Central sequences and C*-algebras

Hemant Pendharkar
University of New Hampshire, Durham

Follow this and additional works at: https://scholars.unh.edu/dissertation

Recommended Citation
https://scholars.unh.edu/dissertation/2098

This Dissertation is brought to you for free and open access by the Student Scholarship at University of New Hampshire Scholars' Repository. It has been accepted for inclusion in Doctoral Dissertations by an authorized administrator of University of New Hampshire Scholars' Repository. For more information, please contact nicole.hentz@unh.edu.
Central sequences and C*-algebras

Abstract
We study central sequences of C*-algebras. We find connections of the central sequences of a C*-algebra and its representations. More specifically, we prove the following results: (a) Characterization of central sequences in certain C*-subalgebras of C(X, Mn), where X is a compact Hausdorff space. We also state the conditions under which central sequences are trivial/hypercentral. (b) A representation of the C*-algebra is in the point norm closure of the set of all equivalence classes of irreducible representation if and only if it is multiplicity free. (c) For a C*-algebra A, all of whose representations are bounded by some fixed number, the following are equivalent: (1) A is a continuous trace C*-algebra. (2) Every central sequence in A is trivial. (3) Irr(A, Mn) is point norm closed in Rep(A, Mn). (4) A can be written as a finite direct sum of C*-algebras of the form C(X, Mn, ~, beta) where ~ is an equivalence relation on X and beta:~→ Un. Un is the set of n x n unitary matrices.

Keywords
Mathematics

This dissertation is available at University of New Hampshire Scholars' Repository: https://scholars.unh.edu/dissertation/2098
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
Bell & Howell Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
This dissertation has been examined and approved.

Don Hadwin
Dissertation Director, Don Hadwin
Professor of Mathematics

Eric Nordgren, Professor of Mathematics

Edward Hinson, Associate Professor of Mathematics

Debajyoti Sinha, Associate Professor of Mathematics

Liming Ge, Associate Professor of Mathematics

July 23, 1999
Date
Dedication

To

my father-in-law V. Ananthnarayan

Richard and Torene

and

my Jyoti
Acknowledgements

I would like to express my deepest gratitude to my dissertation advisor Professor Don Hadwin. His constant encouragement, continued support, and sustained patience guided me through my toughest moments. Not only did he bring me closer to the subject of Mathematics but he also made my research a fun experience. His friendship is something I will cherish lifelong.

This dissertation would not have become a reality without the help and support of a great many people. First and foremost I would like to thank the other members of my committee, Professors Eric Nordgren, Liming Ge, Edward Hinson, and Debajyoti Sinha.

Many a time I have found myself barging into Professor Rita Hibschweiler’s office seeking her advice. I would like to thank her for her time and patience.

I would like to thank Jiankui Li for being a great friend and teacher.

Last but not the least, I would like to thank Jan Jankowski for simplifying my life at the department.
Foreword

This paper is designed to be the author's Ph.D. dissertation. It is organized as follows:

Chapters 1, and 2 contain a short presentation of the specific area of C*-algebras in which the subject of this paper is placed. Chapter 3 contains a description of central sequences in certain C*-algebras. Chapter 4 discusses the relationship between central sequences and multiplicity free representations. This chapter also provides a description of certain C*-subalgebras of Continuous trace C*-algebras.
TABLE OF CONTENTS

Dedication .................................................................................................................... iii
Acknowledgements ..................................................................................................... iv
Foreword ...................................................................................................................... v
Abstract ......................................................................................................................... vii

CHAPTER 1 Preliminaries .......................................................................................... 1
  1.1 Introduction ................................................................................................ 2
  1.2 Definitions and Examples ......................................................................... 4

CHAPTER 2 Representations ................................................................................... 10
  2.1 Spectrum ..................................................................................................... 10
  2.2 Topologies .................................................................................................. 12

CHAPTER 3 Some Results on Central Sequences ............................................ 14

CHAPTER 4 Some Applications .............................................................................. 23
  4.1 Hypercentral Sequences .............................................................................. 23
  4.2 Trivial Central Sequences ............................................................................ 26

Bibliography ................................................................................................................... 36
Abstract

Central Sequences and C*-algebras

By

Hemant Pendidhkar
University of New Hampshire, September, 1999

We study central sequences of C*-algebras. We find connections of the central sequences of a C*-algebra and its representations. More specifically, we prove the following results:

• Characterization of central sequences in certain C*-subalgebras of $C(X, M_n)$, where $X$ is a compact Hausdorff space. We also state the conditions under which central sequences are trivial/hypercentral.

• A representation of the C*-algebra is in the point norm closure of the set of all equivalence classes of irreducible representation if and only if it is multiplicity free.

• For a C*-algebra $A$, all of whose representations are bounded by some fixed number, the following are equivalent:

1. $A$ is a continuous trace C*-algebra.
2. Every central sequence in $A$ is trivial.
3. $\text{Irr}(A, M_n)$ is point norm closed in $\text{Rep}(A, M_n)$.
4. $A$ can be written as a finite direct sum of C*-algebras of the form $C(X, M_n, \sim, \beta)$ where $\sim$ is an equivalence relation on $X$ and $\beta : \sim \rightarrow U_n \cdot U_n$ is the set of $n \times n$ unitary matrices.
Chapter 1

Preliminaries

1.1 Introduction

Classification of C*-algebras and finding C*-algebraic invariants have always interested operator algebraists. In the past, some connections between central sequences and derivations, and automorphism groups of C*-algebras have been established. It turns out that the notion of central sequences is a useful tool for classifying certain C*-algebras.

S. Sakai [13, 14] showed that every derivation of a simple unital C*-algebra is inner. G. Elliot [4] has studied C*-algebras all of whose derivations are inner. An important step towards obtaining a classification theorem for C*-algebras using central sequences was taken by C. A. Akemann and G. K. Pedersen [1]. They established that a separable C*-algebra is a continuous trace C*-algebra if and only if every central sequence in it is trivial. They further showed that a derivation of a separable C*-algebra is implemented by a multiplier if and only if every summable central sequence is trivial. They went on to describe these C*-algebras as the direct sum of a continuous trace C*-algebra and a reduced sum of simple C*-algebras.

J. Phillips [11] established that a certain C* algebra, which is not a continuous trace C*-algebra, has an uncountable outer automorphism group. He also studied the relationship
between the automorphism group of a separable unital C*-algebra and central sequences.

Continuous trace C*-algebras were first described by Fell [3]. His definition is not easy to check; for that matter it is complicated to remember. Furthermore, it is also difficult to determine whether every central sequence is trivial for a given C*-algebra. Our primary aim is to understand separable unital C*-algebras whose central sequences are hypercentral. The goal is also to establish the relationship between central sequences and certain representations of the C*-algebra.

The collection of all equivalence classes of irreducible representation of a C*-algebra is called the spectrum. It is natural to ask, “when are two (irreducible) representations geometrically the same?” Multiplicity theory came along addressing this question. Answering this question thus got reduced to classifying multiplicity free representation in terms of a suitable set of invariants.

In this work, we establish a connection between central sequences and multiplicity free representations. The C*-algebras in question are unital, separable, and have a bound on the dimension of all representations. This class of C*-algebras is a subclass of “completely continuous representation” C*-algebras.

We describe central sequences in certain C*-algebras. We also determine when central sequences are trivial and when they are hypercentral. We study the spectrum, topologized by the point norm topology, of a separable unital C*-algebra, whose representations are bounded.
by some fixed number. Using central sequences, we establish that for such C*-algebras, a representation is in the point norm closure of the set of irreducible representations if and only if it is multiplicity free. We also show that the set of irreducible representations is point norm closed in the spectrum if and only if every central sequence is trivial. Consequently, for any continuous trace C*-algebra, whose representations are bounded by a fixed number, the set of irreducible representations is point norm closed in the spectrum. We give a description of continuous trace C*-algebras whose representations are bounded by a fixed number.
1.2 Definitions and Examples

Let $H$ be a complex Hilbert Space, and $B(H)$ be the algebra of all bounded linear operators on $H$.

