following students’ reasoning more than when working with students’ written work” (p. 213). With respect to the source of the written work, although I used research on students’ and teachers’ understanding of these concepts to inform the work, by designing the work myself I was able to allow for a variety of responses, by ensuring that each hypothetical student’s work contained both correct and incorrect work as well as that the work includes a range of understanding of different concepts and includes unconventional procedures.

Before I introduce the actual problems and student work, I describe the general sequencing of the prompts/questions asked to the participants once they had time to work on the mathematics problem at hand and look at students work. The first prompt that was given in each task sequence immediately after the teachers’ review of student work was, Please take a look at the work and tell me what you notice. Without asking them to describe the students’ strategies or discuss the students’ understandings, this prompt aimed to assess how and what teachers’ attended to on their own. I then asked questions that are slightly modified versions of the prompts used by Jacobs, Lamb, and
Philipp (2010) in their study of elementary teachers’ professional noticing of children’s thinking. If the participant had not already, I would prompt them to *Please describe in detail what you think each student did in response to the problem.* I would then prompt them to *Please explain what you learned about these students’ understandings.* This was asked in an open way such that the participants were allowed to talk about the students’ understanding of whichever ideas/concepts the participants felt were relevant or central to the problem and their work. Once they were allowed enough time to thoroughly elaborate freely on the students’ understandings, I then asked more specifically what they could say about the students’ understanding of specific concepts, the targeted concepts of this study which I determined were central in the problem and work at hand. Finally, the deciding to respond prompt in each task sequence was: *Pretend that you are the teacher of these students, describe some ways you might respond to them, and explain why you chose these responses.* The problems and student work for each task are provided in Appendices C, D, E, F, and G, and the tasks are also included within the interview protocols in Appendices H and I. In the remainder of this section, I provide a description of each of the five task sequences included in the two interviews. More specifically, I provide a description of the mathematics problem the teachers are asked to complete as well as a description of the student work in each task sequence.

**Task sequence one: Arc length.** In the *Arc Length Task Sequence*, participants were first asked to find the length of the arc on a circle with a radius length of three in the xy-plane associated with the angle $\theta = -\pi/12$. Individuals with a strong understanding of radian should be able to correctly identify the length of the arc as $\pi/4$. 
Those without an understanding of radian may still correctly answer this prompt by thinking about the arc length as a portion of the circumference. Participants were then told, “imagine you have assigned this problem to your students, and as they work you hear and see the following.” They were presented with (hypothetical) student work from four students, Allen, Lexi, Annie, and Charlie. In the following I present each student’s work as well as my own analysis of their work and thinking. My analyses are provided for two reasons. Firstly, I hope to illustrate to the reader some of the ideas that were considered in designing these tasks, the ideas that I hoped teachers would attend to. Secondly, such transparency is critical to minimizing bias and maximizing objectivity. It is important to note that “my own analysis” was additionally informed and influenced by the analyses of my peers as well as my advisor. I provide similar descriptions of analyses for each of the task sequences in subsequent sections as well.

Allen

![Allen's Work: Task One](image)

*Allen says, “Well the negative doesn’t really matter if we’re talking about length. So I’m just going to treat it as pi over twelve.”*

He writes:

\[
\text{Circumference: } 2\pi r = 6\pi \\
12x = 6\pi \\
x = \frac{\pi}{2}
\]

Figure 3.1. Allen’s Work: Task One
Allen correctly calculated the circumference. It looks like he then drew the picture of the circle to figure out what portion of the whole circle $\pi/12$ radians is. He first seems to consider $\pi/12$ radians as a portion of $\pi/2$ radians. In doing so, he makes an error when, in his picture, he divides $\pi/2$ radians into 3 sections (instead of 6) of $\pi/12$ radians. Perhaps he is thinking that a quarter of the circle contains $\pi/4$ radians instead of $\pi/2$ radians. This might have been a simple error because he was thinking “quarter,” but he also might hold a significant misconception, that a circle is made up of $\pi$ radians. Without this error, he would have arrived at a correct solution. His approach seems to indicate that he understands arc length as a portion of the circumference. There is no evidence to indicate that he understands arc length as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle.

Lexi writes:

\[ S = \theta r \]

And then says, “I'm not sure if theta should be in degrees or radians. It must be radians because otherwise we'd have a degree symbol in our answer. I don't think there is a real symbol for radians.”

She writes:

\[ S = -\frac{\pi}{18} \times 3 = -\frac{\pi}{6} \]

And then says, “Hmm, it seems weird that I have a negative number with pi in it as a length.”

Figure 3.2. Lexi’s Work: Task One
Lexi correctly recalls the arc length formula; however, because she is not sure if “theta should be in degrees or radians,” it does not seem like she understands why this formula makes sense. That is, she does not seem to understand radian as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle. There is also a lack of evidence that she understands arc length, especially since she incorrectly identified arc length as negative. Her concern about pi in her answer is evidence that she may have a misconception about pi, perhaps as a sort of unit of radian rather than a real number.

Anna

**Figure 3.3. Annie’s Work: Task One**

Annie’s first drawing is evidence that she understands radian as the length of the arc on the unit circle subtended by the angle. It seems she tries to use the idea that all circles are similar to then think about a circle of radius three. She ends up with an incorrect answer of $\pi/36$. She might have (1) meant to correctly multiply $\pi/12$ by three,
but then made an error in her calculations, (2) thought that she needed to divide by three, but I am uncertain why this strategy would make sense to her or (3) it might also be that she does not know what to do with a circle that is not the unit circle, indicating a limited procedural understanding.

Charlie

Charlie has on his paper:


g
d

Figure 3.4. Charlie’s Work: Task One

Although he is incorrect and has no work, Charlie’s answer may be evidence that he recalls/understands that there exists a relationship between radian measure and arc length. In particular, Charlie might understand radian as the length of the arc on a circle subtended by the angle, rather than the length of the arc on the unit circle subtended by the angle.

Task sequence two: Sine and cosine estimation. In the Sine and Cosine Estimation Task Sequence, the participants were first asked to estimate \( \sin 20^\circ \) and \( \cos 20^\circ \) and then later to reevaluate given \( \sin(70^\circ) \approx 0.9397 \). This item is a simpler version (involving an angles less than 90°) of item five on the trigonometry assessment. By asking them to explain their work and reasoning aloud, I gained further insight into their understanding of the sine and cosine functions as well as the relationship between the sine and cosine functions, i.e. their understanding of cofunctions. They were then told, “Three high school students’ were asked the following: Given \( \sin(70^\circ) \approx 0.9397 \), what
can you say, if anything, about sin 20° and cos 20°? They are provided with three students’ work: Alex, Martha, and Cheyenne.

**Alex**

**Alex’s work:**

Since 20° and 70° are complimentary, we know that sin 20° + sin 70° = sin 90° and so sin 20° ≈ 1 - 0.9397

We also know that sin²θ + cos²θ = 1, so sin²(20°) + cos²(20°) = 1.

So cos 20° = ±√(1 - sin²(20°)) ≈ ±√(1 - (1 - 0.9397)²)

And 20° is in the first quadrant, so cos 20° ≈ √(1 - (1 - 0.9397)²)

Figure 3.5. Alex’s Work: Task Two

There is little evidence that Alex understands the sine and cosine functions, except for that he knows cosine values are positive in the first quadrant. In fact, there is some evidence that he has a weak understanding of the sine function— he makes an error, sin 20° + sin 70° = sin 90°. This is also evidence of a lack of knowledge of the sum-product properties of trig functions. He is able to recall the relationship between sine and cosine values via “trig identity”; however it is unclear if he understands the meaning of cofunctions.

**Martha**

**Martha’s work:**

Well 90°=20°+70°, so cos 20° ≈ .9397

Also, 0 < sin 20° < 1 because the sine function gives out values between -1 and 1 and 20° is in the first quadrant.

Figure 3.6. Martha’s Work: Task Two

Martha seems to hold some understanding of the sine and cosine as cofunction since she noted, “Well 90°=20°+70°, so cos 20° ≈ 1-.9397.” She also knows the range of
the sine function and that sine is positive for theta values from 0 to 90 degrees; however, the fact that she cannot reason a closer estimate for \(\sin 20^\circ\) is evidence that perhaps she has a lesser understanding of the sine function.

**Cheyenne**

![Cheyenne's work:]

\[
\sin(0) < \sin(20^\circ) < \sin(30^\circ) \text{ so } \sin(20^\circ) \text{ is between } 0 \text{ and } \sqrt{3}/2
\]

\[
\cos(30^\circ) < \cos(20^\circ) < \cos(0^\circ) \text{ So } \cos(20^\circ) \text{ is between } 1/2 \text{ and } 1
\]

Figure 3.7. Cheyenne’s Work: Task Two

Although she makes an error in swapping \(\cos(30^\circ)\) and \(\sin(30^\circ)\) values, Cheyenne shows evidence of some understanding of the sine and cosine functions in that she has at least some idea of when they are increasing and decreasing and is able to use this (along with her knowledge of special angles) to estimate the sine and cosine values at \(20^\circ\). If she had a stronger understanding of sine and cosine, she would have been aware that \(\sin(30^\circ)\) is less than \(\cos(30^\circ)\). There is no evidence that Cheyenne understands the relationship between sine and cosine.

**Task sequence three: Right triangle.** The problem involved in the *Right Triangle Task Sequence* is relatively simple. They were given a triangle with one 90 degree angle and another labeled, \(\theta\). Values were given for both the length of the hypotenuse as well as the side adjacent to the angle \(\theta\), and they are asked to find \(\sin(\theta)\). Participants were then told, “Imagine you have assigned this problem to your students, and as they work you see the following,” and presented with the work of Joe and Anna. Brown (2006) identified three major ways of knowing sine and cosine: as
coordinates of a point on the unit circle, as horizontal and vertical distances on the unit circle, and as ratios of sides of a right triangle. I designed the following student work such that the two students displayed the latter two ways of knowing sine.

Joe

Figure 3.8. Joe’s Work: Task Three

Even though he is given a reference triangle, Joe does not seem to be thinking of sine and cosine as ratios of the sides. Instead, his picture of a similar triangle with a hypotenuse length of one, is evidence that he is thinking of sine and cosine as horizontal and vertical distances on the unit circle. He then uses the Pythagorean theorem to solve for the sine value, but he makes an error when he forgets to take the square root, which is most likely an error that is not a result of a misconception and might have been avoided if he had written out his work in more detail or written down the Pythagorean theorem.
Anna correctly uses the Pythagorean theorem to solve for the missing side measure. In contrast to Joe’s work, Anna’s work is evidence that she was thinking of sine as a ratio of the sides of the reference triangle - a more predictable strategy, especially since a right triangle was presented. Specifically, she seemed to recall that, “sine=opposite over hypotenuse.” She wrote siny instead of sinθ at the end, but this is most likely just a thoughtless error; however perhaps she holds some confusion about input values of sine being angle measures.

**Task sequence four: Explaining the cosine graph (with the unit circle).** In the Explaining the Cosine Graph (with the Unit Circle) task sequence, the participants were first asked to draw the graph of the cosine function and to explain how they arrived at that graph. They were then asked to make sense of the graph in terms of the unit circle. Explanations of why functions hold certain properties provide valuable insight into individuals’ understanding of these functions (Weber, 2005). Beyond assessing the teachers’ understanding of the cosine function, this item also provides insight into how they perceived the unit circle and it’s usefulness. After they explained their graph twice,
(once however they choose, and once specifically using the unit circle), participants were then asked to review the work of two students, Mike and Alice, who were asked to use the unit circle to draw the graph of the cosine function. I designed the students’ work to represent the two ways of knowing the cosine function in terms of the unit circle: (1) as coordinates of a point on the unit circle and (2) as horizontal and vertical distances on the unit circle (Brown, 2006).

**Mike**

![Mike had drawn:](image)

**Figure 3.10. Mike’s Work: Task Four**

Mike’s use of the unit circle shows evidence that perhaps he thinks of sine and cosine as coordinates of a point on the unit circle. It is possible however that he simply has memorized this image, a sort of reference circle, along with these values, and does not actually understand that these are in fact coordinates, based on the unit circle being centered at the origin. Mike has used the cosine values of different angles (which he knows from the unit circle) to draw the graph of cosine from 0 to pi. He drew arrows at
both ends of the curve indicating that perhaps he understands that the domain of cosine is infinite; however perhaps he has simply learned to always put arrows on graphs (whether correct or not). One cannot decipher whether or not Mike understands the periodic nature of cosine.

Alice

Alice uses the unit circle very differently than Mike. Similarly, she seems to know and use some exact cosine values, specifically at 0, pi/2, pi, 3pi/2, and 2pi; however there is more evidence in her work that she understands how to find these values, rather than memorize them. She drew horizontal line segments indicating she thinks of cosine as horizontal distances on the unit circle. This is further evidenced by her indication of when cosine (or these horizontal line segments) is increasing or decreasing. Her graph does not seem precisely the right shape – this is evidence that it is more likely she did not memorize the graph, but in fact, is truly relying on her

Figure 3.11. Alice’s: Task Four
knowledge of cosine as horizontal distances on the unit circle. Her graph only includes values from 0 to 2π; thus it cannot be determined whether she understands that cosine can be extended to all real numbers.

**Task sequence five: Inverse cosine.** In the *Inverse Cosine Task Sequence*, participants were first asked to evaluate $\cos^{-1}\left(\cos\left(-\frac{11\pi}{4}\right)\right)$. In the LMTT study, participants had significant difficulty working with inverse functions, especially inverse trigonometric functions. Fi (2003) found similar results; in particular preservice teachers often confused inverse trigonometric functions with reciprocal trigonometric functions. Although inverse trigonometric functions were not one of the targeted concepts in this study, this problem additionally required thinking about the cosine function and involves an angle measure that requires thinking about cosine on the unit circle. Participants were provided the work of four students on this same problem.

**Alexis**

Alexis wrote that $\cos^{-1}\left(\cos\left(-\frac{11\pi}{4}\right)\right) = \frac{1}{\cos\left(\cos\left(-\frac{11\pi}{4}\right)\right)} = \frac{1}{\cos^{-2}\left(-\frac{11\pi}{4}\right)}$. Alexis paused and said $-\frac{11\pi}{4}$ is an odd multiple of $\frac{\pi}{4}$ so its cosine value is either positive or negative $\frac{\sqrt{2}}{2}$. She then wrote, $\frac{\sqrt{2}}{2} = 2$.

Joe said, well they are inverses so $\cos^{-1}(\cos(x)) = x$. So $\cos^{-1}\left(\cos\left(-\frac{11\pi}{4}\right)\right) = -\frac{11\pi}{4}$.

**Figure 3.12. Alexis’s Work: Task Five.**

Alexis thinks that the -1 represents the numeric power of cosine rather than representing the inverse. She also makes an error in thinking that $\cos\left(\cos\left(-\frac{11\pi}{4}\right)\right)$ is equivalent to $\cos^{2}\left(-\frac{11\pi}{4}\right)$. Her last statement is correct, and she provides an efficient strategy for evaluating cosine squared. She also correctly identifies
the cosine of $\frac{\pi}{4}$. The last statement is also evidence that she has some understanding of the periodicity of cosine.

**Joe**

Joe said, well they are inverses so $\cos^{-1}(\cos(x)) = x$. So $\cos^{-1}(\cos(\frac{-11\pi}{4})) = \frac{-11\pi}{4}$.

**Figure 3.13. Joe’s Work: Task Five**

Joe is most likely recalling a fact he learned, that a function composed with its inverse is the identity. He does not understand that $\cos^{-1}(x)$ is the inverse of cosine restricted to the interval $[0, \pi]$. It is likely that he does not understand what it means to be invertible, but one cannot say for certain based on this work alone.

**Sam**

Sam said, well that means we want the angle whose cosine value is the cosine of $\frac{-11\pi}{4}$, but that’s an infinite number of angles. So, it’s like, all the angles that are located in the same spot as $\frac{-11\pi}{4}$.

**Figure 3.14. Sam’s Work: Task Five.**

Sam has not taken into account that $\cos^{-1}(x)$ is the inverse of cosine restricted to the domain $[0, \pi]$. He may not think of $\cos^{-1}(x)$ or a function or perhaps he does not understand what it means to be a function. Instead, Sam seems to be solving $\cos\left(\frac{-11\pi}{4}\right) = \cos(x)$ for $x$. Even with this thinking in mind, he makes an error in only considering angles located in the third quadrant, and missing angles in the second quadrant. He also has not shown evidence that he knows how to express, “all the angles that are located in the same spot as $\frac{-11\pi}{4}$” with mathematical notation.
It isn’t necessary to convert radians into degrees here, but Amy does, which is evidence that she might feel more comfortable working with angle measures in degrees. She correctly converts radians into degrees, and correctly identifies that the cosine of \(-495^\circ\) is the same as the cosine of \(-135^\circ\), evidencing that she understands the periodic nature of cosine. Although she cannot find the exact value, she correctly identifies the answer as one angle in the second quadrant, evidence that she most likely understands \(\cos^{-1}(x)\) is the inverse of cosine restricted to the domain [0, \(\pi\]). This is also evidence that she has some knowledge of for which values cosine is positive and for which values cosine is negative. Perhaps she would be able to evaluate the cosine of \(-135^\circ\) if she had realized that it was a multiple of 45°, which might have been more obvious to her if she had left the angle measures in radians.

**Phase Three: Written Reflections**

Once I had finished interviewing a particular participant, i.e. at the conclusion of phase two data collection, I offered her or him the opportunity to participate in phase three. In this stage of research, each participant was asked to write reflectively about his or her experience analyzing students’ work during these interviews. In particular, within one week of his or her final interview, all twelve interview participants were emailed and/or handed in person (in most cases, both) the following set of questions:
1. What mathematical thinking and problem solving did you feel you engaged in while analyzing the students’ work during the interviews, if any? Please explain and provide specific examples.

2. How has your experience analyzing student work during these interviews influenced your knowledge of and/or thinking about trigonometry? If so, which parts of the interviews and how? Please provide specific examples.

3. What were the greatest challenges for you when analyzing the students’ work and thinking? How, in your opinion, could teacher preparation courses and professional development help you overcome these challenges?

4. Any other comments or thoughts you would like to add about the experience or interview tasks in particular?

The first two questions were directly aimed at addressing my second research question, *In what ways does the mathematical work presented to the preservice, in-service, and student teachers challenge and/or further their mathematical thinking?* The third question allowed participants opportunities to discuss the limits to their mathematical knowledge or knowledge of content and students (addressing research question 1e) as well as opportunities to provide input as to how we can improve teachers ability to recognize and respond to students’ thinking in the future, which will be discussed in conclusions. The third question also allowed participants an opportunity to describe ways that the problems and work challenged their thinking, even if overall they did not feel they had a better understanding (again, addressing the second research question). I included a fourth optional free reflection item in the case that a participant wants to, for example, talk about the strength or weaknesses of one particular task or hypothetical student work, which would be discussed in implications, discussing how best to implement these sorts of tasks in teacher preparation courses.

All twelve of the interview participants were also provided with the five tasks and corresponding hypothetical student work from the interviews to aid them in their
recollection and reflection. Eight individuals chose to participate and emailed me their written responses.

**Analysis**

**Analysis of Assessment/Survey Data**

The participants’ responses to the items designed to assess their trigonometric knowledge were analyzed both quantitatively and qualitatively. To analyze the items quantitatively, a rubric was created to score items 1, 2a, 3, 4a, 4b, 5ab, 5c, 6a, and 6b for correctness. The rubric for each item is provided in Appendix K, and a sample rubric (item 2a) is provided in Table 3.3. Rubrics were also created to assess whether participants overall had no, weak, some, or strong understanding of the concepts of (i) radian, (ii) unit circle and (iii) the sine and cosine functions; however in analysis the only rubric that proved to provide additional information about the participants was the rubric for understanding of radian. The rubric used to assess participants’ overall understanding of radian is provided in Table 3.4

Table 3.3.

<table>
<thead>
<tr>
<th>ITEM 2a</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>
Table 3.4.  
Rubric for Scoring of Participants’ Overall Understanding of Radian Based on Responses to Trigonometry Assessment.

<table>
<thead>
<tr>
<th>Radian</th>
<th>General Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>There exists confusion about the relationship between radian and degrees.</td>
</tr>
<tr>
<td>Weak</td>
<td>Consistent evidence that he or she understands the relationship between radians and degrees</td>
</tr>
<tr>
<td>Some</td>
<td>Some evidence exists that he or she might understand radian as the measure of an angle as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle</td>
</tr>
<tr>
<td>Strong</td>
<td>Consistent evidence that he or she understands radian as the measure of an angle as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle</td>
</tr>
</tbody>
</table>

Two researchers applied the rubrics (for each item and for understanding of radian) independently and then compared the scores. There was 86.55% initial agreement on the scoring of individual assessment items. Inter-rater reliability for each individual item is displayed in Table 3.5. There was also 89.47% initial agreement on the scoring of overall understanding of radian. Overall, there was initial disagreement on 25 out of 190 responses. Out of these 25 disagreement, 24 were a difference of only one-point, and one initial disagreement was a two-point difference. In each of the 25 cases of initial disagreement, the scorers reexamined and discussed the participant’s response in detail, and in 100% of the cases, agreed on a final score.

Table 3.5.  
Initial Inter-Rater Reliability: Scoring of Trigonometry Assessment Items

<table>
<thead>
<tr>
<th>Item #</th>
<th>1</th>
<th>2a</th>
<th>3</th>
<th>4a</th>
<th>4b</th>
<th>5ab</th>
<th>5c</th>
<th>6a</th>
<th>6b</th>
</tr>
</thead>
<tbody>
<tr>
<td>% Initial Agreement</td>
<td>78.95</td>
<td>84.21</td>
<td>89.47</td>
<td>100</td>
<td>73.68</td>
<td>84.21</td>
<td>84.21</td>
<td>89.47</td>
<td>94.74</td>
</tr>
</tbody>
</table>

Once scored, descriptive statistics of mean, range, median, maximum, and minimum were used to analyze the data. Note, the rubric I applied produced ordinal
data, and thus I have been careful not to use the averages to make claims about
teachers’ knowledge in general but rather to situate individual participants and subsets
of participants within my sample. The data was quantitatively analyzed in this manner
by participant and item-by-item. In addition, qualitative analysis was conducted to
investigate patterns in the ways participants’ understand radian, unit circle, and
trigonometric functions. Patterns were identified and categorized into themes. Although
new themes did emerge from the data, I also specifically looked for themes previously
found in prior research, such as (i) concept image dominated by degree (Akkoc, 2008),
and (ii) three ways of knowing of sine and cosine: ratios of the lengths of sides on a
right triangle, coordinates of a point on the unit circle, and horizontal and vertical
distances on the unit circle (Brown, 2006).

**Transcription of Interviews**

Analysis of the interview data took place only once all interviews had been
conducted in order to “minimize imposing the generative process of the interviews what
I think I have learned from other participants” (Seidman, 2006, p. 113). Analysis and
interpretation of this data was conducted in several phases. Initial analysis took place as
I transcribed the interview video data (Seidman, 2006). I transcribed all of the interview
data so as not to make “premature judgments about what is important and what is not”
(Seidman, 2006, p.115), and I attempted to capture the precise words of the
participants, including pauses. While transcribing, I wrote notes in a transcription
journal whenever I noticed something of interest or if I thought of a potential code. For
example, during transcription I decided upon new categorical codes such as “predicting
student work or thinking” and “comparing students’ work or thinking.” Once all the
videos were transcribed, I prepared the transcripts for the remainder of analysis by coordinating images of the participants’ written work with the transcribed dialogue that corresponded to their work.

**On Initial Coding and Memo Writing Using Qualitative Data Analysis (QDA) Software**

I used NVivo qualitative data analysis (QDA) software to code the finalized interview transcripts, assessments, and reflective writing responses. According to Charmaz, “coding is the pivotal link between collecting data and developing an emergent theory to explain these data” (2004, p. 506). I utilized the coding process not only to explore and identify patterns or themes in the data, but also, at first, for organizational purposes. Better organization of the data enabled further analysis and strengthened transparency, especially when writing my results. QDA programs like NVivo have been found to be particularly useful in organizing and coding data, especially aiding in retrieval of data (Basit, 2003; Suddaby, 2006). For example, in the first phase of coding, I applied case codes, or codes that corresponded to particular participants or tasks. More specifically, I coded each data source by participant and then aggregated this coded data by experience level: Preservice Teacher, Student Teacher, or In-service Teacher. I also coded all dialogue and work that corresponded to each of the five task sequences, and within each chunk of data corresponding to a given task sequence, I coded dialogue and work that corresponded to each hypothetical student’s work. Later in my analysis, this preliminary work of coding by cases, allowed me to run queries via the software that furthered by analysis. For example, I could run a query for data that was both coded as April and deciding to respond to more closely look at patterns in the way April decided to respond.
Additionally, I used the memo-writing function of the software. As I will discuss later, during the next phases of coding and interpreting, I read each line of the transcript and coded each line or groups of lines using categorical codes and analytic codes; however in some cases, I found it helpful to provide a summary of what happened over the course of a chunk of an interview – for example, noting how one participant started thinking X, interacted with Y, and then thought Z. In this case, I would write a memo detailing the chain of events or summarizing, and then I could code an entire memo or part of that memo corresponding to a particular category or emerging theme.

While the Nvivo software made the process of coding less tedious and improved the quality of my reports, coding was still an intellectual exercise (Basit, 2003). Suddaby (2006) noted, "Qualitative software programs can be useful in organizing and coding data, but they are no substitute for the interpretation of data. The researcher must make key decisions about which categories to focus on, where to collect the next iteration of data and, perhaps most importantly, the meaning to be ascribed to units of data" (p.638). Similarly, after examining two qualitative studies, one using QDA and one not, Basit (2003) concluded, "coding was an intellectual exercise in both the cases. The package did not eliminate the need to think and deliberate, generate codes, and reject and replace them with others that were more illuminating and which seemed to explain each phenomenon better" (p.152) As I will discuss in more detail in the following sections, the brunt of the intellectual work and interpretation took place as I (i) coded the interview, assessment, and reflective writing data by categories that corresponded to my research questions and sub-questions and (ii) examined the data for patterns or themes within these categories, some which I pre-identified from the existent literature,
and some which emerged from the data itself. In the following, I discuss how I conducted these processes in more detail with respect to each research question.

**Analysis: Preliminary Research Question 0 - What content knowledge of radian, unit circle, and the sine and cosine functions do secondary mathematics preservice, in-service, and student teachers possess?**

To address the preliminary research question, I coded assessment and interview data that showed evidence of participants' knowledge of Radian, Unit circle, and the Sine and Cosine Functions. As I analyzed the data, I also found it useful to code evidence of participants' knowledge of Arc length, Cofunctions, Inverse trig, Pi, and Other Concepts. I then attempted to identify themes with respect to the participants' concept images of radian, unit circle, and the sine and cosine functions. I specifically looked for themes identified in my review of the literature. In particular, I categorized instances of evidence of participants' understanding of sine and cosine as Coordinates on Unit Circle, Distances on Unit Circle, or Ratio of Triangle Sides (Brown, 2006). I also categorized instances of evidence of one's concept image of radian being Dominated By Degree (Akkoc, 2008) or evidence of Thinking of Radian in Terms of Pi (Akkoc, 2008). As new themes emerged, I created codes for these themes and then recoded the data using these new codes.

**Analysis: Research Question 1 - How do secondary mathematics preservice, in-service, and student teachers attend to, interpret, and respond to hypothetical students’ written work and thinking in trigonometry?**

Once the case codes were applied to each data source, I examined each source line-by-line to identify particular lines or chunks of data that corresponded to each research question. With respect to the first and main research question, like Jacobs, Philipp and Lamb (2010), I coded data using the following main categorical codes that
correspond to the three skills of noticing: Attending, Interpreting, and Responding. I coded the participants’ responses to the initial, unstructured prompt, “please take a look at the students' work and tell me what you notice,” as attending because this data provided insight into what the participants attended to without imposed structure of focus. All of the participants' responses to the final prompt, “Pretend that you are the teacher of these students. Describe some ways you might respond to them,” were coded as Responding. In addition, some participants began to discuss how they would respond to participants earlier in their analysis of the students’ work and thinking, and this data was also coded as Responding. I applied the interpreting code whenever a participant made a claim that was interpretive in nature, including interpreting students’ thinking about the problem or ways of knowing various concepts. Interpreting data included descriptions of aspects of students' strategies that were not evidenced directly in the written work. Thus, in some cases the interpreting code was applied to data also coded as attending or responding. That is, there was not one particular interview prompt that corresponded to interpreting, although the “describe the students' understandings” was designed to elicit a focus on thinking and ways of knowing.

Within the categories of Interpreting, Responding, and Comparing, I also found it valuable to code this data as Prompted or Unprompted. This was especially important in the case of Interpreting. Recall that in my interview protocol, I asked the participants what they noticed, and then later I asked them to tell me about the students’ understanding. It was important that as the researcher, I distinguish between instances when participants attended to students' thinking on their own (in response to the former
prompt) versus when they attended to thinking because I had asked them to (in response to the latter prompt).

Thus far, my discussion of the coding scheme has been restricted to case codes and categorical codes with respect to research question one. The next step in analysis was to make sense of the data within each category. As I began doing so, it became clear that the noticing of several participants was limited due to a lack of trigonometric knowledge. While this is a finding in and of itself, it is for this reason I chose to provide more nuanced descriptions of the noticing of five particular participants who had adequate knowledge of trigonometry or other areas of mathematics to engage with the students’ work in a meaningful way. In the remainder of this section, I describe the more fine-grained analyses with respect to research question one, taking care to clarify to the reader which analysis took place overall and which methods of analysis were used for the five particular cases.

In an attempt to describe the participants’ attention, I acknowledged whether or not each participant attended to thinking without being prompted. I also looked for patterns what the participants did as they attended to the hypothetical students’ work and thinking. I created codes to describe these patterns, applied the codes, refined them, and repeated this cycle until I settled categories to describe participants’ initial attention. In particular, I coded instances when participants’ initial attention included: (i) a focus on student thinking, (ii) describing, (iii) evaluating, (iv) investigating the mathematics, (v) comparing to self, (vi) comparing students, and (vii) challenged by a students’ claim. In Table 3.6, for each of the seven codes I include: (i) a description of how I recognized data that fell in this category, (ii) some key words which often
correlated with the code, and (iii) an example or examples of data that fell within this category.

Table 3.6.
Description of Codes Applied to Participants’ Initial Attention to Students’ Work and Thinking (Without Structure or Imposed Focus)

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
<th>Sample/Key Words Or Phrases</th>
<th>Example(s)</th>
</tr>
</thead>
</table>
| **Focus on Thinking**       | Makes reference to the student’s thinking or ways of knowing/understanding; describes aspects of a student’s strategy or thinking that is not stated/shown directly in the student’s work. | Is thinking; understands; sees; recognizes; knows; considers. | (T1) “he’s (Charlie’s) seeing uh radians as like a direct correspondence to arc lengths for all circles”  
(T3) “maybe he's (Joe’s) trying to compare it to the unit circle”  
(T5) She (Sam) knows she’s looking for an angle measure. |
| **Evaluating**              | Makes a definitive claim about the validity or correctness of a student’s work or thinking or one aspect of his or her work and thinking. | Correct; wrong; good; true; makes sense; I like; I don’t like; | (T2) “Alex started down the wrong path from the beginning…and even though this is correct, I mean what he does here seems to make sense because he’s solving for cosine of 20, but he’s got an incorrect value here.” |
| **Describing**              | Describes what the student did, i.e., his or her strategy or one aspect of his or her strategy. | She wrote; he drew; first she; he took; she multiplied; | (T4) “Mike drew his unit circle, and put all like the cosine and sine points on there, and then he took his graph and plugged in like each of those points through like zero to pi and then connected the line.” |
| **Challenged by Mathematical Claim** | Notices a mathematical claim that he/she doesn’t understand or know the truth of immediately; notices something he/she can’t make sense of immediately that could be investigated mathematically. | I don’t know (if this is true/correct); I’m not sure about; I don’t know where this came from; I think this is right/wrong; Is that right?: (“hmm” or “umm” indicating a pause to think) | (T1) “She (Lexi) writes this. If that formula is right, I wish I remembered it, because that’s a lot easier”  
(T3) “I don’t know if this (pointing to Anna) is right”  
(T5) “So using the exponent rule she (Alexis) put it in the denominator, brought it down, and then it’s, then she had to equal one over cos squared, which I, you cannot do?” |
| **Investigating the Mathematics** | Engages in mathematical thinking or work to test a student’s claim or make | I’m just going to try; Let me draw; Let me | (T2- Cheyenne’s Work) “So it’s between zero and root three over two…(thinking)... If it's sine - ok so |
sense of his or her work. This often included testing a student's claim using a particular example.

think about this; (long pauses followed by a "oh!")
sine of 30 is (draws unit circle) (incomprehensible) remember if it's shorter. So if it's 30 degrees, it's the shorter one. The shorter one is the one half as the height and the distance is root three over two.

(T2) Um...well, I bet I can try it out and see if it works. If I use my 30-60-90...(drawing)... (T2) "So he (Mike) included more angles"

Better than; like his; more than her; (T4) "Martha also, I mean less writing, but she pretty much says the same thing as he(Alex) does right there"

Comparing to Other Students

Notices and makes reference to differences or similarities between the student's work and thinking and another student's work or thinking on the same problem.

(T5) "She (Amy) thought like me (laughing) and went to degrees"

Comparing to Self

Notices and makes reference to differences or similarities between the student's work or thinking and the participant's own work and thinking.

(T3) "So she (Anna) labeled it unknown, and then, OK, I probably should've just done that.

(T2) "Cheyenne I think uses the unit circle, like I did."

To provide an overall account of the participants' attention (noticing without structure or imposed focus), I performed a frequency count of these seven categories. For each case study participant, I created more detailed descriptions of the extent to which the participant's attention on students' work and thinking.

Within the data coded as unprompted or prompted interpretation, I also identified themes that emerged from the data such as Procedural Correctness = Understanding, Procedural Correctness is NOT Understanding, Diagnosing Students' with Their Own Thinking, and Overuse of the Word Understand. I then used these themes to recode the data. I also made sense of the participants' interpretations by coding the data with respect to the resources the participants' drew on to interpret. This is discussed in more detail later in this section.
I made sense of the *responding* data by categorizing the responses by the types of reasoning used by the participants, as identified by Jacobs and colleagues (Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011) - *student’s mathematical thinking, the child’s affect, and general teaching moves* - as well as by the categories of responses identified by Son and Sinclair (2010) - “show” and “tell” versus “give” or “ask.” I also looked for themes that emerged from the data. I discovered potential codes, applied them, and refined them, and arrived at two additional codes/themes: responses that aim to access additional information about the student, and responses that aim to have a student realize X. In Table 3.7, for each of the eight codes I include: (i) a description of how I recognized data that fell in this category, (ii) some key words which often correlated with the code, and (iii) an example or examples of data that fell within this category.

**Table 3.7.**
*Description of Codes Applied to Participants’ Instructional Responses*

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
<th>Sample/Key Words Or Phrases</th>
<th>Example(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Focus on Student’s Thinking</td>
<td>Deciding to respond on the basis of the participants’ interpretations of the student’s thinking.</td>
<td>NA</td>
<td>(T5) I might ask her (Alexis) if she can explain what she’s looking for here, what the problem is asking her to do, umm, because she doesn’t seem to know that she’s looking for an angle, and that’s gonna be the first step is to make sure she knows she’s looking for an angle.</td>
</tr>
<tr>
<td>Affect</td>
<td>Deciding to respond in order to support or improve a student’s confidence.</td>
<td>Good job; almost there;</td>
<td>(T3) So I would just, um, praise her (Anna) for um setting it up right.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(T4) You know of course, tell him (Mike) he did a good job getting it started</td>
</tr>
<tr>
<td>General Move</td>
<td>Response made to the individual or the group of students with little specificity; response is not specific to the problem or the student.</td>
<td>ask to explain; share with each other; explain your work;</td>
<td>(T1) Charlie I think I would ask him what was he thinking as in, how did he approach this and what does he need a boost to help him to figure it out, a recap of what they went over. (T2) Martha I would just ask to have her explain it</td>
</tr>
</tbody>
</table>

**TYPES OF RESPONSES**  
(Son & Sinclair, 2010)

| Code | Description | Sample/Key Words Or Phrases | Example(s)  
(T# represents task number) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Show and Tell</td>
<td>Response delivers verbal or nonverbal information to the student to hear or see – explaining a procedure, telling a fact, showing an image or example, etc.</td>
<td>Tell; show; explain; say;</td>
<td>(T5) Alexis first thing first is I’d have to explain that that is um, that this does not mean that it’s like the whole function to the negative one, so that that is not, um, what this means.</td>
</tr>
<tr>
<td>Give or Ask</td>
<td>Response involves the student delivering verbal or nonverbal information – to do something with a manipulative, image, example or question that is given to the student.</td>
<td>Ask; Give; Have her draw; have him look at;</td>
<td>(T5) Um, Joe, I would ask him to elaborate more if he could on why he, why this works. (T2) Um, my next move for Alex would be um, try out this relationship. Maybe check out what is the sine of 20, what is the sine of 70, do they add up to sine of 90?</td>
</tr>
</tbody>
</table>

**GOALS OF THE RESPONSES**

| Code | Description | Sample/Key Words Or Phrases | Example(s)  
(T# represents task number) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Aims to Access Additional Information</td>
<td>The purpose of the response is to further the teacher’s knowledge of the student’s strategy or thinking. These responses often involve asking a direct question or prompting a student to complete an additional task so that the teacher can test a hypothesis.</td>
<td>why; how come; what’s her reasoning; where did she come up with;</td>
<td>(T3) I’d probably ask Joe to explain to me exactly what he's thinking so I can see more of where he's coming from, (T4) Uh Alice I would ask what she means by this.</td>
</tr>
<tr>
<td>Aims to Have Student Realize</td>
<td>The purpose of the response is to have the student realize (without explicitly telling them) that an aspect of their work is incorrect or incomplete or to have the student realize/discover</td>
<td>Realize; discover; he’d say “oh!”;</td>
<td>(T1) I might say to Allen um, how many radians are there in a whole circle, just to kinda point him in that direction to help him see this error (T2) Um, my next move for Alex would be um, try out this</td>
</tr>
</tbody>
</table>
For each of the case study participants, I performed a frequency count of all categories, which are included in the results section. Further, for each of my case studies, I created more nuanced descriptions of their responses as well as how his or her responses were or were not a reflection of his or her interpretations. As I did with the interpreting data, I also made sense of the participants’ responses by coding them with respect to the resources the participants’ drew on to respond. This is discussed in more detail below.

As I mentioned, I also coded data that showed evidence of participants’ drawing on particular resources to notice. I used my literature review to identify what types of resources the participants were drawing on. These included Subject Matter Knowledge, Knowledge of Content and Students, Knowledge of Content and Teaching, Knowledge of Content and Curriculum, and Beliefs and Orientations. I also used open coding techniques looking for resources that emerged from the data. As I mentioned previously, I found multiple instances in which differences in participants’ language seemed to be resulting in differences in noticing. After reevaluating the literature, I found that one way to make sense of the data was using the idea of interpretive discourse (van Es, 2011). Thus I recoded the data incorporating Interpretive Language as a potential resource for noticing. I also acknowledged and coded instances in which Limited Subject Matter Knowledge Limited Noticing and instances in which participants drew on their Own Work and Thinking. To help create descriptions of how the participants drew on each resource, I first coded data within each type corresponding to whether the participant drew upon the given resource To Evaluate, To Interpret, or To
Respond. For each case study, I also created written descriptions of how this particular resource informed their interpretations and responses.

**Analysis: Research Question 2 - In what ways does the mathematical work presented to the preservice, in-service, and student teachers challenge and/or further their mathematical thinking?**

To address the second research question, I began by identifying any interview or reflective writing data that indicated that a participant was *Engaging in Mathematical Work or Thinking*. I further distinguished between instances of engagement *Prior to Seeing Student Work* and *After Seeing Student Work*. Within the data coded as *After Seeing Student Work*, I coded for instances of *Cognitive Conflict*, *Thinking is Challenged by Students’ Work* and/or *Expanded Thinking as a Result of Student Work*. In each of these instances, which were not frequent, I then created hypotheses as to HOW each of the case studies’ previous understanding interacted with and was influenced by the students’ work. Unfortunately, I did not observe many instances of expanded thinking. Later, I elaborate on this, and in particular, hypothesize why this might have occurred using the interview data and their reflective responses, and I discuss what this means for teacher preparation and development programs.

