Energy flow and non-guiding center effects in the Earth's magnetotail

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Energy flow and non-guiding center effects in the Earth's magnetotail

Abstract
The purpose of this study is to investigate energy transfer processes in the Earth's magnetotail. An equilibrium, two-dimensional magnetic field model is used to simulate the mid-tail region. Groups of particles are traced in the model magnetic field and the groups are combined to generate a self-consistent two-dimensional current sheet. Two thin current sheets are generated with thicknesses characteristic of a substorm growth phase. One current sheet contains Speiser-type particles only and the other one contains mostly Speiser-type and some trapped particles. It is found that the two current sheets show differences in energy flow processes and pressure anisotropies due to the different nature of orbits that they support. The generalized pressure equation is also investigated. It is found that shearing effects are very important in terms of power flow in the region. The heat flow and most energy flow processes as well as pressure tensor elements have a strong spatial dependence as one moves away from the equatorial plane. Very near the equatorial plane, particles do not obey the guiding center approximations and non guiding center effects determine the energetics of the current sheet.

Keywords
Physics, Fluid and Plasma, Geophysics

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ENERGY FLOW AND NON-GUIDING CENTER EFFECTS IN THE
EARTH'S MAGNETOTAIL

BY

IOANNIS DIMITRIOS KONTODINAS
B.S. Physics, University of Arizona, 1992
M.S. Physics, University of New Hampshire, 1994

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirement for the Degree of

Doctor of Philosophy

in

Physics

May, 1998
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This dissertation is dedicated to

my brother Georgios
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ABSTRACT

ENERGY FLOW AND NON-GUIDING CENTER EFFECTS
IN A THIN, TWO-DIMENSIONAL CURRENT SHEET

by

Ioannis Dimitrios Kontodinas

University of New Hampshire, May, 1998

The purpose of this study is to investigate energy transfer processes in the Earth’s magnetotail. An equilibrium, two-dimensional magnetic field model is used to simulate the mid-tail region. Groups of particles are traced in the model magnetic field and the groups are combined to generate a self-consistent two-dimensional current sheet. Two thin current sheets are generated with thicknesses characteristic of a substorm growth phase. One current sheet contains Speiser-type particles only and the other one contains mostly Speiser-type and some trapped particles. It is found that the two current sheets show differences in energy flow processes and pressure anisotropies due to the different nature of orbits that they support. The generalized pressure equation is also investigated. It is found that shearing effects are very important in terms of power flow in the region. The heat flow and most energy flow processes as well as pressure tensor elements have a strong spatial dependence as one moves away of the equatorial plane. Very near the equatorial plane, particles do not obey the guiding center approximations and non guiding center effects determine the energetics of the current sheet.

Research Advisor: Professor Richard L. Kaufmann

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Chapter I

INTRODUCTION

The Earth's environment inside the magnetosphere offers a lot of interesting phenomena to be studied. The shape of the magnetosphere itself is a result of the interaction of the solar wind and the dipolar magnetic field of the Earth. Figure 1.1, taken from Parks, gives a good overview of the Earth's space environment. The solar wind compresses the magnetosphere on the dayside and stretches it on the nightside in the direction of the solar wind flow creating the configuration called the magnetotail. The solar wind transfers a lot of momentum and energy to the magnetosphere, a large fraction of which is stored in the tail and is dissipated into the atmosphere by substorm mechanisms.

The magnetic field lines in the nightside magnetosphere emanating from high latitudes on Earth are open field lines and stretch out along the length of the magnetotail. The magnetic field lines near the Earth are closed and almost dipolar. In between the far-tail and the near-Earth region is the mid-tail region where the field lines are closed but stretched in a tail-like configuration. The geometry of the magnetic field there requires the existence of a current sheet in the dawn-dusk direction. The current sheet is imbedded inside the plasma sheet, a region of high density plasma. The current sheet is a very important region to study in order to understand the dynamics of substorms and the energy transfer processes in the magnetotail. The focus of this work is to examine the behavior of the current sheet in the mid-tail region between 15 and 20 RE from the Earth in the anti-sunward direction. In particular, we look at energy transfer processes, pressure effects and
Figure 1.1: A schematic diagram of Earth's magnetosphere in the noon-midnight plane
particle orbit characteristics inside the current sheet to gain insight on the dynamics of the magnetotail.

**Background**

The current sheet and the substorm mechanism have been the topics of investigation for years. An extensive list of references on magnetotail studies is given by Chen [1992]. The present study was motivated by previous work which employed the self-consistent orbit tracing method (SCOT), described by Larson and Kaufmann [1996], to study the mid-tail region [Kaufmann et al., 1997]. In two papers by Kaufmann et al. [1997], a self-consistent two-dimensional model current sheet was generated by tracing particle orbits in the magnetotail region. It was used to investigate force balance in the region by looking at the momentum equation. The results were applied to study substorm processes. Substorm effects were also studied in the context of non-guiding center particle orbits. Here we extend their work to study the energy and pressure equations which describe the evolution of energy and the pressure tensor respectively as plasma flows towards the Earth.

The substorm is a very dynamical process that is responsible for energy transfer from the magnetotail to the Earth during times of high magnetic activity. The process goes through a cycle where the magnetic field configuration changes dramatically. The detailed descriptions of the substorm phases may differ from model to model but in general the basic characteristics are the same. Figure 1.2, taken from Mitchell et al. [1990], shows the main features of a substorm. The configuration starts with a quiet time when there is not much activity, the current sheet is relatively thick (a few Earth radii) and the magnetic field lines near the Earth are almost dipolar. In the growth phase, the plasma sheet thins as the
Early growth phase

Deep tail
Current sheet

Quiet

Middle growth phase

Lobe pressure increases,
$R_e$ decreases, $P_{11} = P_{12}^a$ increases

Thin current sheet develops
within thick plasma sheet

Late growth phase

Plasma sheet thin;
on current confined to
thin current sheet region,
becomes dominant current,
current is unstable

Near-earth current disrupts:
Field dipolarizes near earth;
current sheet may remain
further tailward.

Figure 1.2: A schematic diagram of the substorm sequence
lobe pressure increases and the magnetic field lines become more stretched. This a phase when energy is built up in the tail. The current sheet continues to thin until the onset phase where the cross-tail current disrupts and the field dipolarizes again. Most of the built up magnetic field energy is transferred to particles in the magnetosphere. Current sheet thicknesses as small as a few tenths of an Earth radius have been observed [Mitchell et al., 1990; Sergeev et al., 1990].

It is very difficult from a computational standpoint to simulate the complete substorm process since it is a very dynamical event and one would need at least a time-dependent magnetic field model to simulate the changing of the field configuration throughout the event. Instead, most of the investigations use steady state models to study physical processes that are characteristic of a particular configuration. Another computational limitation is the three-dimensionality of the models required to produce the most realistic results. Many studies, including the present work, use a two-dimensional, time independent model to study events in the magnetotail region. Despite the limitations, a lot of physics can be learned about important mechanisms that drive magnetospheric activity.

Here, we choose to study energy flow and pressure effects for a thin current sheet configuration characteristic of a substorm growth phase. The method for generating the current sheet will be described in the next section. An extremely thin current sheet offers some interesting characteristics. First, it is supported by the non-guiding center part of the particle orbits near the equatorial plane. Unlike the guiding center part of the orbit where the particles are magnetized and spiral around magnetic field lines, near the equator where the magnetic field is weak the particles meander back and forth across the equator. Several different types of particle orbits have been described which exhibit non-guiding center behavior around the equator. The earliest one to be identified is the Speiser type particle
orbit [Speiser, 1965]. These are particles that spiral around magnetic field lines, mirror away from the equatorial plane and cross the equator in a meandering fashion shown in Figures 1.3, 4.2 and 4.3. The characteristic feature of Speiser particles is that they carry most of their current in the duskward direction near the equator. Another particle orbit with non-guiding center behavior near the equator is the trapped particle orbit in which particles are trapped near the equator. A most typical representative of this type of orbit is the figure-eight particles which cross the equator in a fashion which resembles a figure 8 when projected onto the y-z plane (Figure 4.6). Figure 8 particles carry current in the negative y direction near the equator. For the current sheet configurations that we examine, only Speiser type and trapped particles are required. A more complete list of non-guiding center orbits can be found in Kaufmann and Lu [1993] and references therein.

In our study, we adopt the Geocentric Solar Magnetospheric coordinate system (GSM) with the x-axis pointing from the Earth towards the sun, the y-axis perpendicular to the plane formed by the x-axis and the Earth's magnetic dipole and pointing from dawn to dusk and the z-axis closing the right-hand system. A very important parameter used throughout this study is the characteristic distance \( z_0 \). It is defined by

\[
z_0 = \frac{mv}{|qB_{xy}(z_0)|},
\]

where \( B_{xy}^2 = B_x^2 + B_y^2 \), \( m \) is the particle mass, \( v \) is the velocity and \( q \) is the charge. This distance represents the point where the z-component of the particle's gyroradius is equal to the particle's distance from the equator. The Speiser type particles carry most of their current inside the characteristic distance \( z_0 \) from the equator. The figure 8 particles carry cross tail current in the -y direction inside \( z_0 \), current in the +y direction from \( z_0 \) to \( 2z_0 \) and zero current near \( z_0 \). If a current sheet's thickness is close to \( z_0 \) then this current sheet can be supported by Speiser particles only. If the thickness of the current sheet is bigger than \( z_0 \) but not much bigger than \( 2z_0 \) then the current sheet requires also...
Figure 1.3: A Speiser type particle
trapped particles to provide the current in the region where Speiser particles carry little current. We generate two instances of thin current sheets that differ in this respect and we look at the differences that they exhibit in energy and pressure considerations.

**Self-consistent Current Sheet**

We employ the self-consistent orbit tracing method (SCOT) used to generate two instances of a two-dimensional self-consistent current sheet. By self-consistent we mean that the particles must carry the current to support the pre-selected magnetic field in which they are traced. The magnetic field is described in detail in chapter II. We use two magnetic field configurations, one for a very thin current sheet (0.08 $R_E$) which is supported exclusively by Speiser particles and another one for a thicker current sheet (0.11 $R_E$) which is supported mostly by Speiser particles but also includes trapped particles. We combined the model magnetic field with a uniform dawn-to-dusk electric field ($E = 0.3 \hat{y}_{GSM} \text{ mV/m}$) which provides the Earthward drift of the plasma.

The region under investigation is the mid-tail region between -20 and -15 $R_E$ in the $x_{GSM}$ axis, -0.4 to 0.4 $R_E$ in the $z_{GSM}$ axis and -10 to 10 $R_E$ in the $y_{GSM}$ axis. We grid the region into 10 boxes in the $x$-direction, 40-boxes in the $z$-direction and only 1 box in the $y$-direction since there is no $y$-dependence and we do not need to keep information in this direction. Both models are symmetric around the equator and to save computer memory space we fold the $z$-direction so that information is kept only in the 20 boxes at positive $z$. Therefore, we keep particle information in 200 boxes that divide our region. Speiser groups are groups of ions that are injected outside our region at $z=1.5 \ R_E$. These particles are initially on Speiser orbits as they are injected far away from the equator but as they drift through the region of interest they may become trapped or change the nature of their
orbit. Each group consists of 5000 particles injected at random pitch angles at a fixed point in space with a Maxwellian energy distribution of a temperature of 15 keV. Since the orbit types mix as the particles drift we use several groups of ions injected at different points on the x-axis, at the same z-distance, to ensure that each x-box in our region is dominated by Speiser orbits to carry the required current at the equator. The trapped particle groups are all injected at the equator to guarantee that the majority of the particle orbits in a group are of trapped type. All particles are followed forward and backward in time to guarantee that no part of the orbit is missed inside the region that we examine. The orbit tracing stops after the particles have drifted enough so that they cannot return to the region.

We calculate the distribution function of all groups of ions inside every box. This is used to calculate all fluid parameters like the mass and current densities. Each group of ions is combined with electrons so that charge neutrality is ensured. The electrons are assumed to obey the guiding center approximations. They are magnetized and spiral around magnetic field lines. We do a polynomial fitting of the electron density to the ion density and a polynomial fitting of the electron temperature to one seventh the value of the ion temperature since this is observed in the region [Baumjohann et al., 1989]. The electron current density is then calculated using the guiding center equations. The total current density for each group of particles is the sum of the ion plus the electron current densities.