**Definition 1.2.1** A C*-algebra is a norm closed self adjoint subalgebra of $B(H)$. Equivalently, it is a Banach algebra $A$ with an involution $*: A \to A$, satisfying

1. $(a^*)^* = a$
2. $(a + b)^* = a^* + b^*$
3. $(\lambda a)^* = \overline{\lambda} a^*$
4. $(ab)^* = b^* a^*$
5. $\|a^* a\| = \|a\|^2$

for every $a, b \in A$, and for every $\lambda \in \mathbb{C}$.

Throughout this paper, C*-algebras will be separable and unital with unit 1.

**Examples:**

1. If $H$ is a Hilbert space then $B(H)$ is a C*-algebra, where the * operation is the adjoint operation. If dimension of $H = n < \infty$, then $B(H)$ is the C*-algebra of all $n \times n$ matrices.
2 If is a compact Hausdorff space then \( C(X) \) the set of all continuous complex valued functions on \( X \), is a \( C^\ast \)-algebra. The * operation is given by complex conjugation. This is a commutative \( C^\ast \)-algebra.

3 If \( X \) is a compact Hausdorff space then \( C(X, M_n) \) is a \( C^\ast \)-algebra.

**Definition 1.2.2** A *representation* of a \( C^\ast \)-algebra \( A \) is a unital *-homomorphism \( \pi : A \to B(H) \) for some Hilbert space \( H \), i.e. \( \pi \) is an algebra homomorphism and

\[
\pi(a^*) = (\pi(a))^*.
\]

**Definition 1.2.3** Central Sequence.

A bounded sequence \( \{a_n\} \) in a \( C^\ast \)-algebra \( A \), is called a *Central Sequence*, provided

\[
\|a_n x - xa_n\| \to 0, \text{ as } n \to \infty, \text{ for each } x \in A.
\]

**Example:**

Suppose \( \{a_n\} \subset A \) is a sequence that converges to 0. Then clearly \( \{a_n\} \) is a central sequence.

Another example of a central sequence is a bounded sequence \( \{a_n\} \) in \( Z(A) \), where \( Z(A) \) is the center of \( A \). Observe that if \( \{a_n\} \) and \( \{b_n\} \) are central sequences then \( \{a_n + b_n\} \) is a central sequence. In particular, a sequence \( \{z_n + w_n\} \) which is the sum of a null-convergent sequence \( \{z_n\} \) and a sequence \( \{w_n\} \) of elements in the center of \( A \), is also a central sequence. For suppose, \( x \in A \). Then
Definition 1.2.4 Trivial central sequence.

A central sequence in a C*-algebra $A$ is called a trivial central sequence provided, it can be written as the sum of a null sequence and a sequence of elements in the center of $A$.

Definition 1.2.5 Hypercentral sequence

A central sequence $\{a_n\}$ in a C*-algebra $A$ is said to be hypercentral provided, for any central sequence $\{b_n\}$, $\|a_n b_n - b_n a_n\| \to 0$ as $n \to \infty$.

Remark: Clearly for a C*-algebra $A$, if all central sequences are trivial, then they are also hypercentral.

Proof: Suppose every central sequence in $A$ is trivial. Let $\{a_n\}$ be such a sequence. To show that $\{a_n\}$ is hypercentral. Let $\{b_n\}$ be any central sequence. Since $\{a_n\}$ is trivial, we can write $\{a_n\} = \{z_n + w_n\}$, the sum of a null sequence $\{z_n\}$ and a sequence $\{w_n\}$ of elements in the center of the C*-algebra. Now,

$$\|b_n (z_n + w_n) - (z_n + w_n) b_n\| \leq \| b_n z_n - z_n b_n \| + \| b_n w_n - w_n b_n \|.$$

The second summand on the right hand side is clearly zero for each $n$, and

$$\| b_n z_n - z_n b_n \| \leq \| b_n z_n \| + \| z_n b_n \|.$$

Now since $\{b_n\}$ is bounded and $\{z_n\}$ is null convergent, both the summands on the right hand side converge to zero as $n \to \infty$. This implies that $\{a_n\}$ is hypercentral.
Questions. Suppose $A$ is a $C^*$-algebra.

1. When is every central sequence in $A$ trivial?
2. When is every central sequence in $A$ hypercentral?

We will discuss the above questions for a certain class of $C^*$-algebras. We now introduce examples that provide the motivation for some of the ideas in this paper.

Example 1.2.6 Let $A = C([0,1], M_2)$, the $C^*$-algebra of all $2 \times 2$ complex matrix valued continuous functions on the closed interval $[0,1]$. The norm of an element $f \in A$ is given by $\| f \| = \sup_{t \in [0,1]} \| f(t) \|$.

Remark: Every central sequence in $A$ is trivial.

Proof: Suppose $\{ f_n \} = \{ \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \}$ is a central sequence in $A$. Then $\{f_n\}$ must asymptotically commute with every element of $A$. In particular, $\{f_n\}$ must asymptotically commute with the constant functions,

$$
p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } q(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

i.e. we have,

$$0 = \lim_{n \to \infty} \| f_n p - pf_n \|$$
\[
\lim_{n \to \infty} \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\| = \lim_{n \to \infty} \left\| \begin{pmatrix} 0 & -b_n \\ c_n & 0 \end{pmatrix} \right\|
\]

and

\[
0 = \lim_{n \to \infty} \| f_n q - q f_n \|
\]

\[
= \lim_{n \to \infty} \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\| = \lim_{n \to \infty} \left\| \begin{pmatrix} c_n & a_n - d_n \\ 0 & c_n \end{pmatrix} \right\|
\]

This implies \( b_n \to 0, c_n \to 0, \) and \( d_n - a_n \to 0, \) as \( n \to \infty. \)

Thus we can write,

\[
f_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 0 & b_n \\ c_n & d_n - a_n \end{pmatrix} + \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix},
\]

which is the sum of a null sequence and a sequence of elements of the center. Thus every central sequence in \( A \) is trivial.

We can get interesting examples by looking at certain elementary subalgebras of \( A. \)

**Example 1.2.7**  Let \( D \) be a unital \( \mathbb{C}^* \)-subalgebra consisting of diagonal elements in \( M_2, \) and let \( B = \{ f \in A : f(0) \in D \}, \) where \( A \) is the algebra in example 1.2.6.

For each \( n, \) define the mapping \( f_n : [0,1] \to [0,1] \) given by,

\[
f_n(t) = \begin{cases} 
0 & 0 \leq t \leq \frac{1}{4n} \\
4nt - 1 & \frac{1}{4n} \leq t \leq \frac{1}{2n}
\end{cases}
\]
Now for any element $A$ in $D$, if $\alpha_n(t) = f_n(t) A$ for $t \in [0,1]$, then the sequence \{\alpha_n\} is a central sequence. It is enough to check in the neighborhood of 0. And in the neighborhood of 0, functions take values that are close to the algebra $D$. Now by the definition of $\alpha_n$, the elements in the sequence take values close to $D$ in the neighborhood of 0 (since $A$ is an element of $D$). Since diagonal matrices commute, it follows that the sequence \{\alpha_n\} asymptotically commutes with every element of $A$. Also observe that the constant function $p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is in $B$. Taking $A = p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$
\| f_n \left( \frac{1}{2n} \right) p \left( \frac{1}{2n} \right) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \| = \| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \| > 0 \quad \text{as} \quad n \to \infty.
$$

By definition, a sequence \{\beta_n\} in $A$, is trivial if and only if it can be written as a sum of a null convergent sequence \{z_n\} and a sequence \{w_n\} of elements in the center of $A$, i.e., \{\beta_n\} is trivial if and only if $\beta_n - w_n = z_n$. In the above example, the elements in the center of $B$ are functions that take values in the center of $M_2$. These are $2 \times 2$ scalar matrix valued continuous functions on $[0, 1]$. It follows that \{\alpha_n\} is a non-trivial central sequence.

