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COMPOSITION OPERATORS ON RIEMANN SURFACES

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DISSERTATION

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in partial fulfillment of
the requirements for the degree of

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6/21/96
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Dedication

To my dear teachers: Maria Stere
Carmen Dumitrescu
Cabrica Andreian-Cazacu

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TABLE OF CONTENTS

Dedication	iv
Acknowledgments	v
Abstract	x
1 Preliminaries	1
1.1 Riemann Surfaces	1
1.2 Differential Forms on Riemann Surfaces	6
1.3 Composition Operators	14
2 Boundedness and Compactness of Composition Operators on the Space $\Lambda_2^1(N)$	20
3 Composition Operators on $\Lambda_{2,a}^1(\mathbb{D})$	28
3.1 Boundedness and Compactness of the Composition Operator Between Spaces of Analytic Differentials on the Disc	28
3.2 Invertibility of the Composition Operator	30
3.3 Some Spectrum Computations	32
4 Composition Operators on $\Lambda_{2,a}^1(N)$	53
4.1 Boundedness and Compactness	53
4.2 About the Invertibility of Composition Operators	62

5 Examples and Questions	67
5.1 Examples	67
5.2 Open Questions	72
Bibliography	74

List of Figures

1-1		2
1-2		3
1-3		14
2-1		23
5-1		71

ABSTRACT

COMPOSITION OPERATORS ON RIEMANN SURFACES

by

Ioana C. Mihaila

University of New Hampshire, September, 1996

General background. Composition operators are defined on a Hilbert (or Banach) spaces of complex valued functions defined on some set X . For the big majority of cases the set X is the unit disc in the complex plane, and the space of functions is one of the Hardy or Bergman spaces (weighted or not). This is due, without doubt, to the richness of those spaces, and the high degree of interest in them. There have been also important papers on the study of the Hardy spaces on the unit ball of n -dimensional complex space.

I have been working with my advisor, Prof. Eric Nordgren, towards generalizing the setting of the composition operators from the spaces above to Hilbert spaces of differential forms on Riemann surfaces. The composition operator then becomes the familiar "pullback", but its properties as an operator between Hilbert spaces have not been taken into consideration before.

Results. We have found a characterization for the boundedness of the composition operator induced by an analytic map on the spaces of measurable square integrable 1-forms on two Riemann surfaces and answered the compactness questions for this case. We also found corresponding characterizations for the operator acting

on the spaces of square integrable analytic forms.

We discovered that if we consider the operator on the space of analytic square integrable forms of the unit disc, it is invertible if and only if the inducing map is invertible. This is also the case on all compact surfaces, with the exception of tori. In Chapter 4 we showed that the invertibility of the operator implies that the inducing function is one to one on more general Riemann Surfaces.

Some results were obtained in computing the spectrum of the operator induced by automorphisms of the unit disc. However, a complete description seems hard to get.

In the last chapter we give two examples of composition operators acting on spaces of square integrable analytic 1-forms on open Riemann surfaces.

Chapter 1

Preliminaries

The purpose of this chapter is to gather the main ingredients necessary for the rest of the paper. Definitions of Riemann surfaces, forms and composition operators are presented. Useful results about them are included too. Because the proofs are well-known, references are substituted for technical details in most of the cases.

1.1 Riemann Surfaces

A *two dimensional manifold* M is a connected Hausdorff topological space such that every x in M has a neighborhood homeomorphic to an open disc in the plane. If M is a two dimensional manifold, a *complex chart* on M is a homeomorphism $\alpha : U_\alpha \longrightarrow V_\alpha$ of an open subset $U_\alpha \subset M$ onto an open subset $V_\alpha \subset \mathbb{C}$. Two charts α and β are *analytically compatible* if the *transition map*

$$\tau_{\alpha\beta} = \beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \longrightarrow \beta(U_\alpha \cap U_\beta) \quad (1.1.1)$$

is biholomorphic (Fig. 1-1).

A *complex atlas* on M is a collection of analytically compatible charts $\mathcal{A} = \{ (\alpha, U_\alpha) \}$ whose domains cover M , i.e. $M = \bigcup_\alpha U_\alpha$.

Two complex atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *analytically equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a complex

Figure 1-1

atlas (i.e. every chart in $\mathcal{A}_1$ is compatible with every chart in $\mathcal{A}_2$).

An *analytic structure* on a two dimensional manifold M is an equivalence class of analytically equivalent atlases.

A *Riemann surface* is a two dimensional manifold with an analytic structure.

Examples [Fo, p. 4-5] 1. *The Riemann Sphere* $\mathbb{P}^1$. Let $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$. Introduce the following topology on $\mathbb{P}^1$: the open sets are the usual open sets $U \subset \mathbb{C}$ together with sets of the form $V \cup \{\infty\}$, where $V \subset \mathbb{C}$ is the complement of a compact set

$K \subset \mathbb{C}$. With this topology $\mathbb{P}^1$ is a compact Hausdorff topological space, homeomorphic to the 2-sphere. Set

$$U_\alpha := \mathbb{C}, \quad U_\beta := \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}.$$

Define the following maps: $\alpha : U_\alpha \rightarrow \mathbb{C}$, $\beta : U_\beta \rightarrow \mathbb{C}$, where α is the identity

map and

$$\beta(z) := \begin{cases} 1/z & \text{for } z \in \mathbb{C}^* \\ 0 & \text{for } z = \infty \end{cases}$$

These maps are homeomorphisms and thus $\mathbb{P}^1$ is a two dimensional manifold. Since U_α and U_β are connected and have non-empty intersection, $\mathbb{P}^1$ is also connected. The complex structure on $\mathbb{P}^1$ is defined by the atlas consisting of the charts α and β . To show that the two charts are analytically compatible note that $\alpha(U_\alpha \cap U_\beta) = \beta(U_\alpha \cap U_\beta) = \mathbb{C}^*$ and $\tau_{\alpha\beta} : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is given by $\tau_{\alpha\beta}(z) = 1/z$, which is biholomorphic.

2. *Tori.* Suppose $\lambda_1, \lambda_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. Let $\Gamma := \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ be the lattice spanned by λ_1 and λ_2 (Fig. 1-2). The lattice Γ induces an equivalence

Figure 1-2

relation on the set of complex numbers by $z \sim z' \pmod{\Gamma}$ if and only if $z - z' \in \Gamma$.

Let $\pi : \mathbb{C} \rightarrow \mathbb{C}/\Gamma$ be the canonical projection. Make the quotient space $\mathbb{C}/\Gamma$ into a

topological space by giving it the quotient topology (i.e. $U \subset \mathbb{C}/\Gamma$ is open if and only if $\pi^{-1}(U) \subset \mathbb{C}$ is open). With this topology $\mathbb{C}/\Gamma$ is a Hausdorff topological space and the quotient map π is continuous. Since $\mathbb{C}$ is connected, $\mathbb{C}/\Gamma$ is connected too. The space $\mathbb{C}/\Gamma$ is compact because it is the image under π of the compact parallelogram $P = \{a\lambda_1 + b\lambda_2 \mid a, b \in [0, 1]\}$. The map π is open. To see this we need to show that $\pi^{-1}(\pi(V))$ is open for every $V \subset \mathbb{C}$. But $\pi^{-1}(\pi(V)) = \bigcup_{\lambda \in \Gamma} (\lambda + V)$ and every $\lambda + V$ is open.

The complex structure on $\mathbb{C}/\Gamma$ is defined in the following way: for any open set $V \subset \mathbb{C}$ such that no two points of V are equivalent under Γ , $U := \pi(V)$ is open and π restricted to V is a homeomorphism onto U . Its inverse is a complex chart on $\mathbb{C}/\Gamma$. We have to show that any two charts $\alpha : U_\alpha \rightarrow V_\alpha$ and $\beta : U_\beta \rightarrow V_\beta$ obtained in this fashion are analytically compatible. For every $z \in \alpha(U_\alpha \cup U_\beta)$ we have $\pi(\tau_{\alpha\beta}(z)) = \alpha^{-1}(z) = \pi(z)$ and thus $\tau_{\alpha\beta}(z) - z \in \Gamma$. Since Γ is discrete and $\tau_{\alpha\beta}$ is continuous, $\tau_{\alpha\beta}(z) - z$ must be constant on every connected component of $\alpha(U_\alpha \cup U_\beta)$. Thus $\tau_{\alpha\beta}$ is holomorphic. But $\tau_{\alpha\beta}^{-1} = \tau_{\beta\alpha}$ and hence $\tau_{\beta\alpha}$ is holomorphic as well. Therefore $\mathbb{C}/\Gamma$ is a Riemann surface. It is homeomorphic to a torus.

Let M and N be Riemann surfaces. A continuous map $\rho : M \rightarrow N$ is *analytic* (*holomorphic*) if for any chart α on M and for any chart β on N with $\rho(U_\alpha) \subset V_\beta$ the function

$$\rho_{\alpha\beta} = \beta \circ \rho \circ \alpha^{-1} : \alpha(U_\alpha) \rightarrow \beta(V_\beta) \quad (1.1.2)$$

is analytic. The map ρ is called *bianalytic (biholomorphic)* if it is bijective and both ρ and ρ^{-1} are analytic. The Riemann surfaces M and N are called isomorphic if there exists a bianalytic map between them.

Remarks. 1. In the case of $N = \mathbb{C}$ (with the structure given by the identity chart) ρ is analytic if and only if $\rho \circ \alpha^{-1} : \alpha(U_\alpha) \rightarrow \mathbb{C}$ is analytic for any (α, U_α) chart on M .

2. A function $f : M \rightarrow \mathbb{C}$ is of class l – denoted by $f \in \mathcal{C}^{(l)}(M)$ – if $f \circ \alpha^{-1}$ is of class l in the usual sense (all its partial derivatives up to order l are continuous).

3. If M , N and S are Riemann surfaces and $\rho : M \rightarrow N$ and $\sigma : N \rightarrow S$ are analytic, then $\sigma \circ \rho : M \rightarrow S$ is analytic.

The following theorem is an easy consequence of the definitions and the corresponding result for analytic functions in the plane:

Theorem 1.1 (Identity theorem [Fo, p.6]) *If M and N are Riemann surfaces and $\rho_1, \rho_2 : M \rightarrow N$ are two analytic maps which coincide on a set $A \subset M$ having a limit point $x \in M$, then $\rho_1 = \rho_2$.*

Definition 1.1 Let M be a Riemann surface and $U \subset M$ an open subset. A *meromorphic* function f on U is an analytic function $f : U' \rightarrow \mathbb{C}$ where $U' \subset U$ is an open subset such that:

- (i) $U \setminus U'$ contains only isolated points.
- (ii) $\forall p \in U \setminus U', \lim_{x \rightarrow p} |f(x)| = \infty$.

The points of $U \setminus U'$ are the poles of f . The set of all meromorphic functions on U is denoted by $\mathcal{M}(U)$.

Definition 1.2 A *triangle* on M is a Jordan domain together with a homeomorphism onto a triangle in the plane. The images of the vertices and sides of the plane triangle via the homeomorphism called vertices and sides of the triangle on M . A 2-dimensional manifold is called *triangulable* if there are triangles $\{T_i\}_i$ on M such that:

$$\begin{aligned} a) \quad & \forall P \in M \exists i \text{ s.t. } P \in T_i \\ b) \quad & \text{If } i \neq j, \quad T_i \cap T_j = \begin{cases} \emptyset & \text{or,} \\ \text{one side} & \text{or,} \\ \text{one vertex} \end{cases} \\ c) \quad & \forall P \in M, \exists V \in \mathcal{V}_P \text{ s.t. } \{i \mid T_i \cap V \neq \emptyset\} \text{ is a finite set.} \end{aligned}$$

where $\mathcal{V}_P$ denotes the set of neighborhoods of P .

By subdividing a triangulation it is always possible to have each triangle contained in the domain of a chart [St, p.193, vol.II].

Theorem 1.2 [Sp, p.239] *Any Riemann surface is triangulable.*

1.2 Differential Forms on Riemann Surfaces

Notations. Let (α, U_α) be a chart on a Riemann surface M . We will denote the real and imaginary parts of α by α^1 and α^2 (i.e. $\alpha = \alpha^1 + i\alpha^2$) and the complex

conjugate of α by $\bar{\alpha}$. If $f \in \mathcal{C}^{(1)}$ is a function $f : U_\alpha \rightarrow \mathbb{C}$,

$$\begin{aligned} D_1^\alpha(f) &:= \frac{\partial}{\partial x}(f \circ \alpha^{-1}) \circ \alpha = \partial_1(f \circ \alpha^{-1}) \circ \alpha \\ D_2^\alpha(f) &:= \frac{\partial}{\partial y}(f \circ \alpha^{-1}) \circ \alpha = \partial_2(f \circ \alpha^{-1}) \circ \alpha \\ D^\alpha(f) &:= \frac{\partial}{\partial z}(f \circ \alpha^{-1}) \circ \alpha = \partial(f \circ \alpha^{-1}) \circ \alpha \\ D^{\bar{\alpha}}(f) &:= \frac{\partial}{\partial \bar{z}}(f \circ \alpha^{-1}) \circ \alpha = \bar{\partial}(f \circ \alpha^{-1}) \circ \alpha. \end{aligned} \quad (1.2.1)$$

where for a complex function F on some complex domain

$$\partial F := \frac{1}{2}(\partial_1 F - i \partial_2 F) \quad \bar{\partial} F := \frac{1}{2}(\partial_1 F + i \partial_2 F). \quad (1.2.2)$$

Let M be a Riemann surface. A *0-form* on M is a complex valued function on M . A *1-form* ω on M is an (ordered) assignment of two functions f^α and g^α to each local coordinate chart $\alpha = \alpha^1 + i\alpha^2$ on M such that the expression $f^\alpha d\alpha^1 + g^\alpha d\alpha^2$ is invariant under coordinate changes. That is, if $\beta = \beta^1 + i\beta^2$ is another local coordinate and $\text{dom } \alpha \cap \text{dom } \beta \neq \emptyset$, then:

$$\begin{pmatrix} f^\beta \\ g^\beta \end{pmatrix}(\beta^{-1}(z)) = \begin{pmatrix} D_1^\beta(\alpha^1) & D_1^\beta(\alpha^2) \\ D_2^\beta(\alpha^1) & D_2^\beta(\alpha^2) \end{pmatrix}(\beta^{-1}(z)) \cdot \begin{pmatrix} f^\alpha \\ g^\alpha \end{pmatrix}(\beta^{-1}(z)) \quad (1.2.3)$$

$\forall z \in \beta(U_\alpha \cap U_\beta)$. The matrix in the above formula is the Jacobian of the transition map $\tau_{\alpha\beta}$ and it is denoted by $\mathcal{J}_{\alpha \circ \beta^{-1}}$. A *2-form* Ω on M is an assignment of a function f^α to each local coordinate α such that the expression $f^\alpha d\alpha^1 \wedge d\alpha^2$ is invariant under coordinate changes. That is:

$$f^\beta(\beta^{-1}(z)) = \det \mathcal{J}_{\alpha \circ \beta^{-1}}(z) \cdot f^\alpha(\beta^{-1}(z)), \quad \forall z \in \beta(U_\alpha \cap U_\beta). \quad (1.2.4)$$

If f is a function on M and ω is a k -form ($k=1,2,3$) on M , then $f\omega$ is a k -form on M too. We will also use the complex notation for forms: if (α, U_α) is a chart on M , let

$$d\alpha := d\alpha^1 + i d\alpha^2 \quad (1.2.5)$$

$$d\bar{\alpha} := d\alpha^1 - i d\alpha^2$$

so the form $f^\alpha d\alpha^1 + g^\alpha d\alpha^2$ is equivalent to the form $u^\alpha d\alpha + v^\alpha d\bar{\alpha}$ where $f^\alpha = u^\alpha + v^\alpha$ and $g^\alpha = i(u^\alpha - v^\alpha)$. This notation leads to a new form of formula (1.2.3): for $\omega = u^\alpha d\alpha + v^\alpha d\bar{\alpha} \equiv u^\beta d\beta + v^\beta d\bar{\beta}$ we get:

$$\begin{pmatrix} u^\beta \\ v^\beta \end{pmatrix} (\beta^{-1}(z)) = \begin{pmatrix} D^\beta(\alpha) & D^\beta(\bar{\alpha}) \\ D^{\bar{\beta}}(\alpha) & D^{\bar{\beta}}(\bar{\alpha}) \end{pmatrix} (\beta^{-1}(z)) \cdot \begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix} (\beta^{-1}(z)) \quad (1.2.6)$$

A form will be called $\mathcal{C}^{(l)}, \mathcal{C}^{(\infty)}$ or measurable if the coefficient functions are of class $\mathcal{C}^{(l)}, \mathcal{C}^{(\infty)}$ or are measurable with respect to some local chart. Since the Jacobian involved in the change of chart formulas is analytic, this definition does not depend on the choice of chart (because the surface is connected, any two charts can be linked by a chain of charts). The set of all measurable k -forms on the Riemann surface M will be denoted by $\Lambda^k(M)$ (or Λ^k if the surface is obvious). It is a complex vector space with the natural addition of k -forms. The space of all $\mathcal{C}^{(l)}$ k -forms on M will be denoted by $\mathcal{C}^{(l),k}(M)$.

