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New methods for modeling accelerated life test data

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New Methods for Modeling Accelerated Life Test Data

BY

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DISSERTATION

Submitted to the University of New Hampshire
in partial fulfillment of
the requirements for the degree of

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in
Mathematics

September 1999
This dissertation has been examined and approved.

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May 28, 1999
Date
Dedication

My research and dissertation are dedicated in memory of Orman A. Hopkins. Eventhough you are no longer with us, I know that you are somewhere watching over me.

To my grandparents, thank you for all of the wonderful memories you have given me through the years. Your love and support make me believe that with a little patience and alot of hard work, anything is possible.

To my father and mother, I appreciate all of the sacrifices that you have made for me over the years. They have not gone unnoticed. You are amazing parents and I love you with all my heart.

To my husband, I never promised you a rose garden, but thank you for sticking by me during the thorny times. Now, we are entering a new stage in our adventure together. I am ready and I know you are too.

To Brian, a sister could not ask for a better brother. I really enjoy and cherish the times that we spend together. Thank you for being my friend as well as my “favorite” brother.

To Jil, thank you for being my best friend, my reality check and my sunshine. Friends Forever and Always.
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ABSTRACT

New Methods for Modeling Accelerated Life Test Data

by

Michelle Hopkins Capozzoli
University of New Hampshire, September, 1999

An accelerated life test (ALT) is often used to obtain timely information for highly reliable items. The increased use of ALTs has resulted in nontraditional reliability data which can not be analyzed with standard statistical methodologies. I propose new methods for analyzing ALT data for studies with

1. two independent populations,

2. paired samples and

3. limited failure populations (LFP).

Here, the Weibull distribution, which can accommodate a variety of failure rates, is assumed for the models I develop. For case (1), a parametric hypothesis test, a Bayesian analysis and a test using partial likelihood are proposed and discussed. For paired samples, I show that there is no exact test for the equality of the survival distributions. Thus, several tests are investigated using a simulation study of their Type I errors. A Bayesian approach that allows for the comparison and estimation of the failure rates is also considered. For computation, Markov Chain Monte Carlo (MCMC) methods are implemented using BUGS.

Certain types of devices (such as integrated circuits) that are operated at normal use conditions are at risk of failure because of inherent manufacturing faults (latent risk factors).
A small proportion of defective units, $p$, may fail over time under normal operating conditions. For the non-defective units, the probability of failing under normal conditions during their “technological lifetime” is zero. Meeker ([29], [31]) called a population of such units a limited failure population (LFP). I propose a new model for LFP in which the number of latent risk factors and the times at which they become fatal depend on the stress level. This model allows for a fraction of the population to be latent risk free. For analyzing this model, I propose a classical as well as a Bayesian approach, which can be very useful when an engineer has expert knowledge of the manufacturing process. In all cases, a real data set is analyzed to demonstrate my procedures.
Chapter 1

Introduction

Many products and materials are designed to be highly reliable under “normal use” conditions. Failures may not occur for many years. This makes it difficult to conduct an experiment under natural operating conditions that would assess a product’s long term performance. In these cases, an accelerated life testing procedure can be useful in obtaining information about the time to failure distribution. With this type of testing, a product is subjected to a higher than usual stress level to obtain failure modes more quickly. Typically, the lifetime of a product can be shortened by applying a higher level of stress such as temperature, relative humidity, pressure, voltage, or vibration than what is usually observed at the normal operating level [37].

Example: Insulating Fluid

For example, Nelson [36] describes an accelerated life test (ALT) which was conducted to investigate the effect of voltage on the distribution of the time to breakdown for an insulating fluid. Under normal operating conditions, it may take thousands of years for the insulating fluid to breakdown. By applying a high level of voltage, the breakdown time can be substantially decreased. Here, the ALT consisted of exposing insulating fluid to one of seven high levels of voltage, 26 kV, 28 kV, 30 kV, 32 kV, 34 kV, 36 kV and 38 kV, and recording the time to breakdown in minutes (see Appendix A). Using the data, engineers wanted to predict the probability of the fluid breaking down at 20 kV, the normal operating condition.
**Censored Data:**

The insulating fluid data is an example of a data set where the exact failure times of the units are known. As with most lifetime data, ALT data can also experience censoring. Censored data can result from several different situations. Interval censoring occurs when units under test are monitored periodically for failures. For each unit, only the interval between inspections containing the failure time is known instead of the exact time to failure. Meeker and LuValle [33] give an example of interval censored data set that resulted from an ALT on printed circuit boards (see Appendix B). There are also situations when some units are removed from the test due to circumstances beyond the experimenter's control or the test may have been terminated before all units fail. In this case, the failure time is only known to be beyond a certain point. Such data are said to be right censored. The censoring time for an experiment can also be fixed. Data resulting from such an experiment are said to be time censored or Type I censored. There are also situations when an experiment is terminated after a certain number of failures have occurred. This results in what is called failure censored or Type II censored data. Nelson [36] provides a comprehensive description of the types of data that can occur from ALTs as well as many examples. It should be noted that the presence of censoring in a data set can complicate the data analysis. While much research has been conducted in this area, methods equipped to handle different kinds of censoring are needed.

The objective of ALTs is to use the data observed at accelerated conditions to draw inferences about the lifetime distribution under normal operating conditions. Inferential methods for these tests may demand specialized models and computational tools depending upon the complexity of the stress-lifetime relation and the presence of censoring in the data.
This chapter will give a general overview of some of the techniques that have been used to model ALT data.

1.1 Traditional Methods for Modeling ALT Data

1.1.1 Classical Methods

Suppose that $T_i$ is a random variable representing the time to failure of a unit operating under the $i$th stress level with probability density function (pdf) $f(t; \theta_i)$, where $\theta_i$ is a vector of the parameters at the $i$th stress level, $i = 1, \ldots, k$. Also, $T_0$ represents the time to failure of a unit operating under usual stress conditions with pdf $f(t; \theta_0)$, where $\theta_0$ is a vector of the parameters at the normal operating conditions. In practice, the distribution is typically assumed to be exponential, Weibull or lognormal. Let $V_i$ denote the magnitude of the $i$th stress level for $i = 1, \ldots, k$ and $V_0$ represent the magnitude of the normal operating stress. The common, classical, parametric approach is to make the following assumptions about the ALT model [24]:

1. The functional form of lifetime distribution, $f(t; \theta_i)$, is the same for all stress levels. Only the values of the parameters of the distribution will differ.

2. The relationship between the stress levels and the parameters of the distribution, $\theta_i = g(V_i; \gamma_1, \gamma_2, \ldots, \gamma_m)$ is known except for one or more of the acceleration parameters, $\gamma_1, \gamma_2, \ldots, \gamma_m$. Typically, $\theta_i$ will be a mean or scale type parameter.

3. The relationship, $\theta_i = g(V_i; \gamma_1, \gamma_2, \ldots, \gamma_m)$, is valid for a certain range of stress levels, and that range contains $V_0$. 

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4. For every \( i \), the stress level applied to a unit, \( V_i \), remains constant during the testing period.

Once an acceleration model is determined, the unknown parameters of the relationship, \( \gamma_1, \gamma_2, \ldots, \gamma_m \), are estimated based on the accelerated test data. Inferences about \( \theta_0 \) for the normal use stress, \( V_0 \), can then be obtained through the assumed acceleration model [37].

The specification of an acceleration model is equivalent to specifying the distribution's parameters as a function of the stress level. However, the determination as to what form this function takes is not simple. Padgett [37] notes that model selection should be based on the physical properties of a unit on test and the type of stresses being applied to cause failures. With this in mind, there have been many models derived using such considerations as kinetic theory and/or quantum mechanics.

The exponential failure distribution with parameter \( \lambda \), denoted by \( \text{Exp}(\lambda) \), will be used for the discussion of several of these models. So, for the \( i \)th stress level, the pdf of \( T_i \) is

\[
f(t; \lambda_i) = \lambda_i \exp(-\lambda_i t) \quad \text{for } t \geq 0,
\]

where \( \lambda_i > 0 \) and the mean time to failure is \( \mu_i = 1/\lambda_i \). For this specific case, the specification of an acceleration model is equivalent to expressing \( \mu_i \) as a function of the stress level \( V_i \).

The most commonly found acceleration model in the literature is the (inverse) power law model, which is derived by considerations of kinetic theory and activation energy ([24], [37]). The model is

\[
\mu_i = \gamma_1 V_i^{-\gamma_2},
\]
where $\gamma_1 > 0$ and $\gamma_2 > 0$ are unknown parameters. Intrinsic to this model is the implication that the mean life of a product decreases with the increase of stress. It has been applied to accelerated life testing of electrical insulations, simple metal fatigue due to mechanical loading and incandescent lamps [36]. In fact, Nelson [36] argues in favor of using this model for the insulating fluid data discussed previously.

The Arrhenius model, based on the Arrhenius Law for simple chemical-reaction rates, is used to express the degradation rate of a parameter of the device as a function of its operating temperature [37]. It is has been used for such products as electrical insulation, semiconductors, battery cells, greases and lubricants [36]. Here, $\lambda_i = \exp(\gamma_1 - \gamma_2/V_i)$, where $\gamma_1 > 0$ and $\gamma_2 > 0$ are unknown parameters.

An alternative to the Arrhenius model is the Eyring Model for a single stress. This model can be obtained from principles of quantum mechanics and is given by $\lambda_i = V_i \exp(\gamma_1 - \gamma_2/V_i)$, where $\gamma_1 > 0$ and $\gamma_2 > 0$ are unknown parameters [24]. Chernoff developed an acceleration model for exponential models with mean failure time $\mu_i = (\gamma_1 V_i + \gamma_2 V_i^2)^{-1}$, where $\gamma_1 > 0$ and $\gamma_2 > 0$ are the unknown parameters [37]. The importance of this model is that $\lambda_i$ is considered to be a quadratic function of the stress level.

Lastly, the Generalized Eyring Model is used when the device under consideration is subjected to two accelerated stresses, specifically a thermal stress and a non-thermal stress. Here, $V_i$ is a two dimensional vector, $V_i = (V_{1i}, V_{2i})$, where $V_{1i}$ is the thermal stress level and $V_{2i}$ is the non-thermal stress level. The model is given by $\lambda_i = \gamma_1 V_{1i} \exp(-\gamma_2/(KV_{1i})) \exp(\tau V_{2i} + \delta V_{2i}/(KV_{1i}))$, where $\gamma_1, \gamma_2, \tau, \delta$, and $\lambda$ are unknown parameters to be estimated and $K$ denotes Boltzmann's constant [24].
As noted earlier, choosing an acceleration model is a complicated task and should be based on the physical properties of the test device and the type of stresses being applied to accelerate failure [37]. Meeker [28] points out that the appropriateness of a model should always be determined through empirical verification. This may typically require standard regression plots, transformation plots, total-time-on-test plots or some other more formal goodness-of-fit test procedure ([28], [36]). In some cases, the relationship is unknown or difficult to verify. There are also models which are only valid for a certain range of stress levels. Beyond these ranges, a new model may have to be assumed, posing a problem with estimation and prediction [24]. In recent years, there have been some developments for modeling and analyzing ALT data where an acceleration model is not explicitly specified. For example, Kvam and Samaniego [23] proposed an exponential model where the levels of stress vary by a scale change. Durham and Padgett [11] developed models for systems under tensile loading that were derived from cumulative damage arguments and incorporate a "system size" or length variable.

Most of the analysis performed on ALT data includes a combination of graphical techniques (e.g. scatter plots) and analytical methods such as regression analysis and maximum likelihood methods [30]. Inference can become quite difficult depending upon the acceleration model, the number of parameters in the lifetime distribution and the presence of censoring in the data. Also, the design of an ALT experiment can affect the precision of the analysis from ALT data. Designing an ALT experiment involves determining the number of stress levels, the selection of the stress levels, and the number of units tested at each stress level [37]. The aspects of designing an ALT are beyond the scope of this thesis and will not be discussed further. The reader is referred to [36], [24], [32] for more details. Here, a test
procedure discussed by Padgett [37] and Mann, Schafer and Singpurwalla [24] will be used to highlight the difficulties in analyzing ALT data. The simplest parametric cases, involving the exponential distribution (Equation (1)) and the (inverse) power law model (Equation (2)) are demonstrated below.

Suppose that life tests are to be conducted at \( k \) accelerated stress levels, \( V_i \) for \( i = 1, \ldots, k \), and that the distribution of a unit on test at level \( V_i \) is \( \text{Exp} (\lambda_i) \) with mean \( \mu_i = 1/\lambda_i \). The accelerated stress levels and the number of units tested at the \( i \)th stress level \( (n_i) \), are determined by some procedure. Usually, the accelerated stresses are chosen so that:

1. They fall within the specified range where the acceleration model is known to be valid.

2. They are sufficiently high to induce failures within a reasonable time interval.

All the test units are then randomly allocated to stress levels. In this type of experiment, all the units within a stress level are tested at the same time. The \( k \) life tests should be performed simultaneously. However, cost and/or apparatus constraints may require that the life test be sequentially performed. To ensure exchangeability of the \( k \) life tests in this case, the sequence in which the \( k \) life tests are performed is randomly selected. Let \( t_{ij} \) represent the \( j \)th failure time under acceleration level \( V_i \), for \( i = 1, \ldots, k \) and \( j = 1, \ldots, n_i \). The resulting data set is then represented by \( \{V_i, n_i, \bar{\mu}_i\} \), for \( i = 1, \ldots, k \) and where \( \bar{\mu}_i = 1/n_i \sum_j t_{ij} \) is the sample mean for the \( i \)th stress level.

The extrapolation between the accelerated stress levels and the normal operating stress level requires an acceleration model. Here, the (inverse) power law model, as seen in Equation (2), is chosen only for illustrative purposes. The goal is to estimate \( \gamma_1 \) and \( \gamma_2 \) from the data set, \( \{V_i, n_i, \bar{\mu}_i\} \) for \( i = 1, \ldots, k \), obtained using the procedure described above, so that
inferences can be drawn for $\mu_0$. Singpurwala [43] showed that, in order to obtain estimators of $\gamma_1$ and $\gamma_2$ that are asymptotically independent, the power rule must be modified slightly to

$$\mu_i = \frac{\gamma_1}{\left(\frac{V_i}{\bar{V}}\right)^{\gamma_2}},$$

for all values of $V_i$ within the specified range. $\bar{V}$ is defined to be the weighted geometric mean of the $V_i$'s, $\bar{V} = \prod_{i=1}^{k}(V_i)^{R_i}$ with $R_i = n_i/\sum_{i=1}^{k} n_i$.

Maximum likelihood methods or least squares methods are commonly used to obtain estimates for $\gamma_1$ and $\gamma_2$. Nelson [36] describes least squares methods for fitting a model with a Weibull or exponential distribution to uncensored ALT data. He notes that this method yields estimates that are not as accurate as those from maximum likelihood fitting. However, they are easier to implement since software packages with least squares regression capabilities are readily available. Both Nelson [36] and Mann, Schafer and Singpurwala [24] discuss maximum likelihood methods. For illustrative purposes, the maximum likelihood method will be demonstrated.

Since $\hat{\mu}_i$ can be thought of as the weighted sum of $n_i$ exponential random variables, $\hat{\mu}_i \sim \text{Gamma}(n_i, n_i/\mu_i)$ with mean $\mu_i$ and variance $\mu_i^2/n_i$. The randomization of the order in which the tests are performed ensures independence of $\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_k$. Therefore, the likelihood function of $\gamma_1$ and $\gamma_2$ can be written as

$$L(\gamma_1, \gamma_2 \mid \hat{\mu}) = \prod_{i=1}^{k} \Gamma^{-1}(n_i) \left[ \frac{n_i}{\gamma_1} \left(\frac{V_i}{\bar{V}}\right)^{\gamma_2} \right]^{n_i} (\hat{\mu}_i)^{n_i-1} \exp\left[ -\frac{n_i \hat{\mu}_i}{\gamma_1} \left(\frac{V_i}{\bar{V}}\right)^{\gamma_2} \right],$$

(4)
where \( \hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_k) \). The maximum likelihood estimators (mles) of \( \gamma_1 \) and \( \gamma_2 \), \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) respectively, are given by solving:

\[
\hat{\gamma}_1 = \frac{\sum_{i=1}^{k} n_i \hat{\mu}_i \left( \frac{V_i}{V} \right)^{\gamma_2}}{\sum_{i=1}^{k} n_i} \tag{5}
\]

and

\[
\sum_{i=1}^{k} n_i \hat{\mu}_i \left( \frac{V_i}{V} \right)^{\gamma_2} \ln \left( \frac{V_i}{V} \right) = 0. \tag{6}
\]

The equations are nonlinear, resulting in the need for numerical methods (such as the Newton-Raphson method) to find estimates of \( \gamma_1 \) and \( \gamma_2 \). Inferences about \( \mu_0 \) can then be made by using the acceleration model to extrapolate between the accelerated stress levels and the normal stress level.