We combine the total current densities of several groups to produce a fit to the goal current density, the one corresponding to the pre-selected magnetic field. We use a least-squares fitting program to do the fit. We only fit the y-component of the current density which is the direction of the cross-tail current. Results of the fitting for both magnetic field models, labeled thin_008 and thin_011 according to their thicknesses are seen in Figures 1.4a and 1.4b respectively. On the x-axis we show all ten x-boxes from -20 R_E on the left.
to -15 R_E on the right. Within each x-box we plot the current density j_y versus z. We see that the fit to the goal current is good in all boxes except the two Earthward boxes in both models. We were unable to produce a good fit in that region despite using several different groups of particles. This could be due to the presence of a chaotic region. Chaotic regions are regions in space where the nature of the particle orbits changes at an unpredictable fashion as they cross the equator. The nature of the orbits is dictated by the adiabaticity parameter, called the Büchner and Zelenyi kappa parameter. It is defined by

\[ \kappa^2 = \frac{R_{\text{min}}}{\rho_{\text{max}}} \]  

[Büchner and Zelenyi, 1989], where \( R_{\text{min}} \) is the minimum magnetic field line radius of curvature and \( \rho_{\text{max}} \) is the maximum particle gyroradius, both found at \( z=0 \). For a certain value of kappa (0.8), it was found that a very thin current sheet cannot support the current required for self-consistency [Kaufmann et al., 1997]. It is suggested that the two most Earthward boxes in our models fall into such a chaotic region although the kappa values are different than those observed in earlier studies. The values of kappa are shown for each x-box.

The last step in our fitting is to create a combined distribution function for each model case. This is done by combining the groups that were selected from the fitting with their weights. The final combined distribution function for the ions is used to calculate all the fluid parameters for the ions that will be used in the energy and pressure equations.
Figure 1.4: Combined $j_y$ vs goal $j_y$ for a) thin_008, b) thin_011

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Chapter II

THE MODEL MAGNETIC FIELD

In the self-consistent orbit tracing method that we employ, the particles are traced in a pre-selected electric and magnetic field. The region of the nightside magnetosphere that we study is the mid-tail region between 15 and 20 \( R_E \) from the center of the Earth in the \(-x_{GSM}\) (anti-sunward) direction, extending 0.4 \( R_E \) in the \( z_{GSM} \) coordinate from either side of the equatorial plane and 10 \( R_E \) in the \( y_{GSM} \) coordinate from either side of the noon-midnight plane. The magnetic field chosen to model this region should have realistic features that are not far off from observational features of the noon-midnight field. Of particular interest to this study are magnetic field configurations that allow for thin current sheets characteristic of substorms [Sergeev et. al., 1993]. We chose this to be the main criterion for our selection of a magnetic field model.

2.1 Magnetospheric models

Many magnetic field models have been employed to study the magnetosphere. The most elaborate are a series of empirical magnetic field models developed by N. A. Tsyganenko and his collaborators [Tsyganenko and Usmanov, 1982; Tsyganenko, 1989; Tsyganenko, 1993; Tsyganenko, 1995; Tsyganenko and Stern, 1996]. Several spacecraft including IMP, HEOS and ISEE provided the data bases from which these models were constructed. The latest model of this series, the T96, improved upon several deficiencies of its predecessors. It incorporated a magnetopause, a ring current, a magnetotail current, the
system of Birkeland region 1 and 2 currents, and effects of IMF interconnection with the magnetosphere [Tsyganenko, 1995; Tsyganenko and Stern, 1996].

These complicated magnetic field models are not always the most desirable to use. All empirical models have thick current sheets because they are generated from averages of measurements taken at different times by different satellites whose location with respect to neutral sheet is unknown. Therefore, no empirical model is very useful to study the growth phase of a substorm when the current sheet becomes extremely thin. Another problem with empirical models is that they tend to have complicated expressions that take a lot of computation time. Simpler models have been constructed to reduce computation time. The modified Harris [1962] magnetic field model is one example. It has been often used to study particle orbits in the Earth's magnetotail [Chen, 1992]. The one-dimensional modified Harris model has been used [Kaufmann and Lu, 1993] to generate self-consistent one-dimensional current sheets. A two-dimensional simplified model called the "standard model" based on an equilibrium tail field [Zwingmann, 1983] was used by Larson and Kaufmann [1996] to study the structure of the magnetotail current sheet. The model's parameters were adjusted so that the relatively thick current sheet has similar features to the \( K_p=4 \) version of the Tsyganenko [1989] (T89) model. The parameters of the Zwingmann model could not be adjusted to produce a relatively thin current sheet.

2.2 A simple 2-D magnetic field model

In this study we wanted a magnetic field model that would be able to generate very thin current sheets on the order of 0.1 \( R_E \) thick. Thicknesses of this order were detected during substorm growth phases [Mitchell et al., 1990], [Sergeev et al., 1993], [Sergeev et al., 1995]. None of the above mentioned magnetic field models that we looked at was able to produce such thin current sheets. As mentioned before, the empirical Tsyganenko mag-
Magnetic field models represent average values over different periods of time including active and quiet times whereas in a substorm growth phase the thinning of the current sheet lasts only a few minutes [Sergeev et. al., 1995].

The only magnetic field model that we found able to generate the desirable current sheet thickness is a 2-D equilibrium model provided to us by people at the University of Iowa (A. Bhattacharjee, personal communication, 1996) that we call the “Iowa” model or the “Iowa Hyperbolic Tangent” model because of the hyperbolic tangent function that appears in the expression of the magnetic field components. It was first used in two-dimensional, MHD simulations to study magnetospheric substorm dynamics [Ma et al., 1995]. It was also used by Kaufmann et al. [1997] to investigate non-guiding center particle orbits in a thin current sheet in the magnetotail. We use the Iowa model in combination with a three-dimensional dipole field and a uniform $B_{zn}$. The Iowa model is used to represent the mid-tail region and the dipole field is used to represent the near-Earth region more realistically. Most of the particles that we are tracing are of Speiser type and mirror near the Earth. A realistic dipole field is therefore important to model the near-Earth region. The uniform component $B_{zn}$ is used to adjust values of the $B_z$ component and the kappa parameter. Thus, we have

$$\mathbf{B}(x) = \mathbf{B}_{\text{Dipole}}(x) + \mathbf{B}_{\text{Iowa}}(x) + B_{zn} \quad (2.1)$$
The Dipole Field

The dipole magnetic field is generated by motion of conducting material in the Earth's interior. There is no net current outside the Earth associated with it. In cartesian coordinates the vector potential of the dipole is

\[ A_{\text{dipole}}(x) = \frac{m}{r^3}(y\hat{x} - x\hat{y}) \]  

(2.2)

where

\[ m = -m\hat{z} = -31100 \text{ nT R}_E^3 \]  

(2.3)

is the dipole moment and \( r^2 = x^2 + y^2 + z^2 \). The associated magnetic field is

\[ B_{\text{Dipole}}(x) = \left( \frac{3mxz}{r^5} \right)\hat{x} + \left( \frac{3myz}{r^5} \right)\hat{y} + \left( \frac{m(3z^2 - r^2)}{r^5} \right)\hat{z} \]  

(2.4)

The Iowa model

The solar wind hits upon the dipolar field of the Earth and compresses it in the dayside region while stretching it in the nightside region. This creates the elongated profile of the magnetotail in the anti-sunward direction. The geometry of the magnetic field in the tail requires a net current. The magnetic field of the northern lobe is directed towards the Earth while the magnetic field of the southern lobe is directed away from the Earth. Particles must carry a net current across the tail in order to sustain the magnetic field's geometry according to Ampere's law. This generates the cross-tail current sheet. The region around the equator that is heavily populated by particles is called the plasma sheet. These particles provide the cross-tail current and also sufficient pressure to balance the magnetic pressure of the lobes.
The current sheet is a very interesting and important region to study. The nature of the particle orbits in the current sheet control many of the dynamics of the magnetosphere. A two-dimensional equilibrium tail field would be useful to study the current sheet. The Iowa magnetic field is such a module. It is given by the vector potential in GSM coordinates

$$A(x) = -B_c a_c \ln \left[ e^{\frac{a_c - \epsilon x}{\epsilon}} \cosh \left( \frac{z}{a_c} \frac{a_c - \epsilon x}{\epsilon} \right) \right] \hat{y} \quad (2.5)$$

where $B_c, a_c, \text{ and } \epsilon$ are parameters discussed later on. Then, $B = \nabla \times A$ and we get

$$B_x = B_c e^{\frac{a_c - \epsilon x}{\epsilon}} \tanh \left( \frac{z}{a_c} \frac{a_c - \epsilon x}{\epsilon} \right) \quad (2.6)$$
$$B_z = B_c \epsilon \left( \frac{a_c}{a_c - \epsilon x} \right)^2 \left[ 1 - \frac{z}{a_c} e^{\frac{a_c - \epsilon x}{\epsilon}} \tanh \left( \frac{z}{a_c} \frac{a_c - \epsilon x}{\epsilon} \right) \right] \quad (2.7)$$

The vector potential has only a $y$-component. Therefore, there is no $y$-component for the magnetic field and the model is two-dimensional. Also, both the vector potential and the magnetic field have no $y$-dependence. This means that the current density vector has only a $y$-component since

$$\mu_0 j = \nabla \times B = \left[ \begin{array}{c} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial x} \end{array} \right] B_x = \hat{x} \left( \frac{\partial B_z}{\partial y} \frac{\partial B_z}{\partial z} \frac{\partial B_x}{\partial x} \right) + \hat{y} \left( \frac{\partial B_x}{\partial x} \frac{\partial B_y}{\partial y} \frac{\partial B_z}{\partial z} \right) \quad (2.8)$$

Because the model is independent of $y$, the only non-zero component for the current density is


\[ \mu_0 j_y = -\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} \]  

(2.9)

**Parameter selection criteria**

From Ampere's law (equations (2.8) and (2.9)) it is obvious that selecting a magnetic field model is equivalent to selecting a model for the current density \( j \). The parameters of the Iowa model, \( B_c, a_c \) and \( \varepsilon \), as well as the uniform component \( B_{zn} \) can all be adjusted to provide us with a desirable configuration. \( B_c \) essentially controls the strength or the magnitude of the magnetic field components and the current density and carries the units of the magnetic field. The parameter \( a_c \) has units of length and for the most part controls the thickness of the current sheet. The parameter \( a_c \) is basically a characteristic scale length in the z-direction. It also affects the magnitude of the field. The other parameter, \( \varepsilon \), is a dimensionless quantity that controls the x-dependence of the model. Within a characteristic scale length \( L_z = 2a_c \) in the z-direction there is variation along the x-direction of the model, the smaller \( \varepsilon \) is the bigger the variation. Finally, \( B_{zn} \) is adjusted mainly for the kappa parameter rather than the \( B_z \) component itself. It is true that the \( B_z \) component is very important to non-adiabatic particle effects and its average value in the region that we study has been an issue of controversy over the past [Fairfield, 1986], [Huang and Frank, 1994], [Rostoker and Skone, 1993]. However, we are not interested in comparing our model's \( B_z \) value to the average \( B_z \) value of any of these studies since our model is extremely stretched and \( B_z \) necessarily smaller than the average measured \( B_z \).

In this study we used two sets of parameters for the Iowa magnetic field model. Table 2.1 shows the values of the parameters. The two cases are labeled thin_008 and
thin_011 according to the thickness of the current sheets they produce. For example, the
thin_008 model produces a current sheet with thickness 0.08 R_E.

<table>
<thead>
<tr>
<th>Model Type</th>
<th>B_c [nT]</th>
<th>a_c [R_E]</th>
<th>ε</th>
<th>B_zn [nT]</th>
</tr>
</thead>
<tbody>
<tr>
<td>thin_008</td>
<td>70</td>
<td>0.05</td>
<td>0.005</td>
<td>1</td>
</tr>
<tr>
<td>thin_011</td>
<td>80</td>
<td>0.07</td>
<td>0.005</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.1: Iowa Model Parameters

Figure 2.1 shows plots of j_y versus z for both cases. In both cases B_zn was set to the same
value, 1 nT. Both sets produce very thin current sheets, but there are differences. The
major difference lies in the ratio of the current thickness to z_0. The parameter z_0 was
defined before as the z-point where the z-component of the particle’s gyroradius is equal
to the particle’s distance from the equator. Table 2.2 shows values for the scale length L_z
(where L_z is the current sheet thickness defined earlier) and the average value over all x-
boxes of the parameter z_0 for both models. Most of the current is contained within a dis-
tance of 2L_z from the equator as can be seen from figure 2.1.

<table>
<thead>
<tr>
<th>Model Type</th>
<th>L_z [R_E]</th>
<th>z_0 [R_E]</th>
</tr>
</thead>
<tbody>
<tr>
<td>thin_008</td>
<td>0.08</td>
<td>0.15</td>
</tr>
<tr>
<td>thin_011</td>
<td>0.11</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Table 2.2: Iowa Models, Characteristic Thicknesses

The ratio of the current sheet thickness to z_0 determines what kind of particles are needed
in order to self-consistently fit the current density profile j_y. A current sheet that contains
most of its current at a distance z_0 from the equator (2L_z = z_0) can be created by using
Figure 2.1: Current density $j_y$ versus $z$ at $x=-17.5\, R_E$

a) thin_011  

b) thin_008
only Speiser particles. The thin_008 instance is such a case. The other model, thin_011, needs trapped particles to fill in the current at z distances where Speiser particles having a small \( z_0 \) \((z_0 < 2L_z)\) cannot contribute. This difference between the two models is essential because the type of orbits plays a major role in energetic effects in the current sheet as we will see later.