Remark. If ω looks like $\omega = f^\alpha d\alpha$ in some chart α and β is another chart such that $\text{dom } \alpha \cap \text{dom } \beta \neq \emptyset$, then $\omega = f^\beta d\beta$ because $\mathcal{J}_{\alpha \circ \beta^{-1}}$ is a diagonal matrix ($\alpha \circ \beta^{-1}$ is analytic, so $D^{\bar{\beta}}(\alpha) = D^{\bar{\beta}}(\bar{\alpha}) = 0$), and from formula (1.2.6) we get:

$$f^\beta = D^\beta(\alpha) \cdot f^\alpha, \quad g^\beta = 0 \quad (1.2.7)$$

Similarly, if $\sigma = g^\alpha d\bar{\alpha}$ in chart α , then $\sigma = g^\beta d\bar{\beta}$ in chart β ($\text{dom } \alpha \cap \text{dom } \beta \neq \emptyset$).

We can now define :

$$\Lambda^{1,0}(M) = \{ \omega \in \Lambda^1(M) \mid \omega = f^\alpha d\alpha, \text{ in each chart } \alpha \} \quad (1.2.8)$$

$$\Lambda^{0,1}(M) = \{ \omega \in \Lambda^1(M) \mid \omega = g^\alpha d\bar{\alpha}, \text{ in each chart } \alpha \} \quad (1.2.9)$$

It is clear that $\Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}$ as vector spaces.

Note. Forms in general are loosely called “differential forms”, although this name isn’t usually applied to the measurable ones. It should be clear from the context which kind of forms we are talking about.

Definition 1.3 Suppose $f \in \mathcal{C}^{(1)}(M)$. We define

$$df := D_1^\alpha(f) d\alpha^1 + D_2^\alpha(f) d\alpha^2 \quad (1.2.10)$$

Then $df \in \Lambda^1$. For complex notation we will use:

$$\Delta f := D^\alpha(f) d\alpha \in \Lambda^{1,0}$$

$$\bar{\Delta} f := D^{\bar{\alpha}}(f) d\bar{\alpha} \in \Lambda^{0,1}$$

Therefore $d = \Delta + \bar{\Delta}$. For $\omega \in \mathcal{C}^{(1)}(M)$, define $d\omega$, $\Delta\omega$ and $\bar{\Delta}\omega$ in the following manner. Let $\omega = \sum_k f_k dg_k$ (the sum in this representation is finite and the g_k ’s are $\mathcal{C}^{(1)}$ functions on M . In particular they could be α , α^1 , etc.). Then

$$d\omega := \sum_k df_k \wedge dg_k$$

$$\Delta\omega := \sum_k \Delta f_k \wedge dg_k$$

$$\bar{\Delta}\omega := \sum_k \bar{\Delta} f_k \wedge dg_k$$

These definitions are independent of the representation of ω [Fo, p.66]. Also, note the following elementary properties:

$$\begin{aligned} d^2 &= \Delta^2 = \bar{\Delta}^2 = 0 \\ d\omega &= \Delta\omega + \bar{\Delta}\omega, \quad \forall \omega \in \mathcal{C}^{(1)} \\ d(f\omega) &= df \wedge \omega + f d\omega \\ \Delta\bar{\Delta} + \bar{\Delta}\Delta &= 0 \end{aligned}$$

We introduce the *conjugation operator* for 1-forms as follows: if $\omega = f^\alpha d\alpha^1 + g^\alpha d\alpha^2$ in some chart α ,

$$^*\omega := -g^\alpha d\alpha^1 + f^\alpha d\alpha^2 \quad (1.2.11)$$

In complex notation, this amounts to :

$$\begin{aligned} \omega &= u^\alpha d\alpha + v^\alpha d\bar{\alpha} \\ ^*\omega &= -i u^\alpha d\alpha + i v^\alpha d\bar{\alpha} \end{aligned}$$

Definition 1.4 *The Pull-Back of Forms.* Every analytic map $\rho : M \longrightarrow N$ between two Riemann surfaces induces morphisms $\rho^* : \Lambda^k(V) \longrightarrow \Lambda^k(\rho^{-1}(V))$, for every V open, $V \subset N$, $k=0, 1, 2$, defined by:

$$\begin{aligned} \rho^*(f) &:= f \circ \rho \\ \rho^*(\sum_j f_j dg_j) &:= \sum_j (\rho^* f_j) d(\rho^* g_j) \\ \rho^*(\sum_j f_j dg_j \wedge dh_j) &:= \sum_j (\rho^* f_j) d(\rho^* g_j) \wedge d(\rho^* h_j) \end{aligned} \quad (1.2.12)$$

The definitions are independent of the local representations chosen for the forms. In particular, for any $f \in \mathcal{C}^{(1)}(N)$ and $\omega \in \mathcal{C}^{(1),1}(N)$ we have:

$$\rho^*(df) = d(\rho^* f)$$

$$\rho^*(d\omega) = d(\rho^* \omega)$$

Definition 1.5 Integration of Forms:

a) *1-forms* Let $\omega \in \Lambda^1(M)$ and $c : [0, 1] \rightarrow M$ be a piecewise differentiable curve such that there exists a partition

$$0 = t_0 < t_1 < \dots < t_n = 1$$

of $[0, 1]$ and charts (α_k, U_{α_k}) with $c([t_{k-1}, t_k]) \subset U_{\alpha_k}$, for $k=1, 2, \dots, n$ and

$\alpha_k^1 \circ c, \alpha_k^2 \circ c \in \mathcal{C}^1([t_{k-1}, t_k])$. We define

$$\int_c \omega := \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left(f^k(c(t)) \frac{d\alpha_k^1(c(t))}{dt} + g^k(c(t)) \frac{d\alpha_k^2(c(t))}{dt} \right) dt \quad (1.2.13)$$

where $\omega = f^k d\alpha_k^1 + g^k d\alpha_k^2$ on U_k . The definition is independent of the choice of partition and charts [Fo, p.69].

b) *2-forms* Let U be a domain in $\mathbb{C}$ and $\Omega \in \Lambda^2(U)$. Then $\Omega = f dx \wedge dy = \frac{i}{2} f dz \wedge d\bar{z}$, for some measurable function f on U . Define

$$\int_U \Omega := \int_U f(x, y) dx dy. \quad (1.2.14)$$

If $V \subset \mathbb{C}$ is open and $\varphi : V \longrightarrow U$ is bianalytic, $\varphi = u + iv$, we have:

$$|\mathcal{J}_\varphi| = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} = |\varphi'|^2$$

$$\text{so } \int_U f \, dx dy = \int_V (f \circ \varphi) |\varphi'|^2 \, dx dy.$$

But $\varphi^*(\Omega) = \varphi^*\left(\frac{i}{2} f \, dz \wedge d\bar{z}\right) = \frac{i}{2} (f \circ \varphi) \, d\varphi \wedge d\bar{\varphi} = \frac{i}{2} (f \circ \varphi) \varphi \, dz \wedge \bar{\varphi}' d\bar{z} = \frac{i}{2} (f \circ \varphi) |\varphi'|^2 \, dz \wedge d\bar{z} = (f \circ \varphi) |\varphi'|^2 \, dx \wedge dy$, so

$$\int_U \Omega = \int_V \varphi^* \Omega. \quad (1.2.15)$$

Now let $\Omega \in \Lambda^2(M)$ and define $\text{supp } \Omega := \overline{\{a \in M \mid \Omega(a) \neq 0\}}$. If there exists a chart (α, U_α) such that $\text{supp } \Omega \subset U_\alpha$, define

$$\int_M \Omega := \int_{U_\alpha} \Omega := \int_{\alpha(U_\alpha)} (\alpha^{-1})^* \Omega.$$

The definition is independent of the chart. Let (β, V_β) be another chart such that $\text{supp } \Omega \subset V_\beta$. Since $\text{supp } \Omega$ is contained in both U_α and V_β we can assume that $U_\alpha = V_\beta$ (by simply restricting the domain of each chart). The transition map $\tau_{\alpha\beta} : \alpha(U_\alpha) \longrightarrow \beta(U_\alpha)$ is holomorphic and we have

$$(\alpha^{-1})^* \Omega = (\beta^{-1} \circ \tau_{\alpha\beta})^* \Omega = \tau_{\alpha\beta}^* ((\beta^{-1})^* \Omega),$$

$$\text{so } \int_{\alpha(U_\alpha)} (\alpha^{-1})^* \Omega = \int_{\beta(U_\alpha)} (\beta^{-1})^* \Omega.$$

Definition 1.6 If $f \in \Lambda^0(M)$, $\omega \in \Lambda^1(M)$, $\Omega \in \Lambda^2(M)$ and (α, U_α) is a chart

such that $\omega = f^\alpha d\alpha + g^\alpha d\bar{\alpha}$ and $\Omega = F^\alpha d\alpha^1 \wedge d\alpha^2$, let:

$$f_\alpha = f \circ \alpha^{-1} \quad (1.2.16)$$

$$\omega_\alpha = f_\alpha^\alpha dz + g_\alpha^\alpha d\bar{z}$$

$$\Omega_\alpha = F_\alpha^\alpha dx \wedge dy.$$

Notice that for Ω as above

$$\begin{aligned} (\alpha^{-1})^*(\Omega) &= (\alpha^{-1})^*(F_\alpha^\alpha d\alpha^1 \wedge d\alpha^2) = \\ &= (F^\alpha \circ \alpha^{-1}) d((\alpha^{-1})^*(\alpha^1)) \wedge d((\alpha^{-1})^*(\alpha^2)) = \\ &= F_\alpha^\alpha dx \wedge dy \end{aligned}$$

$$\text{so} \quad \int_{U_\alpha} \Omega = \int_{\alpha(U_\alpha)} (\alpha^{-1})^* \Omega = \int_{\alpha(U_\alpha)} F_\alpha^\alpha dx dy.$$

To finish the definition of the integral of a 2-form on M , suppose first M is compact.

Then M has a finite triangulation $\{T_i\}_i$ with each triangle contained in the domain of a chart. Define:

$$\int_M \Omega = \sum_i \int_{T_i} \Omega. \quad (1.2.17)$$

The same definition can be used for an arbitrary Riemann surface (knowing $\{T_i\}_i$ is countable) provided the infinite sum converges. Alternatively, it is possible to use a partition of unity to define the integral [Fo, p.77]. The definitions are equivalent and independent of the choice of triangulation (or partition of unity).

1.3 Composition Operators

Consider a set S and let X be a space of functions defined on the set S . If ρ is a function that maps S into itself, one natural question is whether for every function $f \in X$ the result of composing with ρ , namely $f \circ \rho$, is also a member of X .

Definition 1.7 If $f \circ \rho \in X$ for every $f \in X$, the *composition operator* induced by ρ on X is defined by

$$C_\rho : X \longrightarrow X, \quad C_\rho(f) := f \circ \rho. \quad (1.3.1)$$

Figure 1-3

From now on the space X will be at least a Banach space. Additional structure will be considered when needed.

Definition 1.8 A Banach space X of complex valued functions on a set S is called a *functional Banach space on S* if the vector operations are the pointwise operations, X separates the points of S (i.e. $f(x) = f(y)$ for each function in the space X implies $x = y$), and for each x in S the linear functional $f \longrightarrow f(x)$ is continuous.

Note that the continuity of point evaluation functionals implies that pointwise convergence of functions in X is a consequence of norm convergence.

Theorem 1.3 *If the map $\rho : S \longrightarrow S$ induces a composition operator on a functional Banach space X , the composition operator C_ρ is bounded.*

PROOF. This is a standard consequence of the Closed Graph Theorem. Suppose (f, g) is in the closure of $\text{Graph}(C_\rho)$. Take a sequence $f_n \longrightarrow f$ such that $C_\rho f_n \longrightarrow g$. Then, since point evaluations are continuous we have:

$$f_n \longrightarrow f \implies f_n(\rho(x)) \longrightarrow f(\rho(x)), \forall x \in S$$

$$\text{and } f_n \circ \rho \longrightarrow g \implies (f_n \circ \rho)(x) \longrightarrow g(x), \forall x \in S$$

so $C_\rho f = g$. Thus $\text{Graph}(C_\rho)$ is closed and therefore C_ρ is bounded. $\square$

If both ρ and φ define composition operators on X , then so does $\rho \circ \varphi$ and we have

$$C_{\rho \circ \varphi} = C_\varphi C_\rho. \quad (1.3.2)$$

Examples. [CM, p. 9-15] *Composition operators on weighted Hardy spaces.* A Hilbert space $\mathcal{H}$ whose vectors are functions analytic on the unit disc is a *weighted Hardy space* if the monomials $1, z, z^2, z^3 \dots$ constitute a complete orthogonal set of non-zero vectors in $\mathcal{H}$.

Assume that $\|1\| = 1$ and let $\beta(j) = \|z^j\|$. The norm on $\mathcal{H}$ is given by

$$\left\| \sum_{j=0}^{\infty} a_j z^j \right\|^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2$$

and the inner product by

$$\left\langle \sum_{j=0}^{\infty} a_j z^j, \sum_{j=0}^{\infty} c_j z^j \right\rangle = \sum_{j=0}^{\infty} a_j \bar{c}_j \beta(j)^2.$$

The weighted Hardy space with weight sequence $\{\beta(j)\}_{j \geq 0}$ will be denoted $H^2(\beta)$.

Note. The definition of $H^2(\beta)$ spaces is more general. The elements of $H^2(\beta)$ are formal power series $\sum_{j=0}^{\infty} a_j z^j$, which need not converge on $\mathbb{D}$. The condition on the sequence $\{\beta(j)\}_{j \geq 0}$ for $H^2(\beta)$ to consist of analytic functions on the disc is $\liminf_{j \rightarrow \infty} \beta(j)^{1/j} \geq 1$.

The properties of the weighted Hardy space $H^2(\beta)$ depend on the weights $\beta(j)$ and many of those properties can be described in terms of its *generating function*

$$k(z) = \sum_{j=0}^{\infty} \frac{z^j}{\beta(j)^2}.$$

This function is analytic on the unit disc. Moreover, point evaluations are bounded linear functionals on $H^2(\beta)$ and are given by

$$f(w) = \langle f, K_w \rangle, \text{ where } K_w(z) = k(\bar{w}z),$$

$$\|K_w\|^2 = k(|w|^2).$$

- a) For $\beta(j) \equiv 1$ we obtain the classical Hardy space.
- b) For $\beta(j) = (j + 1)^{1/2}$ we obtain the classical Dirichlet space.
- c) For $\beta(j) = (j + 1)^{-1/2}$ we obtain the classical Bergman space. Since this case bears some similarity to the material presented in this paper, we will take a closer look at it.

The Bergman space $L_a^2(\mathbb{D})$ is the set of analytic functions on the unit disc, square integrable with respect to Lebesgue area measure. It is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} \frac{dA(z)}{\pi}.$$

The monomials $1, z, z^2, z^3 \dots$ are orthogonal in $L_a^2(\mathbb{D})$ and $\|z^j\|^2 = 1/(j+1)$. For a function with Taylor expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$ we have

$$\|f\|^2 = \sum_{j=0}^{\infty} \frac{|a_j|^2}{j+1}.$$

The expressions for the evaluation at w in the disc become

$$K_w^a(z) = \frac{1}{(1 - \bar{w}z)^2} \quad \text{and} \quad \|K_w^a\| = \frac{1}{1 - |w|^2}.$$

Since the polynomials are dense in the Bergman space, we can get a nicer formula for the point evaluation at w , namely

$$f(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - w\bar{z})^2} \frac{dA(z)}{\pi}.$$

It is important to notice that for all weighted Hardy spaces (including the Bergman space), the adjoint of the composition operator acts in the following way on the point evaluations:

$$C_{\rho}^* K_w = K_{\rho(w)} \tag{1.3.3}$$

since $\langle f, C_{\rho}^* K_w \rangle = \langle C_{\rho} f, K_w \rangle = \langle f \circ \rho, K_w \rangle = f(\rho(w)) = \langle f, K_{\rho(w)} \rangle$.

Our purpose is to define composition operators for appropriate Banach or Hilbert spaces when the set S is a Riemann surface. The important feature of Riemann surfaces is that locally they look like the complex plane. But if we try to define a normed space of functions on such a surface, there is a serious problem. We expect to be able to preserve the local structure, and therefore we would like to have the norm of a function defined by an integral which should be a lifting of the Lebesgue integral on the surface. Unfortunately, Lebesgue measure and hence the Lebesgue integral cannot be lifted to Riemann surfaces in general. That is because the change of variable formula is in disagreement with the change of chart formula. So functions (with any particular properties) on a Riemann surface do not naturally form a Banach space.

However, both measurable 1-forms and analytic 1-forms that are square integrable produce interesting Hilbert spaces.