**Overall Analytic Framework**

In this analysis section, I have discussed how I characterized and made sense of the participants’ noticing of students' work and thinking as well as the relationship between various resources, especially mathematical knowledge for teaching, and one’s noticing of students' work and thinking. In Figure 3.16, at the end of this chapter, I provide an overall summary, or analytic framework, which guided my analysis and allowed me to answer the research questions.
Case Studies

As I mentioned previously, I chose to provide more nuanced descriptions of the mathematical noticing and mathematical engagement of five particular participants whose mathematical knowledge was sufficient to engage with the students’ work in a meaningful way. In this way, I do not use case studies as a methodology, but rather, as a heuristic, or “an approach that focuses one’s attention during learning, construction, discovery or problem solving” (VanWynsberghe & Khan, 2007, p.2). My research questions aim to investigate how teachers notice and how teachers engage with mathematics in response to students’ work, and these how questions warrant thick descriptions of connections between individuals thinking about concepts, what they notice, how they interpret, and how they respond. A case study approach allowed me to make sense of these relationships and provide such detailed descriptions.
Figure 3.16. Analytic Framework
CHAPTER IV
RESULTS: TRIGONOMETRIC KNOWLEDGE

In this chapter, I present results with respect to participants’ overall trigonometric knowledge. First, I present the overall performance of all 19 participants on the trigonometry assessment. I then share overall results with respect to patterns in the participants’ understanding of (i) radian, (ii) unit circle, and (iii) the sine and cosine functions. Recall, both the trigonometry assessment and the interview tasks targeted these concepts because they are central ideas that exist throughout most trigonometry curricula. In my discussion of the participants’ understanding of these target concepts, I share data from the trigonometry assessment as well as the interview data to support my findings. These results address my preliminary research question. The results reported in this section help situate the interview participants, especially the case studies, within the larger sample. I report findings with respect to the main research questions in Chapters V and VI.

Results of Trigonometry Assessment: An Overall Quantitative Report

As I have mentioned, two researchers (including myself) analyzed the participants’ responses to assessment items 1, 2a, 3, 4a, 4b, 5ab, 5c, 6a, and 6b, (with 86.55% initial agreement and 100% final agreement) using rubrics that are provided in Appendix ?. The final scores awarded to each item for each participant are provided in Table 4.1 along with their total scores and percentage of points awarded. Recall that all items except 4a were scored out of three points, and 4a was scored out of one point. Thus, the total possible number of points that could have been awarded was 25. In Table 4.1, an asterisk next to a name indicates those who participated in the interview
phase of my research, and the names with two asterisks indicate case study participants.

Table 4.1.  
Participants’ Scores on Trigonometry Assessment Items (*Interview Participants)

<table>
<thead>
<tr>
<th>Name</th>
<th>1</th>
<th>2a</th>
<th>3</th>
<th>4a</th>
<th>4b</th>
<th>5ab</th>
<th>5c</th>
<th>6a</th>
<th>6b</th>
<th>Total</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tara PST</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>32%</td>
</tr>
<tr>
<td>Kelly PST</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>10</td>
<td>40%</td>
</tr>
<tr>
<td>Molly PST</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>15</td>
<td>60%</td>
</tr>
<tr>
<td>Zola PST</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>32%</td>
</tr>
<tr>
<td>Arianna PST</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>12%</td>
</tr>
<tr>
<td>Maggie PST</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>16%</td>
</tr>
<tr>
<td>Jess PST</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>16%</td>
</tr>
<tr>
<td>Todd PST</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>7</td>
<td>28%</td>
</tr>
<tr>
<td>Hunter PST</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>20%</td>
</tr>
<tr>
<td>Fred PST</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>9</td>
<td>36%</td>
</tr>
<tr>
<td>Toby PST</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>16</td>
<td>64%</td>
</tr>
<tr>
<td>Mary ST*</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>10</td>
<td>40%</td>
</tr>
<tr>
<td>Connie** ST</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>16</td>
<td>64%</td>
</tr>
<tr>
<td>Elliot** ST</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>23</td>
<td>92%</td>
</tr>
<tr>
<td>Sarah** IST</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>20</td>
<td>80%</td>
</tr>
<tr>
<td>Pat **IST</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>12</td>
<td>48%</td>
</tr>
<tr>
<td>Lana** IST</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>15</td>
<td>60%</td>
</tr>
<tr>
<td>Craig** IST</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>21</td>
<td>84%</td>
</tr>
<tr>
<td>Byron** IST</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>20</td>
<td>80%</td>
</tr>
</tbody>
</table>

I also compared the participants’ overall scores by subgroups of participants in Table 4.2. It is important to note, that while it was helpful for me to compare these subgroups to make sense of my participant sample and the distribution of levels of trigonometric knowledge, I recognize that due to the small sample size and other factors (such as the fact that the individuals who agreed to participate might have done so because they were more familiar with the content), these comparative results are not necessarily indicative of the larger population.
Table 4.2. Summary of Scores on Trigonometry Assessment: Comparing Subgroups of Participants

<table>
<thead>
<tr>
<th>Participants (n)</th>
<th>Mean (%)</th>
<th>Median (%)</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>All (19)</td>
<td>48%</td>
<td>40%</td>
<td>12%</td>
<td>92%</td>
</tr>
<tr>
<td>Preservice Teachers (11)</td>
<td>32.4%</td>
<td>32%</td>
<td>12%</td>
<td>64%</td>
</tr>
<tr>
<td>Student Teachers (3)</td>
<td>65.33%</td>
<td>64%</td>
<td>40%</td>
<td>92%</td>
</tr>
<tr>
<td>Inservice Teachers (5)</td>
<td>70.4%</td>
<td>80%</td>
<td>48%</td>
<td>84%</td>
</tr>
<tr>
<td>Participants without Precalculus Teaching Experience (16)</td>
<td>40.5%</td>
<td>38%</td>
<td>12%</td>
<td>80%</td>
</tr>
<tr>
<td>Participants with Precalculus Teaching Experience (3)</td>
<td>85.33%</td>
<td>84%</td>
<td>80%</td>
<td>92%</td>
</tr>
<tr>
<td>Student Teachers and Inservice Teachers (Excluding PSTs) without Precalculus Teaching Experience (5)</td>
<td>68.4%</td>
<td>60%</td>
<td>40%</td>
<td>80%</td>
</tr>
<tr>
<td>Student Teachers and Inservice Teachers (Excluding PSTs) with Precalculus Teaching Experience (3)</td>
<td>85.33%</td>
<td>84%</td>
<td>80%</td>
<td>92%</td>
</tr>
<tr>
<td>Preservice Teachers</td>
<td>32.4%</td>
<td>32%</td>
<td>12%</td>
<td>64%</td>
</tr>
<tr>
<td>Student Teachers and Inservice Teachers without Precalculus Teaching Experience</td>
<td>68.4%</td>
<td>60%</td>
<td>40%</td>
<td>80%</td>
</tr>
</tbody>
</table>

I first compared Preservice Teachers versus Student Teachers, and Inservice Teachers. Each subgroup scored higher than the next with mean scores of 32.4%, 65.33%, and 70.4% respectively. I then excluded in-service teachers and student teachers who had experience teaching precalculus, and preservice teachers still scored far lower (mean score of 32.4%) than the remaining student teachers and in-service teachers who had no experience teaching precalculus (mean score of 68.4%). While I expected participants with precalculus teaching experience to score highest (mean
score of 85.33%), I assumed that preservice teachers in their third or fourth years of their undergraduate mathematics education program would have seen and engaged with trigonometry content more recently than in-service and student teachers who taught in other content areas, but this was not the case. This result may just be representative of my sample. It may suggest something about the perseverance, mathematical awareness, flexible procedural fluency, and/or general problem solving skills gained through teaching; however it might also suggest that trigonometry concepts are more pervasive in the high school curriculum at the schools of these participants. It is important to note that it is likely that the preservice teachers’ knowledge and procedural fluency would have improved prior to completing their degree program since all but one were currently enrolled in a content course designed to engage them in in-depth analysis of high school mathematics content, including trigonometry, and the assessment was administered prior to their work with trigonometry.

Table 4.3. *Mean Scores for Each Item on Trigonometry Assessment*

<table>
<thead>
<tr>
<th>Item #</th>
<th>1</th>
<th>2a</th>
<th>3</th>
<th>4a</th>
<th>4b</th>
<th>5ab</th>
<th>5c</th>
<th>6a</th>
<th>6b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Score</td>
<td>2.16</td>
<td>1.21</td>
<td>1.37</td>
<td>0.74</td>
<td>1.26</td>
<td>1.26</td>
<td>0.42</td>
<td>2.21</td>
<td>1.26</td>
</tr>
</tbody>
</table>

Mean scores were also calculated for each assessment item, which are provided in Table 4.3. Item 4a was worth one point, while all of the other items were worth three points. Recall, Fi (2003) found that most preservice teachers could convert between radian and degree measurements, and thus it was not surprising that the participants scored relatively high on the first item. That is, participants were successful at converting between radian and degree measure. The participants were also relatively successful on item 6a, identifying the domain, period, and symmetry of the cosine function.
function. Students had the most difficulty responding to 5c, which required them to draw on knowledge of sine and cosine as cofunctions. In fact, 15 out of 19 participants were awarded 0 points for their response on this problem. This is consistent with Fi’s (2003) findings; in particular, he found that none of the fourteen preservice teacher participants in his study had deep knowledge of cofunctions. I elaborate on typical types of responses and successes and challenges the participants had in responding to the items in the following sections in order to contribute to an overall picture the participants’ knowledge of radian, unit circle, and the sine and cosine functions.

**Participants’ Knowledge of Radian**

Radian measure is valued as a “powerful and versatile measure of angles that is not encumbered by unit of measurement” (Fi, 2003, p.94), and those with richer understandings of radian measure have been shown to be more likely to use the unit circle and establish richer connections between the unit circle and other concepts in trigonometry (Akkoc, 2008; Topcu, Kertil, Akkoc, Yilmaz & Onder, 2006). However, research has shown that many preservice and in-service teachers (i) cannot define radian correctly as the ratio of two lengths or as the length of the arc on the unit circle, (ii) have concept images of radian dominated by concept images of degree, and (iii) believe that radian measures are only given in terms of the value or symbol pi (Akkoc, 2008; Fi, 2003; Topcu, Kertil, Akkoc, Yilmaz & Onder, 2006). Again, these issues are problematic as radian measure serves as a common foundation between various trigonometry contexts (Moore, 2009). In the following sections, I compare my own results to the results of these previous studies (Akkoc, 2008; Fi, 2003; Topcu, Kertil, Akkoc, Yilmaz & Onder, 2006), acknowledging the aforementioned common
misconceptions identified by the researchers as well as the ways in which the participants successfully viewed and worked with radian measure.

**Defining Radian**

The Common Core State Standards for Mathematics (CCSSM, 2010) state that students should “understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.” When asked to define radian on the assessment, none of the participants provided this particular unit circle definition on the assessment; however, two participants, Elliot and Byron, did define radian correctly as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle. In particular, Elliot, a student teacher with precalculus teaching experience, defined radian as “the measure of the central angle of a circle that intersects an arc length equal in length to the radius of a circle.” Byron, an in-service teacher with 15 years of precalculus teaching experience, defined radian as an “angle created/subtended by ‘wrapping’ 1 radius on the circumference,” and he provided the drawing in Figure 4.1 to clarify this definition:

![Figure 4.1. Byron’s Definition of One Radian.](image)

In addition, during the interviews both Elliot and Byron recognized radian as the length of the arc on the unit circle subtended by the angle. Elliot and Byron exhibited stronger knowledge of radian appeared stronger than the majority of participants in previous studies (Akkoc, 2008; Fi, 2003; Topcu et al., 2006). Recall that in Fi’s (2003) study, none of the 14 preservice teachers had provided a correct definition of radian, in
Akkoc’s (2008) study only one out of 42 preservice teachers correctly defined radian, and in Topcu and colleagues’ study (2006), none of the 37 preservice or fourteen in-service teachers defined radian correctly.

Further, while they did not define radian correctly, an additional four out of 19 participants in this study did recognize a connection between radian and “length.” For example, Molly defined radian as “A form of measurement used in the unit circle to measure length.” Thus, Molly acknowledged a connection between radian measure and length (although she did not specify what length), and she also thought of radian measure in relation to the unit circle. Another participant, Connie stated that, “I think it comes from measuring arc length or circumference on a unit circle.” That is, Connie did not specifically define radian as the arc length on the unit circle, but she did seem to recognize a connection between the two. The remaining participants defined radian (i) generally as “unit of measure” or “angle measure,” (ii) in relation to degrees, or (iii) as $1/2\pi$ of a circle.

In summary, two of the nineteen participants in this study correctly defined radian in relation to arc length. Both of these participants defined radian as a ratio and also recognized radian as arc length on the unit circle. Further, four additional participants in this study recognized a connection between radian and “length.” The remaining thirteen participants had weaker conceptions of radian. A summary of how the participants defined radian is provided below in Table 4.4.
Table 4.4.  
*Participants’ Definitions of Radian*

<table>
<thead>
<tr>
<th>Defined Radian:</th>
<th>Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>As the ratio of Two Lengths: the length of the arc subtended by the angle and the radius of the circle</td>
<td>Elliot Byron</td>
</tr>
<tr>
<td>As Having to do with length…</td>
<td>Molly Toby Connie Mary</td>
</tr>
<tr>
<td>As a unit of measure or angle measure</td>
<td>Tara Kelly Zola Arianna Maggie Jess Todd Hunter Lana</td>
</tr>
<tr>
<td>In Relationship to Degrees</td>
<td>Pat Sarah</td>
</tr>
<tr>
<td>As 1/2pi of the unit circle</td>
<td>Craig</td>
</tr>
<tr>
<td>None</td>
<td>Fred</td>
</tr>
</tbody>
</table>

**Relationship between Radians and Degrees**

Consistent with Fi’s (2003) findings, most participants (15 out of 19) accurately converted (with the exception of converting 19 radians to degrees) between radians and degrees on item one of the assessment. Further all but two of these 15 participants appeared to use a process that would continue to result in correct conversions in other instances. Nine of the participants who converted correctly multiplied by $180/\pi$ to convert from radians to degrees and/or multiplied by $\pi/180$ to convert from degrees to radians, as is shown in Byron’s work in Figure 4.2. In contrast, one participant, Fred, converted using proportions as is shown in Figure 4.3. Three other participants replaced $\pi$ with 180 in order to convert from radians to degrees and when given degree measures, they broke the angle down into the sum or fraction of familiar radian measures. For example, Toby’s work is shown Figure 4.4. This method seemed to
demonstrate an understanding of the relationship between degrees and radians; however two out of the three participants who used this method did not convert 19 radians to degrees. Since there was no $\pi$ involved, their standard method of replacing $\pi$ with 180, did not work in this case.

Figure 4.2. Byron's Work on Assessment Item One

Figure 4.3. Fred's Work on Assessment Item One

Figure 4.4. Toby's Work on Assessment Item One
Two additional participants converted correctly (with the exception of converting 19 radians to degrees), but it was difficult to determine from their work which method(s) they were using to convert. For example, Pat used a proportion in one instance, and didn’t show adequate work for the others. One participant, Mary could convert correctly, but provided incorrect answers because of an incorrect belief that coterminal angles are equivalent. The remaining four participants, all preservice teachers did not convert correctly; however three of these participants at least knew the relationship between radians and degrees, i.e., 360 degrees is equivalent to $2\pi$ radians or 180 degrees is equivalent to $\pi$ radians. For example, as is shown in Figure 4.5, Hunter knew that 360 degrees is equivalent to $2\pi$ radians, but he wasn’t able to use this fact to convert. Arianna was the only participant who did not seem to know the relationship between radians and degrees; she only appeared to know that the conversion involved “180,” as is shown in her work in Figure 4.6.

![Figure 4.5. Hunter's Work on Assessment Item One](image)

![Figure 4.6. Arianna's Work on Assessment Item One](image)
Recall that Akkoc (2008) argued that many of his participants’ concept images of radian were dominated by degree because they consistently converted from radians to degrees before solving problems. While it was difficult to determine this from the assessment alone, since the problems provided angles in degrees, several interview participants in my own study showed preference for working with degrees, and in some cases, could only work with degrees. For example, when working on Task Five trying to think about \(-11\pi/4\) radians, Jess, a preservice teacher, stated that she “can’t really draw that.” She wasn’t able to convert to degrees herself, but when she saw Amy’s work and her conversion to degrees, Jess was then able to draw the point on the unit circle that corresponds to the angle, \(-11\pi/4\) radians, shown below in Figure 4.7. Note that she also drew the radius corresponding to the angle coterminal with \(11\pi/4\) radians.

![Figure 4.7. Jess’s Work in Task Five](image)

Further, Jess explicitly stated:

> I thought about doing it, but then I couldn't remember (incomprehensible), which I should be able to remember 180 over pi, but, so I think it's a, it's an easier way to see it without like the pi and all the other numbers. Sometimes it's just easier to see as a um, ang- uh degree.

In the above excerpt, Jess admitted that she couldn’t convert to degrees but would have if she could have because it is easier to “see” the angle. In a related situation, when asked about the hypothetical students’ understanding of radian in Task One, Jess stated, “I feel like I can’t really tell how much they know about radians, because there’s
no degree to radian flipping back and forth. I mean I think they understand that radians are another way for degrees.” That is, Jess argued that the evidence of one’s understanding of radian lies in their ability to convert between radian and degrees. Thus, it appeared that Jess only understood radian in terms of its relationship to degrees.

Sarah’s concept image of radian also appeared to be dominated by degree. She defined radian only in terms of its relationship to degree measure, as seen in Figure 4.8.

![Figure 4.8. Sarah’s Definition of Radian.](image)

Further, during the interviews, Sarah consistently converted radians to degrees, explaining “I just gotta convert it into degrees because I think better in terms of degrees.”

In summary, similar to Akkoc’s (2008) and Fi’s (2003) studies, most participants (15 out of 19) in the current study could correctly convert between radians and degrees. To convert correctly, participants (i) multiplied by conversion factors $\frac{180}{\pi}$ and/or $\frac{\pi}{180}$, while others (ii) set up a proportion and solved or (iii) wrote the angle as the sum of or multiple of familiar angle measures and interchanged radian and degree measures of those familiar angles. Of those who could not convert correctly, only one did not acknowledge that $\pi$ radians was equivalent to 180 degrees; the others acknowledged this but did not convert correctly or provided incorrect answers because of an incorrect belief that coterminal angles are equivalent. In addition, consistent with Akkoc’s (2008) findings, several participants showed a preference for degree over radian.
Radian and Pi

Recall that in Akkoc’s (2008) study, the majority of preservice teacher participants considered angles to be given in radians only when they included \( \pi \). Consistent with this finding, in my own study, I found that several participants struggled to convert 19 radians into degrees. Recall that in the first assessment item, I asked the participants to convert radians to degrees and vice versa, and the angles given were \( \frac{5\pi}{2}, \frac{3\pi}{5}, 19, 50^\circ, \) and \( 570^\circ \). Participants appeared confused by the fact that 19 did not have a degree symbol, nor did it include \( \pi \). For example, as seen in Figure 4.4, Toby could convert \( \frac{3\pi}{5} \) radians into degrees, but he said that “19 is a number since there is no degree symbol,” and then didn’t convert in any way. Similarly, Maggie simply wrote a “?.” These participants appeared tentative about radian measures that did not involve \( \pi \). Similarly, on the assessment, when explaining the relationship between radian measure and degree measure, Mary wrote that a “radian is the angle measure in terms of \( \pi \).” Conversely, other participants appeared to believe that angle measures involving \( \pi \) must be in radians. For example, Kelly, a preservice teacher shared this line of thinking and provided her rationale in the following excerpt, during her analysis of Lexi’s work on Task One:

Kelly: I think it’s interesting how she said there’s no real symbol for radians. Um, when you have pi over four, then you know that it's in radians already.

Interviewer: How so?

Kelly: Um, because you are in polar form, so that's just, it's the pi in polar form shows that it's in radians

In the above excerpt, Kelly pointed out that the reason Lexi should know the angle measure is given in radians is because it involves \( \pi \), and thus must be in polar form,
which shows it’s in radians. It should be noted that no other participants referenced polar form in their discussions of radian.

In summary, eight out of 19 participants did correctly convert 19 radians into degrees, and thus, these participants most likely do not perceive radians to only be given in terms of $\pi$. In contrast, like the participants in Akkoc’s (2008) study, several participants in this study considered angles to be given in radians only when they included $\pi$. This is not surprising given the fact that students and teachers almost always work with radian measures involving $\pi$; however this sort of thinking does indicate a limited understanding of radian as a real number and/or a limited understanding of $\pi$. Further, this concept image can prohibit an individual’s understanding of trigonometric functions defined on all real numbers as well as an understanding that the unit circle enables the extension of trigonometric functions to all real numbers.

To conclude this section regarding the participants’ knowledge of radian, in Table 4.5, I provide an overall summary of how participants defined radian as well as which participants: (i) exhibited knowledge of the relationship between radians and degrees, (ii) could convert correctly between radians and degrees, (iii) converted 19 radians correctly into degrees (and thus, are not restricted to radian measures involving $\pi$), and (iv) seemed to exhibit a concept image of radian dominated by degree either because they defined radian in terms of degree measure, could only work with degree measures, or frequently converted radians into degrees before attempting a problem.
<table>
<thead>
<tr>
<th>Name</th>
<th>Defined Radian:</th>
<th>Exhibited Knowledge of the Relationship between radians and degrees</th>
<th>Converted Correctly (excluding 19 radians)</th>
<th>Correctly Convert 19 radians to degrees</th>
<th>Degree dominated Radian</th>
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<td>✓</td>
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<td></td>
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<td>✓</td>
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<td></td>
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<tr>
<td>Molly</td>
<td>As a unit of measure or angle measure</td>
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<td>✓</td>
<td>✓</td>
<td></td>
</tr>
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<td>Zola*</td>
<td>As a unit of measure or angle measure</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Arianna</td>
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<td>✓</td>
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<td>Maggie</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Todd</td>
<td>As a unit of measure or angle measure</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Hunter</td>
<td>As a unit of measure or angle measure</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fred</td>
<td>No Definition Provided</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Toby*</td>
<td>As Having to do with length…</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mary*</td>
<td>As Having to do with length…</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Connie**</td>
<td>As Having to do with length…</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Elliot**</td>
<td>As the ratio of Two Lengths: the length of the arc subtended by the angle and the radius of the circle</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Sarah**</td>
<td>In Relationship to Degrees</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Pat *</td>
<td>In Relationship to Degrees</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lana*</td>
<td>As a unit of measure or angle measure</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Craig**</td>
<td>As 1/2pi of the unit circle</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Byron**</td>
<td>As the ratio of Two Lengths: the length of the arc subtended by the angle and the radius of the circle</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Participants’ Knowledge of the Unit Circle

The unit circle is a central concept of trigonometry that serves as an important tool for reasoning about trigonometric functions, arc length, and angle measure. Further, research has shown that students’ success working with trigonometric functions can be attributed to their use of the unit circle (Weber, 2005). However, research has also shown that typically, students’ and teachers’ use and understandings of the unit circle are superficial (Fi, 2003; Moore, LaForest, & Kim, 2012). For example, Fi (2003) found that while most preservice teachers could identify the unit circle as a circle with radius of one, they had weak understandings of the functionality of the unit circle. In the following sections I discuss how my own findings compare with these findings. I share alternate definitions provided by the participants, as well as the reasons they felt the unit circle is taught in high school. I also discuss the participants’ tendency to view the unit circle as a sort of fixed diagram or reference circle that students memorize or are given to solve trigonometry problems.

Defining Unit Circle

On part (a) of item four on the assessment, the participants were asked, “What is a unit circle? Explain.” Consistent with Fi’s (2003) findings, the majority of participants correctly defined the unit circle. In particular, fourteen out of 19 participants defined the unit circle as a circle with a radius of one. Additional responses included: “circle we use with trig identities that can be measured with radians or degrees”; “used to help students remember their units and signs. Used to evaluate trig functions”; “shows all the sine (and/or cosine) values at various points along the circle (different angle)”; and “a circle measured in radians. For example π, 2π.” As is shown in these responses, many
of the participants who did not define the unit circle as a circle with radius of one tended to provide a response that was not technically a definition, but rather, a description of the circle’s functionality, which will be discussed in a later subsection. These participants’ responses may be a result of misunderstandings about what constitutes a mathematical definition, misunderstandings of the unit circle, or both.

**Unit Circle as a Fixed Reference/Diagram to Be Memorized**

There was evidence throughout both the assessment and interview data that seven out of 19 participants' thought of the unit circle as a sort of fixed diagram or reference circle that students or teachers are either given or need to memorize. In particular, it appeared that these participants believed that the unit circle is a diagram/picture of a circle (with or without a radius of one) that lists certain angles as well as their sine and cosine values. For example, when asked to define unit circle, Mary wrote, “A unit circle shows all the sine (and/or cosine) values at various points along the circle (different angles).” Mary also elaborated on her thinking during the first interview. The following is an excerpt taken from this first interview, in which Mary is discussing Cheyenne’s work on Task Two.

**Mary:** And then for Cheyenne I think, um, I'm thinking this might be a lower level student, just by like the kind of work she's doing. Um, so maybe just bring the unit circle out, so she can look at it. See where she went wrong. Um...

**Interviewer:** So by showing her the unit circle, what, like what exactly would you be showing?

**Mary:** To see, I mean that she just got these values wrong, that sine of 30 is not that; it's one half, and cosine is blah blah blah. They're backwards. Um..

**Interviewer:** So would the, would it be labeled on the unit circle or she would just?
Mary: Yeah I guess. I'm making assumptions that they have some sort of unit circle that they've been memorizing

In the above excerpt, the interviewer was first confused by and interested in Mary’s statement, “maybe just bring the unit circle out.” After asking her to elaborate, Mary eventually clarified that by “the unit circle” she meant a circle with sine and cosine values on it, assuming that they have this “sort of unit circle that they've been memorizing.”

One participant, Elliot shed light on this way of thinking about the unit circle. In the following excerpt, Elliot was referring to Alice’s work in Task Four (See Appendix F); recall that Alice had (hypothetically) been asked to use the unit circle to graph the cosine function. Also, note that the capitalized and italicized A and THE in this excerpt indicate emphasis Elliot placed on these words.

Here she used a circle. Like, and I guess it is the unit circle in the fact that uh she has a one here and a negative one here, um, referring to the x-value. So it does have a radius of one. So it is A unit circle. I, I just think of when you ask THE unit circle, you're asking for kind of the coordinates or angles, or at least one of the two, not just like a random...

Elliot seemed to hold two conflicting ideas of the unit circle: (1) a circle with radius one and (2) a sort of reference circle (or picture) which lists sine and cosine values for specific reference angles. Elliot seemed to acknowledge the latter not as a definition but as part of a contract between teacher and students – “when you ask THE unit circle, you’re asking for kind of the coordinates or angles.” Similarly Pat recalled being asked to memorize the unit circle. In particular he stated that Mike was “memorizing the unit circle, which I remember being asked to do.”
In summary, seven out of 19 participants seemed to perceive the unit circle as a circle (with or without a radius of one) that (i) lists certain angles and/or their sine and cosine values and (ii) is given to or recalled by students. While I would argue this is a perfectly rational belief for these participants to hold given their likely experiences “memorizing the unit circle,” I would also argue that we should aim to expand or shift these beliefs. Recall that Weber (2005) attributed students’ success working with trigonometric functions to their use of the unit circle; however he argued that not all approaches toward teaching with the unit circle result in successful student learning. For example, Kendall and Stacey (1997) found that students who were taught trigonometry using a unit circle model learned less than those students who learned with a right triangle model. Weber (2005) argued that the key to successful student learning using the unit circle was that the students use the unit circle as a tool for reasoning about trigonometric functions and understand the process of and take part in the process of constructing the unit circle and its relationship to trigonometric functions. Thus, I would argue that it is important to persist in shifting and expanding teachers’ current perceptions of the unit circle as (i) a **diagram that lists or shows** and is **given** to students toward perceptions of the unit circle as (ii) a **tool for reasoning** that is **constructed** by students. Moore’s (2010) research suggests that one way to support such a perception is by first supporting students’ understanding of radius as a unit of measure.

**Functionality of the Unit Circle**

The Common Core State Standards for Mathematics (CCSSM, 2010) state that students should be able to (i) “explain how the unit circle in the coordinate plane
enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle” and to (ii) “use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.” Fi (2003) found that while many preservice teachers could define the unit circle, they had weak understandings of the functionality of the unit circle. In particular, Fi (2003) argued that they “did not express the encapsulating power of the unit circle” relative to trigonometric functions of angles greater than 90°, clockwise rotations, periodicity, coterminal angles, proof of identities and formulas, and the notions of even and odd functions” (p. 118). Similarly, the majority of the participants in my own study did not discuss these aspects of “the encapsulating power of the unit circle”; however, two participants’ responses did indicate a more impressive understanding of its functionality.

Craig, an in-service teacher with precalculus experience, noted that the unit circle is “used to understand periodic behavior.” Craig also noted its importance in the “proof of \( \cos \left( \frac{n\pi}{3} \right), \cos \left( \frac{n\pi}{2} \right), \cos \left( \frac{n\pi}{4} \right), \) and \( \cos \left( \frac{n\pi}{6} \right) \).” By including the “n”, it seems Craig had, at least to some extent, indicated the role of the unit circle in extending sine and cosine to all real numbers. Byron, the only other in-service teacher with precalculus teaching experience, explained how one could use the unit circle to prove the Pythagorean Identity. In particular, Byron explained that “if the radius is 1” then “(x,y)” becomes “\((1\cos\theta, 1\sin\theta)\),” and that since a “point consists of an x distance, y distance,” then “\(1 = x^2 + y^2\),” and so therefore, “\(1 = \cos^2(\theta) + \sin^2(\theta)\).” While Byron appeared to use circular reasoning (proving the Pythagorean theorem using the distance formula despite the fact that the distance formula is derived from the Pythagorean theorem) both Craig
and Byron, at least to some extent, seemed to perceive the unit circle as a tool for reasoning about sine and cosine.

There were participants who did not specifically discuss how the unit circle could be used to reason about trigonometric functions but did comment that it was useful because \( \cos(\theta) \) is represented either as the \( x \)-coordinate or horizontal length or that \( \sin(\theta) \) is represented either as the \( y \)-coordinate or vertical length. For example, when asked about why the unit circle is taught in high school, Connie drew the following:

![Figure 4.9. Connie’s Response to Item 4b](image)

Other participants indicated that the unit circle was useful for: (i) finding angles, (ii) memorizing/evaluating values of trigonometric functions, and/or (iii) converting from radians and degrees. Many of these comments seemed rooted in their interpretations of the unit circle as a sort of fixed diagram that lists information, as was discussed in the previous section. For example, Mary defined unit circle as a “circle that shows sine and cosine values,” and then stated that it is taught as a “convenient way to remember oft-used angles’ sin/cos values,” and Jess defined unit circle as “a circle with the radius of 1 that gives the radians and degrees” and then stated that it is taught, “to represent radians compared to degrees.”

In summary, the two in-service teachers with precalculus teaching experience exhibited knowledge of the importance of the unit circle in understanding the periodicity of the trigonometric functions, evaluating trigonometric functions at angles greater than
90 degrees, and in proving trigonometric identities. That is, they discussed the unit circle as a tool for reasoning about trigonometric functions. Consistent with Fi’s (2003) findings, however, the majority of participants did not exhibit knowledge of the functionality of the unit circle. In particular, most (ten of 19) argued that the unit circle was important for remembering/evaluating trigonometric functions and converting between radians and degrees.

To conclude this section regarding the participants’ knowledge of the unit circle, I have included Table 4.6, which summarizes the ways in which the participants’ perceived the unit circle. I did not distinguish between how the participants defined unit circle versus how or why they listed it was important in teaching since many of them discussed usefulness when they had been asked to define the unit circle.
Table 4.6.  
The Participants’ Perception/Knowledge of the Unit Circle

<table>
<thead>
<tr>
<th>Name</th>
<th>Radius of One</th>
<th>Fixed/ Given Diagram</th>
<th>Used to Find Angles</th>
<th>Used to convert Radians ↔ Degrees</th>
<th>Used to Remember / Evaluate Trig Values</th>
<th>Useful since cosine is x, sine is y</th>
<th>Tool for Reasoning about Sine and Cosine</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tara</td>
<td>PST</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Kelly *</td>
<td>PST</td>
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<td>✓</td>
<td></td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>✓</td>
<td>✓</td>
<td></td>
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<td></td>
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<tr>
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<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>✓</td>
<td></td>
<td></td>
<td></td>
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<td>✓</td>
<td></td>
<td></td>
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<tr>
<td>Hunter</td>
<td>PST</td>
<td>✓</td>
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<td>Fred</td>
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<td>✓</td>
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<td></td>
<td></td>
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<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>ST</td>
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<td>✓</td>
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<td>✓</td>
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<td>✓</td>
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<td></td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Byron**</td>
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<td>✓</td>
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<td></td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>

Participants’ Knowledge of the Sine and Cosine Functions

Defining Sine and Cosine

Brown (2006) identified three major ways of knowing sine and cosine: as coordinates of a point on the unit circle, as horizontal and vertical distances on the unit circle, and as ratios of side lengths of a right triangle. On the assessment, five participants exhibited an understanding of sine and cosine as coordinates on the unit circle, two exhibited an understanding of sine and cosine as ratios of the lengths of
sides on a right triangle, and two exhibited an understanding of sine and cosine as vertical and horizontal lengths on a unit circle. Task three during the interviews also specifically revealed whether or not participants understood sine and cosine as ratios of side lengths of a right triangle; all but two of the eleven participants (who participated in the second interview) exhibited correct knowledge of sine and cosine as ratios of the lengths of sides on a right triangle. In addition, the interview prompts, especially Task Four, revealed whether or not participants understood sine and cosine via at least one of the two unit circle definitions. Four participants, Byron, Toby, Craig and Sarah knew sine and cosine as both coordinates and distances. Two participants, Jess and Pat, did not hold either definition.

It was interesting that several participants who understood sine and cosine as coordinates did not seem to exhibit an understanding of sine and cosine as distances and vice versa. In particular, Elliot, Lana, and Zola all exhibited knowledge of sine and cosine as coordinates on the unit circle; however they did not recognize Alice’s strategy on Task Four (See Figure 3.11) in which she considered cosine as the horizontal distance on the unit circle. Conversely, Mary and Connie knew cosine as the horizontal distance on the unit circle; however they were immediately puzzled by Mike’s strategy in Task Four (See Figure 3.10), in which he displayed cosine and sine as the x and y coordinates on the unit circle. Unlike Elliot, Lana, and Zola, however, Mary and Connie were able to eventually make the connection between the two definitions. This will be discussed further in a later section on the mathematics the participants engaged in when attending to the hypothetical student work.
Several participants in this study did not exhibit any of Brown’s (2006) three correct or productive ways of knowing sine and cosine. For example, while Jess drew “the unit circle” to graph cosine during the interviews, when I asked her, “how do you know cosine is zero there?” Jess responded that she had simply memorized the unit circle. She couldn’t tell me what cosine or sine represented on the unit circle, but instead simply seemed to memorize the cosine and sine values based on “the picture.” Jess also had difficulty with the right triangle definition. In particular, even though she recalled “SohCahToa,” when asked to find the sine of theta in the third task during interview two, Jess wrote:

\[
\sin \left( \frac{x}{9.73} \right) = 0
\]

And eventually solved for:

\[
x^2 = 9.73^2 - 4.57^2
\]

\[
x = \sqrt{9.73^2 - 4.57^2}
\]

\[
\sin \left( \frac{\sqrt{9.73^2 - 4.57^2}}{9.73} \right) = 0
\]

As shown above, Jess’s work implies that an angle measure in a right triangle is equal to the sine of the ratio of the opposite side length to the hypotenuse. It is possible that Jess was incorrectly recalling a procedure she had memorized, or Jess might hold a misconception about the domain and range of the sine function. Similarly, Pat, an in-service teacher who also didn’t seem to hold any correct way of knowing sine and cosine, solved for x using the Pythagorean theorem and then stated that “what I would
then do is take the value that I obtained for x and the length of the hypotenuse and use arcsine."

In summary, I examined both the assessment and interview data for instances in which the participants exhibited any of three ways of knowing sine and cosine identified by Brown (2006): as coordinates of a point on the unit circle, as horizontal and vertical distances on the unit circle, and as ratios of side lengths of a right triangle. While four participants did not exhibit any of these three ways of knowing, 15 participants exhibited at least one, and four participants exhibited all three. It was surprising that some participants seemed to acknowledge sine and cosine as lengths on the unit circle but not as coordinates of a point on the unit circle and vice versa. As will be discussed in the case studies, the student work served as a way to make the connection between these two ways of knowing. I will also discuss how holding these different ways of knowing sine and cosine proved to be an important resource for interpreting students’ thinking.

A summary of the ways of knowing sine and cosine exhibited by each participant (throughout their assessment and interviews) is provided below in Table 4.7. Note that those who did not participate in interviews, i.e. those without an asterisk next to their name, were not provided as many opportunities to exhibit their ways of knowing sine and cosine as those who did participate in the interviews.
Table 4.7. 
Ways of Knowing Sine and Cosine Exhibited by Participants

<table>
<thead>
<tr>
<th>Name</th>
<th>Exhibited Knowledge of Sine and Cosine as:</th>
<th>Ratios of Side Lengths of Right Triangle</th>
<th>Lengths on Unit Circle</th>
<th>Coordinates on Unit Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tara</td>
<td>PST</td>
<td>√</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kelly</td>
<td>PST</td>
<td>√</td>
<td>√</td>
<td></td>
</tr>
<tr>
<td>Molly</td>
<td>PST</td>
<td>√</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>Zola*</td>
<td>PST</td>
<td>√</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>Arianna</td>
<td>PST</td>
<td>√</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>Maggie</td>
<td>PST</td>
<td>√</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jess*</td>
<td>PST</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Hunter</td>
<td>PST</td>
<td>√</td>
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</tr>
<tr>
<td>Fred</td>
<td>PST</td>
<td>√</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Toby*</td>
<td>PST</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Mary*</td>
<td>ST</td>
<td>√</td>
<td>√</td>
<td></td>
</tr>
<tr>
<td>Connie**</td>
<td>ST</td>
<td>√</td>
<td>√</td>
<td></td>
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<tr>
<td>Elliot**</td>
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<tr>
<td>Sarah**</td>
<td>IST</td>
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<tr>
<td>Pat*</td>
<td>IST</td>
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<tr>
<td>Lana*</td>
<td>IST</td>
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<tr>
<td>Craig**</td>
<td>IST</td>
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<tr>
<td>Byron**</td>
<td>IST</td>
<td>√</td>
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<td>√</td>
</tr>
</tbody>
</table>

\[ \sin(20^\circ) + \sin(70^\circ) = \sin(90^\circ) ? \]

On the second task in the first interview, participants were given \( \sin(70^\circ) \approx .9397 \) and were asked to find \( \cos(20^\circ) \) and \( \sin(20^\circ) \). Ideally, they would recognize \( \sin(70^\circ) \) is equivalent to \( \cos(20^\circ) \) and then use the Pythagorean theorem to find \( \sin(20^\circ) \); however, three out of twelve interview participants instead found \( \sin(20^\circ) \) by equating \( \sin(90^\circ) \).
with the sum of \( \sin(70^\circ) \) and \( \sin(20^\circ) \) and then subtracting \( \sin(70^\circ) \) from \( \sin(90^\circ) \). For example, Kelly thought aloud, “Well this is 90 degrees, you already have 70; so you have 20 degrees left. So you're able, if you…this so, we already know what uh sine of 90 degrees is, and if you subtract sine of 70 degrees then you get the last proportion in the circle so you'll be able to figure it out.” Four participants who were able to answer this interview prompt correctly and had stronger trigonometric knowledge in general, were still initially uncertain about the validity of hypothetical student Alex’s claim that \( \sin(90^\circ) = \sin(70^\circ) + \sin(20^\circ) \).” For example, Byron who had taught Precalculus for 15 years and had received the third highest score on the assessment first thought aloud, “I feel like this is just not right to me” and then elaborated, “I'm not sure why I feel that way, and I may have the wrong feeling on this.” He seemed to get frustrated, exclaiming, “Ugh, I'm just not, I don't know.” Eventually he realized this claim was incorrect using a counterexample; however it was interesting that this was not immediately clear and that he needed to work through a counter-example to disprove the claim.

In contrast, there were three participants, Elliot, Mary, and Craig who knew immediately that Alex’s claim was false. Mary immediately claimed, “that's not true, because we have to use the formula” and elaborated, “just because the angles add up to 90, doesn't mean that the sine of the angles add up to 90 because, um, they just don't. (laughing) Cuz the pythagorean theorem, like it's, that's not their relationship.” Mary could not explain why Alex’s statement was false; however she did have the intuitive sense that it was not true. Similarly, Craig immediately claimed, “it almost looks
like she wants to have a property that \( \sin(x) + \sin(y) = \sin(x + y) \). Like she wants to distribute sine. Sine is not a multiplier. It’s a function technically, so, or an operation. That’s how I like to explain it to the kids.” While Craig correctly identified Alex’s error, he too was unable to provide an adequate reason for why it was incorrect. In contrast, Elliot’s explained:

he’s changing \( \sin(20) + \sin(70) \) to \( \sin(20 + 70) \) to the sine of 20 degrees plus 70 degrees, and, and that’s cuz, he understands where the angles are, but he doesn’t understand how the sine is affecting it, and I think that goes back to how I was trying to um, explain how our moving goes around the circle […] the sine would move more at the beginning of the circle and less after […] So he (Alex) would have an estimate that would be actually more than, um, once we add his two estimates up.