The magnetic field lines for the two models are shown in figure 2.2. For comparison we also show the T89c model for \( K_p = 4 \) with no dipole tilt. The difference in stretching of the lines between the two thin models and the relatively thicker T89c is apparent. Figures 2.3 and 2.4 show contour plots for the magnetic field components \( B_x \) and \( B_z \) respectively. Figure 2.5 shows plots of the kappa parameter for an energy of 15 keV for the thin_008 model. The kappa parameter for the thin_011 model is within 0.1% of the kappa parameter for the thin_008 model so they are essentially the same.
Figure 2.2: Magnetic field lines in the noon-midnight plane for thin 008, thin 011 and t89c(Kp=4)
Figure 2.3: Bx contours for the thin_012 and thin_007 models.
Figure 2.5: Kappa parameter for model thin_008

\[ \kappa = \sqrt{\frac{\text{maximum Larmor radius}}{\text{minimum curvature radius}}} \]
Chapter III

THE ENERGY EQUATIONS

3.1 Introduction

We study the energy transfer processes in our model plasma sheet. These processes are described by the energy equations. Specifically, we look first at how total energy is conserved in the plasma sheet. We then separate the plasma energy into bulk kinetic energy and internal or thermal energy and investigate how these change as the plasma convects earthward through the region of interest. The thermal energy equation is a very interesting one because it gives us information about the thermodynamics of the plasma sheet. In its most generalized form it describes the evolution of the kinetic (pressure) tensor including all the anisotropic terms that may be important to understanding the heating of the plasma. The pressure equation will be discussed in chapter V.

Most of the notation used follows Rossi and Olbert [1970]. All equations are non-relativistic so that rest energy is not included. Our model has no explicit time dependence so all terms that involve partial derivatives with respect to time are zero in our case. The energy equations in their most general form also include gravitational terms and terms involving the net plasma charge density. These terms are also ignored in our case because gravitational effects are not important in our region of interest and the plasma is neutral with the addition of electrons.

Finally, two things should be kept in mind when studying the energy equations. First, an equation can describe the behavior of a single species (for example, ions) or a
multi-component plasma (for example, ions plus electrons or different ion groups combined). Second, each equation can be represented in different frames of reference. We distinguish between three frames of reference: the frame of reference of a single species, the laboratory frame and the proper or center of momentum frame. In our study, the ions carry most of the momentum and so the ion frame is also the proper frame. Sometimes it is advantageous to represent an energy equation in a particular frame of reference to gain a better understanding of the physical processes involved.

3.2 Total Energy

The Boltzmann equation describes the evolution of the distribution function of a single ion species:

$$\frac{\partial f_a}{\partial t} + v_{a,i} \frac{\partial f_a}{\partial x_i} + F_{a,i} \frac{\partial f_a}{\partial p_i} = \left( \frac{\delta f_a}{\delta t} \right)_{coll}$$

(3.1)

where \( \left( \frac{\delta f_a}{\delta t} \right)_{coll} \) represents particle collisions. In our case the particle collision term is zero and (3.1) becomes the Vlasov equation. From now on the collision term will be dropped. Multiplying (3.1) by the total energy of an ion, \( U_a \), and integrating in momentum space will give the total energy equation for the single ion species \( a \). Summing over all species finally gives the total energy equation for the whole plasma:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \varepsilon_T \right) = -\frac{\partial}{\partial x_i} \left[ \left( \frac{1}{2} \rho V^2 + \varepsilon_T \right) V_i + P_{ij} V_j + q_i^* \right] + \mathbf{E} \cdot \mathbf{j} + \rho \mathbf{g} \cdot \mathbf{V}$$

(3.2)

where the subscripts \( i \) and \( j \) refer to Cartesian components and repeated indices are summed. Quantities with an asterisk superscript are evaluated in the proper frame or center of momentum coordinate system. This is the same as the ion rest frame for the electron-ion plasmas and approximations used here. Equation (3.2) describes the rate of
change of the total energy density of the plasma. If we integrate it over a given volume and apply Gauss’ theorem to the divergence terms on the right-hand side then it’s physical meaning becomes more clear. On the left-hand side we have the rate of change of the total energy, bulk kinetic energy \((1/2 \rho V^2)\) plus thermal or internal energy \(e_T\) of the plasma contained in the given volume. This should be equal to the sum of the following terms: the rate at which the bulk kinetic and the internal energies change due to plasma flow across the boundary surface (first two terms on the right-hand side), the work done due to “pressure flux” gradients (third term), the heat flow across the boundary surface (fourth term), and the work done by the electric and gravitational forces. The quantities \(P\) and \(q^*\) represent the pressure tensor and the heat flux vector respectively:

\[
P_{ij} = \rho \langle w_i w_j \rangle
\]

\[
q_i^* = \frac{1}{2} \rho \langle w^2 w_i \rangle
\]

where \(\rho\) is the mass density and \(w\) is the thermal velocity of the ions \((w=v-V)\).

Neglecting the gravitational term and dropping the derivative with respect to time we get the total energy equation for our model plasma sheet after some rearrangement of the terms:

\[
\frac{\partial}{\partial x_i} \left[ \left( \frac{1}{2} \rho V^2 + e_T \right) V_i \right] + \frac{\partial}{\partial x_i} (P_{ij} V_j) + \frac{\partial}{\partial x_i} q_i^* = E \cdot j
\]

\((3.3)\)

Inside the volume of a box in the region that we investigate, the rate at which work is done by the electric forces should equal the divergence of the bulk kinetic and thermal energy flow through the surface of the box plus the divergence of the heat flow plus the rate at which work is done by pressure effects. Figures 3.1a, 3.1b and 3.1c show how the total energy is balanced in the region of space that we study for the thin_008 model. Figure 3.1a
shows a plot of the left-hand side of (3.3), which includes all the divergences of the energy flow terms (solid line), versus the volume term of the rate of work done by the electric field (dashed line). The x-axis is divided into ten 0.5-Re thick x-boxes from x=-20 Re to x=-15 Re in GSM coordinates. The x-boxes are separated in the plot by vertical dotted lines. Within each x-box there are twenty points corresponding to the twenty z-boxes which divide the region above the equator. Only positive z’s are plotted starting from z=0.0 Re at the left-hand side of the plot of each x-box to z=0.4 Re at the right-hand side of each x-box. The z-boxes are 0.02-Re thick. Therefore, within each of the ten x-boxes that divide the x-axis we have a plot of the divergence of the energy density flux or the rate of change of energy density versus z. The dashed line is the rate at which work is done by the electric field. The dashed line is smoother than the solid line because the current $j_y$ is a smooth function and $E_y$ is constant. The solid line is the sum of all the divergences of (3.3). Taking numerical derivatives of energy flux terms in our model produces terms that are jagged so some smoothing was done to reduce the noise in these terms and produce the solid line in Figure 3.1a. We see that in some boxes the total energy is not well balanced. This happens mostly at the end x-boxes, especially at the two most earthward x-boxes, where we were unable to produce a good fit to the current $j_y$. In Figure 3.1b the same energy terms are averaged over all x-boxes and are plotted versus z. This shows that, in the average x-box, total energy is balanced for all z’s. In Figure 3.1c we average the energy terms in each x-box over z and we plot them versus x. As in Figure 3.1a, we see that although the total energy rates balance well in the middle of our region where the fit is best, they do not balance very well in the earthward region. Balancing equation (3.3) is a good test for our model since the total energy should be conserved for a plasma convecting through a region in space. However, it does not give us any information about which
Figure 3.1: LHS vs RHS of equation (3.3) for the thin_008 model
a) all x-boxes  b) x-averaged box vs z  c) z-averaged box vs x

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energy transfer processes are dominant or how the energy is distributed among different forms. To do that, we need to compare the individual terms of the energy equation separately.

Figures 3.2, (a), (b) show plots of the individual terms of the total energy equation for the same model, thin_008. Again, in part (a) we divide the x-axis into ten x-boxes separated with vertical dotted lines and in each x-box we plot the energy terms of (3.3) versus z. The divergence of the bulk kinetic energy density flux is negligible in our case. It is about two orders of magnitude less than the dominant terms. This is because the bulk velocity on average is about an order of magnitude smaller than the average particle velocity ($10^5$ km/s versus $10^6$ km/s). The divergences of the heat flux vector, the thermal energy density flux and the pressure term are more important because they involve particle kinetic energies instead of bulk kinetic energies. They add up to balance the rate of work done by the electric forces $\mathbf{E} \cdot \mathbf{j}$.

The divergence of the heat flux vector is the smallest of the dominant terms as we see in Figure 3.2. This could be a result of our model being two-dimensional. In a more realistic, three-dimensional model this term might be more dominant since gradients in the y-direction will be non-zero. Figure 3.3 is a vector plot that shows the projection of the heat flux vector on the x-y plane for every box in our grid. The x-axis shows the ten x boxes in GSM coordinates from $x=-20$ R$_E$ to $x=-15$ R$_E$ and along the z-axis we plot the $z_{GSM}$ coordinate from $z=0.0$ R$_E$ to $z=0.4$ R$_E$. For each x box we have twenty vectors, one at each z box. The z-component of the heat flux vector is negligible so we only plot the x-y components of this vector. As is evident from Figure 3.3 most of the heat flow is in the negative $y_{GSM}$ direction inside the current sheet and in the positive $y_{GSM}$ direction at higher z's but there is no y-gradient in our model. In a three-dimensional model the
Figure 3.2: Divergence terms of equation (3.3) for the thin_008 model
Figure 3.3: The heat flux vector for thin_008 model
y-gradient of the heat flux could be an important term. The heat flux vector changes rapidly as one moves in the z-direction. Very near the equatorial plane, up to around 0.1 \( R_E \) the heat flow is in the dawnward (-y\(_{GSM}\)) direction and above that z height it is predominantly in the duskward (+y\(_{GSM}\)) direction. Since this gradient in the z-direction of the y-component of the heat flux vector is not part of the divergence, its effect is not apparent in the total energy equation that we examined.

Summarizing the total energy equation for our model plasma sheet, we see that the rate at which internal energy and heat flow across the boundary of the boxes is equal to the rate at which work is done by the electric forces plus the work done by “pressure forces”. The latter term involves the spatial derivative of pressure times velocity. Work is done on the plasma in a box by adjacent boxes both because pressure elements act on bulk velocity gradients and because gradients of pressure elements (forces) act on bulk velocity to produce acceleration.

Finally, since our model is consistent, the rate of work done by the electric forces inside a fixed box in space should be equal to the rate at which the electromagnetic energy density is changing inside the volume plus the rate at which electromagnetic energy density is flowing across the boundary surface:

\[
E \cdot j = -\frac{\partial}{\partial t} \varepsilon_{em} - \nabla \cdot S
\]  

(3.4)

where the electromagnetic energy density \( \varepsilon_{em} \) is constant in the present model and \( S \) is the Poynting vector. Figure 3.4 shows a plot of \( E \cdot j \) versus \( -\nabla \cdot S \). The plotting scheme for Figure 3.4a is the same as before where we divide the x-axis into ten x-boxes and in each
x-box we plot the energy terms versus z. Poynting's theorem is verified with the exception of the most earthward x box where the fit of \( j_y \) is not very good and our model is not consistent.

### 3.3 Bulk Kinetic and Internal Energy Equations

The total energy equation can be separated into two equations, one describing the bulk kinetic energy and the other one the internal energy. The bulk kinetic energy vanishes in the proper frame. For an observer moving with the plasma, the only form of kinetic energy observed is the internal energy \( e_T \). Thus, it is usually more interesting to study the equation that describes the evolution of the internal energy as seen in a frame moving with the plasma. This provides useful information about the thermodynamics of the system. Appendix A describes how we can derive an equation for the internal energy from the total energy equation. A similar derivation can be done for the bulk kinetic energy.

#### Bulk Kinetic Energy

The bulk kinetic energy equation is

\[
\frac{\partial}{\partial x_i} \left( 2 \rho V^2 V_i \right) = -\nu \frac{\partial P_{ij}}{\partial x_i} + (j \times B) \cdot V + \eta E \cdot V
\]  

Equation (3.5) includes only terms that change the bulk flow energy. Again, all derivatives with respect to time are dropped and gravitational terms are neglected. The last term involves the total charge density \( \eta \) and can be dropped because in our case the plasma is neutral. The left-hand side of (3.5) is the difference in the rate at which bulk kinetic energy flows in and out of a fixed box in space. The first term on the right-hand side shows the rate at which pressure tensor forces do work through bulk acceleration of plasma in the
box. The second term on the right-hand side shows that $\mathbf{j} \times \mathbf{B}$ forces which act along the bulk velocity vector will increase the bulk speed of the plasma. Figures 3.5a and 3.5b show plots of the bulk kinetic energy equation terms for the thin_008 model. Again, the divergence of the kinetic energy flux is negligible compared to the other two terms so the only terms that are significant here are the pressure and magnetic force terms. Figure 3.5a then says that inside a fixed box in space, the rate of work done by the pressure tensor forces is almost balanced by the rate of work done by the $\mathbf{j} \times \mathbf{B}$ forces and that the net rate of flow of bulk energy into the box is very small. Figure 3.5b shows how, on average, these processes are balanced in the z-direction and Figure 3.5c shows how, on average, these processes are balanced in the x-direction.