It turns out, (and follows from the theorem in this paper, which will characterize central sequences) that every central sequence in this algebra is hypercentral.

$$
\begin{align*}
f_n(t) &= -2nt + 2 & \frac{1}{2n} \leq t \leq \frac{1}{n} \\
f_n(t) &= 0 & \frac{1}{n} \leq t \leq 1.
\end{align*}
$$
Chapter 2

Representations

2.1 Spectrum

Definition 2.1.1 Let $\pi : A \rightarrow B(H)$, be a representation of $A$ on a Hilbert space $H$. Then $\pi(A)$ is a C*-subalgebra of $B(H)$. A representation $\pi$ is called nondegenerate, provided the norm closed linear span of $[\pi(A)H] = H$.

Definition 2.1.2. Let $\pi$ and $\sigma$ be two representations of $A$ on Hilbert spaces $H$ and $K$ respectively. Then, $\pi$ and $\sigma$ are said to be equivalent if there is a unitary operator $U : H \rightarrow K$ such that $\sigma(x) = U \pi(x) U^*$ for all $x \in A$. This equivalence is denoted by $\pi \sim \sigma$.

Definition 2.1.3. A non-zero representation $\pi$ is said to be irreducible provided, the C*-algebra $\pi(A)$ is irreducible, i.e. $\pi(A)$ has no non-trivial invariant subspace.

Definition 2.1.4 An invariant subspace $M$ for the C*-algebra $\pi(A)$ is said to be cyclic, provided there exists a vector $\xi$ in $H$ such that the closed linear span of the
vectors of the form $\pi(x)\xi$, $x \in A$, is all of $M$. A representation $\pi$ is called a cyclic representation of $A$ on a Hilbert space $H$ provided, $H$ is a cyclic subspace for $\pi$.

Note: A cyclic representation is nondegenerate.

**Some notations:**

- $\text{Rep}(A)$ will denote the class of all representations of the C*-algebra $A$.
- $\text{Irr}(A)$ will denote the class of all irreducible representations of $A$.
- $\text{Irr}(A, M_n)/\sim$ will denote the set of equivalence classes of irreducible representations into $M_n$ for some fixed $n$.

**Definition 2.1.5** A linear functional $f$ on a C*-algebra $A$ is said to be positive if $f(z^*z) \geq 0, \forall z \in A$. If $f(1) = 1$, then $f$ is called a state.

**Theorem 2.1.6** (Gelfand-Naimark-Segal)

Every C*-algebra $A$ is isometrically *-isomorphic to a C*-algebra of operators on some Hilbert space $H$.

When we write an irreducible representation $\pi$, we mean the unitary equivalence class $[\pi]$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
2.2 Topologies

**Definition 2.2.1** Let $\mathcal{A}$ be a C*-algebra. Then the set of all equivalence classes of irreducible representations of $\mathcal{A}$ (denoted by $\hat{\mathcal{A}}$) is called the *spectrum* of $\mathcal{A}$.

**Definition 2.2.2** An ideal $\mathcal{J}$ in $\mathcal{A}$ is called a *primitive* ideal provided there exists an irreducible representation $\pi$ of $\mathcal{A}$ such that $\mathcal{J} = \text{kernel} \pi$.

**Definition 2.2.3** Let $\text{prim}(\mathcal{A})$ denote the set of all primitive ideals in $\mathcal{A}$. Let $S$ be a subset of the set of $\text{prim}(\mathcal{A})$. We define the closure of $S$ to be the set

$$\{ m : m \supseteq \bigcap_{j \in S} j \}.$$  

This defines a topology on the set of all primitive ideals of $\mathcal{A}$ called the hull-kernel topology.

Consider the mapping,

$$\alpha : \hat{\mathcal{A}} \to \{ \ker \pi : \pi \in \hat{\mathcal{A}} \} \text{ given by } \alpha(\pi) = \ker \pi.$$  

**Remark 2.2.3.1** The range of $\pi$ is given the relative hull-kernel topology, and $\hat{\mathcal{A}}$ is topologized by pulling back this topology using the map $\alpha$. Thus a subset of $\hat{\mathcal{A}}$ is open if and only if it has the form $\alpha^{-1}(S)$ for some open subset $S$ of $\text{prim}(\mathcal{A})$.  

12

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Definition 2.2.4  Point-norm topology on $A$

A sequence of representations $\{\pi_n\}$ of $A$, converges to a representation $\pi$ in the point-norm topology provided $\pi_n(a) \to \pi(a)$, for every $a \in A$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Chapter 3

Some Results on Central Sequences

In this chapter we characterize central sequences in certain C*-algebras. These results will be used later to establish a relationship between central sequences and multiplicity free representations.

Definition 3.1 Let $A$ be a C*-algebra. Let $S$ be a subset of $A$. We define the commutant of $S$ in $A$, denoted by $S'$, to be the set

$$S' = \{ a \in A : as = sa, \forall s \in S \}.$$

In particular the commutant of $A$ in $A$ is the center of $A$.

Theorem 3.2 Let $A$ be a unital C*-subalgebra of $M_k$. Let $X$ be a compact Hausdorff space, let $x_0$ be a limit point of $X$, and let $B$ be the C*-subalgebra of $C(X,M_k)$ where given by,

$$B = \{ f \in C(X,M_k) | f(x_0) \in A \}.$$

Let $\{ f_n \}$ be a bounded sequence in $B$. Then $\{ f_n \}$ is a central sequence in $B$ if and only if

1. for some sequence of scalar-valued functions $\lambda_n$, $\| f_n - \lambda_n I_k \| \to 0$ uniformly on compact subsets of $X \setminus \{ x_0 \}$, and
2. \[ \text{distance} \left( f_n, A' \right) \to 0 \text{ uniformly on } X. \]

**Proof:** Let \( \{ f_n \} \) be a central sequence. Then,

\[ \sup_{x \in X} \| f_n(t)g(t) - g(t)f_n(t) \| \to 0 \text{ for every } g \in B. \]

Suppose distance \( (f_n, A') \) does not go to zero. This implies that \( \sup_{x \in X} \inf_{a \in A'} \| f_n(t) - a \| \) does not go to zero. Thus there exists an \( \varepsilon > 0 \) and a subsequence \( \{ n_k \} \) such that, for every \( k \),

\[ \sup_{x \in X} \inf_{a \in A'} \| f_{n_k}(t) - a \| > \varepsilon, \]

i.e. \( \exists \{ t_{k} \} \subset X, k = 1, 2, 3, \ldots \) such that, \( \inf_{a \in A'} \| f_{n_k}(t_k) - a \| > \varepsilon. \)

Therefore \( \exists \varepsilon' > \varepsilon \) such that, \( \forall a \in A', \| f_{n_k}(t_k) - a \| \geq \varepsilon'. \) Using the compactness argument, we can find a convergent subsequence \( \{ f_{n_k}(t_k) \} \subset M_k \) that converges to some \( w \in M_k. \) Then distance \( (w, A') = \lim_{k \to \infty} \text{distance} \left( f_{n_k}(t_k), A' \right) \geq \varepsilon'. \)

This implies that \( w \not\in A'. \) Therefore we can find an \( a \in A \) such that, \( wa \neq aw, \)

i.e. \( a \lim_{k \to \infty} f_{n_k}(t_k) \neq \lim_{k \to \infty} f_{n_k}(t_k)a. \) This implies that \( \| f_{n_k}(t_k) - a - af_{n_k}(t_k) \| \) does not go to zero. Define \( g \in B \) such that, \( g(t) = a, t \in X. \) Then clearly,

\[ \sup_{x \in X} \| f_{n_k}(t)g(t) - g(t)f_{n_k}(t) \| \geq \| f_{n_k}(t_k)g(t_k) - g(t_k)f_{n_k}(t_k) \| \]

which is bounded away from zero. This is a contradiction since \( \{ f_n \} \) is a central sequence. Hence condition (2) of the theorem is true.