In the excerpt above, in contrast with most (nine out of twelve) interview participants, Elliot not only immediately recognized that Alex’s claim was false, but provided sufficient reasoning for why it was false without the use of a counterexample. In particular, Elliot visualized the unit circle and differentiated from linear thinking by noting that, “the sine would move more at the beginning of the circle and less after.”

In summary, three of the participants seemed to apply linear reasoning to situations involving the sine function. In particular, they attempted to use the idea that \( \sin(A) + \sin(B) = \sin(A + B) \). Some seemed to consider that this would be true when \( A \) and \( B \) are complementary, while others seemed to believe it was true for all angles. Even those who did not initiate this idea could not immediately determine the validity of hypothetical student Alex’s claim that “\( \sin(90^\circ) = \sin(70^\circ) + \sin(20^\circ) \).” In Table 4.8 below, I provide an overview of which of the interview participants: (i) considered that “\( \sin(90^\circ) = \sin(70^\circ) + \sin(20^\circ) \)” prior to seeing Alex’s work, (ii) could immediately determine that
Alex’s statement was false, (iii) eventually determined that Alex’s statement was false, and (iv) could/did not determine the validity of Alex’s statement.

Table 4.8. Participants’ Thinking regarding Alex’s Claim that “\(\sin(90^\circ) = \sin(70^\circ) + \sin(20^\circ)\)”

<table>
<thead>
<tr>
<th>Name</th>
<th>Considered this Method Prior to Seeing Alex’s Work</th>
<th>Immediately Knew it Was False</th>
<th>Eventually Determined it Was False</th>
<th>Did/Could not Determine Validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kelly *</td>
<td>PST</td>
<td>√</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>Zola*</td>
<td>PST</td>
<td>√</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>Jess*</td>
<td>PST</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Toby*</td>
<td>PST</td>
<td></td>
<td>√</td>
<td></td>
</tr>
<tr>
<td>Mary*</td>
<td>ST</td>
<td></td>
<td>√</td>
<td></td>
</tr>
<tr>
<td>Connie**</td>
<td>ST</td>
<td>√</td>
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<td>√</td>
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<tr>
<td>Elliot**</td>
<td>ST</td>
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<tr>
<td>Sarah**</td>
<td>IST</td>
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<tr>
<td>Pat *</td>
<td>IST</td>
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<tr>
<td>Lana*</td>
<td>IST</td>
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<td>Craig**</td>
<td>IST</td>
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<td>Byron**</td>
<td>IST</td>
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<td>√</td>
</tr>
</tbody>
</table>

Sine and Cosine: Cofunctions

Knowledge of cofunctions allows individuals to make sense of the relationship between sine and cosine, as well as other complementary pairs of trigonometric functions (tangent and cotangent; secant and cosecant), and “provides versatility and adaptability” in problem solving as well as proof writing (Fi, 2003, p. 202); however none of the fourteen preservice teachers in Fi’s (2003) study exhibited such knowledge. Consistent with his findings, all eleven preservice teachers in this study were awarded zero out of three points on assessment item 5c. That is, based on the rubric, all of the preservice teachers either provided no correct work, did not answer the question, or
responded that they could not provide a closer approximation when given \( \sin(70^\circ) \) and asked to find \( \cos(340^\circ) \). In addition, only four out of eight of the remaining participants (student teachers and in-service teachers) were awarded more than zero points. Note that ideally the participants would have acknowledged that (i) \( \cos(340^\circ) \) is equivalent to \( \cos(-20^\circ) \) which is (ii) equivalent to \( \cos(20^\circ) \) which is (iii) equivalent to \( \sin(70^\circ) \). Thus, this problem required three steps, or realizations, and the third is the only one that requires an understanding of the relationship between sine and cosine as cofunctions. Thus, their inability to answer this item correctly does not necessarily indicate that they do not understand this relationship; however, the second task in the first interview asked a simpler statement only involving angles in the first quadrant and yielded similar results. In particular, in the second task during the first interview, participants were given \( \sin(70^\circ) \approx .9397 \) and were asked to find \( \cos(20^\circ) \) and \( \sin(20^\circ) \). Only one of the preservice teacher interview participants answered this prompt correctly using knowledge of cofunctions. As was discussed in the previous section, many of them instead found \( \sin(20^\circ) \) by equating \( \sin(90^\circ) \) with the sum of \( \sin(70^\circ) \) and \( \sin(20^\circ) \).

In contrast however, the student teachers and in-service teacher participants in this study held relatively impressive knowledge of cofunctions. In fact, two student teachers and four in-service teachers answered the interview prompt in Task Two correctly using their knowledge of cofunctions. Further, some were able to make sense of this relationship between sine and cosine by drawing on their ways of knowing sine and cosine. For example, Sarah reasoned, “I know that the sine of one of my reference
angles is gonna be the same as the cosine of the other one. Why do I know that? Because…the sine of this angle is opposite over hypotenuse. So if I make that my unit circle, I can put .9397 there. The cosine of 20 is then adjacent over hypotenuse. So I know the cosine of 20 is .9397."

In summary, consistent with Fi’s (2003) findings, ten out of eleven preservice teacher participants did not exhibit knowledge of cofunctions and had difficulty solving problems that required knowledge of cofunctions. In contrast, two of the three student teachers and four of the five in-service teacher participants did exhibit knowledge of the relationship between sine and cosine as cofunctions. In Table 4.9, I have provided an overall summary of the productive knowledge of sine, cosine, and cofunctions exhibited by each participant.
Table 4.9.
Knowledge of Sine, Cosine, and Their Relationship as Cofunctions Exhibited by Participants

<table>
<thead>
<tr>
<th>Name</th>
<th>Exhibited Knowledge of Sine and Cosine as:</th>
<th></th>
<th>Exhibited Knowledge of the Relationship between Sine and Cosine as Cofunctions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ratios of Side Lengths of Right Triangle</td>
<td>Lengths on Unit Circle</td>
<td>Coordinates on Unit Circle</td>
</tr>
<tr>
<td>Tara</td>
<td>PST</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Kelly</td>
<td>PST</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Molly</td>
<td>PST</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Zola*</td>
<td>PST</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Arianna</td>
<td>PST</td>
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<td>✓</td>
</tr>
<tr>
<td>Maggie</td>
<td>PST</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Jess*</td>
<td>PST</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Todd</td>
<td>PST</td>
<td>✓</td>
<td></td>
</tr>
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<td>Hunter</td>
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</tr>
<tr>
<td>Fred</td>
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<td>✓</td>
<td></td>
</tr>
<tr>
<td>Toby*</td>
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<td>✓</td>
<td>✓</td>
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<tr>
<td>Mary*</td>
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<td>Elliot**</td>
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<td>Sarah**</td>
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<td>Pat *</td>
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<td>IST</td>
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<td>Byron**</td>
<td>IST</td>
<td>✓</td>
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Conclusion

In this chapter, I have presented results with respect to the participants’ overall performance on the trigonometry assessment, and, drawing on data from both the assessment and interviews, I have addressed my preliminary research question. In particular, I have discussed the participants’ knowledge of three central concepts in
trigonometry: radian, unit circle, and the sine and cosine functions. With respect to their strengths, most participants (i) exhibited an understanding of the relationship between radian and degree measure, (ii) could correctly convert between radians and degrees, (iii) defined the unit circle as a circle with radius one, and (iv) exhibited at least one correct way of knowing sine and cosine (either as coordinates of a point on the unit circle, as horizontal and vertical distances on the unit circle, or as ratios of side lengths of a right triangle). Further, almost half of participants correctly converted 19 radians into degrees, and thus, they likely did not believe that radians are only expressed in terms of \(\pi\).

On the other hand however, most participants did not exhibit an understanding of (i) radian as the ratio of two lengths or as the length of the arc on the unit circle, (ii) the functionality of the unit circle and it’s significance as a tool for reasoning, or (iii) the relationship between sine and cosine as cofunctions. In addition, many participants (i) appeared to believe that radians are only given terms of pi, (ii) viewed the unit circle as a fixed diagram that is given to students, (iii) did not appear to connect all three ways of knowing sine and cosine, and (iv) used linear thinking to reason about sine and cosine.

While the overall results are consistent with findings from prior research, in this study, participants with teaching experience often exhibited stronger knowledge than expected. For example, recall that two of the three participants with precalculus teaching experience correctly defined radian, whereas in Fi’s (2003), Akkoc’s (2008), and Topcu and colleagues’ (2006) studies, only one out of a collective 107 participants correctly defined radian. The inservice teacher participants also scored significantly higher on the assessment in general, and in particular, two out of five described the unit
circle as a tool for reasoning about trigonometric functions, and three out of five were able to connect all three ways of knowing sine and cosine.

Recall that in order to (i) assess which resources, especially mathematical knowledge, teachers draw upon in the work of noticing students’ work and thinking (addressing my first research question) and to (ii) understand how engaging with student work encourages teachers to think more deeply about mathematics (addressing my second research question), it was important to obtain insight into the mathematical knowledge of my participants, i.e., the findings I have presented in this chapter. In the next chapter, I address my main two research questions, and within these discussions, I explore the value and role of this content knowledge in noticing.
CHAPTER V

RESULTS: NOTICING STUDENTS’ MATHEMATICAL WORK AND THINKING

What Happened When Teachers Didn’t Know Enough Mathematics

As was discussed in the previous chapter, most participants, especially the preservice teacher participants, did not have strong trigonometric knowledge. In this section, I discuss how these weaknesses in participants’ subject matter knowledge severely limited their noticing as well as some ways in which they compensated for these weaknesses. This section is included not only to share one important aspect of the findings, but also to explain and justify my decision to focus in more detail on the noticing of five case study participants who had stronger trigonometric knowledge.

Their Attention to and Evaluation of Correctness

The participants’ lack of knowledge hindered their ability to evaluate student’s work in three ways: (i) they avoided making claims about the correctness of students’ work, (ii) they could not make decisive claims despite attention to correctness, or (iii) they made incorrect evaluations. With respect to the first of these three ways, on items where they lacked content knowledge Jess, Zola, Mary, Pat, and Toby, and Connie avoided issues of correctness and instead simply described the students’ work. Mary, Toby, and Connie did so in Task Five, and Pat did so in tasks three and four. Jess and Zola were especially persistent in avoiding issues of correctness and each did so for more than half the students. For example, toward the end of the discussion of Task One, the interviewer eventually asked Jess to talk specifically about the correctness of the hypothetical students’ work, and even then Jess stated, “Allen, Lexi, and Annie, show more effort in ideas towards the problem than Charlie. They put more effort and
tried to think of reasons why it would work. More Annie and Allen, because they drew diagrams, and he (Allen) wrote circumference, and then she (Lexi) just wrote $S = \theta r$ without much to, to determine.” That is, even when asked specifically to evaluate, Jess, compensated by judging the work based on superficial aspects like how much work was written or whether or not they included diagrams as opposed to evaluating the mathematical claims or thinking. In other instances, Jess would refer to work as “interesting” when ignoring the correctness of claims or strategies. For example, Jess’s initial unstructured noticing of Alex’s work on Task Two was simply, “I think it’s interesting how he (Alex) related the $\cos^2 \theta$, he related to that theorem.” She described his work further and then again concluded, “So that’s interesting.” Similarly, instead of attending to the correctness of Alexis’ work on Task Five, Zola stated that it was “a different way of looking at it.”

As I mentioned previously, in contrast to an avoidance of issues of correctness, in some cases, participants would attend to correctness but due to their lack of knowledge, they couldn’t make decisive claims. This was the case for Zola, Jess, Toby, Pat, Sarah, Mary, Kelly, and Connie with respect to at least two of the hypothetical students’ work. In an example, with respect to Alexis’ work on Task Five (See Appendix G), Mary claimed:

Mary: Alexis flipped it onto the bottom, so like one over cosine, instead of cosine to the negative one. I don't think that works with, does it work with cosine? I don't remember if that works or not.

Interviewer: The?

Mary: The negative one, like can you put it on the bottom? Cuz and this is notation, but negative one denotes an inverse, not necessarily the multiplicative inverse cuz it's ambiguous notation. So (laughing) I'm not sure that that is legal. Maybe it is, maybe it isn't. I can't remember.”
As is shown in the excerpt above, Mary was focused on the correctness of Alexis’ strategy, but because of Mary’s limited knowledge of the inverse of cosine (or arccosine) as well as the conventional notational of $\cos^{-1}$, she could not determine whether or not Alexis’ claim was in fact valid.

Thus far, I have discussed how participants’ lack of knowledge hindered their ability to evaluate student’s work in two ways: (i) avoidance of issues of correctness and (ii) inability to make decisive claims despite attention. As I mentioned previously, in some instances in which participants did not have enough mathematical knowledge, they also (iii) made incorrect evaluative claims. For example, several participants incorrectly evaluated Annie’s work on the first task (See Appendix C). In particular, Craig, Kelly, and Jess, all incorrectly claimed that Annie’s first sketch was incorrect; in other words, all three participants claimed that arc length was not equivalent to radian measure on a unit circle. In contrast however, for the most part, participants appeared aware of the limits to their knowledge and did not make invalid definitive claims about the correctness of the work. In many instances however, because participants could not make sense of the students’ strategies, they would dismiss their thinking as nonsensical. These instances are discussed in the next section with respect to how limited knowledge influenced one’s attention and interpretations of thinking.

In summary, when the participants’ lack of mathematical knowledge affected their attention to correctness, it did so in several ways. Three ways in which their limited mathematical knowledge was manifested in their attention to correctness were: (i) avoidance of issues of correctness, (ii) an inability to make decisive claims about correctness, or (iii) incorrect claims made by the participants’ about correctness. Some
ways in which the participants compensated for their limited mathematical knowledge were by focusing on aspects of the work or thinking for which they could determine correctness or focusing on superficial qualities of the work such as the amount of work shown or the level of detail or explanation provided. In the next subsection I discuss how limited mathematical knowledge influenced participants’ attention to and interpretations of the hypothetical students’ thinking.

**Their Attention to and Interpretations of Thinking**

On items in which participants’ lacked content knowledge, it was difficult for them to make meaningful claims regarding students’ thinking or ways of knowing. In fact, of the eleven participants who participated in both interviews, the three who scored lowest on the trigonometry assessment, attended to thinking the least. Without structured prompts, Jess only attended to thinking for one of the 15 students, Zola for two, and Mary for six. In other instances in which participants lacked the content knowledge to make sense of students’ strategies, they claimed that the students’ had “limited understandings,” or “didn’t understand” and were not able to discuss what the students’ strategies might have been or how the students understood various concepts. In particular their (i) limited ways of knowing sine and cosine, (ii) lack of knowledge of radian as the arc length of an angle on the unit circle or as the ratio of two lengths, (iii) lack of knowledge of cofunctions, as well as (iv) limited knowledge of inverse functions and inverse cosine in particular, severely limited their interpretations of students’ thinking. In the following, I discuss four examples of participants’ dismissal of students’ thinking due to their own lack of content knowledge; each example corresponds to one of the four common weaknesses in content listed above.
Example 1: Limited Ways of Knowing Sine and Cosine. In general, participants who did not hold all three ways of knowing sine and cosine identified by Brown (2006) (as coordinates of a point on the unit circle, as horizontal and vertical distances on the unit circle, and as ratios of side lengths of a right triangle) had difficulty interpreting students’ thinking, especially in tasks three and four. As I mentioned in Chapter IV, in-service teacher, Lana, understood sine and cosine as coordinates on the unit circle but, like several participants, did not seem to connect this coordinate definition with the idea of vertical and horizontal lengths on a unit circle. I also mentioned in Chapter IV that this limited Lana’s ability to make sense of Alice’s strategy in Task Four. Similarly, this lack of knowledge limited Lana’s interpretation of Joe’s thinking in Task Three, whose work (See Appendix E) was designed to represent knowledge of sine as a vertical length on the unit circle. Joe’s only error was that he did not take a square root at the end of his work; however, Lana claimed that:

Joe, his understanding is very limited. [...] why did he, um, not solve for sine, so I don't know if he got stuck. Maybe he just doesn't know what to do from there, um, or if he forgot what he was looking for. Maybe he just thought he had to find the unknown side and you know, but yeah this, Joe I don't see a whole lot of understanding.

As shown above, Lana dismissed Joe as having “limited” understanding and in particular, interpreted that Joe didn’t actually know that he was supposed to “solve for sine.” In this case, Lana had enough knowledge to answer the question correctly but not have enough knowledge to interpret Joe’s unconventional strategy. In particular, because Lana did not have knowledge of sine as a vertical length on the unit circle, she could not interpret Joe’s understandings of sine and instead dismissed his understanding as “limited.”
Example 2: Lack of knowledge of radian as the arc length of an angle on the unit circle or as the ratio of two lengths. Participants who did not have knowledge of radian as the arc length of an angle on the unit circle or as the ratio of two lengths typically had difficulty making sense of Lexi, Annie, and Charlie’s strategies in Task One, and thus frequently dismissed their thinking as nonsensical or limited. For example, recall that Jess did not understand radian measure of an angle as arc length on the unit circle. Subsequently she could not think of any reasonable explanations for Charlie’s result. Instead Jess claimed that Charlie “just took the angle theta,” that he “just put the over 12 and dropped the negative.” Further, when asked specifically about Charlie’s understandings, Jess claimed, “Charlie doesn’t have much of a conceptual understanding.” Similarly, Jess claimed that Lexi just haphazardly multiplied the two numbers because they were the numbers that were given.

Example 3: Lack of knowledge of cofunctions. As I acknowledged in Chapter IV, many of the participants didn’t have knowledge of the relationship between sine and cosine as cofunctions, and thus they had difficulty interpreting students’ thinking in the second task, especially Martha’s claim that (given \( \sin(70^\circ) \approx 0.9397 \)) “90=20+70, so \( \cos 20^\circ = 0.9397 \).” For example, Kelly claimed that, “I don’t get how they understand that this implies that.” Later in the interview, anytime the interviewer prompted her to talk about Martha’s understandings or strategy, Kelly commented on aspects of Martha’s work or focused on the fact that Martha “didn’t really explain fully her thought.” Eventually, at the end of their discussion, the interviewer asked Kelly to again specifically address why Martha may have done this, and Kelly reasoned that Martha might have been thinking about 30-60-90 triangles.
Example 4: Limited knowledge of inverse functions and inverse cosine.

While participants’ knowledge of inverse trigonometric functions was not investigated on the written assessment or addressed in Chapter IV, almost all of the participants had difficulty with Task Five involving knowledge of both inverse trigonometric functions as well as inverse functions in general. For example, Toby did not have sufficient knowledge to correctly answer the prompt; in particular, he did not seem to know that the range of arccosine is restricted to $[0,\pi]$. As a consequence, Toby could not make sense of and thus could not see the strengths in Amy’s thinking on Task Five (See Appendix G) as shown in his statements below.

I don't know if she’s thinking it's negative because that's negative or just because she knows that it's in the uh, third quadrant [...] she must think that it reflects around the x-axis or something, and that's why she's looking at the angle in the second quadrant.

In the above excerpt Toby struggled to make sense of Amy’s claim that because the cosine of $-\frac{11\pi}{4}$ is negative, the arccosine must be an angle in the second quadrant. Toby couldn’t make sense of this because he himself did not know or could not recall the restricted range. Thus, Toby concluded that Amy must have interpreted inverse as a reflection “around the x-axis.” Toby was also unable to see the strengths in Sam’s work and thinking on this task (See Appendix G) and dismissed it, claiming, “I don't think (laughing) that he um, I'm not sure that he understands the actual problem itself.”

In this subsection I have discussed how limited mathematical knowledge influenced many of the teachers’ attention to and interpretations of the hypothetical students’ thinking. In particular, (i) those with the lowest trigonometric knowledge tended to attend to thinking less and (ii) limited trigonometric knowledge made it difficult for participants’ to make sense of students’ strategies and thus, identify potential strengths.
in students' thinking. Later, in the description of five case study participants' noticing, I elaborate further on the role of mathematical knowledge when interpreting. In the next subsection, I will discuss how limited mathematical knowledge influenced the participants’ proposed instructional responses.

**Instructional Responses**

As I have discussed, limited trigonometric knowledge inhibited participants’ abilities to identify correctness, errors and misconceptions, and also inhibited their ability to make sense of the students’ thinking. As a consequence, many of the interview participants with weak trigonometric knowledge proposed general instructional responses to the hypothetical students, simply asking the students’ to explain what they did. In fact one third of Jess’ responses and all of both Kelly and Zola’s proposed instructional responses were coded as “general moves.” For example, in response to Joe’s work on Task Five (See Appendix G), Jess proposed, “I would ask him to elaborate more if he could on why he, why this works, if it does work um to get this. I would discuss um, why they thought that this is true, that these quantities are true and how they knew that.” Note that in her response, Jess was not specific to the student or the particular problem. Zola even lumped the hypothetical students in Task One together, explaining, “I would say we need to go back over how to do it. (laughing) Um, maybe come up with a different example and walk them through it, and then ask them where they think they went wrong with their original one.” Again Zola’s response was a general move, not specific to the problem or students.

In spite of weaknesses in their mathematical knowledge, some responses were non-general and informed by participants' interpretations of students' thinking; however,
recall that in many cases, the participants’ interpretations of students’ thinking were limited or flawed because of their own weak trigonometric knowledge. In a sort of domino effect, the participants’ own weaknesses in their mathematical knowledge were often also subsequently reflected in their proposed responses to students. For example, recall that Lana didn’t appear to understand sine as a vertical length on the unit circle and thus, she thought that Joe, in Task Three, had “limited” understanding and that, in particular, Joe didn’t know that he had to solve for sine. Thus, Lana’s response to Joe was, “I would ask him, um, what you know, if he even remembers what the sine of an angle is, um, is equal to, and if he doesn't remember I'd probably um, give it to him.” That is, while Lana’s response was made on the basis of her interpretation of his thinking, those interpretations were limited by her own limited knowledge of sine and cosine, and therefore, Lana was unable to build on Joe’s thinking in her instructional response.

While participants’ instructional responses to the hypothetical students were influenced by weaknesses in the participants’ own mathematical knowledge, most of these responses were categorized as “give or ask” and aimed to access additional information about students. Because of this, fortunately, their responses were not as potentially detrimental to the students as they might have been. For example, based on the participants’ flawed interpretations and evaluations, they might have said to the students something like, “No. This is wrong. Start over this way.” Thus, while participants who didn’t have enough mathematical knowledge were not able to build on students’ thinking, they also did not necessarily reroute students and put a stop to students’ unconventional strategies.
Conclusion

In the previous chapter, I discussed the participants’ knowledge of trigonometry and in particular, I highlighted many of the weaknesses in the participants’ trigonometric knowledge. In this section, I have further explained how these weaknesses limited their ability to notice students’ work and thinking. In particular, gaps in the participants’ knowledge often contributed to a decreased attention on both correctness and thinking, interpretations of thinking that were not well-grounded in the work, and responses that were not specific to or well founded in the students’ work or thinking. Thus, when choosing case study participants, I chose to focus on those who appeared to have stronger trigonometric knowledge. In particular, I chose to focus in more detail on the noticing of the five interview participants who scored the highest on the trigonometry assessment.
Case Studies

Introduction

The two main research questions that guided this study aimed to investigate how teachers notice and how teachers engage with mathematics in response to students’ work. These how questions warranted thick descriptions of connections between individuals thinking about concepts, what they noticed, how they interpreted, and how they responded. A case study approach allowed me to make sense of these relationships and provide detailed descriptions. As was discussed previously, many of the interview participants did not have adequate trigonometric knowledge to engage with the written work in meaningful ways. Thus, I chose to focus on five particular participants whose knowledge was sufficient in most cases: Elliot, Craig, Byron, Sarah, and Connie. While these five participants were similar in that they exhibited above average subject matter knowledge, there were differences in the ways they noticed, the resources they drew upon to notice, how they drew on various resources, as well as the extent to which their mathematical thinking was furthered by their analysis of students’ work. Both their similarities and differences allowed each case to illuminate the others and shed light on the possible ways in which teachers engage in noticing as well as the types of experiences and support that should be provided to preservice and in-service teachers in order to improve their ability to engage in responsive teaching that builds on students’ thinking.

In the following sections, for each participant, I first provide background information regarding his or her teaching experience and self-reported teaching style. I then address each of my three research questions with respect to that particular
participant. Specifically, I discuss (0) each participant’s knowledge of radian, the unit circle, and the sine and cosine functions, (1) how each participant attended to, interpreted, and responded to the hypothetical students’ written work and thinking in trigonometry, as well as (2) in what ways the hypothetical students’ work challenged and/or furthered each participant’s mathematical thinking. Following the discussions of each case study participant, I address the main two research questions, looking across the five cases.
Elliot

At the time of data collection, Elliot was a student teacher and had completed more than half of his full year student teaching experience. He was currently teaching Pre-Algebra, Precalculus, and Calculus. To give the reader insight into Elliot’s teaching style, note that Elliot claimed, “a teacher is a facilitator and guide for the students. Since they are also a source of knowledge, they need to be open to sharing and assisting in the learning process while motivating students to work and strive to become educated individuals.” In the following sections I address each of my three research questions with respect to this one particular participant, Elliot.

Question 0: What content knowledge of radian, unit circle, and the sine and cosine functions did Elliot possess? Elliot received the highest score of all participants on the trigonometry assessment, 23 out of 25 points, or 92%. His high achievement might be due, at least in part, to the fact that he was one of only three participants who had experience teaching Precalculus and one of two participants who were teaching Precalculus concurrently with data collection. While Elliot exhibited strong knowledge of most of the trigonometric concepts addressed in this study, his knowledge of sine and cosine as well as the unit circle did prove (during the interviews) to be limited or insufficient for interpreting the students' work in some cases. In the following subsections, I provide a brief description of Elliot’s knowledge of radian and arc length, unit circle, as well as the sine and cosine functions, including their relationship as cofunctions. Later, I discuss how this mathematical knowledge influenced his mathematical noticing of students' work and thinking.
**Radian and arc length.** Elliot was one of two participants who correctly defined radian on the written assessment. In particular, Elliot defined radian as “the measure of the central angle of a circle that intersects an arc length equal in length to the radius of a circle,” and he used this knowledge to quickly and easily calculate arc length, explaining, “right off when I think of arc length I think of trying to find the radius, um, of a circle which we've got right here is radius 3, and I try to find the, um, angle measure in radians.” Elliot simultaneously understood radian measure as the arc length corresponding to the given angle on the unit circle.

**Unit circle.** Recall from Chapter IV, that Elliot defined unit circle correctly, but that Elliot also seemed to hold two conflicting ideas of the unit circle: (1) a circle with radius one and (2) a sort of reference circle (or picture) which lists sine and cosine values. Elliot seemed to acknowledge the latter not as a definition but as part of a contract between teacher and students – “when you ask THE unit circle, you’re asking for kind of the coordinates or angles.” Finally, Elliot also explained, on his assessment, that the unit circle is important because “since our radius is one our trig functions, which are learned in geometry became simplified to the x or y coordinate. This helps students who are trying to learn trig functions without a calculator.”

**Sine and cosine.** Elliot exhibited an understanding of sine and cosine as both coordinates of points on a unit circle as well as ratios of the lengths of sides on a right triangle. Elliot’s preference seemed to be for the unit circle definition, which he utilized to make sense of various problems and aspects of students’ work. For example, when prompted to draw the graph of cosine, he began sketching and commented aloud, “uh, we're gonna go all the way around the circle.” He later stated, “obviously I don't think
about the unit circle every time I do these graphs anymore, but I think that's how it should be explained or to help students get, get to graph it." It was surprising that despite Elliot's strong understanding and use of the “coordinates” definition, he did not appear to ever use the similar definition of sine and cosine as horizontal and vertical lengths in the unit circle, and he had difficulty interpreting student work that was designed to represent this sort of thinking.

Overall, as has been discussed in this section, Elliot had very strong knowledge of the targeted trigonometry concepts and could answer all of the interview prompts correctly. The only weakness exhibited by Elliot was that he did not seem to understand sine and cosine as horizontal and vertical lengths in the unit circle. Within the next section, I include a discussion of how this mathematical knowledge influenced his mathematical noticing of students’ work and thinking.

**Question 1: How did Elliot attend to, interpret, and respond to hypothetical students’ work and thinking in trigonometry?** In this section, I discuss findings regarding how Elliot (i) attended to, (ii) interpreted, and (iii) responded to students’ work and thinking.

*Elliot’s initial attention.* In this section, I answer the following: (i) *what did Elliot do when asked to notice?*, and (ii) *to what extent was his focus on the students’ thinking?*. An overview of how Elliot initially attended to the hypothetical students’ work and thinking is provided below in Table 5.1.
### Table 5.1.
**Summary of How Elliot Initially Attended to the Students’ Work and Thinking**

<table>
<thead>
<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Thinking</th>
<th>Describing</th>
<th>Evaluating</th>
<th>Challenged by Mathematics</th>
<th>Investigating</th>
<th>Comparing to Others</th>
<th>Comparing to Own</th>
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<tr>
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</table>

As is shown in this table, without being prompted specifically to do so, Elliot focused significantly on correctness of work and thinking. That is, Elliot made definitive evaluative claims for all fifteen students; further, all of the evaluative claims were mathematically valid. Also note that in the one instance in which Elliot appeared to notice work for which he was uncertain of the correctness, he investigated the mathematics. In particular, Elliot was challenged and engaged in further mathematical thinking in response to Alice’s work on Task Four which is discussed in later sections. Elliot did not appear to attend to similarities and differences between the hypothetical students’ work and thinking as much as he attended to the similarities and differences.
between the work and his own thinking. In particular, as shown in the table, without any specific prompt to do so, Elliot made claims comparing the students’ work and thinking to his own for over one third of the hypothetical students. Most importantly, Elliot attended to the thinking of eleven out of fifteen of the hypothetical students. His interpretations of students’ thinking, including the resources he drew on to interpret students’ thinking, are included in the next sections.

**Elliot’s interpreting.** As I mentioned, Elliot was cautious when describing students’ thinking. In particular he was hesitant to equate procedural correctness with correct understanding. For example, during the second interview, in response to Anna’s work on Task Three, Elliot noted, “you are able to do the work, but do you understand?” In another example, Elliot noted that Lexi “obviously remembers a formula probably that was given to her in class” but that “maybe she doesn’t fully understand as much of the concept as say, Allen does.” Similarly, Elliot was hesitant to equate incorrect work with misconceptions. For example, with respect to Cheyenne’s work on Task Two (See Appendix D), Elliot posed the questions, “was this a mistake, by accident, or is this something she always thinks?” Here, as in several instances, Elliot differentiated between errors and misconceptions, contributing to his overall cautious differentiation between procedural correctness and thinking.

Thus far, I have discussed Elliot’s (i) cautiousness to avoid equating procedural correctness with correct thinking and more specifically, and (ii) his differentiation between errors and misconceptions. In the following sections I continue the discussion of Elliot’s interpreting by addressing the following question (which corresponds to research sub-question 1b): *What resources did Elliot draw upon to interpret students’*
thinking? In particular, I explain how Elliot drew upon his mathematical knowledge, his own work and thinking, his interpretive language, and his pedagogical content knowledge, to interpret the hypothetical students' thinking.

Resource for interpreting: Elliot’s subject matter knowledge. Elliot’s mathematical knowledge enabled his interpretations of students’ thinking in most cases. For example, because he understood that radian and arc length “are the same on a circle with a radius of 1,” Elliot was one of only a few participants to not dismiss Charlie’s simple response of $\pi/12$ (See Appendix A) and to recognize that Charlie might be “really focused on the unit circle.” Elliot elaborated that Charlie might have “used $S$ equals, um, the angle theta times one thinking it was the unit circle.” This is precisely what the work was designed to represent.

Similarly, Elliot also drew on his mathematical knowledge to evaluate the work of and then interpret students’ thinking in Task Five, including Sam’s (See Appendix G). In particular, Elliot (i) drew on his knowledge of the range of inverse cosine to evaluate Sam’s work, noting that Sam failed to restrict his answers and (ii) also drew on his knowledge of cosine to evaluate that Sam had “failed to, to find the other spot,” and then finally (iii) drew on his knowledge of what it means to be invertible (not specific to trigonometry) to interpret Sam’s thinking or misconception, noting, “what he seems to fail to understand was when we’re talking about inverse functions, we have to have a one-to-one function.”

As was mentioned previously, Elliot relied heavily on his understanding of sine and cosine as coordinates on the unit circle, and he also used his understanding of sine and cosine as ratios of the lengths of sides of a right triangle; however, he did not seem
to acknowledge sine and cosine as the horizontal and vertical lengths on a unit circle. This became evident in his analysis of Joe’s work in Task Three (See Appendix E) and Alice’s work in Task Four (See Appendix F). In particular, in this case his mathematical knowledge was limited and in turn, he struggled to interpret the students’ thinking. Elliot immediately recognized that Joe had created a similar triangle to get “a hypotenuse of one to think of the unit circle,” and Elliot also identified the error in Joe’s work, noting that he should have taken the square root at the end; however Elliot insisted that Joe “has to divide at the very end uh by the hypotenuse.” Thus it appeared Elliot did not recognize sine as the vertical side of the triangle and consequently could not sufficiently interpret Joe’s work or thinking.

After several prompts to attend to, interpret, and respond to Joe’s work, Elliot eventually seemed to make the connection between the definitions. He explored,

I kind of make it on the unit circle, and then once I kind of make it on the unit circle, I can think about it as like oh now I know x is cosine and y is sine. (writes “(x, y)”). So then I can think about this is like, oh now we have… now let’s think of this side as the y so all we’re looking for is the question mark now. We don’t have to worry about doing question mark over one. We’re just looking um for the y-value or this side and so that’s what I, I think would be what someone might have shown him or he came up with. It’s kind of a good idea. Um…but yeah. I don’t know if that’s true.

Elliot seemed to make sense of Joe’s strategy, realizing that he didn’t have to “worry about doing the question mark over one,” but that the sine was in fact the vertical side; however, Elliot seemed uncertain with his findings in the end, and almost immediately afterwards, he was unable to interpret Alice’s work in Task Four (See Appendix F) in which she utilized the horizontal lengths of the unit circle to construct the graph of cosine. To some extent it did appear that Joe’s work challenged Elliot’s current mathematical understandings and motivated him to build on and make connections to
his previously held conceptions of sine and cosine. This is discussed further in the section regarding the ways that the hypothetical students' work challenged and/or furthered Elliot's thinking.

Resource for interpreting: Elliot’s interpretive language. Recall that van Es (2011) argued that learning a new discourse that is more interpretive in nature is central to noticing. I argue that while Elliot did appear to have such interpretive language for talking about students’ thinking, it was not as developed as participants like Byron or Sarah, which will be discussed in later sections. In particular, Elliot’s focus on thinking was limited by his inconsistent use of the word “understand.” In some (positive) instances he used “understand” descriptively to talk about students’ different ways of knowing. For example, he noted that, “Annie understands that radian measure of, of an angle here, pi over 12, is equal to the arc length.” In other instances however, Elliot used the term “understand” in evaluative terms (he did understand X versus she did not understand Y) or to describe procedural correctness. For example, Elliot also said that Annie “understood the unit circle.” It is unclear what Elliot meant by understanding the unit circle. Elliot did not describe how Annie understood the unit circle; instead Elliot simply evaluated that yes, she understood it. In another example of less productive use of the word “understand,” Elliot noted that “Charlie doesn't understand the difference when you’re changing from a unit circle to a um, to a circle without radius one.” Similarly, Elliot noted that Lexi “understands, um, how to plug it in, um, the formula to start.” In both these instances, Elliot used the term “understand” to refer to procedural correctness rather than different ways of knowing.
Resource for interpreting: Elliot’s own work and thinking. There was one instance in which it was clear that Elliot drew on his own work and thinking about the problem to interpret thinking. In particular, Elliot claimed that Martha (in Task Two) (whose work or result resembled his own) (See Appendix D) was thinking the same way he had. Martha simply wrote that since the sum of 70 and 20 is 90, that the sine of 70 degrees is equal to the cosine of 20 degrees. Elliot had an advanced understanding of cofunctions rooted in his understanding of sine and cosine as coordinates on the unit circle, and he attributed the same thinking/understanding to Martha:

she seemed to understand like hey look if we knew we were off the x-axis, um, with the 20 degrees we could go off of the y-axis with the 70 degrees, kinda like thinking that's 20 degrees off the y-axis, um, to kinda understand, I know it's not true reference angle, but like I like thinking about it as a reference angle, because you're still like pulling out a triangle out of the, out of the unit circle. Um, so I think like she understands some of how they interact together.

Elliot also noted that “she didn’t seem to say that” but that that’s what she probably did since, “that’s kind of what I did, um, here.” In this instance, I would argue that Elliot has over-attributed his own thinking to Martha.

Resource for interpreting: Elliot’s Pedagogical Content Knowledge (PCK). Recall that PCK consists of Knowledge of Content and Teaching (KCT), Knowledge of Content and Students (KCS), and Knowledge of Content and Curriculum (KCC). Elliot often drew simultaneously on his knowledge of common teaching strategies and his KCS to interpret students’ thinking. For example, earlier I discussed how Elliot’s mathematical knowledge informed his interpretation of Charlie’s simple response of $\pi/12$ in Task One (See Appendix C). Below, I provide an expanded quote that shows Elliot drawing
simultaneously on his knowledge of teaching and students in addition to his mathematical knowledge to make claims about Charlie’s thinking.

Charlie is, is uh, really focused on the unit circle. Um, a lot of times teachers start with the unit circle, and they start with the concept that, um, well your radian measure of your angle is equal to your arc length because the radius is one, but it kind of gets left by the wayside slowly when the kids are, are listening to the teacher talk, or they kind of block that out, and so he kind of maybe used S equals, um, the angle theta times one.

That is, Elliot drew on his KCT, or knowledge that teachers start with “your radian measure of your angle is equal to your arc length because the radius is one” as well as his corresponding KCS, or knowledge that kids “kind of block that out” and think arc length is equal to “the angle theta times one,” to interpret that Charlie is “really focused on the unit circle.”

Elliot made similar comments regarding the rationale behind Joe’s work in Task Five (See Appendix G). He noted that what Joe and most students think is that when you compose a function with its inverse, “the value that started with always comes out. It just always does.” Further Elliot rationalized this misconception based on his knowledge that typically, “beginning examples we show them always do that.” Here Elliot drew on his knowledge of the sequence and way in which inverse functions are presented as well as his knowledge of common student errors and thinking to explain Joe’s oversimplification of the problem and subsequent error. There were several other instances during which Elliot explained students’ work by citing the way certain content was “typically” taught or by making comparisons to his own students. This was impressive since Elliot had been student teaching for less than a year.
In one instance, Elliot appeared to draw on his knowledge of Content and Curriculum as well. He noted that in Task Three, Anna (See Appendix E) may have just made “a really quick careless mistake”; however he also considered that Anna may have:

saw the y was blank, and we're solving for a variable, and if y's blank, we must be looking for sine of, of y, you know? We don't usually look for, you know, theta's before, precalc, you know, theta's just come in precalc. Then all of a sudden kids don't know what to do because they're funny shaped or something

In the excerpt above, Elliot drew on his knowledge of content and curriculum, “we don’t usually look for, you know, theta’s before precalc, you know, theta’s just come in precalc,” as well as his knowledge of content and students, “kids don’t know what to do because they’re funny shaped” to interpret that Anna might have thought “if y’s blank, we must be looking for the sine of, of y.”

Thus far, in addressing the first main research question, I have discussed how Elliot attended to and interpreted the hypothetical students’ work and thinking, including the resources that influenced Elliot’s interpretations. In the final subsection corresponding to the first main research question, I discuss the third skill of noticing. In particular, I discuss Elliot’s proposed instructional responses to these hypothetical students.

**Elliot’s responding.** In this section, I discuss how Elliot responded to students’ work and thinking. In particular, I address the following questions (which correspond to research sub-questions 1d and 1c): (i) what were the goals of Elliot’s proposed instructional responses and how were they informed by his interpretations of students’ thinking, and (ii) what other resources did Elliot draw on to respond to students? An
overview of how Elliot responded to the hypothetical students’ work and thinking is provided below in Table 5.2,

Table 5.2.

<table>
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<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Student’s Thinking</th>
<th>Affect</th>
<th>General Move</th>
<th>Show and Tell</th>
<th>Give or Ask</th>
<th>Aims to Access</th>
<th>Aims to Have Student Realize</th>
<th>Aims to Have Student Realize</th>
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<tr>
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</tbody>
</table>

As is shown in the table, fourteen out of fifteen of Elliot’s proposed hypothetical instructional responses were made, at least to some extent, on the basis of his interpretations of the students' thinking. For example, Elliot’s response to Allen on Task One (See Appendix C) was:

I think the first question has to be like, OK like, how many radians, um, are in the circle [...] and if he says pi right off then you know where he went wrong. You just have to make sure he understands it's not pi, it's 2 pi. Maybe you show how you get that. Um, maybe you relate it back to circumference which is a formula that he's really really, um, maybe
stuck in his head from an earlier class [...]. If he says two pi [...] the question becomes, did you think [...] pi over 12 plus pi over 12 plus pi over 12 equals pi over two? And if that's where he went wrong, then you have to talk about adding or multiplying radian measures for pi over 12 type of fractions to, to make sure he's getting good at splitting up circles. [...]it's one of those two simple things.