The data resulting from many lifetime experiments are subject to censoring. Type I censoring is seen most often in practice, but theoretical models (such as the one demonstrated above) can be quite complicated. In certain cases, Type II censoring can be theoretically more tractable. The methods described above can be used with one slight modification to accommodate Type II censoring for the exponential-power law model (Equation (1) and Equation (3)). For each of the \( i \) stress levels, the life test is terminated after \( r_i \) failures and the respective times to failure, \( t_{i1}, \ldots, t_{ir_i} \), are recorded. The resulting data set is then \( \{V_i, n_i, r_i, \hat{\mu}_i\} \), for \( i = 1, \ldots, k \) where \( \hat{\mu}_i \) is an estimator for \( \mu_i \), the mean time to failure for the \( i \)th stress level. Epstein [13] showed that the unique minimum variance unbiased
efficient estimator for \( \mu_i \) is

\[
\mu_i = r_i \left( \sum_{j=1}^{r_i} t_{ji} + (n_i - r_i) * t_{ri} \right),
\]

which has a Gamma\( (r_i, r_i/\mu_i) \). Now, the analysis directly follows the uncensored data case.

For further examples of the traditional methods of analysis, Mann, Schafer and Singpurwalla [24] give a detailed derivation of the methods of analysis when the lifetime distribution is exponential (Equation (1)) and the acceleration model is assumed to be the Arrhenius or the Erying model. They also consider non-parametric techniques which will not be discussed here. Nelson [36] extensively covers the standard methods of analysis for ALTs, as well as gives practical examples. Viertl [46] also provides an overview of a larger class of statistical methods that do not seem to be widely used because of practical reasons. He also includes a discussion on Bayesian methods, which will be discussed in the next section.

1.1.2 Bayesian Methods

The goals of the analysis of ALT data involve prediction at the normal operating stress level. This prediction depends on the prior understanding of the stress-failure relationship. Also, frequentist methods rely heavily on asymptotics. Large sample sizes can be prohibitively expensive and/or time consuming. Efficient use of available prior information can be cost effective and may reduce the number of units needed for testing. Therefore, Bayesian methods have the potential to be useful and effective in making good decisions in ALTs. In fact, Bayes rule offers a natural and logical blending of prior or expert knowledge and information from the data obtained through accelerated testing. Suppose that the lifetime distribution
of a unit is $S(t \mid \theta(V))$, where $\theta(V) = g(V; \gamma)$ represents the acceleration model, and that $Y$ is the accelerated life test data. Using the Bayesian paradigm, inferences can be drawn from the posterior distribution,

$$p(\gamma \mid Y) \propto L(\gamma \mid Y) p(\gamma),$$

where $L(\gamma \mid Y)$ is the likelihood which is representing the information from the accelerated test and $p(\gamma)$ represents prior knowledge about the parameters of the lifetime model.

There has not been enough research done in the way of developing Bayesian models for ALT data. The analysis of ALT data within a Bayesian framework is usually limited to placing priors on the parameters of the same models used in frequentist methods. The problem with this type of modeling is that the acceleration model's parameters may not have meaningful physical interpretations for the units on test. This makes it difficult to state what a “prior belief” may be.

In order to obtain an estimate for $S(t \mid V)$, Vierdt [46] points out that there are several methods that are available. The first method uses the posterior Bayes estimator of $\gamma$ given by

$$\hat{\gamma} = E_{\gamma}[\gamma \mid Y],$$

where $E_{\gamma}[\cdot \mid Y]$ means the expectation is taken with respect to the posterior density $p(\gamma \mid Y)$. $S(t \mid V)$ can then be estimated by $\hat{S}(t \mid V) = S(t \mid g(V; \hat{\gamma}))$ for $t \geq 0$. Another approach is to use

$$\tilde{g}(V) = E_{\gamma}[g(V; \gamma) \mid Y].$$

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$S(t \mid V)$ can then be estimated by $\tilde{S}(t \mid V) = S(t \mid \tilde{g}(V))$ for $t \geq 0$. Lastly, the most accurate method is when the survival function is estimated via $\tilde{S}(t; V) = E[Y \mid g(V; \gamma)]$ for $t \geq 0$.

A non-parametric Bayesian approach proposed by Proschan and Singpurwalla [38] does not require a distributional assumption or the specification of an acceleration model. There is a loss of statistical precision because of the non-parametric nature of their model. However, their methodology diminishes the chance of selecting an incorrect acceleration model.

Let the $k$ stress levels be such that $V_i$ is less severe than $V_{i+1}$ for $i = 0, \ldots, k-1$ and $V_0$ is the normal use stress level. Because the severity of the stresses increases, they point out that it is reasonable to assume that the hazard rates (i.e. $\lambda(t)dt = P(t < T < t + dt \mid T > t)$) also increase, so that for any $t \geq 0$,

$$\lambda_0(t) < \lambda_1(t) < \ldots < \lambda_k(t). \quad (11)$$

Using the accelerated life test data, the goal is to find a Bayes estimate for $\lambda_i$, say $\hat{\lambda}_i$, for $i = 1, \ldots, k$ such that for some $0 < L < \infty$ and all $t \in [0, L]$,

$$\hat{\lambda}_1(t)^{st} \leq \hat{\lambda}_2(t)^{st} \leq \ldots \leq \hat{\lambda}_k(t). \quad (12)$$

The notation $X \leq Y^{st}$ denotes the fact that $X$ is stochastically smaller than $Y$. Proschan and Singpurwalla [38] use a discretized model for the $\lambda_i(t)$'s to obtain these Bayes estimators.

As an alternative approach to analyzing ALT data, several authors have used Kalman filter models ([34], [3], [25]). In particular, Meinhold and Singpurwalla [34] discuss the case when the lifetime distribution for the $i$th stress level is Exp$(1/\lambda_i)$ with mean $\mu_i = \lambda_i$, for
\( i = 1, \ldots, k \), and the acceleration model is the power law, as seen in Equation (2). They assume that the stress levels are \( V_{k+1} < V_k < \ldots < V_2 < V_1 \), where \( V_{k+1} \) is the normal operating stress level.

Suppose that \( T_{i1}, \ldots, T_{in_i} \) are the failure times for the \( i \)th stress level, \( i = 1, \ldots, k \). The mean time to failure for the \( i \)th stress level is given by

\[
\bar{T}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} T_{ij}.
\]  

(13)

Meinhold and Singpurwalla show that \( Y_i^* = \ln(\bar{T}_i) \) has an extreme-value distribution with variance \( \psi'(n_i) \) and mean \( \ln(n_i) - \ln(\lambda_i) + \psi(n_i) \), where \( \psi(\cdot) \) and \( \psi'(\cdot) \) are the digamma and the trigamma functions, respectively. They also note that the extreme-value distribution closely approximates the normal distribution for reasonably large \( n_i \). Thus, the observation equation can be specified in terms of \( Y_i = \ln(n_i) + \psi(n_i) - Y_i^* \) as

\[
Y_i = \ln(\lambda_i) + \nu_i,
\]  

(14)

where \( \nu_i \sim \text{Normal}(0, \psi'(n_i)) \). The system equations are motivated by the acceleration model which they assume is the power law (Equation (2)). By indexing \( \gamma_2 \) by \( i \), Meinhold and Singpurwalla propose that the equations are as follows

\[
\ln(\lambda_i) = \ln(\lambda_{i-1}) + \gamma_{2,i} \ln\left(\frac{V_{i-1}}{V_i}\right) + \nu_i
\]  

(15)

and

\[
\gamma_{2,i} = \gamma_{2,i-1} + \nu_i,
\]  

(16)
where \((u_i, v_i)' \sim \text{Normal}(0, \Sigma)\) and \(u_i = v_i \ln(1/V_i)\). The Kalman filter procedure is started after the initial values of \(\lambda_0\) and \(\gamma_{2,0}\) are specified. Meinhold and Singpurwalla [34] provide a comprehensive discussion of this model.

Mazzuchi and Singpurwalla [25] apply Kalman filter models to several other test scenarios. In particular, they develop the model for the case when the accelerated life test is conducted for a finite period and the proportion of survivals is used to obtain the observation equation. They also discuss the situation of running the accelerated life test at each stress level for a fixed time period. Here, the observation equation is obtained using order statistics. Blackwell and Singpurwalla [3] propose using these models with correlated observation errors. They develop their ideas for the exponential-power law model (Equation (1) and Equation (2)).

Lastly, Mazzuchi and Soyer [26] propose a Bayesian procedure that is based on dynamic general linear models. Their procedure does not require large numbers of items to be tested at each accelerated stress level and does accommodate censoring. To illustrate their approach, they assume that the lifetime distribution is exponential (Equation (1)) and that the power law (Equation (2)) is the acceleration model. However, the procedure can easily be extended to other acceleration models such as the Arrhenius model. Moreover, they extend their approach to the Weibull distribution [27].

1.1.3 Markov Chain Monte Carlo Methods

Bayesian analysis often entails integrating over the posterior distribution, as seen in Equation (8), in order to draw inferences about model parameters. This integration may turn out to be analytically intractable. One of the tools used to perform such integration is the Markov
Chain Monte Carlo method (MCMC) which is essentially Monte Carlo integration using Markov Chains (see Chapters 3 and 5 in [45] for a review). Algorithms such as Metropolis-Hastings ([35], [21]) and the Gibbs sampler ([15], [45]) are popular MCMC methods. Here, the implementation of the Gibbs algorithm is briefly described. Other related MCMC tools are very similar to the Gibbs sampler.

Suppose \( p(\theta | Y) \) is a joint posterior where \( \theta = (\theta_1, \ldots, \theta_d) \) is a vector of parameters. The Gibbs sampling algorithm enables us to sample from \( p(\theta | Y) \). Given an initial starting point of \( \theta^{(0)} = (\theta_1^{(0)}, \theta_2^{(0)}, \ldots, \theta_d^{(0)}) \), this algorithm iteratively samples from the conditional posteriors as follows:

1. Sample \( \theta_1^{(i+1)} \) from \( p(\theta_1 | \theta_2^{(i)}, \ldots, \theta_d^{(i)}, Y) \)
2. Sample \( \theta_2^{(i+1)} \) from \( p(\theta_2 | \theta_1^{(i+1)}, \theta_3^{(i)}, \ldots, \theta_d^{(i)}, Y) \)
3. \[ \vdots \]
4. Sample \( \theta_d^{(i+1)} \) from \( p(\theta_d | \theta_1^{(i+1)}, \ldots, \theta_{d-1}^{(i+1)}, Y) \).

The vectors \( \theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(t)}, \ldots \) are a realization of a Markov Chain and are used in the Monte Carlo integration. It can be shown that the joint distribution of \( \theta^{(t)} = (\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_d^{(t)}) \) converges to \( p(\theta_1, \ldots, \theta_d | Y) \) as \( t \to \infty \) [45]. Convergence of the distribution can be monitored using methods described by Cowles and Carlin [7].

Any integrable function, say \( g(\theta) \), can then be estimated using Monte Carlo integration as follows

\[
E[g(\theta) | Y] \approx \frac{1}{N} \sum_{j=1}^{N} g(\theta_{(j)}),
\]

(17)
where $\theta_{(1)}, \ldots, \theta_{(N)}$ are independent samples from $p(\theta | Y)$. It can be shown that [45]

$$
\frac{1}{N} \sum_{j=1}^{N} g(\theta_{(j)}) \approx \int g(\theta)p(\theta | Y)d\theta.
$$

In practice, the strong assumption of strict independence of $\theta_{(1)}, \ldots, \theta_{(N)}$ may not be needed (see [45]). The Gibbs sampler is easily implemented using a software package called Bayesian Inference Using Gibbs Sampling (BUGS), produced by MRC Biostatistics Unit, Institute of Public Health, Cambridge, UK [44]. BUGS implements the Gibbs sampling algorithm and allows sampling from the posterior distribution using different priors for the models parameters. Convergence analysis is performed using CODA, Convergence Diagnosis and Output Analysis [2], which is a menu-driven set of S-Plus functions.

### 1.2 Non-Traditional Accelerated Life Tests

The data resulting from ALTs can usually be analyzed using the standard methods discussed in Chapter 1. However, there are situations where these methods are not applicable. Examples of such circumstances are discussed below.

**Two Sample Problem:**

In a reliability engineering application that was studied by Zimmer and Deely [47], the comparison of two potential suppliers of a specific unit is required so that the unit with the smaller failure rate is purchased. Because of the unit's high reliability, an experiment under normal operating conditions is not feasible. The comparison of the quality of this unit from two different suppliers can be ascertained by placing units from both suppliers on test at a specified accelerated stress level. The failure data from each supplier can then be used to
select the supplier with the smaller failure rate. The analysis entails comparing failure rates (or the mean times to failure) from two independent ALTs with equal accelerated stress levels.

**Paired Samples:**

Another problem where standard ALT methods are not applicable occurs when the ALT data is paired. With repairable systems, a pair could represent the time to the first failure of a particular unit and the time to the second failure of the same unit after repair. Here, an engineer would like to know if the unit is "as good as new". This requires comparing the mean time to failure of paired samples. For units which are highly reliable, an accelerating stress could be applied to ensure failure times within a reasonable time interval. Again, the techniques discussed in the previous section no longer apply here.

**Limited Failure Population:**

As a final example, Meeker and LuValle [33] present a data set on lifetimes of printed circuit boards tested using relative humidity (RH) as the accelerating stress (see Appendix B). 72 circuit boards were tested at each of four stress levels, 49.5% RH, 62.8% RH, 75.4% RH and 82.4% RH. The goal of the study was to investigate the effect of relative humidity on the time to failure distribution and to predict the reliability of the circuit boards under the humidity level present in usual operating conditions. On the surface, this appears to be a standard data set from an ALT. However, a plot of the empirical cumulative distribution function (ECDF) reveals that certain assumptions may be violated (Figure 1-1). A standard life test model assumes that at some point all units will fail (i.e. \( P(T < +\infty) = 1 \)). At the 49.5% RH, 62.8% RH, the plot plateaus which implies that a fraction of the population will never fail during their "technological life" or the duration of a unit's usefulness in the
field (i.e. $P(T = \infty) > 0$). Meeker([29], [31]) called such a population a *limited failure population* (LFP). Another important aspect of the plot of the ECDF is that at the higher stress levels, all of the circuit boards fail. This indicates that beyond a certain stress level, a different acceleration model may need to be assumed. This can make prediction at the normal operating stress level difficult.

With today's rapidly advancing technology, manufacturers are being faced with developing new, highly reliable products in short periods of time. Thus, ALTs are being used more often and for a broad spectrum of products and materials. The standard models are often not applicable as seen in the above examples. In addition, certain assumptions can be limiting. The simplified assumption of the lifetime distribution following an exponential distribution is most often not true. Nelson [36] points out that in his experience, the exponential assumption is only valid 15% of the time. Flexible alternatives, such as the Weibull distribution, are more useful. Verifying an acceleration model can be difficult if not
impossible. For example, products used in the military and space travel are subjected to environments that are difficult to simulate or may not be known. To handle these situations, new techniques need to be developed.

Chapters 2 and 3 of this thesis focus on developing models for independent and paired samples from ALTs. Previous models proposed for bivariate accelerated life tests have assumed a bivariate exponential distribution. Here, these ideas are expanded to a bivariate Weibull distribution. Furthermore, the models can accommodate the situation when the exact nature of the acceleration model is not known to statisticians. Chapter 4 proposes a model for accelerated life tests when the data is from a limited failure population. In all cases, both classical and Bayesian techniques will be considered. Chapter 5 will present some conclusions and note areas of future research.
Chapter 2

Comparing Two Independent Data Sets

2.1 Introduction

For data obtained from two independent ALTs, a goal of the analysis can be the comparison of the failure rates (or the mean times to failure). For example, a reliability engineer may be interested in comparing two potential suppliers of a device so that the device with the smaller failure rate is purchased. The high reliability of this device may make it impossible to conduct an experiment under normal use conditions. An ALT performed by both suppliers at a specified accelerated stress level, $V$, may produce failure times within a reasonable time period. The ALT data from each supplier can then be used to select the supplier with the smaller failure rate.