Theorem 1.4 *The space $\Lambda_2^1(M)$ of measurable 1-forms on the Riemann surface M which satisfy the following*

$$\|\omega\|_M^2 = \int_M \omega \wedge \ast \bar{\omega} < \infty \quad (1.3.4)$$

is a Hilbert space, with inner product given by

$$\langle \omega_1, \omega_2 \rangle := \int_M \omega_1 \wedge \ast \bar{\omega}_2. \quad (1.3.5)$$

PROOF. We need only to show that $\Lambda_2^1(M)$ is complete. Suppose $\{\omega_n\}_{n \geq 1}$ is a Cauchy sequence in $\Lambda_2^1(M)$, i.e. $\lim_{m,n \rightarrow \infty} \|\omega_m - \omega_n\| = 0$. On each chart (α, U_α) , each ω_n has

a representation $f_n^\alpha d\alpha + g_n^\alpha d\bar{\alpha}$ and the coefficients f_n^α and g_n^α have square integrable corresponding images in $L^2(\alpha(U_\alpha))$. This is complete, so there are functions f^α and g^α on U_α such that $f_n^\alpha \circ \alpha^{-1} \rightarrow f^\alpha \circ \alpha^{-1}$ and $g_n^\alpha \circ \alpha^{-1} \rightarrow g^\alpha \circ \alpha^{-1}$ in $L^2(\alpha(U_\alpha))$. Let ω be the measurable 1-form that has representation $f^\alpha d\alpha + g^\alpha d\bar{\alpha}$ in chart U_α . It is easy to see that this recipe defines indeed a 1-form on M .

Since $\lim_{m,n \rightarrow \infty} \|\omega_m - \omega_n\| = 0$, by first doing the limit when $m \rightarrow \infty$, we obtain $\lim_{n \rightarrow \infty} \|\omega_n - \omega\| = 0$, which means that $\omega_n - \omega \in \Lambda_2^1(M)$. Hence $\omega \in \Lambda_2^1(M)$. $\square$

Note. If $\omega = u d\alpha + v d\bar{\alpha}$ in some chart (α, U_α) , we have

$$\omega \wedge {}^*\bar{\omega} = i(u\bar{u} + v\bar{v}) d\alpha \wedge d\bar{\alpha} = 2(|u|^2 + |v|^2) d\alpha^1 \wedge d\alpha^2 \quad (1.3.6)$$

Let M and N be two Riemann surfaces and $\rho : M \rightarrow N$ an analytic map. For $\omega \in \Lambda_2^1(M)$, define $C_\rho(\omega) := \rho^*(\omega)$, the pullback of the form ω . The question which we will try to answer in the next chapter of this paper is “when does C_ρ define a bounded operator between the spaces $\Lambda_2^1(N)$ and $\Lambda_2^1(M)$?”; in other words, when is C_ρ a composition operator?

Chapter 2

Boundedness and Compactness of Composition Operators on the Space

$$\Lambda_2^1(N)$$

Definition 2.1 Let $\rho : M \longrightarrow N$ be an analytic map. The counting function $N_\rho : N \longrightarrow \mathbb{N} \cup \{\infty\}$ is defined by

$$N_\rho(w) := \#\{z \in M \mid \rho(z) = w\}$$

where the number of z above is counted with appropriate multiplicity.

Lemma 2.1 *Suppose that the counting function induced by an analytic map between two Riemann surfaces is essentially bounded. Then it is bounded with bound the essential supremum.*

PROOF. Let $m = \text{ess sup } N_\rho = \inf_{A(S)=0} (\sup\{N_\rho(w) \mid w \in N \setminus S\})$. Let $m' > m$. If there is a point $w^* \in N$ with $N_\rho(w^*) = m'$, there are points $z_1, z_2, \dots, z_k$ in the preimage of w^* with the sum of the orders of these points equal to m' . Hence there is a neighborhood of w^* on which $N_\rho \geq m'$ (the neighborhood is covered by at least m' sheets). But the measure of any such neighborhood is not zero, which contradicts the fact that the essential supremum of N_ρ is m . □

We will start by proving the main result in the special case that the Riemann surfaces M and N are both the unit disc $\mathbb{D}$, and then we will use the idea of this proof in the general case.

Theorem 2.1 *Let $\mathbb{D}$ be the unit disc and let $\rho : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic and not constant. Then C_ρ is a bounded operator on $\Lambda_2^1(\mathbb{D})$ if and only if the counting function $N_\rho : \mathbb{D} \rightarrow \mathbb{N} \cup \{\infty\}$ is bounded.*

PROOF. Let $\omega \in \Lambda_2^1(\mathbb{D})$. Then $\omega = f(z) dz + g(z) d\bar{z}$. The norm of ω is given by

$$\|\omega\|^2 = 2 \int_{\mathbb{D}} (|f(z)|^2 + |g(z)|^2) dA(z) < \infty. \quad (2.1.1)$$

The composition operator acts on $\Lambda_2^1(\mathbb{D})$ by

$$C_\rho(\omega) = (f \circ \rho)(z) \rho'(z) dz + (g \circ \rho)(z) \overline{\rho'(z)} d\bar{z} \quad (2.1.2)$$

$$\text{so} \quad \|C_\rho(\omega)\|^2 = 2 \int_{\mathbb{D}} (|(f \circ \rho)(z)|^2 + |(g \circ \rho)(z)|^2) |\rho'(z)|^2 dA(z). \quad (2.1.3)$$

Let $\mathbb{D}_0 = \{z \in \mathbb{D} \mid \rho'(z) \neq 0\}$. The set $\mathbb{D} \setminus \mathbb{D}_0$ is at most countable. If $z \in \mathbb{D}_0$, there is some $r_z > 0$ such that ρ is one to one on $D(z, r_z)$. So for any integrable function h on the disc we have

$$\int_{D(z, r_z)} (h \circ \rho) |\rho'|^2 dA = \int_{\rho(D(z, r_z))} h dA.$$

The discs $D(z, r_z)$ form a cover for $\mathbb{D}_0$ and we can pick a countable subcover

$\{D(z_n, r_n) \mid n \in \mathbb{N}\}$. Let

$$\begin{aligned} D_1 &= D(z_1, r_1) \\ D_2 &= D(z_2, r_2) \setminus D_1 \\ &\dots\dots\dots \\ D_n &= D(z_n, r_n) \setminus \bigcup_{k=2}^n D_{k-1} \quad \forall n \geq 2 \end{aligned}$$

so $\{D_n \mid n \in \mathbb{N}\}$ is a pairwise disjoint Borel cover of $\mathbb{D}_0$. Then

$$\begin{aligned} \int_{\mathbb{D}} h \circ \rho \cdot |\rho'|^2 dA &= \int_{\mathbb{D}_0} h \circ \rho \cdot |\rho'|^2 dA = \\ \sum_{n=1}^{\infty} \int_{D_n} h \circ \rho \cdot |\rho'|^2 dA &= \sum_{n=1}^{\infty} \int_{\rho(D_n)} h dA = \\ \int_{\mathbb{D}} h \sum_{n=1}^{\infty} \chi_{\rho(D_n)} dA &= \int_{\mathbb{D}} h N_{\rho} dA, \end{aligned}$$

since $\sum_{n=1}^{\infty} \chi_{\rho(D_n)}(w) = \#\{z \in \mathbb{D}_0 \mid \rho(z) = w\} = N_{\rho}(w)$ for $w \notin \rho(\mathbb{D} \setminus \mathbb{D}_0)$.

Applying this string of equalities to $h = |f|^2 + |g|^2$, we get

$$\|C_{\rho}(\omega)\|^2 = 2 \int_{\mathbb{D}} (|f|^2 + |g|^2) N_{\rho} dA. \quad (2.1.4)$$

Therefore $C_{\rho}(\omega) \in \Lambda_2^1(\mathbb{D})$ for every $\omega \in \Lambda_2^1(\mathbb{D})$ if and only if $N_{\rho} \in L^{\infty}(\mathbb{D})$. The norm of the composition operator is bounded by

$$\|C_{\rho}\| \leq \sqrt{\sup N_{\rho}} \quad (2.1.5)$$

Equality is possible, as we will see in an example. $\square$

Now we are ready to prove a similar result in the general case. The idea is to use compatible triangulations (in a sense that will be explained precisely in the proof of the theorem) to apply the procedure of the previous proof to pairs of triangles on the two surfaces that are mapped one into the other.

Figure 2-1

Theorem 2.2 *Let M and N be Riemann surfaces and let $\rho : M \rightarrow N$ be an analytic map (Fig. 2-1). Then the composition operator C_ρ is a bounded operator from $\Lambda_2^1(N)$ to $\Lambda_2^1(M)$ if and only if the counting function N_ρ is bounded.*

PROOF. *Case 1.* We'll assume first that the surfaces M and N admit finite triangulations. Consider triangulations $\{U_\alpha\}_{\alpha \in A}$ of M and $\{V_\beta\}_{\beta \in B}$ of N , where A and B are finite sets, with the following properties:

a) Each U_α is contained in the domain of some chart on M and each V_β is contained in the domain of some chart on N . Since there might exist several charts containing a given triangle, we pick one (using the Axiom of Choice) and then we restrict it to the interior of the triangle. To simplify the notation, call α the chart defined by the triangle U_α , and call β the one defined by V_β .

b) For every α there is a β such that $\rho(U_\alpha) \subset V_\beta$.

Now let $\omega \in \Lambda_2^1(N)$; so ω looks like $f^\beta d\beta + g^\beta d\bar{\beta}$ on V_β . Then

$$\|\omega\|^2 = 2 \sum_{\beta \in B} \int_{\beta(V_\beta)} (|f_\beta^\beta|^2 + |g_\beta^\beta|^2) dA < \infty. \quad (2.1.6)$$

We have

$$[C_\rho(\omega)]_\alpha = f_\beta^\beta \circ \rho_{\alpha\beta} \cdot \rho'_{\alpha\beta} dz + g_\beta^\beta \circ \rho_{\alpha\beta} \cdot \overline{\rho'_{\alpha\beta}} d\bar{z}, \quad (2.1.7)$$

where $\rho_{\alpha\beta} = \beta \circ \rho \circ \alpha^{-1} : U_\alpha \longrightarrow V_\beta$ is defined for each pair α and β such that $\rho(U_\alpha) \subset V_\beta$ (note that $f_\beta^\beta \circ \rho_{\alpha\beta} = (f^\beta \circ \rho)_\alpha$).

For each $\alpha \in A$, let $U_{\alpha 0} = \{u \in U_\alpha \mid \rho'_{\alpha\beta}(\alpha(u)) \neq 0\}$ and as in the proof of Theorem 2.1 write $U_{\alpha 0} = \bigcup_n U_{\alpha n}$ (disjoint union) such that $\rho_{\alpha\beta}$ is one to one on $U_{\alpha n}$.

So we have:

$$\begin{aligned} \|C_\rho(\omega)\|^2 &= 2 \sum_{\alpha \in A} \sum_{n \in \mathbb{N}} \int_{\alpha(U_{\alpha n})} \left(|f_\beta^\beta \circ \rho_{\alpha\beta}|^2 + |g_\beta^\beta \circ \rho_{\alpha\beta}|^2 \right) (z) |\rho'_{\alpha\beta}(z)|^2 dA(z) \\ &= 2 \sum_{\alpha \in A} \sum_{n \in \mathbb{N}} \int_{\rho_{\alpha\beta}(\alpha(U_{\alpha n}))} \left(|f_\beta^\beta|^2 + |g_\beta^\beta|^2 \right) (z) dA(z) \\ &= 2 \sum_{\alpha \in A} \sum_{n \in \mathbb{N}} \int_{\beta(V_\beta)} \chi_{\rho_{\alpha\beta}(\alpha(U_{\alpha n}))}(z) \left(|f_\beta^\beta|^2 + |g_\beta^\beta|^2 \right) (z) dA(z). \end{aligned}$$

Let $w \in \beta(V_\beta^-)$. Define

$$N_{\rho_{\alpha\beta}}(w) = \#\{z \in \alpha(U_\alpha) \mid \rho_{\alpha\beta}(z) = w\}. \quad (2.1.8)$$

Then for $w \notin \rho(U_\alpha \setminus U_{\alpha 0})$, we have $N_{\rho_{\alpha\beta}}(w) = \sum_n \chi_{\rho_{\alpha\beta}(\alpha(U_{\alpha n}))}(w)$ (where, again, $U_\alpha \setminus U_{\alpha 0}$ is at most countable for each α). So

$$\begin{aligned} \|C_\rho(\omega)\|^2 &= 2 \sum_{\alpha \in A} \int_{\beta(V_\alpha)} (|f_\beta^\beta|^2 + |g_\beta^\beta|^2) N_{\rho_{\alpha\beta}} dA \\ &= 2 \sum_{\beta \in B} \sum_{\rho(U_\alpha) \subset V_\beta} \int_{\beta(V_\beta)} (|f_\beta^\beta|^2 + |g_\beta^\beta|^2) N_{\rho_{\alpha\beta}} dA. \end{aligned}$$

$$\begin{aligned} \text{Now } (N_\rho)_\beta(w) &= N_\rho \circ \beta^{-1}(w) = N_\rho(\beta^{-1}(w)) = \#\{u \in M \mid \rho(u) = \beta^{-1}(w)\} = \\ &= \sum_{\rho(U_\alpha) \subset V_\beta} \#\{u \in U_\alpha \mid \rho(u) = \beta^{-1}(w)\} = \\ &= \sum_{\rho(U_\alpha) \subset V_\beta} \#\{z \in \alpha(U_\alpha) \mid \beta(\rho(\alpha^{-1}(z))) = w\} = \\ &= \sum_{\rho(U_\alpha) \subset V_\beta} N_{\rho_{\alpha\beta}}(w), \text{ and therefore} \end{aligned}$$

$$\|C_\rho(\omega)\|^2 = 2 \sum_{\beta \in B} \int_{\beta(V_\beta)} (|f_\beta^\beta|^2 + |g_\beta^\beta|^2) (N_\rho)_\beta dA. \quad (2.1.9)$$

This means that $N_\rho|_{V_\beta}$ is essentially bounded for every β (and therefore bounded). But $N_\rho|_{V_\beta}$ is bounded for every β if and only if N_ρ is bounded. The conclusion of the theorem follows.

Case 2. For arbitrary Riemann surfaces M and N we can use countable triangulations instead of the finite ones. Notice that we're integrating only positive differentials, so all the above integrals and sums make perfect sense and hence the same proof holds, provided we can show that (2.1.9) implies N_ρ is bounded. If it isn't, there exists a sequence of triangles, each containing an open set V_n , such that

for every $n \geq 1$ we have $N_\rho|_{V_n} \geq n^2$. Assume that V_n is contained in the domain of the chart β_n . Define the following measurable 1-form: on each set V_n , let ω be represented by $\frac{1}{\sqrt{N_\rho} \sqrt{A(\beta_n(V_n))}} d\beta_n$, and let ω be zero outside the union of the sets V_n - where $A(\beta_n(V_n))$ is the Lebesgue area measure of $\beta_n(V_n)$. Then ω is square integrable with

$$\|\omega\|^2 = 2 \sum_{n \geq 1} \int_{\beta_n(V_n)} \frac{1}{N_\rho \cdot A(\beta_n(V_n))} \circ \beta_n^{-1} dA \leq 2 \sum_{n \geq 1} \frac{1}{n^2}$$

But $C_\rho(\omega)$ is not square integrable since

$$\|C_\rho(\omega)\|^2 = 2 \sum_{n \geq 1} \int_{\beta_n(V_n)} \frac{N_\rho}{N_\rho \cdot A(\beta_n(V_n))} \circ \beta_n^{-1} dA. \quad \square$$

Note. We have actually shown that

$$\|C_\rho(\omega)\|^2 = \int_N N_\rho \omega \wedge \overline{\omega}. \quad (2.1.10)$$

Corollary 2.1 *The operator $C_\rho^* C_\rho$ is the multiplication by N_ρ .*

Corollary 2.2 *The norm of the composition operator C_ρ is bounded by*

$$\|C_\rho\| \leq \sqrt{\sup N_\rho}. \quad (2.1.11)$$

Corollary 2.3 *If ρ is a bijection, C_ρ is an isometry.*

Example. For $M = N = \mathbb{D}$, and $\rho(z) = z^n$ we obtain $\|C_\rho\| = \sqrt{n} = r(C_\rho)$ (the spectral radius of C_ρ).

As a corollary to the boundedness theorem we will discuss the compactness of the composition operator just defined.

Theorem 2.3 *The composition operator C_ρ defined in the preceding theorem between the spaces of square integrable measurable 1-forms of two Riemann surfaces M and N is not compact.*

PROOF. Since ρ is analytic, there exists an open set $V \subset \rho(M)$. By choosing V small enough we can assume that it lies inside the domain of some chart α on N . Let S be the space of square integrable measurable 1-forms on V extended to be zero outside V . Choose ω_n to be an orthonormal sequence in S (let $\omega_n = f^n d\alpha + g^n d\bar{\alpha}$). This implies that $\omega_n \xrightarrow{w} 0$ (weak convergence). But

$$\|C_\rho(\omega_n)\|^2 = 2 \int_V (|f^n|^2 + |g^n|^2) N_\rho \geq \|\omega_n\|^2 = 1.$$

So $C_\rho(\omega_n)$ does not converge strongly to zero, and therefore the operator is not compact. □

Chapter 3

Composition Operators on $\Lambda_{2,a}^1(\mathbb{D})$

We already have boundedness and compactness conditions for composition operators on the spaces of square integrable differentials. But what can be said about the operator C_ρ (where as before $\rho : M \rightarrow N$ is analytic) on the space of *analytic* square integrable differentials? In this chapter we will try to find some of the properties of these operators in the special case when $M = N = \mathbb{D}$. Throughout this chapter ρ will be an analytic function, $\rho : \mathbb{D} \rightarrow \mathbb{D}$.