As is shown in the excerpt, Elliot indicated that Allen’s error might have been the result of (i) a misconception that there are only pi radians in a circle or (ii) errors when adding fractions or dividing two pi by pi over twelve. On the basis of these interpretations, Elliot first formed “give or ask” responses to determine which of those two issues informed Allen’s error. Note also that within his response to Allen, Elliot included both a “give or ask” response and a “show and tell” response. Of all of the case study participants, Elliot included “show and tell” responses the most frequently.

As in the example above, because Elliot was cautious to differentiate between students’ work and their thinking, Elliot frequently proposed “give or ask” responses aimed to access additional information about the students’ thinking. For example, Elliot’s response to Mike in Task Four (See Appendix F) was “you graphed me half a period. So what you did, do it again with the rest of it,” because Elliot wanted to determine if Mike knew how to do this but just didn’t finish or if “he has no idea what these two things are, um, the third and fourth quadrant,” because in that case, “then you have a problem.” Similarly, Elliot proposed asking questions to determine if Cheyenne’s error was a “mistake, by accident” or if it was “something she always thinks.”

In addition, five of Elliot’s responses were intended to lead students to some realization. Elliot advocated for the use of counter-examples, and in particular, he seemed to suggest the significance of cognitive conflict, noting, “like getting students to just be like mind-boggled like whoa what just happened there? Something I thought I
definitely knew is not the same as something else I definitely know, you know, and it's the confusion that makes them have to, kind of resolve that, and I think that's where learning comes from for them.” Elliot also advocated for discovery through student collaboration. In Task Two (See Appendix D), Elliot proposed:

Martha can talk about what she did correctly […] then maybe everybody could have some of their positive things that were correct on the board, and then you could start like really combining them to say like […] we know we can look at these cofunctions right? […] and then does Alex's Pythagorean theorem with both of these? And then it does, and they're like woah ok

As shown in the above excerpt, Elliot noticed that each student in Task Two exhibited some correct work or thinking as well as some incorrect work or thinking, and thus Elliot proposed they learn from each other. One could consider that in the examples I have provided, Elliot might have drawn on his orientations toward teaching or an aspect of his knowledge of content and teaching, namely knowledge of ideal teaching to make these decisions to use counter-examples, discovery learning, and student collaboration.

In response to Joe's work on Task Five (See Appendix G), Elliot called upon his own teaching experience to propose a sequence of events that might correct Joe's misconception.

I think you have to go back to kind of like when I was drawing this with the um, with some like kids today in class, like ok we're restricting from zero to pi for a reason right? We have to pass the vertical line test. […]You found a value, […] negative root two over two, and you were assuming it was here right in the third quadrant, but you don't get to use it like if this was like an imaginary dotted line, you have to find a place on the solid line that it actually um, it's at the same value on this y-axis, right?

At the end of his response, Elliot admitted that his current teaching toolbox was limited noting, “I think that might help him, but I mean some kids, you tell them that and it still doesn't stick. So I don't have a better way yet. Um, try to think of one but that's the,
Thus far, I have discussed (0) Elliot’s knowledge of radian, the unit circle, and the sine and cosine functions as well as (1) how Elliot attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry. In the following section, I discuss (2) the ways in which the hypothetical students’ work challenged and/or furthered Elliot’s mathematical thinking. I then provide an overall summary regarding this case study of Elliot.

**Question 2: In what ways did the hypothetical students’ work on trigonometry problems challenge and/or further Elliot’s mathematical thinking?**

Elliot’s relevant mathematical knowledge was strong prior to engaging in the interview tasks; however there were situations during the interviews when Elliot’s mathematical ideas were challenged. As I’ve mentioned earlier, despite Elliot’s understanding and use of the “coordinates” definition, he did not appear to ever use the similar definition of sine and cosine as horizontal and vertical lengths on the unit circle on the assessment or in working out the mathematical problems during the interviews, and Elliot had difficulty interpreting student work that was designed to represent this sort of thinking, like Joe’s (in Task Three) and Alice’s (in Task Four). Recall that as I discussed in a previous section, after several prompts to attend to, interpret, and respond to Joe’s work, Elliot eventually seemed to make the connection between the definitions. He explored,

I kind of make it on the unit circle, and then once I kind of make it on the unit circle, I can think about it as like oh now I know x is cosine and y is sine. (writes “(x, y)”). So then I can think about this is like, oh now we have... now let's think of this side as the y so all we're looking for is the question mark now. We don't have to worry about doing question mark over one. We're just looking um for the y-value or this side and so that's what I, I think would be what someone might have shown him or he came up with. It's kind of a good idea. Um...but yeah. I don't know if that's true.
Elliot seemed to realize that the length of the vertical side on the unit circle was equivalent to the “y-value” on the unit circle. Elliot did seem uncertain with his findings in the end, and almost immediately afterwards, he was unable to interpret Alice’s work in Task Four in which she utilized the horizontal lengths of the unit circle to construct the graph of cosine; however to some extent it appeared that Joe’s work challenged Elliot’s current mathematical understandings and motivated him to build on and make connections to his previously held conceptions of sine and cosine. In his written reflection, Elliot also acknowledged that, the students’ work challenged his own thinking, explaining, “I think that I engaged in a lot of mathematical thinking and problem solving while analyzing the student work.”

The students’ work also seemed to encourage Elliot to think about his own students as well as his teaching. Elliot seemed to think about issues of teaching and students in response to Alexis’s work on Task Five (See Appendix G):

Yeah so I mean this goes back uh back to, to Alexis, um, and, and really, I mean it's a hard thing for students to understand because nobody explains it and explains it real well, and it's just notation being different. Like it, it's not like an explanation. It's like no, that's not the same thing, and it's just mathematically how we notate things.

In this case, Elliot grappled with the notion of teaching a concept versus teaching convention. In particular, it appeared Elliot was recognizing that the idea that we use an exponent of negative one to represent the multiplicative inverse in some instances but the inverse function in others is not something he could explain but rather “just mathematically how we notate things” and that without an explanation, “it’s a hard thing for students to understand.”
Similarly, the student work in Task Four (See Appendix F) and the mathematics Elliot engaged in in response to those students’ work influenced him to think about “how switching from the unit circle to a graph of a trig function may be difficult.” That is, Mike and Alice’s work encouraged Elliot to think about the difficulties students have connecting a rotation on the unit circle with a point on the graph of cosine, as identified by Brown (2006). Elliot noted that the work influenced him to think about other difficulties students may have in trigonometry, noting, “I really thought more deeply about how difficult even and odd functions might be to students” and “I also learned that the unit circle can be a very tough concept for students to understand because of how it is explained and then seems to change for them.” In this last statement, it is not clear exactly what Elliot is referring to; it might be the instance I recounted earlier in which Elliot states that Charlie’s response on task (See Appendix C) one may have been because a teacher introduced radian measure as the arc length of an angle on a unit circle, and Charlie assumed this was true for any circle, not just the unit circle.

**Summary: Elliot.** Recall that Elliot was a student teacher with one year of precalculus teaching experience who scored the highest on the trigonometry assessment. He had sufficient trigonometric knowledge to answer all mathematical prompts, but he did not seem to have an understanding of sine and cosine as horizontal and vertical lengths on the unit circle, and consequently, his thinking about sine and cosine was challenged by some of the students’ work. He reported that analyzing the student work forced him to engage in additional mathematical thinking as well.

When presented with students’ work, Elliot focused on students’ thinking without being prompted, and he was also cautious not to equate procedural correctness with
correct thinking; in particular, Elliot differentiated between errors and misconceptions. Elliot drew on several resources to interpret students’ thinking. He often held multiple ways of knowing a particular concept, and he was able to identify and make sense of the important ideas in each problem; thus, he was often able to recognize the important ideas behind students’ work. While I argue that Elliot had sufficient interpretative language to talk about the students’ thinking, he used the term “understand” in inconsistent ways, and his language for talking about students’ ways of knowing was not as developed as participants like Byron and Sarah. Elliot’s own work and thinking also served as a resource for interpreting, but he over-attributed his own thinking to Martha, arguing that because their work was the same, so was their thinking. Finally, he drew on his knowledge of common teaching and common students’ thinking to rationalize the students’ end products and help justify his claims about their thinking.

Elliot almost always decided to respond, at least to some extent, based on his interpretations of the hypothetical students’ thinking. Further, his cautiousness to differentiate between correctness of work and correctness of thinking was reflected in his responses, especially his “give or ask” responses designed to test various hypotheses about the students’ thinking. Many of Elliot’s “give or ask” responses also aimed to have the student realize his or her own errors and misconceptions. He specifically advocated for counter-examples as a result of his orientations, perhaps informed by his pedagogical content knowledge. In addition, Elliot proposed a greater number of “show and tell” responses than any of the other case study participants.
Craig

At the time of data collection, Craig had been a high school mathematics teacher for eight years. On his survey, Craig claimed that his “teaching style is constantly evolving,” and elaborated, “I started as a teacher-led classroom, and with technology have incorporated more investigations and problem solving into the classroom.” Craig had taught Algebra I, Algebra II, Probability and Statistics, an integrated mathematics class, and four years of Precalculus. In the following sections I address each of my three research questions with respect to this one particular participant, Craig.

**Question 0: What content knowledge of radian, unit circle, and the sine and cosine functions did Craig possess?** Out of 19 participants, Craig received the second highest score on the trigonometry assessment, 21 points out of 25, or 84%. Craig exhibited above average procedural trigonometric knowledge and fluency as was evidenced by his assessment score and the fact that, with the exception of a couple of arithmetic errors, Craig was able to correctly answer all of the mathematical prompts during the interview. As he interacted with the students’ work however, it became clear that Craig had a somewhat superficial or narrow understanding of some of the trigonometric concepts, especially radian and the relationship between sine and cosine as cofunctions. In the following subsections, I provide a brief description of Craig’s knowledge of radian, arc length, unit circle, and the sine and cosine functions (including their relationship as cofunctions). Later, I talk about how he drew on this knowledge to professionally notice students’ work and thinking.

**Radian and arc length.** Recall that Craig could accurately convert between radians and degrees on the assessment; however, both on the assessment and during
the interviews, Craig did not seem to understand radian as (i) the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle OR as (ii) the length of the arc on the unit circle subtended by the angle. Instead, Craig defined radian as “a unit of measure describing $1/2\pi$ of the central angle of a circle.” Similarly, during the interviews, Craig explained that he couldn’t say anything about a particular student’s understanding of radian because “she’s not looking at it as like a piece of an entire circle.” Craig also used similar proportional reasoning to find arc length. Both on the assessment and during the first interview, he found arc length of a given angle correctly by finding the circumference of the circle and then multiplying by the angle measure in radians divided by $2\pi$ (or the angle measure in degrees divided by $360^\circ$). In a later section, I provide additional evidence and details with respect to Craig’s understanding of radian while discussing how he drew on his subject matter knowledge to interpret thinking.

Unit circle. Craig correctly defined the unit circle as “a circle with a radius of 1 unit and center at the origin (0,0).” When asked why the unit circle was taught in trigonometry, Craig listed three reasons: (i) “in order to solve and develop trig ratios where the radius is 1,” (ii) “the proof of $\cos\left(\frac{n\pi}{3}\right)$, $\cos\left(\frac{n\pi}{2}\right)$, $\cos\left(\frac{n\pi}{4}\right)$, $\cos\left(\frac{n\pi}{6}\right)$, and their sine counterparts is much easier and can use previous geometry knowledge to complete,” and (iii) “to understand periodic behavior.” Recall that Craig was one of only two participants who indicated that the unit circle is a tool for reasoning about trigonometric functions. Consistent with this view, Craig himself used the unit circle to help make sense of problems, especially those involving sine and cosine.
**Sine and cosine.** Craig exhibited all three ways of knowing sine and cosine: as coordinates on a unit circle, horizontal and vertical lengths on a unit circle, and as ratios of the length of the sides of a right triangle. Craig seemed to rely on the unit circle definitions more heavily when solving mathematical problems and making sense of students' work. For example, during the first interview, when asked for an estimate for the cosine of and sine of 20 degrees, Craig explained, “so if you just draw a unit circle, I know 20 degrees is, is there. So if that's 20 degrees, sine is the, if it's the unit circle, sine is the y, x is the cosine, and so you know, well cosine of 20 degrees is greater than sine of 20 degrees.”

Craig also knew, $\sin(90° - \theta) = \cos(\theta)$, but he seemed to think of this as a particular “property” or “formula” which he memorized, and he did not seem to understand why this was true or have an understanding of the prefix co- in cofunctions. I make this claim based on his uncertainty regarding this “property,” and his comments with respect to the students in Task Two. For example, when responding to Alex (See Appendix D), Craig noted, “So I would have him - I'm assuming that it's ($\sin(90° - \theta) = \cos(\theta)$) been taught or discovered the correct formula or equation.” This statement appeared to indicate that Craig’s knowledge of $\sin(90° - \theta) = \cos(\theta)$ was knowledge of a “correct formula or equation” and not of the important relationship between sine and cosine. Further, when I asked Craig to comment on the students’ understanding of the relationship between sine and cosine, he did not mention “cofunctions.” Instead he posed that there were two relationships at hand defined by two equations, the Pythagorean identity and $\sin(90° - \theta) = \cos(\theta)$. Craig struggled to find the language to talk about Martha’s understanding of the relationship between sine and cosine, noting
that, “Martha has a definite understanding that uh the complements have a relationship.” Again, this is evidence that at the very least, Craig did not have knowledge of the term cofunctions.

Overall, as has been discussed in this section, Craig could solve all of the interview tasks correctly, had strong knowledge of sine, cosine, and the unit circle, as well as a procedural understanding of co-functions, but Craig only understood radian as a portion of circumference and not in relation to arc length. Within the next section, I include a discussion of how this mathematical knowledge influenced his mathematical noticing of students' work and thinking.

**Question 1: How did Craig attend to, interpret, and respond to hypothetical students’ work and thinking in trigonometry?** In this section, I discuss findings regarding how Connie (i) attended to, (ii) interpreted, and (iii) responded to students’ work and thinking.

**Craig's initial attention.** In this section, I answer the following: (i) what did Craig do when asked to notice?, and (ii) to what extent was his focus on the students’ thinking?. An overview of how Craig initially attended to the hypothetical students’ work and thinking is provided below in Table 5.3,
Table 5.3.
Summary of How Craig Initially Attended to the Students’ Work and Thinking

<table>
<thead>
<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Thinking</th>
<th>Describing</th>
<th>Evaluating</th>
<th>Challenged by Mathematics</th>
<th>Investigating</th>
<th>Comparing to Others</th>
<th>Comparing to Own</th>
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<tr>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
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<td></td>
<td></td>
</tr>
<tr>
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</tr>
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</table>

While Craig was able to make inferences about students’ thinking when prompted, without being prompted to do so specifically, Craig only focused on students’ thinking for seven out of 15 students, as is shown in the table above. Instead, Craig, most often described the students’ work and also evaluated the correctness of the work. In the following example in which Craig attended to Alex’s work on Task Two (See Appendix D), Craig’s descriptions are in italics and his evaluations are underlined.

Um, so he’s saying because they’re complementary then the sine of 20 degrees plus the sine of 70 degrees must equal the sine of 90 degrees, uh, which is not true, uh so sine of 20 degrees is approximately...ok. So then, well so...alright so then he got...ok so he’s trying to solve for...where, oh! because he’s using his previous wrong assumption, ok,
into that. That's why that looks weird, and then, well at least he knows it's the right quadrant....He was right on the complementary part, uh but then that's not a real property. So then because it's not a real property, that's obviously wrong. Uh then he tries to make another connection, so - oh that's him trying to solve for sine of 20 degrees. Now he's trying to solve for cosine of 20 degrees. So he's using the Pythagorean identity which is correct. He's plotting it correctly at first. So he's gonna subtract off the sine of 20 degrees and then take the square root. You get the plus or minus which is fine for now. Um, and then he uses the proviso wrong assumption form here and here to plug into that so that that answer is going to be incorrect; however he does know that it's gonna be positive because he knows that 20 degrees is in the first quadrant, and cosine in the first quadrant is, is positive there.

In the example above, Craig attended to and correctly evaluated all aspects of Alex's work but made no interpretive remarks or reference to why Alex might have made his errors that he did, i.e., the misconceptions that might have supported his errors.

As is also shown in the table, there were instances during Craig’s initial attention for which he appeared challenged by the mathematics in the students’ strategies but did not investigate and thus resolve these issues. For example, in response to Annie’s work, Craig noted that on Task One when Annie (See Appendix C) moved from a circle with a radius of one to a circle with radius three, “she's made it a third smaller, whereas it should be larger, maybe not necessarily three times larger, we can explore that, but definitely not smaller.” That is, Craig knew that the arc length corresponding to an angle on a circle of radius three should be larger than the arc length corresponding to that same angle on a circle of radius one; however he was not sure whether or not it would be precisely three times larger. It was interesting that he did not investigate this further, since he demonstrated the procedural fluency to calculate arc length.

Craig’s interpreting. Craig often equated procedural correctness with understanding or errors with misconceptions. For example, on Task Three (See
Appendix E) Craig claimed that because (i) “Anna um, seems to know how to use the Pythagorean Theorem correctly” and because (ii) Anna “knows how to solve for the sine of theta,” that consequently, (iii) “Anna understands, mathematically, what's going on.” Similarly, Craig claimed that Alex had an understanding of the relationship “between sine squared and cosine squared” because he had used the Pythagorean Identity.

When comparing Alex and Cheyenne’s work on Task Two (See Appendix D), Craig also claimed that “their understanding might have been very similar if not the same, because they were each wrong in one assumption.” That is, even though their errors were different, the level of their understandings were the same, because they each had one error. Some of these claims might be a result of Craig’s lack of interpretive language, in particular, his use of the word “understand,” which is discussed in a later section.

In the following sections, I continue the discussion of Craig’s interpreting by addressing the following question (which corresponds to research sub-question 1b):

What resources did Craig draw upon to interpret students’ thinking

In his reflections, Craig claimed, “in my view of it, my experiences in analyzing student work shaped the way I responded to the questions while being interviewed.” While Craig may have drawn on his pedagogical content knowledge gained from his teaching experience indirectly when interpreting students’ thinking, I only observed one instance in which Craig appeared to draw on his pedagogical content knowledge in a direct way. This was surprising since Craig had taught eight years and taught precalculus four times. Craig’s interpretations of students’ thinking seemed to be mainly informed by two resources: (i) his subject matter knowledge and (ii) his own thinking. In the following subsections I will
discuss how each informed his interpretations as well as how his lack of interpretive language seemed to limit his interpretations.

*Resources for interpreting: Craig’s subject matter knowledge.* When Craig attempted to interpret students’ thinking, his trigonometric knowledge both enabled and inhibited his ability to interpret students’ thinking. In Task One, he was not able to make sense of the students’ strategies because of his own limited understanding of radian. In particular, Craig noted that Charlie in Task One (See Appendix C) “seems like he’s just taking the number and bringing it down,” and with respect to Annie’s understanding of radian, “she doesn’t really mention anything about radians there.” Craig elaborated:

she at least knows that it’s an angle, that theta means angle, but as opposed to a degree versus radian, does not assume that, or doesn’t appear that she knows anything about it, because she’s not using it as a ratio. So she’s not looking at it as like a piece of an entire circle.

In the above excerpt, Craig’s understanding of radian as a portion of the central angle directly informed his interpretation of Annie’s thinking. Further, Craig articulated that he could not infer anything about Annie’s understanding of radian since she did not indicate the angle as a portion of a circle or show radian measure in relation to degrees.

Because Craig understood sine and cosine as horizontal and vertical lengths on the unit circle, Craig was one of the only participants who could make sense of Alice’s work in Task Four (See Appendix F), acknowledging that, “these horizontal lines, that's actually kind of an interesting way of looking at it, um, because it's the x-value, so she's seeing that it gets shorter and shorter and shorter which means that, the, the horizontal length is actually gonna end up being the height for the graph.” This was precisely what the work was designed to represent.
It was interesting that despite Craig’s understanding of sine and cosine as coordinates and lengths on the unit circle, he was unable to successfully interpret Joe’s thinking on Task Three (See Appendix E). Recall that Joe’s work was designed to represent a student who approached a right triangle trigonometry problem, by thinking about the unit circle definition of sine and cosine. Craig acknowledged that, “maybe he's trying to compare it to the unit circle. He's trying to make it with a hypotenuse of one, so in order to do that, uh, he makes it a similar triangle.” Despite the fact that Craig noticed Joe’s use of the unit circle, Craig did not access the rationale or thinking behind this strategy, noting that, “he didn't try to solve the problem, or answer what was asked of him.” In fact, when asked to discuss Joe’s understanding of the sine function in particular, Craig responded, “So I don't learn anything about the sine function from Joe. I learn that he can try to find all sides to a triangle.” This was surprising, especially given that (i) Craig was able to very quickly recognize Alice’s similar thinking in Task Four (See Appendix F) during the same interview and (ii) Craig recognized that Joe probably created a similar triangle with a hypotenuse of one to “compare it to the unit circle.”

I hypothesize that these contrasting examples may be a result of a rigidity in Craig’s knowledge of sine and cosine. In other words, while he understood sine as a vertical length on the unit circle, he had difficulty applying this understanding in a right triangle situation. I further hypothesize that if I had Joe draw the triangle inscribed in a circle, that Craig might have been able to say more about Joe’s thinking. The other possibility is that this difference in interpretation has less to do with the rigidity of Craig’s subject matter knowledge, and more of a reliance on his own thinking about the given problem. On this particular problem, Task Three, Craig solved the problem himself by
recalling “SOHCAHTOA,” relying on his understanding of sine as the ratio of the lengths of the opposite side and hypotenuse of the right triangle, whereas in Task Four, when prompted to graph the cosine function, he immediately drew the unit circle, noting, “So at zero cosine is the x so it's at one.” Thus perhaps he was more likely to recognize Alice’s thinking because it was more similar to his own in Task Three. Craig drew on his own thinking to interpret the students’ thinking in more obvious ways, which I discuss in the next section.

*Resources for interpreting: Craig’s own thinking.* As I mentioned, there were seven instances, during which Craig’s initial attention included a focus on the student’s thinking. In the following excerpt, I provide one such example, Craig’s initial attention to Amy’s work on Task Five. In addition to showing an immediate focus on this student’s thinking, the example was also chosen because it highlights how Craig drew on his own thinking to interpret the students’ thinking.

I'm probably just projecting here [...] there's a slight chance that she might understand the whole domain and range thing and that it has to be in the first or second quadrant, but I'm not sure. [...] that's off my assumptions not hers, because she didn't actually state that. So that could've just been either a lucky guess or something entirely different then what I'm thinking.

Here Craig recognized that he was drawing on his own understanding to interpret the students’ thinking; unlike Elliot however, we see that Craig was more cautious to diagnose students’ with his own thinking. In fact, I observed no instances in which Craig diagnosed students’ with his own thinking without the existence of evidence.

*Craig’s interpretive language.* In this section, I present the argument that Craig lacked interpretive language for talking about students’ thinking. When asked to talk about Alex’s understandings in Task Two (See Appendix D) Craig noted that “she does
understand the Pythagorean identity, sine squared plus cosine squared equals one” and that “really the only concept she didn't understand is she didn't understand that that's \((\sin(20°) + \sin(70°) and \sin(90°))\) not equal.” Here, as in several instances, Craig seemed to refer to properties or facts as concepts that one might or might not understand.

There were a few instances in which Craig referred to students' ways of knowing in descriptive terms. For example, when discussing Annie's understanding of radian, Craig commented that, “she's not looking at it as like a piece of an entire circle.” In most cases however, instead of talking about students' different ways of knowing or understanding in descriptive terms (“she understands X as…”), Craig talked about students' understanding in evaluative terms (good or bad) or comparative terms (better/worse understanding than another participant). For example, in response to Mike and Alice's work in Task Four, Craig noted that Mike had “a better understanding of the period” (comparative) and “that they both have an understanding of points, like the correct points” (evaluative). In these cases, Craig did not talk about how these students understood the “period” or “points.” Instead he simply commented on whether or not they understood them and who understood them better. To the reader or listener it is unclear what he meant by this. Perhaps he believed there is one correct way of knowing period, or perhaps he is again, using the term, understand, to refer to procedural correctness. Thus, I argue that Craig had not acquired adequate language for talking about students' thinking, or at the very least, students' understandings.

Thus far, in addressing the first main research question, I have discussed how Craig attended to and interpreted the hypothetical students' work and thinking, including
the resources Craig drew on to interpret. In the final subsection corresponding to the first main research question, I discuss the third skill of noticing. In particular, I discuss Craig’s proposed instructional responses to these hypothetical students.

**Craig’s responding.** In this section, I discuss how Craig responded to students’ work and thinking. In particular, I address the following questions (which correspond to research sub-questions 1d and 1c): (i) what were the goals of Craig’s proposed instructional responses and how were they informed by his interpretations of students’ thinking, and (ii) what other resources did Craig draw on to respond to students? An overview of how Craig responded to the hypothetical students’ work and thinking is provided below in Table 5.4.

**Table 5.4. Summary of How Craig Responded to the Hypothetical Students**

<table>
<thead>
<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Student’s Thinking</th>
<th>Affect</th>
<th>General Move</th>
<th>Show and Tell</th>
<th>Give or Ask</th>
<th>Aims to Access</th>
<th>Aims to Have Student Realize</th>
<th>Aims to Have Student Access</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Allen</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Lexi</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Annie</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
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<td>✓</td>
<td></td>
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<td>✓</td>
<td>✓</td>
</tr>
<tr>
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<td></td>
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<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>2</td>
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<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>2</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>Joe</td>
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<td>✓</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>Anna</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Mike</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>4</td>
<td>Alice</td>
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<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>Alexis</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>Joe</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>Sam</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>Amy</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
Twelve out of fifteen of Craig’s proposed hypothetical instructional responses were made, at least to some extent, on the basis of his interpretations of the students’ thinking. Two thirds of Craig’s responses were categorized as “Give or Ask,” as opposed to “Show and Tell,” which was consistent with Craig’s self-reported transition from a teacher-led classroom to a more investigation-based classroom. Also, consistent with his self-reported discovery approach, many of Craig’s “Give or Ask” responses were intended to lead students to some realization. In two of these instances, in addition to drawing on his interpretations of the students’ thinking, Craig seemed to also draw on his pedagogical content knowledge when deciding to respond to students. For example, in responding to Amy in Task Five (See Appendix G), Craig drew on knowledge of content and teaching, most specifically his knowledge of ideal teaching, to respond to Amy by using a particular representation of function, the “function box” like the one Craig drew in Figure 5.1.

![Figure 5.1. Craig’s Sketch of a Function Box](image)

Craig, explained:

one way to do the cosine of the cosine thing is to kind of look at just like a function box. So what goes in the cosine function? An angle goes in, so some sort of thing with pi usually or degrees, and what comes out is something between negative one and one [...] then to go back into cosine again that needs to be an angle and it's not. So that right away should give you pause right there.
Craig’s wording, “one way to do the cosine of cosine thing is to…,” seemed to indicate that this is a strategy he has used before, and he also knew the advantage to using this representation with students, noting, “that right away should give you pause.” In another similar example, Craig drew on his knowledge of the advantages of counter-examples, his subject matter knowledge, and his orientations to respond to Alex in Task Two (See Appendix D). In deciding to respond to Alex, Craig wrote the example below:

\[ \sin(90^\circ) + \sin(60^\circ) \]

He then explained, “So I would give him just this and hopefully he could do the rest of it.” He later said, “I would ask him what he, by his own equation what would it equal? It would be sine of 90. Then I would have him say well do these out and do they equal each other, and hopefully he says no. If he says no, he should hopefully realize that his final answer couldn’t be correct.” Craig drew on his knowledge of the advantages of counter-examples to choose his general strategy, drew on his subject matter knowledge to pick a particular example that satisfied the necessary conditions, and finally Craig’s orientation towards building on students’ thinking supported his use of counter-examples.

Many of Craig’s responses were also aimed at accessing additional information about the students’ thinking. These responses often reflected his concern for not over-projecting his own thinking onto the students. For example, his response to Charlie in Task One (See Appendix C) was to “just ask him to start from the beginning and explain it,” and his response to Sam and Amy in Task Five (See Appendix G) was to “ask for a little information…like they look like they have some ideas, but I don’t wanna project
what I know about it into what I think that they should know or what I want them to know."

Thus far, I have discussed (0) Craig’s knowledge of radian, the unit circle, and the sine and cosine functions as well as (1) how Craig attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry. In the following section, I discuss (2) the ways in which the hypothetical students’ work challenged and/or furthered Craig’s mathematical thinking. I then provide an overall summary regarding this case study of Craig.

**Question 2: In what ways did the hypothetical students’ work on trigonometry problems challenge and/or further Craig’s mathematical thinking?**

Unfortunately, while there were opportunities for Craig to “grow” mathematically, this growth did not seem to occur. As I mentioned, Craig did not understand radian as (i) the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle OR as (ii) the length of the arc on the unit circle subtended by the angle. Thus, there was potential for Annie, or Charlie’s work, especially Annie’s, to encourage him to expand his conception of radian. I hypothesize that Craig might have experienced cognitive conflict if he had used his own proportional thinking to check Annie’s initial claim that the arc length of the angle in the unit circle was precisely the angle measure, but he quickly dismissed it, noting “So she's equating central angle to arc length. So she's got the wrong idea for that,” and it was also both interesting and disappointing that he did not question why she might have thought to draw a unit circle first. Due to his quickness to dismiss her work as incorrect and due to his focus on other, less significant, aspects of the work, Craig did not experience cognitive conflict.
example highlights the fact that presenting students’ with challenging work is not sufficient for challenging their thinking.

Despite not seeing any mathematical growth, Craig did claim that after the interview experience, he “will have a better understanding of the different ways that students think.” He also claimed that he became aware of “the idea of a teacher projecting what they think the student should know into an answer, when the student doesn’t explicitly write out their reasoning” and, thus he “will, in the future, work with students on better clarifying their answers and reasoning.”

**Summary: Craig.** Recall that Craig had been a teacher for eight years and had taught precalculus for four of those years. He had enough trigonometric knowledge to score an 84% on the assessment and to answer all of the mathematical interview prompts correctly; however because he didn’t understand radian in relation to arc length, he had difficulty interpreting students’ work in the first task. Craig didn’t engage in a significant amount of new mathematics in response to the students’ work, and thus he did not seem to think any differently about radian as the interviews went on.

Craig focused on students’ thinking the least of all of the participants. In fact, without specific prompting, Craig’s initial attention was only focused on the students’ thinking for 6 out of the 15 students. Instead Craig’s initial attention was on the work, which he described and evaluated, drawing on his mathematical knowledge. He also seemed to focus on correctness of the students’ work when ranking the students.

When he did interpret students’ thinking, Craig’s subject matter knowledge as well as his own thinking informed his interpretations. Unlike Elliot, Craig did not always differentiate between correctness of work and thinking but was more cautious when
drawing on his own thinking. In particular, he was aware that he might be “just projecting.” Perhaps Craig might have devoted more focus to students’ thinking, but he appeared to not have adequate language for describing students’ different ways of understanding. Instead, Craig talked about “understanding” in evaluative or comparative terms.

Almost all of Craig’s proposed instructional responses were made on the basis of his interpretations of the students’ thinking. They were mainly “give or ask” responses designed to either have the student realize something or elicit additional information about the students’ thinking, informed by his cautiousness with regards to projecting his own thinking onto his students. To inform his responses Craig also drew on his subject matter knowledge as well as his knowledge of content and teaching, most specifically his knowledge of content and ideal teaching.
Byron

At the time of data collection, Byron had taught high school mathematics for 17 years and had taught precalculus for 15 of those 17 years. He had also taught an honors precalculus class twice. On his background survey, Byron claimed that he was a “traditional teacher” or “lecturer w/examples.” He elaborated that he “call(s) upon students to answer or come to board show. Some group work, but mostly independent work – worksheets/bookwork at seat.” Byron reasoned, “I think that students need to have the tools before they can use them. A teacher provides/illustrates how to use and what the tools are. The student given a basis/context can then use them correctly. I believe in constructivism. NOT Blind discovery.” In the following sections I address each of my three research questions with respect to this one particular participant, Byron.

**Question 0: What content knowledge of radian, unit circle, and the sine and cosine functions did Byron possess?** Recall that out of 19 participants, Byron received the third highest score on the precalculus assessment, 20 points out of 25, or 80%. In the following subsections, I provide a brief description of Byron's knowledge of radian and arc length, unit circle, as well as the sine and cosine functions, including their relationship as cofunctions. Later, I discuss how this mathematical knowledge influenced his mathematical noticing of students' work and thinking.

**Radian and arc length.** Recall that on the assessment, Byron defined radian as the “angle created/subtended by ‘wrapping’ 1 radius on the circumference,” and he provided the picture in Figure 4.1 to clarify this definition. Later, during his explanation of the relationship between radians and degrees, Byron also explained that given a central angle, “You can view that as subtending an arc length,” and that “radian measure would
then be how many radii fit into that arc length.” Thus, Byron seemed to hold an understanding of radian as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle.

While Byron understood radian measure as “how many radii fit into that arc length,” it was interesting that he did not appear to draw directly on this understanding when solving problems that required him to find arc length. Instead he called upon the formula, and I observed no evidence that indicated he understood why this formula worked. For example during the interview when asked to find arc length, he began by stating, “There’s a formula. Arc length equals the radius of the circle times the uh angle, and the angle has to be in radians, and this angle measure is in radians. So the radius is three. So its three times negative pi twelfths.” Similarly on the assessment, Byron justified his work by writing the following in Figure 5.2.

![Figure 5.2. Byron’s Explanation When Finding Arc Length](image)

**Unit circle.** Byron correctly defined unit circle as a “circle with radius of 1 unit” on the assessment and similarly during the interviews, explained, “the unit circle is called the unit circle cuz it’s radius is one.” Recall that Byron was one of two participants who indicated, to some extent, the importance of the unit circle for reasoning about trigonometric functions. In particular, Byron explained that “if the radius is 1” then “(x,y)” becomes “(1cosθ, 1sinθ),” and that since a “point consists of an x distance, y distance,” then “1 = x² + y²,” and so therefore, “1 = cos²(θ) + sin²(θ).”

**Sine and cosine.** Byron exhibited all three ways of knowing sine and cosine: as
coordinates on a unit circle, horizontal and vertical lengths on a unit circle, and as ratios of the length of the sides of a right triangle. In contrast to Elliot and Craig, Byron's concept image of sine and cosine did not seem to be dominated by one of these three. Byron drew on these ways of thinking simultaneously and in support of each other. For example, when drawing the graph of the cosine function, Byron explained:

The cosine on the unit circle - so the unit circle is called the unit circle cuz it's radius is one. So therefore when I take any point here, ok, x, y, and this would be the angle in standard position. The cosine of the angle is gonna be the adjacent. If we're talking about a right triangle here is gonna be the adjacent over the hypotenuse - well the hypotenuse is one. So it's really just the adjacent side over one. So that being said, since it's an ordered pair x-y, that would be x over one. So the cosine of theta is gonna be x. So when we get to pi over two, we see we don't really actually have a triangle, or if you think about it as a really tight little triangle, the x-coordinate here is zero, and the y-coordinate in one.

In the above example, Byron referenced all three ways of knowing cosine: (i) “the adjacent,” (ii) “adjacent over hypotenuse,” and (iii) “x.” Byron also seemed to coordinate the three together. In another example, when asked to come up with estimations for certain sine and cosine values during the interviews, Byron immediately reasoned, “Oh boy, I'm not sure. Um...I mean (drawing triangle) if we're talking right triangle trig, then we're talking about this being 20 degrees and if this is y, and this is x, and this is r, then I'm saying that the sine of 20 degrees is gonna be y over r, and the cosine of 20 degrees is gonna be x over r.” It is interesting that Byron used r to represent the length of the hypotenuse, unlike most who use h or “hyp”; this corroborates evidence that Byron holds both unit circle and right triangle conceptions of sine and cosine and draws on them simultaneously.

Byron also understood the relationship between sine and cosine as cofunctions. When he was asked to estimate sine and cosine values, he immediately tried to think of
the relationship between the two and claimed “the sine of 20 degrees is gonna be equal to the cosine of 70 degrees, and the cosine of 20 degrees is gonna be equal to the sine of uh, 70 degrees, so there’s a relationship.” Byron also had knowledge of the correct terminology, noting that, “sine and cosine are cofunctions.”

Overall, as has been discussed in this section, Byron had very strong and flexible knowledge with respect to the targeted trigonometry concepts and could answer all of the interview prompts correctly. Within the next section, I include a discussion of how this mathematical knowledge influenced his mathematical noticing of students’ work and thinking.

**Question 1: How did Byron attend to, interpret, and respond to hypothetical students’ work and thinking in trigonometry?** In this section, I discuss findings regarding how Byron (i) attended to, (ii) interpreted, and (iii) responded to students’ work and thinking.

**Byron’s initial attention.** In this section, I answer the following: (i) what did Byron do when asked to notice?, and (ii) to what extent was his focus on the students’ thinking?. An overview of how Byron initially attended to the hypothetical students’ work and thinking is provided below in Table 5.5.
Table 5.5.
Summary of How Byron Initially Attended to the Students’ Work and Thinking

<table>
<thead>
<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Thinking</th>
<th>Describing</th>
<th>Evaluating</th>
<th>Challenged by Mathematics</th>
<th>Investigating</th>
<th>Comparing to Others</th>
<th>Comparing to Own</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Allen</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Lexi</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Annie</td>
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<td>✓</td>
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<td>✓</td>
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<td></td>
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<tr>
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<tr>
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<td>Alex</td>
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<td>✓</td>
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<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
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<td>✓</td>
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<td>✓</td>
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<td></td>
</tr>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Joe</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
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<tr>
<td>3</td>
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<td>✓</td>
<td>✓</td>
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</tr>
<tr>
<td>4</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Alice</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Alexis</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
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<td></td>
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<td>✓</td>
<td>✓</td>
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</tbody>
</table>

As is shown in this table, without being prompted specifically to do so, Byron attended to two thirds of the hypothetical students’ thinking. His interpretations of students’ thinking as well as the resources he drew on to interpret students’ thinking are included in the next sections. When attending to the students’ work and thinking Byron also made definitive evaluative claims for fourteen out of fifteen students, and all of the evaluative claims were accurate. Also note that, in all three of the instances for which Byron noticed an aspect of the work for which he was uncertain of the validity, he investigated, and then was able to evaluate. For example, when he examined Cheyenne’s work on Task Two (See Appendix D), Byron pondered, “sine of 30, I’m
pretty sure is one half. Maybe I'm wrong”; he then drew a right triangle and it became clear to him that “if this is one then this is twice that and this is that. Yeah. So sine of 30 is one half.” Thus, he correctly determined Cheyenne’s error. Only Alex’s work on Task Two seemed to challenge Byron in any significant way; this is discussed further in the section on “Effects of the Interview Experience.” As is shown in the table above, one should also note that Byron rarely compared the students’ work and thinking to his own or to the other students when initially attending to the students’ work or thinking.

**Byron’s Interpreting.** As I previously noted, Byron spent significant time focusing on students’ thinking. Further, when interpreting students’ thinking Byron distinguished between procedural correctness and thinking or understanding of concepts. For example, when asked about Lexi’s understanding of radian based on her work on Task One (See Appendix C), Byron noted that she is looking at it “in terms of the formula and r and theta. You put numbers in, and you calculate, and I don’t, I think she's seeing it more in a formulaic sense and not necessarily putting it into the unit circle or seeing it as an arc length as like Annie is.” In another example, Byron noticed Anna’s error in Task Three (See Appendix E) in which she wrote sine of y instead of sine of theta, and considered whether it was “a careless error” or a “conceptual error.”

Another distinguishing factor about Byron’s interpretations of students’ thinking is that he not only described the student’s thinking about the problem and concepts at hand, but also made general claims about the students’ thinking and tendencies. In particular, he contrasted both Allen and Lexi in Task One (See Appendix C) as well as Mike and Alice in Task Four (See Appendix F), in terms of the way they probably
approach/view mathematics in general. For example, when comparing Allen and Lexi (without prompting to compare or rank them), Byron claimed the following:

> Lexi is very theory oriented [...] she's probably really good at memorizing, and she could probably hold lots of formulas in her head [...] she probably uses them correctly most, most of the time, um, but she doesn't see things sort of in a conceptual manner. Like, she's more right handed than left-handed, and Allen is more left-handed, maybe more geometry oriented.

Byron further argued that Allen had “more of a sense of being able to sort of expand his views mathematically” and that he “probably has more potential to grow mathematically than Lexi.” He reasoned this was because:

> She (Lexi) probably sees math as very black and white, which you know, maybe in some ways math is black and white, but I think Allen sort of has more of a malleable sense of the math, and with that he can adjust it and grow and make changes. Uh, this might be more hard for someone like Lexi to do that are very formula driven and not, very rigid maybe in their thinking, which would give rise to why she sort of just plugged and chugged [...] Allen has a sense of trying to make bigger connections and try to pull things together even though he, you know, kind of makes mistakes.