Zimmer and Deely [47] discuss such a data set, reproduced in Table 2.1. Each column represents the failure times from an ALT performed by a supplier. Unlike the traditional experiments discussed in Chapter 1, each supplier is only testing devices at one fixed accelerated stress level, $V$. This makes it impossible to estimate the parameters of an acceleration model and to extrapolate between the accelerated stresses and the usual operating stress. Instead, the results from the accelerated condition are assumed to be indicative of what occurs during usual operating conditions. Also, the focus here is on the comparison of two suppliers rather than the prediction of the lifetime at the normal operating condition.

One approach for analyzing this data is to perform a two sample t-test. However, this...
Table 2.1 Failure Times

<table>
<thead>
<tr>
<th>Supplier 1</th>
<th>Supplier 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.0</td>
<td>1.4</td>
</tr>
<tr>
<td>8.0</td>
<td>5.0</td>
</tr>
<tr>
<td>5.2</td>
<td>0.6</td>
</tr>
<tr>
<td>6.3</td>
<td>0.5</td>
</tr>
<tr>
<td>7.7</td>
<td>2.3</td>
</tr>
<tr>
<td>5.3</td>
<td>6.0</td>
</tr>
<tr>
<td>9.6</td>
<td>3.6</td>
</tr>
<tr>
<td>4.0</td>
<td>10.0</td>
</tr>
<tr>
<td>9.0</td>
<td>3.0</td>
</tr>
<tr>
<td>7.0</td>
<td>6.5</td>
</tr>
</tbody>
</table>

method should not be used for data that do not come from normally distributed populations unless sample sizes are large. Non-parametric methods, such as the Wilcoxon Rank Sum test, fail to use all the information present in the data. They are also not as powerful as a parametric test. Nelson [36] proposes that the same class of models be fitted independently to each sample. The pairs of estimates can then be compared by one of three methods:

1. normal approximate confidence intervals,

2. likelihood ratio confidence intervals or

3. the likelihood ratio test.

However, all of these methods rely on asymptotic approximations requiring large sample sizes and ensuring large sample sizes for the ALT experiments can be costly in practice.

Zimmer and Deely [47] approach this problem from a Bayesian perspective. The method that they propose allows researchers to not only compare the suppliers failure rates but also to estimate them. Their ALT model, which shall be referred to as the exponential model, is as follows:
Modeling Assumptions of Zimmer and Deely:

1. The failure time, $T$, under non-accelerated conditions follows an exponential distribution, which has the pdf:

$$f(t \mid \lambda) = \lambda \exp(-\lambda t), \quad \lambda > 0, \quad t > 0.$$  \hfill (1)

2. $S_V(t) = S_u(\theta t)$, \hfill (2)

where $S_V(t)$ and $S_u(t)$ are the survival functions under accelerated testing conditions and under normal use conditions, respectively. The acceleration parameter, $\theta$, is constant over time and is the same for both suppliers.

The parameter $\theta$ represents the effect of the accelerated stress on the unit. Equation (2) indicates that the effect of the accelerated stress is a decreased failure time via a scale change of the lifetime variable. Also, assumptions 1 and 2 imply that $S_V(t) = \exp[-(\lambda \theta t)]$.

Hence, the lifetime distribution under the accelerated condition is Exp($\lambda \theta$). The goal is to compare and estimate the failure rates, $\lambda_1$ and $\lambda_2$ for suppliers 1 and 2 from the data under accelerated conditions.

The only data available is from the ALT where units were tested at one fixed stress level, $V$. This causes $\lambda$ and $\theta$ to become confounded. They are not identifiable with the data from a single supplier. To overcome this obstacle, Zimmer and Deely [47] note that $T_i \sim \text{Gamma}(n_i, \theta \lambda_i)$ for $i = 1, 2$, where $T_i = \sum_{j=1}^{n_i} T_{ij}$ is the sum of the failure times and $n_i$ is the number of items tested for supplier $i$. They show that the distribution of
the ratio, $T_2/T_1$, is free of the nuisance parameter $\theta$. Inference is based on the likelihood for $T_2/T_1$. The comparison of the failure rates is achieved by computing the posterior probability, $P(\lambda_1 < c \lambda_2 \mid data)$, where $0 < c \leq 1$. This posterior probability can measure the magnitude of the difference in performances between the selected supplier and the competitor. The Bayes estimates of the failure rates are obtained through the posterior means, $E[\lambda_1 \mid data]$ and $E[\lambda_2 \mid data]$.

The above procedure assumes that the failure time distribution is exponential, which has limited use because of its constant failure rate. A more flexible survival model is the Weibull distribution which can accommodate increasing, decreasing or constant failure rates. In this chapter, an extension of Zimmer and Deely's [47] methods will be explored using both classical and Bayesian methodology. It will be assumed that the failure time follows a Weibull distribution. The exponential model of Zimmer and Deely [47] is a special case of the Weibull model proposed here. This assumption of a Weibull distribution can be relaxed further by considering a proportional hazards model [8]. Inference for this model will be based on the partial likelihood [8]. To demonstrate the proposed methods, the data set of Zimmer and Deely [47] will be reanalyzed.

2.2 Model and Assumptions

The proposed modeling assumptions are as follows:

**Weibull Model Assumptions**

1. The failure time, $T$, under non-accelerated conditions follows a Weibull distribution,
which has the pdf:

\[
f(t \mid \beta, \alpha) = \beta \alpha^\beta t^{\beta-1} \exp[-(\alpha t)^\beta], \quad t > 0.
\]  

(3)

2. \[S_V(t) = S_u(\theta t),\] (4)

where \(S_V(t)\) and \(S_u(t)\) are the survival functions under accelerated testing conditions and under normal use conditions, respectively. The acceleration parameter \(\theta\) is assumed to be constant over time and the same for both suppliers.

3. The shape parameter, \(\beta\), is assumed to be the same for both suppliers.

4. The scale parameter, \(\alpha\), may be different for the suppliers.

The interpretation of the acceleration parameter \(\theta\) and Equation (4) are the same as they are in the exponential model of Zimmer and Deely [47]. The parameter \(\theta\) still represents the effect of the stress on the device, while Equation (4) indicates that the effect of the accelerated stress is a decreased failure time via a scale change of the lifetime variable. The added assumption that the shape parameter, \(\beta\), is the same for each supplier is analogous to the assumption in a two-sample t-test that the underlying scale parameter is the same for both samples. Lastly, assumptions 1 and 2 now imply that \(S_V(t) = \exp[-(\alpha t)^\beta]\). This is equivalent to saying that the lifetime distribution in the accelerated condition is Weibull with shape parameter \(\beta\) and scale parameter \(\alpha \theta\), denoted by \(\text{Weibull}(\beta, \alpha \theta)\). The goal is to compare the survival distributions for supplier 1 and 2 under the accelerated conditions. The results of the analysis are then assumed to be indicative of the comparative behavior at
the non-accelerated conditions.

The observed data are from only one accelerated stress. Therefore, the parameters, \( \alpha \) and \( \theta \), are not identifiable by looking at the data from one supplier. The following notation is needed to further this discussion.

**Notation**

- \( i \) index for supplier; \( i = 1, 2 \)
- \( n_i \) number of items tested by supplier \( i \)
- \( j \) index for items; \( j = 1, 2, \ldots, n_i \) for \( i = 1, 2 \)
- \( T_{ij} \) failure time of item \( j \) for supplier \( i \)
- \( T_{i(1)} \) \( \min_j \{ t_{ij} \} \) for \( i = 1, 2 \)
- \( \alpha_i \) scale parameter for supplier \( i \)

To overcome the confounding of the parameters, the minimum of the failure times, \( T_{i(1)} \) for \( i = 1, 2 \), was considered. This statistic was selected since the distribution of the minimum of Weibull random variables is again Weibull. Hence, \( T_{i(1)} \sim \text{Weibull}(\beta, \alpha_i n_i^{\frac{1}{\beta}} \theta) \) for \( i = 1, 2 \). In addition, the distribution of the ratio of the failure times, \( R = T_{2(1)}/T_{1(1)} \) is free of \( \theta \). For this model, a closed form for distribution of \( R \) can be obtained using a transformation of variables.

Let \( (U, V) = h(T_{2(1)}, T_{1(1)}) = (T_{2(1)}/T_{1(1)}, T_{1(1)}) \). The two-dimensional function \( h \) is invertible, so \( (T_{2(1)}, T_{1(1)}) = g(U, V) = (g_1(U, V), g_2(U, V)) = (UV, V) \). The joint pdf of \( (U, V) \) is then obtained from the following formula [1]:

\[
   f_{UV}(u, v) = f_{T_{2(1)}, T_{1(1)}}(g_1(u, v), g_2(u, v)) \mid J_g(u, v) ,
\]

\[\text{(5)}\]
where $|J_g(u,v)|$ is the absolute value of the Jacobian of $g$ given by

$$J_g(u,v) = \det \begin{pmatrix} \frac{\partial}{\partial u} g_1(u,v) & \frac{\partial}{\partial v} g_1(u,v) \\ \frac{\partial}{\partial u} g_2(u,v) & \frac{\partial}{\partial v} g_2(u,v) \end{pmatrix}.$$ 

For this situation, the joint distribution of $(U, V)$ is as follows

$$f_{U,V}(u,v) = \beta^2 (\alpha_1 n_1^{1/\theta})^\beta (\alpha_2 n_2^{1/\theta})^\beta u^{\beta-1}v^{2\beta-1} \exp[-(\alpha_1 n_1^{1/\theta} v)^\beta - (\alpha_2 n_2^{1/\theta} uv)^\beta]. \quad (6)$$

By integrating out $V$ in Equation (6) using the substitution $x = V^{\beta}$, the pdf of $U = T_{2(1)}/T_{1(1)}$ is given by

$$f_U(u) = \frac{\beta \xi^\beta u^{\beta-1}}{[1 + (\xi u)^\beta]^2} \quad \text{for } u > 0, \quad (7)$$

where $\xi = (\alpha_2 n_2^{1/\beta})/\alpha_1 n_1^{1/\beta} > 0$ and $\beta > 0$. This distribution is free of $\theta$. It is also a log-logistic distribution which is a special case of the Burr Type XII distributions. By performing another transformation of variables, $D = -\ln U$, the logistic distribution with location parameter, $\ln(\xi)$, and scale parameter, $1/\beta$, denoted by $D \sim \text{logistic} (\ln(\xi), 1/\beta)$, is obtained which has the pdf:

$$f(d) = \frac{\beta \exp\{-\beta[\ln(\xi)]\}}{[1 + \exp\{-\beta[\ln(\xi)]\}]^2} \quad \text{for } -\infty < d < \infty, \quad (8)$$

where $\beta > 0$, and $-\infty < \ln(\xi) < \infty$.

The logistic distribution obtained above can also be derived by using moment generating functions. Instead of using the ratio, $T_{2(1)}/T_{1(1)}$, the statistic, $D = \ln R = \ln T_{2(1)} - \ln T_{1(1)}$, is considered. For $i = 1, 2$, $Y_i = \ln T_{i(1)}$ follows an extreme value distribution with location
parameter, \( \ln(\xi_i) = \ln(\alpha_i n_i^{\frac{1}{\beta}} \theta) \) and scale parameter \( \delta = \frac{1}{\beta} \), denoted by \( \text{EV}(\ln(\xi_i), \delta) \) [36].

The pdf for \( Y_i \) is:

\[
f(y) = \left( \frac{1}{\delta} \right) \exp \left( \frac{y - \ln(\xi_i)}{\delta} \right) \exp \left[ - \exp \left( \frac{y - \ln(\xi_i)}{\delta} \right) \right],
\]

where \(-\infty < y < \infty\), \( \delta > 0 \) and \(-\infty < \ln(\xi_i) < \infty\). By Theorem 1 and Theorem 2, \( D = Y_2 - Y_1 \sim \text{logistic}(\ln(\xi_2), \frac{1}{\beta}) \) with pdf as seen in Equation (8) where \( \xi = \xi_2/\xi_1 \).

**Theorem 1:** If \( E_1 \sim \text{EV}(\xi_1, \delta) \) and \( E_2 \sim \text{EV}(\xi_2, \delta) \), then the difference between the two independent extreme value random variables, \( D = E_1 - E_2 \), has the moment generating function:

\[
M_D(s) = \exp[s(\xi_2 - \xi_1)] \Gamma(1 - s\delta) \Gamma(1 + s\delta).
\]

**Proof:**

\[
M_D(s) = E[\exp(sD)] = E[\exp[s(E_2 - E_1)]] = E[\exp(sE_2) \exp(-sE_1)]
\]

Since \( E_1 \) and \( E_2 \) are independent,

\[
M_D(s) = E[\exp(sE_2)]E[\exp(-sE_1)].
\]

\( E_1 \) and \( E_2 \) are extreme value random variables. Therefore, the moment generating function for \( E_i, i = 1, 2 \) is:

\[
M_{E_i}(s) = E[\exp(sE_i)] = \exp(s\xi_i) \Gamma(1 - \delta s).
\]
Hence,

\[ M_D(s) = M_{E_2}(s)M_{E_1}(s) = \exp(s\xi_2) \Gamma(1 - \delta s) \exp(-s\xi_1) \Gamma(1 + \delta s). \]

The above equation simplifies to:

\[ M_D(s) = \exp[s(\xi_2 - \xi_1)] \Gamma(1 - s\delta) \Gamma(1 + s\delta). \]

**Theorem 2:** If \( E_1 \sim EV(\xi_1, \delta) \) and \( E_2 \sim EV(\xi_2, \delta) \), then the difference between the two extreme value random variables, \( D = E_1 - E_2 \), follows a logistic distribution with location parameter \( \xi_2 - \xi_1 \) and scale parameter \( \delta \).

**Proof:**

Suppose \( X \sim \text{logistic}(0, 1) \) with probability distribution

\[ f_X(x) = \exp(-x)[1 + \exp(-x)]^{-2} \]

and moment generating function:

\[ M_X(t) = \Gamma(1 - t)\Gamma(1 + t). \]

So, \( Z = X\delta + (\xi_2 - \xi_1) \sim \text{logistic}((\xi_2 - \xi_1), \delta) \) has moment generating function:

\[ M_Z(s) = E[\exp\{s[X\delta + (\xi_2 - \xi_1)]\}] = \exp[s(\xi_2 - \xi_1)] E[\exp(sX\delta)] \]

\[ = \exp[s(\xi_2 - \xi_1)] \Gamma(1 - s\delta) \Gamma(1 + s\delta) \]
Since D has the same moment generating function, \( D \sim \text{logistic} \left((\xi_2 - \xi_1), \delta\right) \).

### 2.3 A Parametric Test for Equality of Distributions

The equality of the distributions for \( T_{1(1)} \) and \( T_{2(1)} \) at stress level \( V \) can be tested by considering the hypotheses, \( H_0 : \alpha_1 = \alpha_2 \) versus \( H_a : \alpha_1 \neq \alpha_2 \). Since \( D = \ln(T_{2(1)}) - \ln(T_{1(1)}) \) where \( T_{i(1)} \) is the minimum of the failure times for \( i = 1, 2 \), the data collapses to a single value. The only way to test for equality is to compare the observed value of \( D \) calculated from the data to the mean of \( D \) under the null hypothesis, which in this case is zero. This suggests that the test statistic should be

\[
D^* = \beta \left[ D - \ln \left( \frac{n_2}{n_1} \right)^{\frac{1}{2}} \right]
\]

which simplifies to \( D^* = \beta D \) if \( n_1 = n_2 \). Under the null hypothesis, \( D^* \sim \text{logistic} \left(0, 1\right) \). The p-value can then be found by calculating,

\[
p = 2 \left( \frac{\exp\left(-|D^*|\right)}{1 + \exp\left(-|D^*|\right)} \right).
\]

Critical values can also be calculated by

\[
d_p = -\ln\left(\frac{1}{p} - 1\right),
\]

where \( p = P(D^* < d_p) \). Hence, the null hypothesis is rejected when \( |D^*| \geq d_{1-\frac{\alpha}{2}} \), where \( \alpha \) is the significance level. It should be noted that the behavior of this test can also be investigated since the distribution under the alternative hypothesis is known.
Most often, $\beta$ is unknown and a pooled estimate of $\beta$ will need to be obtained in order to perform the hypothesis test. The mles for each supplier can be easily calculated using the Weibull plot option in Minitab. A estimate for $\beta$ could then be the weighted average

$$\bar{\beta} = \frac{n_1\hat{\beta}_1 + n_2\hat{\beta}_2}{n_1 + n_2}, \quad (12)$$

where $\hat{\beta}_i$ for $i = 1, 2$ is the mle for the $i$th supplier. This estimate, $\bar{\beta}$, can be used to replace $\beta$ in the test statistic for an approximate test.