**Internal Energy**

The internal energy equation is derived in Appendix A from the total energy equation. In the ion frame this is

$$\frac{d\varepsilon_T}{dt} = -\varepsilon_T \frac{\partial V_i}{\partial x_i} - p \frac{\partial V_i}{\partial x_i} - \frac{\partial q_i^*}{\partial x_i} + \mathbf{E}^* \cdot \mathbf{j}^*$$  \hspace{1cm} (3.6)

This is also equation (A.14) from Appendix A. The quantities in asterisc indicate the center of momentum frame of reference which in this case is equivalent to the ion frame of reference. Equation (3.6) describes how the internal or thermal energy of the plasma (ions plus electrons) changes as one moves with the plasma. This is indicated by the total derivative of $\varepsilon_T$ in the left-hand-side of (3.6). In contrast, the bulk flow energy in the previous section was done in the laboratory frame since the bulk kinetic energy is zero as seen by an observer moving with the flowing plasma. The right-hand-side of (3.6) describes the...
Figure 3.5: Terms of equation (3.5) for the thin_008 model
   a) all x-boxes  b) x-averaged box vs z  c) z-averaged box vs x

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physical processes responsible for the change of the internal energy. These are now described separately.

The first term on the right-hand-side of (3.6) is the rate of change of the internal energy due to change in density of the plasma as a box of plasma moves earthward with the convection speed. From the mass continuity equation

\[ \frac{1}{n} \frac{dn}{dt} + \frac{\partial V_i}{\partial x_i} = 0 \]  

we see that the divergence of the bulk velocity is equivalent to the rate of change of the density. If plasma enters an x-box with higher \( V_x \) than it leaves the box the plasma density inside the box will increase. This effect is described pictorially for the ions in Figure 5.1a in the context of the pressure equation. A similar argument can be made for the z-gradient of the \( V_z \) bulk velocity component.

Figure 3.6 shows plots of the compressional term for both thin_008 and thin_011 models in all x-boxes (part a) as well as the x-averaged box vs z (part b) and the z-averaged box vs x (part c). The pattern that is displayed by this term follows the pattern of the divergence of the bulk velocity. For both models, the compressional term of the internal energy equation is positive inside the current sheet (from \( z=0 \) to about \( z=z_0 \)) and negative outside the current sheet. This is because in both cases, the plasma is compressed inside the current sheet and rarefied outside. There are differences between the two models, mainly in the tailward region where thin_011 carries a lot of trapped particles and the compressional term becomes very negative outside the current sheet in that region. The differences between the two models due to the nature of orbits will be examined in greater
Figure 3.6: Compression term for equation (3.6)

a) all x boxes  
b) average box versus z  
c) average box versus x
detail in chapter V when we look at the same physical effects (compression, shearing etc) for the pressure equation.

Figure 3.7 shows the next term on the right-hand-side of (3.6). This term involves pressure tensor elements multiplied by gradients of the bulk velocity. For the diagonal pressure tensor elements \((i=j)\) this term describes again a compressional effect. From a fluid mechanics point of view, the term \(P_{ii} \frac{\partial V_i}{\partial x_i}\) can be interpreted as the rate at which work is done by the stress which acts on the \(i\)-plane of a box and stretches it along the \(i\)-direction so as to change the density of plasma in the box. The off-diagonal pressure tensor elements combine with “cross-directional” derivatives \((i \neq j)\) of the bulk velocity to produce a shearing effect. Now, refer to Figure 5.1b which shows schematically an example with shearing. In this case, it is the \(z\)-gradient of the \(V_x\) bulk velocity component which distorts the shape of a unit volume since the top part of the unit volume moves faster than the bottom part. Even though the density of the unit volume does not change, its shape does and that can change the internal energy of the system. The term \(P_{ij} \frac{\partial V_j}{\partial x_i}\) is often called the “stress power” [Lai et al., 1993] and it represents the rate at which work is done to change the volume and shape of a box of unit volume. In figure 3.7 we cannot distinguish between the compressional and the shearing effects because they are combined. To look at them separately we need the generalized pressure equation.

Figure 3.8 shows a plot of the divergence of the heat flux vector \(q\) for both models. The heat flux vector is mostly in the \(y\)-direction and there is no \(y\)-gradient of \(q_y\) to contribute to the divergence. The \(z\)-gradient of \(q_y\) only enters into the generalized pressure equation as one looks at derivatives of the heat tensor \(Q\). The divergence of the heat flux as is seen in Figure 3.8 does not have a clear pattern other than the fact that it decreases in absolute magnitude to zero as one moves Earthward (Figure 3.8c).
Figure 3.7: The pressure term of equation (3.6) for thin_011 and thin_008 models
a) all x-boxes  b) average over z  c) average over x

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Figure 3.8: Heat flux term of equation (3.6) for the thin_011 and thin_008 models
a) all x-boxes  b) x-average box vs z  c) z-average box vs x

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The last term of equation (3.6), $\mathbf{E}^* \cdot \mathbf{j}^*$ is the joule-work done by the electric field as seen by an observer moving with the plasma acting on the current seen in the same reference frame. The electric field and current in the frame of reference moving with the plasma (proper frame) are given by equations (A.11) and (A.12) respectively and $\mathbf{j}^*$ is called the conduction current as opposed to the total current $\mathbf{j}$. Figure 3.9 shows plots of the electric work done on the plasma. This term is almost the same for both models since they have the same electric field and carry almost the same $j_y$. It is smaller than the other terms of (3.6) and is mostly concentrated inside the current sheet where it balances the compressional and shearing terms.

Finally, we look how the internal energy equation balances for both models. We rewrite (3.6) as

$$\frac{de_T}{dt} + \varepsilon_T \frac{\partial V_i}{\partial x_i} + P \frac{\partial V_j}{\partial x_i} + \frac{\partial q^*_i}{\partial x_i} = \mathbf{E}^* \cdot \mathbf{j}^*$$

(3.8)

Figures 3.10 and 3.11 show how (3.8) balances in all boxes and in the x and z directions for models thin_008 and thin_011 respectively. The balance of this equation is better in the middle boxes than in the earthward and tailward parts of the region especially for the thin_008 model. The tailward region in thin_011 does not balance very well outside the current sheet due to the compression and shearing term of that model. This is a physical effect because that region in space contains many more trapped particles in the thin_011 model than in the thin_008. The difference between absolute magnitudes of the two sides of the internal energy equation for thin_008 (Figure 3.10) is not much larger than the difference seen in the bulk kinetic energy equation (Figure 3.5). The terms in the bulk kinetic energy equation are bigger and the difference is a smaller percentage of these terms.
Figure 3.9: The electric term of equation (3.6) in the thin_011 and thin_008 models
a) all x-boxes  b) average over x versus z  c) average over z versus x

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Figure 3.10: LHS vs RHS of equation (3.8) for the thin_008 model
a) all x-boxes  b) x-average box vs z  c) z-average box vs x

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Figure 3.11: LHS vs RHS of equation (3.9) for the thin_011 model
  a) all x-boxes  b) x-average box vs z  c) z-average box vs x

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Chapter IV

THE ANISOTROPIC PRESSURE TENSOR

4.1 The pressure tensor

Since the orbits of the ions do not obey the guiding center approximations near the equator, it is necessary to look at the complete ion pressure tensor \( P \). In the cartesian system of GSM coordinates the ion pressure tensor is

\[
P = \begin{bmatrix}
P_{xx} & P_{xy} & P_{xz} \\
P_{yx} & P_{yy} & P_{yz} \\
P_{zx} & P_{zy} & P_{zz}
\end{bmatrix}
\]  \hspace{1cm} (4.1)

where

\[
P_{ij} = m \int [v_i - V_i][v_j - V_j] f(r, v) \, dv
\]  \hspace{1cm} (4.2)

are the individual ion pressure tensor elements [Rossi and Olbert, 1970] calculated by integrating the ion distribution function. Studies have shown that in a relatively thick, equilibrium current sheet the pressure tensor becomes nearly isotropic [Cowley, 1978; Nötzel et al., 1985; Kaufmann et al., 1996]. Observations seem to support this result [Stiles et al., 1978; Nakamura et al., 1991] at least for most of the time in a quiet-time plasma sheet. In the two thin current sheets that we developed, the pressure tensor exhibits anisotropies due to the nature of orbits near the equator and the stretching of the field lines.
The diagonal elements

Figures 4.1 (a) and (b) show plots of the diagonal elements of the pressure tensor for models thin_008 and thin_011 respectively. Along the x-axis we plot the ten x-boxes separated with vertical lines and within each x-box we plot the pressure elements versus z. We see that in each x-box the pressures are highest at the lowest z-box (z=0.02 R_E) and they increase as one moves Earthward. We also see that the P_{yy} and P_{zz} components are almost the same but the P_{xx} component is about 20-25% bigger on average everywhere inside the current sheets. Both current sheets are dominated by Speiser particles (all particles in the thin_008 current sheet started on Speiser orbits). Figures 4.2 and 4.3 show parts of two typical Speiser orbits projected onto the x-z plane. These particles were injected outside the region shown, at x=-12.3 R_E, z=1.5 R_E, with different pitch angles. The particle in Figure 4.2 was injected with a small pitch angle (10 degrees) whereas the particle in Figure 4.3 was injected with a large pitch angle (80 degrees). Figures 4.2b and 4.3b are enlargements of the parts of the orbits that cross the equator. The axes of the enlarged views have the same scale and we can see that the particle injected with a small pitch angle (Figure 4.2) enters and leaves the neutral sheet (first and last crossings) with much greater v_x velocity than v_z velocity. The cartesian velocity components are more nearly equal when a particle comes towards the equator with a large pitch angle. The ion in Figure 4.3 encounters the neutral sheet with about as much v_x as v_z. Since in a combined group we have a mixing of pitch angles, the overall v_x distribution will be bigger than the v_z distribution near the equator and at positions where v_y is small (which is true at the first and last crossings or equivalently where the particle enters and leaves the neutral sheet).
Figure 4.1: Diagonal pressure tensor elements for a) thin_008 and b) thin_011
Figure 4.2: Orbit plots for a Speiser particle in the x-z plane
(injected at x=-12.3 R_E, z=1.5 R_E, pitch angle=10°)
Figure 4.3: Orbit plots for a Speiser particle in the x-z plane
(injected at x=-12.3 Re, z=1.5 Re, pitch angle=80°)
Between the entry and exit points the orbit is more complicated. The particle meanders back and forth across the equator, crossing it a number of times that is primarily dependent on the particle’s kappa parameter. At these intermediate crossings where a lot of cross-tail current is carried (positive $v_y$) the $v_x$ and $v_z$ distributions are more isotropic.

Figure 4.4 shows plots of the velocity distribution function for a single group of five thousand Speiser particles at $x=-17.75$ RE and $z=0.02$ RE. The middle row shows slices of $f(v_x, v_y, v_z)$ at several fixed $v_y$ values. At $v_y=0$ we see the anisotropy of the particle distribution towards larger $v_x$ than $v_z$ values. At positive $v_y$ this anisotropy disappears. This supports the explanation of the $P_{xx} - P_{zz}$ anisotropy based on particle orbits.

Figure 4.5 shows a similar plot at $x=-16.75$ RE and $z=0.02$ RE but for the combined distribution function of model thin_008. The box was chosen arbitrarily and is populated by a mixture of orbits since some of the particles that were injected on Speiser orbits became trapped as they drifted. However, the Speiser type features are still dominant. We can still see for example the $P_{xx} - P_{zz}$ anisotropy at zero $v_y$.

The thin_011 model is less anisotropic than the thin_008 because it contains more trapped particles. One feature of the trapped particles, particularly those of the figure-8 type, is that they carry a lot of negative current in the $y_{GSM}$ direction near the equator and not much current in the $x_{GSM}$ direction. Figure 4.6 shows part of a figure 8 orbit and Figure 4.7 shows distribution function plots for a group of trapped particles at $x=-18.25$ RE, $z=0.02$ RE. The particles were injected in the same $x$-box at $z=0$ RE. The high concentration of particles at negative $v_y$ is obvious and we see that $P_{zz} > P_{xx}$. This will reverse the anisotropy created by Speiser particles so that thin_011 which contains trapped particles.
Figure 4.4: Ion distribution function plots for a group of Speiser particles
Figure 4.5: Distribution function plots for the combined thin_008 group
Figure 4.6: Orbit plots for a trapped particle in all planes (injected at $x=-18.25\, R_E$, $z=0\, R_E$, pitch angle=45°)
Figure 4.7: Distribution function plots for a group of trapped particles
will be less anisotropic than thin_008. This is apparent in Figures 4.1a and 4.1b. However, thin_011 is still a very thin current sheet which is dominated by Speiser particles and still exhibits anisotropy.