These overall claims about Lexi and Allen seem to be informed by Byron’s own orientations, namely the way he views mathematics and what it means to do mathematics as well as his beliefs about learning. Similarly, in Task Three, Byron argued, “Mike is more um det- more um analytical, and I would say Alice is more um I can't think of the correct appropriate word probably, but she's more um heuristic, or she has more of a visual sense.” He predicted further that, “in terms of high school math” or “in terms of science and scientific work, and higher level math, Mike’s probably gonna retain - you would hope that he would be more along those lines.” Even in Byron’s written reflection he commented that “it was clear with all the tasks – graphing,
estimating trig values and other trig tasks that there was two views of the students. There was a visual understanding and an analytic understanding."

Thus far, I have discussed Byron’s tendencies (i) to distinguish between procedural correctness and thinking and (ii) to not only described the students’ thinking about the problems and concepts at hand but also to make general claims about the students’ thinking. In the following sections I continue the discussion of Byron’s interpreting by addressing the following question (which corresponds to research sub-question 1b): What resources did Byron draw upon to interpret students’ thinking? In particular, I explain how Byron drew upon his interpretive language, his subject matter knowledge, and his pedagogical content knowledge, to interpret the hypothetical students’ thinking.

Resources for interpreting: Byron’s interpretive language. In this subsection, I present the argument that Byron used interpretive language that allowed him to be more precise in describing students’ thinking. In particular, he used language such as “she understood/sees/thinks of X to be…” instead of only referring to understanding in evaluative terms (good/bad), comparative terms (better or worse) or as existent (“she understands X”) or nonexistent (“he didn’t understand Y”). For example, when asked to discuss each student’s understanding of radian in particular, in the first task, Byron explained that Allen “sees the radian maybe in terms of a fraction, and he's dividing the circle up into fractional pieces,” whereas Annie “is seeing the, the radian measure as arc length, uh, the angle I mean as an, as an arc length.” Similarly, when asked about Anna’s understanding of the sine function (in Task Three), instead of claiming, like that Anna has a good understanding of the sine function, Byron described that she
“understand(s) the idea that the sine is the ratio of - the sine of an angle is the ratio of sides.” Similarly, when asked about Alice’s understanding of cosine (in Task Four), Byron claimed that she was “understanding that it’s the, the length of x” on the unit circle. At first, I hypothesized, as some might, that Byron simply noticed more detail with respect to students’ understanding than most; however, after careful attention to the data, I argue that his interpretive language, and most importantly his perception and use of the word “understand,” was different from the other case study participants, and that this was at least one of the reasons why Byron was able to discuss students’ thinking in more detail.

Resources for interpreting: Byron’s subject matter knowledge. When Byron attempted to interpret students’ thinking, in almost all cases, his trigonometric knowledge enabled his interpretation of students’ thinking. For example, as was discussed, Byron understood radian as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, and thus Byron was able to hypothesize that in Task One, Charlie (See Appendix C) was “seeing the angle and the arc length as being equal,” noting that, “we’re not on a unit circle of one which we can see it does come out there to be that.”

I’ve discussed how Byron also held all three ways of knowing sine and cosine and appeared to hold them simultaneously and in support of one another. By drawing on these understandings Byron was able to make sense of Alice’s work and thinking on Task Four (See Appendix F). In particular, he noted that she was:

sort of seeing a length you know an x length sort of shrinking and growing, and isn't necessarily thinking of it in terms of angle measures and ordered pairs and thinking of the cosine of theta as x over r
In the excerpt above, Byron not only acknowledged how Alice WAS thinking about cosine (as “an x length”) but also acknowledged how Alice WAS NOT thinking about cosine (as “ordered pairs” or as “x over r”). Thus, in a positive way, Byron’s understandings of sine and cosine were very much reflected in and influential on how he interpreted Alice’s thinking. Byron also understood the relationship between sine and cosine as cofunctions, and thus identified that in Task Two, Martha (See Appendix D) “has the cofunction piece down.”

It was interesting that despite Byron’s flexible understandings of sine and cosine, he was not able to make sense of Joe’s work on Task Three (See Appendix E). That is, although he had adequate mathematical knowledge to interpret Joe’s thinking, he was unable to do so. This might be because this sort of work was unfamiliar to, and surprised, Byron. He noted, “I have to say, from what I've experienced, I'm - I'm not so familiar seeing Joe's work.” As I will discuss in the next subsection, Byron drew heavily on his pedagogical content knowledge, especially his knowledge of content and students, to interpret the hypothetical students' thinking. Thus, it was interesting that the one case in which he could not make sense of the student’s work was the one case that he identified as “unfamiliar.”

*Resources for interpreting: Byron’s Pedagogical Content Knowledge (PCK).* Of all the participants, Byron drew on his PCK most frequently when interpreting students’ thinking. This was not very surprising since Byron had taught Precalculus for 15 years. Recall that PCK consists of Knowledge of Content and Teaching (KCT), Knowledge of Content and Students (KCS), and Knowledge of Content and Curriculum (KCC). Most often, Byron drew on his knowledge of content and students (KCS). For example, he
made the following claims regarding Allen in the first task (See Appendix C):

> he's mistaken the quantity of pi over 12s that are gonna be in his full circle. So that's the mistake that he's making, which is a pretty common error, um, students make. They forget that it's 2pi all the way around and not just pi around

Note that Byron went beyond simply drawing on knowledge of a common error. In contrast, Byron drew on knowledge of common thinking/reasoning behind that error: students think it’s “pi around” and forget “that it’s 2pi all the way around.” He also drew on his KCS to make more general claims about Allen’s thinking, explaining, “he's sort of seeing it in a geometrical sense, and I really see that a lot, and I think even as a, as, uh a student of trigonometry, you sort of, there's sort of this geometrical sense to it.”

Similarly, when shown Anna’s work on Task Three (See Appendix E), Byron explained that it was “familiar” and drew on his KCS to interpret the possible thinking behind the error Anna made when she wrote sin(y). Byron explained, “I don't believe students understand the difference between an angle and a side involving trig. Um they constantly sort of get the two mixed up,” and he reasoned further, “they see an a radian measure, and they think it is a ratio of sides. So there's this constant interchange between angles and sides.” This had not been something I had considered in my own analysis of the students’ work. Byron himself was even aware that he drew on KCS to notice. He explained in his written reflection that, “my experiences in analyzing student work shaped the way I responded to the questions while being interviewed.”

There was also one instance for which Byron drew on his knowledge of content and teaching (KCT) to interpret student’s thinking. Recall that Byron interpreted that in Task Four (See Appendix F), Alice was “sort of seeing a length you know an x length sort of shrinking and growing, and isn't necessarily thinking of it in terms of angle
measures and ordered pairs and thinking of the cosine of theta as x over r.” He was able to interpret her thinking about cosine based on his own ways of knowing sine and cosine, but his interpretations also seemed informed by his KCT. When he first looked at Alice’s work on Task Four (See Appendix F), Byron reasoned aloud: “Alice, what are you doing? Um..you are looking [...] I like to call this sort of the um, the animated version, um, and if you, and if - you get a good picture of this if you look at a lot of the um online uh applets or the java scripts or the geogebra, and you see that as x as a length.” Along with his subject matter knowledge, Byron’s knowledge of a strategy for teaching about trigonometric functions using a dynamic representation enabled him to make sense of Alice’s work, and ultimately, interpret her thinking.

Thus far, in addressing the first main research question, I have discussed how Byron attended to and interpreted the hypothetical students’ work and thinking, including the resources Byron drew on to interpret. In the final subsection corresponding to the first main research question, I discuss the third skill of noticing. In particular, I discuss Byron’s proposed instructional responses to these hypothetical students.

**Byron’s responding.** In this section, I discuss how Byron responded to students’ work and thinking. In particular, I address the following questions (which correspond to research sub-questions 1d and 1c): (i) *what were the goals of Byron’s proposed instructional responses and how were they informed by his interpretations of students’ thinking*, and (ii) *what other resources did Byron draw on to respond to students?* An overview of how Byron responded to the hypothetical students’ work and thinking is provided below in Table 5.6.
Twelve out of fifteen of Byron’s proposed hypothetical instructional responses were made, at least to some extent, on the basis of his interpretations of the students’ thinking. For example, recall that Byron interpreted that Allen’s (See Appendix C) misconception was that he was thinking that it was “just pi around” and not “two pi around” the circle. Based on this interpretation, Byron’s proposed response to Allen was:

I would say, you know it's two pi, not just pi so when you're working with your pi over twelve, we're not trying to get it to be um, pi, but we gotta get it to be two pi. So it's not twelve pi over twelve. It's gotta be 24 pi over twelve because we want two pi. We don't want pi.
In this example, while Byron’s reasoning was based in his interpretations of Allen’s thinking, unlike the other case study participants, Byron told Allen directly that this was incorrect and that “it’s two pi, not just pi” and that “it’s gotta be 24 pi over twelve.” In fact, Byron proposed a relatively high number (seven) of “show and tell” responses compared to the other case study participants (with the exception of Elliot), which was consistent with Byron’s claim that he was a “traditional teacher” or “lecturer with examples.” Despite being a “lecturer” however, Byron did form more (ten) “give or ask” responses than he did “show and tell” responses. Most often, his “give or ask” responses were designed to elicit additional information about the student’s thinking. For example, in his response to Joe in Task Five (See Appendix G) Byron explained:

for Joe, probably would need to have a little bit more discussion about you know, um, kinda wanna figure out a little bit more about what's, what he's thinking here, and maybe have him talk through why he did what he did and then bring in the idea of like well ok you know, is this gonna work for the, you know what can you tell me about the cosine inverse in terms of it's, it's um range, and how did you - how are we gonna work with that? And see if he knows his familiar angle measures to evaluate the cosine of that, and then take the cosine inverse of that.

In the above excerpt, Byron proposed asking a series of questions to Joe in order to “figure out a little bit more about what’s, what he’s thinking here.” In another example, when Byron was asked to respond to Cheyenne in Task Two (See Appendix D) he responded:

I think she’s just memor- working off a memory and memorization. I'm not so sure. I'd ask her more about like what does the sine represent? What does the cosine represent? Could you draw me a right triangle and talk about that a bit or draw the unit circle? You know draw the unit circle and sort of say, OK, there we go. There's 20 degrees. So what can you tell me and we'll say this is one. We're on the unit circle so what can you tell me? And then maybe have her walk through a bit with that.
In the above excerpt, Byron interpreted that Cheyenne was memorizing facts incorrectly, but he decided to pose questions to verify that there isn’t a larger misconception underlying her error.

Three of Byron’s “give or ask” responses did not aim to access additional information about the students but instead were intended to lead students to some realization. For example, Byron’s response to Amy on Task Five (See Appendix G) was “you know what familiar angle is negative 135 degrees in the family with? Ok, now draw me the unit circle. Let's plot it. Now what would the reference angle be?.” Byron further explained that he would ask these questions because “she knows she needs to be in quadrant two,” and so his goal is simply “to get her on the track of 45 degrees.” That is, Byron asked Amy to answer questions and draw the angle to realize that her “reference angle” should be 45 degrees.

In the case of Joe in Task Three (See Appendix E), the student whose work and thinking Byron could not make sense of, his response was the least detailed and did not focus on Joe’s thinking. Instead he decided on a “show and tell” response that only addressed one aspect of Joe’s work, an error that Byron could identify involving the Pythagorean theorem. In particular, he claimed “I'd say remember that the Pythagorean identity is squared quantities and we're looking for the sine of theta, not the, not sine squared.”

Thus far, I have discussed (0) Byron’s knowledge of radian, the unit circle, and the sine and cosine functions as well as (1) how Byron attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry. In the following section, I discuss (2) the ways in which the hypothetical students’ work
challeted and/or furthered Byron’s mathematical thinking. I then provide an overall summary regarding this case study of Byron.

**Question 2: In what ways did the hypothetical students’ work on trigonometry problems challenge and/or further Byron’s mathematical thinking?**

Since Byron had adequate trigonometric knowledge to evaluate and interpret the students’ work and thinking, it was not surprising that he was not challenged mathematically in significant ways. There was only one instance in which he seemed to realize something new. Alex, in Task Two (See Appendix D), claimed that the sum of the sine of 20 degrees and the sine of 70 degrees was the sine of 90 degrees, and Byron had difficulty at first determining whether or not this was true. He pondered. “I feel like this is just not right to me” and “I'm not sure why I feel that way, and I may have the wrong feeling on this.” He seemed to get frustrated, exclaiming, “Ugh, I'm just not, I don't know.” Eventually, using a counter example, he determined that the statement was false. Later, he indicated that he did in fact learn something, explaining, “Alex's work, she's making the mistake which I have come to realize that you can't add the (laughing), taking the sine of complements and adding them together doesn't equal the sine of the angle.”

**Summary: Byron.** Recall that Byron had been a high school teacher for 17 years and taught precalculus for 15 of those 17 years. Byron scored the third highest score (82%) on the precalculus assessment. Byron answered all of the mathematical prompts in the interviews correctly and Byron understood radian as the “‘angle created/subtended by ‘wrapping’ 1 radius on the circumference” and held all three ways of knowing sine and cosine. Byron’s subject matter knowledge was challenged by the
students’ work in one particular case in which he was forced to examine the validity of
\[ \sin(20^\circ) + \sin(70^\circ) = \sin(90^\circ). \]

Without prompting, Byron’s attention included a focus on thinking for two thirds of
the hypothetical students. When interpreting students’ thinking, Byron distinguished
between procedural correctness and thinking, and he not only described the students’
thinking about the problems and concepts at hand but also made general claims about
students’ thinking and tendencies. Frequently, Byron discussed students having an
“analytical sense” or a “visual sense.” Byron used interpretive language that allowed him
to be precise in his descriptions of the students’ ways of knowing or understanding, and
he also drew on his subject matter knowledge, knowledge of content and students, and
knowledge of content and teaching to interpret students’ thinking.

The majority of Byron’s proposed hypothetical instructional responses were
made, at least to some extent, on the basis of his interpretations of the students’
thinking. Consistent with Byron’s admission that he was a “lecturer,” Byron proposed
seven “show and tell” responses; however more of his responses were coded as “give
or ask.” Most of these were aimed at eliciting additional information about the students,
which might have been a result of Byron’s cautiousness to differentiate between
procedural correctness and underlying thinking. Byron expressed that students who
engage in problem solving have more potential for growth and development than
students who simply apply formulas and plug in values. It was interesting however, that,
similar to the situation with Elliot, Byron’s rankings of the students did not reflect this
belief.
Sarah

At the time of data collection, Sarah had taught high school mathematics for 19 years. While she had never taught precalculus, she had taught geometry all 19 years (at three different levels, up to honors), and, as will be discussed at length in the following sections, Sarah drew on her geometry knowledge to respond to and notice students’ work on the trigonometry problems. Sarah often referred to herself as a geometry teacher, but she had also taught five years of Algebra I and six years of Algebra II. During the interviews, Sarah indicated that she had “been doing a lot more reading lately and paying attention to conferences and stuff like that and seeing the shift in math education.” It seemed as though Sarah was at a point in her career in which she was trying to balance “traditional” versus “progressive” as she put it, and wasn't sure where to place herself along that spectrum. She noted when responding to students, “the traditional person in me is thinking, well let me just respond to what they have and there we go. That’s it. That’s what you messed up on”; whereas “the more progressive person in me is saying alright well I, let me, let me ask this, let me ask that.” That is, the more “traditional” version of herself would simply say, “that’s what you messed up on” whereas the more “progressive” version of herself would not tell, but ask. This was consistent with her self-reported teaching style; Sarah indicated that in the classroom she saw herself as both a “presenter” as well as a “facilitator” when students are “working solo or in groups on problems.” In the following sections I address each of my three research questions with respect to this one particular participant, Sarah.

Question 0: What content knowledge of radian, unit circle, and the sine and cosine functions did Sarah possess? Out of 19 participants, Sarah was awarded the third highest score (tied with Byron) on the precalculus assessment, 20 points out of 25,
or 80%. Sarah did not have the breadth or depth of trigonometry knowledge that Byron had, but as was mentioned previously, Sarah had significant experience teaching geometry, and was able to draw on her geometry knowledge to solve the trigonometry problems, in most cases. Her thinking and knowledge with respect to radian, unit circle, and the sine and cosine functions are discussed in the following subsections.

_Radian and arc length._ During the interviews and on the assessment Sarah demonstrated that she could convert easily between radians and degrees. Like the participants in Akkoc’s (2008) study, as well as several participants in this study, Sarah’s concept image of radian seemed to be dominated by degree. For example, Sarah defined radian measure in terms of its relationship to degree measure, and each time she was faced with a problem that involved angles measured in radians, she would say something similar to, “I just gotta convert it into degrees because I think better in terms of degrees.” Sarah did not exhibit any awareness of a connection between radian measure and arc length, and Sarah consistently exhibited a view of arc length as a portion of a circle’s circumference.

_Unit circle._ Sarah defined unit circle as “a circle w/radius 1.” Sarah also wrote that, “unit circle is taught to help students learn trigonometric ratios. Right triangles can be drawn, the hyp is 1, the coz is horz leg & sin is vert leg (can’t remember why) but the rt triangles helped w/calculations such as sin30.” That is, like many other participants, Sarah felt that the main reason the unit circle is taught to students is to find/learn/prove the sine and cosine values of 30°, 45°, and 60°. Sarah did not discuss how the unit circle enables the extension of trigonometric functions to all real numbers or that the unit
circle can be used to explain properties of trigonometric functions such as symmetry and periodicity

**Sine and cosine.** Sarah exhibited all three ways of knowing sine and cosine: as coordinates on a unit circle, horizontal and vertical lengths on a unit circle, and as ratios of the length of the sides of a right triangle. She appeared to rely most heavily on her understanding of sine and cosine as ratios of the lengths of the sides of a right triangle, and almost always used this definition to make sense of the situation on the unit circle. Her work estimating sine and cosine of 20 degrees on the second interview task highlights the way she tended to think about sine and cosine on the unit circle. Sarah initially drew a right triangle with one 20° angle and proceeded to describe sine and cosine in terms of the lengths of its sides. Her work is shown below in Figure 5.3.

![Figure 5.3. Sarah's Work on Task Two: Part One](image)

Sarah then contrasted this “geometry” approach to what she felt was more of a “trigonometry” approach using the unit circle. She explained,

> if I were to go a little bit more in depth, since we’re talking about trigonometry, if we think unit circle, so 20 degrees would be maybe about right here, and I know that, again what I remember from high school is the if it’s a unit circle, um, the cosine is - I remember the cosine as being the x-coordinate […] because if, if the cosine is adjacent over hypotenuse, and it’s a unit circle, that means my denominator is one […] and then since sine is opposite over hypotenuse […] sine of 20 degrees would be
this over one. So that’s why that was always the um, uh, the y-coordinate of this point.

This excerpt highlights that Sarah was aware of sine and cosine as coordinates on a unit circle but needed to keep reassuring herself using her understanding of sine and cosine in terms of a right triangle.

While Sarah did rely heavily on her right triangle definition, she frequently proved she was able to draw on the different ways of knowing sine and cosine in conjunction with each other. For example, Sarah was able to explain the relationship between sine and cosine as cofunctions using both the right triangle definition and an understanding of sine and cosine as horizontal and vertical lengths on the unit circle. In particular she argued, “I know that the sine of one of my reference angles is gonna be the same as the cosine of the other one. Why do I know that? Because, the sine of this angle is opposite over hypotenuse. So if I make that my unit circle, I can put .9397 there. The cosine of 20 is then adjacent over hypotenuse. So I know the cosine of 20 is .9397.”

Overall, Sarah had knowledge of cofunctions as well as all three ways of knowing sine and cosine and defined unit circle correctly as a circle with radius one, but she only understood radian as a portion of circumference and not in relation to arc length. Further, Sarah’s knowledge of inverse cosine was not sufficient to answer Task Five correctly. Within the next section, I include a discussion of how her mathematical knowledge influenced his mathematical noticing of students’ work and thinking.

**Question 1: How did Sarah attend to, interpret, and respond to hypothetical students’ work and thinking in trigonometry?** In this section, I discuss findings regarding how Connie (i) attended to, (ii) interpreted, and (iii) responded to students’ work and thinking.
Sarah’s Initial Attention. In this section, I answer the following: (i) what did Sarah do when asked to notice?, and (ii) to what extent was her focus on the students’ thinking?. An overview of how Sarah initially attended to the hypothetical students’ work and thinking is provided below in Table 5.7.

Table 5.7. Summary of How Sarah Initially Attended to the Students’ Work and Thinking

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<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Thinking</th>
<th>Describing</th>
<th>Evaluating</th>
<th>Challenged by Mathematics</th>
<th>Investigating</th>
<th>Comparing to Others</th>
<th>Comparing to Own</th>
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As is shown in this table, without being prompted specifically to do so, Sarah placed significant attention and focus on correctness. In particular, Sarah made definitive evaluative claims for thirteen of the fifteen hypothetical students. While Sarah often didn’t immediately know the correctness of the work, she frequently investigated or explored these claims further until she could determine whether or not they were
correct. These instances are discussed in more detail in a later subsection regarding the effects of the interview experience. In part because of her investigations, all of Sarah’s evaluative claims made during her initial attention to the students’ work and thinking were mathematically valid. As is shown in the table, Sarah also attended to the similarities and differences across the students within a task as well as the similarities and differences between her own work and thinking and the students’. In a later subsection, I discuss how Sarah drew on her own thinking to interpret the thinking of the hypothetical students. Without being prompted to do so specifically, Sarah attended to the thinking of two thirds of the hypothetical students. Her interpretations of students’ thinking, including the resources she drew on to interpret students’ thinking, are included in the next sections.

**Sarah’s interpreting.** As I previously noted, Sarah spent significant time focusing on students’ thinking. Further, Sarah was very cautious when describing their thinking. She often considered multiple explanations for the students’ work, and was hesitant to equate procedural correctness with understanding. For example, in Task Three (See Appendix E), Sarah considered that “even though he (Joe) didn’t quite get the right answer, um he maybe has a deeper understanding trigonometry wise because, this says to me, he was thinking of the unit circle.” Also, in Task Five (See Appendix G), Sarah considered two possible explanations for Amy’s work. She considered, “if she knows that the cosine is negative, maybe she knows that we’re talking about some point on the left hand part of the unit circle, maybe, or maybe that’s just a lucky guess, because it’s either gotta be positive or negative.” Similarly, Sarah considered that Mike’s work in Task Four (See Appendix F) might have been a result of “conceptual
understanding” or “memorization.” She explained that Mike “I think has a pretty decent conceptual understanding because he was able, or maybe, maybe this was just memorization (pointing to the circle). I don’t know. Maybe he’s just really good (snapping fingers) at remembering all of those um values all the way around the unit circle.” Sarah also indicated that incomplete work, just meant that she needed to “press further” and not necessarily that the student did not have adequate knowledge to complete the task.

Thus far, I have discussed Sarah’s cautiousness when describing students’ thinking. In the following sections I continue the discussion of Sarah’s interpreting by addressing the following question (which corresponds to research sub-question 1b): **What resources did Sarah draw upon to interpret students’ thinking?** In particular, I explain how Sarah drew upon her interpretive language, subject matter knowledge, and her own work and thinking, to interpret the hypothetical students’ thinking.

**Resources for interpreting: Sarah’s interpretive language.** In this subsection, I present the argument that Sarah used interpretive language that allowed her to describe students’ thinking. Sarah had difficulty using the word “understand” to describe students’ thinking. For example, Sarah indicated that, “Mike *understands about the coordinates of many points.*” In this case, Sarah used “understand” to refer to the presence of “many points” in Mike’s work in Task Four (Appendix F). Sarah also used “understanding” in evaluative or relative terms (better/good or worse/bad). For example, Sarah claimed that Joe in Task Three (See Appendix E), “maybe has a *deeper understanding trigonometry wise.*” Note that Sarah simply claimed that Joe *did* understand, but not *how* Joe understood trigonometry. In another example, in Task
Three, (See Appendix E) Sarah stated that “Anna has a **really good understanding of** what the sine ratio is,” again using understanding to be evaluative, but then expands by saying, “She **knows in a right triangle it is** the ratio of the opposite leg to the hypotenuse.” Thus, she used “understanding” to be evaluative and the term “knows” to be descriptive. Similarly, Sarah indicated that Alice in Task Four (See Appendix F) had a “**conceptual understanding of**, of what cosine is, um that **she knows that, it is** the horizontal leg of, of the triangle.” Again, Sarah used “understanding” to be evaluative and “knows” to be descriptive. Sarah used several other words/phrases such as “recognize X as,” “thinks of Y as,” “Has a conception of Z as,” and “considers W to be.” For example in the first task (See Appendix A), Sarah (i) compared Lexi’s thinking to her own by claiming, “she’s not **considering that arc length is** a portion of the circumference”; (ii) argued that “Allen and Annie have similar **conceptions of what radian is**”; and (iii) explained that it was “not clear in my mind that they **think of arc length as** a portion of circumference.”

**Resources for interpreting: Sarah’s subject matter knowledge.** Sarah’s subject matter knowledge was highly influential in her interpretations. Sarah’s own knowledge of sine and cosine and the unit circle seemed to be the most helpful to her when interpreting thinking. In particular, she was one of the few participants to recognize that both Joe in Task Three (See Appendix E) as well as Alice in Task Four (See Appendix F), were thinking of cosine as “the horizontal leg” on the unit circle. Thus, Sarah’s mathematical knowledge allowed her to see the validity of their work. Sarah even stated that compared to Mike in Task Four, Alice “maybe has a more conceptual understanding of, of what cosine is, um that she knows that, it is the horizontal leg of, of
the triangles.” Sarah also drew on her knowledge of the relationship between sine and cosine as cofunctions to interpret the students’ thinking on Task Two (See Appendix B). For example, she noticed that “Martha definitely understands the relationship of, the cofunctions”; whereas “Cheyenne didn’t make that connection at all, because she never even used the fact that the sine of 70 was .9397.” While Sarah’s knowledge of sine and cosine enhanced her interpretations of students’ thinking in tasks three and four, as I have discussed here, in the next paragraphs I discuss how weaknesses in her knowledge of inverse trigonometry and radian measure limited her interpretations.

While Sarah had strong and flexible knowledge of sine and cosine, her knowledge of inverse functions, and inverse cosine in particular, was limited, and this made it more difficult for Sam to interpret the students’ thinking in Task Five (See Appendix G). It was interesting however, that while Sarah did not have adequate subject matter knowledge to correctly answer the mathematical problem or to evaluate the students’ work, she was still able to draw on her knowledge to make claims about their thinking. For example, she wasn’t certain that Joe was incorrect but explained that he “understands how inverses behave; they undo each other,” but “he didn’t put together that there’s - that you have to consider what quadrant you’re in.”

Another weakness of Sarah’s with respect to her subject matter knowledge was that her understanding of radian measure was limited to its relation to degree measure, and due to this, Sarah had difficulty interpreting the students’ work in Task One (See Appendix A). In particular, Sarah did not understand radian as (i) the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle OR as (ii) the length of the arc on the unit circle subtended by the angle, and as a
consequence, Sarah had difficulty interpreting Lexi, Annie, and Charlie’s thinking on the first task (See Appendix C). For example, Sarah had trouble making sense of Lexi’s work, posing at first that “she hasn’t clued into the fact that no matter how big or small the circle is… it would still be um, a negative pi over 12 for the angle measure,” but then Sarah stopped and considered, “but she kinda is taking that into consideration, cuz she’s got the radius here.” Eventually Sarah gave up and noted, “yeah I’m not sure what to say about that one other than the fact that she didn’t consider that it’s a portion of the circumference.” Note that in this last claim, while Sarah did not make sense of Lexi’s work, she did note that Lexi was not thinking about arc length the way she herself thought of arc length, as a “portion of the circumference.” This was one of many cases in which Sarah drew on her own work or thinking to make claims about the students’ thinking as is discussed in the following subsection.

*Resources for interpreting: Sarah’s own work and thinking.* Recall that Sarah considered arc length as a “portion of the circumference” of a circle. When Sarah’s mathematical knowledge was not sufficient to interpret how the students were thinking of arc length in Task One, Sarah was still able to claim that Lexi, Annie, and Charlie (See Appendix C) were not thinking of arc length as a “portion of the circumference.” In particular she argued:

*I learned that um since they're finding arc length, I would want them - I would want them to convey that that's a portion of the circumference. Um, I saw, Allen considered that. Um, I didn't see that in Lexi or Annie, or I have no idea what Charlie did. So I mean just at first glance, I think that Allen was the most advanced in his thinking.*

Here, Sarah argued that success on this problem would mean that the student had demonstrated an understanding of arc length similar to her own.
On Task Five (See Appendix G), Sarah’s own discomfort with radians allowed her to notice the significance of Amy’s choice to convert to degrees. Sarah noted, “Amy, like I did, likes to work in terms of degrees instead of radians.” Similarly, because Anna’s work on Task Three resembled her own, Sarah claimed, “Anna thinks like I think.” While in this case Sarah might have over-attributed her own thinking to Anna, in other situations Sarah was more cautious to diagnose students’ with her own thinking. For example, in Task Five, Sarah considered that Alexis might have been “kind of thinking like me, uh, with right triangles, or maybe Alexis just knows ok where you know pi- this is.” In summary, in drawing on her own work and thinking to interpret, Sarah (i) sometimes over-diagnosed students’ with her own thinking yet other times cautiously proposed her own thinking as just one possibility, (ii) sometimes showed a preference towards work and thinking, yet not always, and (iii) drew on her own thinking to compensate for an inability to make sense of the students’ work.

Thus far, in addressing the first main research question, I have discussed how Sarah attended to and interpreted the hypothetical students’ work and thinking, including the resources Sarah drew on to interpret. In the final subsection corresponding to the first main research question, I discuss the third skill of noticing. In particular, I discuss Sarah’s proposed instructional responses to these hypothetical students.

**Sarah’s responding.** In this section, I discuss how Sarah responded to students’ work and thinking. In particular, I address the following questions (which correspond to research sub-questions 1d and 1c): (i) what were the goals of Sarah’s proposed instructional responses and how were they informed by her interpretations of students’ thinking, and (ii) what other resources did Sarah draw on to respond to students? An
overview of how Sarah responded to the hypothetical students' work and thinking is provided below in Table 5.8.

Table 5.8. Summary of How Sarah Responded to the Hypothetical Students

<table>
<thead>
<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Student's Thinking</th>
<th>Affect</th>
<th>General Move</th>
<th>Show and Tell</th>
<th>Give or Ask</th>
<th>Aims to Access</th>
<th>Aims to Have Student Realize</th>
<th>Aims to Have Student Realize</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Allen</td>
<td>√</td>
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<td>1</td>
<td>Lexi</td>
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<td>1</td>
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<tr>
<td>1</td>
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<td></td>
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<td>√</td>
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<tr>
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<td>Cheyenne</td>
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<td>√</td>
<td></td>
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<tr>
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<td>Amy</td>
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</table>

Nine out of fifteen of Sarah’s proposed hypothetical instructional responses were made, at least to some extent, on the basis of her interpretations of the students’ thinking. For example, Sarah had interpreted that on Task Two, Allen (See Appendix D) had thought that pi over 12 was “one twelfth of the circle” and on the basis of this interpretation, Sarah’s response to Allen was “I might say to Allen um, how many radians are there in a whole circle, just to kinda point him in that direction to help him see this error.” This was also one of six instances in which Sarah asked questions or gave a student a task
in order for the student to realize something. Note that fourteen out of fifteen responses were coded as “give or ask.” In fact, Sarah never proposed telling or showing a student something directly. Five of Sarah’s “give or ask” responses were also designed to access additional information about the students’ thinking. For example, within the fifth task (See Appendix G), Sarah proposed asking a series of questions to hypothetical student, Sam, because “maybe Sam knows a little bit more about this, and I just need to press for that.” Similarly, in Task Three, Sarah’s response to Joe (See Appendix E) was:

I would ask him to kind of fill in some of the blanks, to see where this came from, because you don’t always know. I mean, sometimes they write down something and it’s just a lucky coincidence

Sarah’s response to Joe above highlights not only one example of her trying to access additional information about a student, but it also shows how Sarah drew on her knowledge of content and students, particular knowledge that students’ correct work does not always imply correct thinking, in deciding to do so. While she didn’t share this reasoning each time she decided to ask questions to elicit additional information, it might be that this particular aspect of Sarah’s knowledge of content and students, gained through her own experience with students, might have informed many of her decisions to probe further. In Sarah’s reflection, she commented that one challenge for her “was deciding if student errors were because of lack of understanding of the trigonometry concepts or if they were due to something else, like dumb arithmetic mistakes” She acknowledged herself that experience had taught her to recognize this difference, explaining that “As a young teacher I might have ‘taken off points’ for both types of errors in the same way. But now when I evaluate students’ work in my own classes, they would lose far less credit for a silly arithmetic mistake than an error with
the geometry concept that I’m trying to assess.”

In addition to drawing on this important aspect of her knowledge of content and students, Sarah appeared to also draw on her beliefs or orientations when deciding how to respond to students. When asked to respond, Sarah explained:

the traditional person in me is thinking, well let me just respond to what they have and there we go. That’s it. That’s what you messed up on. (laughing) But the more progressive person in me, is saying alright well I, let me, let me ask this, let me ask that, and explain this a little bit more. Can you figure this out? So I’m really not telling them what to do, but I want to give them a little bit of a nudge to kid of just get them - to press them further, to see what they know [...] maybe that’s the case with some students is there is more stuff in there, in their head, but by maybe not asking a question in the right way, we’re not really getting at what they really know

Recall that Sarah had been attending conferences recently and claimed to be reading more about “progressive” mathematics education. Perhaps as a result, Sarah seemed to be deciding to provide opportunities for students’ to either realize new information or prove to her that they know more than they showed based on a more “progressive” orientation, which seemed to include that (i) students’ thinking is valued over student work and (ii) students should not be told, but instead discover their own mistakes to some extent. Sarah’s newfound “progressive” stance is also highlighted in her response to Lexi and Annie on Task One (See Appendix C):

I want to help them recognize that arc length is a portion of circumference, um, but I don't wanna just come out and say those words. [...] I'm thinking of things that I learned at the conference. [...] students get the message if the teacher eventually is just gonna give them what the answer is, then they're not gonna think very hard. [...] I want to choose my words carefully to kind of lead them in the direction of considering you know, what, what is arc length. What does it mean? [...] that might lead them towards oh it’s a part of a circle’s circumference.

As is shown above, Sarah’s response to Lexi and Annie seemed to be informed by her
beliefs including, “if the teacher eventually is just gonna give them what the answer is, then they’re not gonna think very hard.”

Sarah’s response to Annie and Lexi is also one of the few instances for which Sarah’s responses included a focus on her own thinking instead of the students’ thinking. Recall that Sarah had difficulty interpreting Lexis’s, Annie’s, and Charlie’s thinking and that this was, at least in part, because Sarah did not understand radian in relation to arc length. Recall also that Sarah compensated for this, by focusing on how the students did not think like she herself thought; that is she commented on how they were not thinking of arc length as a “portion of the circumference.” As a consequence, Sarah’s responses to Annie and Lexi (as shown above) were made on the basis of Sarah’s own thinking instead of these students’. In particular, Sarah said that she would “want to help them recognize that arc length is a portion of circumference,” and she would have a “discussion that might lead them towards oh it’s a part of a circle’s circumference, and then ok well how do you find circumference?” Thus, to summarize, in this case, although Sarah had enough knowledge to answer the problem correctly, she did not have the knowledge necessary to make sense of these students’ strategies and thinking, and thus her responses to these students redirected the students’ thinking instead of building on their thinking.

Thus far, I have discussed (0) Sarah’s knowledge of radian, the unit circle, and the sine and cosine functions as well as (1) how Sarah attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry. In the following section, I discuss (2) the ways in which the hypothetical students’ work challenged and/or furthered Sarah’s mathematical thinking. I then provide an overall
summary regarding this case study of Sarah.

**Question 2: In what ways did the hypothetical students’ work on trigonometry problems challenge and/or further Sarah’s mathematical thinking?**

Sarah was often challenged by the students’ work, and on several occasions was stirred to think more deeply about something or realize something new or forgotten. As I’ve mentioned previously, despite the fact that Sarah answered the mathematical prompt correctly, Sarah had the most difficulty making sense of the students’ work in Task One (See Appendix C). It seemed as though she might recognize something new in response to Lexi’s work when she explained, “I was thinking like no matter how big or how small the circle is, it would still be um, a negative pi over 12 for the angle measure, but she kinda is taking that into consideration, cuz she’s got the radius here,” but unfortunately, she quickly gave up, stating, “yeah, I’m not sure what to say.” While she wasn’t able to expand her thinking about radian or arc length during that interview, Sarah must have continued thinking about their work post-interview, perhaps consulting an outside resource, because in her reflection she noted, “As I try to recall how I analyzed Annie’s work, I think I may have commented about how she labeled the angle and arc the same in the first circle, which didn’t make sense as it seemed as though she was mixing up arc length and arc measure. In hindsight that was a correct relationship for the unit circle.”

As I have discussed, there were multiple occasions when the student work did prompt Sarah to engage in further mathematical thinking during the interviews. For example, on Task Four, Sarah investigated Mike’s (See Appendix F) use of sine and cosine as coordinates on the unit circle to plot the graph of the cosine function. At first
Sarah thought, “he just plotted his x and y coordinates,” but then she realized that wasn’t the case, and pondered:

So really you’ve got three different quantities that we’re talking about here. We’ve got the radian measure or the degree measure, and then he’s got an x-coordinate and a y-coordinate. So…[...] alright, now I gotta go back and think about […] so what do the coordinates mean when we graph a trig function? […] oh dear. I’m hittin’ a brain cramp

She then explored further and explained:

he’s got ordered pairs here, and I’m thinking oh! ordered pairs, that’s what I’ve gotta graph. […] he’s got an x-coordinate and a y-coordinate. So I’m automatically jumping into thinking he’s trying to plot these, but it’s really, this is the radian measure is what’s on the x-axis, and it’s this x-value of the coordinate that represents the cosine. [...] the x-value is the radian measure; the y-value is the, the cosine.

She went back-and-forth about this, and eventually concluded:

I was just getting so hung up on the fact that I need to use both of those coordinates! But the cosine of theta is just the x-coordinate, and so that’s what you need to plot the graph. If he wanted to plot the sine function, then we would still use these same radian values, but it would be the y-coordinate that’s representing the sine. So Mike confused me a little bit.

Thus, while Sarah was able to successfully graph the cosine function and explain the graph in terms of the unit circle, she did not use the idea of sine and cosine as coordinates on the unit circle, and thus in investigating and making sense of Mike’s work, she had to grapple with new challenges that did not exist for her in simply answering the mathematical prompt. These challenges were consistent with Brown’s (2006) findings that students often struggle to connect a rotation on the unit circle with a point on the graph of cosine. The hypothetical work was successful, in this case, in forcing Sarah to think about what values on the horizontal axis represented on a cosine graph. She commented on this in her reflection, admitting, “I got confused about the meaning of the coordinates of each point. It took me a while to realize that since the
graph is of the cosine function, the x coordinates were the radian measures around the unit circle and the y coordinates were the cosine values.”

Sarah was also challenged by some of the work on Task Five (See Appendix G). For example, in response to Alexis she asked “why does it matter if it’s an odd multiple of pi over four?” and then she did some investigating, exclaiming, “Wait a second, because multiples of pi over four, you can be, oh hold on, gotta back up and remember some trig.” She also explained that she just had to make sense of it herself for her own sanity. Eventually she agreed with Alexis, explaining, “so here’s - here’s all of our multiples of pi over four, and so yeah, if we do the even multiples it’s here and here and here. So we want the odd multiples.” Sarah also eventually considered that Amy might be on the right track, and that although her own answer was in the third quadrant, perhaps the correct answer is in the second.

**Summary: Sarah.** Recall that Sarah was the most experienced teacher with 19 years of high school mathematics teaching experience. While she had not taught Trigonometry, she had taught geometry for all of those years, and using this knowledge she received the third highest score on the trigonometry assessment, an 82%. Sarah had a preference for degrees over radian and only seemed to define or think of radian measure in terms of its relationship to degree and not in relation to arc length. Sarah held all three ways of knowing sine and cosine, but relied most heavily on her understanding of sine and cosine as ratios of the lengths of the sides on a right triangle. Her trigonometric knowledge was often challenged by the students’ work, and as a result she often engaged in additional mathematical work or thinking (beyond that required to answer the prompt).
Sarah’s initial attention typically included a focus on the student’s thinking. Sarah had interpretive language for talking about students’ thinking; while she had difficulty talking about “understanding” she had access to other language that allowed her to engage in similar discussions. Sarah also drew on her subject matter knowledge and her own work and thinking to interpret the students’ thinking. When drawing on her own work and thinking, Sarah sometimes over-diagnosed students’ with her own thinking yet other times cautiously proposed her own thinking as just one possibility, and Sarah drew on her own thinking to compensate for an inability to make sense of the students’ work. In responding to the students, Sarah formed “give or ask” responses, which drew on her interpretations of students’ thinking, mathematical knowledge, knowledge of content and students, and her developing “progressive” orientations.
At the time of data collection, Connie was a student teacher. While her undergraduate degree prepared her for secondary certification, Connie’s student teaching placement was in a seventh-grade mathematics classroom. Connie described her goals for teaching to be “to guide students to discover and to help them make connections between the old and new knowledge – to engage them and keep them curious.” In the following sections I address each of my three research questions with respect to Connie.