2.4 Bayesian Inference

If $\rho = \ln(\alpha_2/\alpha_1)$, the likelihood for Weibull model can be expressed as

$$L(\rho, \beta | d) \propto \frac{\beta \exp\{-\beta[d - \rho - \frac{1}{\beta}\ln\left(\frac{n_2}{n_1}\right)]\}}{(1 + \exp\{-\beta[d - \rho - \frac{1}{\beta}\ln\left(\frac{n_2}{n_1}\right)]\})^2}. \quad (13)$$

Inference using the Bayesian paradigm is then based on the posterior distribution

$$p(\rho, \beta | d) \propto L(\rho, \beta | d) \ast \pi(\rho, \beta), \quad (14)$$

where $\pi(\rho, \beta)$ is the joint prior for the parameters $\rho$ and $\beta$. It will be assumed that $\pi(\rho, \beta) = \pi_1(\rho) \ast \pi_2(\beta)$. The selection of $\pi_1(\rho)$ and $\pi_2(\beta)$ will be discussed in the next section.

Comparison of the distributions of $T_{1(1)}$ and $T_{2(1)}$ at the stress level $V$ is ascertained by obtaining the posterior distribution of $\rho$, $p(\rho | d)$, and calculating a 95% credible region for $\rho$. If this region contains the value zero, then there is no difference in the distributions of $T_{1(1)}$ and $T_{2(1)}$. 

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2.4.1 Selecting Priors

Several priors for $\rho = \ln\left(\frac{n_2}{n_1}\right)$ and $\beta$ were considered. The first choice was to put a flat prior on both parameters. Unfortunately, the resulting posterior distribution was not proper. The next prior that was considered was Jeffrey's prior.

**Result 1:** If $D \sim \text{logistic} \left( \rho + \frac{1}{\beta} \ln\left(\frac{n_2}{n_1}\right), \frac{1}{\beta} \right)$ then the information matrix for $\rho + \frac{1}{\beta} \ln\left(\frac{n_2}{n_1}\right)$ and $\frac{1}{\beta}$ is

$$I(\rho + \frac{1}{\beta} \ln\left(\frac{n_2}{n_1}\right), \frac{1}{\beta}) = \beta^2 \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & (3 + \pi^2)/9 \end{bmatrix}.$$  

**Proof**

Suppose $X \sim \text{logistic} (\mu, \sigma)$. The information matrix for $\mu$ and $\sigma$ [10] is

$$I(\mu, \sigma) = \frac{1}{\sigma^2} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & (3 + \pi^2)/9 \end{bmatrix}.$$  

(15)

Hence, the information matrix for $\rho + \frac{1}{\beta} \ln\left(\frac{n_2}{n_1}\right)$ and $\frac{1}{\beta}$ is

$$I(\rho + \frac{1}{\beta} \ln\left(\frac{n_2}{n_1}\right), \frac{1}{\beta}) = \beta^2 \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & (3 + \pi^2)/9 \end{bmatrix}.$$  

(16)

**Result 2:** If $D \sim \text{logistic} \left( \rho + \frac{1}{\beta} \ln\left(\frac{n_2}{n_1}\right), \frac{1}{\beta} \right)$ then Jeffrey's prior is

$$\pi(\rho + \frac{1}{\beta} \ln\left(\frac{n_2}{n_1}\right), \beta) = \beta \sqrt{\frac{3 + \pi^2}{27}}.$$
Proof:

The determinant of the information matrix is:

\[ \det(I) = \beta^2 \left[ \frac{1}{3} \cdot \frac{3 + \pi^2}{9} - 0 \right] = \frac{\beta^2}{27} (3 + \pi^2). \]

Hence, Jeffrey's prior is:

\[ \pi \left( \rho + \frac{1}{\beta} \ln \left( \frac{n_2}{n_1} \right), \beta \right) = \sqrt{\frac{\beta^2}{27} (3 + \pi^2)} = \beta \sqrt{\frac{3 + \pi^2}{27}}. \quad (17) \]

It can be shown that the resulting posterior is again not proper if the prior for \( \beta \) is improper. It appeared that in order to obtain a proper posterior a proper prior for \( \beta \) needed to be selected. To avoid the high influence of a subjective prior on the data analysis, one would like to choose a prior so that the overall shape of the prior is flat with decreasing tails. Therefore, the flexible gamma prior for \( \beta, \beta \sim \text{Gamma}(\eta, \psi) \) was considered. For \( \rho \), any noninformative prior is appropriate. This led to the selection of \( \rho \sim \text{Normal}(\mu, \sigma^2) \) where \( \mu \) is the mean, \( \sigma^2 \) is the variance and is chosen to be very large (i.e. \( \sigma^2 > 10000 \)).

2.4.2 Markov Chain Monte Carlo Methods

The joint posterior for this model is very complicated. Therefore, sampling based techniques, such as the Gibbs Sampler ([15], [45]), were employed to sample from this distribution. Using the priors \( \beta \sim \text{Gamma}(\eta, \psi) \) and \( \rho \sim \text{Normal}(\mu, \sigma^2) \), the conditional posteriors required to implement Gibbs sampling are

\[
p(\rho \mid \beta, d) = \frac{p(\rho, \beta \mid d)}{p(\beta \mid d)} = \frac{L(\rho, \beta \mid d) \cdot \pi_1(\rho) \cdot \pi_2(\beta)}{\int L(\rho, \beta \mid d) \cdot \pi_1(\rho) \cdot \pi_2(\beta) d\rho}
\]
\[
\beta \exp \left[ -\beta (d - \rho) - \frac{1}{2} \left( \frac{\alpha - \mu}{\sigma} \right)^2 \right] \nonumber
\]
\[
\frac{1}{(1 + \exp \{-\beta (d - \rho - \frac{1}{2} \ln(\frac{\sigma^2}{\eta_1}))\})^2}
\]
(18)

and

\[
p(\beta \mid \rho, d) = \frac{p(\rho, \beta \mid d)}{p(\beta \mid d)} = \frac{L(\rho, \beta \mid d) \cdot \pi_1(\rho) \cdot \pi_2(\beta)}{\int L(\rho, \beta \mid d) \cdot \pi_1(\rho) \cdot \pi_2(\beta) d\beta}
\]
\[
\propto \beta^\rho \exp[-\beta (\eta + d - \rho - \frac{1}{2} \ln(\frac{\sigma^2}{\eta_1}))]\nonumber
\]
\[
\frac{1}{1 + \exp[-\beta (d - \rho - \frac{1}{2} \ln(\frac{\sigma^2}{\eta_1}))]^2}
\]
(19)

To illustrate how the model can easily be adapted by practitioners, a small program (see Appendix C) was developed using BUGS. BUGS implements the Gibbs sampling algorithm and allows sampling from the posterior distribution using different priors for \( \rho \) and \( \beta \). To demonstrate these procedures, the Zimmer and Deely data was analyzed. The output can be seen in the Examples section.

2.5 Cox model and the Partial Likelihood

The hazard function for the \( i \)th supplier for the Weibull model under accelerated conditions is

\[
h_i(t) = \beta (\alpha_i \theta)^{\beta} t^{\alpha_i - 1} \quad \text{for} \quad i = 1, 2.
\]
(20)

Therefore, the ratio of the two hazard functions is given by

\[
\frac{h_2(t)}{h_1(t)} = \left( \frac{\alpha_2}{\alpha_1} \right)^\beta
\]
(21)

which is free of \( t \) (i.e. the ratio of the two hazard functions is constant over time). The Weibull model, as seen in Equation (20), is actually a special case of a more general class of
models called the proportional hazards model or Cox model [8]. The Cox model in this two
sample situation does not assume a distributional form for the lifetime distribution, only
that the distribution has a proportional hazards structure (i.e. \( \frac{h_2(t)}{h_1(t)} \) is free of \( t \)). The basic
model is as follows:

\[
    h(t \mid Z) = h_0(t) \exp(\gamma Z),
\]

(22)

where \( h_0(t) \) is an arbitrary baseline hazard rate, \( \gamma \) is a regression parameter vector and \( Z \)
is a vector of covariates. In order to compare the suppliers, a binary covariate \( Z \) can be
defined such that

\[
    Z = \begin{cases} 
    0 & \text{for supplier 1} \\
    1 & \text{for supplier 2} 
    \end{cases}
\]

(23)

This implies that \( h(t \mid Z = 0) = h_0(t) \) and \( h(t \mid Z = 1) = h_0(t) \exp(\gamma) \). Thus, the ratio of
the hazards is

\[
    \frac{h_2(t)}{h_1(t)} = \frac{h(t \mid Z = 1)}{h(t \mid Z = 0)} = \exp(\gamma)
\]

(24)

which is free of \( t \). Comparing the suppliers can then be accomplished by testing the hypoth­
thesis \( H_0 : \gamma = 0 \) versus \( H_\alpha : \gamma \neq 0 \).

Inference is based on the partial likelihood (PL) [9] rather than the full likelihood. The
advantage to this method can be seen by realizing that the PL depends only on \( \gamma \). The
previously proposed hypothesis test and the Bayesian inference are based on \( T_{i(1)} \) for \( i = 1, 2 \)
which is the minimum of the failure times for each supplier. This results in a loss of
information and a loss of power. Therefore, a test based on the partial likelihood statistic
may have more power since it is using more of the information that is present in the data.

The PL is also free of \( \theta \), since the acceleration parameter represents a rescaling of the time

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line and the rank statistic is invariant to the time-scale parameter $\theta$. The partial likelihood is given as follows:

$$PL(\gamma) = \prod_{i \in D} \frac{\exp(\gamma z_i)}{\sum_{j \in R(y_i)} \exp(\gamma z_j)},$$  \hspace{1cm} (25)$$

where $D$ is the set of failures, $R(y)$ is the risk set at time $y$, $y_i$ is the $i$th failure time. Testing the hypothesis $H_0 : \gamma = 0$ using the PL of Equation (25) can be easily implemented using the PHREG procedure in SAS or using the coxph command in Splus.

### 2.6 Examples

The data set presented by Zimmer and Deely [47] will be reanalyzed to demonstrate the three procedures discussed above. For the parametric hypothesis test, $D = \ln(T_{2(1)}) - \ln(T_{1(1)}) = -2.079$. Minitab found the mles for $\beta_1$ and $\beta_2$ to be $\hat{\beta}_1 = 4.37162$ and $\hat{\beta}_2 = 1.32222$. From Equation (12), the values give a pooled estimate for $\beta$ to be $\bar{\beta} = 2.84692$. The test statistic is then $D^* = -5.9187$. The approximate p-value was found to be $p = .0054$. According to this procedure, there does appear to be a significant difference in the failure rates for the two suppliers at a significance level of .05. This is consistent with the results found by Zimmer and Deely [47].

For the Bayesian approach, the Unix version of BUGS was implemented since the Windows version does not support the logistic distribution. Table 2.2 contains the posterior mean, standard deviation, 2.5% percentile and the 97.5% percentile for $\beta$ and $\rho$. These estimates were obtained by assuming the locally noninformative priors: $\pi_1(\rho) \sim $ Normal $(0, 100)$ and $\pi_2(\beta) \sim $ Gamma $(.03, .03)$. The interval between the 2.5% percentile and 97.5% per-
Table 2.2 Summary Statistics for $\beta$ and $\rho$

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>standard deviation</th>
<th>2.5% percentile</th>
<th>97.5% percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>6.622</td>
<td>13.75</td>
<td>0.0239</td>
<td>47.75</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-1.920</td>
<td>4.038</td>
<td>-10.82</td>
<td>8.214</td>
</tr>
</tbody>
</table>

Table 2.3 Results for Cox Model and PL Analysis

<table>
<thead>
<tr>
<th>Test</th>
<th>Test Statistic</th>
<th>Degrees of Freedom</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Likelihood Ratio</td>
<td>2.745</td>
<td>1</td>
<td>.0975</td>
</tr>
<tr>
<td>Wald</td>
<td>2.801</td>
<td>1</td>
<td>.0942</td>
</tr>
<tr>
<td>Score</td>
<td>2.920</td>
<td>1</td>
<td>.0875</td>
</tr>
</tbody>
</table>

The centile for $\rho$ includes zero. This indicates that there is no difference in the failure rates at a 95% posterior probability.

To test the hypothesis $H_0 : \gamma = 0$ using the PL, as seen in Equation (25), the PHREG procedure in SAS was used. This procedure conducts three tests, a likelihood ratio test, the Wald test and the Efficient score test. The results are in Table 2.3. At a .05 significance level, there does not appear to be a significant difference in the failure rates. This is consistent with the results of the Bayesian inference. However, these results differ from the parametric hypothesis test and Zimmer and Deely's [47] results. The inconsistency in the results for the Bayesian case could be contributed to the fact that this method takes into consideration the lack of prior knowledge about the parameters. For the PL method, the proportional hazards assumption model may not be appropriate. A plot of the Schoenfeld residuals, see Figure 2-1, was constructed using S-plus. If the proportional hazards assumption is valid the plot should be a random walk. This is clearly not the case, which indicates that the proportional hazards assumption is not valid.
Figure 2-1 Plot of the Residuals to Check Proportional Hazards Assumption
Chapter 3

Paired Observations from a Conditionally Independent Weibull Distribution

3.1 Introduction

Experiments resulting in paired data are frequently conducted in medical and engineering research. For example, Gross and Lam [19] present a data set on the length of time until patients achieve relief from headaches (Table 3.1). Each patient receives a standard treatment and a new treatment on separate occasions. The time to relief for each treatment is the recorded response. The goal of this study was to compare the treatments and determine if there is a difference in the lengths of the relief times. Paired experiments also occur in reliability studies. With repairable systems, a pair could represent the time to the first failure of a particular unit and the time to the second failure of the same unit after repair. Here, an engineer would like to compare the paired failure times and determine if the system is “as good as new.”

The idea of pairing is also applicable to ALTs. Expanding on the reliability example, it may be that the test units are also being exposed to an accelerating stress, say $V$. The effect of the accelerating stress on the failure mode may not be of primary interest, unlike the printed circuit board ALT discussed in Chapter 1 [33]. The accelerating stress is only being applied to induce failures within a reasonable time interval. The focus of the experiment still hinges on determining if the failure distributions for before and after repair are equivalent.
under normal operating conditions. Methods developed to analyze the resulting data would assume that the comparative behavior of the units under accelerated stress is indicative of what will occur during normal operating conditions.

Traditionally, data from paired experiments are analyzed by performing a paired t-test. However, this method should not be used for data that is not normally distributed unless sample sizes are large. Non-parametric approaches, such as the signed rank test, could be applied. However, these types of methods do not fully use the information present in the data and are less powerful than parametric methods. The inadequacy of these methods to effectively analyze paired data presents the need for the development of new parametric methodology.

A popular parametric model for survival data is the exponential distribution. Since this data is paired, a bivariate exponential distribution needs to used for model development. This poses a problem since there is no obvious bivariate extension of the exponential distribution for correlated data. The univariate exponential distribution is characterized by the
memoryless property or $P(T > t + s \mid T > s) = P(T > t)$ where $t, s \geq 0$. The extension of this property to the bivariate case is 

\[ P(T_1 > t_1 + s_1, T_2 > t_2 + s_2 \mid T_1 > s_1, T_2 > s_2) = P(T_1 > t_1, T_2 > t_2) \]

for $t_1, t_2, s_1, s_2 \geq 0$. Block and Basu [4] point out that in order to have an absolutely continuous bivariate exponential distribution with the memoryless property, the marginals cannot be assumed to be exponentially distributed. They instead propose an absolutely continuous bivariate exponential model (ACBVE) where the marginals are mixtures or weighted averages of exponential distributions. For the paired variables $(T_1, T_2)$, their proposed joint density is as follows:

\[
f(t_1, t_2) = \begin{cases} 
\frac{\lambda_1 \lambda_2 + \lambda_{12}}{\lambda_1 + \lambda_2} \exp(-\lambda_1 t_1 - (\lambda_2 + \lambda_{12}) t_2) & \text{if } t_1 < t_2 \\
\frac{\lambda_2 \lambda_1 + \lambda_{12}}{\lambda_1 + \lambda_2} \exp(-(\lambda_1 + \lambda_{12}) t_1 - \lambda_2 t_2) & \text{if } t_1 > t_2
\end{cases}
\]

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, and $\lambda_1, \lambda_2, \lambda_{12} > 0$. Block and Basu [4] demonstrate two derivations for their bivariate model as well as discuss its properties and inference procedures under their model.