**Fire-hose anisotropy**

A similar examination of the individual orbits can explain why $P_{xx} > P_{yy}$. Everywhere except very close to $z=0$, the $P_{xx}$ component of the pressure tensor is almost parallel to the magnetic field and the other two diagonal components are almost perpendicular to the magnetic field. The anisotropy described above then ($P_{xx} > P_{zz} = P_{yy}$) is the fire-hose anisotropy. Cowley [1978] argued that when such anisotropy occurs, a thin, non-adiabatic current sheet forms embedded within a broader, adiabatic one. It was suggested [Nötzol et al., 1985] that such thin current sheets will probably be unstable. The condition for fire-hose stability is

$$P_{//} - P_{\perp} \leq \frac{B^2}{\mu_0}$$

(4.3)

The two current sheets that are described here meet this criterion yet they are thin and highly non-adiabatic near the equator. Figure 4.8 shows a plot of $P_{//} - P_{\perp}$ versus $\frac{B^2}{\mu_0}$ for the thin_008 model which is the thinnest of the two. We see that, at the neutral sheet, $P_{//} - P_{\perp}$ is almost equal to $\frac{B^2}{\mu_0}$ whereas everywhere else the current sheet is stable. Of course, this configuration might be unstable to other kinds of instabilities like the tearing instability [Schindler, 1974] but it is suggested here that thin current sheets like the thin_008 and thin_011 models may not be uncommon and they could survive long enough...
Figure 4.8: \( P/\bar{P} \) versus \( B/\bar{B}_0 \) in the thin_008 model.

Pressure [N/m²]

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to have significant effects in the course of a substorm.

**Off-diagonal elements**

The biggest off-diagonal pressure component is $P_{xz}$. In one-dimensional models, which do not have gradients in the x-direction, the $\frac{\partial P_{xz}}{\partial z}$ pressure force must balance the Earthward magnetic field line tension force. Figure 4.9 shows a plot of $P_{xz}$ for both thin models. The feature seen in both cases is that the value of $P_{xz}$ in almost every x-box increases from a very low value at the equator to a maximum at around $z=z_0$ and then it drops back down at higher $z$'s. This is characteristic of Speiser type particles. If we look back at Figures 4.2 and 4.3 that show parts of the Speiser orbits we see that during the multiple crossings at the equator the particles can go either in the negative or the positive x and z-directions, and that there is no correlation between $v_x$ and $v_z$. Also, away from the equator, the particles will spiral around the magnetic field lines and in general will go into both positive and negative z-directions even though the $v_x$ velocity component remains the same. The $v_x v_z$ product, which determines $P_{xz}$, therefore averages to zero. On the other hand, right where the particles enter and leave the current sheet, at $z \sim z_0$, $v_x$ and $v_z$ are strongly correlated.

Figure 4.10 shows the other two off-diagonal terms, $P_{xy}$ and $P_{yz}$ for the two models. These two pressure elements are smaller than $P_{xz}$ by one and two orders of magnitude respectively. The bigger of the two, $P_{xy}$ will be examined in the next chapter in the context of the generalized pressure equation.
Figure 4.9: $P_{xz}$ for a) thin_008 and b) thin_011
Figure 4.10: Pxy and Pyz for a) thin_008 and b) thin_011

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Chapter V

THE GENERALIZED PRESSURE EQUATION

5.1 Physical description

The pressure equation, in its most general form, describes the evolution of the kinetic tensor of a fluid. For a single component fluid, the kinetic tensor calculated in the frame of reference of the species, is also the partial pressure tensor of the species. The pressure equation then for a single ion component, in the frame of reference of the ions, describes the evolution of the pressure tensor components of the ions. In Appendix B the pressure equation is derived from Boltzmann's equation. In one of its forms [Rossi and Olbert, 1970] it is

\[
\frac{n}{dt} \frac{d}{dt} \left( \frac{\mathbf{K}_{jk}^{(a)}}{n} \right) + \left[ \frac{\partial \mathbf{V}_j}{\partial x_i} - \Omega_o \alpha_{ji} \right] \mathbf{K}_{ik}^{(a)} + \left[ \frac{\partial \mathbf{V}_k}{\partial x_i} - \Omega_o \alpha_{ki} \right] \mathbf{K}_{lj}^{(a)} = -\frac{\partial}{\partial x_i} Q_{ijk}^{(a)} \tag{5.1}
\]

This is also equation (B.24) of Appendix B in which the collision term was neglected because our model is collisionless. The quantity \( \mathbf{K}_{jk}^{(a)} \) is the j-k component of the kinetic tensor of a single ion species \( a \) (here H\(^+\)) calculated in the frame of reference of the ions as indicated by the superscript \( (a) \). Since this is also the partial pressure of the ions, \( P_{jk} \), we can write (5.1) as

\[
\frac{n}{dt} \frac{d}{dt} \left( \frac{P_{jk}}{n} \right) + \left[ \frac{\partial \mathbf{V}_j}{\partial x_i} - \Omega_o \alpha_{ji} \right] P_{ik} + \left[ \frac{\partial \mathbf{V}_k}{\partial x_i} - \Omega_o \alpha_{ki} \right] P_{ij} = -\frac{\partial}{\partial x_i} Q_{ijk}^{(a)} \tag{5.2}
\]
where it is understood that the pressure terms refer only to the ions and not to the whole plasma, ions plus electrons. For convenience, no subscripts or superscripts are used to indicate that. Note also that the index $i$ is contracted due to summation notation so that (5.2) describes the evolution of the $jk$ component of the ion pressure tensor. The terms of (5.2) are now described.

The first term of (5.2) involves the total derivative of the ion pressure component $P_{jk}$ divided by the density $n$. Since $P_{jk} = \rho \langle w_j w_k \rangle$ where $w$ is the thermal velocity of the ions, the first term of (5.2) gives the change of the correlation of the $j$ and $k$-components of the thermal velocity as seen by an observer moving with the ions. All the other terms of (5.2) describe the reasons why this change occurs. The next two brackets in (5.2) are similar to each other. They involve the pressure elements $P_{ik}$ and $P_{ij}$ where $i$ is summed over all indices. Thus, the $j$ and $k$-components of the thermal velocity are coupled with every other component. The first term in each bracket involves gradients of the bulk velocity which indicate bulk mechanical processes, either tensile or shearing as we will see shortly. The second terms in the brackets are the magnetic effects. They are written here in a compact form but they are basically the rates at which work is done by different components of the Lorentz force. The dimensionless, antisymmetric tensor

\[
\alpha_{ij} = \frac{1}{B} \begin{bmatrix}
0 & B_x & -B_y \\
-B_y & 0 & B_x \\
B_y & -B_x & 0
\end{bmatrix}
\] (5.3)

was used for convenience to write

\[
(v \times b)_i = \alpha_{ij} v_j
\] (5.4)
and $\Omega_0 = \frac{qB}{m}$ is the ion gyrofrequency. Finally, the term on the right-hand side of (5.2) is
the rate at which heat flows out of the boxes of our spatial grid. The third-rank heat tensor $Q$ is described in Appendix C. Each term of (5.2) will be examined in more detail in the
next section when we apply (5.2) to a specific pressure tensor element.

The generalized pressure equation is an equation of state. It gives more specific
information about the system than the internal energy equation does. It describes how the
energy is redistributed among degrees of freedom of the system. It does not say what is the
rate at which some form of energy is transformed to another for the whole system. Rather,
it gives the rate at which some form of energy is redistributed from one degree of freedom
to another within the system. In other words, it describes how the degrees of freedom of
the system are correlated. It could be, for example, that due to bulk shearing effects, the
coupling of the $x$ and $z$ thermal motions changes so that thermal energy is shifting from on
degree of freedom to the other. The total thermal energy of the system may or may not
change under such a process but this internal process is important if we are to understand
how energy of one form becomes available in the system.

To summarize very briefly, equation (5.2) says that the correlation of any two com­
ponents of the thermal velocity changes from box to box in a frame of reference moving
with the ions because mechanical, magnetic and heating effects change the rate at which
these two components couple with all other components. The generalized pressure equa­
tion is a tensor equation which gives more specific information about the energetics of
each degree of freedom of a system unlike the energy equation which gives information
about the energetics of the whole system. The pressure equation of course can be summed
over all degrees of freedom by setting $i=j$ and one can then retrieve the thermal energy
equation.
5.2 Example: the $P_{xx}$ element.

We apply (5.2) to the diagonal pressure tensor element $P_{xx}$. The purpose is not to solve (5.2) but rather to look at its terms and get an insight about the physical processes they describe. Later on we will apply (5.2) to the two model cases, thin_008 and thin_011.

For $P_{xx}$, (5.2) becomes

$$n \frac{d}{dt} \left( \frac{P_{xx}}{n} \right) + \left[ \frac{\partial V_x}{\partial x_i} - \Omega_0 \alpha_{xi} \right] P_{ix} + \left[ \frac{\partial V_x}{\partial x_i} - \Omega_0 \alpha_{xi} \right] P_{ix} = -\frac{\partial}{\partial x_i} Q_{ixx} \quad (5.5)$$

**Mechanical effects**

In (5.5), look for a moment only at the terms which involve bulk velocity gradients. They describe rates of change of pressure elements due to mechanical effects. Summing over $i$ in (5.5) we have

$$n \frac{d}{dt} \left( \frac{P_{xx}}{n} \right) = -2 \left[ \frac{\partial V_x}{\partial x} P_{xx} + \frac{\partial V_x}{\partial z} P_{xz} \right] \quad (5.6)$$

First, note that $\frac{P_{xx}}{n}$ has units of energy. It is proportional to the "x-part" of the average thermal energy. In other words, $\frac{P_{xx}}{n} = m \langle w_x w_x \rangle = m \langle w_x^2 \rangle = 2 \langle \epsilon_x \rangle$ where $\epsilon$ and $w$ are the thermal energy and thermal velocity respectively. Dividing by $2n$ we can write (5.6) as

$$\frac{d}{dt} \left( \frac{1}{2} m w_x^2 \right) = -\frac{\partial V_x}{\partial x} m \langle w_x w_x \rangle - \frac{\partial V_x}{\partial z} m \langle w_x w_z \rangle \quad (5.7)$$

To see the mechanical effects, consider a box of plasma of unit volume in Figure 5.1. In part (a) we see the effect of having an $x$-gradient of the $V_x$ bulk velocity component. The right-hand side of the box moves slower in the $x$-direction than the left-hand side and the box becomes compressed in the $x$-direction. As we move with the plasma, we see changes...
Figure 5.1: Mechanical analog for a) compression, b) shearing
Arrows show velocity vectors $V_x$
in the correlation of $w_x$ with itself because the density is changing. In part (b) of Figure 5.1 we see the effect of having a $z$-gradient of the $V_x$ component of the bulk velocity. If $V_x$ increases with $z$ then the top part of the box moves faster than the bottom. The shape of the box will get distorted even though its volume and density remain the same. This is a shearing effect which changes the coupling of the $x$ and $z$ thermal motions.

Therefore, (5.7) says that the rate at which the average thermal energy in the $x$-degree of freedom of a particle of unit volume changes is equal to: a) the rate at which the correlation of the $x$-component of the thermal velocity with itself changes because of compression in the $x$-direction plus b) the rate at which the correlation of the $x$ and $z$-components of the thermal velocity changes due to shape distortion of the unit volume in the $x$-$z$ plane. The shearing effect redistributes thermal motion from one degree of freedom to another.

**Magnetic effects**

Now we go back to (5.5) and look only at the magnetic terms. We have

\[
\frac{d}{dt} \left( \frac{P_{xx}}{n} \right) = 2\Omega_0 \alpha_{xix_i} \frac{P_{ix}}{n}
\]

(5.8)

Again divide by $2n$ and sum over $i$ to get

\[
\frac{d}{dt} \left( \frac{1}{2} m w_x^2 \right) = qB \left[ \alpha_{xx} \langle w_x w_x \rangle + \alpha_{xy} \langle w_x w_y \rangle + \alpha_{xz} \langle w_x w_z \rangle \right]
\]

(5.9)

Use (5.3) for the tensor $\alpha_{ij}$. The term $\alpha_{xx}$ is zero identically. The term $\alpha_{xz}$ gives $B_y$ and in general is not zero but we will neglect it here since our models are two-dimensional and the dipole contribution is negligible in the region we keep information. Only $\alpha_{xy}$ survives
then which gives $B_z$ and is responsible for the coupling of the $x$ and $y$-directions. Therefore,

$$
\frac{d}{dt}\langle \frac{1}{2} m w_x^2 \rangle \equiv q B_z \langle w_y w_x \rangle
$$

(5.10)

The right-hand side is

$$
q B_z \langle w_y w_x \rangle = \int q B_z w_y w_x f dv = \int q (w \times B)_x w_x f dv
$$

(5.11)

where $q (w \times B)_x$ is the $x$-component of the Lorentz force. Then (5.10) says that the rate at which the average $x$-component of the thermal energy changes is equal to the rate at which the $x$-component of the Lorentz force does work on average to change the thermal velocity in the $x$-direction. Note that the $x$-component of the Lorentz force is due to thermal motion in the $y$-direction. This motion becomes "magnetized" in a sense by $B_z$ and couples with the thermal motion in the $x$-direction. Two degrees of freedom in this case exchange thermal and magnetic energies.

**Heating effects**

The right-hand side of (5.5) involves gradients of the heat tensor components $Q_{ixx}$.