**Question 0: What content knowledge of radian, unit circle, and the sine and cosine functions did Connie possess?** Out of 19 participants, Connie was awarded the fourth highest score on the precalculus assessment, 16 points out of 25, or 64%. During the interviews Connie was able to successfully answer the mathematical prompts in Task One, Task Three, and Task Four, but she had difficulty answering the mathematical prompts in tasks two and five. More specifically, (i) given the sine of 70 degrees, Connie could not evaluate the cosine of 20 degrees, and (ii) she could not evaluate $\cos^{-1}(\cos \left(\frac{-11\pi}{4}\right))$. In the following subsections, I provide a description of Connie’s understanding of radian and arc length, the unit circle, as well as sine and cosine, including the relationship between sine and cosine as cofunctions.

**Radian and arc length.** When completing the assessment, Connie first defined radian simply as a “unit of measure.” Later, after thinking about and responding to the third question on the assessment, which required her to calculate various arc lengths, she added to her definition of radian, “Based on my work in #3, I think it comes from measuring arc length or circumference of a unit circle.” That is, while Connie couldn’t
articulate specifically that radian measure was the same as *the length of the arc on the unit circle subtended by the angle*, she did seem to notice a connection between arc length on a unit circle and radian measure. When asked to find the arc length during the interview, she recalled her thinking on the assessment, and explained:

this got me thinking about radian and how I realized I don’t quite exactly know, so that made me think radians measuring, it’s measuring like the arc length of a unit, or it comes from measuring the arc length of a unit circle

She explored this relationship further, and then explained:

I was just thinking that if we’re looking at the unit circle, the, the angle measure in radians tells us the arc length, because you would have 2 pi radians to go around the whole circle […] and I know that the circumference is 2 pi, cuz 2pi times the radius which is one

Connie concluded:

so I was thinking that the, the radian then corresponds to the arc length on a unit circle, and then because this circle’s three times the, the radius is three times that of the unit circle. The radius is three, so I said the arc length would be three times the arc length of the unit circle, with that same angle measure

While her reasoning was correct, Connie expressed that she was not confident in her work or answer until later when she arrived at the same answer by dividing the circumference of the circle by 24. Coming into the research study, it seemed Connie understood arc length as a portion of the circumference and only thought of radian as angle measure; however after engaging in the mathematical prompts on the assessment and Task One but prior to seeing any students’ work, Connie gained or regained an understanding of radian measure as *the length of the arc on the unit circle subtended by the angle*. 
It is important to note two other aspects of Connie’s thinking related to radian. Firstly, she showed slight preference for degree measure over radian measure. In many instances, when given an angle in radians, Connie would first convert to degrees, such as when looking at Mike’s work on Task Four, stating “so pi over six, which is thirty degrees” or in Task Five, stating, “So this part is just cosine of five pi over four…Um, and that’s, that’s 45 degrees.” Secondly, Connie seemed to experience confusion about co-terminal angles in Task Five. In particular, she noted that she had answered “five pi over four,” but Joe has answered “negative 11 pi over four,” which at first she said was, “not equal,” but then she exclaimed “It is the same angle! Hold on. (laughing) It’s the same angle. It’s just - they’re calling - yeah. It’s the same angle. So I - I think he’s missing, like I’m missing, all the others - all the other angle names for that same angle.” Thus instead of recognizing that these co-terminal angles have the same location on the unit circle and thus, the same cosine value, Connie concluded that these angles are in fact the same, and just have different “names.”

**Unit circle.** Connie defined unit circle as “a circle with a radius of 1 unit.” When asked why the unit circle was taught and/or why it was useful in the learning and teaching of trigonometry, Connie wrote and drew the following, in Figure 5.4:

![Figure 5.4. Connie’s Response to Assessment Item 4b: Purpose of Teaching Unit Circle.](image-url)
That is Connie seemed to indicate that the unit circle was helpful in relating the definition of sine and cosine on a right triangle with the definition of sine and cosine as vertical and horizontal lengths on the unit circle.

**Sine and cosine.** As is shown in the Figure 5.4, Connie understood sine and cosine both as the vertical and horizontal lengths on the unit circle as well as the ratios of the lengths of sides on a right triangle and was able to coordinate the two conceptions. Connie seemed to draw on her understanding of sine and cosine as lengths on the unit circle most often when solving problems and making sense of students’ work. For example, when asked to estimate the sine and cosine of 20 degrees, Connie thought aloud:

sine of 20 is gonna be equal to whatever the height is here, and cosine of 20 degrees will be the uh length. [...] it looks like cosine of 20 degrees is gonna be bigger than sine of 20 degrees.

In the excerpt above Connie used her knowledge that the sine is the vertical length and cosine the horizontal on the unit circle to approximate sine and cosine of 20 degrees relative to each other. As mentioned previously, in this same task, Task Two, given the sine of 70 degrees, Connie could not evaluate the cosine of 20 degrees. That is, it appeared she did not hold an understanding of the relationship between sine and cosine as cofunctions. As will be discussed in later sections, her lack of understanding was evidenced further in her inability to make sense of Martha’s work on the same problem.

Overall, as has been discussed, at the start of the study Connie held two ways of knowing sine and cosine, correctly defined the unit circle as a circle with radius one, and through her work on Task One, Connie gained or regained an understanding of radian
measure as the length of the arc on the unit circle subtended by the angle; however
Connie did not have knowledge of the relationship between sine and cosine as
cofunctions. As will be discussed in a later section, Connie also gained knowledge of
sine and cosine as coordinates on the unit circle through her investigations of student
work. Within the next section, I include a discussion of how this mathematical
knowledge influenced his mathematical noticing of students' work and thinking.

**Question 1: How did Connie attend to, interpret, and respond to**
**hypothetical students' work and thinking in trigonometry?** In this section, I discuss
findings regarding how Connie (i) attended to, (ii) interpreted, and (iii) responded to
students' work and thinking.

**Connie's initial attention.** In this section, I answer the following: (i) *what did
Connie do when asked to notice?*, and (ii) *to what extent was her focus on the students’
thinking?*. An overview of how Connie initially attended to the hypothetical students’
work and thinking is provided below in Table 5.9
Table 5.9.
Summary of How Connie Initially Attended to the Students’ Work and Thinking

<table>
<thead>
<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Thinking</th>
<th>Describing</th>
<th>Evaluating</th>
<th>Challenged by Mathematics</th>
<th>Investigating</th>
<th>Comparing to Others</th>
<th>Comparing to Own</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td></td>
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<tr>
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<tr>
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</tr>
<tr>
<td>2</td>
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<tr>
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<td>✓</td>
<td></td>
<td></td>
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<tr>
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<tr>
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<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>

As is shown above, when asked initially to think aloud and describe what she noticed about the students, Connie included a focus on students’ thinking for eight of the fifteen hypothetical students. Further, Connie typically provided a description of what the students had done and almost always attempted to evaluate the students’ work. On four occasions, she also investigated the mathematics in the problem. For example, she couldn’t evaluate Cheyenne’s work on Task Two (See Appendix D), until she drew a right triangle and investigated the sine and cosine values at 30 and 60 degrees for herself. In some instances, gaps in Connie’s mathematical knowledge limited her
evaluations. For example, also in Task Two, Connie described that Alex “took his sine of 90 which is one an subtracted um, sine of 70 degrees to get … that equation for the cosine of 20 degrees,” but then she could not evaluate the correctness, “because I don’t, I don’t know if this part is true, sine of 20 degrees plus sine of 70 degrees is equal to sine of 90 degrees… I just don’t know if that’s - I don’t remember, um, I don’t remember - yeah, I don’t remember if we can do that.” Even after investigating this on her own, she could not decisively say whether or not Joe’s claim was true. Gaps in Connie’s knowledge limited her interpretations more extensively than her evaluations; these limitations are discussed in a later subsection. More generally, Connie’s interpretations of students’ thinking, including the resources she drew on to interpret students’ thinking, are included in the next sections.

Connie’s interpreting. Connie could be characterized as extremely cautious when making claims about students’ thinking. Most specifically she (i) recognized that the written work gave her limited access to students’ thinking and (ii) differentiated between correctness and understanding or thinking. Connie often considered more than one possible train of thought or reasoning that would result in the same work. For example, in response to Cheyenne’s error in her work on Task Two (See Appendix D), Connie considered, “either she mislabeled her triangle, um, or maybe she has it memorized, and she might have forgot, or maybe she just picked the adjacent side instead.” On Task Three, Connie considered that Anna (See Appendix E) could have just “put the wrong variable” but knew “it’s supposed to be theta,” OR “she really thought it would be sine of y there, y is a side and not an angle, so that is a problem.” Similarly, in Task Three, Connie considered possible rationale for Joe’s work (See Appendix E) to
find a similar triangle with hypotenuse length one, explaining, “I don’t know if he thought he had to have a hypotenuse of one, or if he just thought it would be nicer at the end to not have a denominator.” Connie consistently differentiated between correctness of work and understanding and thinking. While Connie claimed that Lexi’s work on Task One was correct, except for the sign, she considered “I don't know that she knows what it like, what it is we’re looking for, other than she can pick out the right formula to use.” Consistent with my findings with regards to Connie’s cautiousness and distinctions between correct work versus understanding or errors versus misconceptions, Connie indicated on her written reflection that in analyzing the student work, a “challenge was trying to figure out the students’ reasoning behind some of their steps,” and that she “wanted to ask the students questions about their work to find out if something they did was a careless mistake or if it was the result of misunderstandings.”

Thus, Connie recognized that the written work gave her limited access to students’ thinking and was able to differentiate between procedural correctness and understanding. In the following sections, I continue the discussion of Connie’s interpreting by addressing the following question (which corresponds to research sub-question 1b): What resources did Connie draw upon to interpret students’ thinking? In particular, I explain how Connie drew upon her interpretive language, subject matter knowledge, and her own work and thinking, to interpret the hypothetical students’ thinking.

Resources for interpreting: Connie’s interpretive language. It is difficult to characterize Connie’s interpretive language for talking about students’ thinking, because while she was able to discuss students’ thinking, her language was limited. Connie was
able to make claims about students’ understandings and thinking using certain language (such as “knowing”). However, when asked specifically to talk about the students’ “understandings,” she often struggled and didn’t focus on the students’ thinking, especially when she was asked to discuss the students’ understandings of a particular concept. For example, as I mentioned previously, without prompting to specifically talk about the students’ understandings, Connie claimed that Mike in Task Three (See Appendix E) “knows the cosine has to be equal to the x-value of each of these points at every angle”; however when asked later about Mike’s understanding of cosine, Connie stated “he knows how to get, um, the value of the cosine of an angle.” That is, she referred to procedural correctness instead of Mike’s way of knowing cosine. In some instances, the interviewer asked Connie about the students’ understandings, and when Connie would simply talk about their work, the interviewer had to repeatedly prompt, did that tell you anything about their understandings? For example, with respect to the work in Task Two (See Appendix D), after repeatedly being asked to talk about understandings, Connie said, “I think really showing each step and not, um…shows that he or she knew, knew where to go with that.” Thus, again Connie referred to procedural correctness when asked to talk specifically about understandings.

While Connie often referred to procedural correctness when specifically prompted to discuss “understandings,” she did possess language that allowed her to talk about students’ thinking, which she did most often without being prompted, when she was first presented with the work. Most often she talked about students knowing something. For example, Connie stated that in Task Four (See Appendix F), Mike “knows the cosine has to be equal to the x-value of each of these points at every
angle” and also, “Alice knows that she’s looking at the x-values. That’s what the horizontal lines are telling me.” In addressing Cheyenne’s work on Task Two (See Appendix B), Connie stated, “Cheyenne showed that she knows they’re continuous functions, and, and that it’s all increasing on this interval for zero to 30 degrees, um…Cheyenne didn’t use anything about the relationship between sine and cosine. So I don’t know if she knows anything about that or not.” Thus, while Connie’s language may have been more limited than Byron’s or Sarah’s, she did possess interpretive language, which enabled her to talk more descriptively about students’ different ways of knowing than participants like Craig, who did not appear to have sufficient interpretive language.

Resources for interpreting: Connie’s subject matter knowledge. As did all the other case study participants, Connie drew on her mathematical knowledge to interpret students’ thinking. For example, Connie understood cosine as horizontal lengths on the unit circle; thus, as was discussed previously, Connie was able to interpret Alice’s thinking in Task Four (See Appendix F). Even so, in many cases, Connie’s mathematical knowledge limited her ability to interpret the students’ thinking. Connie herself indicated in her written reflection, “I think my biggest challenges when analyzing the students’ work and thinking were the instances when I could not remember how to solve the problem or part of the problem (for example, not completely understanding radian measure), and so there was very little for me to say about the work.”

As I mentioned, Connie did not understand the relationship between sine and cosine as cofunctions and thus when asked about Martha’s (See Appendix D) understanding of the relationship between sine and cosine, Connie stated, “I don’t know
where this is coming from, cosine of 20 degrees is approximately equal to the given sine of 70 degrees…So I’m not sure if she knows anything about the relationship between them.” Similarly, Connie did not know the range of arccosine, nor did she, more generally, seem aware that a function must be injective to have an inverse, and thus, in addressing Amy’s work on Task Five (See Appendix G), Connie noted that she could not tell where Amy’s statement, ‘the answer has to be an angle in the second quadrant’ had come from. Connie also seemed to make a few claims about students’ thinking regarding continuity that seemed to indicate that she might equate continuity with (i) being increasing and/or (ii) being defined on the entire set of real numbers. For example, in Task Four (See Appendix F), Connie stated, “Alice does not have arrows on here. Um, and so I can’t really assume that she knows that it’s a continuous function,” and in in Task Two, Connie stated that Cheyenne “knows that sine is a continuous function.” In summary, Connie’s mathematical knowledge was highly influential on her interpretations of students’ thinking. In some instances it enabled her interpretations, whereas in many, it prohibited her from engaging with aspects of the students’ work and thus their thinking, or caused her to make claims that were inconsistent with the work.

*Resources for interpreting: Connie’s own work and thinking.* In part because Connie often attempted each mathematical prompt using multiple strategies due to a lack of confidence, her own work and thinking was perhaps her most valuable resource for interpreting students’ thinking. As was consistent with her cautiousness in other regards, Connie was also cautious not to over-diagnose students with her own thinking. As she put it in her written reflection:

I tried to be objective, but often found myself comparing the students’ work to my own thinking as I was solving the problems. […] I wanted to figure
out what each student knew and understood without assuming that the students were thinking the same things I was thinking when solving the problem.

Recall that when solving the mathematical prompt in Task One (See Appendix C), Connie attempted the problem two ways, one that was aligned with Allen’s strategy and one that was aligned with Annie’s strategy. As a consequence, her own work and thinking was particularly useful in interpreting the students’ thinking on this task. The following excerpt, Connie’s interpretations of Charlie’s thinking, is an example of how Connie drew cautiously on her own work and thinking to talk about the students’ thinking:

I really don’t know what he (Charlie) was thinking, whether it was uh…if he forgot about the radius three, we’re talking about a bigger circle, or if he thought it would be the same as the unit circle, or if he even thought about a unit circle at all uh, or maybe he was just writing down.

In the above excerpt, Connie suggests three possibilities (and drew on her own work and thinking as well as her mathematical knowledge in the first two): (i) Charlie knew radian measure was equivalent to arc length on the unit circle and then forgot to consider the radius of three, (ii) Charlie thought that radian measure was equivalent to arc length on any circle, or (iii) he just wrote down the angle without much thought.

Similar to Sarah, in Task Five on the inverse or arccosine, Connie had gaps in mathematical knowledge, but she used her own thinking to decide on goals for the task, most specifically the idea she wanted to see in the students’ work, which in-turn influenced her interpretations. In particular, while she was uncertain about her final answer to the mathematical prompt, which was in fact incorrect, Connie was confident in her decision that the answer should be an angle. This was a helpful discovery for her in solving the problem, as is shown in the excerpt below:
I’m thinking one over cosine. One over cosine is secant. I don’t know - I don’t think that’s the same thing. Maybe it is the same thing. (laughing) […] oh wait, wait wait […] when you have inverse cosine, that’s because this is a length, and not an angle, I think. (laughing) Because I’m gonna get out a, this, I’m gonna get an angle measure.

In the above excerpt Connie considered that inverse cosine was the same as secant, but when she realized that what she was looking for was an angle, she knew that the inverse cosine could not be equivalent to secant. Thus, in her interpretations of the students’ thinking, while her mathematical knowledge limited her ability to interpret much of their work and thinking, she was able to focus on whether or not they knew they were looking for an angle. In particular, Connie claimed that, “Joe, Sam, and Amy all know they’re looking for an angle” whereas “Alexis doesn’t seem to know that she’s looking for an angle.” In summary, Connie was successful in drawing on her own work and thinking in part because (i) she was careful not to over-attribute her own reasoning to the students, (ii) she had attempted the problems using more than one strategy and thus had varied work and thinking to draw on, and (iii) she used her own thinking to make claims about students' thinking even when she herself could not successfully answer the prompt.

Thus far, in addressing the first main research question, I have discussed how Connie attended to and interpreted the hypothetical students' work and thinking, including the resources Connie drew on to interpret. In the final subsection corresponding to the first main research question, I discuss the third skill of noticing. In particular, I discuss Connie’s proposed instructional responses to these hypothetical students.

Connie’s responding. In this section, I discuss how Connie responded to
students’ work and thinking. In particular, I address the following questions (which correspond to research sub-questions 1d and 1c): (i) what were the goals of Connie’s proposed instructional responses and how were they informed by his interpretations of students’ thinking, and (ii) what other resources did Connie draw on to respond to students? An overview of how Connie responded to the hypothetical students’ work and thinking is provided below in Table 5.10.

Table 5.10.
Summary of How Connie Responded to the Hypothetical Students

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<thead>
<tr>
<th>Task #</th>
<th>Hypothetical Student</th>
<th>Focus on Student’s Thinking</th>
<th>Affect</th>
<th>General Move</th>
<th>Show and Tell</th>
<th>Give or Ask</th>
<th>Aims to Access</th>
<th>Aims to Have Student Realize</th>
<th>Aims to Have Student Realize</th>
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<tr>
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<tr>
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Nine out of fifteen of Connie’s proposed hypothetical instructional responses were made, at least to some extent, on the basis of her interpretations of the students’ thinking. For example, Connie decided to respond to Alexis on the basis of Connie’s
interpretations of Alexis’s thinking. Recall that even though Connie drew on her own work and thinking to interpret that Alexis “doesn’t seem to know that she’s looking for an angle,” later, when asked to respond to Alexis, Connie stated, “I might ask her if she can explain what she’s looking for here, what the problem is asking her to do, umm, because she doesn’t seem to know that she’s looking for an angle, and that’s gonna be the first step is to make sure she knows she’s looking for an angle.” Thus, Connie decided to respond to Alexis on the basis of her interpretation that Alexis “doesn’t seem to know that she’s looking for an angle.” As in this example, Connie most often sought to access additional information as a result of her cautiousness. In particular, recall that Connie was careful to differentiate between errors and misconceptions. As a consequence, when asked to respond to Cheyenne in Task Two (See Appendix D), Connie stated:

I would ask Cheyenne how she got sine of 30 degrees and the cosine of 30 degrees [...] I’d ask her to draw the triangle and label it, [...] see if she’s, if she’s drawing the triangle wrong [...] or if she just accidentally picked the wrong one or if she memorized wrong.

As in several other instances, her cautiousness in interpreting the students’ thinking was reflected in her response (above). In particular, Connie aimed to access additional information to determine if Cheyenne’s work was a result of simply memorizing wrong or a larger issue of “drawing the triangle wrong.” As Connie stated in her written reflection, “I wanted to ask the students questions about their work to find out if something they did was a careless mistake or if it was the result of misunderstandings.”

Fourteen out of fifteen of Connie’s proposed instructional responses were categorized as “give or ask” responses, and in most of these cases, Connie’s goal was to access additional information regarding the student’s work or thinking. As I’ve
discussed, most of these instances were designed to test several hypotheses about students’ thinking; however in some instances she sought to access additional information because gaps in her mathematical knowledge limited her ability to make sense of the student work. For example, in response to Martha on Task Two (See Appendix D) she stated, “Um I would ask Martha how she’s using this 90 is equal to 20 plus 70, because I have no idea where the rest of, of, well where this comes from.” In addition to trying to access additional information about the students’ thinking, two of Connie’s responses aimed to have the student realize some fact or idea.

Thus far, I have discussed (0) Connie’s knowledge of radian, the unit circle, and the sine and cosine functions as well as (1) how Connie attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry. In the following section, I discuss (2) the ways in which the hypothetical students’ work challenged and/or furthered Connie’s mathematical thinking. I then provide an overall summary regarding this case study of Connie.

Question 2: In what ways did the hypothetical students’ work on trigonometry problems challenge and/or further Connie’s mathematical thinking?

There were instances during the interviews in which Connie was challenged by the students’ work and was stirred to think differently about something or realize something new or forgotten. As was mentioned previously, Connie often initially attended to the students’ work by investigating the mathematics involved in their strategies since she was initially uncertain of the validity or unsure of the correct answer to the prompt herself. For example, prior to seeing the student work, Connie only demonstrated an understanding of sine and cosine as distances on the unit circle and as ratios of the
lengths of sides on a right triangle but not as coordinates on a unit circle, and thus, when she initially attended to Mike’s work on Task Three (See Appendix E), she was confused about the points he had labeled on the unit circle. Connie commented at first that, “so he graphed the points wrong,” but then she investigated:

ok. Well he graphed…(pausing/thinking)…[...] um, ok, sorry. (laughing) I don’t know why - it’s like - ok, and then he’s got square root of two over two for pi over four. Um, pi over three was one half […] So he’s picking out his x, x-coordinate, um in all of those, because on the unit circle, cosine of, of x, cosine of theta would be equal to the x-coordinate. So those are the ones that he’s graphing.

Through her investigation of the mathematics in Mike’s work, and using her previously held understanding of sine and cosine as distances on the unit circle, she is able to eventually realize that “cosine of theta would be equal to the x-coordinate.” Connie herself noted in her written reflection that, “Analyzing student work and solving the problems myself helped me remember and clarify some things about trigonometry that I had forgotten.” She recalled an example in which she was challenged mathematically and was able to “see” something new:

for the question about finding the arc length of an arc on a circle with a radius of 3 and angle measure of $-\pi/12$, I did not remember a formula and had to figure out the arc length by thinking about the circumference and comparing the given circle to a unit circle. When analyzing the student work for this problem, I was able to see connections in the students’ responses to my thinking about how to solve this problem, but also the formula that I’m sure I was taught at some point.

In summary Connie’s mathematical thinking was challenged and influenced by the students’ work beyond the challenge and influence of the mathematical prompt on its own. This additional mathematical engagement can be, at least in part, attributed to the fact that (i) Connie was uncertain of her own answers to the mathematical prompts and
thus (ii) she investigated the validity of the mathematics involved in many students’ strategies.

**Summary:** Connie. Connie was a student teacher who earned a 64% on the trigonometry assessment, the fourth highest score of all the scores, but the lowest of the five case study participants. Connie was able to successfully answer three out of five of the mathematical prompts during the interviews. She had difficulty with the idea of cofunctions in Task Two and inverse trigonometric functions in Task Five. Perhaps because of this, Connie often investigated the mathematics involved in students’ work in an attempt to make sense of the strategy or to evaluate. As a result, there were instances during the interviews in which Connie was challenged by the students’ work and was stirred to think differently about something or realize something new or forgotten.

Connie’s initial attention included a focus on the students’ thinking for around half of the students. She also spent significant time describing and evaluating the work. When making claims about students’ thinking Connie was extremely cautious, recognizing her limited access to students’ thinking as well as making distinctions between correctness and thinking. She was also cautious “to figure out what each student knew and understood without assuming that the students were thinking the same things I was thinking when solving the problem.” Connie did, however, draw on her own thinking to posit one of multiple possibilities for students’ thinking. Connie also drew on her subject matter knowledge and interpretive language to interpret students’ thinking. Connie’s responses were informed by her interpretations and the same
cautiousness she used in making claims about students’ thinking. As a result, many of her responses aimed to access additional information about the students’ thinking.
CHAPTER VI

RESULTS: CROSS-CASE DISCUSSION

In the previous chapter, I discussed each of my three research questions with respect to each of the five case study participants, Elliot, Craig, Byron, Sarah and Connie. In this chapter, I extend this discussion across all five cases. Recall that out of the 19 participants in this study, these five case study participants (Elliot, Craig, Byron, Sarah and Connie) were awarded the top scores (23, 21, 20, 20, and 16, respectively) out of 25 points on the trigonometry assessment. Further, Elliot, Craig and Byron were able to answer each of the five mathematical prompts correctly during the interviews, while both Sarah and Connie could not correctly evaluate, \( \cos^{-1}(\cos\left(\frac{-11\pi}{4}\right)) \) in Task Five, and Connie could not use the fact that, \( \sin(70^\circ) \approx 0.9397 \) to find \( \sin 20^\circ \) in Task Two. Recall that at the time of data collection, Elliot and Connie were student teachers, and Sarah, Byron, and Craig were in-service teachers. In addition, recall that Elliot, Byron, and Craig all had experience teaching Precalculus to some extent.

In this cross-case discussion, I address each of my main two research questions across all five cases. In particular, I compare, contrast, and discuss as a whole (1) how these participants attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry, as well as (2) in what ways the hypothetical students’ work challenged and/or furthered their mathematical thinking.

Question 1: How Did These Teachers Attend to, Interpret, and Respond to Hypothetical Students’ Work and Thinking in Trigonometry?

In this section, I discuss findings regarding the case study participants with respect to my first main research question. In particular, in the following subsections, I
discuss how they (i) attended to, (ii) interpreted, and (iii) responded to students’ work and thinking.

**Initial Attention**

In this section, I address the first two research sub-questions (within my first main research question) with respect to the case study participants. In particular, I answer the following: (i) what did they do when asked to notice?, and (ii) to what extent was their focus on the students’ thinking?. An overview of how the five case study participants initially attended to the hypothetical students’ work and thinking is provided below in Table 6.1. In particular the table displays the number of times (out of 15) each participant’s initial attention included: a focus on thinking, describing the work or thinking, noticing a mathematical claim that they couldn’t make sense of (that could potentially be investigated mathematically), actually investigating the mathematics, making comparisons to other students, and making comparisons to his or her own work or thinking.

**Table 6.1. Summary of How The Case Study Participants Initially Attended to the Hypothetical Students’ Work and Thinking**

<table>
<thead>
<tr>
<th></th>
<th>Focus on Thinking</th>
<th>Describing</th>
<th>Evaluating</th>
<th>Challenged by Mathematics</th>
<th>Investigating</th>
<th>Comparing to Others</th>
<th>Comparing to Own</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliot</td>
<td>11</td>
<td>15</td>
<td>15</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>Craig</td>
<td>7</td>
<td>14</td>
<td>14</td>
<td>7</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Byron</td>
<td>10</td>
<td>14</td>
<td>14</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
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<tr>
<td>Sarah</td>
<td>10</td>
<td>15</td>
<td>13</td>
<td>10</td>
<td>9</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Connie</td>
<td>8</td>
<td>15</td>
<td>12</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
As is shown above and has been discussed in the previous sections regarding each individual case study participant, these teachers attended a great deal to the students’ strategies and gave significant attention to the correctness of the students’ work and thinking. In fact, each case study participant made definitive evaluative claims with respect to at least twelve out of fifteen hypothetical students. Elliott and Byron rarely encountered a piece of mathematical work or a claim for which they could not immediately determine its correctness; however, in those rare instances, they were able to investigate the mathematics, and make a final evaluation. Sarah was challenged by the students’ work and claims the most frequently of all of the case study participants, but she also investigated the mathematics the most frequently.

With the exception of Craig, all of the case study participants initially attended to students’ thinking to a greater extent than the average participant in this study. In particular, on average, the interview participants’ initial attention included a focus on students’ thinking for approximately half of the students, but four out of five of the case study participants attended to thinking for more than half of the students, and in fact, three out of five of the case study participants attended to at least two thirds of the students’ thinking. Further, since the case study participants represent a subset of participants with relatively strong content knowledge, this suggests that individuals with stronger subject matter knowledge might attend to students’ thinking more frequently than those with weak subject matter knowledge. However, adequate subject matter knowledge did not ensure attention to thinking. The case study participants’ interpretations of students’ thinking, including the resources they drew on to interpret students’ thinking, are included in the next sections.
Interpreting

As I have discussed, the case study participants spent significant time focusing on students’ thinking. In this section, I compare and contrast how they interpreted students’ thinking, including a discussion of the resources they drew upon to interpret students’ thinking. In particular, I include subsections within this section corresponding to the four main resources the case study participants appeared to draw on to interpret students’ thinking: their subject matter knowledge, their interpretive language, their own work and thinking, and their pedagogical content knowledge. In each of these subsections I compare and contrast how the participants drew on these resources (if they did) to interpret students’ thinking. Before I discuss any of these resources however, I will first discuss overall distinguishing characteristics of the case study participants’ interpretations.

Differentiating between correctness and thinking/understandings. Four out of five of the case study participants were careful to differentiate between procedures and thinking, especially correctness of students’ work and students' thinking or understandings, whereas, in contrast, Craig often equated correctness of work with correctness of thinking. In the following, I provide examples that highlight the contrast between Craig’s interpretations and the remaining participants’ more cautious interpretations. In the first example, recall that in Task Five (See Appendix G), Craig claimed that Mike “understood things very well.” In contrast however, Sarah more cautiously considered that Mike’s work in Task Four (See Appendix F) might have been a result of “conceptual understanding” or a result of “memorization.” In a second example, recall that on Task Three (See Appendix E) Craig claimed that because “Anna
um, seems to know how to use the Pythagorean Theorem correctly” and “knows how to solve for the sine of theta,” that consequently, Anna “understands, mathematically, what's going on.” In contrast, Elliot differentiated between Anna’s work and her thinking, noting, “you are able to do the work, but do you understand?” These first two examples highlight how in contrast to Craig, four out of five of the case study participants were not 100% satisfied with correct work; they considered that there were multiple ways and degrees of understanding and thinking that could result in the same correct work.

The other case study participants were also more cautious than Craig in that they considered and differentiated between the seriousness of particular errors. As I’ve discussed in the previous example, Craig noted that Anna (See Appendix E) understood “mathematically, what's going on.” Craig also acknowledged Anna’s error in her final answer and claimed that she was “just not um keeping the thetas and y's in line.” That is, Craig simply described Anna’s error. In contrast however, Byron considered that Anna might be “working off a memory and memorization” but Byron was “not so sure”; thus, he wanted to assess what sine and cosine represented to Anna to verify that there wasn’t a larger misconception underlying her error. Similar to Byron, Connie considered that Anna could have just “put the wrong variable” but knew “it’s supposed to be theta,” OR “she really thought it would be sine of y there, y is a side and not an angle, so that is a problem.” That is, in contrast to Craig, Byron and Connie considered the seriousness of the error by considering that the error might be just an error or it might be a result of a more serious misconception. In a second example, Craig claimed that in Task Two (See Appendix D), Alex and Cheyenne’s “understanding might have been very similar if not the same, because they were each wrong in one
assumption.” By equating Cheyenne and Alex’s understanding based on having the same number of errors in their work, Craig (i) failed to distinguish between correctness of work and thinking and (ii) failed to consider the relative seriousness of each error. In contrast, Connie considered that Cheyenne either “mislabeled her triangle, um, or maybe she has it memorized, and she might have forgot, or maybe she just picked the adjacent side instead.” That is, Connie differentiated between Cheyenne’s work and thinking, noting that there were many possible avenues of thinking that could have resulted in the same error. Connie also seemed to consider the seriousness of the error; she considered that Cheyenne might have just “accidentally switched them” or “she doesn’t know how to label a triangle to find cosine and sine of 30 degrees.”

In this section I have summarized how Sarah, Byron, Elliot, and Connie differentiated between work and thinking, but Craig did not. These results in conjunction with the fact that Craig attended to students’ thinking the least of all the case study participants, were surprising in light of previous findings exhibiting that elementary teachers with more experience showed more engagement with students’ thinking than those with less (Jacobs, Lamb, & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011). Recall that Craig had taught for eight years, including four years of precalculus, whereas Elliot and Connie were student teachers in their first year of teaching; however both Elliot and Connie attended to thinking more frequently and with more caution. These results may be explained by Craig’s limited access to interpretive language, which was previously discussed and elaborated on in a later section.

**Making generalizations.** Byron’s interpretations of students’ thinking were uniquely different than the other case study participants in that in addition to interpreting
the student’s thinking about the problem and concepts at hand, Byron also made
general claims about the students’ thinking and tendencies based on their work on the
problem. In particular, recall that Byron hypothesized and compared the way Allen and
Lexi as well as Mike and Alice thought about mathematics and approached
mathematics problems in general. Recall that Byron claimed that Lexi was more “theory
oriented” and “sees math as very black and white,” whereas Allen was more “geometry
oriented” and “has more of a malleable sense of the mathematics.” Similarly, recall that
Byron claimed that Mike was more “analytical” whereas Alice had more of a “visual
understanding.” Byron’s tendency to make generalizations about students based on
their work might be, at least in part, due to the fact that he has had the most experience
working with students on trigonometry problems. In particular, recall that Byron had
taught high school mathematics for 17 years and had taught precalculus for 15 of those
17 years, and thus, perhaps Byron had a better sense than the other case study
participants of students’ typical learning trajectories based on their responses.

Thus far, I have discussed how four out of five of the case study participants
were careful to differentiate between procedures and thinking in contrast to Craig who
often equated correctness of work with correctness of thinking and also, as a
consequence, did not seem to consider the relative seriousness of errors or the
underlying misconceptions. I also discussed how Byron’s interpretations of students’
thinking were uniquely different than the other case study participants in that in addition
to interpreting the student’s thinking about the problem and concepts at hand, he also
made general claims about the students’ thinking and tendencies based on their work
on the problems. In the following section, I continue the discussion of their interpreting
by addressing the following question (which corresponds to research sub-question 1b):

What resources did they draw upon to interpret students’ thinking?

Resources for interpreting. In the remainder of this section regarding the participants’ interpretations of students thinking, I discuss how their interpretations were informed by four resources: their subject matter knowledge, their own work and thinking, their pedagogical content knowledge, and their interpretive language.

Subject matter knowledge. As I have shown, one obvious influential resource for interpreting students’ thinking was the participants’ subject matter knowledge. For example, while Elliot, Craig, Byron, Sarah, and Connie could all answer the mathematical prompt in Task One (See Appendix C) correctly, differences in their understanding of radian resulted in differences in their interpretations of Charlie’s thinking. Elliot, Byron, and Connie all had knowledge of radian as the arc length on the unit circle corresponding to that given angle, and thus were able to deduce that Charlie may have been thinking about the unit circle in some way. For example, Elliot recognized that Charlie might be “really focused on the unit circle,” and Connie noted that it was possible that Charlie “thought it would be the same as the unit circle.” In contrast, Craig and Sarah did not understand radian as (i) the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle OR as (ii) the length of the arc on the unit circle subtended by the angle. Instead, Craig defined radian as “a unit of measure describing $1/2\pi$ of the central angle of a circle,” Sarah only understood radian measure in relation to degree measure, and both Craig and Sarah understood arc length as a portion of circumference and did not perceive it’s relationship to radian measure. Thus, Craig claimed that Charlie was not engaging in
any meaningful mathematical thinking but rather “just taking the number and bringing it down,” and Sarah could not interpret Charlie’s thinking, noting, “I have no idea what Charlie did.”

Similarly, while Elliot, Craig, Byron, Sarah, and Connie could all answer the mathematical prompt in Task Four (See Appendix F) successfully, differences in their understanding of sine and cosine resulted in differences in their interpretations of Alice’s thinking. Recall that Sarah, Byron, Connie and Craig all understood sine and cosine as horizontal and vertical lengths on the unit circle, and thus they were able to notice and interpret Alice’s thinking about sine and cosine as lengths. For example, when asked about Alice’s understanding of Cosine, Sarah noted that Alice had a “conceptual understanding of, of what cosine is, um that she knows that, it is the horizontal leg of, of the triangles perhaps going around the circle.” In contrast, despite Elliot’s understanding and use of the “coordinates” definition, he did not appear to ever use the similar definition of sine and cosine as horizontal and vertical lengths in the unit circle, and thus he had difficulty interpreting Alice’s thinking in Task Four. In particular, when asked specifically about Alice’s understanding of the cosine function, Elliot claimed:

she’s understanding here that the cosine’s one, here the cosine’s zero, here the cosine’s negative one, here the cosine’s zero, but you know, she didn't put the ordered pair. She just put one and like zero, you know um, and things like that. [...] So that would be my instinct to say like this kind of shows less understanding uh and more just oh I know what the critical points are.

In the above excerpt, Elliot did not talk about the lines Alice drew but noticed that she labeled the x- and y-intercepts on the unit circle with just the cosine value and not the “ordered pair”; Elliot did not consider that she was not using a coordinate definition of cosine but rather thinking of cosine as horizontal lengths on the unit circle. Even after
the interviews, after he had time to look at the work on his own, Elliot questioned, “What was ‘Alice’ thinking when she was drawing those horizontal lines in the unit circle? That still has me stumped.”

In the two examples I have presented thus far (Task One and Task Four) as well as in most instances, subject matter knowledge was a significant resource for interpreting students' thinking. In particular, interpreting students' thinking required more subject matter knowledge than was required to simply solve the mathematical prompt. That is, in order for the teachers to successfully engage in noticing of students' thinking, they not only needed to be able to successfully do the mathematics, but they needed to hold flexible mathematical knowledge, including multiple ways of knowing various concepts such as sine and cosine or radian.

In some cases, despite having sufficient mathematical knowledge, teachers’ still could not interpret a hypothetical students' strategy. In particular, it was interesting that despite Byron and Craig's flexible understandings of sine and cosine, they were not able to make sense of Joe's work on Task Three (See Appendix E). For example, Craig claimed that Joe “didn't try to solve the problem” and that “I don't learn anything about the sine function from Joe.” This might be because this sort of work was unconventional or unfamiliar to these experienced precalculus teachers. Byron even noted that, “I have to say, from what I've experienced, I'm - I'm not so familiar seeing Joe's work.” Thus, while having deep and flexible mathematical knowledge was a necessity for interpreting students’ thinking and increased one’s likelihood to interpret students’ thinking, it did not appear to be sufficient. That is, interpreting students’ thinking seemed to require more than subject matter knowledge.
Their own work and thinking. With the exception of Byron, the case study participants’ own solution strategies and thought processes also informed their interpretations of students’ thinking. In some cases, participants drew on their thinking in negative ways, limiting their interpretations, by over-diagnosing students with their own thinking in instances when there existed minimal or no evidence that the students shared their thinking. For example, because Anna’s work on Task Three (See Appendix E) resembled her own, Sarah claimed, “Anna thinks like I think.” In the following, I highlight an additional example of how Elliot over-attributed his own thinking to Martha.

Recall that, in Task Two, the hypothetical student, Martha (See Appendix D), wrote that since the sum of 70 and 20 is 90, the sine of 70 degrees is equal to the cosine of 20 degrees. Thus, it would appear that Martha knew of the relationship between sine and cosine as cofunctions; however, it is not clear to what extent she would be able to explain this relationship. That is, she may have simply remembered this fact about sine and cosine as it was told to her; however, because Elliot spent time making sense of the relationship between sine and cosine as cofunctions using the unit circle in response to the mathematical prompt in Task Two, he assumed Martha must have reasoned similarly, explaining:

she seemed to understand like hey look if we knew we were off the x-axis, um, with the 20 degrees we could go off of the y-axis with the 70 degrees, kinda like thinking that’s 20 degrees off the y-axis, um, to kinda understand, I know it’s not true reference angle, but like I like thinking about it as a reference angle, because you’re still like pulling out a triangle out of the, out of the unit circle. Um, so I think like she understands some of how they interact together.
Here, it seemed Elliot assumed he and Martha shared similar thinking, with almost no evidence that this was in fact the case. Similarly, because Anna’s work on Task Three (See Appendix E) resembled her own, Sarah claimed, “Anna thinks like I think.”

In most cases, however, participants used their own thinking to expand and improve their interpretations rather than limiting them, by positing their own thinking as one possible line of reasoning while considering alternatives. For example, in Task Five, Sarah noted that Alexis might have been “kind of thinking like me, uh, with right triangles, or maybe Alexis just knows ok where you know pi- this is.” Similarly, in response to Amy in Task Five (See Appendix G), Craig stated:

Um, it well, I'm probably just projecting here, but it seems like she, there’s a slight chance that she might understand the whole domain and range thing and that it has to be in the first or second quadrant, but I'm not sure. [...] that could've just been either a lucky guess or something entirely different then what I'm thinking.

Craig drew on his own thinking to talk about Amy’s thinking but was cautiously aware that (i) this was only one possibility and (ii) the evidence in Amy’s work was not sufficient to make any definitive claims in this respect. Similarly, Connie was quite cautious not to “project” her own thinking onto students. She even noted in her reflection that she “tried to be objective” and that she “wanted to figure out what each student knew and understood without assuming that the students were thinking the same things I was thinking when solving the problem.”