Gross and Lam [19] extend Block and Basu's research [4] by proposing a test for the equality of correlated survival distributions when the joint pdf is the ACBVE. This likelihood ratio test is based on the distribution of the ratio $T_2/T_1$ and requires that the mles of the parameters be determined numerically. Gross and Lam [19] also point out that when $\lambda_{12} = 0$, the ACBVE corresponds to independent survival distributions with the joint pdf

\[
f(t_1, t_2) = \lambda_1 \lambda_2 \exp\{-\lambda_1 t_1 + \lambda_2 t_2\} \text{ for } t_1 \geq 0, t_2 \geq 0,
\]

where $\lambda_1 > 0$ and $\lambda_2 > 0$. The equality of the independent survival distributions is tested...
using likelihood ratio testing procedures.

The tests proposed by Gross and Lam [19] depend on asymptotic theory. As an alternative, Cantor and Knapp [5] developed an exact parametric test for the equality of mean survival times based on data from a bivariate exponential distribution where a random pairing parameter is associated with each pair. Their conditional joint pdf for $(T_1, T_2)$, is expressed as:

$$f(t_1, t_2 | \theta) = \lambda_1 \lambda_2 \theta^{-2} \exp[-\theta^{-1}(\lambda_1 t_1 + \lambda_2 t_2)] \quad \text{for } t_1 \geq 0, t_2 \geq 0, \quad (3)$$

where $\lambda_1$, $\lambda_2$ and $\theta > 0$. Here, $\theta$ is a value of a random variable $\Theta$ that represents the susceptibility or the resistability of a unit to failure. The parameter $\theta$ is assumed to be the same for a given pair but can differ from pair to pair.

In order to test the equality of the two survival distributions, Cantor and Knapp [5] consider the ratio, $R = T_2 / T_1$. The random variable $R$, which no longer depends on the pairing parameter $\theta$, has the following pdf:

$$f_R(r) = \frac{\eta}{(1 + \eta r)^2} \quad \text{for } r > 0, \quad (4)$$

where $\eta = \lambda_2 / \lambda_1 > 0$. This is the same distribution obtained by Gross and Lam [19] for the ratio of the survival distributions in the independent case, see Equation (2). However, Cantor and Knapp [5] develop an exact parametric test for the hypotheses $H_0$: $\eta = 1$ versus $H_a$: $\eta \neq 1$ instead of using likelihood ratio testing procedures.
Cantor and Knapp [5] perform the transformation $U = 1 + R$ to obtain their test statistic

$$2n \ln \left[ \prod_i (1 + r_i) \right]^{1/n} = 2 \ln \left[ \prod_i (u_i) \right],$$

(5)

where $r_i = t_{2i}/t_{1i}$ is the ratio of the two component survival times. This test statistic has a chi-squared distribution with $2n$ degrees of freedom under the null hypothesis. The rejection region can be obtained by calculating the conventional chi-square critical values.

While Cantor and Knapp's test is an exact parametric test, it is not label invariant because there is no one to one correspondence between $2n \ln \left[ \prod_i (1 + t_{2i}/t_{1i}) \right]^{1/n}$ and $2n \ln \left[ \prod_i (1 + t_{1i}/t_{2i}) \right]^{1/n}$. The test may give different results with the same data set depending upon how the component survival times are labeled. This can be seen by comparing the results obtained by Cantor and Knapp [5] and those obtained when the test statistic is instead calculated using the ratio $T_1/T_2$. The authors perform their test on a data set representing the duration of relief from itching of a skin disease. For each patient, a different medication is applied to each arm. They calculate their test statistic to be 8.66. The lower $X^2_{.025,20}$ is 9.59 while the upper $X^2_{.975,20}$ is 34.17. Thus, they conclude that there is a difference in mean duration of relief. However if the test is instead performed using the ratio $T_1/T_2$, the test statistic is calculated to be 27.57. This value clearly does not fall in the rejection region indicating that the null hypothesis is not rejected.

Furthermore, Salvia and Bollinger [39] propose three other tests for comparing survival distributions when Cantor and Knapp's bivariate exponential model, see Equation (3), is assumed. The tests are an F-test, the Neyman-Pearson one tail test and a likelihood ratio test. The F-test assumes that $\theta$ is fixed and the same for all pairs. This test is not suitable.
for this model since Cantor and Knapp's model (Equation (3)) assumes that $\theta$ can differ from pair to pair. Both the Neyman-Pearson test and the likelihood ratio test are based on asymptotic results which require large sample sizes. Salvia and Bollinger [39] analyze the same data reported in Cantor and Knapp [5]. They also point out that Cantor and Knapp [5] performed a sign test and a signed rank test both of which do not reject the null hypothesis.

The results (Table 3.2) emphasize the difficulty in selecting a test procedure for Cantor and Knapp's bivariate exponential distribution, Equation (3). Unfortunately, the authors do not calculate and compare the powers of these tests.

As an alternative to the classical techniques, Zimmer and Deely propose using Bayesian methods for the comparison of survival data from the bivariate exponential distribution as seen in Equation (3). Inference is based on the likelihood for $T_2/T_1$ so that there is no confounding of the parameters. The comparison of the failure rates is achieved by computing the posterior probability, $P(\lambda_1 < c \cdot \lambda_2 \mid data)$, where $0 < c \leq 1$, while estimates of the failure rates are obtained through the posterior means, $E[\lambda_1 \mid data]$ and $E[\lambda_2 \mid data]$. The main advantage of the Bayesian approach is the ability to estimate the failure rate of each unit. This can not be obtained using classical methods.
The methodologies proposed above are all based on a bivariate extension of the exponential distribution which can be a restrictive assumption in practice. Here, the more flexible Weibull distribution (which can accommodate a variety of failure rates) is used for model development. It will be shown that there is no exact test for the equality of the survival distributions in this situation. Thus, several testing methods will be compared by investigating the Type I error using a simulation study for various sample sizes and model parameter values. A Bayesian approach will also be considered, which will allow for the comparison and estimation of the failure rates of two units. Markov Chain Monte Carlo methods will be implemented using BUGS for the Bayesian computation. To demonstrate these procedures, the data set on the relief times for headaches discussed by Gross and Lam [19] will be analyzed using both the classical and the Bayesian procedures.

3.2 A Bivariate Weibull Distribution

Suppose \((T_{1j}, T_{2j})\) are the survival times for the \(j\)th pair, for \(j = 1, 2, \ldots, n\) and that an unobservable random variable \(\Theta\), with pdf \(h(\theta)\), is associated with each pair. The random variable \(\Theta\) can be interpreted as representing the resistability of an object to failure which will be the same for each pair but can differ from pair to pair. The pdf, \(h(\theta)\), is unknown but is assumed to be continuous and positive only for non-negative values of \(\theta\). If an accelerating stress, say \(V\), is also being applied to the units on test, the parameter representing the effect of the stress will be confounded with the pairing parameter. The effect of the accelerating stress in this situation is not of interest. It is only being applied to ensure reasonable failure times. Therefore, the assumption will be made that the conclusions drawn from the accelerated conditions are indicative of the behavior of the units under normal operating
conditions. In addition the following assumptions will be made.

**Modeling Assumptions:**

1. The component survival times, $T_{1j}$ and $T_{2j}$, are independent given the random effect of pairing and accelerated condition $\theta_j$.

2. The distributions of $T_{1j}$ and $T_{2j}$ given $\theta_j$ are

   $$T_{ij} \mid \theta_j \sim \text{Weibull}(\beta, \alpha_1 \theta_j); \text{ for } i = 1, 2.$$  \hfill (6)

The dependence of $T_{1j}$ and $T_{2j}$ is attributed to the random effect $\theta_j$. Therefore, the above assumptions are equivalent to saying that knowing the unobservable $\theta_j$ makes the conditional distributions of $T_{1j}$ and $T_{2j}$ independent.

### 3.3 Joint Distribution

To simplify the notation, the subject index, $j$, will be suppressed. Assumption 1 and 2 imply that the joint pdf of $T_1$, $T_2$, and $\Theta$ can be expressed as the product of the conditional distributions of $T_1$, $T_2$ and $h(\theta)$,

$$f(t_2, t_1, \theta \mid \beta, \alpha_1, \alpha_2) = g_W(t_1 \mid \alpha_1 \theta, \beta)g_W(t_2 \mid \alpha_2 \theta, \beta)h(\theta)$$ \hfill (7)

where $g_W(t \mid a, b)$ is the Weibull density as seen in Chapter 2, Equation (3). The joint pdf of $T_1$ and $T_2$ is then obtained by integrating over $\theta$:

$$f(t_1, t_2 \mid \beta, \alpha_1, \alpha_2) = \int g_W(t_1 \mid \alpha_1 \theta, \beta)g_W(t_2 \mid \alpha_2 \theta, \beta)h(\theta)d\theta.$$ \hfill (8)
Through further integration, the marginal pdf's of $T_1$ and $T_2$ can be expressed as:

$$f(t_1 | \beta, \alpha_1) = \int \beta(\alpha_1 \theta)^{\beta} t_1^{\beta-1} \exp[-(\alpha_1 \theta t_1)^{\beta}] h(\theta) d\theta,$$

(9)

$$f(t_2 | \beta, \alpha_2) = \int \beta(\alpha_2 \theta)^{\beta} t_2^{\beta-1} \exp[-(\alpha_2 \theta t_2)^{\beta}] h(\theta) d\theta$$

(10)

As long as the first two inverse moments of $\Theta$ exist, it is possible to obtain expressions for the expected value, the variance and the covariance of $T_1$ and $T_2$ using the pdf's in Equation 10 and Equation 11. They are as follows

$$E[T_1] = \frac{1}{\alpha_1} \Gamma(1 + \frac{1}{\beta}) \int \frac{1}{\theta} h(\theta) d\theta$$

$$E[T_2] = \frac{1}{\alpha_2} \Gamma(1 + \frac{1}{\beta}) \int \frac{1}{\theta} h(\theta) d\theta$$

$$Var[T_1] = \frac{1}{\alpha_1^2} \Gamma(1 + \frac{2}{\beta}) \int \frac{1}{\theta^2} h(\theta) d\theta - \left[ \frac{1}{\alpha_1} \Gamma(1 + \frac{1}{\beta}) \int \frac{1}{\theta} h(\theta) d\theta \right]^2$$

$$Var[T_2] = \frac{1}{\alpha_2^2} \Gamma(1 + \frac{2}{\beta}) \int \frac{1}{\theta^2} h(\theta) d\theta - \left[ \frac{1}{\alpha_2} \Gamma(1 + \frac{1}{\beta}) \int \frac{1}{\theta} h(\theta) d\theta \right]^2$$

$$Cov[T_1, T_2] = \frac{1}{\alpha_1 \alpha_2} (\Gamma(1 + \frac{2}{\beta}))^2 \left( \int \frac{1}{\theta^2} h(\theta) d\theta \right)^2 \left( \int \frac{1}{\theta^2} h(\theta) d\theta - 1 \right)^2$$

3.4 Distribution of $T_2/T_1$

In order to develop a test for the hypothesis of equality of the two survival distributions, the ratio of the failure times, $R = T_2/T_1$, will be considered to avoid confounding of the parameters. It will be shown that the distribution of $R$ is free of $\theta$. For this model, a closed form for the distribution of $R$ can be obtained using a transformation of variables.
Let \( (U, V) = h(T_2, T_1) = (T_2/T_1, T_1) \). The two-dimensional function \( h \) is invertible, so \( (T_2, T_1) = g(U, V) = (g_1(U, V), g_2(U, V)) = (UV, V) \). The joint pdf of \( (U, V) \) is then obtained from the following formula [1]:

\[
f_{U,V}(u,v) = f_{T_2,T_1}(g_1(u,v), g_2(u,v)) | J_g(u,v) |
\]

(11)

where \( | J_g(u,v) | \) is the absolute value of the Jacobian of \( g \) given by

\[
J_g(u,v) = \det \begin{pmatrix}
\frac{\partial g_1}{\partial u}(u,v) & \frac{\partial g_1}{\partial v}(u,v) \\
\frac{\partial g_2}{\partial u}(u,v) & \frac{\partial g_2}{\partial v}(u,v)
\end{pmatrix}.
\]

For this situation, the joint distribution of \( (U, V) \) is as follows

\[
f_{U,V}(u,v) = \beta^2(\alpha_1 \theta)^{\beta}(\alpha_2 \theta)^{\beta} u^{\beta-1}v^{2\beta-1} \exp[-(\alpha_1 \theta v)^{\beta} - (\alpha_2 \theta uv)^{\beta}].
\]

(12)

By integrating out \( V \) in Equation (12) using the substitution \( x = V^\beta \), the pdf of \( U = T_2/T_1 \) is given by

\[
f_U(u) = \frac{\beta(\xi)^{\beta}u^{\beta-1}}{[1 + (\xi u)^{\beta}]^2} \text{ for } u > 0,
\]

(13)

where \( \xi = \alpha_2/\alpha_1 > 0 \) and \( \beta > 0 \). This is a log-logistic distribution which is a special case of the Burr Type XII distributions. Furthermore, the logistic distribution is obtained by using the transformation, \( D = -\ln U \) where \( D \) has pdf

\[
f(d) = \frac{\beta \exp\{-\beta[d - \ln(\xi)]\}}{[1 + \exp\{-\beta[d - \ln(\xi)]\}]^2} \text{ for } -\infty < d < \infty,
\]

(14)
The logistic distribution obtained above can also be derived using the methodology of Chapter 2. Instead of using the ratio $T_2/T_1$, the statistic, $D = \ln R = \ln T_2 - \ln T_1$, is considered. For $i = 1, 2$, $Y_i = \ln T_i \sim EV (\ln(\xi_i), \delta)$ where $\xi_i = \alpha_i \theta$ and $\delta = \frac{1}{2} \beta$ [36]. The pdf for $Y_i$ is as follows:

$$f(y_i) = (\frac{1}{\delta}) \exp\left(\frac{y_i - \ln(\xi_i)}{\delta}\right) \exp[- \exp\left(\frac{y_i - \ln(\xi_i)}{\delta}\right)]$$ (15)

where $-\infty < y_i < \infty$, $\delta > 0$ and $-\infty < \ln(\xi_i) < \infty$. Since $D$ is the difference between two extreme value random variables, Theorem 1 and Theorem 2 in Chapter 2 imply that $D \sim \text{logistic}\left(\ln(\xi), \frac{1}{2}\right)$ where the density of $D$ is (14) and $\xi = \xi_2/\xi_1$. Next, a parametric method for testing the equality of distributions will be developed for cases when $\beta$ is known or unknown.

### 3.5 A Test for Equality of Distributions

#### 3.5.1 Case 1: $\beta$ Is Known

Testing the equality of the distributions of $T_1$ and $T_2$ is equivalent to testing if the mean for the distribution of $D = \ln(T_2/T_1)$ is equal to zero. Therefore, the null hypothesis $H_0 : \alpha_1 = \alpha_2$ implies that $D \sim \text{logistic}\left(0, \frac{1}{\beta}\right)$. The similarity between the logistic and normal distributions suggests the test statistic,

$$M = \frac{\bar{D}}{\sigma_D \sqrt{n}},$$ (16)
where $\sigma_D$ is the population standard deviation of $D_1, \ldots, D_n$ and $\sigma_D = \pi/(\sqrt{3} \beta)$. Thus, $M$ follows the distribution of the standardized mean of a sample of size $n$ from logistic $(0, \frac{\beta}{3})$.

Goel [18] derived the cumulative distribution function of the standardized mean of samples from a logistic population by applying the Laplace transformation inversion method for convolutions of Polya type functions. In addition, he calculated tables for the cdf for various values of $n$. George and Mudholkar [16] obtained an expression for the distribution of a convolution of independent and identically distributed logistic random variables by directly inverting the characteristic function. Both formulas contain a term $(1 - \exp(x))^{-k}$ for $k = 1, \ldots, n$. When $n$ is large, there is a precision problem for the computation at the values of $x$ near zero.