3.1 Boundedness and Compactness of the Composition Operator Between Spaces of Analytic Differentials on the Disc

We can write any analytic differential form on the disc as $\omega = f dz$. Therefore $\Lambda_{2,a}^1(\mathbb{D})$ is isomorphic to the Bergman space on the disc (with $\|\omega\|_{\Lambda_{2,a}^1(\mathbb{D})}^2 = 2\pi \|f\|_{L_a^2}^2$). So we have:

$$\begin{aligned} C_\rho &: L_a^2 \rightarrow L_a^2 \\ C_\rho(f) &= f \circ \rho \cdot \rho'. \end{aligned} \tag{3.1.1}$$

Definition 3.1 A finite Borel measure μ on $\mathbb{D}$ is a *Carleson measure* if

$$\sup_{\substack{\zeta \in \partial\mathbb{D} \\ 0 < \delta \leq 2}} \frac{\mu(S(\zeta, \delta))}{\delta^2} < \infty,$$

where $S(\zeta, \delta) = \{z \in \mathbb{D} \mid |z - \zeta| < \delta\}$. If in addition

$$\lim_{\delta \rightarrow 0} \sup_{\zeta \in \partial\mathbb{D}} \frac{\mu(S(\zeta, \delta))}{\delta^2} = 0,$$

μ is a *vanishing* (or *compact*) Carleson measure.

Theorem 3.1 a) The operator C_ρ is bounded if and only if $N_\rho dA$ is a Carleson measure.

b) Assume that C_ρ is bounded. Then C_ρ is compact if and only if $N_\rho dA$ is a vanishing Carleson measure.

PROOF. Recall that

$$\|C_\rho f\|^2 = \int_{\mathbb{D}} N_\rho |f|^2 dA.$$

This theorem is now an immediate consequence of the following:

Theorem 3.2 [Zh, p.109-112] Let μ be a finite positive Borel measure on $\mathbb{D}$ and $p \geq 1$. Then:

a) μ is Carleson if and only if $L_a^p \subseteq L^p(\mathbb{D}, \mu)$ and $i_p : L_a^p \longrightarrow L^p(\mathbb{D}, \mu)$ is bounded (where i_p is the inclusion).

b) If μ is Carleson, i_p is compact if and only if μ is vanishing Carleson. $\square$

Note. The theorem implies the condition $\int_{\mathbb{D}} N_{\rho} dA < \infty$. This condition says that the composition operator maps the constant functions into the Bergman space and that the measure $N_{\rho} dA$ is finite.

Corollary 3.1

$$\|C_{\rho}\| = \sup \left\{ \left(\int_{\mathbb{D}} |f|^2 N_{\rho} dA \right)^{1/2} \mid f \in L_a^2, \|f\| = 1 \right\}.$$

3.2 Invertibility of the Composition Operator

Lemma 3.1 *If both ρ and φ generate composition operators on the space of analytic square integrable differentials of the unit disc, then so does $\varphi \circ \rho$, and*

$$C_{\varphi \circ \rho} = C_{\rho} C_{\varphi}. \quad (3.2.1)$$

PROOF. For any $f \in L_a^2$ we have:

$$C_{\varphi \circ \rho}(f) = (\varphi \circ \rho)'(f \circ \varphi \circ \rho) = \rho'(\varphi' \circ \rho)(f \circ \varphi \circ \rho) = C_{\rho} C_{\varphi}(f). \quad \square$$

Note. This result extends to arbitrary Riemann surfaces.

Definition 3.2 A function h defined on the unit disc is called an *inner function* if it is a bounded analytic function such that $\lim_{r \rightarrow 1-} |h(re^{i\theta})| = 1$ a.e. with respect to the Lebesgue measure on the circle.

Theorem 3.3 C_{ρ} is invertible on $\Lambda_{2,a}^1(\mathbb{D})$ if and only if ρ is invertible on $\mathbb{D}$.

PROOF. Because of the previous lemma, we need to prove only that if C_ρ is invertible, then so is ρ . First, we show that ρ is one to one.

Suppose there are points z_1 and z_2 in the disc such that $\rho(z_1) = \rho(z_2)$. Notice that ρ' does not vanish at any point z in the disc, otherwise the constant functions wouldn't be in the range of C_ρ . Choose a constant c such that $\rho'(z_1) - c\rho'(z_2) = 0$ and then choose a function $g \in L_a^2$ such that $g(z_1) - cg(z_2) \neq 0$. But then $f \circ \rho \cdot \rho' \neq g$ for every $f \in L_a^2$, and therefore C_ρ is not onto.

Let's notice that, as a consequence of the fact that ρ is injective, ρ' is in the Bergman space. Since ρ' is obviously analytic, we need only show that it is square integrable. But

$$\int_{\mathbb{D}} |\rho'|^2 dA = \int_{\rho(\mathbb{D})} dA = A(\rho(\mathbb{D})).$$

The right side of this equality represents the Lebesgue area measure of $\rho(\mathbb{D})$, which is less than or equal to the area of the disc, therefore finite. Next we show that ρ has to be an inner function. Suppose not, then there is a set E of positive 1-dimensional Lebesgue measure on the boundary such that $|\rho| < 1$ on E (ρ is a bounded analytic function and hence it has radial limits a.e.; consider ρ extended to the boundary [Ru, p.234]). The fact that C_ρ is invertible means that for every $g \in L_a^2$ there is a unique $f \in L_a^2$ such that

$$\rho' \cdot f \circ \rho = g. \quad (3.2.2)$$

Since ρ' doesn't vanish, we can divide by ρ' in this equation. We obtain

$$f \circ \rho = \frac{1}{\rho'} \cdot g.$$

The left side is a square integrable analytic function (because the classical composition operator is defined on the Bergman space), and since this equation has to be satisfied for every function $g \in L_a^2$, $1/\rho'$ is a multiplier of L_a^2 and hence has to be bounded (it is analytic). So $1/\rho'$, and therefore ρ' , has radial limits a.e. on the unit circle, and hence on E . But the equality (3.2.2) cannot hold if we choose $g \in L_a^2$ such that it doesn't have a radial limit at a point in E where ρ' does. This contradiction shows that ρ must be inner.

By [CM, p.152], a one to one inner function is a disc automorphism. $\square$

3.3 Some Spectrum Computations

The study of the spectra of classical composition operators has been the subject of a number of papers, the most important by Nordgren, Kamowitz and Cowen. Their results show the relation between the properties of the composition operator and the behavior of ρ near its fixed points. The same kind of relationship exists in our case.

The results in this section – far less extensive than the ones known for the classical composition operator – were obtained using some techniques from Cowen's paper [Co₁].

Definition 3.3 Let A be an operator on a Hilbert space $\mathcal{H}$. The *spectrum* of A is

the set $\sigma(A)$ of complex λ for which $A - \lambda I$ is not invertible.

The spectrum of an operator can be written as a union as follows:

$$\sigma(A) = \sigma_a(A) \cup \Gamma(A). \quad (3.3.1)$$

The set $\sigma_a(A)$ is the *approximate point spectrum* of A , i.e. the set of $\lambda \in \mathbb{C}$ for which there exists a sequence $\{f_n\}_{n=0}^{\infty}$ of unit vectors such that $\|(A - \lambda)f_n\| \rightarrow 0$.

The *point spectrum* $\sigma_p(A)$ is the subset of λ 's for which there is a unit vector f such that $Af = \lambda f$. The set $\Gamma(A)$ is called the *compression spectrum*, and it's the set of complex λ 's for which the closure of the range of $A - \lambda I$ is a proper subspace of $\mathcal{H}$.

We have the following relationships between the spectra of an operator and its adjoint:

$$\begin{aligned} \sigma_p(A^*) &= \overline{\Gamma(A)} \\ \sigma(A) &= \sigma_a(A) \cup \overline{\sigma_a(A^*)}, \end{aligned}$$

where the bar means complex conjugation.

Definition 3.4 A point $b \in \overline{\mathbb{D}}$ is a *fixed point* of ρ if $\lim_{r \rightarrow 1^-} \rho(rb) = b$.

Each ρ has at least one fixed point [Co₁, p.79].

Theorem 3.4 (Denjoy-Wolff, [Bu₂, p.400], [Co₁], [De], [Wo]). *If ρ is not a disc automorphism, then there is a unique fixed point $a \in \overline{\mathbb{D}}$ of ρ such that $|\rho'(a)| \leq 1$.*

If $|a| = 1$, then $0 < \rho'(a) \leq 1$, and if $|a| < 1$, then $0 \leq |\rho'(a)| < 1$.

Definition 3.5 The fixed point whose existence was established in the theorem above is called the *Denjoy-Wolff point* of ρ .

Lemma 3.2 *The action of the adjoint of the composition operator on the point evaluations K_z^a ($z \in \mathbb{D}$) is given by the formula:*

$$C_\rho^* K_z^a = \overline{\rho'(z)} K_{\rho(z)}^a. \quad (3.3.2)$$

PROOF. For every $f \in L_a^2$

$$\langle f, C_\rho^* K_z^a \rangle = \langle C_\rho f, K_z^a \rangle = f(\rho(z)) \cdot \rho'(z) = \langle f, \overline{\rho'(z)} K_{\rho(z)}^a \rangle. \quad \square$$

Definition 3.6 A nonconstant sequence $\{z_k\}_{k=0}^\infty$ is a *B-sequence* for ρ if $\rho(z_k) = z_{k-1}$ for $k \geq 1$. A point $b \in \partial\mathbb{D}$ is a *B-point* for ρ if $\lim_{r \rightarrow 1^-} \rho(rb) = b$ and there is a B-sequence for ρ converging to b .

A nonconstant sequence $\{z_k\}_{k=0}^\infty$ is an *F-sequence* for the analytic function ρ if $\rho(z_k) = z_{k+1}$ for all $k \geq 0$.

Note. [Co₁] If $\{z_k\}_{k=0}^\infty$ is a B-sequence for ρ , $\lim_{k \rightarrow \infty} |z_k| = 1$. If $\{z_k\}_{k=0}^\infty$ is an F-sequence, $\lim_{k \rightarrow \infty} z_k = a$, the Denjoy-Wolf point of ρ .

In the case of a B-sequence with $z_k \rightarrow b$, we have $\lim_k \frac{1 - |z_{k-1}|}{1 - |z_k|} \geq \rho'(b)$. If $\{z_k\}_{k=0}^\infty$ is an F-sequence and $|a| = 1$, then $\lim_k \frac{1 - |z_{k+1}|}{1 - |z_k|} \geq \rho'(a)$.

Lemma 3.3 [Co₁, p.88]. *If $b \in \partial\mathbb{D}$ is a fixed point of ρ with $\rho'(b) > 1$ and ρ is analytic in a neighborhood of b , then there are uncountably many B-sequences for ρ converging to b with $\lim_k \frac{1 - |z_{k-1}|}{1 - |z_k|} = \rho'(b)$.*

The following results have the purpose of connecting B-sequences to basic sequences equivalent to orthonormal sets in L_a^2 , which are defined below.

Definition 3.7 A sequence $\{f_k\}_{k=0}^{\infty}$ contained in a Hilbert space $\mathcal{H}$ is called a *basic sequence equivalent to an orthonormal set* if there is an orthonormal basis $\{e_k\}_k$ in $\mathcal{H}$ and a bounded operator A on $\mathcal{H}$ such that $Ae_k = f_k$ for every $k \geq 0$.

The next lemma is well known.

Lemma 3.4 Let $\mathcal{H}$ be a Hilbert space and $\{f_n\}_{n=0}^{\infty}$ a linearly independent sequence in $\mathcal{H}$. The following are equivalent:

(i) There is an orthonormal basis $\{e_n\}_n$ in $\mathcal{H}$ and a bounded operator A on $\mathcal{H}$ such that $Ae_n = f_n$ for every $n \geq 0$.

(ii) The matrix $B := \{\langle f_n, f_m \rangle\}_{m,n \geq 0}$ is bounded on l^2 .

Furthermore, A is invertible if and only if B is invertible in l^2 , and in this case the above imply

(iii) there exist positive constants m and M such that

$$m \sum_{k=0}^{\infty} |\alpha_k|^2 \leq \left\| \sum_{k=0}^{\infty} \alpha_k f_k \right\|^2 \leq M \sum_{k=0}^{\infty} |\alpha_k|^2$$

whenever $\{\alpha_k\}_{k=0}^{\infty} \in l^2$.

PROOF. (i) $\Rightarrow$ (ii). Suppose A is bounded and $Ae_n = f_n$ for some basis $\{e_n\}_n$.

Then $\langle f_n, f_m \rangle = \langle Ae_n, Ae_m \rangle = \langle A^*Ae_n, e_m \rangle$, so $(\langle f_n, f_m \rangle)_{m,n}$ is the matrix of A^*A , and is therefore bounded and invertible.

(ii) $\Rightarrow$ (i). Suppose $(\langle f_n, f_m \rangle)_{m,n}$ is the matrix of a bounded operator B with respect to an orthonormal basis $\{e_n\}_n$. For two vectors x and y in a Hilbert space, the tensor product $x \otimes y$ is given by $(x \otimes y)(z) := \langle z, y \rangle x$. Define $A_n = \sum_{k=0}^n f_k \otimes e_k$

for $n \geq 0$. Then $A_n^* = \sum_{k=0}^n e_k \otimes f_k$, and we have:

$$\begin{aligned} A_n^* A_n &= \sum_{l,k} (e_l \otimes f_l)(f_k \otimes e_k) = \sum_{l,k} e_l \otimes f_l(\langle \cdot, e_k \rangle f_k) \\ &= \sum_{l,k} \langle f_k, f_l \rangle e_l \otimes e_k, \end{aligned}$$

so the matrix of $A_n^* A_n$ agrees with the matrix of B in the upper left hand corner and has zeros everywhere else. Consequently, $A_n^* A_n \leq B$, and hence $\{A_n\}_{n \geq 0}$ is uniformly bounded. Every weak operator topology convergent subnet of $\{A_n\}_{n \geq 0}$ has the same limit $A = (\langle f_n, e_m \rangle)_{m,n}$, and thus A is in fact the weak operator topology limit of $\{A_n\}_{n \geq 0}$. We have

$$A(e_n) = \sum_{k=0}^{\infty} \langle f_n, e_k \rangle e_k = f_n.$$

(i) $\Rightarrow$ (iii). Suppose A is invertible and $\{\alpha_k\}_{k \geq 0}$ is a sequence in l^2 . Let $u = \sum_{k=0}^{\infty} \alpha_k e_k$

and $v = Au = \sum_{k=0}^{\infty} \alpha_k f_k$. Then

$$\frac{1}{\|A^{-1}\|} \|u\| \leq \|Au\| \leq \|A\| \|u\|$$

So (iii) holds with $m = 1/\|A^{-1}\|$ and $M = \|A\|$. □

Notation. We will denote by $\widehat{K_z^a}$ the unit vector $\frac{K_z^a}{\|K_z^a\|}$.

Definition 3.8 $\{z_k\}_{k=0}^{\infty}$ is an *interpolating sequence* for $H^{\infty}(\mathbb{D})$ if for every bounded sequence $\{c_k\}_{k=0}^{\infty}$ of complex numbers, there is a function $f \in H^{\infty}(\mathbb{D})$ such that $f(z_k) = c_k$, for all $k \geq 0$.

Lemma 3.5 *Let $\{z_n\}_{n=0}^\infty$ be an interpolating sequence for $H^\infty(\mathbb{D})$. Then there exists B invertible and bounded on L_a^2 such that $Be_n = \widehat{K_{z_n}^a}$.*

PROOF. Equivalently we will prove that $\left(\langle \widehat{K_{z_n}^a}, \widehat{K_{z_m}^a} \rangle\right)_{m,n}$ is bounded and invertible.

We have:

$$\langle \widehat{K_{z_n}^a}, \widehat{K_{z_m}^a} \rangle = \frac{(1 - |z_n|^2)(1 - |z_m|^2)}{(1 - \overline{z_n}z_m)^2}. \quad (3.3.3)$$

Let K_z denote the point evaluations on the Hardy space $H^2(\mathbb{D})$ and let $\widehat{K_z}$ be the corresponding normalized vectors. Their inner product in $H^2(\mathbb{D})$ is given by:

$$\langle \widehat{K_{z_n}}, \widehat{K_{z_m}} \rangle = \frac{\sqrt{(1 - |z_n|^2)(1 - |z_m|^2)}}{1 - \overline{z_n}z_m}.$$

By lemma 3.2 in [Co₁, p.88] there exists an invertible A and an orthonormal sequence $\{e_n\}_{n=0}^\infty$ in some Hilbert space $\mathcal{H}$ such that $Ae_n = \widehat{K_{z_n}}$. Let $C := A^*A = \{\langle \widehat{K_{z_n}}, \widehat{K_{z_m}} \rangle\}_{m,n}$.

For two matrices $X = (x_{ij})_{i,j}$ and $Y = (y_{ij})_{i,j}$, denote by $X * Y$ the Schur product matrix $(x_{ij}y_{ij})_{i,j}$. We then have $C * C = \left(\langle \widehat{K_{z_n}^a}, \widehat{K_{z_m}^a} \rangle\right)_{m,n}$. So we have to prove that $C * C$ is bounded.