The third-rank tensor $Q$ is described in Appendix C. In the frame of reference of the ions where the total velocity is equal to the thermal velocity, the $Q$ tensor is

$$
Q_{ijk} = m \int w_i w_j w_k f dp
$$

(5.12)

The heat flux vector $q$ is derived from the heat tensor $Q$ by

$$
q_j = \frac{1}{2} Q_{iji} = \frac{1}{2} m \int w_i^2 w_j f dp
$$

(5.13)
Unlike the heat flux vector, the physical meaning of the tensor $Q_{ijk}$ is not so clear. In the case we examine here, two indices are the same, $j=k=x$ and

$$Q_{ixx} = m \int w_i^2 w_x f dp$$

(5.14)

Then, $Q_{ixx}$ can be interpreted as a quantity proportional to the flux in the $i$-th direction of the $x$-part of the thermal energy. When all three indices are different this is not a meaningful description any more. In that case, $Q_{ijk}$ describes the correlation of the thermal motions in all three directions. The gradients of (5.14) which appear on the right-hand side of (5.5) are

$$\frac{\partial}{\partial x_i} Q_{ixx} = -\left[ \frac{\partial}{\partial x} Q_{xxx} + \frac{\partial}{\partial y} Q_{yxx} + \frac{\partial}{\partial z} Q_{zxx} \right]$$

(5.15)

Since there is no $y$-dependence in our model, (5.15) becomes

$$\frac{\partial}{\partial x_i} Q_{ixx} = -\left[ \frac{\partial}{\partial x} Q_{xxx} + \frac{\partial}{\partial z} Q_{zxx} \right]$$

(5.16)

Now, we go back to (5.5) and look only at the heating terms:

$$n \frac{d \langle P_{xx} \rangle}{dt} = -\left[ \frac{\partial}{\partial x} Q_{xxx} + \frac{\partial}{\partial z} Q_{zxx} \right]$$

(5.17)

Again dividing by $2n$ we get

$$\frac{d}{dt} \langle \frac{1}{2} mw_x^2 \rangle = -\left[ \frac{\partial}{\partial x} \langle \frac{1}{2} mw_x^2 w_x \rangle + \frac{\partial}{\partial z} \langle \frac{1}{2} mw_x^2 w_z \rangle \right]$$

(5.18)

Equation (5.18) says that the rate of change of the average thermal energy of the $x$-degree of freedom is equal to the rate at which the fluxes of this energy in the $x$ and $z$-directions change in the respective directions. The second term on the right-hand side of (5.18) says...
for example that if particles change their $w_z$ thermal velocity as they move up from the equator, they will experience a change in their thermal energy of the $x$-degree of freedom.

To summarize (5.5) we have: the rate at which the average thermal energy of the $x$-degree of freedom changes as we move with the plasma is equal to a) the rate at which the correlation of the $x$-component of the thermal velocity with itself changes because of compression of the plasma in the $x$-direction, plus b) the rate at which the correlation of the $x$ and $z$-components of the thermal velocity changes due to shearing between the $x$ and $z$-directions, plus c) the rate at which the $x$-component of the Lorentz force does work on average to change the thermal motion in the $x$-direction plus d) the rate at which the $x$-part of the thermal energy flows out of the boxes in the $x$ and $z$-directions. This is the physical meaning of the generalized pressure equation as applied to the pressure element $P_{xx}$ in the absence of $y$-gradients and when $B_y$ is zero. A similar description can be made for all pressure tensor elements.

5.3 Application: thin_008 versus thin_011

We apply (5.5) to the two models, thin_008 and thin_011. Figure 5.2 shows the compressional, shearing, magnetic and heat terms of (5.5) for both models. The terms are plotted from $z=0$ to $z=0.4 \, R_E$ in each $x$-box. The $x$-boxes are again separated by vertical lines. The main thing to notice here is that some terms are bigger than others and there are patterns in the $x$-direction. The magnetic and heating terms are the most dominant inside the characteristic thickness of the current sheets (from $z=0$ to $z-z_0$) and they almost cancel each other. This means that some magnetic work turns into heat. The compressional term is small in that region but becomes more important away from the equator. The shearing effect is the smallest in this particular case but it was found to be more significant for off-diagonal pressure tensor elements. We now look at all the terms separately.
Figure 5.2: Terms for equation 5.5 in a) thin_008 b) thin_011 models

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The compressional term

Figure 5.3 shows the compressional term, \( \frac{\partial V_x}{\partial x} P_{xx} \), for the two models. Since this term displays a consistent pattern in the x-direction, it is averaged over all ten x-boxes and plotted versus z. There is not much difference between the two models except that in the thin_011 (the thicker of the two models and the one that requires trapped particles) the compressional term is bigger in magnitude because the pressure element \( P_{xx} \) is bigger. In both models the compressional term is very small near the equator and becomes significant only after \( z \approx z_0 \). This is due to the fact that the x-component of the bulk velocity does not change much in the x-direction in the neutral sheet, \( \frac{\partial V_x}{\partial x} \) is small there. Figure 5.4 shows a plot for the x-component of the bulk velocity, \( V_x \), for both models. On average, \( V_x \) decreases as we move earthward. This is true in the regions where the particles follow guiding-center orbits and the drift in the x-direction is \( V_x \approx E_y / B_x \). Near the equator the particles do not obey guiding center motion and we do not expect the same behavior. In both models the plasma is not getting compressed along the x-direction very near the equator and the energy does not change much in this region from that effect. The compressional term is positive at higher z values because \( \frac{\partial V_x}{\partial x} \) is negative. Guiding center motion is valid there and so the plasma is decelerated in the x-direction and becomes compressed. Therefore, moving with the plasma, an observer would see an increase of the energy in the x-degree of freedom for the same volume of plasma.

The shearing term

Figure 5.5a shows the shearing term, \( \frac{\partial V_x}{\partial z} P_{zx} \), for the two models in all x-boxes and Figure 5.5b shows the same term averaged over all x-boxes and plotted versus z. The shearing term is less than the compressional term for this particular equation. This is
Figure 5.3: The compressional term for a) thin_008 and b) thin_011

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Figure 5.4: Bulk velocity $V_x$ in a) thin_008, b) thin_011
Figure 5.5: The shearing term for thin_008 and thin_011 models
a) all x-boxes, b) average x-box versus z
because $P_{xx} > P_{xz}$ by an order of magnitude even though $\left| \frac{\partial V_x}{\partial z} \right| > \left| \frac{\partial V_x}{\partial x} \right|$. The shearing term displays a pattern in its variation along the $x$-direction due to the pattern of $\frac{\partial V_x}{\partial z}$.

Going back to the plot of $V_x$ (Figure 5.4), we see that in both models $V_x$ increases sharply with $z$ near the equator in the first four to five $x$-boxes (tailward region) but it decreases with $z$ near the equator in the last four $x$-boxes (earthward region). This is characteristic of particles going through a chaotic region and changing the nature of their orbits at an unpredictable fashion as they cross the equator. In this case there is a chaotic region around $x = -18 R_E$ where the kappa parameter is approximately 0.18. The chaotic region is between two resonant regions in which particles retain the orbit characteristics between mirrorings. Such chaotic and resonant regions and their dependence on kappa have been identified before [Anderson et al., 1997; Kaufmann et al., 1997].

The first resonant region has small $V_x$ at the equator. A number of Speiser type particles in a wide range of pitch angles were observed to carry negative velocity near the equator at this resonant region whereas other Speiser particles at different pitch angles carried a positive velocity. Figure 5.6 shows two Speiser orbits that are very common in this region. The two particles were injected at different pitch angles. The arrows at the plots show the relative drift at the equator, the difference in the $x$-direction between the entry and exit points of the orbits. Therefore, there is a lot of cancellation of $V_x$ in this region due to the mixing of the orbits which accounts for the small value of $V_x$ at the equator. The second resonant region (earthward region) is dominated by Speiser particles which carry mostly positive $x$-velocity and $V_x$ is big at the equator in that region. The number of crossings of the equator changes after the particles go through the chaotic region. This number depends also on the kappa parameter [Chen, 1992; Ashour-Abdalla et al., 1993; Kaufmann and Lu, 1993]. The two resonant regions are therefore different in respect to the nature of
Figure 5.6: Speiser type orbits injected at $X=-12.3 \, R_E$, $Z=1.5 \, R_E$ with pitch angle a) $40^\circ$ b) $60^\circ$.
the particle orbits they support.

The thin_011 model has a lot of trapped particles in the tailward region. These do not carry much velocity in the x-direction as can be seen from Figure 4.6a. This explains why $V_x$ is even smaller at the equator of the first resonant region for the thin_011 model than it is for thin_008. This difference between the two models produces different physical effects. If we go back to Figure 5.5b and look at the average shearing term we notice two things: First, for most z-boxes the shearing term is negative because $\frac{\partial V_x}{\partial z}$ is positive. However, near the equator, thin_008 has a positive shearing term whereas thin_011 has a negative shearing term due to the fact that contribution from the first resonant region dominates. In other words, near the equator and within the characteristic thickness of the current sheets the two models exhibit different behaviors in regards to the shearing effect when averaged across the whole x-region. One configuration (thin_008) will tend to increase the thermal energy of the x-degree of freedom around the equator due to shearing distortions in the x-z plane whereas the other configuration (thin_011) will tend to decrease it.

The magnetic term

The magnetic term, $2\Omega_0 \alpha_{xi} P_{ix} = \frac{2qB_x}{m} P_{yx}$, shown in Figure 5.7 is another example of non-guiding center effects within the current sheet. Figure 5.8 shows the pressure element $P_{yx}=P_{xy}$ which dictates the pattern. Again, we see that inside the characteristic thickness of the current sheets there is a switch in the pattern at about $x=-18$ RE after transition through a chaotic region. $P_{xy}$ is positive near the equator in the first resonant region but more so for the thin_011 model which contains trapped particle orbits in this region. Looking at particle orbits and plots of distribution functions it is almost impossible.
Figure 5.7: The magnetic term for thin_008 and thin_011 models
a) all x-boxes, b) average x-box versus z

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Figure 5.8: Pressure element $P_{xy}$ for thin_008 and thin_011 models
to see the feature that gives the pattern of $P_{xy}$. The slightest effect could change that pattern because $P_{xy}$ is a very small number (about two orders of magnitude smaller than the diagonal elements). However, it is interesting to note that for both magnetic field configurations there are regions near the equator with strikingly different patterns of $P_{xy}$ and in addition the same regions in space differ in $P_{xy}$ depending on the ratio of trapped to Speiser particles. These differences show in the magnetic term which is significant even though $P_{xy}$ is very small. On average, throughout the whole x-region, the magnetic term near the equator is negative for thin_008 and almost zero for thin_011 (Figure 5.7b). In thin_008, the correlation between the x and y-components of the thermal velocity is negative and changes at a rate given by the ion gyrofrequency. In the context of equation (5.10), near the equator the x-component of the Lorentz force does negative work on average to reduce the thermal motion in the x-direction or the thermal energy of the x-degree of freedom. In the thin_011 model, this effect can actually be reversed at the neutral sheet and the Lorentz force can act on average to increase the thermal motion.

**The heating term**

Figure 5.9a shows the heat term, $-\left[\frac{\partial}{\partial x}Q_{xxx} + \frac{\partial}{\partial z}Q_{zxx}\right]$, for both models in all x-boxes and Figure 5.9b shows the same term averaged over x. The structure is very obvious in Figure 5.9b. In both models, the heat flux is concentrated inside the thickness of the current sheet and drops to zero outside. Most of the contribution to this term comes from the z-gradient of $Q_{zxx}$. The term $Q_{zxx}$ is proportional to the flux in the z-direction of x-part of the thermal kinetic energy of the system whereas $Q_{xxx}$ is the flux in the x-direction of the same energy. Figure 5.10 shows both these terms for both models. The flux of the x-part of the thermal energy is bigger in the x-direction than it is in the z-direction. A similar result was found for the total heat flux vector components $q_x$ and $q_z$. However, the gradi-
ents in the z-direction are stronger than the gradients in the x-direction and they dominate in the divergence terms of the tensor $Q$.

The thermal energy flux in the y-direction is the most dominant of all $Q$ terms which results in the dominance of the $q_y$ term over the other two components of the heat flux. Figure 5.11 compares $Q_{yxx}$ with $Q_{xxx}$ and $Q_{zxx}$. In this particular equation, there are no y-gradients of $Q_{yxx}$ to enter into the heat term. There are strong z-gradients of $Q_{yxx}$ mostly inside the thickness of the current sheet which agrees with the result we obtained for the heat flux component $q_y$. 

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Figure 5.9: The heat term of equation 5.5 in thin_008 and thin_011 models
a) all x-boxes  b) x-average
Figure 5.10: Q tensor elements in a) thin_008 and b) thin_011 models
Figure 5.11: Q tensor elements in a) thin_008 and b) thin_011 models
Chapter VI

CONCLUSIONS

We created a two-dimensional self-consistent thin current sheet to study the energy and pressure equations in the mid-tail magnetosphere. We examined the terms of the energy equations which describe energy change processes and the terms of the pressure equation which describe how the elements of the pressure tensor change as one moves with the plasma. Finally, we looked at the non-adiabatic behavior of particles inside the thin current sheet and the effects of thickening of the current sheet.