The distribution can also be approximated using a normal distribution, the student $t$ distribution, the Edgeworth series expansion or Cornish-Fisher series expansion ([16], [20]). In particular, Gupta and Han [20] discussed approximating the distribution by the Edgeworth series expansion correct to order $n^{-\nu/2}$, $\nu = 4, 6, 8$. The critical values for the rejection region using the Edgeworth series expansion to order $n^{-3}$ can be approximated with the following expansion of $F_n(z)$:

$$
F_n(z, \nu = 6) = \Phi(z) - \phi(z)\{\left[ \frac{1}{4!}\left(\frac{6}{5}\right)H_3(z)\right]n^{-1} + \left[ \frac{1}{6!}\left(\frac{48}{7}\right)H_5(z) + \frac{35}{8!}\left(\frac{6}{5}\right)^2H_7(z)\right]n^{-2} + \left[ \frac{1}{8!}\left(\frac{432}{5}\right)H_9(z) + \frac{210}{10!}\left(\frac{48}{7}\right)\left(\frac{6}{5}\right)^2H_7(z)\right]n^{-3}\} + O(n^{-\frac{7}{2}}),
$$

(17)
where $\phi(z)$ and $\Phi(z)$ are the standard normal pdf and cdf respectively and $H_j(z)$'s are the Hermite polynomials of degree $j$, which are orthogonal and can be obtained using the following recursion formula:

$$H_j(z) = xH_{j-1}(z) - (j - 1)H_{j-2}(z), j = 1, 2, 3, \ldots$$

(18)

The first thirty Hermite polynomials are given in Table III in Draper and Tierney [12].

Gupta and Han [20] compared this approximation as well as approximations by the standard normal and the standardized Student's $t$ to the exact distribution given in Goel [18]. This was accomplished by calculating the approximations of the cdf for different values and subtracting the results from the exact values obtained by Goel [18]. They showed that the approximation using the Edgeworth series expansion correct to order $n^{-3}$ is superior to the other two with a maximum error of about 0.0001. Thus, the distribution seen in Equation (16) can be well approximated and critical values for a hypothesis test can be found.

3.5.2 Case 2: $\beta$ Is Not Known

When $\beta$ is unknown, one might consider the test statistic for the null hypothesis $H_0: \alpha_1 = \alpha_2$

$$K = \frac{\bar{D}}{s_D \sqrt{n}},$$

(19)

where $s_D$ is the estimated standard deviation of $D_1, \ldots, D_n$. The null distribution of this test statistic is very complicated, making it necessary to approximate it. If the data were normal, the test statistic's null distribution would be the $t$-distribution. The test would then be the $t$-test. Arnold [1] points out that the $t$-test is asymptotically insensitive to the normal
assumption. The logistic distribution is well behaved in the sense that it is similar to the normal distribution except that it has heavier tails. For these well behaved distributions, the central limit theorem usually becomes valid for relatively small sample sizes, so using a t distribution to approximate the null distribution of $K$ should be adequate.

The parametric bootstrap method is an alternative method for testing $H_0: \alpha_1 = \alpha_2$ when $\beta$ is unknown. Implementation of this algorithm requires the estimation of $\beta$. A simple estimate of $\beta$, $\tilde{\beta}$, can be obtained by calculating

$$\tilde{\beta} = \frac{\bar{z}}{\sqrt{3}s_D}$$  (20)

where $s_D$ is the sample standard deviation of the $D_1, \ldots, D_n$. The parametric bootstrap method is then implemented by performing the following steps:

1. Initialize $m$ (i.e. let $m = 0$).

2. Draw $D_1^*, \ldots, D_n^*$ from logistic $(0, \frac{1}{\tilde{\beta}})$.

3. Calculate the mean of $D_1^*, \ldots, D_n^*$, denoted by $\bar{D}^*$.

4. Compare $\bar{D}^*$ to $\hat{\mu}$, the sample mean of $D_1, \ldots, D_n$ (i.e. $|\bar{D}^*| \geq |\hat{\mu}|$).

5. If the step 3 is true, then $m = m + 1$.

6. Repeat steps 1 through 4 $B$ times.

Once this has been repeated $B$ times, a p-value can be found by calculating $p = m/B$ where $m$ is the number of times a more severe value than the $\hat{\mu}$ is observed.

The alternative estimate for $\beta$ is the mle for $\beta$, $\tilde{\beta}$. This requires implementing an iterative
method such as Newton-Raphson. For this estimate of $\beta$, the parametric bootstrap method is implemented by performing the following steps:

1. Initialize $m$ (i.e. let $m = 0$).

2. Drawing $D_1^*, \ldots, D_n^*$ from logistic $(0, \frac{1}{\beta})$.

3. Calculate the mle for the mean, denoted by $\hat{\mu}^*$.

4. Compare $\hat{\mu}^*$ to $\hat{\mu}$ (i.e. $|\hat{\mu}^*| \geq |\hat{\mu}|$), where $\hat{\mu}$ is the mle for the mean using the observed data.

5. If step 3 is true, then $m = m + 1$.

6. Repeat steps 1 through 4 $B$ times

Once this has been repeated $B$ times, a p-value can be found by calculating $p = m/B$ where $m$ is the number of times a more severe value than $\hat{\mu}$ is observed.

To develop a final method for comparing the survival distributions, the large sample distribution of the maximum likelihood estimates of the parameters for the logistic distribution was considered. Let $D \sim \text{logistic}(\mu, \sigma)$ where $\mu = \ln(\alpha_2/\alpha_1)$ and $\sigma = 1/\beta$. The large sample distribution for the mles of $\mu$ and $\sigma$ is

$$\left(\begin{array}{c} \hat{\mu} \\
\hat{\sigma}
\end{array}\right) \approx_{\text{Normal}} \left(\begin{array}{c} \mu \\
\sigma
\end{array}\right), \frac{1}{n}I(\mu, \sigma)^{-1} \tag{21}$$

where $I(\mu, \sigma)$ is the Fisher's Information matrix as seen in Chapter 2, Equation (12). The testing of the equality of the distributions of $T_1$ and $T_2$ can then be conducted by testing
\( H_0 : \mu = 0 \) versus \( H_a : \mu \neq 0 \). The test statistic is given by

\[
H = \frac{\hat{\mu}}{\frac{\sqrt{3}}{\sqrt{n}\beta}}
\]

where \( \frac{\sqrt{3}}{\sqrt{n}\beta} \) is the asymptotic standard deviation of \( \hat{\mu} \). The null distribution of \( H \) is approximately standard normal so p-values and critical values can easily be obtained. Next, the behavior of these tests will be compared to other parametric tests by investigating Type I error rates in a simulation study.

3.6 Simulation Study

A simulation study of the Type I error was performed comparing the t-test, the parametric bootstrap with \( \hat{\beta} \) (Bootstrap 1), the parametric bootstrap with \( \hat{\beta} \) (Bootstrap 2) and the test based on the asymptotic distribution of the mle. In all cases, the significance level \( \alpha \) was fixed at .05000 while the number of samples, \( n \), and the scale parameter were varied. Here, \( n = 10, 20, \) and 30 and \( \beta = .01, .1, .5, 1, 2, 10, \) and 100. For the parametric bootstrap methods, \( B = 2000 \) iterations were performed. The results of the 10,000 simulations can be found in Table 3.3, 3.4 and 3.5. All of the tests were rejecting more than they should except for several of the t-tests when \( n = 30 \). The Type I error also appears to be invariant over \( \beta \) since there does not seem to be any apparent trends across values of \( \beta \). Overall, the t-test’s observed Type I error was consistently lower than the other tests and was approximately .05 for all cases. Thus, this simulation study indicates that the t-test is superior to the other tests I considered here for the case when \( \beta \) is unknown.
Table 3.3 Observed Type I Error Rates For Sample Size 10

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Table 3.4 Observed Type I Error Rates For Sample Size 20

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Table 3.5 Observed Type I Error Rates For Sample Size 30

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3.7 Bayesian Inference

As with the methods developed above, comparison of the survival distributions of \( T_1 \) and \( T_2 \) at the accelerated condition is accomplished through determining if \( \alpha_1 = \alpha_2 \). This is ascertained by calculating the posterior probability, \( P(\alpha_1 < c \* \alpha_2 \mid data) \) where \( 0 < c \leq 1 \).

However, the real incentive for using Bayesian methods is the capability of estimating \( \alpha_1 \) and \( \alpha_2 \). Estimates for \( \alpha_1 \) and \( \alpha_2 \) can not be obtained using classical methods because of the confounding of the parameters. These estimates are obtained by computing the posterior means, \( E[\alpha_1 \mid data] \) and \( E[\alpha_2 \mid data] \).

Let \( \ln(\alpha_2/\alpha_1) = \ln(\alpha_2) - \ln(\alpha_1) = a_2 - a_1 \). Then, \( D_j \sim \text{logistic}(a_2 - a_1, \beta) \) for \( j = 1, \ldots, n \). Therefore, the likelihood for this model is

\[
L(a_2 - a_1, \beta \mid d_1, \ldots, d_n) = \prod_{j=1}^{n} \frac{\beta \exp\{-\beta[d_j - (a_2 - a_1)]\}}{[1 + \exp\{-\beta[d_j - (a_2 - a_1)]\}]^2}. \tag{23}
\]

Following the Bayesian paradigm, inference is based on the posterior distribution,

\[
p(a_1, a_2, \beta \mid d_1, \ldots, d_n) \propto L(a_2 - a_1, \beta \mid d_1, \ldots, d_n) \* \pi(a_1, a_2, \beta), \tag{24}
\]

where \( \pi(a_1, a_2, \beta) \) is the joint prior for the parameters \( a_1, a_2 \) and \( \beta \). It will be assumed that \( \pi(a_1, a_2, \beta) = \pi_1(a_1) \* \pi_2(a_2) \* \pi_3(\beta) \). Since a published data set is being used for illustrative purposes, prior information is not available. Therefore, locally noninformative priors are being used. Here, the gamma density for \( \pi_3(\beta) \) and the normal densities for \( \pi_1(a_1) \) and \( \pi_2(a_2) \) are chosen due to their flexibility in representing a wide variety of prior beliefs.

The joint posterior distribution for this model is very complicated. In order to sample
from this distribution, sampling based techniques, such as the Gibbs sampler ([15], [45]) were employed. Using priors $\pi_1(a_1) \sim \text{Normal}(\mu_1, \sigma_1^2)$, $\pi_2(a_2) \sim \text{Normal}(\mu_2, \sigma_2^2)$ and $\pi_3(\beta) \sim \text{Gamma}(\psi, \tau)$, the conditional posteriors required to implement the Gibbs sampler are

$$p(a_1 | a_2, \beta, d_1, \ldots, d_n) = \frac{p(a_1, a_2, \beta | d_1, \ldots, d_n)}{p(a_2, \beta | d_1, \ldots, d_n)}$$

$$= \frac{L(a_2 - a_1, \beta | d_1, \ldots, d_n) \ast \pi_1(a_1) \ast \pi_2(a_2) \ast \pi_3(\beta)}{\int L(a_2 - a_1, \beta | d_1, \ldots, d_n) \ast \pi_1(a_1) \ast \pi_2(a_2) \ast \pi_3(\beta) da_1} \ast \beta^n \sqrt{2\pi} \sigma_1 \exp\{-\beta \sum_{j=1}^{n} [d_j - (a_2 - a_1)] - \frac{1}{2} \left( \frac{a_1 - \mu_1}{\sigma_1} \right)^2 \}, \quad (25)$$

and

$$p(a_2 | a_1, \beta, d_1, \ldots, d_n) = \frac{p(a_1, a_2, \beta | d_1, \ldots, d_n)}{p(a_1, \beta | d_1, \ldots, d_n)}$$

$$= \frac{L(a_2 - a_1, \beta | d_1, \ldots, d_n) \ast \pi_1(a_1) \ast \pi_2(a_2) \ast \pi_3(\beta)}{\int L(a_2 - a_1, \beta | d_1, \ldots, d_n) \ast \pi_1(a_1) \ast \pi_2(a_2) \ast \pi_3(\beta) da_2} \ast \beta^n \sqrt{2\pi} \sigma_2 \exp\{-\beta \sum_{j=1}^{n} [d_j - (a_2 - a_1)] - \frac{1}{2} \left( \frac{a_2 - \mu_2}{\sigma_2} \right)^2 \}, \quad (26)$$

and

$$p(\beta | a_1, a_2, d_1, \ldots, d_n) \sim \text{Gamma} \left( n + \psi - 1, \sum_{j=1}^{n} [d_j - (a_2 - a_1)] + \tau \right). \quad (27)$$

To illustrate how the model can be easily adapted by practitioners, a small program was developed using BUGS, see Appendix D. With this software, it is possible to sample from the posterior distributions using a normal prior for $a_1$ and $a_2$ and a gamma prior for $\beta$. BUGS also allows for the computation of the posterior densities of the functions of the parameters. This enables the user to estimate the posterior means of $a_1 = \exp(a_1)$ and $a_2 = \exp(a_2)$. 

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In addition, the posterior probability, \( P(\alpha_1 > \alpha_2 \mid \text{data}) \), is calculated using a step function which takes on the value 1 if \( \alpha_1 - \alpha_2 \geq 0 \) and zero otherwise. The posterior mean of the step function is simply the proportion of iterations for which \( \alpha_1 - \alpha_2 \geq 0 \) (i.e. \( P(\alpha_1 > \alpha_2 \mid \text{data}) \)). This program will be used to analyze a data set in the next section.

### 3.8 Example of the Procedure

To demonstrate the procedures proposed above, the data set that appeared in Gross and Lam [19], see Table 3.1, will be analyzed. The data describes the length of time required for patients with headaches to achieve relief. Each patient receives a standard treatment and a new treatment on separate occasions. Here, the data has not been subjected to an accelerated stress level. The goal of the study is to determine if there is a difference in the mean relief times of the two treatments.

Since \( \beta \) is unknown, a t-test was performed. This results in an approximate p-value of 0.0319, which implies that for \( \alpha = .05 \), the null hypothesis is rejected. There does appear to be a difference in the mean relief times of the treatments.

For the Bayesian analysis, the Unix version of BUGS was used since the Windows version does not support the Logistic distribution. Table 3.6 contains the posterior mean, standard deviation, 2.5% percentile and the 97.5% percentile for the posteriors of \( \alpha_1, \alpha_2, \beta \) and \( \rho \). These estimates were obtained by assuming the locally noninformative priors: \( \pi_1(\alpha_1) \sim \text{Normal}(0, 100), \pi_2(\alpha_2) \sim \text{Normal}(0, 100) \) and \( \pi_3(\beta) \sim \text{Gamma}(0.03, 0.03) \).
Table 3.6 Summary Statistics for $\alpha_1$, $\alpha_2$, $\beta$ and $\rho$

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>standard deviation</th>
<th>2.5% percentile</th>
<th>97.5% percentile</th>
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<td>$\alpha_1$</td>
<td>317.2</td>
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<td>.3623</td>
<td>2116.0</td>
</tr>
<tr>
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<td>292.2</td>
<td>569.7</td>
<td>.3322</td>
<td>1945.0</td>
</tr>
<tr>
<td>$\beta$</td>
<td>14.71</td>
<td>4.157</td>
<td>7.548</td>
<td>23.77</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-.08263</td>
<td>.04126</td>
<td>-.1686</td>
<td>-.005675</td>
</tr>
</tbody>
</table>

The interval between the 2.5% percentile and 97.5% percentile for the posterior of $\rho = \alpha_2 - \alpha_1$ does not include zero. This indicates that there is a difference in the mean relief time for the headache data at a 95% posterior probability. In addition, $P(\alpha_1 > \alpha_2 | data)$ was estimated to be .9815.
Chapter 4

Modeling Data from a Limited Failure Population

4.1 Introduction

Many types of devices (such as integrated circuits) that are operated at normal use conditions can be at risk of failure because of inherent manufacturing faults (called latent risk factors). After production, units that are obviously defective are removed from the production lot via inspection or burn in, but a small proportion of undetected defective units, \( p \), may remain. Under normal operating conditions, these units fail over time according to a time to failure distribution \( F(t) \). For the non-defective units, the probability that they fail under normal conditions during their "technological lifetime" is essentially zero. Meeker ([29], [31]) called this type of population a limited failure population (LFP). To model this behavior, Meeker ([29], [31]) suggests using a mixture distribution, where a random unit from an LFP has the cumulative distribution function (cdf), 

\[
P(T \leq t) = G(t | p) = pF(t).
\]

Notice that as \( t \to \infty \), the cdf \( G(t | p) \) approaches \( p \). For a reliability engineer whose product follows such a model, knowledge of the parameter \( p \) and the time-to-failure distribution is important for evaluating the manufacturing process (quality control) and the design of the product [29].