The operator C is bounded and positive (by construction and the results in the previous lemma), and hence there are positive constants δ and M such that $\delta I \leq C \leq MI$, where I is the identity on $\mathcal{H}$. Since $C : \mathcal{H} \rightarrow \mathcal{H}$ is positive and invertible, there exists a positive invertible D such that $C = D^2$. Therefore $C \otimes C : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is positive and invertible too, with $C \otimes C = (D \otimes D)^2$

(see [Pa]). This means that there are positive constants δ' and M' such that

$$\delta' I \leq C \otimes C \leq M' I. \quad (3.3.4)$$

Since $\{e_n\}_{n=0}^\infty$ is an orthonormal basis for $\mathcal{H}$, $\{e_n \otimes e_m\}_{n,m=0}^\infty$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{H}$. Let $\mathcal{M} = \bigvee_m e_m \otimes e_m$. Assuming that $C = (c_{ij})_{i,j}$ relative to the basis $\{e_n\}_{n=0}^\infty$, we have:

$$\begin{aligned} \langle (C \otimes C)(e_n \otimes e_n), e_m \otimes e_m \rangle &= \langle C e_n \otimes C e_n, e_m \otimes e_m \rangle = \\ &= \langle C e_n, e_m \rangle \langle C e_n, e_m \rangle = c_{mn}^2, \end{aligned}$$

so $C \otimes C$ compressed to $\mathcal{M}$ has matrix $C * C$. If we compress the operators to $\mathcal{M}$, the inequality (3.3.2) becomes:

$$\delta' I_{\mathcal{M}} \leq C * C \leq M' I_{\mathcal{M}}$$

and therefore $C * C$ is bounded and invertible. $\square$

Corollary 3.2 [Co₁, p.88]. *If $\{z_k\}_{k=0}^\infty$ is a B-sequence with $\frac{1 - |z_k|}{1 - |z_{k-1}|} \leq r < 1$ for all $k \geq 1$, then $\widehat{K_{z_k}^a}$ is a basic sequence in L_a^2 equivalent to an orthonormal set. In particular, this is the case when $z_k \rightarrow b$, where b is a B-point for ρ with $\rho'(b) > 1$.*

PROOF. [Newman] If $\{z_k\}_{k=0}^\infty$ satisfies the hypothesis of the corollary, it is an interpolating sequence, and therefore the previous lemma applies. If $z_k \rightarrow b$, then $\lim_k \frac{1 - |z_k|}{1 - |z_{k-1}|} \leq \rho'(b)^{-1} < 1$. $\square$

Remark. If $\{z_k\}_{k=0}^\infty$ is a B-sequence, equation (3.3.2) gives

$$C_\rho^* \widehat{K_{z_k}^a} = \overline{\rho'(z_k)} \cdot \frac{1 - |z_k|^2}{1 - |z_{k-1}|^2} \cdot \widehat{K_{z_{k-1}}^a}. \quad (3.3.5)$$

The next two theorems will give us some information about the spectrum of the composition operator by looking at its adjoint. The first one assumes only the existence of a B-point and finds that some circle with radius less than or equal to one has to be in the spectrum, while the second one assumes the existence of a fixed boundary point other than the Denjoy-Wolff point and draws additional conclusions.

Theorem 3.5 *Let b be a B-point for ρ with B-sequence $z_k \rightarrow b$. If the limit $r = \lim_k \frac{1 - |z_k|}{1 - |z_{k-1}|} < 1$, (recall that $r \leq \rho'(b)^{-1}$), then $C_\rho^* - \lambda$ is not bounded from below for $|\lambda| = r\rho'(b)$.*

PROOF. Since by Corollary 3.2 $\widehat{K_{z_k}^a}$ is a basic sequence in L_a^2 equivalent to an orthonormal set, there are constants m and M such that

$$m \sum_{k=0}^{\infty} |\alpha_k|^2 \leq \left\| \sum_{k=0}^{\infty} \alpha_k \widehat{K_{z_k}^a} \right\|^2 \leq M \sum_{k=0}^{\infty} |\alpha_k|^2 \quad (3.3.6)$$

whenever $\{\alpha_k\}_{k=0}^{\infty} \in l^2$. Fix $n > 0$ and define

$$\alpha_k = \lambda^{k-n-1} \cdot \frac{1 - |z_n|^2}{1 - |z_k|^2} \cdot \prod_{l=1}^k \frac{1}{\rho'(z_l)}. \quad (3.3.7)$$

Let λ be such that $|\lambda| < r\rho'(b)$ and let $w_\lambda = \sum_{k=n+1}^{\infty} \alpha_k \widehat{K_{z_k}^a}$. Since

$$\lim_{k \rightarrow \infty} \frac{|\alpha_k|}{|\alpha_{k-1}|} = \lim_{k \rightarrow \infty} |\lambda|^2 \cdot \frac{1 - |z_{k-1}|^2}{1 - |z_k|^2} \cdot \frac{1}{|\rho'(z_k)|^2} = \frac{|\lambda|^2}{r^2 |\rho'(b)|^2},$$

the series for w_λ converges. Therefore

$$\begin{aligned}
C_\rho^* w_\lambda &= \sum_{k=n+1}^{\infty} \alpha_k \overline{\rho'(z_k)} \cdot \frac{1 - |z_k|^2}{1 - |z_{k-1}|^2} \cdot \widehat{K}_{z_{k-1}}^a \\
&= \sum_{k=n+1}^{\infty} \lambda^{k-n-1} \cdot \frac{1 - |z_n|^2}{1 - |z_{k-1}|^2} \cdot \prod_{l=1}^{k-1} \frac{1}{\rho'(z_l)} \widehat{K}_{z_{k-1}}^a = \\
&= \sum_{k=n+1}^{\infty} \lambda \alpha_{k-1} \widehat{K}_{z_{k-1}}^a = \lambda \alpha_n \widehat{K}_{z_n}^a + \lambda w_\lambda \\
&= \prod_{l=1}^n \frac{1}{\rho'(z_l)} \widehat{K}_{z_n}^a + \lambda w_\lambda.
\end{aligned}$$

So if we let $|\lambda_0| = r\rho'(b)$, we have

$$\begin{aligned}
\left\| (C_\rho^* - \lambda_0) \frac{w_\lambda}{\|w_\lambda\|} \right\| &\leq \left\| (C_\rho^* - \lambda) \frac{w_\lambda}{\|w_\lambda\|} \right\| + |\lambda - \lambda_0| = \\
&= \left\| \prod_{l=1}^n \frac{1}{\rho'(z_l)} \widehat{K}_{z_n}^a \right\| \cdot \|w_\lambda\|^{-1} + |\lambda - \lambda_0|
\end{aligned}$$

Next, we claim that for $\lambda \rightarrow \lambda_0$ we have $\sum_{k=0}^{\infty} |\alpha_k|^2 \rightarrow \infty$, and therefore $\|w_\lambda\| \rightarrow \infty$.

Let $\epsilon > 0$. Since $\lim_k \frac{1 - |z_k|^2}{1 - |z_{k+1}|^2} = \frac{1}{r}$ and $\lim_k |\rho'(z_k)| = \rho'(b)$, we can choose $n \in \mathbb{N}$ such that for $k \geq n$

$$\frac{1 - |z_k|^2}{1 - |z_{k+1}|^2} \geq \frac{1}{r + \epsilon} \quad \text{and} \quad \frac{\rho'(b)}{|\rho'(z_k)|} \geq 1 - \frac{\epsilon^2}{r^2} \quad (3.3.8)$$

Let λ be such that $|\lambda - \lambda_0| < \epsilon$, $|\lambda| < r\rho'(b)$ and $|\lambda|^2 > r^2\rho'(b)^2 - \epsilon^2$. From the definition of α_k we get

$$\sum_{k=n+1}^{\infty} |\alpha_k|^2 = \sum_{k=1}^{\infty} |\lambda|^{2(k-1)} \left(\frac{1 - |z_n|^2}{1 - |z_{n+k}|^2} \right)^2 \cdot \prod_{l=1}^{n+k} \frac{1}{|\rho'(z_l)|^2}.$$

By using the inequalities obtained above for our choice of k and λ , we have:

$$\begin{aligned}
\sum_{k=n+1}^{\infty} |\alpha_k|^2 &\geq \sum_{k=1}^{\infty} \frac{1}{(r + \epsilon)^{2k}} \cdot \frac{(r^2|\rho'(b)|^2 - \epsilon^2)^{k-1}}{\prod_{l=1}^{n+k} |\rho'(z_l)|^2} = \\
&= \prod_{l=1}^n |\rho'(z_l)|^2 \cdot \sum_{k=1}^{\infty} \frac{1}{(r + \epsilon)^{2k}} \cdot \frac{(r^2|\rho'(b)|^2 - \epsilon^2)^{k-1}}{\prod_{l=n+1}^{n+k} |\rho'(z_l)|^2}.
\end{aligned}$$

Notice that $\frac{(r^2|\rho'(b)|^2 - \epsilon^2)^{k-1}}{\prod_{l=n+1}^{n+k} |\rho'(z_l)|^2} = \prod_{l=n+1}^{n+k} \left(\frac{r^2\rho'(b)}{|\rho'(z_k)|} - \frac{\epsilon^2}{|\rho'(z_l)|^2} \right)$ which by using the second of the inequalities (3.3.7), is greater or equal to $\prod_{l=n+1}^{n+k} (r^2 - \epsilon^2) = (r^2 - \epsilon^2)^{k-1}$.

Therefore:

$$\begin{aligned} \sum_{k=n+1}^{\infty} |\alpha_k|^2 &\geq \prod_{l=1}^n |\rho'(z_l)|^2 \cdot \sum_{k=1}^{\infty} \frac{(r^2 - \epsilon^2)^{k-1}}{(r + \epsilon)^{2k}} = \\ &= \prod_{l=1}^n |\rho'(z_l)|^2 \cdot \sum_{k=1}^{\infty} \frac{(r - \epsilon)^{k-1}}{(r + \epsilon)^{k+1}} = \\ &= \frac{1}{(r + \epsilon)^2} \cdot \prod_{l=1}^n |\rho'(z_l)|^2 \cdot \sum_{k=1}^{\infty} \left(\frac{r - \epsilon}{r + \epsilon} \right)^{k-1}. \end{aligned}$$

The last expression converges to ∞ as $\epsilon \rightarrow 0$. Hence, $\sum_{k=0}^{\infty} |\alpha_k|^2 \rightarrow \infty$. Therefore $C_\rho^* - \lambda_0$ is not bounded from below on the subspace spanned by $\{\widehat{K_{z_k}^a} \mid k \geq n\}$. $\square$

Theorem 3.6 *Assume ρ is not a disc automorphism. Suppose that ρ has at least one fixed point b on the boundary and that ρ is analytic in its neighborhood. If $\rho'(b) > 1$, then $\sigma(C_\rho)$ (the spectrum of the composition operator) contains the unit circle and intersects every circle of radius $r < 1$.*

PROOF. Let b be the fixed point on the boundary as in the hypothesis. By Lemma 3.3

we can choose a double sequence $\{z_k^j\}_{k,j=0}^{\infty}$ such that:

- for each j the sequence $\{z_k^j\}_{k=0}^{\infty}$ is a B-sequence converging to b ,

- $\lim_{k \rightarrow \infty} \frac{1 - |z_k^j|}{1 - |z_{k-1}^j|} = \rho'(b)^{-1} < 1$,

- $\rho'(z_k^j) \neq 0$ for any $j, k \geq 0$,

- the set $\{z_k^j\}_{k,j=0}^{\infty}$ does not satisfy the Blaschke condition. Let $r < 1$. Fix j

and k_0 and let $\{z_n\}_{n=0}^{\infty}$ be the B-sequence defined by $z_n = z_{n+k_0}^j$. Then we have

$\lim_n \frac{1 - |z_n|}{1 - |z_{n-1}|} = \rho'(b)^{-1} < 1$ and so $\{\widehat{K_{z_n}^a}\}_n$ is a basic sequence in L_a^2 equivalent to an orthonormal set. For λ with $|\lambda| = r$, set

$$w_\lambda = \sum_{n=1}^{\infty} \lambda^{n-1} \prod_{l=1}^n \frac{1}{\rho'(z_l)} \cdot \frac{1 - |z_0|^2}{1 - |z_n|^2} \cdot \widehat{K_{z_n}^a}$$

Then series defining w_λ converges if and only if $\sum_n |\alpha_n|^2 < \infty$, where

$$\alpha_n = \lambda^{n-1} \prod_{l=1}^n \frac{1}{\rho'(z_l)} \cdot \frac{1 - |z_0|^2}{1 - |z_n|^2}$$

Since $\lim_n \frac{|\alpha_n|^2}{|\alpha_{n-1}|^2} = \frac{|\lambda|^2}{\rho'(b)^2} \cdot \lim_n \frac{1 - |z_{n-1}|^2}{1 - |z_n|^2} = \frac{|\lambda|^2}{\rho'(b)^2} \cdot \rho'(b)^2 = |\lambda|^2 < 1$, the series is convergent. Then

$$\begin{aligned} C_\rho^* w_\lambda &= \sum_{n=1}^{\infty} \lambda^{n-1} \prod_{l=1}^n \frac{1}{\rho'(z_l)} \cdot \frac{1 - |z_0|^2}{1 - |z_n|^2} \cdot \overline{\rho'(z_n)} \cdot \frac{1 - |z_n|^2}{1 - |z_{n-1}|^2} \cdot \widehat{K_{z_{n-1}}^a} \\ &= \lambda \sum_{n=1}^{\infty} \lambda^{n-2} \prod_{l=1}^{n-1} \frac{1}{\rho'(z_l)} \cdot \frac{1 - |z_0|^2}{1 - |z_{n-1}|^2} \cdot \widehat{K_{z_{n-1}}^a} \\ &= \lambda w_\lambda + \widehat{K_{z_0}^a}. \end{aligned}$$

So, $(C_\rho^* - \lambda)w_\lambda = \widehat{K_{z_0}^a}$. If $\sigma(C_\rho) \cap \{\lambda \mid |\lambda| = r\} = \emptyset$, then $(C_\rho^* - re^{i\theta})^{-1}$ exists for each θ and we can define $Q := \frac{1}{2\pi} \int_0^{2\pi} (C_\rho^* - re^{i\theta})^{-1} d\theta$.

$$\begin{aligned} Q \widehat{K_{z_0}^a} &= \frac{1}{2\pi} \int_0^{2\pi} w_{re^{i\theta}} d\theta \\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} r^{n-1} e^{i(n-1)\theta} \prod_{l=1}^n \frac{1}{\rho'(z_l)} \cdot \frac{1 - |z_0|^2}{1 - |z_n|^2} \cdot \widehat{K_{z_n}^a} d\theta \\ &= \frac{1 - |z_0|^2}{1 - |z_1|^2} \cdot \frac{1}{\rho'(z_1)} \widehat{K_{z_1}^a}. \end{aligned}$$

So $QK_{z_0}^a = \frac{1}{\rho'(z_1)} K_{z_1}^a$, and therefore $C_\rho^* Q K_{z_0}^a = \frac{1}{\rho'(z_l)} C_\rho^* K_{z_1}^a = K_{z_0}^a$. Because $C_\rho^* Q = Q C_\rho^*$ and since z_0 is an arbitrary element of a non-Blaschke sequence, C_ρ^* is

invertible, hence C_ρ is invertible. By Theorem 3.3, ρ is a disc automorphism, which contradicts our hypothesis. Therefore $\sigma(C_\rho) \cap \{\lambda \mid |\lambda| = r\} \neq \emptyset$ for every $r < 1$.

The fact that $\sigma(C_\rho) \supset \{\lambda \mid |\lambda| = 1\}$ is a consequence of the previous theorem (put $r = \rho'(b)^{-1}$). $\square$

Theorem 3.7 *If $\{z_k\}_{k=0}^\infty$ and $\{z'_k\}_{k=0}^\infty$ are B-sequences for ρ with $z_0 = z'_0$, but $b = \lim_{k \rightarrow \infty} z_k \neq \lim_{k \rightarrow \infty} z'_k = b'$, then every λ such that*

$$|\lambda| < \lim_{k \rightarrow \infty} \frac{1 - |z_k|}{1 - |z_{k-1}|} \leq \lim_{k \rightarrow \infty} \frac{1 - |z'_k|}{1 - |z'_{k-1}|} \leq 1$$

is an eigenvalue of C_ρ^ .*

PROOF. Assume that $z_1 \neq z'_1$. Because of the growth conditions on the sequences, $\{z_k\}_{k=0}^\infty \cup \{z'_k\}_{k=0}^\infty$ is an interpolating set [Co₁, p.91], so $\{\widehat{K_{z_n}^a}\}_{k=0}^\infty \cup \{\widehat{K_{z'_k}^a}\}_{k=0}^\infty$ is a basic sequence for L_a^2 equivalent to an orthonormal set. Let

$$w_\lambda = \sum_{n=1}^\infty \lambda^{n-1} \prod_{l=1}^n \frac{1}{\rho'(z_l)} \cdot \frac{1 - |z_0|^2}{1 - |z_n|^2} \cdot \widehat{K_{z_n}^a} - \sum_{n=1}^\infty \lambda^{n-1} \prod_{l=1}^n \frac{1}{\rho'(z'_l)} \cdot \frac{1 - |z'_0|^2}{1 - |z'_n|^2} \cdot \widehat{K_{z'_n}^a}.$$

As in the proof of the previous theorem, the series for w_λ is convergent and

$$C_\rho^* w_\lambda = \lambda w_\lambda + \widehat{K_{z_0}^a} - \widehat{K_{z'_0}^a} = \lambda w_\lambda.$$

Therefore $(C_\rho^* - \lambda)w_\lambda = 0$. $\square$

Remark. Clearly, for the hypothesis of the previous theorem to be satisfied, there have to be at least two fixed points on the boundary, other than the Denjoy-Wolff point, and the function ρ cannot be one to one. If all the conditions are met, what we get is that the disc centered at zero with radius equal to the smaller of the two limits

is contained in the spectrum of the composition operator (because of the relationship between the spectrum of an operator and its adjoint). The next theorem shows which part of the spectrum (under less restrictive conditions) cannot come from eigenvalues. In particular, if ρ is analytic in a neighborhood of b , an eigenvalue λ of the composition operator has to be outside the open disc with radius $\rho^{-1}(b)$ ($\rho^{-1}(b) \leq 1$).