A hyperbolic tangent analytical magnetic field model, called the Iowa model, was used to create two instances of a thin current sheet. The model is given by an expression of the y-component (in cartesian GSM coordinates) of the vector field. This guarantees no y-component of the magnetic field vector which makes the model two-dimensional. All quantities are independent of the y-coordinate. The Iowa model has three parameters which control the magnitude and the spatial variation of the magnetic field vector and the current density which is related to the magnetic field through Ampere’s law. We selected two sets of parameters to produce two thin current sheets, the thin_008 model with a current sheet thickness of 0.08 RE and the thin_011 model with a current sheet thickness of 0.11 RE. Both thicknesses are characteristic of a substorm’s growth phase.

The Iowa model was used to simulate the region in the magnetotail between 15 and 20 RE from the Earth in the anti-sunward direction and -0.4 to 0.4 RE from the equator. We
combined the Iowa model with a three-dimensional dipole to simulate the region closer to the Earth. The region of interest in the magnetotail was gridded into 200 boxes. We traced groups of particles in the pre-selected magnetic field and kept information about the particle orbits inside each box. The particles had to carry the same current density $j_y$ which is required by Ampere’s law to produce the pre-selected magnetic field model. This makes the model self-consistent. The two current sheets have most of the cross-tail current near the equator. Speiser type particles which mirror high above the equator and meander across the equatorial plane carry their current near the equator. Therefore, the two current sheets were constructed using Speiser particles. Groups of particles were injected at high $z$-values and at different $x$-points to guarantee that all our $x$-boxes are dominated by Speiser particles. Thin_008 was constructed exclusively by such groups of particles. Thin_011 was constructed predominantly by Speiser particles but required also one group of trapped particles. This is because thin_011 requires current at a region where Speiser particles do not carry much current (above the characteristic distance $z_0$) and where only trapped particles carry positive $j_y$. The ion groups were combined with electrons which are here assumed to obey the guiding center approximations.

The number density of electrons was fitted to the number density of the ions to ensure charge neutrality and the electron temperature was fitted to one-seventh the temperature of the ions. The electron current was calculated using the guiding center equations. A combination of different groups of particles was used for each model case to fit the total $j_y$ to the current density corresponding to the magnetic field so that self-consistency is ensured. The fit was good in most of our region, especially the middle $x$-boxes. The $x$-boxes on the two sides of the region, especially the earthward side, were not fit very well.
This could be a characteristic of a chaotic region where the particles cannot support a thin
current sheet.

The distribution functions for the combined groups were calculated in both cases
and the ion fluid parameters used in the energy equations were calculated from the com-
bined ion distribution functions. We examined the energy equations for both our model
cases. We looked separately at the total, bulk kinetic and thermal energy equations. The
total energy was found to be conserved in the boxes where the fit was good. The highest
rates of energy change occur within the current sheet where the ion orbits are non-adiab-
atic. The bulk kinetic energy and the thermal energy equations were not balanced so well.
This was due to numerical effects rather than physical effects. Most of the terms required
taking numerical derivatives and that produced a lot of jaggedness or noise in our results.
A smoothing subroutine was used to reduce the noise in most of the terms but that did not
make the terms any more accurate. Where the energy terms were small, taking numerical
derivatives and smoothing introduced a high percentage error. For example, the error in
the bulk kinetic energy equation is the same as the error in the thermal energy equation but
percentage wise the error is smaller for the bulk kinetic energy terms because they are big-
ger in magnitude.

Taking numerical error into consideration then we looked at the balancing of the
bulk kinetic and thermal energy equations. First, we see that the highest energy change
rates for the convecting plasma are the rate at which work is done by the $j \times B$ forces to
change the bulk velocity and the rate at which work is done by the pressure forces to
change the bulk velocity. These are competing processes and they almost cancel in our
case so that the rate at which bulk kinetic energy flows out of the boxes is negligible. From
the thermal energy equation we saw that the thermal energy of the plasma increases as one
moves with the plasma at a rate which is mostly due to the stress power. The stress power is the rate at which work is done to change the volume and shape of a particle of unit volume. The divergence of the heat flux was not found to be very big for the two dimensional current sheets but it could be important in a three dimensional configuration. The other term in this equation is the rate at which work is done by the conduction currents and the electric field calculated in the frame of reference of the ions. This term is not much bigger in magnitude than the divergence of the heat flux. The final remark for the thermal energy equation is that all physical processes described by its terms could be important to provide a change rate for the thermal energy in a realistic situation.

We then looked at non-guiding center effects on the pressure tensor elements. It was found that the diagonal component $P_{xx}$ was greater than the other two diagonal components, $P_{zz}$ and $P_{yy}$ in both current sheets. Outside the current sheet where the particles are field aligned, this anisotropy is essentially the fire-hose anisotropy ($P_{//} > P_{\perp}$) because the magnetic field lines are very stretched and almost aligned with the $x$-direction. Inside the current sheet, the anisotropy is due to the non-guiding center behavior of Speiser particles. For example, Speiser particles enter and leave the equatorial plane with greater $x$-velocity than $z$-velocity on average and so $\langle w_x w_x \rangle > \langle w_z w_z \rangle$ or $P_{xx} > P_{zz}$. For trapped particles this anisotropy is reversed because they do not have as much $x$-velocity as $z$-velocity when they cross the equator and so $P_{zz} > P_{xx}$. The thicker current sheet (thin_011) has a group of trapped particles and that reduces the anisotropy created by the Speiser particles. Pressure anisotropy could be used as a diagnostic tool to detect current sheet thinning in a course of a substorm.

Finally, we looked at the generalized pressure equation for the ions which describes the change in the correlation of any two thermal velocity components. It was
suggested that the generalized pressure equation is the most general equation of state as it describes how internal energy is distributed among the three degrees of freedom of the system. We looked at each individual term of the pressure equation and described its physical meaning. We applied it to the case of the diagonal pressure element $P_{xx}$ and calculated the terms for both model cases, thin_008 and thin_011. We found that the terms of the pressure equation display patterns in the $x$ and $z$-directions. The pattern in the $z$-direction is due to the characteristic distance $z_0$ and the non-adiabatic effects of ion orbits. The pattern in the $x$-direction is related to the kappa parameter. The nature of the orbits (in particular, the number of equatorial crossings) changes from the tailward to the Earthward region as the particles go through a chaotic region in the middle. Both models display the same pattern in the $x$-direction. In the tailward region, the rate of heat flow dominates all other rates which change the thermal energy in the $x$ direction. In the earthward region it is the rate at which work is done by the $x$-component of the Lorentz force that is dominant in changing the thermal motion in the $x$-direction. We also found differences between the two models inside the current sheet due to the presence of trapped particles in the thin_011. The different nature of the trapped particles modifies the correlation of thermal velocity components and that affects the way internal energy is distributed from one degree of freedom to the other.
LIST OF REFERENCES


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Appendix A: Derivation of the Internal Energy Equation

The internal energy equation can be derived from the total energy equation with the use of
the momentum equation and the mass continuity equation. The equation derived here
describes the change of the internal or thermal energy \( e_T \) as seen by an observer moving
with the fluid. This is given by the total or convective derivative of the internal energy:

\[
\frac{de_T}{dt} = \frac{\partial e_T}{\partial t} + V_i \frac{\partial e_T}{\partial x_i} \quad \quad \text{(A.1)}
\]

Rewriting the second term on the right-hand side gives

\[
\frac{de_T}{dt} = \frac{\partial e_T}{\partial t} + \frac{\partial}{\partial x_i}(V_i e_T) - \frac{\partial}{\partial x_i} e_T \quad \quad \text{(A.2)}
\]

The total energy equation is

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + e_T \right) = -\frac{\partial}{\partial x_i} \left[ \left( \frac{1}{2} \rho V^2 + e_T \right) V_i \right] - \frac{\partial}{\partial x_i} (P_{ij} V_j) - \frac{\partial q_i^*}{\partial x_i} + \mathbf{E} \cdot \mathbf{j} + \rho g \cdot \mathbf{V} \quad \text{(A.3)}
\]

We use (A.2) and (A.3) to eliminate \( \frac{\partial}{\partial x_i} (V_i e_T) \) giving

\[
\frac{de_T}{dt} = \frac{\partial e_T}{\partial t} - e_T \frac{\partial V_i}{\partial x_i} - \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + e_T \right) - \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i \right) - \frac{\partial q_i^*}{\partial x_i} + \mathbf{E} \cdot \mathbf{j} + \rho g \cdot \mathbf{V} \quad \text{(A.4)}
\]

The nonrelativistic momentum equation is

\[
\rho \left( \frac{dV}{dt} - \mathbf{g} \right) + \frac{\partial P_{ij}}{\partial x_i} = \eta \mathbf{E} + \mathbf{j} \times \mathbf{B} \quad \text{(A.5)}
\]
where \( P \) is the pressure tensor. The \( \frac{\partial}{\partial x_i} (P_{ij} V_j) \) term in (A.4) is expanded

\[
\frac{\partial}{\partial x_i} (P_{ij} V_j) = V_j \frac{\partial P_{ij}}{\partial x_i} + P_{ij} \frac{\partial V_j}{\partial x_i}
\]  

(A.6)

We take the dot product of (A.5) with \( V \) and eliminate \( V \cdot \frac{\partial P_{ij}}{\partial x_i} \) between (A.5) and (A.6).

This gives

\[
\frac{\partial}{\partial x_i} (P_{ij} V_j) = \left[ \eta E + j \times B - \rho \left( \frac{dV}{dt} - g \right) \right] \cdot V + P_{ij} \frac{\partial V_j}{\partial x_i}
\]  

(A.7)

Substituting (A.7) in (A.4) we get

\[
\frac{d\epsilon_T}{dt} = \frac{\partial \epsilon_T}{\partial t} - \epsilon_T \frac{\partial V_i}{\partial x_i} - \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \epsilon_T \right) - \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i \right) \\
- \left[ \eta E + j \times B - \rho \left( \frac{dV}{dt} - g \right) \right] \cdot V - P_{ij} \frac{\partial V_j}{\partial x_i} - \frac{\partial q_i^*}{\partial x_i} + E \cdot j - \rho g \cdot V
\]  

(A.8)

Because of the mass continuity equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho V_i) = 0
\]  

(A.9)

it can be easily shown that all the terms in (A.8) involving the mass density vanish. We are then left with

\[
\frac{d\epsilon_T}{dt} = - \epsilon_T \frac{\partial V_i}{\partial x_i} - P_{ij} \frac{\partial V_j}{\partial x_i} - \frac{\partial q_i^*}{\partial x_i} \\
+ E \cdot j - \eta E \cdot V - (j \times B) \cdot V
\]  

(A.10)

We can further simplify (A.10) with the use of the following two relations:
\[ E^* = E + V \times B \]  
\[ j^* = j - \eta V \]  
(A.11)  
(A.12)

The first relation gives the electric field at the proper frame, \( E^* \) versus the electric field at the laboratory frame, \( E \). The second relation does the same for the currents \( j, j^* \) and \( \eta V \) which are the total, conduction and convection current densities respectively. Multiplying (A.11) with (A.12) we have

\[ E^* \cdot j^* = E \cdot j - (j \times B) \cdot V - \eta E \cdot V \]  
(A.13)

This is the rate at which electromagnetic energy is transferred to thermal energy. Substituting in (A.10) we get

\[ \frac{de_T}{dt} = -\varepsilon_T \frac{\partial V_i}{\partial x_i} - P_{ij} \frac{\partial V_j}{\partial x_i} - \frac{\partial q_i^*}{\partial x_i} + E^* \cdot j^* \]  
(A.14)

This is the final form of the thermal energy equation. A similar derivation can be performed for the bulk kinetic energy. The conservation equation of the bulk flow energy in the proper frame of reference is

\[ \frac{d}{dt} \left( \frac{1}{2} \rho V^2 \right) = \left( \frac{1}{2} \rho V^2 \right) \nabla \cdot V - V \frac{\partial P_{ij}}{\partial x_i} + (j \times B) \cdot V + \eta E \cdot V + \rho g \cdot V \]  
(A.15)
Appendix B: The Generalized Pressure Equation

We derive the most general equation for the pressure tensor of a single species. The distribution function $f_a$ of the species $a$ satisfies Boltzmann’s equation

$$\frac{\partial f_a}{\partial t} + v_{ai} \frac{\partial f_a}{\partial x_i} + F_a \frac{\partial f_a}{\partial p_i} = \left( \frac{\delta f_a}{\delta t} \right)_{\text{coll}} \quad (B.1)$$

where the right-hand side term represents particle collisions (in a collisionless plasma, this term is zero and (B.1) is called the Vlasov equation). For convenience we drop the subscript $a$ but it is understood that all quantities refer to a single species. Multiplying (B.1) by $mv_jv_k$, where $m$ is the mass of a particle of the species $a$ and integrating over all velocities we get

$$\int mv_jv_k \frac{\partial f}{\partial t} dv + \int mv_jv_k v_i \frac{\partial f}{\partial x_i} dv + \int mv_jv_k \frac{F_i}{m} \frac{\partial f}{\partial v_i} dv = \int mv_jv_k \left( \frac{\delta f_a}{\delta t} \right)_{\text{coll}} dv \quad (B.2)$$

For convenience, we drop the right-hand side of (B.2). In our case it is zero since the plasma that we study is collisionless but we will add it at the end for generality. The first term on the left-hand side of (B.2) is

$$\int mv_jv_k \frac{\partial f}{\partial t} dv = \int \frac{\partial}{\partial t} [mv_jv_k f] dv - \int f \frac{\partial}{\partial t} [mv_jv_k] dv$$

$$= \frac{\partial}{\partial t} mn \langle v_jv_k \rangle - mn \langle \frac{\partial}{\partial t} \langle v_jv_k \rangle \rangle \quad (B.3)$$

where the bracket notation means averaging over velocity space according to
\( \langle \psi \rangle = \frac{1}{n} \int \psi f dv \) for any variable \( \psi \) with \( n \) being the number density of the species. The last term in (B.3) is zero because \( v \) and \( t \) are independent variables. The first term on the right-hand side of (B.3) is the time derivative of the kinetic tensor. Therefore,

\[
\int mv_j v_k \frac{\partial f}{\partial t} dv = \frac{\partial}{\partial t} K_{jk}
\]  \hspace{1cm} (B.4)

where \( K_{jk} \) is the kinetic tensor of the species in the laboratory frame of reference.