It remains to show that condition (1) is true. We want to show that distance \( (f_n(t), M_k' ) \to 0 \) uniformly on compact subsets of \( X \setminus \{ x_0 \}. \) Suppose not. This implies that there exists a neighborhood \( U_{x_0} \) of \( x_0 \) such that on \( X \setminus U_{x_0}, \) distance \( (f_n(t), C \cdot I_k) \) does not go to zero uniformly. i.e. \( \sup_{x \in X \cup U_{x_0}} \inf_{a \in M_k(CY)} \| f_n(t) - a \| \) does not go to zero.
Therefore there exists an \( \varepsilon > 0 \), and a subsequence \( \{n_j\} \) such that for each \( k \),

\[
\sup_{t \in X \cup U} \inf_{a \in M_k} \| f_{n_j}(t_k) - a \| > \varepsilon,
\]

i.e. distance \( (f_{n_j}(t_j), I_k) > \varepsilon \).

Again using the compactness argument, we can find a convergent subsequence which we will still call \( \{f_{n_j}(t)\} \), that converges to some \( w \in M_k \). Then,

distance \( (w, C \cdot I_k) = \lim_{j \to \infty} \) distance \( (f_{n_j}(t_j), I_k) > \varepsilon \). Thus \( w \in M_k \). We can find an \( a \in M_k \) such that \( aw \neq wa \). This implies that \( \| f_{n_j}(t_j)a - af_{n_j}(t_j) \| \) does not go to zero.

Define using Urysohn's lemma, a continuous function \( h : X \to [0,1] \) such that,

\[
h(t) = 1, \quad t \in X \setminus U_{x_0} \text{ and } h(x_0) = 0, \quad \text{where } U_{x_0} \text{ is a neighborhood of } x_0.
\]

And now define,

\[
g : X \to M_k \text{ given by } g(t) = h(t)a.\text{ Then clearly } g \in B, \text{ and}
\]

\[
\sup_{t \in X} \| f_{n_j}(t)g(t) - g(t)f_{n_j}(t) \| \geq \| f_{n_j}(t_j)g(t_j) - g(t_j)f_{n_j}(t_j) \| \geq \varepsilon.
\]

This is a contradiction, since \( \{f_n\} \) is a central sequence.

Conversely, let us suppose that conditions (1) and (2) hold. We will show that \( \{f_n\} \) is a central sequence. Let \( g \in B \), and let \( \varepsilon > 0 \). Then using continuity we can find a neighborhood \( U \) of \( x_0 \) such that, whenever \( t \in U \), we have \( \| g(t) - g(x_0) \| < \varepsilon \). Also by condition (1), for \( t \in X \setminus U \), we have \( \| f_n(t)g(t) - g(t)f_n(t) \| \to 0 \) uniformly. Now,

\[
\| f_n(t)g(t) - g(t)f_n(t) \| = \| f_n(t)(g(t) - g(x_0) + g(x_0)) - (g(t) - g(x_0) + g(x_0))f_n(t) \|
\]

Using boundedness of the sequence \( \{f_n\} \), we can find some positive number \( M \) such that \( \| f_n \| \leq M, \forall n \). By condition (2), we know that the sequence takes values close to the
commutant of $A$. It follows now from triangle inequality that $\{f_n\}$ is a central sequence.

**Theorem 3.3** Let $A$ be a unital $C^*$-subalgebra of $M_k$. Let $x_0$ be a limit point of $X$. Also let $B = \{f \in C(X, M_k) : f(x_0) \in A\}$. Then every central sequence in $B$ is hypercentral if and only if $A'$ is abelian.

**Proof:** Suppose $A'$ is abelian. Let $\{f_n\}$ and $\{g_n\}$ be two central sequences. Let $U$ be a neighborhood of $x_0$. We know that, $\{f_n\}$ and $\{g_n\}$ are close to scalars when restricted to $X \setminus U$ (since $X \setminus U$ is a compact subset in $X$). Hence they commute asymptotically when restricted to the compact subset $X \setminus U$. In the neighborhood $U$ of $x_0$, the functions take values close to the commutant of $A$, i.e.,

$$\|f_n g_n - g_n f_n\| \leq \sup_{t \in X \setminus U} \|f_n(t)g_n(t) - g_n(t)f_n(t)\| + \sup_{t \in U} \|f_n(t)g_n(t) - g_n(t)f_n(t)\|.$$

The two summands are uniformly small. This gives that the sequence is hypercentral.

Conversely, suppose that the commutant of $A$ is not abelian. Let $\{t_n\}$ be a sequence in $X$ such that $t_n \to x_0$ and let $F_{t_n}$ be the neighborhoods of $t_n$, $n = 1, 2, 3...$ that do not contain $x_0$. Choose $a, b$ in the commutant of $A$, such that $ab \neq ba$. Also, using Urysohn's lemma, define continuous functions

$$f_n : X \to [0,1]$$

such that $f_n \mid_{X \setminus F_{t_n}} = 0$ and $f_n(t_n) = 1$.

Then the sequences $\alpha_n(t) = f_n(t)a$ and $\beta_n(t) = f_n(t)b$ are central sequences that do not commute asymptotically. For,
\[ \| \alpha_n \beta_n(t_n) - \beta_n \alpha_n(t_n) \| = \| ab - ba \| \quad n = 1, 2, 3, \ldots \]

which is clearly not zero. Hence if commutant of \( A \) is abelian, then every central sequence is hypercentral.

**Theorem 3.4** Let \( A \) be a unital \( \mathbb{C}^* \)-subalgebra of \( M_k \). Let \( x_0 \) be a limit point of \( X \). Also let \( B = \{ f \in C(X, M_k) : f(x_0) \in A \} \). Then every central sequence in \( B \) is trivial if and only if \( A' \) consists of scalar multiples of the identity. (In other words; The algebra \( B \) has nontrivial central sequences precisely when \( A \) is a proper subalgebra of \( M_k \).

**Proof:** Suppose \( A \) is not all of \( M_k \). We can use the functions defined above in the second half of the proof of Theorem 3.4 to construct a non-trivial central sequence. Choose a non-scalar matrix \( a \in A' \) and define \( \alpha_n(t) = f_n(t)a \). Then \( \{ \alpha_n(t) \} \) is a central sequence. This sequence is not uniformly close to the scalars since,

\[ \| \alpha_n(t_n) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \| = \| a - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \| > 0, \quad n = 1, 2, 3, \ldots \] (as \( a \) is a non-scalar matrix).

Conversely, suppose every central sequence is trivial. We will show that \( A \) is all of \( M_k \). Using the Theorem 3.2, we know that if \( \{ f_n \} \) is a central sequence, then

1. \( \| f_k(t) - \lambda_k(t)I_n \| \to 0 \) uniformly on compact subsets of \( X \setminus \{ x_0 \} \) and
2. \( \text{distance}(f_n(t), A') \to 0 \) uniformly on \( X \).

But by the hypothesis, we get that distance \( (A', \text{Scalars}) \) goes to zero. This implies that \( A' \) is the scalar multiples of the identity.
Corollary 3.5 (To Theorem 3.2) Let $A$ be a unital $C^*$-subalgebra of $M_k$. Let $C$ be the $C^*$-subalgebra of $C((\{1-\frac{1}{j}\} \cup \{1\}), M_k)$, $j = 1, 2, 3, \ldots$ given by

$$C = \{ f : \{1 - \frac{1}{j}\} \cup \{1\} \to M_k \mid f(1) \in A \}.$$ 

Let $\{f_n\}$ be a bounded sequences in $C$. Then $\{f_n\}$ is a central sequence in $C$ if and only if there exists a sequence of scalar valued functions $\lambda_n$ such that

1. $\| f_n(t) - \lambda_n(t) I_k \| \to 0$ uniformly on compact subsets of $\{1 - \frac{1}{j}\}$, and

2. $\text{distance} (f_n(t), A') \to 0$ uniformly on $\{1 - \frac{1}{j}\} \cup \{1\}$.

Remark 3.5.1 When $X = \{1 - \frac{1}{j} : j = 1, 2, 3, \ldots\} \cup \{1\}$ and $x_0 = 1$, then

$X \setminus \{x_0\} = \{1 - \frac{1}{j} : j = 1, 2, 3, \ldots\}$, and compact subsets of $X \setminus \{x_0\}$ are just finite sets.