Sarah and Connie also drew on their own work and thinking in a novel way to compensate for gaps in subject matter knowledge. For example, while Sarah’s knowledge of radian prohibited her from making sense of the students’ strategies in Task One (See Appendix C), when asked to talk about their understanding, Sarah
compared the students to herself and interpreted that Lexi, Annie, and Charlie (See Appendix C) were not thinking of arc length as a “portion of the circumference” (like Sarah had). Further, Sarah seemed to incorporate her own thinking or way of understanding arc length into her goals for the problem. Sarah explained that what she wanted to see in students’ work was that they “convey that that’s (arc length is) a portion of the circumference” like Sarah herself had. This seems potentially quite detrimental to the students’ learning since Sarah is only allowing one avenue of thought. Connie’s similar use of her own work and thinking to inform goals for the task, discussed below, seemed less potentially detrimental.

Recall that Connie could not correctly or confidently answer the prompt in Task Five, but she did know that her answer should be an angle, and this was a helpful and meaningful discovery for her as shown below:

I’m thinking one over cosine. One over cosine is secant. I don’t know - I don’t think that’s the same thing. Maybe it is the same thing. (laughing) […].…oh wait, wait wait […] when you have inverse cosine, that’s because this is a length, and not an angle, I think. (laughing) Because I’m gonna get out a, this, I’m gonna get an angle measure.

In this excerpt, Connie thought that inverse cosine was the same as secant, but when she realized that what she was looking for was an angle, she knew that the inverse cosine could not be equivalent to secant. Thus, this became something she looked for in students’ work and thinking. Connie claimed that, “Joe, Sam, and Amy all know they’re looking for an angle” whereas “Alexis doesn’t seem to know that she’s looking for an angle.” In this example, Connie’s goals for the task were informed by her own work and thinking on the problem which in turn informed the way she interpreted the students’ thinking.
In this section, I have discussed how the case study participants’ drew on their own work and thinking regarding the particular problems to interpret the thinking of the hypothetical students. In particular, I discussed how they drew on their own thinking in three different ways: (i) over-attributing their own thinking to a student, (ii) posing their own thinking as one possible way in which a student was thinking, and (iii) comparing a student’s thinking to his or her own thinking (typically to compensate for weaknesses in their own subject matter knowledge). These findings indicate that being able to solve and think about a problem multiple ways is valuable when interpreting thinking. The findings also suggest that teachers are often tempted to assume their students share the same thinking and understandings as they do, but when teachers are aware of the temptations, they are able to draw on their own thinking more productively.

**Pedagogical Content Knowledge (PCK).** While participants may have drawn on PCK gained from their teaching experience indirectly when interpreting students’ thinking, I only observed two participants, Byron and Elliot drawing on their PCK in direct ways in multiple instances. This was interesting since Elliot had only been teaching for one year. Recall that PCK consists of Knowledge of Content and Curriculum, Knowledge of Content and Teaching (KCT) and Knowledge of Content and Students (KCS). Byron seemed to have a wealth of KCS with respect to trigonometry. In particular, Byron explicitly called out “common” or “familiar” errors in Allen’s work in Task One, Lexi’s work in Task One, Anna’s work in Task Three, Joe’s work in Task Five, and Alexis’s work in Task Five. In addition to having knowledge of these “common” or “familiar” errors, Byron appeared to have knowledge of the common ways of thinking or misconceptions that informed these errors, and he used this knowledge to
inform his interpretations. For example, recall that Byron noticed an error in Anna’s work on Task Three (See Appendix E), namely, that Anna wrote “sin(y)” instead of “sin(\(\theta\)).” Byron noted that this error was “familiar” and drew on his KCS to interpret the possible thinking behind the error, explaining, “I don't believe students understand the difference between an angle and a side involving trig. Um they constantly sort of get the two mixed up,” and he reasoned further, “they see an, a radian measure, and they think it is a ratio of sides. So there's this constant interchange between angles and sides.”

Similarly, Recall that Byron made the following claim about Allen in Task One (See Appendix C):

So he's mistaken the quantity of pi over 12s that are gonna be in his full circle. So that's the mistake that he's making, which is a pretty common error, um, students make. They forget that it's 2pi all the way around and not just pi around. So they, they only go half way and then they work with that in thinking that they've gone the full way around the unit circle.

Note that Byron went beyond simply drawing on knowledge of a common error. Byron was drawing on knowledge of common thinking/reasoning behind that error: students think it’s “pi around” and forget “that it’s 2pi all the way around.” Byron also drew on his KCS to make more general claims about Allen’s thinking, explaining, “he's sort of seeing it in a geometrical sense, and I really see that a lot, and I think even as a, as, as, uh a student of trigonometry, you sort of, there's sort of this geometrical sense to it.”

There were instances in which Elliot drew on his KCS as well, but in contrast to Byron, Elliot often drew on his KCT and KCS simultaneously. In particular, Elliot used his knowledge of common teaching strategies in support of his knowledge of common student errors and thinking to inform his interpretations. For example, recall that in Task One (See Appendix C), Elliot acknowledged that Charlie’s answer was incorrect, and he
also had noted that Charlie’s answer was the arc length corresponding to the given angle on the unit circle. Further, Elliot drew on his knowledge of content and common teaching when he noted, “a lot of times teachers start with the unit circle, and they start with the concept that, um, well your radian measure of your angle is equal to your arc length because of the radius is one.” He drew on his knowledge of common student thinking in stating, “it kind of gets left by the wayside slowly when the kids are, are listening to the teacher talk, or they kind of block that out,” to interpret that Charlie is just “focused on the unit circle.” In some instances, in addition to drawing on his PCK, Elliot seemed to be reflective about his PCK, perhaps bringing these ideas to the forefront of his mind for the first time. For example, recall Elliot’s consideration of Alexis’s work on Task Five (See Appendix G):

I mean it's a hard thing for students to understand because nobody explains it and explains it real well, and it's just notation being different. Like it, it's not like an explanation. It's like no, that's not the same thing, and it's just mathematically how we notate things.

In this case, Elliot seemed to grapple with the notion of teaching a concept versus teaching convention. In particular, Elliot recognized that he could/would not “explain” why an exponent of negative one holds a different meaning in different situations, but instead, it is, “just mathematically how we notate things.” Further, Elliot noted that since an explanation is not given to students, “it's a hard thing for students to understand.”

I only identified one instance in which it was clear that a participant was drawing on his or her knowledge of content and curriculum to interpret thinking. In particular, Elliot considered Anna’s error in Task Three (See Appendix E):

maybe it's something way more confusing, uh, and (Anna) saw the y was blank, and we're solving for a variable, and if y's blank, we must be looking for sine of, of y, you know? We don't usually look for, you know, theta's
As is shown in the above excerpt, to interpret the thinking behind Anna’s error, Elliot drew on his knowledge of content and curriculum, namely that, “we don’t usually look for, you know, theta’s before precalc, you know, theta’s just come in precalc,” as well as his knowledge of content and students, namely that, “kids don’t know what to do because they’re funny shaped,” to interpret that Anna might have thought “if y’s blank, we must be looking for the sine of, of y.”

In this section, I discussed how two participants, Byron and Elliot, drew on their KCS, KCT, and in one instance, KCC, to aid in interpreting students’ thinking. I discussed how both Byron and Elliot drew frequently on their KCS. In particular, they drew on their knowledge of common errors as well as knowledge of student thinking and/or misconceptions that typically inform these common errors, to interpret the thinking of hypothetical students. Additionally, I discussed how one aspect of Elliot’s KCT, his knowledge of common teaching, also informed his interpretations of the hypothetical students’ thinking. In particular, Elliot drew on the way concepts are typically taught to explain how and rationalize why the hypothetical students would develop certain misconceptions. These findings suggest the importance of providing preservice teachers with opportunities to gain exposure to common teaching strategies in addition to common student difficulties, thinking, and misconceptions.

**Interpretive language.** Recall that van Es (2011) argued that learning a more interpretive discourse, “in which the teachers’ goal is to make sense of student thinking,” is “central to noticing” (p.135). While her claim was not supported by evidence, in my
own study, I found that the types of interpretive language the participants used, as well as how the participants used interpretive language, had the potential to contribute to or limit their interpretations of students’ thinking. Recall that, like Jacobs, Lamb, and Philipp (2010), I asked the participants, “Please explain what you learned about these students’ understandings.” I also asked the participants to talk specifically about the students’ understandings of certain concepts. The participants appeared to have difficulties with the word, “understand,” and seemed to use and view the word, “understand”, inconsistently.

Some participants used the word, “understand”, to refer to procedures or the correctness of work. For example, while Connie did talk about students’ thinking and ways of knowing, when specifically asked to talk about “understandings,” she often talked about their work or procedures. Similarly, there were instances in which Elliot and Craig specifically used “understand” to talk about procedures. For example, Elliot noted that Lexi “understands, um, how to plug it in, um, the formula to start.” Similarly, Craig noted that Alex, in Task Two (See Appendix D), “does understand the Pythagorean identity, sine squared plus cosine squared equals one.” In both of these instances, the teachers spoke of understanding, but referred to the fact that the students did something correctly.

In other, perhaps somewhat related, instances, the case study participants used the term “understand” in an evaluative sense (he did understand X, she did not understand Y) or comparative sense (he has a worse understanding, she has a better understanding). For example, in an evaluative sense, Elliot claimed that Annie “understood the unit circle” (evaluative). Here, it is unclear to the reader or listener what
Elliot meant by understanding the unit circle. He did not describe her way of knowing the unit circle; rather, he evaluated that yes, she understood it. Similarly, Craig comparatively noted that Mike, in Task Four (See Appendix F), had “a better understanding of the period” than Alice. Note that similarly to Elliot, Craig did not talk about how these students understood the “period” of the trigonometric function. Instead, Craig simply commented on whether or not they understood the “period” and who understood them better. To the reader or listener it is unclear what he means by this. Perhaps he believed there is one correct way of knowing period, or perhaps he is again, using the term “understand” to refer to procedural correctness.

In some cases, participants used the term “understanding” in even more general or broad ways (not referring to a particular concept or idea) while still using the term to be comparative rather than descriptive. For example, Sarah noted that in Task Three, Joe (See Appendix E) “maybe has a deeper understanding trigonometry wise” than Anna. Similarly, with respect to Task Two (See Appendix D), Craig stated, “I think Martha has the most understanding” and with respect to the other two students, “I think their understanding might have been very similar.” Again, Craig has not described how the students’ understood, but instead compared “understandings.” In this instance he did not refer to their understanding of a particular concept, rather, Craig seemed to be referring to relative levels of overall “understanding.”

In contrast, there were some instances in which the participants would use the word “understand” to describe the students’ ways of knowing. For example, when asked about Anna’s understanding of the sine function (in Task Three), instead of claiming, that Anna has a good understanding of the sine function, Byron described that she
“understand(s) the idea that the sine is the ratio of - the sine of an angle is the ratio of sides.” Similarly, when asked about Alice’s understanding of cosine (in Task Four), Byron claimed that she was “understanding that it’s the, the length of x” on the unit circle. Similarly, in Task One (See Appendix C), Elliot noted that, “Annie understands that radian measure of, of an angle here, pi over 12, is equal to the arc length.”

When the participants were more descriptive about students’ ways of knowing, they often used language that didn’t involve “understand.” For example, participants described how the students were “seeing,” “considering,” “thinking,” “knowing,” or “recognizing.” Specifically, in the first task, Sarah noted that Annie was, “not seeing this as a portion out of the whole circle,” and that Lexi was, “not considering that arc length is a portion of the circumference.” Byron claimed that Annie, “is seeing the, the radian measure as arc length, uh, the angle I mean as an, as an arc length.” Byron also explained that Allen “sees the radian maybe in terms of a fraction, and he’s dividing the circle up into fractional pieces.” Elliot noted that it was possible that Charlie, “always thinks that the […] arc length of the circle is equal to the radian measure.”

In all of these cases, the participants referred to how the students did or did not see, consider, or think of different concepts. That is, they had sufficient interpretive language to talk about the students’ ways of knowing.

In this section, I have discussed the term “understand” was used inconsistently: sometimes used to refer to procedures or correctness of work; some times used in an evaluative or comparative way; and, in some instances, used to describe ways of knowing. When the participants did describe students’ thinking and ways of knowing, most often they did not use “understand” language, but instead used other interpretive
language, such as “seeing” or “considering”. These findings suggest that teacher preparation and development programs should provide teachers with opportunities to develop their own interpretive language that allows them to be descriptive, rather than evaluative, about students’ thinking and ways of knowing. The findings also suggest that teacher educators should either shift away from talking about “understandings” or think more carefully about how we use this term and how it is interpreted.

Thus far, in addressing the first main research question, I have discussed how the case study participants attended to and interpreted the hypothetical students’ work and thinking, including the resources they drew on to interpret. In the final subsection corresponding to the first main research question, I discuss the third skill of noticing. In particular, I discuss the case study participants’ proposed instructional responses to the hypothetical students.

**Responding**

In this section, I discuss how the case study participants responded to students’ work and thinking. In particular, I address the following questions (which correspond to research sub-questions 1d and 1c): (i) *what were the goals of their proposed instructional responses and how were they informed by interpretations of students’ thinking*, and (ii) *what other resources did they draw on to respond to students?* An overview of how the case study participants responded to the hypothetical students’ work and thinking is provided below in Table 6.2. In particular the table displays the number of times (out of 15) each participant decided to respond to a hypothetical student on the basis of the student’s thinking or the student’s affect, as well as the number of times they produced a general move. It also displays the number of times
(out of 15) each participant’s response to a hypothetical student included a “give or ask” or a “show and tell” response as well as how many times their response to a hypothetical student aimed to access additional information about the student’s thinking or have the student realize something.

Table 6.2.  
Summary of How The Case Study Participants Responded to the Hypothetical Students

<table>
<thead>
<tr>
<th></th>
<th>Focus on Student’s Thinking</th>
<th>Affect</th>
<th>General Move</th>
<th>Show and Tell</th>
<th>Give or Ask</th>
<th>Aims to Access</th>
<th>Aims to Have Student Realize</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliot</td>
<td>14</td>
<td>2</td>
<td>0</td>
<td>12</td>
<td>11</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>Craig</td>
<td>12</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>14</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>Byron</td>
<td>12</td>
<td>1</td>
<td>7</td>
<td>10</td>
<td>10</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Sarah</td>
<td>9</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>14</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Connie</td>
<td>9</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>14</td>
<td>11</td>
<td>2</td>
</tr>
</tbody>
</table>

As is shown in Table 6.2, the case study participants with stronger trigonometry knowledge were more likely to demonstrate their knowledge to their students. That is, both Elliot and Byron, the two participants who showed the strongest trigonometric knowledge, proposed “show and tell” responses the most frequently. Further, while not as significant of a difference, both Elliot and Byron also proposed “give or ask” responses the least frequently of all the case study participants.

It is also interesting that the case study participants with less teaching experience were more likely to propose responses aimed to access additional information about the students’ thinking. That is, the two student teachers (Elliot and Connie) aimed to access additional information about approximately two thirds of the
students, whereas the more experienced teachers aimed to access additional information about less than half of the students. I hypothesize that, due to their lack of experience, Elliot and Connie felt less confident in their abilities to interpret students’ thinking, and as a consequence, wanted to access additional information. This might suggest that while teaching experience is beneficial in that it provides teachers with certain pedagogical content knowledge, teachers with less experience may be less likely to assume they know what their students are thinking and thus, provide more opportunities for students to explain their thinking; however it may also be the case that these particular individuals will remain cautious throughout their teaching careers. Future research is needed to investigate this hypothesis further.

As has been discussed previously, most of the case study participants’ responses were made, at least to some extent, on the basis of their interpretations of the students’ thinking. For example, recall that Connie responded to Alexis in Task Five (See Appendix G) in the following way: “I might ask her if she can explain what she’s looking for here, what the problem is asking her to do, umm, because she doesn’t seem to know that she’s looking for an angle, and that’s gonna be the first step is to make sure she knows she’s looking for an angle.” Thus, Connie decided to respond to Alexis on the basis of her interpretation that Alexis “doesn’t seem to know that she’s looking for an angle.” Note that in this instance, Connie’s interpretations of Alexis’s thinking were informed by Connie’s own work and thinking on the problem, and thus in turn, her response is also a reflection of her own work and thinking. In fact, since most of the participants’ responses were made on the basis of their interpretations, and their interpretations were informed by pedagogical content knowledge, subject matter
knowledge, interpretive language, and their own work and thinking, in a sort of domino effect, I would argue that these resources all influenced the teachers’ responses in addition to their interpretations.

The participants’ drew directly on other resources (besides their interpretations of the students’ thinking) as well when responding to the students. For example, some participants appeared to draw on their pedagogical content knowledge and/or their orientations towards teaching. For example, recall that one particular aspect of Sarah’s KCS, namely, her knowledge that, “sometimes they (students) write down something and it’s just a lucky coincidence” encouraged her to frequently elicit additional information from the students. In most instances, it was difficult to differentiate between knowledge of content and teaching and the participants’ beliefs or orientations toward teaching. For example, recall that Craig responded to Amy by using a particular representation of function, the “function box” like the one Craig drew in Figure 5.1. Craig, explained:

one way to do the cosine of the cosine thing is to kind of look at just like a function box. So what goes in the cosine function? An angle goes in […] and what comes out is something between negative one and one […] but then to go back into cosine again that needs to be an angle and it's not. So that right away should give you pause right there.

Craig’s wording, “one way to do the cosine of cosine thing is to,” seemed to indicate this was a strategy he had used before, and he also knew the advantage to using this representation with students, noting, “that right away should give you pause.” Similarly, Elliot advocated for the use of counter-examples, noting that,

going students to just be like mind-boggled like whoa what just happened there? Something I thought I definitely knew is not the same as something else I definitely know, you know, and it's the confusion that makes them have to, kind of resolve that, and I think that's where learning
comes from for them.

In instances such as these two, it was difficult to determine if these participants were drawing on their knowledge of ideal teaching or their orientations and beliefs about teaching, or both. In the case of Sarah, it was clearer that she was drawing on her orientations or beliefs to respond. For example, Sarah always posed a “give or ask” response and this was informed by her orientations towards teaching, such as when she stated:

   So the traditional person in me is thinking, well let me just respond to what they have and there we go. That’s it. That’s what you messed up on. (laughing) But the more progressive person in me, is saying alright well I, let me, let me ask this, let me ask that, and explain this a little bit more. Can you figure this out?

Sarah seemed to be deciding to provide opportunities for students to do and explain more, based on a more “progressive” orientation.

The participants’ level of cautiousness to make definitive claims about the students’ thinking also played a significant role in their decisions to respond. In particular, they would “give or ask” in order to determine additional information about the students’ thinking in order to test their hypotheses about the students’ thinking. For example, recall that Connie was careful to differentiate between errors and misconceptions, and as a consequence, when asked to respond to Cheyenne in Task Two (See Appendix D), Connie stated:

   I would ask Cheyenne how she got sine of 30 degrees and the cosine of 30 degrees […] I’d ask her to draw the triangle and label it, […] see if she’s, if she’s drawing the triangle wrong […] or if she just accidentally picked the wrong one or if she memorized wrong.

Connie aimed to access additional information to determine if Cheyenne's work was a result of simply memorizing wrong or a larger issue of “drawing the triangle wrong.”
Similarly, recall that Elliot proposed asking questions to determine if Cheyenne’s error was a “mistake, by accident” or if it was “something she always thinks.” As I noted previously, Connie and Elliot, the two case study participants with the least amount of teaching experience, proposed responses aimed to access additional information about the students’ thinking more frequently than those with more teaching experience.

Finally, the case study participants’ inability to interpret the students’ thinking or make sense of their strategies also informed their responses. In particular, this is one additional reason that there were so many “give or ask” responses. For example, in response to Martha on Task Two (See Appendix D) Connie stated, “Um I would ask Martha how she’s using this 90 is equal to 20 plus 70, because I have no idea where the rest of, of, well where this comes from.” Similarly, Sarah simply asked Charlie in Task One (See Appendix C) to “explain.” As I explained earlier, Sarah compensated for her inability to make sense of the students’ work in Task One by focusing on how the students did not think like she herself thought; that is she commented on how they were not thinking of arc length as a “portion of the circumference.” This was reflected in her responses to Annie and Lexi; Sarah would, “want to help them recognize that arc length is a portion of circumference,” and she would have a “discussion that might lead them towards oh it’s a part of a circle’s circumference, and then ok well how do you find circumference?” In this example, Sarah has rerouted their thinking and not allowed for unfamiliar solution strategies. In contrast, in the previous example, Connie could not make sense of the students' work and thinking, but she allowed the students an opportunity to explain themselves and did not, at least immediately, reroute their thinking.
Summary

Thus far, I have addressed my first main research question. In particular, I have discussed how the case study participants attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry. For the most part, the case study participants were successful in evaluating the students’ work, frequently attended to students’ thinking, frequently decided to respond on the basis of their interpretations, and drew on a wide range of resources to interpret and respond to students' work and thinking. The case study participants’ overall noticing seemed to be most significantly influenced by their subject matter knowledge, interpretive language and overall cautiousness. Although a different group of participants might have offered different responses, these findings reflect the role and importance of certain types of knowledge, experience, and characteristics for noticing and suggest that teacher preparation and professional development programs provide opportunities to develop these resources for noticing.

Question 2: In What Ways Did The Hypothetical Students’ Work On Trigonometry Problems Challenge and/or Further Their Mathematical Thinking?

While the case study participants’ subject matter knowledge was challenged frequently, there did not appear to be an overwhelming number of instances in which the student work seemed to further the participants’ thinking. The hypothetical student work which seemed to have the most success in furthering the participants’ thinking were Mike’s work in Task Four (See Appendix F) as well as Alex’s work in Task Two (See Appendix D). In the following subsections, I discuss the ways in which these
students’ work challenged and/or furthered the participants’ thinking. Following these sections, I also discuss how and hypothesize why other student

**Mike’s Work on Task Four**

Both Connie and Sarah’s mathematical knowledge appeared to be furthered through their exploration of Mike’s work on Task Four. Recall that prior to engaging with Mike’s work, Connie only demonstrated an understanding of sine and cosine as distances on the unit circle and as ratios of the lengths of sides on a right triangle but not as coordinates on a unit circle. Thus, she was perplexed by Mike’s work at first, noting, “so he graphed the points wrong”; however Connie investigated:

> ok. Well he graphed...(pausing/thinking)...[...] um, ok, sorry. (laughing) I don’t know why - it’s like - ok, and then he’s got square root of two over two for pi over four. Um, pi over three was one half [...] So he’s picking out his x, x-coordinate, um in all of those, because on the unit circle, cosine of, of x, cosine of theta would be equal to the x-coordinate. So those are the ones that he’s graphing.

Through her investigation of the mathematics in Mike’s work, and using her previously held understanding of sine and cosine as distances on the unit circle, she was able to eventually realize that “cosine of theta would be equal to the x-coordinate.”

In contrast, Sarah did demonstrate an understanding of sine and cosine as coordinates on a unit circle prior to engaging with Mike’s work; however Mike’s work still furthered her thinking about the relationship between this unit circle definition and the graph of cosine. Recall that at first Sarah thought, “he just plotted his x and y coordinates” on the graph, but then she realized that wasn’t the case, and pondered:

> So really you’ve got three different quantities that we’re talking about here. We’ve got the radian measure or the degree measure, and then he’s got an x-coordinate and a y-coordinate. So...[...] alright, now I gotta go back and think about [...] so what do the coordinates mean when we graph a
trig function? Cuz we’re talking about the number of radians - oh dear. I’m hittin’ a brain cramp.

She then explored further and explained:

I’m getting hung up on the fact that he’s got ordered pairs here, and I’m thinking oh! ordered pairs, that’s what I’ve gotta graph. [...] he’s got an x-coordinate and a y-coordinate. So I’m automatically jumping into thinking he’s trying to plot these, but it’s really, this is the radian measure is what’s on the x-axis, and it’s this x-value of the coordinate that represents the cosine. [...] the x-value is the radian measure; the y-value is the, the cosine.

She went back-and-forth about this, and eventually concluded:

I was just getting so hung up on the fact that I need to use both of those coordinates! But the cosine of theta is just the x-coordinate, and so that’s what you need to plot the graph. If he wanted to plot the sine function, then we would still use these same radian values, but it would be the y-coordinate that’s representing the sine. So Mike confused me a little bit.

Thus, while Sarah was able to successfully graph the cosine function and explain the graph in terms of the unit circle, she did not use the idea of sine and cosine as coordinates on the unit circle, and thus in investigating and making sense of Mike’s work, she had to grapple with new challenges that did not exist for her in simply answering the mathematical prompt. These challenges were consistent with Brown’s (2006) findings that students often struggle to connect a rotation on the unit circle with a point on the graph of cosine. The hypothetical work was successful, in this case, in forcing Sarah to think about what values on the horizontal axis represented on a cosine graph. She commented on this in her reflection, admitting, “I got confused about the meaning of the coordinates of each point. It took me a while to realize that since the graph is of the cosine function, the x coordinates were the radian measures around the unit circle and the y coordinates were the cosine values.”
I argue that two factors seemed to be important to Sarah’s and Connie’s investigations of Mike’s work: (i) Sarah and Connie were able to answer the mathematical prompt correctly themselves and thus, could evaluate that Mike’s graph was correct, but (ii) Mike’s work represented a different strategy, or way of knowing cosine. I hypothesize that if Mike’s graph of cosine included a major error, Sarah and Connie may have been distracted by or focused on that error, but since Sarah and Connie identified that his graph was correct, there was reason to believe that his strategy was reasonable, and thus, they investigated further, and eventually expanded their thinking about cosine as well as the relationship between it’s graph and the unit circle.

**Alex’s Work on Task Two:** “*Since 20° and 70° are complementary, we know that sin 20°+sin 70° = sin 90° and so sin 20° ≈ 1 - 0.9397.*”

At first, Sarah, Byron, and Connie all were unsure of the validity of Alex’s claim that, “sin 20°+sin 70° = sin 90°” Sarah reacted, “I don’t know about this. Yeah. I don’t- I don’t remember my trig identities at all.” Connie noted that, “I don’t know if this part is true, sine of 20 degrees plus sine of 70 degrees is equal to sine of 90 degrees... I just don’t know if that’s - I don’t remember.” Similarly, Byron noted that, “I feel like this is just nor right to me,” but that he “may have the wrong feeling on this” and, “Ugh, I'm just not, I don't know.” They all then investigated this claim using values other than 20 degrees and 70 degrees. Sarah and Byron both looked at another example, “sin(30°)+ sin(60°)= sin(90°)” and found that this statement was untrue, and thus, Alex’s claim was false. Sarah concluded, “So, that's an incorrect statement. Um, he's just thinking that because they're complementary you can take the sine of them, and that would still be uh - sine of
everything - and that would still be a true relationship, um, and I don’t think it is.”

Similarly, Byron concluded, “Alex's work, she's making the mistake which I have come to realize that you can't add the (laughing), taking the sine of complements and adding them together doesn't equal the sine of the angle.” Unfortunately, Connie did not investigate Alex’s claim further, and did not resolve her uncertainty.

As I noted in earlier sections, Alex’s claim seemed to be one that nine out of twelve interview participants were unsure of. I would argue that it is for this reason as well as the fact that (i) the claim was clearly stated and reasoned by Alex, and (ii) the underlying idea of the claim was easily investigated with a counter-example, that participants were able to further their thinking about sine and cosine in response to Alex’s work. As was demonstrated in Connie’s case however, Alex’s claim in and of itself was not sufficient for promoting such change. A certain level of investigative persistence on the part of the teacher was also necessary. Future research might investigate what beliefs and potential dispositional characteristics that support such persistence. Connie was also the only one of the case study participants who was not able to correctly answer the mathematical prompt in Task Two, and thus, perhaps because the others had more confidence regarding what the correct process should be, they were more comfortable investigating Alex’s claim.

**Work That Challenged but Did Not Further Their Thinking**

Since Craig and Sarah did not have knowledge of radian as either as (i) the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle OR as (ii) the length of the arc on the unit circle subtended by the angle, Annie’s work had the potential to expand their thinking about radian; however it did not appear to do
so during the interviews. Sarah simply noted that Annie perhaps “had a certain algorithm to figure out what that arc length was, but they didn't relate it to circumference.” Instead of investigating Annie's strategy further, Sarah kept focusing on the fact that Annie was not thinking like she herself had; that is, “Annie is also not considering that this is a portion of circumference.” Craig was quicker to just dismiss Annie's strategy altogether, noting, “So she’s equating central angle to arc length. So she's got the wrong idea for that.” Craig misevaluated Annie's sketch as incorrect, and thus he did not investigate her strategy further. I hypothesize Craig and Sarah might have experienced cognitive conflict and as a consequence, furthered their thinking about radian, if they had used their own knowledge of arc length as a portion of circumference to check Annie’s initial claim that the arc length of the angle in the unit circle was precisely the angle measure. This again demonstrates that the students’ work itself was not sufficient for promoting growth in subject matter knowledge. It should be noted that while Sarah did not further her thinking with respect to radian during the interviews, she did reflect later that, “As I try to recall how I analyzed Annie’s work, I think I may have commented about how she labeled the angle and arc the same in the first circle, which didn’t make sense as it seemed as though she was mixing up arc length and arc measure. In hindsight that was a correct relationship for the unit circle.” Thus, at the very least, Sarah’s experience analyzing this work, prompted her to perhaps look up new mathematics.

Joe’s work on Task Three (See Appendix E) as well as Alice’s work on Task Four (See Appendix F) also challenged, but did not necessarily seem to further Elliot’s mathematical thinking. Recall that despite Elliot’s understanding and use of the
“coordinates” definition of sine and cosine, he did not appear to hold similar definitions of sine and cosine as horizontal and vertical lengths in the unit circle. Thus, Elliot had difficulty making sense of these two students’ work, which was designed to represent knowing of sine and cosine as these lengths. Elliot repeatedly insisted that Joe “has to divide at the very end uh by the hypotenuse.” As I’ve mentioned, Elliot did eventually seem to make the connection between the definitions. He explored,

I kind of make it on the unit circle, and then once I kind of make it on the unit circle, I can think about it as like oh now I know x is cosine and y is sine. (writes “(x, y)”). So then I can think about this is like, oh now we have... now let’s think of this side as the y so all we’re looking for is the question mark now. We don’t have to worry about doing question mark over one. We’re just looking um for the y-value or this side and so that’s what I, I think would be what someone might have shown him or he came up with. It’s kind of a good idea. Um...but yeah. I don’t know if that’s true.

Elliot did seem uncertain with his findings in the end, and almost immediately afterwards, he was unable to interpret Alice’s work in Task Four.

Summary

In this section I have discussed that while the case study participants’ subject matter knowledge was challenged frequently, there weren’t as many instances in which the student work seemed to further the participants’ thinking. It was also shown that the students’ work was not sufficient, and that a certain level of investigative persistence on the part of the teacher was also necessary. As I’ve noted previously, future research might investigate the beliefs and potential dispositional characteristics that support such persistence. The participants’ mathematical thinking was furthered most often in situations when (i) the participant could correctly answer the mathematical prompt themselves, yet (ii) a particular student’s work represented a different strategy than their own, and/or (iii) the student’s work was detailed and clearly reasoned. I hypothesize that
these participants would have been more likely to benefit from this experience if they
had also been given time to collaborate with other teachers. Consistent with this
hypothesis, Sarah suggested, “I thought also about how having the opportunity to
engage in these discussion with a colleague might help us to better understand our
students’ thought processes,” and Elliot suggested, “Multiple perspectives could have
helped me overcome the challenges that came up.”

Other Benefits of the Interview Experience

In their reflections, the participants seemed to view their interview experiences as
valuable both to their subject matter knowledge and pedagogical content knowledge. In
particular, the participants expressed a desire to include these experiences in teacher
preparation programs. For example, Sarah noted, “How wonderful it would be for
preservice math teachers to take a class in which they were tasked with analyzing
student work such as what I did during the interviews! It would not only help to clarify
any misconceptions and refresh their memory about high school topics, but it could also
give them practice with scoring student work.” Similarly, in response to the question,
“Any other comments or thoughts you would like to add about the experience or
interview tasks in particular?,” Elliot reflected:

this was a great experience and it helped me understand the topics I am
teaching and students misconceptions more thoroughly. I believe that a
class that was made out of these types of activities could help students
learn in college […] Understanding high school math and problems
students may have seems to be much more important to being a good
teacher than learning advanced mathematics such as Abstract Algebra or
the Gen Eds that we are forced to take. These interview tasks force us to
think quickly on the spot and analyze some important work, I believe they
were very helpful to my maturation as a teacher.
Elliot also noted that the work influenced him to think about other difficulties students may have in trigonometry, noting, “I really thought more deeply about how difficult even and odd functions might be to students” and “I also learned that the unit circle can be a very tough concept for students to understand because of how it is explained and then seems to change for them.” While Elliot acknowledged increased thinking about content and students, Craig acknowledged increased thinking about his teaching, noting that the interviews made him think about “the idea of a teacher projecting what they think the student should know into an answer, when the student doesn’t explicitly write out their reasoning” and, this he “will, in the future, work with students on better clarifying their answers and reasoning.” In contrast, Sarah and Connie acknowledged increased thinking about subject matter. More specifically, Sarah noted that “I found that the more I talked about something, the more I convinced myself one way or the other about the validity of a statement,” and Connie noted that, “Analyzing student work and solving the problems myself helped me remember and clarify some things about trigonometry that I had forgotten.” Finally, Byron noted that he “found that if you do not have a clear understanding of the results/answer, then it makes the analysis of the student’s work very difficult” and elaborated, “It is hard to ‘see’ others work when you do not have it solid in your own mind.”

**Conclusion**

In the Cross-Case Discussion, I have presented findings that address each of my main two research questions. In particular, I have discussed (1) how the teachers attended to, interpreted, and responded to hypothetical students’ written work and thinking in trigonometry, as well as (2) in what ways the hypothetical students’ work
challenged and/or furthered their mathematical thinking. The case studies, as a whole, suggest that teachers with stronger subject matter knowledge are not only able to evaluate students’ work more accurately, but they also are more likely to attend to students thinking.

The findings also suggest the importance of various resources and factors that are influential on one’s interpreting: (i) subject matter knowledge, (ii) interpretive language, (iii) one’s own thinking about problems, (iv) knowledge of common teaching, (v) knowledge of common students errors and misconceptions, as well as (vi) one’s cautiousness to differentiate between (a) correctness of work and thinking, (b) one’s own thinking and the students’ thinking, and (c) the relative seriousness of different errors. Further, the findings suggest that two of these resources/factors are not only influential but also necessary for interpreting. In particular, they suggest that engaging in meaningful discussions of students’ thinking requires (i) more subject matter knowledge than is required to simply solve the mathematical prompts as well as (ii) interpretive language for talking about students’ thinking. In addition, important resources and influential factors for deciding to respond to students included not only (i) one’s interpretations of students’ thinking, but also (ii) knowledge of ideal teaching, (iii) one’s beliefs and orientations, and (iv) one’s cautiousness to make decisive claims about students’ thinking. Further, teachers with less experience were more likely to access additional information about students’ thinking, and teachers with more subject matter knowledge were more likely to “show and tell”.

With respect to the second main research question, I discussed the ways in which the hypothetical students’ work did or did not further the teachers’ mathematical
thinking. In particular, the participants mathematical thinking was more likely to be furthered in situations when (i) the participant could correctly answer the mathematical prompt themselves, yet (ii) a particular student’s work represented a different strategy than their own, and/or (iii) the student’s work was detailed and clearly reasoned. I also hypothesized that the teachers would have been more likely to benefit from this experience if they had also been given time to collaborate with other teachers. Finally, I discussed the participants’ praise of the experience and the self-reported benefits of engaging in analysis of work and thinking, including recalling or learning new mathematics as well as learning and thinking about students and teaching in new ways.

In the next and final chapter, Chapter VII, I provide an overall summary of the important findings from this study, the implications for the mathematics education community, the limitations of the study, as well as future directions for research.
CHAPTER VII

CONCLUSION

Introduction

In Chapters IV, V, and VI, I presented findings with respect to my research questions, and in this chapter, I provide concluding remarks with respect to my three research questions. In the first section, I provide an overall summary of the important findings and conclusions. In the second section I discuss the significance of these findings to the mathematics education community, including both mathematics education research as well as mathematics teacher preparation and development. Finally, I discuss limitations of the study as well as directions for future research.

Summary and Conclusions from Findings

In the previous chapters, I have discussed my results with respect to my three research questions: (0) What content knowledge of radian, unit circle, and the sine and cosine functions do secondary mathematics teachers possess?; (1) How do secondary mathematics teachers attend to, interpret, and respond to hypothetical students’ work and thinking in trigonometry?; and (2) In what ways does the hypothetical students’ work on trigonometry problems challenge and/or further the preservice, in-service, and student teachers' mathematical thinking? In this section, I summarize the overarching important findings. Most of the summaries and conclusions presented in this chapter correspond to the case study analysis; however, I begin by discussing conclusions with respect to all 19 participants’ knowledge of trigonometry as well as the importance of this knowledge for noticing, based on the data from the twelve interview participants.

Teachers’ Knowledge of Radian

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While fifteen out of 19 participants exhibited an understanding of the relationship between radian and degree measure and could correctly convert between radians and degrees, only two participants defined radian correctly as the ratio of two lengths or as the length of the arc on the unit circle. Even so, two out of 19 was a relatively high ratio given Fi’s (2003), Akkoc’s (2008), and Topcu and colleagues’ (2006) findings, in which only one out of a collective 107 participants correctly defined radian. Four additional participants in this study did not define radian correctly but did at least recognize that radians have some connection with length. In contrast, nine out of 19 participants defined radian more generally as a unit of angle measure. In addition, four out of 19 participants demonstrated a concept image of radian dominated by their concept image of degree, as described by Akkoc (2008), and about half of the participants in this study exhibited a belief that radians are expressed only in terms of pi.

**Teachers’ Knowledge of the Unit Circle**

While fourteen out of nineteen participants defined the unit circle as a circle with radius one, seven of of them also exhibited a perception of the unit circle as a fixed diagram or picture that is given to students and did not exhibit a strong understanding of the functionality of the unit circle and it’s significance as a tool for reasoning. This is problematic in light of Weber’s (2005) claim (supported by his findings) that successful student learning using the unit circle is dependent on the students’ use of the unit circle as a tool for reasoning about trigonometric functions; students need to understand the process of and take part in the process of constructing the unit circle and its relationship to trigonometric functions. In contrast to Weber’s finding, participants in this study claimed that the unit circle was taught (i) to facilitate conversion between radians and
degrees or show radians compared to degrees, and/or (ii) to show or help students remember/evaluate values of trigonometric functions.

**Teachers’ Knowledge of Sine and Cosine**

While fifteen out of nineteen participants exhibited at least one correct way of knowing sine and cosine (either as coordinates of a point on the unit circle, as horizontal and vertical distances on the unit circle, or as ratios of side lengths of a right triangle), most did not appear to connect all three ways of knowing. In particular, several participants seemed to acknowledge sine and cosine as lengths on the unit circle but not as coordinates of a point on the unit circle and vice versa. The hypothetical student work, especially Mike and Alice’s work in Task Four (See Appendix F), seemed to be a powerful tool for challenging the participants’ ways of knowing sine and cosine, and in some cases, encouraged the participants to make the connection between the two unit circle definitions.

I also touched upon two important findings related to students’ understandings of sine and cosine. First, I found that three of the participants used linear thinking to reason about sine and cosine. In particular, they used the faulty idea that, \( \sin(A) + \sin(B) = \sin(A+B) \). Second, consistent with Fi’s (2003) findings, I found that ten out of eleven of the preservice teachers did not have an understanding of the relationship between sine and cosine as cofunctions. In contrast however, the student teacher and in-service teacher participants in this study held stronger knowledge of cofunctions. In fact, two student teachers and four in-service teachers answered the interview prompt in Task Two correctly using their knowledge of cofunctions. In addition, some participants,
like Elliot and Sarah, were able to explain this relationship by drawing on their ways of knowing sine and cosine.

The Importance of Subject Matter Knowledge in Noticing Students’ Work and Thinking

In my discussions of how teachers noticed, I have talked about various resources for noticing. While several resources have proved to be valuable for noticing, strong subject matter knowledge was a necessity. The findings suggest that teachers with stronger subject matter knowledge are not only able to evaluate students’ work more accurately, but they also are more likely to attend to students’ thinking and make sense of students’ strategies. Further, as was shown with the case studies, the knowledge required to solve a mathematics problem is not always sufficient for interpreting students’ thinking on that problem. That is, noticing of students’ work and thinking requires more subject matter knowledge than is required to simply do the mathematics. In particular, participants were more likely to make sense of students’ strategies and thinking when they themselves had deep and flexible mathematical knowledge of the concepts at play, including multiple ways of knowing sine, cosine, and radian. Further, I hypothesize that these findings are not specific to trigonometry; that is, these issues are likely prevalent in other mathematics subjects as well as, more generally, the STEM disciplines.