Theorem 3.8 *Let $\{z_k\}_{k=0}^\infty$ be a B -sequence for ρ with $z_k \rightarrow b$, $\lim_{k \rightarrow \infty} \frac{1 - |z_k|}{1 - |z_{k-1}|} = r$ and $\rho'(z_k) \neq 0$, $\forall k \geq 0$. If $f \in L_a^2$ is an eigenvector for the composition operator induced by ρ with $f(z_0) \neq 0$, and λ is the corresponding eigenvalue, then $|\lambda| \geq r$.*

PROOF. If $C_\rho f = \lambda f$, for every $k > 0$ we have $\rho'(z_k)f(z_{k-1}) = \lambda f(z_k)$, or, equivalently, $f(z_k) = \lambda^{-1}\rho'(z_k)f(z_{k-1})$. By iterating we obtain

$$f(z_{m+k}) = \lambda^{-k} \left[\prod_{l=m+1}^{m+k} \rho'(z_l) \right] f(z_m).$$

This implies that $f(z_k) \neq 0$, $\forall k \geq 0$. But $\widehat{K_{z_k}^a} \rightarrow 0$ weakly in L_a^2 , so for every function $f \in L_a^2$, $\widehat{K_{z_k}^a}(f) \rightarrow 0$. This implies that $(1 - |z_k|^2)f(z_k) \rightarrow 0$ as $k \rightarrow \infty$.

Let $\tilde{r} < r$ and let M be such that for $m > M$ we have $\frac{1 - |z_m|}{1 - |z_{m-1}|} \geq \tilde{r}$ and $|\rho'(z_k)| > 1$. Then

$$\begin{aligned} (1 - |z_{m+k}|^2)|f(z_{m+k})| &= \\ &= \frac{1 - |z_{m+k}|}{1 - |z_m|} \cdot \prod_{j=1}^k \frac{1 - |z_{m+j}|}{1 - |z_{m+j-1}|} \cdot |\lambda|^{-k} \cdot |f(z_m)| \cdot \prod_{l=m+1}^{m+k} |\rho'(z_l)| \geq C \frac{\tilde{r}^k}{|\lambda|^k}, \end{aligned}$$

where $C > 0$ is a constant that does not depend on k . Since the left side of the equality converges to zero, we must have $|\lambda| > \tilde{r}$. But $\tilde{r}$ was an arbitrary constant

less than r , and therefore $|\lambda| \geq r$. $\square$

The next theorem finds the eigenvalues of the composition operator in the case that the function ρ has its Denjoy-Wolff point inside the unit disc.

Theorem 3.9 *If ρ is not a disc automorphism and $a \in \mathbb{D}$ (the Denjoy-Wolff point of ρ), then $\overline{\rho'(a)^n}$ is an eigenvalue of C_ρ^* for $n \geq 1$.*

If $\rho'(a) \neq 0$, λ is an eigenvalue for C_ρ if and only if $\lambda = (\rho'(a))^n$, where n is a natural number, $1 \leq n < N$ (N natural number or ∞). If $\rho'(a) = 0$, C_ρ has no eigenvalues.

PROOF. Let $\psi(z) = \frac{z+a}{1+\bar{a}z}$. Then $C_{\psi^{-1}} = C_\psi^{-1}$ and $C_\psi C_\rho C_\psi^{-1} = C_{\psi^{-1}\rho\psi}$. But the Denjoy-Wolff point of $\psi^{-1}\rho\psi$ is 0, and $(\psi^{-1}\rho\psi)'(0) = \rho'(0)$, so we may assume $a = 0$. Therefore $\rho(z) = \alpha z + \dots$, where $\alpha = \rho'(0)$. For every n we have:

$$C_\rho(z^n) = \rho'(z) \cdot \rho^n = (\alpha + \dots)(\alpha z + \dots)^n.$$

The sequence $\{z^n \sqrt{n+1}\}_{n=0}^\infty$ is an orthonormal basis for L_a^2 , and the preceding equation shows that the matrix of C_ρ is lower triangular relative to this basis. Moreover,

$$\begin{aligned} \langle C_\rho z^n \sqrt{n+1}, z^n \sqrt{n+1} \rangle &= \langle \rho'(z) \cdot \rho^n \sqrt{n+1}, z^n \sqrt{n+1} \rangle = \\ &= (n+1) \langle \alpha^{n+1} z^n + \dots, z^n \rangle = \alpha^{n+1}. \end{aligned}$$

Thus $\bar{\alpha}^{n+1}$ is an eigenvalue of C_ρ^* for every $n \geq 0$, and therefore $\rho'(a)^n \in \Gamma(C_\rho)$ (the compression spectrum of C_ρ) for every $n \geq 1$.

If $\rho'(a) \neq 0$, there is an analytic function σ on $\mathbb{D}$ such that $\sigma \circ \rho = \rho'(a)\sigma$. Moreover, a function f satisfies the equation $f \circ \rho = \lambda f$ if and only if $\lambda = (\rho'(a))^n$ and $f = c\sigma^n$, for some constant c [Co₁, p.93]. If $F = f' \in L_a^2$, taking derivatives we get $C_\rho F = \lambda F$. So in this case $(\rho'(a))^n$ is an eigenvalue of C_ρ for $n \geq 1$ ($n = 0$ corresponds to $f = c$, i.e. $F = 0$).

Can 1 be an eigenvalue for C_ρ ? If the answer is yes, there is a nonzero function $f \in L_a^2$, $f = \sum_n a_n z^n$, such that for every z we have $\rho'(z)f(\rho(z)) = f(z)$. Recalling that $\rho'(z) = \alpha + \alpha_1 z + \dots$, we must have:

$$(\alpha + \alpha_1 z + \dots)[a_0 + a_1(\alpha z + \dots) + a_2(\alpha z + \dots)^2 + \dots] = a_0 + a_1 z + \dots$$

Setting the corresponding coefficients equal, we get $a_0\alpha = a_0$, etc. Notice that by Schwarz's lemma, $|\alpha| \neq 1$, for otherwise ρ is a disc automorphism. If $\alpha \neq 1$, we get $a_0 = 0$, then since $\alpha^2 \neq 1$ we get $a_1 = 0$, and so on. Therefore $f \equiv 0$, which is a contradiction. So 1 is not an eigenvalue for our composition operator.

If $\rho'(a) = 0$, the equation $C_\rho f = \lambda f$ implies $f(a) = 0$. By writing ρ as $\rho(z) = \beta z^2 + \dots$ we obtain:

$$(2\beta z + \dots)[a_0 + a_1(\beta z^2 + \dots) + a_2(\beta z^2 + \dots)^2 + \dots] = a_0 + a_1 z + a_2 z^2 \dots$$

So $a_0 = 0$, $a_1 = 0$, etc. therefore $f \equiv 0$. Hence C_ρ has no eigenvalues in this case.

□

If the Denjoy-Wolff point a of ρ is in $\partial\mathbb{D}$ and $\rho'(a) < 1$, then there is an analytic

function $\sigma : \mathbb{D} \rightarrow \mathbb{D}$, such that $\Phi \circ \sigma = \sigma \circ \rho$ with

$$\Phi(z) = \frac{(1 + \rho'(a))z + (1 - \rho'(a))}{(1 - \rho'(a))z + (1 + \rho'(a))}$$

The symbols Φ and σ will denote these maps in the following paragraphs.

Proposition 3.1 *If ρ has Denjoy-Wolff point $a \in \partial\mathbb{D}$, with $\rho'(a) < 1$, then every F-sequence for ρ is an interpolating sequence. In particular, each F-sequence for ρ gives rise to an invariant subspace of C_ρ^* on which it is similar to a weighted shift.*

PROOF. The first statement, that the F-sequence is an interpolating sequence, is proved in [Co₁, p 94].

Let $\{z_k\}_{k=0}^\infty$ be an F-sequence for ρ , and let $V := \bigvee_{k=0}^\infty \{\widehat{K_{z_k}^a}\}$ be the space spanned by the point evaluations corresponding to the points of the sequence. Then $\{\widehat{K_{z_k}^a}\}_{k=0}^\infty$ is a basic sequence in V equivalent to an orthonormal basis. Also

$$C_\rho^* \widehat{K_{z_k}^a} = \overline{\rho'(z_k)} \cdot \frac{1 - |z_k|^2}{1 - |z_{k-1}|^2} \cdot \widehat{K_{z_{k+1}}^a}.$$

Hence C_ρ^* restricted to V is similar to a shift with weights $\overline{\rho'(z_k)} \cdot \frac{1 - |z_k|^2}{1 - |z_{k-1}|^2}$. $\square$

The following theorem states that for ρ with Denjoy-Wolff point on the boundary and derivative at this point strictly less than one, the spectrum has a circular symmetry.

Theorem 3.10 *If ρ has Denjoy-Wolff point $a \in \partial\mathbb{D}$, with $\rho'(a) < 1$, then for every real θ , C_ρ is similar to $e^{i\theta}C_\rho$. In particular, if λ is in the spectrum of C_ρ , then $e^{i\theta}\lambda$ is in the spectrum of C_ρ .*

PROOF. Let $\theta \in \mathbb{R}$ and define $\beta = \frac{\theta}{\log \rho'(a)}$, and $F(z) = \exp \left(i\beta \log \frac{1+z}{1-z} \right)$. Then $F(\Phi(z)) = e^{i\theta} F(z)$, and for $z \in \mathbb{D}$ we have $e^{-\pi\beta/2} \leq |F(z)| \leq e^{+\pi\beta/2}$. Define now $f = F \circ \sigma$. Then $f \circ \rho = e^{i\theta} f$ and f and $1/f$ are in $H^\infty(\mathbb{D})$. If we denote by M_f the operator of multiplication by f , we obtain:

$$\begin{aligned} (M_f^{-1} C_\rho M_f)(h) &= (M_f^{-1} C_\rho)(f \cdot h) = M_f^{-1}(\rho' \cdot f \circ \rho \cdot h \circ \rho) = \\ &= M_f^{-1}(\rho' \cdot e^{i\theta} f \cdot h \circ \rho) = e^{i\theta}(\rho' \cdot h \circ \rho) = (e^{i\theta} C_\rho)(h), \end{aligned}$$

for every $h \in L_a^2$, which proves the similarity of the operators. $\square$

Corollary 3.3 *If ρ satisfies the hypothesis of the preceding theorem, each eigenvalue of C_ρ has infinite multiplicity.*

PROOF. The previous construction gives infinitely many linearly independent bounded analytic functions f_k such that $f_k \circ \rho = f_k$ (obtained for $\theta = 2k\pi$). If $h \in \ker(C_\rho - \lambda)$ and $h \neq 0$, we have:

$$\begin{aligned} (C_\rho - \lambda)(f_k h) &= f_k \circ \rho \cdot h \circ \rho \cdot \rho' - \lambda f_k h = f_k \cdot h \circ \rho \cdot \rho' - \lambda f_k h = \\ &= f_k(h \circ \rho \cdot \rho' - \lambda h) = f_k(C_\rho - \lambda)h = 0. \end{aligned}$$

So $h \cdot f_k \in \ker(C_\rho - \lambda)$. $\square$

Corollary 3.4 *If ρ is analytic with Denjoy-Wolff point $a \in \partial\mathbb{D}$, $\rho'(a) < 1$, and $b \in \partial\mathbb{D}$ is another fixed point of ρ , with ρ analytic in a neighborhood of b , then $\overline{\mathbb{D}} \subset \sigma(C_\rho)$.*

PROOF. Apply circular symmetry in Theorem 3.6. $\square$

So far we have looked at the spectrum of the composition operator induced by an analytic map ρ , other than a disc automorphism. One would expect a more complete description to be possible in the case that ρ is a disc automorphism.

If ρ is a disc automorphism, the counting function N_ρ is identically equal to one and therefore the composition operator is an isometry. Hence $\|C_\rho\| \leq 1$. Therefore $\sigma(C_\rho) \subset \overline{\mathbb{D}}$. But the same is true for ρ^{-1} , and so the spectrum is contained in the unit circle.

Case 1. If the Denjoy-Wolff point $a \in \mathbb{D}$, the first part of Theorem 3.9 holds (the proof works for ρ automorphism too), and so $\sigma(C_\rho) \supset \{\rho^n(a) \mid n \geq 1\}$. Could 1 (which corresponds to $n = 0$) be an eigenvalue?

We can assume that $\rho(z) = \alpha z$, and $\alpha \neq 1$, $|\alpha| = 1$. Then $C_\rho f = f$ if and only if $\alpha \cdot f(\alpha z) = f(z)$, i.e.

$$\alpha(a_0 + a_1 \alpha z + a_2 \alpha^2 z^2 + \dots) = a_0 + a_1 z + \dots$$

By setting the coefficients of corresponding powers of z equal, we get:

$$\alpha a_0 = a_0$$

$$\alpha^2 a_1 = a_1$$

$$\dots\dots\dots$$

$$\alpha^{k+1} a_k = a_k$$

This is possible for $\alpha = \varepsilon_k$, where ε_k is a k -th root of unity and $f(z) = g(z^k)$, where g is a function in L_a^2 with $g(0) = 0$.

We can actually find the spectrum in this case. The set $\{z^n \mid n \in \mathbb{N}\}$ is a spanning set of orthogonal vectors for L_a^2 . $C_\rho(z^n) = \alpha \rho^n(z) = \alpha^{n+1} z^n$. Since the operator is diagonal with respect to this basis, the spectrum is the closure of the set of eigenvalues. Hence $\sigma(C_\rho) = \overline{\{\rho'(a)^n \mid n \geq 1\}}$. Notice that if $\alpha = e^{i\theta}$ and θ is not a rational multiple of π , the spectrum is the whole unit circle; otherwise it consists of a finite number of points. $\square$

Case 2. The Denjoy-Wolff point is on the boundary and the transformation is hyperbolic. The spectrum of the operator is the unit circle.

PROOF. For ρ hyperbolic we have $\rho'(a) < 1$, so Theorem 3.10 applies and so the spectrum is the whole unit circle. $\square$

Case 3. The Denjoy-Wolff point is on the boundary of the disc and ρ is parabolic. Unfortunately, Theorem 3.10 can no longer be applied, since for ρ parabolic we have $\rho'(a) = 1$.

Some symmetry results can be obtained. Since all parabolic automorphisms are conjugate, we may assume that the fixed point of ρ is 1. Define $\gamma(z) = i \frac{1+z}{1-z}$ and notice that there is a real number b such that $\gamma \circ \rho \circ \gamma^{-1} = z + b$. The function $g_\theta(z) = \exp\left(-\frac{\theta}{b} \frac{1+z}{1-z}\right)$ is bounded for θ of the same sign as b and satisfies the equation $g_\theta \circ \rho = e^{i\theta} g_\theta$.

If C_ρ has an eigenvalue λ , i.e. there is an eigenfunction f such that $f \circ \rho \cdot \rho' = \lambda f$,

then

$$C_\rho(fg_\theta) = \rho' \cdot f \circ \rho \cdot g_\theta \circ \rho = \lambda e^{i\theta} fg_\theta,$$

so every point on the unit circle is an eigenvalue. But C_ρ is unitary, and hence eigenfunctions corresponding to distinct eigenvalues are orthogonal. This means that we have uncountably many mutually orthogonal vectors, which is impossible in a separable Hilbert space.

We will prove that the spectrum of the operator is the unit circle. Before doing that, we need to quote some results.

Definition 3.9 Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. A subspace $\mathcal{M}$ of $\mathcal{H}$ *reduces* A if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under A .

Definition 3.10 Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. A is called *reductive* if every invariant subspace of it is reducing.

Theorem 3.11 [We] *A unitary operator with spectrum strictly contained in the unit circle is reductive.*

Assume that the spectrum of the composition operator is not the whole unit circle.

Consider the following subspace of $\Lambda_{2,a}^1(\mathbb{D})$:

$$\mathcal{M} := \{f \in L_a^2 \mid f(\rho^n(0)) = 0, \forall n \geq 0\}$$

The subspace is not trivial because the sequence $\{\rho^n(0)\}_{n \geq 0}$ is a Blaschke sequence

and the Blaschke product g formed with it belongs to $\mathcal{M}$. Clearly $\mathcal{M}$ is an invariant subspace for C_ρ . However, $C_\rho^*(g) = C_\rho^{-1}(g)$ is not in $\mathcal{M}$, and thus C_ρ is not reductive.