Uniform convergence on finite sets is the same as pointwise convergence. Therefore condition (1) above can be written as follow:

1. There exists scalar valued functions $\left\{\lambda_n(j)\right\}_{j=1}^\infty$ such that

$$\| f_n(1 - \frac{1}{j}) - \lambda_n(j) \| \to 0 \text{ as } n \to \infty \text{ for every } j.$$ 

Corollary 3.6 Every central sequence in $C$ is hypercentral if and only if $A'$ is abelian.
Corollary 3.7  Every central sequence in \( C \) is trivial if and only if \( A' \) consists of scalar multiples of the identity.

Definition 3.8  Let \( A \) be any C*-algebra. Let \( \pi_1, \pi_2, \pi_3, \ldots : A \to M_k \) be disjoint irreducible representations such that \( \pi_n(a) \to \pi(a) \forall a \in A \).

Then we define \( \rho(A) = \{ \pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \ldots : a \in A \} \).

Theorem 3.9  Let \( \rho(A) \) be as defined above in 3.8. Then \( \rho(A) \) is *-isomorphic to the C*-algebra \( C \) in Corollary 3.5.

Proof: Define the mapping \( \phi : \rho(A) = \{ \pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \ldots \} \to C = \{ f : \{1,\frac{1}{n}\} \cup \{1\} \to M_k \mid f(l) \in \pi(A) \} \), given by

\[
\phi(\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \ldots) = f_a \quad \text{where,} \quad f_a(1 - \frac{1}{n}) = \pi_n(a), \quad n = 1, 2, 3, 4, \ldots \quad \text{and} \quad f_a(1) = \pi(a).
\]

Then the map is clearly well defined and is a C*-homomorphism. Also the map is one to one. For suppose, \( f_a = f_b, \ a, b \in A \). Then

\[
f_a(1 - \frac{1}{n}) = f_b(1 - \frac{1}{n}), \quad \text{for each} \ n. \quad \text{This implies that} \quad \pi_n(a) = \pi_n(b), \quad \text{for each} \ n. \quad \text{Hence}
\]

\[
(\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \ldots) = (\pi(b) \oplus \pi_1(b) \oplus \pi_2(b) \oplus \pi_3(b) \oplus \ldots).
\]

Thus the C*-algebra \( \phi(\rho(A) \) is the C*-subalgebra

\[
\{ f_a : \{1,\frac{1}{n}\} \cup \{1\} \to M_k \mid f_a(l) \in \pi(A) : a \in A \} \quad \text{of} \quad C.
\]

We will now show that \( \rho(A) \supseteq C \).
Since $\pi_1$ is disjoint from $\pi$, there exists an $a \in A$ such that $\pi(a) = 0$ and $\pi_1(a) = 1$. Also since $\pi_n(a) \to \pi(a) = 0$ there exists an $N$ such that $n \geq N$ implies that $\|\pi_n(a)\| < \frac{1}{2}$. Let $g$ be a continuous function on the reals such that $g\big|_{\{0,\frac{1}{4}\}} = 0$ and $g\big|_{\{\frac{1}{4}, \infty\}} = 1, \varepsilon > 0$ (small).

Let $b = g(a * a)$. Then we get that $\pi_1(b) = 1$. And for $n \geq N$ we have $\|\pi_n(a * a)\| < \frac{1}{4}$.

Therefore

$$\|\pi_n(b)\| = \|\pi_n(g(a * a))\| = \|g(\pi_n(a * a))\| = 0.$$  

Using disjointness again for $2 \leq j \leq N$, we can find a $c \in A$ such that $\pi_1(c) = 1$ and $\pi_j(c) = 0$. Let $x = bc$. Then $\pi_1(x) = 1$ and $\pi_j(x) = 0, j = 2, 3, 4, \ldots$. Now using the $x$ above we get that

$$\rho(x)\rho(A) = \{0 \oplus \pi_1(a) \oplus 0 \oplus 0 \oplus \ldots | a \in A\}.$$

$$= \{0 \oplus T \oplus 0 \oplus \ldots | T \in M_k\}.$$

$$= B_1 \text{ (say).}$$

Thus $\rho(A) \supseteq B_1$. Continuing the same way for each $\pi_j, j = 2, 3, 4, \ldots$ we get that the set $B_j = \{0 \oplus 0 \oplus \ldots \oplus T \oplus 0 \oplus \ldots | T \in M_k\} \subseteq \rho(A)$. Now since $\rho(A)$ is a C*-algebra, we get that $\rho(A)$ contains the norm closed span of $D = \{0 \oplus T_1 \oplus T_2 \oplus \ldots | T_n \to 0\}$.

Now suppose $f : \{1 - \frac{1}{n}\} \cup \{1\} \to M_k$ such that $f(1) \in \pi(A)$. Write

$$f = (s, s_1, s_2, \ldots)$$

where $s_n \to s \in \pi(A)$. Choose a $w \in A$ such that $\pi(w) = s$. Then
clearly $f - \rho(w) \in D \subseteq \rho(A)$. This implies that $f \in D \subseteq \rho(w) + \rho(A) \subseteq \rho(A)$. Thus $\rho(A)$ is isomorphic to $C$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Chapter 4

Some Applications

4.1 Hypercentral Sequences

Definition 4.1.1 A representation \( \pi \) of a C*-algebra \( A \) is said to be multiplicity free provided \( \pi(A)' \) is abelian.

Definition 4.1.2 Suppose \( \pi \) is a representation of a C*-algebra \( A \) on a Hilbert space \( H \). Then the dimension of the representation \( \pi \) is the dimension of the Hilbert space \( \pi(A)H \).

Theorem 4.1.3 Let \( A \) be a unital separable C*-algebra. Let \( N \) be a fixed natural number such that, dimension \( \pi \leq N \) for every irreducible representation \( \pi \) of \( A \). Then the following are equivalent:

(1) Every central sequence in \( A \) is hypercentral.

(2) If a representation \( \pi \) is in the point norm closure of \( \text{Irr}(A,M_k), k \leq N \), in the set of all representations of \( A \), then \( \pi \) is multiplicity free.

Proof: Suppose every central sequence is hypercentral. Let \( \pi \) be a representation of
A, in the point norm closure of \( \text{Irr}(A, M_k) \) for some \( k \leq N \). If \( \pi \) belongs to \( \text{Irr}(A, M_k) \), then there is nothing to prove, so suppose it does not. Then we can let \( \{\pi_n\} \) be a sequence of pairwise disjoint irreducible representations in \( \text{Irr}(A, M_k) \) such that, \( \pi_n(a) \rightarrow \pi(a), \forall a \in A \). To show that \( \pi \) is multiplicity free, by Definition 4.1.1, we must show that \( \pi(A)' \) is abelian.

Consider the C*-algebra

\[
\rho(A) = \{\pi(a) \otimes \pi_1(a) \otimes \pi_2(a) \otimes ... : a \in A \}.
\]

Then by Theorem 3.8, \( \rho(A) \) is *-isomorphic to a C*-subalgebra of

\[
C = \{f : \{1 - \frac{1}{n}\} \cup \{l\} \rightarrow M_k \mid f(l) \in \pi(A)\}.
\]

But then, every central sequence in \( \rho(A) \) is hypercentral. Using Theorem 3.9 and Corollary 3.6, we get that \( \pi(A)' \) is abelian.

Conversely, let us suppose that not every central sequence is hypercentral.

We will show that \( \pi(A)' \) is not abelian for some \( \pi \). Let \( \{a_n\} \) be a central sequence that is not hypercentral. Then there exists a central sequence \( \{b_n\} \) and an \( \varepsilon > 0 \) be such that, after restricting to a subsequence if necessary, \( \|a_n b_n - b_n a_n\| > \varepsilon \).