While having deep and flexible mathematical knowledge was a necessity for interpreting students’ thinking and increased one’s likelihood to interpret students’ thinking, it did not appear to be sufficient. That is, interpreting students’ thinking seemed to require more than subject matter knowledge. For example, engaging in meaningful
discussions of students’ thinking also required interpretive language, as is summarized in the next section. Recall that because participants with weak trigonometric knowledge did not engage with or notice the students’ work and thinking in meaningful ways, I focused in most detail on the noticing of five case participants who scored the highest on the trigonometry assessment. The remaining summaries and important conclusions presented here were, for the most part, obtained via investigation of these five case study participants.

**Interpretive Language and Noticing Students’ Work and Thinking**

As I have discussed, when I first began analyzing the data, I observed (i) inconsistencies in the ways the participants used the word “understand” as well as (ii) differences among participants in the types and extent of the language they used to describe students’ thinking and ways of knowing. In particular, very few participants used the word “understand” to be descriptive about students’ thinking (ex. she understands cosine as…); instead they used “understand” in (i) an evaluative way (he did understand X, she has a good understanding of Y) or in (ii) a comparative way (he has a weaker understanding, she has a better understanding), and/or (iii) to talk about procedures or correctness of work. When participants were more descriptive about students’ ways of knowing, they often used language that didn’t involve the word “understand”; instead, they described how the students were “seeing,” “considering,” “thinking,” or “recognizing”. These findings suggest, as van Es (2011) noted, that noticing, at least noticing aloud, requires a special discourse for talking about students’ thinking. Further, the findings suggest that acquiring such language is not easy. For example, even Craig, a teacher with strong subject matter knowledge and eight years of
teaching experience, struggled to talk interpretively, and used the word “understand” both frequently and inconsistently. These findings have implications for teacher preparation, which will be discussed in a later section.

**Cautiousness in Noticing Students’ Work and Thinking**

The idea of cautiousness persisted throughout my analysis and findings. In particular, the ways in which the participants interpreted and responded to students’ work and thinking could be both categorized and explained by their relative cautiousness with regards to (i) differentiating between correctness of work and thinking, (ii) differentiating between the relative seriousness of errors, and (iii) considering multiple ways of thinking or understanding that might produce the same work or result. Sarah seemed to indicate that aspects of her cautiousness had developed over time through her experiences teaching. In particular, she noted that she had difficulty determining whether “errors were because of lack of understanding of the trigonometry concepts or if they were due to something else, like […] arithmetic mistakes,” and Sarah acknowledged that experience had taught her to recognize this difference, explaining, “As a young teacher I might have ‘taken off points’ for both types of errors in the same way.” In contrast however, Connie and Elliot were both student teachers, and despite their lack of teaching experience, showed significant cautiousness. In fact, Connie was arguably the most cautious of all five case study participants when interpreting students’ thinking, consistently debating whether errors were serious or not, considering multiple ways the students might have thought about the problems, and being careful not to over-project her own thinking. Further, Elliot’s and Connie’s responses aimed to access additional information about the students’
thinking more frequently then the in-service teachers’ (Craig, Byron, and Sarah) responses. I hypothesize that due to their lack of experience, Elliot and Connie may have felt less confident in their abilities to interpret students’ thinking, and thus provided more opportunities for students to communicate their thinking; however it may also be the case that Elliot and Connie will continue to be cautious throughout their teaching career.

Additional Resources for Noticing Students’ Work and Thinking

In addition to one’s subject matter knowledge, interpretive language, and cautiousness, several other resources were valuable when noticing students’ work and thinking. In particular, in their interpretations of students’ thinking, the case study participants appeared to also draw on their (i) own work and thinking, (ii) knowledge of content and common teaching, and (iii) knowledge of common student errors and misconceptions. Also, in deciding to respond to the students, the participants appeared to draw on their (i) interpretations of students’ thinking, (ii) knowledge of content and ideal teaching, and (iii) beliefs and orientations about teaching. In the remainder of this section, I summarize how the participants’ drew on each of these resources to inform their noticing.

Own work and thinking. The case study participants’ own solution strategies and thought processes also informed their interpretations of students’ thinking. As I have discussed, some over-attributed their own thinking to students, while others were more cautious and posed their own thinking as one of several possible ways in which a student was thinking. In addition, some participants made their own thinking or strategy part of the goals or expectations for successful completion of the task and thus,
compared the students’ thinking to their own. The findings suggest that (i) the ability to solve and think about a problem multiple ways is valuable when interpreting thinking and (ii) teachers should be aware and reflective regarding temptations to (a) assume their students share their same thinking and understandings as they do and/or (b) value students’ work that looks like their own over work that is nonconventional.

**Knowledge of Content and (common and ideal) Teaching.** My findings also suggest that one’s Knowledge of Content and Teaching, or KCT, is a valuable resource for both interpreting and deciding to respond to students’ work and thinking. Recall that KCT is one aspect of Pedagogical Content Knowledge (PCK), which is a category of Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008). As I have discussed, within KCT, I found it useful to further distinguish between knowledge of content and **common** teaching and knowledge of content and **ideal** teaching. In particular, I found that participants drew on their knowledge of content and common teaching, or knowledge of how certain content is typically taught along with the advantages or disadvantages of teaching content that way, to inform their interpretations of students’ thinking and in particular, to help explain the evolution of students’ thinking. In contrast, participants drew on their knowledge of content and ideal teaching, or knowledge of ways of teaching particular content that is more likely to result in successful student learning, to inform their decisions to respond.

**Knowledge of common student errors and misconceptions.** One’s Knowledge of Content and Students, or KCS, was also a valuable resource for interpreting students’ work and thinking. Again, recall that KCS is a category of one’s pedagogical content knowledge (Ball, Thames, & Phelps, 2008). The findings from this
study suggest that teachers attend to familiar or common errors and draw on their knowledge of the types of student thinking and/or misconceptions that typically inform these errors, in order to interpret students’ thinking. In this study, this knowledge was exhibited and drawn on mainly by participants with experience teaching precalculus, namely Byron and Elliot.

**Beliefs and orientations.** Research has established that teachers’ practices are shaped by their own beliefs and orientations (Aguirre & Speer, 2000; Handal, 2003; Philipp, 2007; Thompson, 1992), and with respect to noticing in particular, Schoenfeld (2011) argued that, “what you attend to- what you notice-is in large measure a function of your orientations.” In this study, it is likely that the participants’ attention, interpretations, and decisions to respond, were in many ways a result of their beliefs and orientations; however, it was not a direct part of my study and thus, it was often difficult to determine precisely what these teachers’ beliefs and orientations were as well as how and when they were influencing their decision making. Despite this fact, there were instances in which the participants’ beliefs and orientations were clearly influential, especially on their decisions to respond to students. In particular, Elliot seemed to draw on his beliefs that learning occurs through cognitive conflict, and Sarah seemed to respond based on her evolving “progressive” orientations to tell less, and ask more, allowing students more opportunities to explain.

**Using Student Work as a Means to Improve and Strengthen Teachers’ Knowledge**

While prior research has provided some evidence that investigating students’ work can improve teachers’ understanding of mathematical content knowledge, there had been a call for further investigation of this “reversed relationship,” looking beyond
whether or not growth occurs to looking at how it occurs (Crespo, 2000). In this study, I found that growth was more likely to occur in situations when the participant had enough knowledge to correctly answer the mathematical prompt themselves, yet a particular student’s work represented a different strategy that was new, to them. For example, the work that was most successful at furthering the participants’ mathematical thinking was Alice and Mike’s work on Task Four (See Appendix F). Recall that several participants demonstrated an understanding of sine and cosine as distances on the unit circle but not as coordinates on a unit circle, and vice versa. Mike and Alice’s work each represented one of the ways of knowing, and thus challenged the participants to connect these two ways of knowing. In a similar scenario, in Task One, several participants could answer the prompt correctly, yet Annie, and Charlie’s work in Task One represented a way of knowing radian (as arc length on a unit circle) not previously held by most of the interview participants; in contrast to Task Four however, participants did not experience furthered mathematical thinking in response to the students’ work in Task One. I hypothesize two distinctions between the these tasks that might account, at least in part, for the difference in the participants’ investigation: (1) because Annie and Charlie’s work was less detailed than Alice and Mike’s work, it was more difficult to investigate the students’ strategies and/or (2) because Alice and Mike’s graphs were mostly correct, this gave participants reason to think their strategies might have been valid and caused the participants to investigate further, whereas in Annie and Charlie’s work, the participants might have been distracted or dismissive due to the students’ errors and incorrect results. Thus, I hypothesize that perhaps correct and well-detailed work that represents unconventional strategies is the most successful at expanding
teachers’ subject matter knowledge. Further, both Elliot and Sarah (as well as additional interview participants) suggested that they would have benefited from the experience more if they had also been given time to collaborate with other teachers.

In their reflections, some participants also reported that the interview experiences were valuable both to their subject matter knowledge and their pedagogical content knowledge (although they did not use these particular terms). In particular, some noted that (i) they believe more experiences analyzing students’ work and thinking should be provided in undergraduate mathematics education programs, (ii) the experiences encouraged them to think more deeply about mathematics concepts, and (iii) the experiences encouraged them to think more deeply about students’ difficulties and their own teaching.

**Significance to Mathematics and STEM Education Community**

This study and its findings contribute to the field of mathematics education, and more broadly, STEM education, in two ways. First, this study contributes to the current research body regarding teachers’ noticing of students’ work and thinking as well as students’ and teachers’ knowledge of trigonometry. Second, the findings regarding how teachers notice as well as how the teachers’ mathematical thinking was challenged and/or furthered by the students’ work shed light on the types of experiences and opportunities that should be provided in both teacher preparation and professional development programs. In the following subsections, I provide more detail as to how this study contributes to the field in these two respects.
**Contributions to Research**

This study has produced findings that contribute to current research in three areas. First, while not the main focus of my research, this study contributes to current research (Akkoc, 2008; Brown, 2006; Fi, 2003; Moore, 2009, 2013; Moore, LaForest, & Kim, 2012; Topcu et al., 2006; Tuna, 2013; Weber, 2005, 2008) on students' and teachers' understandings of trigonometry content. In particular, this study provides additional evidence of findings discussed in prior research, including the persistence of concept images of radian dominated by degree, beliefs that radians are only expressed in terms of pi, weak understandings of the functionality of the unit circle, radian and cofunctions, and the existence of three ways of knowing sine and cosine. Further, this study has shown that some teachers have knowledge of sine and cosine as vertical and horizontal lengths on the unit circle, some teachers have knowledge of sine and cosine as coordinates on the unit circle, and yet, they don't always connect these two ways of knowing. In addition, most research on teachers' understanding of trigonometry (Akkoc, 2008; Fi, 2003; Moore, LaForest, & Kim, 2012; Tuna, 2013) has focused on preservice teachers’ understandings, and my own results suggest that in-service teachers’ understandings might not be as limited as preservice teachers.

Second, this study has expanded upon prior research on teachers' noticing of students' work and thinking (Haltiwanger & Simpson, 2014; Jacobs, Lamb, & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle 2011; Jacobs & Philipp, 2010; Roth McDuffie, 2014; Schack et al., 2013; van Es, 2011) by providing findings with respect to (i) secondary mathematics teachers, as opposed to elementary teachers and (ii) a new content area, trigonometry. More significantly, this study also contributes a framework
for explaining what types of knowledge and resources enable or inhibit noticing of students’ work and thinking while more generally characterizing different ways in which secondary mathematics teachers analyze and respond to students’ work. Further, findings suggest that for future research, it is imperative to take into account participants’ subject matter knowledge as well as the ways in which they use interpretive language. As I have discussed, these two factors significantly influenced the ways in which the teachers in my study noticed, and more significantly, weaknesses in subject matter knowledge and/or interpretive language prevented participants from engaging in meaningful noticing of students’ work and thinking. Finally, in terms of methodology, my own interview prompts were less restrictive than those in previous studies of teachers’ noticing (Haltiwanger & Simpson, 2014; Jacobs, Lamb, & Philipp, 2010; Jacobs & Philipp, 2010; Jacobs, Lamb, Philipp, & Schappelle, 2011) and provided more insight into what participants attend to and how they respond on their own.

A third way in which this study has enhanced current research is by contributing to budding discussions (Crespo, 2000; Francisco & Maher, 2011; Herbel-Eisenmann & Phillips, 2005; Kazemi & Franke, 2004) of how the study of student solutions stimulates teachers’ engagement in mathematics. In particular, the findings suggest that simply having teachers engage in the analysis of students’ work will not ensure improved mathematical thinking on the part of the teachers and that teachers may be more likely to engage in further mathematical thinking if they themselves can solve the problems correctly. Further the findings suggest that using correct and detailed work that represents an unconventional strategy might have the most potential for encouraging engagement in mathematics.
Implications for Teacher Preparation and Development

The findings presented in this paper have significant implications for teacher preparation and development in that they highlight the importance of having strong subject matter knowledge, pedagogical content knowledge, and interpretive language in order to successfully notice and build on students’ work and thinking. In particular, the findings suggest that teacher preparation and development programs should provide preservice and in-service teachers with opportunities to: (1) develop effective ways to talk about student thinking (perhaps not centered around the word “understand”), (2) develop deep and flexible subject matter knowledge including multiple ways of knowing important concepts, (3) gain exposure to common student thinking, misconceptions, and difficulties, (4) gain exposure to both common and ideal teaching strategies, and (5) develop a “cautiousness” when interpreting students’ thinking. While my own study focused on noticing of students’ work and thinking in trigonometry, the issues I’ve uncovered are likely prevalent across various mathematics content areas, grade levels, as well as, more broadly, the STEM disciplines, and thus, I argue that these types of opportunities be provided to preservice and in-service teachers across all grade levels and STEM disciplines. The findings highlight that providing these opportunities is essential for them to engage in reform-minded teaching that is informed by and builds upon their students’ thinking.

In addition, the findings emphasize that preservice teachers should be provided opportunities to analyze students’ work and thinking, not only to gain experience and skill in noticing, but also to enhance their subject matter knowledge. As was evidenced in this study, in many cases the participants’ knowledge was challenged by the
students’ work to a greater extent than it had been by the mathematics problem itself. Further, in their reflections, teachers indicated that they find this work both interesting and valuable, which suggests that these opportunities would be motivating and engaging for mathematics education students. In particular, based on differences between the work that furthered their thinking and the work that did not, this study also indicates that teacher educators might have the most success by engaging teachers in analysis of student work that is mostly correct, relatively detailed, and illustrates unconventional strategies. In addition, based on the teachers’ reflections, opportunities to collaboratively notice would be appreciated and perhaps strengthen their engagement.

Limitations of the Study

In this section, I discuss three limitations of this study. First, the research participants volunteered to participate, and thus, the participants may possess knowledge and/or certain other qualities not characteristic of the general population of secondary mathematics preservice, in-service, and student teachers. For example, individuals who felt less comfortable with trigonometry or participants who were less interested in students’ work and thinking might have been less likely to volunteer to participate in this study.

A second limitation of this study was that while the interview setting allowed me to capture the teachers’ noticing of students’ work and thinking in ways I could not have in an actual classroom setting, the fact that this was a non-authentic noticing experience may have influenced their responses. For example, in the interview setting, the participants may have been more likely to make “progressive” claims based on what
they thought a mathematics educator might want to hear. Also, in this non-authentic setting, the teachers most likely had more time to devote to each particular student’s work and thinking than they normally would have in a regular classroom setting. In addition, the hypothetical student work presented to the participants was designed to represent different ways of thinking, and thus, in reality, teachers are less likely to see work that represents solution strategies so different from their own. Further, the teachers may have responded differently if they were noticing their own students in their own classroom on a task they had chosen. For example, I suspect that in an actual classroom setting participants would have been more likely to respond on the basis of students’ affect, since they would be simultaneously interpreting students’ emotions in the classroom. I also suspect that if the teachers had chosen the tasks for their students, their own goals for the tasks as well as their current knowledge of those particular students would have also served as resources for interpreting the students’ thinking.

A third limitation of this study is the fact that my access to what the participants attended to, how they interpreted, and how they decided to respond was limited to their spoken dialogue. For example, their pedagogical content knowledge may have informed their interpretations and responses in more ways than I had accounted for; however I could not determine this unless they provided some spoken indication. Further, while I did frequently and repeatedly ask them to think aloud, the participants may have attended to other aspects of students’ work and thinking, yet did not verbalize it. Thus, I had to rely on the participants’ think aloud dialogue to determine how they made certain decisions.
Directions for Future Research

This study suggests several possible directions for future research. First, it would be informative to establish the generalizability of this study’s findings. Since the participants of this study all attended the same university (at varying times), it would be valuable to replicate this study with participants with more varying backgrounds to compare how they notice as well as the influential power of the various resources for noticing identified in this study.

Research that looks more closely at any of the individual influential factors for noticing would also be valuable. For example, this study has begun describing how one’s interpretive language and one’s varying forms of cautiousness influence one’s noticing. Future research might continue to investigate the role of these two factors as well as investigating how teacher preparation and professional development programs can successfully develop these skills.

Future studies might also look more closely at the important role of subject matter knowledge in noticing. For example, a recent study conducted by Haltiwanger and Simpson (2014) found significant differences between how sophomore level preservice teachers and senior level preservice teachers responded to students’ work. They suggested that these differences are likely a result of increased field experiences and teacher education courses; however, it would be interesting to also investigate to what extent gains in their subject matter knowledge accounted for these differences. Further, as I have noted, most research on noticing is done at the elementary school level. Since, at the elementary school level, teachers may be more likely to answer the mathematics problems correctly themselves, the role of subject matter knowledge for
noticing may be overlooked; however, this study shows that more knowledge is required to interpret thinking than is required to simply solve the problems correctly. In particular, teachers must have deep flexible knowledge as well as multiple ways of knowing the concepts involved in the problem. Thus, it would be interesting to investigate the role of subject matter knowledge when noticing young children’s work and thinking.

The findings of this study suggest hypotheses with respect to the role of both teaching experience and subject matter knowledge in deciding to respond. It would be informative to assess these findings with a larger more varied population. For instance, it would be interesting to see if in general, there is a tendency for teachers with greater subject matter knowledge to “show and tell” and to look at the hypothesis that teachers with less experience tend to provide more opportunities for students to explain their thinking. Another study might continue the investigation into which characteristics of problems or student work encourage or support teachers’ engagement in mathematical thinking. In particular, it would be interesting to assess the hypothesis that using correct and detailed work that represents an unconventional strategy has the most potential for encouraging teachers’ engagement in mathematics.
APPENDIX A

TRIGONOMETRY ASSESSMENT

Name: ____________________________________

Trigonometry Assessment

Please answer all questions to the best of your ability and show as much work as possible.

1. Convert the given angles in radian measure to degrees and vice versa: $\frac{5\pi}{2}$, $\frac{3\pi}{5}$, 19, 50°, 570°. Show your work.

2. (a) Define radian.

(b) Explain the relationship between radian and degree angle measures.

3. Given that the following angle measurement $\theta$ is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius length of 2 inches, 2.4 inches, and 2.9 inches (figure not to scale). Please explain your work and thinking in solving for the length of one arc.
4. (a) What is a unit circle? Explain.

(b) In your opinion, why is the unit circle taught in trigonometry? Can you name some specific ways that the unit circle is useful in the learning and teaching of trigonometry?

5. Approximate (a) \( \sin 340^\circ \) and (b) \( \cos 340^\circ \). Please explain how you reached these approximations. Could you find a closer approximation? If so, how? If not, why not?

(c) Could you provide a closer approximation if you were given \( \sin 70^\circ \)? Please show and explain.

6. (a) Identify the (i) domain, (ii) period and (iii) symmetry of the cosine function.

(b) Use the unit circle to explain and defend each of your responses in part (a).
APPENDIX B

TEACHING EXPERIENCE, STYLE, AND BELIEF ITEMS

Please circle one: Prospective Teacher  Student Teacher  Teacher

If you are a prospective teacher, please list any mathematics teaching related experience you have had:

If you are a teacher please list the number of years you have taught:

_______________

If you are a teacher OR student teacher, please list the classes you have taught as well as the number of times you have taught them:

<table>
<thead>
<tr>
<th>Classes you have taught</th>
<th>Number of times you have taught this class</th>
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1. Please describe your teaching style. What is your role as the teacher?

2. (a) To what extent do you agree that students are rational decision makers capable of determining for themselves what is right and wrong (please circle one).

   strongly agree  agree  no opinion/ not sure  disagree  strongly disagree

   (b) Please briefly explain your choice.
Problem Posed to Participant and Hypothetical Student: Find the length of the arc on a circle with radius 3 in the x-y plane associated with the angle \( \theta = -\frac{\pi}{12} \)

Student Work:

Allen says, “Well the negative doesn’t really matter if we’re talking about length. So I’m just going to treat it as \( \frac{\pi}{12} \).”

He writes:

\[
\begin{align*}
\text{Circumference: } 2\pi r &= 6\pi \\
12x &= 6\pi \\
x &= \frac{\pi}{2}
\end{align*}
\]

Lexi writes:

\[
S = \theta r
\]

And then says, “I’m not sure if \( \theta \) should be in degrees or radians. It must be radians because otherwise we’d have a degree symbol in our answer. I don’t think there is a real symbol for radians.”

She writes:

\[
S = -\frac{\theta}{x} \cdot 3 = -\frac{\pi}{4}
\]

And then says, “Hmm, it seems weird that I have a negative number with \( \pi \) in it as a length.”
By the time you check on Annie, she is finished and has written:

Charlie has on his paper:
APPENDIX D

TASK TWO PROBLEM AND STUDENT WORK

Problem Posed to Participant: What can you tell me about sin 20°? What about cos 20°? Can you give me an approximation? Explain. Suppose you are given sin(70°) ≈ 0.9397. Would this change your approximation? Explain.

Problem Posed to Hypothetical Student: Given sin(70°) ≈ 0.9397, what can you say, if anything, about sin 20° and cos 20°?

Student Work:

Alex’s work:
Since 20° and 70° are complimentary, we know that sin 20°+sin 70° = sin 90° and so sin 20° ≈ 1 - 0.9397
We also know that sin²θ + cos²θ = 1, so sin²(20°) + cos² (20°) = 1.
So cos 20° = ±√(1 - sin²(20°)) ≈ ±√(1 - (1 - 0.9397)²)
And 20° is in the first quadrant, so cos 20° ≈ √(1 - (1 - 0.9397)²)

Martha’s work:
Well 90°=20+70, so cos 20° ≈ .9397
Also, 0 < sin 20° < 1 because the sine function gives out values between -1 and 1 and 20° is in the first quadrant.

Cheyenne’s work:
sin(0)<sin(20°)<sin(30°) so sin(20°) is between 0 and √3/2
cos(30°) < cos (20°) < cos(0°) So cos (20°) is between 1/2 and 1
Problem Posed to Participant and Hypothetical Student: Given the following triangle find $\sin(\theta)$.

Student Work:

Joe has written:

Anna has written:
Problem Posed to Participant: Draw the graph of the cosine function. Explain your work. How did you arrive at that graph? Please explain how this cosine graph makes sense using the unit circle.

Problem Posed to Hypothetical Student: Construct the graph of the cosine function using the unit circle.

Student Work:

Mike had drawn:

Alice had drawn:
APPENDIX G

TASK FIVE PROBLEM AND STUDENT WORK

Problem Posed to Participant and Hypothetical Student: Evaluate the following:
\[ \cos^{-1}(\cos \left(\frac{-11\pi}{4}\right)) \].

Student Work:

Alexis wrote that
\[ \cos^{-1}(\cos \left(\frac{-11\pi}{4}\right)) = \frac{1}{\cos \left(\frac{-11\pi}{4}\right)} = \frac{1}{\cos^2 \left(\frac{-11\pi}{4}\right)} \].

Alexis paused and said \(-\frac{11\pi}{4}\) is an odd multiple of \(\frac{\pi}{4}\) so it’s cosine value is either positive or negative \(\sqrt{2}/2\). She then wrote, \(\frac{1}{\sqrt{2}} = 2\).

Joe said, well they are inverses so \(\cos^{-1}(\cos (x))=x\). So \(\cos^{-1}(\cos \left(\frac{-11\pi}{4}\right)) = \frac{-11\pi}{4}\).

Sam said, well that means we want the angle whose cosine value is the cosine of \(\frac{-11\pi}{4}\), but that’s an infinite number of angles. So, it’s like, all the angles that are located in the same spot as \(\frac{-11\pi}{4}\).

Amy first wrote, \(\frac{-11\pi}{4} \cdot \frac{180}{\pi} = -495\).

She then wrote, \(\cos^{-1}(\cos(-495^\circ)) = \cos^{-1}(\cos(-135^\circ))\). She said, “I’m not sure what the exact value of \(\cos(-135^\circ)\) is but I know it’s negative, so the answer has to be an angle in the second quadrant.”
APPENDIX H

INTERVIEW ONE PROTOCOL

1. Find the length of the arc on a circle with radius 3 in the x-y plane associated with the angle \( \theta = -\pi/12 \)

2. Imagine you have assigned this problem to your students, and as they work you hear and see the following:

   **Allen** says, “Well the negative doesn’t really matter if we’re talking about length. So I’m just going to treat it as \( \pi/12 \).”

   He writes:

   ![Diagram showing arc calculation]

   Circumference: \( 2\pi r = 6\pi \)

   \[ 12x = 6\pi \]

   \[ x = \frac{\pi}{2} \]

   **Lexi** writes:

   \[ S = \theta r \]

   And then says, “I’m not sure if theta should be in degrees or radians. It must be radians because otherwise we’d have a degree symbol in our answer. I don’t think there is a real symbol for radians.”

   She writes:

   \[ S = \frac{-\pi}{12} \cdot 3 = -\frac{\pi}{4} \]

   And then says, “Hmm, it seems weird that I have a negative number with pi in it as a length.”
By the time you check on Annie, she is finished and has written:

Charlie has on his paper:

a) Please take a look at the students’ work and tell me what you notice.

b) Please describe in detail what you think each student did in response to the problem.

c) Please explain what you learned about these students’ understandings.

d) What can you say about each student’s understanding of radian in particular?

(If at any point the interviewee says they do not have enough information to assess understanding, ask them, what they would do to further assess this understanding)

e) Pretend that you are the teacher of these students. Describe some ways you might respond to them. Explain why you chose these responses.
3. (a) What can you tell me about \( \sin 20^\circ \)? What about \( \cos 20^\circ \)? Can you give me an approximation? Explain.
   (b) Suppose you are given \( \sin(70^\circ) \approx 0.9397 \). Would this change your approximation? Explain.

4. Three high school students’ were asked the following: Given \( \sin(70^\circ) \approx 0.9397 \), what can you say, if anything, about \( \sin 20^\circ \) and \( \cos 20^\circ \)? They gave the following responses:

   **Alex’s work:**
   
   Since 20° and 70° are complimentary, we know that \( \sin 20^\circ + \sin 70^\circ = \sin 90^\circ \) and so \( \sin 20^\circ \approx 1 - 0.9397 \).
   
   We also know that \( \sin^2 \theta + \cos^2 \theta = 1 \), so \( \sin^2(20^\circ) + \cos^2(20^\circ) = 1 \).
   
   So \( \cos 20^\circ = \pm \sqrt{1 - \sin^2(20^\circ)} \approx \pm \sqrt{1 - (1 - 0.9397)^2} \).
   
   And 20° is in the first quadrant, so \( \cos 20^\circ \approx \sqrt{1 - (1 - 0.9397)^2} \).

   **Martha’s work:**
   
   Well 90° = 20° + 70°, so \( \cos 20^\circ \approx 0.9397 \).
   
   Also, \( 0 < \sin 20^\circ < 1 \) because the sine function gives out values between -1 and 1 and 20° is in the first quadrant.

   **Cheyenne’s work:**
   
   \( \sin(0) < \sin(20^\circ) < \sin(30^\circ) \) so \( \sin(20^\circ) \) is between 0 and \( \frac{\sqrt{3}}{2} \).
   
   \( \cos(30^\circ) < \cos(20^\circ) < \cos(0^\circ) \) So \( \cos(20^\circ) \) is between \( \frac{1}{2} \) and 1.

   a) Please take a look at the students’ work and tell me what you notice.

   b) Please describe in detail what you think each student did in response to the problem.

   c) Please explain what you learned about these students’ understandings.

   d) What can you say about each student’s understanding of the sine and cosine functions in particular?

   e) What can you say about each student’s understanding of the relationship between the sine and cosine functions?

   **(If at any point the interviewee says they do not have enough information to assess understanding, ask them, what they would do to further assess this understanding)**

   f) Pretend that you are the teacher of these students. Describe some ways you might respond to them. Explain why you chose these responses.
1. Given the following triangle find $\sin(\theta)$.

2. Imagine you have assigned this problem to your students, and as they work you see the following:

**Joe** has written:

**Anna** has written:

\[ a) \text{ Please take a look at the students' work and tell me what you notice.} \]

\[ b) \text{ Please describe in detail what you think each student did in response to the problem.} \]

\[ c) \text{ Please explain what you learned about these students' understandings.} \]
d) What can you say about each student’s understanding of sine and cosine in particular?

(If at any point the interviewee says they do not have enough information to assess understanding, ask them, what they would do to further assess this understanding)

e) Pretend that you are the teacher of these students. Describe some ways you might respond to them. Explain why you chose these responses.

3. (a) Draw the graph of the cosine function. Explain your work. How did you arrive at that graph?
(b) Please explain how this cosine graph makes sense using the unit circle.

4. Imagine you have asked your students to construct the graph of the cosine function using the unit circle, and as they work you see the following:

Mike had drawn:

Alice had drawn:
a) Please take a look at the students’ work and tell me what you notice.

b) Please describe in detail what you think each student did in response to the problem.

c) Please explain what you learned about these students’ understandings.

d) What can you say about each student’s understanding of the cosine function in particular?

e) What can you say about each student’s understanding of the unit circle?

(If at any point the interviewee says they do not have enough information to assess understanding, ask them, what they would do to further assess this understanding)

f) Pretend that you are the teacher of these students. Describe some ways you might respond to them. Explain why you chose these responses.

5. Evaluate the following: \( \cos^{-1}(\cos\left(\frac{-11\pi}{4}\right)) \). Please explain your work.

6. Imagine you have asked your students to do the same, and you see/hear:

Alexis wrote that \( \cos^{-1}\left(\cos\left(\frac{-11\pi}{4}\right)\right) = \frac{1}{\cos\left(\cos\left(-\frac{11\pi}{4}\right)\right)} = \frac{1}{\cos^2\left(-\frac{11\pi}{4}\right)} \). Alexis paused and said \( -\frac{11\pi}{4} \) is an odd multiple of \( \frac{\pi}{4} \) so it’s cosine value is either positive or negative \( \sqrt{2}/2 \). She then wrote, \( \frac{1}{2} = 2 \).

Joe said, well they are inverses so \( \cos^{-1}(\cos(x))=x \). So \( \cos^{-1}(\cos\left(-\frac{11\pi}{4}\right)) = -\frac{11\pi}{4} \).

Sam said, well that means we want the angle whose cosine value is the cosine of \( -\frac{11\pi}{4} \), but that’s an infinite number of angles. So, it’s like, all the angles that are located in the same spot as \( -\frac{11\pi}{4} \).

Amy first wrote, \( -\frac{11\pi}{4} \cdot \frac{180}{\pi} = -495 \). She then wrote, \( \cos^{-1}(\cos(-495^\circ)) = \cos^{-1}(\cos(-135^\circ)) \). She said, “I’m not sure what the exact value of \( \cos(-135^\circ) \) is but I know it’s negative, so the answer has to be an angle in the second quadrant.”

a) Please take a look at the students’ work and tell me what you notice.
b) Please describe in detail what you think each student did in response to the problem.

c) Please explain what you learned about these students’ understandings.

d) What can you say about each student’s understanding of the cosine function in particular?

e) What can you say about each student’s understanding of inverse cosine?

f) What can you say about each student’s understanding of what it means to be invertible?

(If at any point the interviewee says they do not have enough information to assess understanding, ask them, what they would do to further assess this understanding)

g) Pretend that you are the teacher of these students. Describe some ways you might respond to them. Explain why you chose these responses.
APPENDIX J

REFLECTION QUESTIONS

Please respond to the following questions with as much detail and specificity as possible. I have provided you with the tasks and mathematical work to refresh your memory of your experience during the interviews.

5. What mathematical thinking and problem solving did you feel you engaged in while analyzing the students' work during the interviews, if any? Please explain and provide specific examples.

6. How has your experience analyzing student work during these interviews influenced your knowledge of and/or thinking about trigonometry? If so, which parts of the interviews and how? Please provide specific examples.

7. What were the greatest challenges for you when analyzing the students' work and thinking? How, in your opinion, could teacher preparation courses and professional development help you overcome these challenges?

8. Any other comments or thoughts you would like to add about the experience or interview tasks in particular?
### Item 1

<table>
<thead>
<tr>
<th>Score</th>
<th>General Criteria</th>
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<tbody>
<tr>
<td>0</td>
<td>No response OR nothing correct.</td>
</tr>
<tr>
<td>1</td>
<td>Consistent incorrect conversion (for example 180° equivalent to 2π radians or 360° equivalent to π radians) OR some evidence of correct work or thinking.</td>
</tr>
<tr>
<td>2</td>
<td>All correct (or minor calculation errors) except 19 is treated as degrees instead of radians.</td>
</tr>
<tr>
<td>3</td>
<td>All correct conversions (including 19 treated as radians) OR minor errors that are clearly calculation mistakes.</td>
</tr>
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### Item 2a

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<tbody>
<tr>
<td>0</td>
<td>Did not respond OR nothing correct, OR all incorrect except for identifying radian as “angle measure” (for example, identifies radian as an angle measure in terms of pi).</td>
</tr>
<tr>
<td>1</td>
<td>Defined radian measure only in terms of degree measure, but did so correctly; evidence of some correct work or thinking (other than identifying radian as an “angle measure”).</td>
</tr>
<tr>
<td>2</td>
<td>Mentioned a connection to “radius length” or “arc length” and “radian” but did not provide a complete/correct definition (for example, “one radian is the arc length of that angle”).</td>
</tr>
<tr>
<td>3</td>
<td>Identified radian measure of an angle as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle. (The wording does not need to be precise. For example, 3 points would be rewarded for “arc length on the unit circle.”)</td>
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### Item 3

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<tr>
<td>0</td>
<td>No response OR nothing correct.</td>
</tr>
<tr>
<td>1</td>
<td>Evidence of some correct work or thinking. (Example: recalls circumference formula incorrectly, but rest of the work is correct).</td>
</tr>
<tr>
<td>2</td>
<td>Minor errors (non-calculation); mostly correct ideas. (Example: major arc lengths given instead of minor arc lengths)</td>
</tr>
<tr>
<td>3</td>
<td>Provided correct responses (2(35π/180) inches ≈ 1.22 inches; 2.4(35π/180) inches ≈ 1.47 inches; and 2.9(35π/180) inches ≈ 1.77) OR made errors that were clearly calculation mistakes.</td>
</tr>
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### Item 4a

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<tbody>
<tr>
<td>0</td>
<td>Did not respond OR did not define unit circle as a circle radius one</td>
</tr>
</tbody>
</table>
(example: described the unit circle as a reference circle or provide a picture of a circle with sine and cosine values labeled but not necessarily a radius of one)

<table>
<thead>
<tr>
<th>Score</th>
<th>General Criteria</th>
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<tbody>
<tr>
<td>1</td>
<td>Identified unit circle as a circle of radius one.</td>
</tr>
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</table>

**Item 4b -**

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<th>Score</th>
<th>General Criteria</th>
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<tr>
<td>0</td>
<td>Did not respond OR did not answer the question OR provided an incomplete response. (Example: “to find the values of sine and cosine functions”; convert between radians and degrees)</td>
</tr>
<tr>
<td>1</td>
<td>Indicates usefulness of finding the values of sine and cosine functions AND/OR converting between radians and degrees.</td>
</tr>
<tr>
<td>2</td>
<td>Explained how the unit circle helps understand radian measure AND/OR explains how the unit circle helps them find sine and cosine values (of more than special angles)</td>
</tr>
<tr>
<td>3</td>
<td>Explained that the unit circle extends trig functions to all real numbers AND/OR explained that the unit circle can be used to explain a property of trigonometric functions.</td>
</tr>
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**Item 5ab**

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<thead>
<tr>
<th>Score</th>
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<tbody>
<tr>
<td>0</td>
<td>Did not respond, OR responses do not make sense OR did not answer the question.</td>
</tr>
<tr>
<td>1</td>
<td>Evidence of some correct work or thinking. Approximates as sin30 or cos30 and does not think about the sign for 340.</td>
</tr>
<tr>
<td>2</td>
<td>Used the facts that sin30°=1/2 and cos30°=√3/2 to correctly identify a range these values fall in OR mentioned the use of these special angles to approximate but could not recall their sine and cosine values OR did not do out the work, but discussed some of the ideas below (3), if given more time or a protractor, etc. (Give 2 even if they don’t write sine value as negative)</td>
</tr>
<tr>
<td>3</td>
<td>Constructed a right triangle containing an angle of 20° and then compared the opposite side with the hypotenuse to see its about 1/3 AND acknowledged sin340° must be approximately -1/3 (or somewhere close), OR constructed a unit circle and ray at 340°, AND noted that the y-value is about -1/3 (or somewhere close) OR used these methods, but was off because of poor drawings. Similar for cosine. (If they show correct work for one, sine or cosine, give three points)</td>
</tr>
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**Item 5c**

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<tbody>
<tr>
<td>0</td>
<td>Did not respond, OR provided no correct work OR did not answer the question OR responded that they could not provide a closer approximation given this information.</td>
</tr>
</tbody>
</table>
| 1     | Evidence of some correct work or thinking. (Example: acknowledging
complimentary angles; example: correctly noting $\cos 20^\circ = \cos 340^\circ$
OR Used an incorrect statement, $\sin 70^\circ + \sin 20^\circ = \sin 90^\circ$, to solve for $\sin 20^\circ$, but all the following work is correct.

<table>
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<tbody>
<tr>
<td>2</td>
<td>Identified $\cos 340^\circ$ correctly, but could not find $\sin 340^\circ$ OR included more than half correct work OR explained specifically how one could solve using sum-difference identities but could not recall them specifically.</td>
</tr>
<tr>
<td>3</td>
<td>Correctly identified the values of $\sin 340^\circ$ AND $\cos 340^\circ$ in terms of $\sin 70^\circ$ OR made errors that are clearly calculation mistakes.</td>
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**Item 6a**

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<tbody>
<tr>
<td>0</td>
<td>Did not respond OR no evidence of correct work or thinking.</td>
</tr>
<tr>
<td>1</td>
<td>Identified one of three correctly OR some evidence of correct thinking in at least one of his or her responses.</td>
</tr>
<tr>
<td>2</td>
<td>Identified two of three correctly.</td>
</tr>
<tr>
<td>3</td>
<td>Identified all three correctly.</td>
</tr>
</tbody>
</table>

**Item 6b**

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<th>Score</th>
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<tbody>
<tr>
<td>0</td>
<td>Did not respond; nothing new or explanatory.</td>
</tr>
<tr>
<td>1</td>
<td>Successfully used the unit circle to explain one OR explained three but none completely OR explained two but only one completely. (example of incomplete reasoning: the period is $2\pi$, and you can see the whole circle is $2\pi$ radians).</td>
</tr>
<tr>
<td>2</td>
<td>Successfully used the unit circle to explain two of three correctly, OR explained all three but only explained one or two completely (example of incomplete reasoning: the period is $2\pi$, and you can see the whole circle is $2\pi$ radians).</td>
</tr>
<tr>
<td>3</td>
<td>Successfully used the unit circle to explain all three completely.</td>
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</table>

**Radian**

<table>
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<tbody>
<tr>
<td>0</td>
<td>There exists confusion about the relationship between radian and degrees.</td>
</tr>
<tr>
<td>Weak</td>
<td>Consistent evidence that he or she understands the relationship between radians and degrees</td>
</tr>
<tr>
<td>Some</td>
<td>Some evidence exists that he or she might understand radian as the measure of an angle as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle</td>
</tr>
<tr>
<td>Strong</td>
<td>Consistent evidence that he or she understands radian as the measure of an angle as the ratio of two lengths: the length of the arc subtended by the angle and the radius of the circle, OR as the length of the arc on the unit circle subtended by the angle</td>
</tr>
</tbody>
</table>
APPENDIX L

INSTITUTIONAL REVIEW BOARD APPROVAL LETTER

University of New Hampshire

Research Integrity Services, Service Building
51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

30-Jan-2014

Chaar, May
Mathematics, Kingsbury Hall
CSS Box 13642
530 Titus Avenue
Manchester, NH 03103

IRB #: 5903
Study: Secondary Teachers' Professional Noticing of Written Mathematical Strategies
Approval Date: 28-Jan-2014

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Expedited as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 110.

Approval is granted to conduct your study as described in your protocol for one year from the approval date above. At the end of the approval period, you will be asked to submit a report with regard to the involvement of human subjects in this study. If your study is still active, you may request an extension of IRB approval.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, Responsibilities of Directors of Research Studies Involving Human Subjects. (This document is also available at http://unh.edu/research/irb-application-resources.) Please read this document carefully before commencing your work involving human subjects.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Director

cc: File
    McCrone, Sharon
LIST OF REFERENCES


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