This proves that the spectrum is the unit circle. $\square$

Chapter 4

Composition Operators on $\Lambda_{2,a}^1(N)$

In this chapter we will study the composition operator induced by an analytic map between the spaces of analytic square integrable 1-forms of two Riemann surfaces. We will see that the results for compact surfaces are very different from those for non-compact surfaces. Throughout this chapter $\rho : M \rightarrow N$ will be an analytic map between the Riemann surfaces M and N .

4.1 Boundedness and Compactness

Let's try to understand first what happens if at least one of the Riemann surfaces is compact. Notice that if M is compact, then so is N : non-constant analytic maps are both open and continuous, so the image of M under ρ has to be both open and compact, which is impossible unless N is the image and is compact itself.

If N is compact, the space of analytic differentials is finite dimensional.

Proposition 4.1 *On a compact Riemann surface every analytic 1-form is square integrable.*

PROOF. Choose a finite triangulation of the surface of N , with each triangle T_α contained in the domain of a chart α (i.e. $N = \bigcup_{\alpha \in F} T_\alpha$, where F is a finite set).

Let ω be an analytic 1-form on N with representation $f^\alpha d\alpha$ in the chart α . Then

$$\|\omega\|^2 = 2 \sum_{\alpha \in F} \int_{\alpha(T_\alpha)} |f^\alpha|^2 dA < \infty. \quad \square$$

Hence $\Lambda_{2,a}^1(N)$ has finite dimension, equal to the genus g_N of the surface N . Therefore boundedness and compactness are clear as long as $\Lambda_{2,a}^1(M)$ is not trivial. In particular, if M is compact too, the composition operator has a $g_N \times g_M$ matrix.

The case when $M = N$ and ρ is an automorphism is known (see [FK]). The number of automorphisms is finite and they all have finite order. If $\rho^n = I$ (where $n = \text{ord } \rho$), then the minimal polynomial of the matrix of C_ρ divides $X^n - 1$, and so all its roots are distinct. Therefore the matrix can be diagonalized and the only possible eigenvalues are the n^{th} roots of unity. If ρ has no fixed points, all roots of unity will be eigenvalues.

For the remaining part of this section, let's assume that both surfaces are open. There is an unpleasant fact about open Riemann surfaces, nevertheless. Even though the spaces of analytic 1-forms are infinite dimensional, the spaces of square integrable ones may be zero dimensional! There is a very simple example of this, the complex plane. From now on we shall consider only surfaces whose spaces of square integrable analytic 1-forms are nontrivial.

Before talking about boundedness and compactness in the general case, let's notice one more equivalent condition for the boundedness of C_ρ on the Bergman space of the disc. Carleson measure is not defined on Riemann surfaces. This new condition will

allow us to talk about boundedness and compactness of the composition operator on arbitrary Riemann surfaces.

Let $P : L^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D})$ be the orthogonal projection defined by the formula:

$$Pf(z) = \int_{\mathbb{D}} K(z, w) f(w) dA(w) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} dA(w)$$

Given $\varphi \in L^\infty(\mathbb{D})$ we can define $T_\varphi : L_a^2 \longrightarrow L_a^2$, the Toeplitz operator on L_a^2 , by $T_\varphi(f) = P(\varphi f)$, and we have $\|T_\varphi\| \leq \|\varphi\|_\infty$.

The definition can be extended to finite measures on $\mathbb{D}$ [Zh, p.106] by

$$T_\mu f(z) = \int_{\mathbb{D}} K(z, w) f(w) d\mu(w)$$

The operator will be unbounded in general, but densely defined ($H^\infty \subset \text{dom } T_\mu$).

If $\text{supp } \mu$ is compact, then T_μ is bounded and compact.

If $d\mu \ll dA$, let $\varphi = \frac{d\mu}{dA}$. Then $T_\mu = T_\varphi$.

Notice that for $f, g \in L_a^2$

$$\langle T_\mu f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} d\mu(z)$$

This can be used as an alternative definition of the Toeplitz operator.

Theorem 4.1 [Zh, p.11-114] *Let μ be a finite positive Borel measure on $\mathbb{D}$.*

1. *The following are equivalent:*

(a) T_μ is bounded on L_a^2

(b) μ is a Carleson measure on L_a^2 .

2. The following are equivalent:

(a) T_μ is compact on L_a^2

(b) μ is a vanishing Carleson measure on L_a^2 .

Corollary 4.1 *The composition operator C_ρ and the Toeplitz operator T_{N_ρ} are simultaneously bounded (or compact), under the assumption that the counting function is in $L^1(dA)$.*

Remark. It is clearly necessary to have $\int_{\mathbb{D}} N_\rho dA < \infty$ for the boundedness of C_ρ , and this also insures that the Toeplitz operator T_{N_ρ} is densely defined. But if we have boundedness of C_ρ , we actually get more than that. There exists a constant $c > 0$ such that

$$\int_{\mathbb{D}} N_\rho |f|^2 dA \leq c \int_{\mathbb{D}} |f|^2 dA, \quad \forall f \in L_a^2.$$

Thus the integral on the left side is finite for all $f \in L_a^2$. This means $\int_{\mathbb{D}} N_\rho \omega \wedge \bar{\omega} < \infty$ for all $\omega \in \Lambda_{2,a}^1(\mathbb{D})$.

So instead of asking for $\int_{\mathbb{D}} N_\rho dA < \infty$, we could ask that $\int_{\mathbb{D}} N_\rho \omega_0 \wedge \bar{\omega}_0 = \int_{\mathbb{D}} N_\rho |f_0|^2 dA < \infty$, for some particular differential $\omega_0 = f_0 dz \in \Lambda_{2,a}^1(\mathbb{D})$ without zeros. We can then write any other differential $\omega \in \Lambda_{2,a}^1(\mathbb{D})$ as $\omega = g \omega_0$, where g is analytic on $\mathbb{D}$ and $g \in L^2(\omega_0 \wedge \bar{\omega}_0)$. So $\Lambda_{2,a}^1(\mathbb{D})$ is also isomorphic to $L_m^2(\omega_0 \wedge \bar{\omega}_0)$ (the space of square integrable meromorphic functions with respect to the measure $\omega_0 \wedge \bar{\omega}_0$) via $\omega \mapsto g$. The Toeplitz operator is now defined by asking that the

equality

$$\langle T_{N_\rho} f, g \rangle = \int_{\mathbb{D}} N_\rho f \bar{g} \omega_o \wedge \overline{* \omega_o} \quad (4.1.1)$$

be satisfied for as many $f, g \in L^2_m(\omega_o \wedge \overline{* \omega_o})$ as possible. It is again densely defined and we get the equivalence between the boundedness (or compactness) of C_ρ and T_{N_ρ} , this time under the assumption that the counting function is in $L^1(\omega_o \wedge \overline{* \omega_o})$.

In conclusion, one “ L^1 - type” condition is necessary to get the Toeplitz operator T_{N_ρ} at least densely defined in the case of the disc, and then prove the equivalence between the boundedness of the Toeplitz and the composition operators. Once we have the boundedness established all these conditions are satisfied, and the operator is actually well defined.

We will start the discussion about boundedness in the general case by defining *point evaluations* on the space $\Lambda^1_{2,a}(N)$, thus making this space look like a functional Hilbert space (although it is not a space of functions).

Definition 4.1 Let M be a Riemann surface and $\omega \in \Lambda^1_{2,a}(M)$. For each point $z \in M$ and each chart α on M with domain $U_\alpha \ni z$, define:

$$K_{z,\alpha}(\omega) := f^\alpha(z) = f^\alpha_\alpha(\alpha(z)), \quad (4.1.2)$$

where $\omega = f^\alpha d\alpha$ in the chart α .

Note. It would be perhaps more appropriate to call these functions *point-chart evaluations* since we get one for each pair (z, α) with $z \in U_\alpha$.

Proposition 4.2 *For each point-chart pair (z, α) as in the above definition, $K_{z,\alpha} : \Lambda_{2,a}^1(M) \longrightarrow \mathbb{C}$ is a bounded linear functional.*

PROOF. To see that $K_{z,\alpha}$ is linear, take $\omega_1 = f_1^\alpha d\alpha$ and $\omega_2 = f_2^\alpha d\alpha$ analytic square integrable 1-forms and let $a, b \in \mathbb{C}$. Then

$$K_{z,\alpha}(a\omega_1 + b\omega_2) = (af_1^\alpha + bf_2^\alpha)(z) = aK_{z,\alpha}(\omega_1) + bK_{z,\alpha}(\omega_2).$$

Now we have to show that $K_{z,\alpha}$ is bounded. Let $\{\omega_n\}_{n \geq 0}$ be a net of forms in $\Lambda_{2,a}^1(M)$ converging in norm to zero. That is, $\int_M \omega_n \wedge \overline{\omega_n} \rightarrow 0$. Assume than in the chart α the form ω_n is expressed as $f_n^\alpha d\alpha$. Then

$$\int_M \omega_n \wedge \overline{\omega_n} \geq \int_{U_\alpha} \omega_n \wedge \overline{\omega_n} = 2 \int_{\alpha(U_\alpha)} |f_n^\alpha \circ \alpha^{-1}|^2 dA.$$

Therefore $f_n^\alpha(z) \rightarrow 0$, and hence $K_{z,\alpha}$ is continuous (since in the Bergman space norm convergence implies pointwise convergence). $\square$

The next lemma establishes the connection between the point evaluations at a point of the Riemann surface arising from two different charts.

Lemma 4.1 *Suppose w is a point on the Riemann surface M and β and γ are two charts such that $w \in U_\beta \cap U_\gamma$. Then*

$$K_{w,\beta} = \overline{D^\beta(\gamma)(w)} K_{w,\gamma}. \quad (4.1.3)$$

PROOF. Let $\omega \in \Lambda_{2,a}^1(M)$ have representations $f^\beta d\beta$ and $f^\gamma d\gamma$ in the charts β and γ . Since the point evaluations are bounded linear functionals, their action on

$\Lambda_{2,a}^1(M)$ can be represented via an inner product (Riesz's Theorem). We have:

$$\langle \omega, K_{w,\beta} \rangle = K_{w,\beta}(\omega) = f^\beta(w) = D^\beta(\gamma)(w) \cdot f^\gamma(w),$$

where the last equality comes from the change of chart formula (1.2.6). Replacing $f^\gamma(w)$ by $\langle \omega, K_{w,\gamma} \rangle$ in the line above we obtain:

$$\langle \omega, K_{w,\beta} \rangle = D^\beta(\gamma)(w) \langle \omega, K_{w,\gamma} \rangle = \langle \omega, \overline{D^\beta(\gamma)(w)} K_{w,\gamma} \rangle.$$

Since this is true for every ω in the space, the conclusion of the lemma follows. $\square$

Now we will try to establish boundedness and compactness conditions for the composition operators acting between the spaces of square integrable analytic differentials of two arbitrary open Riemann surfaces.

Theorem 4.2 *If N_ρ satisfies $\int_N N_\rho \omega \wedge \star \bar{\omega} < \infty$ for every $\omega \in \Lambda_{2,a}^1(N)$, the composition operator C_ρ is bounded.*

PROOF. Since by (2.1.10) $\|C_\rho \omega\|^2 = \int_N N_\rho \omega \wedge \star \bar{\omega}$, the condition imposed on the counting function N_ρ insures that the operator is defined on $\Lambda_{2,a}^1(N)$. We will show that the operator is closed, and therefore bounded (by the Closed Graph Theorem).

Take a sequence $\omega_n \rightarrow \omega$ in $\Lambda_{2,a}^1(N)$ such that $C_\rho \omega_n \rightarrow \eta$ in $\Lambda_{2,a}^1(M)$. Let $z \in M$, not a branch point for ρ . There exists a neighborhood of z on which ρ is one to one, and we can assume that it is also contained in the domain of a chart α . Furthermore, we can assume that its image under ρ is contained in the domain of a chart β . Denote this neighborhood by U_α and its image by V_β .

Suppose that each form ω_n has the representation $f_n^\beta d\beta$ on V_β and that ω has the

representation $f^\beta d\beta$ on V_β . Since $\omega_n \rightarrow \omega$, we have that $f_n^\beta(\rho(z)) \rightarrow f^\beta(\rho(z))$ for every $z \in U_\alpha$. On the other hand, $C_\rho \omega_n$ and $C_\rho \omega$ have representations $(f_n^\beta \circ \rho) \cdot D^\alpha(\beta \circ \rho) d\alpha$ and $(f^\beta \circ \rho) \cdot D^\alpha(\beta \circ \rho) d\alpha$ on U_α . If $\eta = h^\alpha d\alpha$ on U_α , we get that $(f_n^\beta \circ \rho)(z) \cdot D^\alpha(\beta \circ \rho)(z) \rightarrow h^\alpha(z)$ for every $z \in U_\alpha$. Hence $(f^\beta \circ \rho) \cdot D^\alpha(\beta \circ \rho) = h^\alpha$ on U_α . By the Identity Theorem, $C_\rho \omega = \eta$, and therefore the operator is closed. $\square$

Note that by (2.1.10)

$$C_\rho : \Lambda_{2,a}^1(N) \rightarrow \Lambda_{2,a}^1(M)$$

is bounded if and only if $\exists c > 0$ s.t. $\forall \omega \in \Lambda_{2,a}^1(N)$

$$\int_N N_\rho \omega \wedge {}^*\bar{\omega} \leq c \int_N \omega \wedge {}^*\bar{\omega}. \quad (4.1.4)$$

Corollary 4.2

$$\|C_\rho\| = \sup \left\{ \left(\int_N N_\rho \omega \wedge {}^*\bar{\omega} \right)^{\frac{1}{2}} \mid \omega \in \Lambda_{2,a}^1(N), \|\omega\| = 1 \right\}.$$

Corollary 4.3 *If ρ is invertible, the composition operator C_ρ is unitary.*

PROOF. If ρ is invertible, N_ρ is constant and equal to one. $\square$

For any two differential forms $\lambda, \omega \in \Lambda_{2,a}^1(N)$ we have $\lambda = f\omega$ where f is a meromorphic function on N . If φ is a positive function that satisfies

$$\int_N \varphi \omega \wedge {}^*\bar{\omega} < \infty \quad \forall \omega \in \Lambda_{2,a}^1(N) \quad (4.1.5)$$

we have:

$$\begin{aligned}
 \left| \int_N \varphi \lambda \wedge \star \bar{\omega} \right| &= \left| \int_N \varphi f \omega \wedge \star \bar{\omega} \right| \\
 &\leq \left[\int_N (\varphi^{\frac{1}{2}})^2 \omega \wedge \star \bar{\omega} \cdot \int_N |\varphi^{\frac{1}{2}} f|^2 \omega \wedge \star \bar{\omega} \right]^{\frac{1}{2}} \\
 &= \left[\int_N \varphi \omega \wedge \star \bar{\omega} \cdot \int_N \varphi \lambda \wedge \star \bar{\lambda} \right]^{\frac{1}{2}} < \infty
 \end{aligned}$$

Assume that C_ρ is bounded. We can now define a Toeplitz operator on $\Lambda_{2,a}^1(N)$ by letting $T_{N_\rho} \lambda$ be the unique element in $\Lambda_{2,a}^1(N)$ that satisfies

$$\langle T_\varphi \lambda, \omega \rangle = \int_N N_\rho \lambda \wedge \star \bar{\omega}. \quad (4.1.6)$$

Proposition 4.3 *The Toeplitz operator T_{N_ρ} defined as above is bounded and positive.*

PROOF. Let $\omega \in \Lambda_{2,a}^1(N)$. Then $\langle T_{N_\rho} \omega, \omega \rangle = \int_N N_\rho \omega \wedge \star \bar{\omega} \geq 0$. $\square$

Corollary 4.4

$$T_{N_\rho} = C_\rho^* C_\rho \quad (4.1.7)$$

PROOF. We have already noticed that for every $\omega \in \Lambda_{2,a}^1(N)$ we obtain $\langle C_\rho^* C_\rho(\omega), \omega \rangle = \|C_\rho(\omega)\|^2 = \langle T_{N_\rho} \omega, \omega \rangle$. By using the polarization identity we see that $\langle C_\rho^* C_\rho(\omega), \lambda \rangle = \langle T_{N_\rho} \omega, \lambda \rangle$ for all $\omega, \lambda \in \Lambda_{2,a}^1(N)$. $\square$

We can now state the connection between the compactness of the composition operator and that of the Toeplitz operator on the space of analytic square integrable 1-forms.

Theorem 4.3 *The operator T_{N_ρ} is compact if and only if the operator C_ρ is compact.*

PROOF. The statement is a consequence of the following:

Proposition 4.4 [Zh, p.11]. *A bounded linear operator A is compact if and only if A^*A is compact.* $\square$

As in the case of the disc, we can find a representation of the Toeplitz operator in terms of the point-evaluations. For every $\omega \in \Lambda_{2,a}^1(N)$ and every chart (β, V_β) on N with $z \in V_\beta$, define

$$g^\beta(z) = \int_N N_\rho \omega \wedge \overline{*K_{z,\beta}} \quad (4.1.8)$$

Because of the relation (4.1.3), $g^\beta d\beta$ defines a 1-form on N .