Using GNS construction, for each \( n \) we can find an irreducible representation \( \pi_n \) such that

\[
\|\pi_n(a_n b_n - b_n a_n)\| = \|a_n b_n - b_n a_n\| > \varepsilon
\]

where \( \pi_n : A \rightarrow M_{k(n)}, 1 \leq k(n) \leq N \). \{k(n)\} is a sequence in \{1,2,3,...,N\}. Let \{k\} be the constant subsequence. Passing to the corresponding subsequence, we get that

\[
\pi_n : A \rightarrow M_k, n = 1,2,3,...
\]
By using the compactness argument we get that (after passing to a subsequence if required) \( \pi_n(a) \to \pi(a) \) for some \( \pi \) and for each \( a \in A \). As before, we can define the C*-algebra \( \rho(A) = \{ \pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \ldots : a \in A \} \). Also note, the central sequences \( \{ \rho(a_n) \}, \{ \rho(b_n) \} \) are not hypercentral. Now once again using the Theorem 3.9, we get that \( \pi(A)' \) is not abelian.
4.2 Trivial central sequences

In this section, we give the description of a certain subclass of continuous trace C*-algebras. Akemann and Pedersen [1] have characterized continuous trace C*-algebras using central sequences. Their result states that a separable C*-algebra is continuous trace if and only if every central sequence in it is trivial. The C*-algebra need not be unital.

Lemma 4.2.1 Suppose $A$ is a separable unital C*-algebra, and suppose $\pi$ is a finite dimensional irreducible representation of $A$. Let $E \subseteq \text{Irr}(A)$ such that $\pi \notin E^-$ (closure in the spectrum, see remark 2.2.3.1). Then there is an $a \in A$ such that $\pi(a) = 1$ and for each $\rho \in E$, $\rho(a) = 0$.

Proof. Since $\pi \notin E^-$, it follows from definition 2.2.3 that $\cap_{\rho \in E} \ker \rho \nsubseteq \ker \pi$.

Hence there exists a $b \in A$, with $\pi(b) \neq 0$ and for every $\rho \in E$, $\rho(b) = 0$. However via unitary equivalence, we can assume that $\pi : A \to M_n$. Since $\pi$ is irreducible, we must have $\pi(A) = M_n$. Since $0 \neq \pi(b) \in M_n$, $\exists s_1, t_1, \ldots, s_n, t_n \in M_n$ such that

$$\sum_{k=1}^n s_k \pi(b) t_k = 1.$$ 

Since $\pi$ is onto, there exists $c_1, d_1, \ldots, c_n, d_n \in A$ such that

$$\pi(c_k) = s_k, \pi(d_k) = t_k, 1 \leq k \leq n.$$ 

Let $a = \sum_{k=1}^n c_k b d_k$. Then $\pi(a) = 1$. And for each $\rho \in E$, $\rho(a) = 0$. We can get $0 \leq a \leq 1$ by replacing $a$ with $f(a^* a)$, where

26
\( f(t) = \min\{t, 1\}. \)

**Definition 4.2.2** Suppose \( X \) is a compact Hausdorff space, \( n \) is a natural number and \( \sim \) is an equivalence relation on \( X \) such that \( \sim \) is a closed subset of \( X \times X \). Also suppose \( \beta : \sim \to U_n \), where \( U_n \) is the set of unitaries in \( M_n \), is a map such that

\[
\forall (x, y), (y, z) \in \sim, \beta(x, y)\beta(y, z) \in \beta(x, z)C.
\]

Then we say that \((X, \sim, \beta, n)\) is an *equivalence system*. We define,

\[
C(X, \sim, \beta, M_n) = \{ f \in C(X, M_n) : f(x)\beta(x, y) = \beta(x, y)f(y) \}.
\]

We say \((X, \sim, \beta, n)\) is *regular*, if for every \( x \in X \), for every \( T \in M_n \), \( \exists f \in C(X, \sim, \beta, M_n) \) such that \( f(x) = T \).

Note: We do not yet know whether every equivalence system is regular.

**Lemma 4.2.3** Suppose \((X, \sim, \beta, n)\) is a regular equivalence system, and \( B \) a unital C*-subalgebra of \( C(X, \sim, \beta, M_n) \) such that,

1. for every \( x \in X \) and every \( T \in M_n \) there is a \( g \in B \) such that \( g(x) = T \), and
2. for every \( x, y \in X \), if \( x \neq y \), there exists a \( g \in B \) with \( g(x) = 0, g(y) = 1 \).

Then \( B = C(X, \sim, \beta, M_n) \).

**Proof:** Note that any pure state \( \varphi \) of \( C(X, \sim, \beta, M_n) \) can be extended to a pure state of \( C(X, M_n) \), and can be represented by an \( x \in X \) and a unit vector \( e \in C^* \) as

\[
\varphi(f) = (f(x)e, e), \forall f \in C(X, M_n).
\]
However, since \( f \mapsto f(x) \) is an irreducible representation on \( C(X, \sim, \beta, M_n) \) (because of regularity), it follows that every functional \( \varphi \) represented in terms of a pair \((x, \epsilon)\) as above, is a pure state on \( C(X, \sim, \beta, M_n) \). Hence the set of pure states on \( C(X, \sim, \beta, M_n) \), being a continuous image of the compact set \( X \times \{ \epsilon \in C^n : \| \epsilon \| = 1 \} \), is closed in the weak* topology. Suppose now that \( \varphi, \psi \) are pure states on \( C(X, \sim, \beta, M_n) \) with

\[ \varphi(f) = (f(x)u, u) \quad \text{and} \quad \psi(f) = (f(y)v, v) \]

for \( x, y \in X \) and \( u, v \) are unit vectors in \( C^n \).

Suppose \( \varphi \neq \psi \) on \( C(X, \sim, \beta, M_n) \). Then there exists an \( f \in C(X, \sim, \beta, M_n) \) such that \( \varphi(f) \neq \psi(f) \). We want to find a \( g \in B \) such that \( \varphi(g) \neq \psi(g) \). If \( x \) is not equivalent to \( y \), then by (2), there exists a \( g \in B \) with \( g(x) = 0 \) and \( g(y) = 1 \). Hence \( \varphi(g) = 0 \) and \( \psi(g) = 1 \). So \( \varphi(g) \neq \psi(g) \). On the other hand, if \( x \sim y \), then by (1), we can choose a \( g \in B \) with \( g(x) = f(x) \). Since \( g \in C(X, \sim, \beta, M_n) \) and \( x \sim y \), it follows that \( g(y) = f(y) \). Hence \( \varphi(g) = \varphi(f) \neq \psi(f) = \psi(g) \). Hence \( B \) separates the closure of the pure states on \( C(X, \sim, \beta, M_n) \). And by the Stone-Weierstrass theorem of J. Glimm [6], \( B = C(X, \sim, \beta, M_n) \).

**Proposition 4.2.4** Suppose, \( A \) is a unital \( \text{C}^* \)-algebra and \( n \) is a natural number.

The following are equivalent:

1. Every irreducible representation of \( A \) is \( n \)-dimensional.

2. \( A \) is \( \ast \)-isomorphic to \( C(X, \sim, \beta, M_n) \) for some regular equivalence system \((X, \sim, \beta, n)\).

**Proof:** (2) implies (1).
It follows from the characterization of the pure states of $C(X,\sim,\beta,M_n)$ given in the proof of Lemma 4.2.3, combined with the uniqueness part of the GNS construction that every irreducible representation of $C(X,\sim,\beta,M_n)$ is unitarily equivalent to a representation given in terms of an $x \in X$ by $f \mapsto f(x)$. Hence every irreducible representation of $C(X,\sim,\beta,M_n)$ is $n$-dimensional.

(1) implies (2).