The second term of the left-hand side of (B.2) is

\[
\int mv_j v_k \frac{\partial f}{\partial x_i} dv = \int \frac{\partial}{\partial x_i} [mv_j v_k f] dv - \int f \frac{\partial}{\partial x_i} [mv_j v_k f] dv 
\]  \hspace{1cm} (B.5)

The second term on the right-hand side of (B.5) vanishes because \( x \) and \( v \) are independent variables. Using the definition of the third-rank \( Q \) tensor \( Q_{ijk} = m \int v_j v_k f dv \) (B.5) becomes

\[
\int mv_j v_k v_i \frac{\partial f}{\partial x_i} dv = \frac{\partial}{\partial x_i} Q_{ijk}
\]  \hspace{1cm} (B.6)

The third term on the left-hand side of (B.2) is

\[
\int mv_j v_k \frac{F_i}{m} \frac{\partial f}{\partial v_i} dv = \int v_j v_k F_i \frac{\partial f}{\partial v_i} dv 
\]  \hspace{1cm} (B.7)

The first integral on the right-hand side of (B.7) vanishes because by the divergence theorem, it is equal to the term in the bracket evaluated at infinity in velocity space where the
distribution function is zero. The second term on the right-hand side of (B.7) can be expanded further to give

\[ \int m v_j v_k \frac{F_i}{m \partial v_i} dv = -\int f \left( \frac{\partial F_i}{\partial v_i} (v_j v_k) + F_i \frac{\partial}{\partial v_i} (v_j v_k) \right) dv \]  (B.8)

Since \( F_i = m g_i + eE_i + e(v \times B) \) is independent of \( v \), the first term on the right-hand side of (B.8) vanishes. For the second term use \( \frac{\partial}{\partial v_i} (v_j v_k) = v_k \delta_{ij} + v_j \delta_{ik} \) in (B.8) to get

\[ \int m v_j v_k \frac{F_i}{m \partial v_i} dv = -\int f (F_j v_k + F_k v_j) dv \]  (B.9)

From (B.4), (B.6) and (B.9) we have

\[ \frac{\partial}{\partial t} K_{jk} + \frac{\partial}{\partial x_i} Q_{ijk} = \int f (F_j v_k + F_k v_j) dv \]  (B.10)

where all the quantities refer to a single species \( a \) and they are calculated in the laboratory or stationary frame.

We wish to transform equation (B.10) to the frame of reference of the species \( (a) \). The transformation for the kinetic tensor is

\[ K = K^{(a)} + mnVV \]  (B.11)

where \( K^{(a)} \) indicates that the kinetic tensor is evaluated at the reference frame \( (a) \) of the species \( a \) and \( V \) is the bulk velocity of the species \( a \). The transformation for the third-rank tensor \( Q \) is

\[ Q_{ijk} = Q_{ijk}^{(a)} + K_{ij}^{(a)} V_k + K_{jk}^{(a)} V_i + K_{ki}^{(a)} V_j + mn V_i V_j V_k \]  (B.12)
Substitute (B.11) and (B.12) into (B.10) and also \( F_i = mg_i + eE_i + e(\mathbf{v} \times \mathbf{B})_i \) to get

\[
\frac{\partial}{\partial t} \left[ K_{jk}^{(g)} + mnV_jV_k \right] + \frac{\partial}{\partial x_i} \left[ Q_{ijk}^{(g)} + K_{ij}^{(g)}V_k + K_{jk}^{(g)}V_i + K_{ki}^{(g)}V_j + mnV_iV_jV_k \right] = \left( mg_j + eE_j \right) \int v_k f dv + \left( mg_k + eE_k \right) \int v_j f dv + \int e(\mathbf{v} \times \mathbf{B})_j v_k f dv + \int e(\mathbf{v} \times \mathbf{B})_k v_j f dv
\]

We can rewrite the following terms as

\[
\frac{\partial}{\partial t} K_{jk}^{(g)} + \frac{\partial}{\partial x_i} K_{jk}^{(g)} V_i = \frac{d}{dt} K_{jk}^{(g)} + K_{jk}^{(g)} \frac{\partial}{\partial x_i} V_i = (B.14)
\]

\[
\frac{\partial}{\partial x_i} (K_{ij}^{(g)} V_k) = \frac{\partial V_k}{\partial x_i} K_{ij}^{(g)} + V_k \frac{\partial K_{ij}^{(g)}}{\partial x_i} = (B.15)
\]

\[
\frac{\partial}{\partial x_i} (K_{kl}^{(g)} V_j) = \frac{\partial V_j}{\partial x_i} K_{kl}^{(g)} + V_j \frac{\partial K_{kl}^{(g)}}{\partial x_i} = (B.16)
\]

We can also write

\[
(v \times \mathbf{B})_i = \alpha_{ij} v_j \quad (B.17)
\]

with the help of the dimensionless antisymmetric tensor

\[
\alpha_{ij} = \frac{1}{B} \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix} = (B.18)
\]

With the help of (B.14), (B.15), (B.16), (B.17) and noting that \( V_k = \frac{1}{n} \int v_k f dv \) equation (B.13) becomes
\[
\frac{d}{dt}K_{jk}^{(q)} + K_{jk}^{(q)} \frac{\partial V_i}{\partial x_i} + \frac{\partial}{\partial t}(\rho V_j V_k) + V_k \frac{\partial K_{ij}^{(q)}}{\partial x_i} \\
+ K_{ij}^{(q)} \frac{\partial V_k}{\partial x_i} + V_j \frac{\partial K_{ik}^{(q)}}{\partial x_i} + K_l^{(q)} \frac{\partial V_l}{\partial x_i} + \frac{\partial}{\partial x_i}(\rho V_i V_j V_k) \\
-(mg + eE)_j n V_k -(mg + eE)_k n V_j \\
-eB \int \alpha_{ji} v_i v_k f \, dv - eB \int \alpha_{ki} v_i v_j f \, dv + \frac{\partial}{\partial x_i}Q_{ijk}^{(q)} = 0
\]

The first two terms can be combined with the use of the continuity equation

\[
\frac{dn}{dt} + n \frac{\partial V_i}{\partial x_i} = 0 	ext{ to give}
\]

\[
\frac{d}{dt}K_{jk}^{(q)} + K_{jk}^{(q)} \left[ \frac{1}{n} \frac{\partial n}{\partial t} \right] = n \frac{d}{dt} \left( \frac{K_{ik}^{(q)}}{n} \right)
\]

The magnetic terms can be written as

\[
eB \alpha_{ji} \int v_i v_k f \, dv = eB \alpha_{ji} \frac{K_{ik}}{m} = \Omega_o \alpha_{ji}[K_{ik}^{(q)} + mnV_i V_k]
\]

where \( \Omega_o = \frac{eB}{m} \) is the gyrofrequency.

We also write

\[
\frac{\partial K_{ij}^{(q)}}{\partial x_i} = \frac{\partial}{\partial x_i}[K_{ij} - mnV_i V_j] = \frac{\partial K_{ij}}{\partial x_i} - \frac{\partial}{\partial x_i}(\rho V_i V_j)
\]

Use the collisionless momentum equation in (B.22) to substitute for \( \frac{\partial K_{ij}^{(q)}}{\partial x_i} \) and get

\[
\frac{\partial K_{ij}^{(p)}}{\partial x_i} = -\frac{\partial}{\partial t}(\rho V_j) + n(mg + eE)_j + e \int (v \times B)_j f \, dv - \frac{\partial}{\partial x_i}(\rho V_i V_j)
\]

Using (B.20), (B.21), (B.23) and some algebra, equation (B.19) can be cast in the form
Equation (B.24) is the generalized pressure equation without collisions. It describes the evolution of the kinetic tensor \( K^{(q)}_j \) of a single species \( a \) in the species frame of reference \((a)\), also known as the partial pressure of the species. In the presence of collisions we have an extra term

\[
\frac{d}{dt} \left( \frac{K^{(q)}_j}{n} \right) + \left[ \frac{\partial V_j}{\partial x_i} - \Omega_o \alpha_{ji} \right] K^{(q)}_j + \left[ \frac{\partial V_k}{\partial x_i} - \Omega_o \alpha_{ki} \right] K^{(q)}_i = -\frac{\partial}{\partial x_i} Q^{(q)}_{jik} \tag{B.24}
\]

\[
\frac{d}{dt} \left( \frac{K^{(q)}_j}{n} \right) + \left[ \frac{\partial V_j}{\partial x_i} - \Omega_o \alpha_{ji} \right] K^{(q)}_j + \left[ \frac{\partial V_k}{\partial x_i} - \Omega_o \alpha_{ki} \right] K^{(q)}_i = -\frac{\partial}{\partial x_i} Q^{(q)}_{jik} + \left( \frac{\delta K^{(a)}_{jk}}{\delta t} \right)_{\text{coll}} \tag{B.25}
\]
Appendix C: The third-rank heat tensor Q

The third-rank tensor Q is defined in Rossi and Olbert (eq. 10.149):

\[ Q_{a,ijk} = m_a \int v_{a,i}, v_{a,j}, v_{a,k} f_a \, dv \]  

(C.1)

for a single fluid component \( a \) as observed in the laboratory (or eulerian) frame. The integration is over the total velocities (thermal plus bulk) of the particles and \( f_a \) is the distribution function calculated in the laboratory frame. The same tensor can be written in the frame of reference of the fluid, \( a \), as:

\[ Q_{a,ijk}^{(a)} = m_a \int v_{a,i}^{(a)}, v_{a,j}^{(a)}, v_{a,k}^{(a)} f_a(v) \, dv(v) \]  

(C.2)

All quantities are now evaluated at the frame of reference moving with this fluid component and the velocities are now just the thermal velocities \( v_{a,i}^{(a)} = v_{a,i} - V_{a,i} \). One can transform from the frame of the single fluid component to the laboratory frame using the following relation:

\[ Q_{a,ijk} = Q_{a,ijk}^{(a)} + K_{a,ijk} V_{a,i} + K_{a,ijk} V_{a,j} + K_{a,ijk} V_{a,k} \]  

(C.3)

The velocities \( V \) are the bulk velocities and the \( K \) s are the components of the kinetic tensor \( K \). In the proper frame of reference (denoted by *) the total tensor Q is:

\[ Q_{ijk}^{**} = \sum_a m_a \int v_{a,i}^{**}, v_{a,j}^{**}, v_{a,k}^{**} f_a^{**} \, dv^{**} \]  

(C.4)
where the summation is over all fluid components $a$ comprising the plasma and every fluid component is evaluated at the proper frame of reference. The heat flux vector then can be written in terms of this tensor if we set $i=k$:

\[
q^*_j = \frac{1}{2} Q^*_{jj} = \frac{1}{2} \sum_a m_a \int v^*_a v^*_j f^*_a dv^*
\]  

\[Q_{ijk} = m \int w_i w_j w_k f dv \]  

Summation over $i$ is assumed so that the third-rank tensor is reduced to a first-rank tensor or a vector.

In this kinetic simulation the plasma consists of a single ion species ($\text{H}^+$) and an electron component. The proper frame of reference of the plasma is essentially the frame of reference of the ions so all the quantities denoted with an asterisk are evaluated in the ion frame. In this frame, the velocities of the ions are the thermal velocities. The components of the heat flux vector and the $Q$ tensor are calculated only for ions according to:

\[
q^*_j = \frac{1}{2} m \int w^2 w_j f dv
\]

\[
Q_{ijk} = m \int w_i w_j w_k f dv
\]

where $w$ is the thermal velocity of the ions, $f$ is the ion distribution function and $m$ the ion mass.