Meeker [29] points out that inference from a life test on a LFP is difficult. Some defective units may not fail by the end of the test. These defective units can not be physically distinguished from non-defective units (which will never fail), making it difficult to estimate \( p \). In addition, life tests also need to be run long enough so that at least a certain proportion
of the defective units fail. Otherwise, \( p \) may become confounded with the parameters of the distribution \( F(t) \). In this situation, ALTs can be very useful in order to draw inferences on the proportion of defective parts and the lifetime distribution under normal operating conditions. The accelerated conditions can ensure that most of the defective units will fail without having to conduct lengthy life tests. In the literature, much has been done in the way of developing methodology for life tests but there has been little research in applying ALTs to the LFP.

Meeker and LuValle [33] applied their LFP model to an ALT data set. This data set is the result of a humidity-accelerated life test on printed circuit boards (see Appendix B). They noted that traditional approaches of analysis such as fitting Weibull distribution and other lifetime models do not accurately predict the time-to-failure distribution of circuit boards operating at normal conditions. Lifetime models, such as the Weibull, assume that \( P(T < +\infty) = 1 \). However, a significant fraction of circuit boards under a normal humidity (stress) level remain virtually failure free during their technological life (i.e. \( P(T = \infty) > 0 \)). It was shown in Chapter 1 that a plot of the observations versus the ECDF (Figure1-1) supports the assumption that \( P(T = \infty) > 0 \). Meeker and LuValle [33] fit the following LFP regression model to the first three relative humidity levels of the printed circuit board data

\[
F_{T}(t \mid \beta_0^\mu, \beta_1^\mu, \beta_0^p, \beta_1^p, \sigma) = p(x) \Phi \left[ \frac{\log(t) - \mu(x)}{\sigma} \right],
\]

(1)

where \( x \) denotes the relative humidity level,

\[
\mu(x) = \beta_0^\mu + \beta_1^\mu \logit(x/100),
\]

(2)
\[
\text{logit}(p(x)) = \beta_0 + \beta_1 \text{logit}(x/100),
\]

and \( \Phi \) is the cdf for the standard smallest extreme distribution. Since Meeker and Lu-Valle [33] felt that there was no physical basis for this model, they also proposed an ALT model based on the chemical kinetic models of the failure process. This model is very specific to their example and cannot be generalized to any other limited failure population. Their statistical kinetic regression model is as follows

\[
F_T(t \mid \beta_0^{k_t}, \beta_1^{k_t}, \beta_0^{k_t}, \beta_1^{k_t}, \sigma) = \Phi \left[ -\frac{\log[k_1 + k_2(1 - \exp[-(k_1 + k_2)t])^{-1} - 1] + 6]}{\sigma} \right],
\]

where \( \Phi \) is the standard smallest extreme value (or normal) cdf,

\[
k_1 = \exp[\beta_0^{k_t} + \beta_1^{k_t} \text{logit}(x/100)]
\]

and

\[
k_2 = \exp[\beta_0^{k_t} + \beta_1^{k_t} \text{logit}(x/100)].
\]

The model does capture the leveling off of the ECDF's at the lower relative humidity levels

and does provide a slightly better fit than the LFP regression model (see Equation (1)). However, the model does not fit the early failures at all. At the 75.4% relative humidity level, they point out that the results are essentially the same as one would obtain fitting a Weibull distribution to the data.

In this chapter, a new model is presented for a LFP which investigates how the number of latent risk factors and the times at which they become fatal are dependent on the stress

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level. Both Bayesian and classical methods of analysis are developed for this model. It is shown that the Bayesian approach can be very useful since an engineer will have expert knowledge of the manufacturing process. Therefore, it is possible to find a useful prior for the relationship between the expected number of latent risk factors under the normal stress level and also under accelerated stress levels. Proper use of such prior information can reduce the cost of ALT without sacrificing the level of accuracy in prediction by avoiding testing too many units. For illustrative purposes, the data set given in Meeker and LuValle [33] (see Appendix B) is analyzed and will be described in the next section.

4.2 ALT on Printed-Circuit-Boards

For this experiment, there were 72 circuit boards tested at each of four stress levels, 49.5% RH, 62.8% RH, 75.4% RH and 82.4% RH. Meeker and LuValle [33] noted that due to problems with the test equipment, there were several circuit boards that did not yield useful information. Therefore, the resulting data set consisted of 70 boards at the stress levels of 49.5% RH, 75.4% RH and 82.4% RH and 68 boards at the 62.8% RH level. The boards were monitored periodically for failures, so the data is interval censored. Meeker and LuValle [33] used the midpoint of the interval to represent the time to failure for a unit that failed in that interval, as suggested by Nelson [36]. This representation will also be used in this thesis. There were also several circuit boards that did not fail, resulting in right-censored data. In particular, there were 48 censored observations at 4,078 hours in the 49.5% RH and 11 at 3,067 hours at the 62.8% RH. All of the circuit boards failed at 75.4% RH and 82.4% RH. Furthermore, the original data set was rescaled for the analysis so that the failure time of a printed circuit board is reported in weeks.
The experiment was designed to study a specific failure mode, the formation of conductive anodic filaments (salt bridges) between the copper plated-through holes (CPTH) in the printed circuit boards. Due to deficiencies during the manufacturing process, chlorine salts remain in the circuit boards. With the presence of moisture, heat, electrical charge and salt, anions are created which react with copper ions to produce copper compounds. These copper compounds precipitate from solution and form conductive filaments that grow from the positively charged CPTH towards the negatively charged CPTH. These failure causing conductive filaments (salt bridges) can be thought of as the latent risk factors. The number of potentially fatal conductive filaments depends upon (among other factors) the relative humidity level (or stress level).

A proper ALT model should be able to identify the effects of the different stress levels on the latent risks and the times at which they become fatal. The circuit board data is an adequate example of an ALT data set that involves failures caused by latent risk factors (often manufacturing faults). The model proposed here can be useful when dealing with an ALT involving products with an unknown number of potentially fatal manufacturing faults. In practice, a statistician involved in the manufacturing process would expect to have useful prior information on the nature and extent of these two types of effects of accelerated stress level on the failure process.

4.3 Model

Let \( n_h \) denote the number of circuit boards tested at the \( h \)th relative humidity level, \( h = 1, 2, 3, 4 \), and \( M_{ih} \) be the number of potentially fatal unobservable salt bridges (latent risk factors) on the \( i \)th circuit board at the \( h \)th relative humidity level, where \( i = 1, \ldots, n_h \).
In addition, let $X_{ijh}$ represent the time when the $j$th salt bridge (latent risk factor) on the $i$th circuit board at the $h$th relative humidity level becomes fatal, where $j = 1, \ldots, M_{ih}$ and $i = 1, \ldots, n_h$. It is assumed that $P(X_{ijh} < +\infty) = 1$. Thus, $T_{ih} = \min\{X_{1ih}, \ldots, X_{M_{ih}h}\}$ is the observable time to failure of the $i$th circuit board at the $h$th relative humidity level.

It should be noted that when $M_{ih} = 0$, $T_{ih} = \infty$. This indicates that it is possible that some of the circuit boards may never fail. In addition, the following assumptions will be made about the distributions for $X_{ijh}$ and $M_{ih}$.

**Assumptions**

1. $X_{ijh}$ are independent and identically distributed with continuous cdf $F^*_h$ for $j = 1, \ldots, M_{ih}$, $i = 1, \ldots, n_h$ and $h = 1, 2, 3, 4$.

2. $F^*_h$ is a proper cdf (i.e. as $t \to \infty$, $F^*_h(t) = 1$).

3. $M_{ih}$ are iid Poisson with parameter $\theta_h$ for $i = 1, \ldots, n_h$ and $h = 1, 2, 3, 4$.

As noted above, one of the properties of this model is that a proportion of the circuit boards will not fail. This is reflected in the survival function of $T_{ih}$.

**Model**

For $h = 1, 2, 3, 4$,

$$S(t) = P(T_{ih} > t) = \exp(-\theta_h F^*_h(t)). \quad (7)$$

It is easily seen that $\lim_{t \to \infty} S(t) = \exp(-\theta_h)$. The quantity $\exp(-\theta_h)$ represents the
proportion of circuit boards that will not fail (risk free proportion) at the $h$th relative humidity level. The hazard function for this model is

$$\lambda(t \mid h) = \theta_h f_h^*(t)$$

which has the proportional hazards structure. In order to accommodate right censoring, it is necessary to present the following notation. For each relative humidity level $h$, let $y_h = (y_{1h}, \ldots, y_{nh})$ and $\delta_h = (\delta_{1h}, \ldots, \delta_{nh})$ where

$$y_{ih} = \begin{cases} T_{ih} & \text{if } T_{ih} < C_h \\ C_h & \text{if } T_{ih} \geq C_h \end{cases}$$

$C_h$ is the censoring time for the $h$th relative humidity level, and

$$\delta_{ih} = \begin{cases} 1 & \text{if } T_{ih} = Y_{ih} \\ 0 & \text{otherwise} \end{cases}$$

Hence for $h = 1, 2, 3, 4$, the likelihood for $\theta_h$ and $F_h^*$ is

$$L(\theta_h, F_h^* \mid y_h, \delta_h) \propto \prod_{i=1}^{nh} \left\{ \exp[-\theta_h F_h^*(y_{ih})] \right\}^{1-\delta_{ih}} \left\{ \theta_h f_h^*(y_{ih}) \exp[-\theta_h F_h^*(y_{ih})] \right\}^{\delta_{ih}}.$$  \hspace{1cm} (9)

### 4.3.1 Model 1

Assuming that $X_{ijh}$ follows an exponential distribution with parameter $\lambda_h$, for $h = 1, 2, 3, 4$, the likelihood for this model is given by
\[ L(\theta_h, \lambda_h \mid y_h, \delta_h) \propto \theta_h^{d_h} \lambda_h^{d_k} \exp\left\{ -\lambda_h \sum_{i=1}^{n_h} y_{ih} \delta_{ih} - \theta_h \sum_{i=1}^{n_h} [1 - \exp(-\lambda_h y_{ih})] \right\}, \]  

where \( d_h = \sum_{i=1}^{n_h} \delta_{ih} \). Under the Bayesian paradigm, inferences should be based on the posterior

\[ p(\theta_h, \lambda_h \mid y_h, \delta_h) \propto L(\theta_h, \lambda_h \mid y_h, \delta_h) * \pi(\theta_h, \lambda_h) \]  

where \( \pi(\theta_h, \lambda_h) \) is the joint prior for the parameters \( \theta_h \) and \( \lambda_h \). Here, proper priors are used for \( \pi(\theta_h, \lambda_h) \) and it is further assumed that \( \pi(\theta_h, \lambda_h) = \pi_1(\theta_h) \pi_2(\lambda_h) \).

The joint posterior distribution for this model is very complicated. Hence, sampling based methods, such as the Gibbs sampler ([15], [45]), to sample from the joint distribution are employed. Using priors \( \pi_1(\theta_h) \sim \text{Gamma}(a_h, b_h) \) and \( \pi_2(\lambda_h) \sim \text{Gamma}(u_h, v_h) \), the conditional posteriors required for implementing the Gibbs algorithm are as follows.

\[ p(\theta_h \mid \lambda_h, y_h, \delta_h) \sim \text{Gamma}\left( d_h + a_h, b_h + \sum_{i=1}^{n_h} [1 - \exp(-\lambda_h y_{ih})] \right) \]  

and

\[ p(\lambda_h \mid \theta_h, y_h, \delta_h) \propto \lambda_h^{d_h + u_h - 1} \exp\left\{ -\lambda_h (v_h + \sum_{i=1}^{n_h} y_{ih} \delta_{ih}) + \theta_h \sum_{i=1}^{n_h} \exp(-\lambda_h y_{ih}) \right\}. \]

Sampling \( \theta_h \) from \( p(\theta_h \mid \lambda_h, y_h, \delta_h) \) is straightforward. It is obviously more difficult to sample from \( p(\lambda_h \mid \theta_h, y_h, \delta_h) \). Alternative algorithms can be implemented such as importance sampling [45] or the adaptive rejection algorithm of [17] provided the density and \( \pi_1(\theta_h) \) is log-concave.
Since a published data set is being used for illustrative purposes, it is not possible to obtain information on the manufacturing process. Hence, priors that are locally noninformative have to be used. More specifically, gamma densities for $\pi_1(\theta_h)$ and $\pi_2(\lambda_h)$ are used because they are flexible enough to represent a wide variety of prior beliefs.

The analysis of this data set can be viewed only as a demonstration of the proposed methodology. In practical applications, a statistician with access to the manufacturing site can develop a realistic informative prior for $\theta_h$ using his/her prior information on the expected number of manufacturing faults in any unit. One method for obtaining an informative prior is by using failure mode analyses such as optical and scanning electron microscopy and x-ray-dispersive spectroscopy on a failed unit. If a circuit board is partitioned into sections of equal width, the number of conductive filaments present in a single section of a failed unit can be counted using failure mode analyses. By multiplying the number of conductive filaments in a partition by the number of partitions, an estimate for the expected number of latent risk factors is obtained. Similarly, any prior knowledge on the incubation distribution of each of these faults can be used to model a prior for $\lambda_h$. Moreover, the parameters $\lambda_h$ and $\theta_h$ have useful physical interpretations for manufacturers and reliability engineers which enable them to develop priors on them. Another alternative is to use automatic prior elicitation using stage-o data [6].

A small program was developed using BUGS. The main reason for using a BUGS program is to demonstrate how easily these methods can be adapted by practitioners in industry for similar problems using existing software packages. With this software package, it is possible to sample from the posterior distributions using the gamma priors for $\lambda_h$ and $\theta_h$, $h = 1, 2, 3, 4$. 

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The BUGS program that was implemented is a simple variation of an example found in the examples manual of BUGS [44]. The example is concerned with the Bayesian inference of the Cox's model with the intensity function

$$I_i(t) = Y_i(t)\lambda_0(t)\exp(\beta'z_i),$$  \hspace{1cm} (14)\]

where $Y_i$ is the observed process taking the value of 1 or 0 according to whether or not object $i$ is observed at time $t$ and $\lambda_0(t)\exp(\beta'z_i)$ is the familiar Cox regression model. Here, the baseline hazard function is

$$\lambda_0(t) = \theta_h\lambda_h\exp(-\lambda_h t).$$  \hspace{1cm} (15)\]

Each relative humidity level was fit separately, since they were assumed to be independent (see Appendix E for the program at the 49.4%RH).

Table 4.1 and Table 4.2 contain the results of the implementation of the BUGS program for the printed circuit board data. The posterior means, standard deviations, 2.5% percentile and the 97.5% percentile for $\lambda_h$ and $\theta_h$ are reported for each relative humidity level. The values for $\theta_h$ increase as the relative humidity level increases implying that on average there are more active salt bridges at the higher relative humidity levels. The values of $\lambda_h$ are also decreasing implying that on average the time until one of the salt bridges becomes fatal is increasing. This was not an expected result. Upon further discussion, it was realized that the parameter space is restricted so that $0 < \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4 < \infty$ and $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 < \infty$. A model with a restricted parameter space can be analyzed using Bayesian
methods. However, the implementation of a program that will handle this situation is quite complicated. The general idea behind the implementation of such a program would be to run BUGS in conjunction with a Rejection/Acceptance algorithm.

4.3.2 Model 2

Model 2 is a simplification of Model 1, where it is assumed that $\lambda_h = \lambda$ for $h = 1, 2, 3, 4$ and $\theta_h$ is allowed to vary with humidity. This implies that the humidity level does not affect the distribution of the time to which the conductive filaments become fatal. For this model, the likelihood function is

$$L_2(\lambda, \theta, \delta | y, \delta) \propto \prod_{h=1}^{4} L(\theta_h, \lambda | y_h, \delta_h),$$  \hspace{1cm} (16)

where $L(\theta_h, \lambda | y_h, \delta_h)$ is the likelihood in Section 3.1 with $\lambda_h = \lambda$ for $h = 1, 2, 3, 4$, $y = (y_1, \ldots, y_4)$, $\delta = (\delta_1, \ldots, \delta_4)$ and $\theta = (\theta_1, \ldots, \theta_4)$. Again, proper priors are used so that
Using the priors $\pi_0(\lambda) \sim \text{Gamma}(a, b)$ and $\pi_h(\theta_h) \sim \text{Gamma}(u_h, v_h)$ for $h = 1, 2, 3, 4$, the conditional posterior distributions needed for the implementation of the Gibbs algorithm are as follows. For $h = 1, 2, 3, 4$,}

\[
p(\lambda \mid \theta, y, \delta) \propto \lambda^{d+\lambda-1} \exp[-\lambda(b + \sum_h \sum_i y_{ih} \delta_{ih})] \exp\left[\sum_h \theta_h \sum_i \exp(-\lambda y_{ih})\right]
\]

(17)

and

\[
p(\theta_h \mid \lambda, \theta_{(-h)}, y_h, \delta_h) \propto \text{Gamma}(d_h + u_h, v_h + \sum_i [1 - \exp(-\lambda y_{ih})])
\]

(18)

where $d = \sum_h \sum_{i=1}^{n_h} \delta_{ih}$ and $\theta_{(-h)}$ is the vector of the $\theta$ parameters excluding $\theta_h$. It is important to note that in practice, the statistician needs to elicit the prior for $\lambda$ using the prior knowledge on the incubation time of a manufacturing fault at any stress level. Using a slightly modified version of the previous BUGS program (see Appendix F), the posterior mean, standard deviation, the 2.5% percentile and the 97.5% percentile were obtained for $\lambda, \theta_1, \theta_2, \theta_3$ and $\theta_4$ (see Table 4.3). For this model, $\pi(\lambda) \sim \text{Gamma}(0.377, 1)$ and $\pi_1(\theta_1) \sim \text{Gamma}(2, 8)$. 