Suppose every irreducible representation of $A$ is $n$-dimensional. Suppose $\pi$ is an $n$-dimensional representation of $A$. Then $\pi$ is the direct sum of irreducible (thus $n$-dimensional) representations. Hence $\pi$ is irreducible. Thus

$$\text{Irr}(A,M_n) = \text{Rep}(A,M_n).$$

Let $X = \text{Irr}(A,M_n)$ with the point norm topology. Since

$$\text{Irr}(A,M_n) = \text{Rep}(A,M_n),$$

$X$ must be compact. Suppose $\pi, \rho \in X$. We say that $\pi \sim \rho \iff \exists$ a unitary $u, \exists \pi(.) = u^* \rho(.) u$. Thus $\sim$ is an equivalence relation on $X$. If $\pi_\lambda \sim \rho_\lambda$ for some nets $\{\pi_\lambda\}, \{\rho_\lambda\}$ and $\rho_\lambda \to \rho$ and $\pi_\lambda \to \pi$, then there exists a net of unitarys $\{u_\lambda\}$ such that $u_\lambda^* \pi_\lambda(.) u_\lambda = \rho_\lambda$. But the set $\{u \in M_n : u \text{ is a unitary}\}$ is compact. So there exists a convergent subnet $u_\lambda \to u$ for some unitary $u$. Thus $u^* \pi(.) u = \rho$. Hence $\sim$ is closed in $X \times X$. For each $\pi, \rho \in X$, choose $\beta(\pi,\rho) = u_{\pi,\rho}$, unitary, i.e. $u_{\pi,\rho}^* \pi(.) u_{\pi,\rho} = \rho(.)$. Then suppose that $\rho \sim \tau$. Then $u_{\rho,\tau}^* \rho(.) u_{\rho,\tau} = \tau(.)$. Hence $u_{\rho,\tau}^* u_{\pi,\rho}^* \pi(.) u_{\pi,\rho} u_{\rho,\tau} = \tau(.)$. But we also have $u_{\pi,\tau}^* \pi(.) u_{\pi,\tau} = \tau(.)$. Thus

$$u_{\pi,\tau}^* u_{\rho,\tau}^* u_{\pi,\rho}^* \pi(.) u_{\pi,\rho} u_{\rho,\tau} u_{\pi,\tau} = (u_{\pi,\rho}^* u_{\rho,\tau} u_{\pi,\tau})^* \pi(.) (u_{\pi,\rho} u_{\rho,\tau} u_{\pi,\tau}^*) = \pi(.).$$
But range $\pi = M_n$ so $u_{\pi,\nu}u_{\nu,\pi} \in C$. Hence $u_{\pi,\nu}u_{\nu,\pi}u_{\pi,\nu} \in Cu_{\pi,\nu}$. Now define the mapping $\alpha : A \to C(X, M_n)$ by $\alpha(a)\pi = \pi(a)\pi$. Then clearly $\alpha(A) \subseteq C(X, M_n, \sim, \nu)$.

However $\alpha(A)$ satisfies the hypothesis of Lemma 4.2.3. Thus

$$\alpha(A) = C(X, \sim, \beta, M_n).$$

Now $\ker \alpha = \cap_{x \in \text{irr}(A, M_n)} \ker \pi = \cap_{x \in \text{irr}(A)} \ker \pi = \{0\}$. Thus $\alpha$ is one to one. It follows that $\alpha$ is a $*$-isomorphism from $A$ onto $C(X, \sim, \beta, M_n)$.

**Proposition 4.2.5** Suppose $(X, \sim, \beta, n)$ is a regular equivalence system with $X$ metrizable. Then every central sequence in $C(X, \sim, \beta, M_n)$ is trivial.

**Proof.** Suppose $\{f_k\}$ is a central sequence in $C(X, \sim, \beta, M_n)$.

Claim: $\lim_{k \to 0} \sup_{x \in X} \text{distance} (f_k(x), C \cdot 1) = 0$.

Assume via contradiction that the claim is false. Then, by restricting to an appropriate subsequence if necessary, we can assume that there is an $\epsilon > 0$ and a sequence $\{x_k\}$ in $X$ such that $\text{distance} (f_k(x_k), C \cdot 1) \geq \epsilon$, for $k = 1, 2, 3, \ldots$. Using the compactness of $X$, we can also assume that $x_k \to x$ for some $x \in X$. Suppose $y \in X$.

Then $\{f_k(y)\}$ is a central sequence in $M_n$, and hence $\text{distance} (f_k(y), C \cdot 1) \to 0$.

Thus there exists a positive integer $k_0$ such that, $k \geq k_0 \Rightarrow \text{distance} (f_k(y), C \cdot 1) < \epsilon$.

If $x_k \sim y$, then $f_k(x_k) \sim f_k(y)$, $f_k \in C(X, \sim, \beta, M_n)$,

so $\text{distance} (f_k(y), C \cdot 1) = \text{distance} (f_k(x_k), C \cdot 1)$. It follows that, if $x_k \sim y$, then $k < k_0$. Thus only finitely many $x_k \sim y$ for any particular $y \in X$. It follows that we
can find a subsequence, which we will still call \( \{x_k\} \) such that for all \( 1 \leq k < j \) we have, \( x \) is not equivalent to \( x_k \) and \( x_k \) is not equivalent to \( x_j \). It follows from the proof of Proposition 4.2.4 that we can assume \( X = \text{Irr}(C(X, \sim, \beta, M_n)) \) and \( \sim \) is the same as unitary equivalence. Thus \( x_j, j = 1, 2, 3, \ldots \) are inequivalent representations of \( C(X, \sim, \beta, M_n) \) and \( x_k \to x \) in the point norm topology. It follows that no \( x_k \) is in the closure (in the spectrum of \( C(X, \sim, \beta, M_n) \)) of \( \{x_j : j \neq k\} \). Thus, by Lemma 4.2.3, for each \( k \) there is a \( g_k \in C(X, \sim, \beta, M_n) \) such that,

1. \( g_k(x_k) = 1 \)
2. \( g_k(x_j) = 0 \) for \( j \neq k \).

Define \( \pi : (C(X, \sim, \beta, M_n) \to \prod_{k=1}^{\infty} M_n \) (C*-product, i.e. bounded sequences) by \( \pi(f) = \{f(x_k)\}_{k=1}^{\infty} \). By considering \( \pi(g_k) \) we see that the range of \( \pi \) contains

\[ \sum_{k=1}^{\infty} M_n = \{\{A_k\} \in \prod M_n : \|A_k\| \to 0\} \]. On the other hand, since \( x_k \to x \), it follows that the range of \( \pi \) is included in \( \{\{A_k\} \in \prod M_n : \lim A_k \right\} \) exists. \( \} := B \). We will show that the range of \( \pi \) is \( B \).

Suppose \( \{A_k\} \in B \), and \( A_k \to A \). It follows from regularity that there is an \( f \in C(X, \sim, \beta, M_n) \) such that \( f(x) = A \). \( f(x_k) \to f(x) = A \). It follows that

\[ \{A_k\} - \pi(f) \in \sum M_n \subset \text{range } \pi. \] Hence \( \{A_k\} \in \pi(f) + \text{range } \pi = \text{range } \pi. \) Hence

\[ \text{range } \pi = B \]. However, if \( D \) is the commutative C*-algebra of all convergent sequences in \( C \), then \( B \) is isomorphic to \( M_n(D) \) or \( M_n(C(Y)) \) for some compact Hausdorff space \( Y \). Hence range \( \pi \) has the property that every central sequence is
trivial. Also the center of $B$ is the set of all convergent sequences of scalar multiples of the identity matrix. Since $\{f_k\}$ is a central sequence in $C(X, \sim, \beta, M_n)$, $\{\pi(f_k)\}$ must be a central sequence in $B$, and thus must be trivial. Hence there is a natural number $k$ such that, for some $\{\lambda_j\cdot1\}$ in the center of $B$,

$$\|\pi(f_k) - \lambda_j\cdot1\| < \frac{\varepsilon}{2}.$$ 

In particular, looking at the $k^{th}$ coordinate gives $\|f_k - \lambda_j\| < \frac{\varepsilon}{2}$; which contradicts distance $(f_k(x_k), C\cdot1) \geq \varepsilon$. This proves the claim, i.e.