Proposition 4.5 *The 1-form $T_{N_\rho}(\omega)$ defined by the Toeplitz operator is equal to g^β (defined above) in each chart (β, V_β) .*

PROOF. Let $T_{N_\rho}(\omega) = t^\beta d\beta$ in the chart β . For every $z \in V_\beta$, we have $t^\beta(z) = \langle T_{N_\rho} \omega, K_{z,\beta} \rangle = \int_N N_\rho \omega \wedge \overline{*K_{z,\beta}} = g^\beta(z)$. $\square$

4.2 About the Invertibility of Composition Operators

The problem that we will be addressing in this section is again the connection between the invertibility of the composition operator and the invertibility of the map inducing the operator. By (3.2.1) we know that if ρ is an invertible map between two Riemann surfaces, the composition operator induced is invertible as well, and we have:

$$C_\rho^{-1} = C_{\rho^{-1}} \quad (4.2.1)$$

So the question that we need to answer is whether the invertibility of the composition operator implies the invertibility of the inducing map. As in the previous section,

there are two very different cases: the case that both surfaces are open, and the case that both surfaces are compact.

Case 1. Let M and N be compact Riemann surfaces and $\rho : M \longrightarrow N$ an analytic map. Since the space of analytic differentials on a compact surface of genus g is finite dimensional with dimension g , it is clear that the composition operator can be invertible only if the two surfaces have the same genus.

Theorem 4.4 (Riemann-Hurwitz, [Fo, p.140]) *Suppose $\rho : M \longrightarrow N$ is an n -sheeted analytic map between compact surfaces M and N with total branching order b . Let g be the genus of M and g' be the genus of N . Then*

$$g = \frac{b}{2} + n(g' - 1) + 1 \quad (4.2.2)$$

As a corollary we can prove the following.

Theorem 4.5 *Let M and N be compact Riemann surfaces with (same) genus $g \geq 2$. Then the composition operator induced by an analytic map $\rho : M \longrightarrow N$ is invertible if and only if ρ is invertible.*

PROOF. Use the Riemann-Hurwitz formula to get

$$\frac{b}{2} + (n - 1)(g - 1) = 0$$

Equality can only hold if $b = 0$ (no branching points) and $n = 1$. The latter means that the map is one to one. But a nonconstant map between compact Riemann surfaces is always onto, hence ρ is invertible. $\square$

Remark. So what happens if we have a map between two surfaces of genus one? Since the spaces of analytic differential forms are one dimensional, the composition operator is invertible. But the map ρ doesn't have to be one to one. We will construct such a map ρ from a torus with periods λ_1 and λ_2 onto itself. Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be the lifting of ρ to the complex plane (which is the universal cover of the torus). Then f satisfies an equation of the type $F(z + \lambda) = F(z) + \mu$ (for every $\lambda \in \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$ there is a $\mu \in \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$ such that equality holds for every complex number z). By taking derivatives we obtain that F' must be doubly periodic. Since F is complex valued, F' has to be a constant. Therefore $F(z) = az + b$ with $a \in \mathbb{C}$. As an example, if $a = 2$, the induced map ρ is four to one. So tori are the exceptions in the case of compact surfaces.

Case 2. Let M and N be open Riemann surfaces. In Chapter 2 we analyzed the particular case when both surfaces were the unit disc. However, the expression that gives the pull-back of a differential form depends on the chosen charts and we can no longer talk about boundary in the sense that we talked about the boundary of the disc. These are the things that make the problem more difficult in the general case.

Recall now that in the case of the disc the adjoint of the composition operator acted like a weighted translation on the point evaluations. On the other hand, C_ρ is invertible if and only if C_ρ^* is, so it would be helpful to know the action of C_ρ^* on the point evaluations.

Lemma 4.2 *Let $\rho : M \longrightarrow N$ be analytic such that $\rho(z) = w$, and let α and β be charts on M and N such that $z \in U_\alpha$ and $w \in V_\beta$ (where U_α and V_β are the domains of the two charts). The adjoint of the composition operator acts on the point evaluations as follows:*

$$C_\rho^* K_{z,\alpha} = \overline{\rho'_{\alpha\beta}(\alpha(z))} K_{\rho(z),\beta}. \quad (4.2.3)$$

PROOF. Let $\omega \in \Lambda_{2,a}^1(N)$ be a form on N with representation $g^\beta d\beta$ in the chart β . Then $C_\rho(\omega)$ will be a form in $\Lambda_{2,a}^1(M)$ and we want to find its representation in chart α .

$$C_\rho(g^\beta d\beta) = (g^\beta \circ \rho) d(\beta \circ \rho) = (g^\beta \circ \rho) D^\alpha(\beta \circ \rho) d\alpha.$$

Using the notation $\rho_{\alpha\beta}$ for $\beta \circ \rho \circ \alpha^{-1}$ and replacing $D^\alpha(\beta \circ \rho)$ from (1.2.1) we get:

$$C_\rho(g^\beta d\beta) = (g^\beta \circ \rho) \cdot (\rho'_{\alpha\beta} \circ \alpha) d\alpha.$$

Now we can evaluate $C_\rho^*(K_{z,\alpha})$.

$$\begin{aligned} \langle \omega, C_\rho^*(K_{z,\alpha}) \rangle &= \langle C_\rho(\omega), K_{z,\alpha} \rangle \\ &= (g^\beta \circ \rho)(z) \cdot \rho'_{\alpha\beta}(\alpha(z)) = \rho'_{\alpha\beta}(\alpha(z)) \langle \omega, K_{\rho(z),\beta} \rangle \\ &= \langle \omega, \overline{\rho'_{\alpha\beta}(\alpha(z))} K_{\rho(z),\beta} \rangle \end{aligned}$$

This proves that $C_\rho^* K_{z,\alpha} = \overline{\rho'_{\alpha\beta}(\alpha(z))} K_{\rho(z),\beta}$. □

Note. If we choose a chart γ with $\rho(z) \in V_\gamma$ (the domain of γ), the above formula becomes $C_\rho^* K_{z,\alpha} = \overline{\rho'_{\alpha\gamma}(\alpha(z))} K_{\rho(z),\gamma}$. Because of the previous lemma, the two are compatible.

Theorem 4.6 *Let $\rho : M \longrightarrow N$ be an analytic map between two open Riemann surfaces. Assume that the analytic square integrable 1-forms on M separate points. If the composition operator induced between the spaces of analytic square integrable 1-forms is invertible, the map ρ is one to one.*

PROOF. Suppose not, then there are points x and y in M such that $\rho(x) = \rho(y)$. Hence $K_{\rho(x),\beta} = K_{\rho(y),\beta}$, for every chart β with domain containing $\rho(x)$. Choose charts α and $\tilde{\alpha}$ in the neighborhoods of x and y . Let μ be a constant such that $\overline{\rho'_{\alpha\beta}(\alpha(z))} - \mu \overline{\rho'_{\tilde{\alpha}\beta}(\tilde{\alpha}(z))} = 0$. We have

$$C_{\rho}^*(K_{x,\alpha} - \mu K_{y,\tilde{\alpha}}) = \left(\overline{\rho'_{\alpha\beta}(\alpha(z))} - \mu \overline{\rho'_{\tilde{\alpha}\beta}(\tilde{\alpha}(z))} \right) K_{\rho(x),\beta} = 0.$$

Since C_{ρ}^* is invertible, its kernel must consist of the zero form alone, and so $K_{x,\alpha} - \mu K_{y,\tilde{\alpha}} = 0$. But this is not possible since on M we can choose $\omega \in \Lambda_{2,a}^1(M)$ with $\omega(x) = 0$ and $\omega(y) \neq 0$. $\square$

Note. The question is, how restrictive is the condition that analytic square integrable 1-forms separate points on M ? If we don't ask for the forms to be square integrable, 1-forms do separate points on every open Riemann surface [Fo, p. 204-205]. However, not all these forms are square integrable. But there are surfaces on which the analytic square integrable 1-forms separate points, for example the surfaces obtained from compact Riemann surfaces by removing a finite number of closed discs.

Chapter 5

Examples and Questions

We have shown in Chapter 3 how our theory works when we have an analytic map from the unit disc into itself. However, we are interested in looking at our composition operators on spaces of forms defined on more general Riemann surfaces.

5.1 Examples

The purpose of this section is to provide two classes of examples of Riemann surfaces and corresponding composition operators defined on the spaces of square integrable analytic 1-forms.

Example 1. Open regions of the complex plane. Any open connected subset of the complex plane is a Riemann surface. The analytic structure can be defined with just one chart, the inclusion function. The space of analytic 1-forms is isomorphic to the space of analytic functions via the map $f dz \longrightarrow f$. Obviously, not all of these forms will be square integrable. In fact, if the regions consist of the whole plane, with only a finite number of points removed, none of them will! But if the region is obtained by removing a set with nonempty interior from the complex plane, then the space of square integrable analytic forms becomes interesting. In the case of the disc, we obtained the Bergman space. Let's try to see what happens if we consider annuli,

together with analytic self-maps.

Let $r > 1$ and consider the annulus $A = \{z \in \mathbb{C} \mid 1/r < |z| < r\}$. We need an analytic function $\rho : A \rightarrow A$ to induce the composition operator. It seems reasonable to ask that the index of the image under ρ of the unit circle be non zero – otherwise we wouldn't be taking advantage of the special structure of the annulus. The following theorem (an application of Schwarz's Lemma) tells us exactly how many maps satisfy such a condition.

Theorem 5.1 [Bu₁, p.348] *Let $0 < r_j < R_j < \infty$ and $A_j = \{z \in \mathbb{C} \mid r_j < |z| < R_j\}$ for $j = 1, 2$. Let $r_1 < r < R_1$ and define γ to be the circle centered at zero with radius r . Suppose $\rho : A_1 \rightarrow A_2$ is holomorphic and set $n = \text{Ind}_{\rho \circ \gamma}(0)$. Then:*

$$|n| \leq \frac{\log(R_2/r_2)}{\log(R_1/r_1)}$$

and if equality holds, $\rho(z) = cz^n$ for some complex constant c .

According to this theorem, a self map of the annulus can only be of the form $\rho(z) = cz$ or $\rho(z) = c/z$, where the constant is chosen appropriately to make the range contained in the domain. For our annulus A , it must have absolute value one. In either case, ρ is invertible and so is the composition operator induced by it.

(i) Let $\rho : A \rightarrow A$, $\rho(z) = e^{i\theta}z$. Then, for every analytic square integrable function f on A we have $C_\rho(f)(z) = e^{i\theta}f(e^{i\theta}z)$. We can compute the spectrum of the operator. The set $\{z^n\}_{n \in \mathbb{Z}}$ is an orthogonal basis for the space of square integrable analytic functions on A and $C_\rho(z^n) = e^{i(n+1)\theta}z^n$. So the composition

operator C_ρ is diagonal with respect to this basis and the spectrum is the closure of the set $\{e^{in\theta} | n \in \mathbb{Z}\}$. If θ is a rational multiple of π , the spectrum is finite, otherwise it's the whole unit circle.

(ii) Let $\rho : A \longrightarrow A$, $\rho(z) = 1/z$. Then, for every analytic square integrable function f on A we have $C_\rho(f)(z) = -1/z^2 \cdot f(1/z)$. Again, we can compute the spectrum of the operator. Let's first find the point spectrum. If $\lambda \neq 0$ is an eigenvalue for C_ρ , we have the following functional equation:

$$-\frac{1}{z^2}f\left(\frac{1}{z}\right) = \lambda f(z).$$

By replacing z with $1/z$ in the line above, we get:

$$f\left(\frac{1}{z}\right) = -\frac{z^2}{\lambda}f(z).$$

Combining the two equations we obtain:

$$f(z) = \frac{f(z)}{\lambda^2}$$

for every z in A . Therefore the only possible eigenvalues are 1 and -1 . Both of them are actual eigenvalues with infinite multiplicity. For, let g be a square integrable analytic function on A . Then $f_1(z) = \frac{1}{z} \left(g(z) - g\left(\frac{1}{z}\right) \right)$ is an eigenfunction corresponding to $\lambda = 1$ and $f_{-1}(z) = \frac{1}{z} \left(g(z) + g\left(\frac{1}{z}\right) \right)$ is an eigenfunction corresponding to $\lambda = -1$.

The operator C_ρ is isometric and onto, therefore unitary. Hence $C_{\frac{1}{z}}^* = C_{\frac{1}{z}}^{-1} = C_{\frac{1}{z}}$. So the operator is also selfadjoint. Hence $\sigma(C_\rho) = \sigma_p(C_\rho) = \{-1, 1\}$. $\square$

Example 2. Open Riemann surfaces with finitely generated fundamental group.

These surfaces are obtained from compact surfaces by removing a finite number of points and/or discs [Be, p.268]. There is not much difference between a compact surface and the same surface with a finite number of points removed (the spaces of analytic square integrable 1-forms will be the same on both), so we will consider Riemann surfaces obtained from compact ones by removing at least a closed disc. These surfaces will have enough analytic square integrable 1-forms. The reason is that any analytic form from the original surface multiplied with a function analytic and bounded on the new surface will be a square integrable analytic form on the new surface. But it is easy to produce analytic bounded forms on the new surface by taking meromorphic functions on the original surface with as many poles as desired in the interior of the disc to be removed. We will now consider a very simple example of this situation.

Consider two hyperbolic automorphisms g and h of the unit disc such that g takes the geodesic L_1 onto L_2 and h takes the geodesic L_3 onto L_4 , as in Fig. 5-1. If we choose the points at which these geodesics meet the unit circle to be $\frac{k\pi}{6}$ for $k = 1, 2, 4, 5, 7, 8, 10, 11$, we can easily compute the following expressions for g and h :

$$g(z) = \frac{2z + \sqrt{3}}{\sqrt{3}z + 2} \quad h(z) = i \frac{2z + i\sqrt{3}}{\sqrt{3}z + 2i}.$$

Denote by G the group generated by g and h , and by R_0 the region between the four geodesics and inside the unit disc. By Poincare's Theorem [Be, p.249], the group G

Figure 5-1

is discrete and R_0 is a fundamental region for G that generates a Riemann surface T . Notice the pairs of equivalent points under the side-pairing transformations g and h . The surface T is topologically a torus with a disc removed. Denote by π the projection map from the disc to T .

Define $r : \mathbb{C} \rightarrow \mathbb{C}$ by $r(z) = iz$. We will show that the rotation r restricted to the unit disc induces a self map of the surface T . First, notice the following relations: $rg = hr$, $rh = g^{-1}r$, $rg^{-1} = h^{-1}r$ and $rh^{-1} = gr$. From these we deduce that for any transformation $t_1 \in G$ there is a transformation $t_2 \in G$ such that $rt_1 = t_2r$. Therefore r induces a map $\rho : T \rightarrow T$, defined in the following way: for $z \in T$, let $w \in \pi^{-1}(z)$ and put $\rho(z) = (\pi \circ r)(w)$.

Does ρ induce a composition operator? Before answering that question, let's show that ρ is not the identity. If it were, we would have $\pi(iw) = \pi(w)$ for every $w \in \mathbb{D}$. This is equivalent to $r \in G$, which is absurd (all four geodesics would then be equivalent under the group G). So ρ is non-trivial. Since r is invertible, so is ρ . Hence the composition operator exists and is invertible as well. Since $r^4 = I$ (identity), the same is true for the composition operator C_ρ . As a consequence, $\sigma(C_\rho) \subset \{1, -1, i, -i\}$.

□

Remark. The self maps of T induced by I , r , r^2 and r^3 are the only invertible ones. For a map $\rho : T \rightarrow T$ to be invertible, its lifting F to the unit disc has to map each region congruent to R_0 modulo the group G one to one and onto another such region. So F has to be a composition of symmetries of R_0 with permutations of the congruent regions among themselves. But I , r , r^2 and r^3 are the only analytic symmetries of R_0 , and for any transformation $t \in G$ (that is, for any permutation of the fundamental regions), the map $r^k t$ generates the same ρ as r^k .

5.2 Open Questions

We have so far found characterizations for the boundedness and compactness of the composition operators on the spaces of measurable 1-forms and on the spaces of analytic square integrable 1-forms of Riemann surfaces. We have also discussed the invertibility of the composition operators defined on spaces of analytic square integrable 1-forms. Some spectrum computations were possible for special cases when

the composition operator was induced by a self map of a Riemann surface. Here is a list of related questions that need answers:

- Does the invertibility of the composition operator induced by a map between open Riemann surfaces imply the invertibility of the map?

- In the case of mapping a surface into itself, what can be said about the spectrum of the operator? We hope that something can be said for surfaces more complicated than the disc, although the problem is not easy on the disc either.

- Can these operators be completely characterized in the case of analytic square integrable forms on compact Riemann surfaces? The answer is known if the operator is induced by an automorphism.

- What specific information can we get about the operator on various classes of open Riemann surfaces? This depends on the richness of the spaces of analytic square integrable 1-forms on those Riemann surfaces.

- Can the adjoints of these operators be computed, at least in some of the cases?

- Can more interesting examples be fully understood? In particular, it would be nice to have a concrete example of a composition operator induced by an analytic map between surfaces of infinite genus. Unfortunately, these maps themselves are not very well known.

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