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>SD</th>
<th>2.5% CI</th>
<th>97.5% CI</th>
</tr>
</thead>
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<td>$\theta_1$</td>
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<td>$\theta_2$</td>
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<td>$\theta_3$</td>
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</tr>
<tr>
<td>$\theta_4$</td>
<td>39.56</td>
<td>18.21</td>
<td>13.01</td>
<td>82.33</td>
</tr>
</tbody>
</table>
4.4 Model Assessment and Selection

In order to assess model adequacy, a graphical approach based on the conditional predictive ordinate (CPO) [14] was employed. Formally, the CPO for the uncensored $i$th observation at the $h$th relative humidity level ($y_{ih}$) is defined as the cross validated predictive density $f_h(y_{ih} \mid Y_{(-i)}) = E[f_h(y_{ih} \mid \lambda_h, \theta_h) \mid Y_{(-i)})$, where $Y_{(-i)}$ stands for the rest of the data after deleting the $i$th observation. Due to a simplification by Gelfand, Dey and Chang [14], the CPOs from MCMC samples from the posterior given $Y$ can be easily computed, using

$$CPO(y_{ih}) = \left\{ E \left[ \{\lambda_h \theta_h \exp(-\lambda_h y_{ih})\}^{-1} \exp\left( \theta_h [1 - \exp(-\lambda_h y_{ih})] \right) \mid Y \right] \right\}^{-1}. \quad (19)$$

The CPO plot for all $i = 1, \ldots, n_h$ and $h = 1, 2, 3, 4$ represents the influence of the $i$th observation. A large CPO indicates agreement between the observation and the model. The plot for Model 1 can be found in Figure 4-1.
For the higher relative humidity levels, the CPO values are only slightly decreasing over time whereas the decrease for 49.5% RH is more pronounced. Overall, it appears that Model 1 fits the data better at the beginning of a board's lifetime. Also, there does not appear to be any highly influential observations. Similar behavior is also found in the plot of the CPO's for Model 2 (see Figure 4-2).

In order to compare the two models, a cross validation approach based on the CPO is used. Large values of the ratio

$$R_{ih}(1, 2) = \frac{CPO_1(ih)}{CPO_2(ih)}$$

indicate that the $i$th observation supports Model 1 over Model 2. Figure 4-3 is the plot of the ratios versus the observation number. From the plot, it appears that the observations support Model 1.

Sensitivity analysis should always be performed to see how the choice of priors is affecting
the analysis. Because the above assessment appears to support Model 1 as a viable model, it was decided to perform a sensitivity analysis on this model. The results in Table 4.4 and Table 4.5 show that the analysis is sensitive to the choice of priors especially at the higher levels. This is not surprising since a plot of the likelihood at 75.4% RH showed that it was multimodal. In practice, a statistician could use an engineers knowledge to choose priors so that reasonable estimates of the parameters were attained.

4.5 Maximum Likelihood Estimation

If it is assumed that \( X_{ijh} \) follows an exponential distribution with parameter \( \lambda_h \) for \( h = 1, \ldots, 4 \), the likelihood for this model is given by (10). The maximum likelihood estimators of \( \theta_h \) and \( \lambda_h \), \( \hat{\theta}_h \) and \( \hat{\lambda}_h \), are given by the solution of the following equations.

\[
\hat{\theta}_h = \frac{d_h}{\sum_{i=1}^{n_h} [1 - \exp(-\hat{\lambda}_h y_{ih})]}
\]
### Table 4.4 Sensitivity Analysis for Model 1: $\theta$

<table>
<thead>
<tr>
<th>RH</th>
<th>Prior for $\theta$</th>
<th>Prior for $\lambda$</th>
<th>$\theta$</th>
<th>$SD(\theta)$</th>
<th>2.5%</th>
<th>97.5%</th>
</tr>
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<tbody>
<tr>
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and

\[
\frac{d_{ih}}{\lambda_{ih}} + \sum_{i=1}^{n_{ih}} [\hat{\theta}_{ih} y_{ih} \exp(-\hat{\lambda}_{ih} y_{ih}) - y_{ih} \delta_{ih}] = 0.
\]

The second equation is nonlinear, so numerical techniques, such as the Newton-Raphson method, need to be employed in order to obtain the maximum likelihood estimators of \( \theta_{ih} \) and \( \lambda_{ih} \). To implement the Newton-Raphson method, initial values need to be obtained. For the first two relative humidity levels of 49.5% RH and 62.8% RH, the initial values for \( \theta \) and \( \lambda \) can be computed from the data. An initial value for \( \theta \) is obtained by calculating the negative of the logarithm of the proportion of circuit boards that do not fail, while \( \lambda \) can be estimated using quantiles. Initial values can not be obtained for the higher relative humidity levels since all of the circuit boards fail. Upon further investigation, the likelihoods at the higher relative humidity levels were found to be multimodal implying that the estimates of \( \theta \) and \( \lambda \) are not unique and highly dependent on the initial values. The problems highlighted here indicate that classical methods of analysis are not appropriate for this model.
Chapter 5

Conclusions and Future Research

In this thesis, I have developed methods for analyzing some of the non-traditional data sets obtained from accelerated life tests. It has been shown that these methods can be implemented by reliability engineers and biostatisticians in the field using existing software packages. Here, conclusions and future research directions are given for each of these different types of data.

5.1 Two Independent Samples

A Weibull model for two sample ALT data, as seen in Chapter 2, Equation (3) and Equation (4), is proposed, which the exponential model of Zimmer and Deely [47] is a special case. Three methods for comparing the failure distributions at normal operating conditions have been developed. Each of the methods have been demonstrated by reanalyzing data from Zimmer and Deely [47].

Future research would include comparing the power of the parametric test to the test based on the PL statistic for different sample sizes. The focus here is in determining how much power is lost when the PL method is used. The PL statistic is based on the ranks of the failure times and is less powerful than parametric tests especially when sample sizes are small. However, the parametric test that was developed in Chapter 2 is based on the minimum of the failure times which results in a loss of information and power. Comparison of these methods by performing a simulation study on the power of these tests will be
conducted in future research. Other areas of interest include the development of models and methodologies incorporating other acceleration models as well as developing methods which can accommodate multicomparisons (i.e. more than 2 suppliers).

5.2 Paired Samples

A bivariate Weibull model is proposed that is applicable to ALTs when it is assumed that the behavior of the units on test at the accelerated level is indicative of what will occur during normal operating conditions. Testing the equality of the two component failure (survival) distributions was shown to be adequately approximated by the t-test. I have shown that the Bayesian analysis can be easily implemented using MCMC simulations via existing software such as BUGS.

This model should also be extended to accommodate multivariate data. The specific details of this extension are currently being researched. In addition, the following questions are open for further investigation:

- How can the testing procedures in Chapter 3.5 and 3.7 be modified to accommodate censoring?

- Can the assumption that the shape parameter $\beta$ is the same for the conditional Weibull distributions of $T_1$ and $T_2$, as seen in Chapter 3, Equation (6), be relaxed?

- Can these methods be extended to accommodate data from more than one stress level?
5.3 Limited Failure Population

A new model was presented that investigates the effect of the accelerated stress on the number of latent risk factors and on the times at which these risk factors become fatal to the unit. It has been shown that Bayesian analysis using MCMC techniques are feasible and associated model assessment methods are demonstrated.

Further research includes the consideration of more flexible models than the model in Chapter 4, Equation (7) and the comparison of these models. In particular, other distributions, such as the Weibull or lognormal distribution, for modeling the time at which a latent risk factor becomes fatal will be considered. Other extensions of this model are presently being explored. The plot of the observations versus the ECDF (Figure 1-1) suggested that the time at which a latent risk becomes fatal may be different for the lower relative humidity levels and higher relative humidity levels. To further explore this issue, Weibull and exponential Kaplan-Meier estimates of the cdf were plotted for all relative humidity levels. This plot indicated that the assumption of the exponential distribution for the time at which a latent risk became fatal is appropriate for the lower relative humidity levels. But, a Weibull distribution seems more appropriate for the 75.4% RH and 82.4% RH. Another possibility is that there are two types of latent risks. For the lower relative humidity levels, the printed circuit boards follow the proposed model in Chapter 4, Equation (7). At higher humidity levels, there is a second type of latent risk which also cause failures. This model can be written as an additive hazard form

\[ \lambda_A(t) = \lambda_{1A}(t) + \lambda_{2A}(t) \]  

(1)
where $\lambda_{1h}(t)$ can be modeled as in Chapter 4 Equation (8) and $\lambda_{2h}$ can be modeled separately. A useful choice for $\lambda_{2h}(t)$ is $\lambda_{2h}(t) = \lambda_2 \cdot 1_{\{t > \eta_h\}}$ where $\eta_h > 0$ is a threshold parameter which depends on the relative humidity level.

Also, suitable regression models where the $\theta$'s and the $\lambda$'s are functions of the stress level $h$ and $\theta(h)$ and $\lambda(h)$ are known except for the regression parameters are presently being explored. Certain model selection procedures such as the Bayes factor and the $L$-criterion (see Sinha and Dey 1997, for a review of model selection methods in reliability) can be used. Lastly, the data from Meeker and LuValle (1995) was interval censored. Neither I nor they address this issue. Sinha, Chen and Ghosh (1997) showed it is possible to handle interval censoring very effectively from survival/reliability data in a Bayesian framework. A long term research goal is to incorporate modeling procedures for LFP data under interval censoring.
Bibliography


Appendices
## Appendix A

### Times to Breakdown of an Insulating fluid in Minutes

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# Appendix B

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Appendix C

BUGS Program for Zimmer and Deely Data Set

#Zimmer and Deely BUGS Program for Unix
model itch

const

N=1; #number of observations

var

   d[N], #observed data
   rho, #location parameter
   beta; #scale parameter

data d in "Bugs/voll/itch/itch.bug";
inits in "Bugs/voll/itch/itch.in";

{
   #model

   for(i in 1:N)
   {
      d[i] ~ dlogis(rho,beta);
   }
   #priors

   beta ~ dgamma(.03,.03);
   rho ~ dnorm(0,.01);
}
Appendix D

BUGS Program for Gross and Lam Data Set

#Gross and Lam BUGS Program for Unix
model relief

const

N=10; #number of observations

var

d[N], #observed data
alpha1, #estimate for alpha1
alpha2, #estimate for alpha1
a1, #ln(alpha1)
a2, #ln(alpha2)
rho, #location parameter
beta, #scale parameter
prob; #posterior probability

data d in "Bugs/voll/relief/relief.bug";
inits in "Bugs/voll/relief/relief.in";

{
    #model

    for(i in 1:10)
    {
        d[i]~ dlogis(rho,beta);
    }
a1~dnorm(0,.01);
a2~dnorm(0,.01);
rho<-a2-a1;
    #priors

    beta~dgamma(.03,.03);

    alpha1<- exp(a1);
    alpha2<- exp(a2);
    prob <- step(alpha1-alpha2);
}
Appendix E

BUGS program for Model 1 at 49.5%RH

# WINBUGS Program for Model 1 49.5% RH level
# Initial value file circuit.in theta=.3773 and Lambda=.0006
# Data file is circuit1.dat on Valencia disk
# Data file in hours is circuit1h.dat on valencia disk
{
  # Set up data

  for(i in 1:N) {
    for(j in 1:T) {
      # risk set = 1 if obs.t >= t
      Y[i,j] ← step(obs.t[i]-t[j] + eps);

      # counting process jump = 1 if obs.t in [ t[j], t[j+1] )
      # i.e. if t[j] <= obs.t < t[j+1]
      dN[i, j] ← Y[i, j] * step(t[j + 1] - obs.t[i] - eps) * fail[i];
    }
  }

  # Model
  for(j in 1:T) {
    for(i in 1:N)
      dN[i, j] ~ dpois(Idt[i, j]); # Likelihood
      Idt[i, j] ← Y[i, j] * theta * lambda * exp(-lambda * t[j]); # Intensity
  }

  # Priors
  theta ~ dgamma(4,8);
  lambda ~ dgamma(2,27);

  # Model Checking
  for(i in 1:N){
    like[i] ← pow(theta * lambda * exp(-lambda * obs.t[i]), fail[i]) * exp(-theta * (1-exp(-lambda * obs.t[i])));
    p.inv[i] ← 1/like[i];
  }
}
Appendix F

BUGS program for Model 2 for Printed Circuit Boards

#Unix BUGS Program for Model 2
model circuit;

const

N =278, # number of circuit board
T = 137,
eps = 0.000001; # used to guard against numerical imprecision in step function

var

obs.t[N], # observed failure or censoring time for each patient
t[T+1], # unique failure times + maximum censoring time
dN[N,T], # counting process increment
Y[N,T], # 1=subject observed; 0=not observed
Idt[N,T], # intensity process
fail[N], # failure = 1; censored = 0
Z1[N], # covariate
Z2[N], # covariate
Z3[N], # covariate
beta1, # regression coefficient
beta2, # regression coefficient
beta3, # regression coefficient
theta1, # Poison parameter for level 1
theta2, # Poison parameter for level 2
theta3, # Poison parameter for level 3
theta4, # Poison parameter for level 4
lambda, # parameter of exponential distribution
like[N-59],
p.inv[N-59];

data obs.t, fail, Z1, Z2, Z3 in "circuitM2.dat", t in "cfailtimeM2.dat";
inits in "circuit.in";
{  
  # Set up data
  for(i in 1:N) {
    for(j in 1:T) {
      # risk set = 1 if obs.t i= t
      Y[i,j] <- step(obs.t[i] - t[j] + eps);

      # counting process jump = 1 if obs.t in [ t[j], t[j+1] )
      # i.e. if t[j] <= obs.t < t[j+1]
      dN[i,j] <- Y[i,j]*step(t[j+1] - obs.t[i] - eps)*fail[i];
    }
  }

  # Model
  for(j in 1:T) {
    for(i in 1:N) {
      dN[i,j] ~ dpois(Idt[i,j]); # Likelihood
      Idt[i,j] = Y[i,j]*exp(betal*Z1[i]+beta2*Z2[i]+beta3*Z3[i])*thetal*lambda*
               exp(-lambda*t[j]); # Intensity
    }
  }

  # Priors
  lambda ~ dgamma(1.5,1);
  thetal ~ dgamma(2,2);
  betal ~ dunif(0.1,1);
  beta2 ~ dnorm(0,1);
  beta3 ~ dnorm(0,1);

  # Estimating theta2, theta3 and theta4
  theta2 ~ exp(betal)*thetal;
  theta3 ~ exp(beta2)*thetal;
  theta4 ~ exp(beta3)*thetal;
}
# Model Checking

```r
for(i in 1:22) {
    like[i] <- pow(theta1*lambda*exp(-lambda*obs.t[i]*fail[i])*exp(-theta1*(1-exp(-lambda*obs.t[i]))));
    p.inv[i] <- 1/like[i];
}
for(i in 71:127) {
    like[i-48] <- pow(theta2*lambda*exp(-lambda*obs.t[i]*fail[i])*exp(-theta2*(1-exp(-lambda*obs.t[i]))));
    p.inv[i-48] <- 1/like[i-48];
}
for(i in 139:208) {
    like[i-59] <- pow(theta3*lambda*exp(-lambda*obs.t[i]*fail[i])*exp(-theta3*(1-exp(-lambda*obs.t[i]))));
    p.inv[i-59] <- 1/like[i-59];
}
for(i in 209:278) {
    like[i-59] <- pow(theta4*lambda*exp(-lambda*obs.t[i]*fail[i])*exp(-theta4*(1-exp(-lambda*obs.t[i]))));
    p.inv[i-59] <- 1/like[i-59];
}
```

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