$$\lim_{k \to \infty} \sup_{x \in X} \text{distance } (f_k(x), C\cdot1) = 0.$$ 

Suppose $T \in M_n$, and choose $\lambda \in C$ such that $\|T - \lambda\| = \text{distance } (T, C\cdot1)$. Then if $\tau_n$ is the normalized trace on $M_n$,

$$|\tau_n(T) - \lambda| \leq \|\tau_n\| \|T - \lambda\| = \text{distance } (T, C\cdot1),$$ 

so, $\|T - \tau_n(T)\| \leq \|T - \lambda\| + |\lambda - \tau_n(T)| < 2 \text{distance } (T, C\cdot1)$. It follows that if $f \in C(X, \sim, \beta, M_n)$ then, the function $\tau_n f : x \mapsto \tau_n(f(x))\cdot1$ is also in $C(X, \sim, \beta, M_n)$ since unitarily equivalent matrices have the same trace. Also $\tau_n f$ is in the center of $C(X, \sim, \beta, M_n)$. Moreover, we have

$$\|f - \tau_n f\| = \sup_{x \in X} \|f(x) - \tau_n(f(x))\cdot1\| \leq \sup_{x \in X} \text{distance } (f(x), C\cdot1).$$ 

It follows that $\{\tau_n f_k\}$ is a sequence in the center of $C(X, \sim, \beta, M_n)$ and it follows from the claim that $\|f_k - \tau_n f_k\| \to 0$. Hence $\{f_k\}$ is a trivial sequence.

**Theorem 4.2.6** Suppose $N$ is a natural number and $A$ is a unital separable C*-
algebra, such that every element of \( \text{Irr}(A) \) has dimension at most \( N \). Then the following are equivalent:

(1) \( A \) is a continuous trace C*-algebra.

(2) Every central sequence in \( A \) is trivial.

(3) \( \text{Irr}(A, M_k) \) is point norm closed in \( \text{Rep}(A, M_k) \), \( k \leq N \).

(4) \( A \) is a finite direct sum of C*-algebras of the form \( C(X, M_k, \alpha, \beta) \).

**Proof:** (1) is equivalent to (2). [1, Theorem 2.4].

(2) implies (3).

Let \( \pi \) be in the point norm closure of irreducible representations of \( A \). Then we can find a sequence of irreducible representations \( \{ \pi_n \} \) such that \( \pi_n(a) \to \pi(a), \forall a \in A \). Again we can define the algebra \( \rho(A) \) as in the proof of 4.4. Using the hypothesis we get that every central sequence in \( \rho(A) \) is trivial. Now using Theorem 3.8, we get that \( \pi(A)' \) consists of scalar multiples of identity or \( \pi \) is onto. This implies that \( \pi \) is irreducible.

(3) implies (4).

Let \( J = \{ k : 1 \leq k \leq N, \text{Irr}(A, M_k) \neq 0 \} \). It follows from (3) that \( \text{Irr}(A, M_k) \) is point norm compact in \( \text{Rep}(A, M_k) \) for each \( k \in J \), since \( \text{Rep}(A, M_k) \) is point norm compact. Suppose \( k, j \in J, k \neq j \). Suppose \( \pi \in \text{Irr}(A, M_k) \) and \( \rho \in \text{Irr}(A, M_j) \).
Then there exist an \( a \in A, 0 \leq a \leq 1 \) such that \( \pi(a) = 0, \rho(a) = 1 \), since \( \rho, \pi \) are clearly disjoint representations. Let \( U_\rho = \left\{ \tau \in \text{Irr}(A, M_j) : \| \rho(a) - \tau(a) \| < \frac{1}{3} \right\} \).

Since \( \rho(a) = 1 \) and \( a \geq 0 \) we have for every \( \tau \in U_\rho \), \( 0 \leq \tau(a) \) and \( 1 - \tau(a) \leq \frac{1}{3} \). Thus \( \frac{2}{3} \leq \tau(a) \). If \( 0 \leq g \leq 1 \) is a continuous function defined on the real numbers such that \( g|_{[0,1/3]} = 0, g|_{[2/3,\infty]} = 1 \). Then \( \pi(g(a)) = 0 \) and \( \tau(g(a)) = 1, \forall \tau \in U_\rho \). Note, \( U_\rho \) is point-norm open. Denote \( g(a) \) by \( x_\rho \). Note that \( x_\rho \geq 0 \). Fix \( \pi \).

Then \( \{ U_\rho : \rho \in \text{Irr}(A, M_j) \} \) forms an open cover of \( \text{Irr}(A, M_j) \). Thus there exists \( \rho_1, \rho_2, \ldots, \rho_s \) such that \( \text{Irr}(A, M_j) \subseteq \bigcup_{n=1}^{s} U_{\rho_n} \). Let \( x = x_{\rho_1} + x_{\rho_2} + \ldots + x_{\rho_s} \).

Then \( \pi(x) = 0 \) and for each \( \rho \in \text{Irr}(A, M_j), \rho(x) \geq 1 \) (since \( \rho \in U_{\rho_n} \), and \( \rho(x_{\rho_n}) = 1 \), for some \( m \)). Again using our function \( g \), we have \( \pi(g(x)) = 0 \) and for each \( \rho \in \text{Irr}(A, M_j), \rho(g(x)) = g(\rho(x)) = 1 \). Call \( g(x) = y_{k,l} \). Let \( y_\pi = g \sum_{i \neq j, j \neq k} y_{k,j} \).

Then \( y_\pi \geq 0, \pi(y_\pi) = 0 \). And for every \( j \in J \setminus \{k\}, \forall \rho \in \text{Irr}(A, M_j), \rho(y_\pi) = 1 \).

Now let

\[
V_\pi = \left\{ \sigma \in \text{Irr}(A, M_k) : \| \sigma(y_\pi) - \pi(y_\pi) \| \leq \frac{1}{3} \right\}.
\]

Then \( V_\pi \) is a point norm neighborhood of \( \pi \) in \( \text{Irr}(A, M_k) \). Since \( \pi(y_\pi) = 0 \), we have

\[
0 \leq \sigma(y_\pi) \leq \frac{1}{3}, \forall \sigma \in V_\pi \text{. Let } z_\pi = g(y_\pi) \text{. Then for every } \sigma \in V_\pi,
\]

\[
\sigma(z_\pi) = \sigma(g(y_\pi)) = g(\sigma(y_\pi)) = 0
\]

and \( \forall j \in J \setminus \{k\}, \forall \rho \in \text{Irr}(A, M_j) \).

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ \rho(z) = \rho(g(y_\pi)) = g(\rho(y_\pi)) = 1. \]

Then \( \{ V_\pi : \pi \in \text{Irr}(A, M_k) \} \) is an open cover and has a finite sub-cover, say \( V_{\pi_1}, \ldots, V_{\pi_r} \).

Let \( b = z_{\pi_1} \cdot z_{\pi_2} \cdot \ldots \cdot z_{\pi_r} \). Then for every \( \pi \in \text{Irr}(A, M_k) \), \( \pi(b) = 0 \). And for every \( j \in J \setminus \{ k \} \), for every \( \rho \in \text{Irr}(A, M_j) \), \( \rho(b) = 1 \).

Since every element of \( \text{Irr}(A) \) is unitarily equivalent to an element of \( \bigcup_{k \in J} \text{Irr}(A, M_k) \), it follows that \( \bigcap_{\pi \in \bigcup_{k \in J} \text{Irr}(A, M_k)} \ker \pi = \{ 0 \} \). For each \( j \in J \), define \( \pi_j = \sum_{\pi \in \text{Irr}(A, M_j)} \pi \) and let \( \rho = \sum_{j \in J} \pi_j \). Then \( \rho \) is one-to-one.

Hence \( A \) is isomorphic to \( \rho(A) \), and the preceding arguments show that \( \rho(A) = \sum_{j \in J} \pi_j(A) \). Moreover, it follows from the definition that for each \( j \in J \), every irreducible representation of \( \pi_j(A) \) is \( j \)-dimensional. Thus (4) now follow from Proposition 4.2.4.

(4) implies (2). This follows from Proposition 4.2.5.
Bibliography


