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An investigation into students' conceptual understandings of the graphical representation of polynomial functions

Judith Ellen Curran
University of New Hampshire, Durham

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An investigation into students' conceptual understandings of the graphical representation of polynomial functions

Abstract

Mathematics educators are realizing the impact that technology is having on the way mathematical functions can be represented and manipulated. The increased use of graphing technology in the classroom is paralleled by an increased emphasis on the role of the graphical representation of a function to solve problems. These changes together with a recognition of the significance and complexity of developing a rich understanding of the graphical representations of polynomial functions are the motivation behind this research.

The study was designed to explore students' conceptual understandings of the graphs of polynomial functions. Guided by a constructivist approach to conceptual change, the investigations were primarily directed towards determining how the students' ability to interpret the graphs of polynomial functions of degree greater than two depends and builds on their understandings of the graphs of linear and quadratic functions.

Based on three case studies of students within an Algebra II class in a large high school in Northern New England, the inquiry was qualitative in nature. Clinical interviews were used to probe into the students' developing understandings of the graphs of linear, quadratic, and cubic functions. Based on findings from the clinical interviews, teaching episodes were designed in an attempt to enhance connections between the classes of polynomial functions. Data used in the analysis came from multiple sources: videotapes of the classroom instruction, collected student work, videotaped clinical interviews and teaching episodes, student journal entries, and field notes.

Results indicate that the students in the study exhibited links between their understandings of the graph of a cubic function and their understandings of the graphs of linear and quadratic functions. These connections, however, were often hindered by the classroom instructional emphases and activities as well as by conventional notation and terminology.

The focus of the teaching episodes was on enhancing the connections between the classes of polynomial functions by building polynomial functions from products of linear expressions. This method did foster connections between the classes of polynomial functions, as well as connections between the algebraic and graphical representations.

Suggested modifications to the curriculum, implications for pedagogy, and avenues for future research are given.

Keywords
Education, Mathematics, Education, Educational Psychology

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AN INVESTIGATION INTO
STUDENTS' CONCEPTUAL UNDERSTANDINGS
OF THE
GRAPHICAL REPRESENTATION
OF POLYNOMIAL FUNCTIONS

BY

JUDITH E. CURRAN
B.S., Bridgewater State College, 1977
M.A., State University of New York at Stonybrook, 1978

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy

in

Mathematics Education

December, 1995
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November 15, 1995
Date

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DEDICATION

Jim, you've really been with me all along the way . . .
encouraging me, giving me hope, and smiling at me
through the faces of Jimmy and Jack.
You have been the wind in my sails, and a rudder that has kept me on course.
This work is because of you, and with all my heart I dedicate it to you.
ACKNOWLEDGEMENTS

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I would like to also express my appreciation to the other members of my committee who have given hours of their time and valuable direction: Dr. Joan Ferrini-Mundy, Dr. Edward Hinson, Dr. Sharon Nodie Oja, and Dr. Donovan VanOsdel.

I am grateful for my loving family and friends who have been there to offer assistance, and have rallied me on through all the ups and downs of the past years. Suzy, thanks for helping me come up with a title!

I have been fortunate in having mathematics teachers that have instilled within me a special appreciation and love for mathematics. Mr. Martin Badoian, my high school mathematics teacher, through his enthusiasm and dedication to his students and the teaching profession, gave me the desire to pursue the field of mathematics education. I am very grateful for such a wonderful model and friend.

Jim and Jack, you are two great kids...you've allowed me to keep at this, and I appreciate that so much. I know you're as glad as I am that it is done! I love you tons.

Todd, you're amazingly patient. Thank you for all your love and support, the chocolate and the flowers. I am thankful that God has brought you and Rachel into our lives.
TABLE OF CONTENTS

DEDICATION ....................................................................................................................................iv

ACKNOWLEDGEMENTS ....................................................................................................................v

LIST OF FIGURES ...........................................................................................................................xiii

LIST OF TABLES ............................................................................................................................xv

ABSTRACT ......................................................................................................................................xvi

CHAPTER 

<table>
<thead>
<tr>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION ................................................................. 1</td>
</tr>
<tr>
<td>The Centrality Of The Function Concept Within The K-12 Mathematics Curriculum ................................................................. 1</td>
</tr>
<tr>
<td>The Relevance Of The Function Concept In Today’s World ......................... 2</td>
</tr>
<tr>
<td>Overview Of The Current Study ........................................................... 3</td>
</tr>
<tr>
<td>Research Questions ............................................................................ 6</td>
</tr>
<tr>
<td>II. RATIONALE ............................................................................ 7</td>
</tr>
<tr>
<td>Conceptual Knowledge and Understanding ........................................ 7</td>
</tr>
<tr>
<td>The Shift to Cognitive Psychology ................................................... 7</td>
</tr>
<tr>
<td>Qualitative Research and Constructivism ........................................... 9</td>
</tr>
<tr>
<td>Conceptual Understandings of the Function Concept .......... 9</td>
</tr>
<tr>
<td>The Complexity of the Function Concept ......................................... 9</td>
</tr>
<tr>
<td>Reevaluating the Ways in Which Functions are Used and Taught .......... 9</td>
</tr>
<tr>
<td>The Function Concept as a Foundation for Calculus ..................... 10</td>
</tr>
<tr>
<td>Algebra Reform And The Impact Of Technology ....................... 11</td>
</tr>
<tr>
<td>Topic</td>
</tr>
<tr>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>The Place of Polynomial Functions in the Curriculum</td>
</tr>
<tr>
<td>The Merits of “Factoring”</td>
</tr>
<tr>
<td>Research Assumptions</td>
</tr>
<tr>
<td>The Relationships Between Roots, X-Intercepts, Zeros and Factors</td>
</tr>
<tr>
<td>Summary</td>
</tr>
<tr>
<td>III. Theoretical Framework</td>
</tr>
<tr>
<td>The Theory Of Conceptual Change</td>
</tr>
<tr>
<td>The Role of Affective, Situational, and Contextual Factors on the Learning Process</td>
</tr>
<tr>
<td>Constructivism and Conceptual Change</td>
</tr>
<tr>
<td>The Individuality of the Learner</td>
</tr>
<tr>
<td>Classifying Student Understandings of the Function Concept</td>
</tr>
<tr>
<td>Concept Image</td>
</tr>
<tr>
<td>Action, Process, Object Conceptions of Function</td>
</tr>
<tr>
<td>Summary</td>
</tr>
<tr>
<td>IV. Background and Related Work</td>
</tr>
<tr>
<td>Making The Function Concept Prominent: An Historical View</td>
</tr>
<tr>
<td>The Mathematical Development of the Function Concept</td>
</tr>
<tr>
<td>The Function Concept’s Prominence in the K-12 Mathematics Curriculum</td>
</tr>
<tr>
<td>The Complexity Of The Function Concept</td>
</tr>
<tr>
<td>Research on Student Views, Attitudes and Understanding of the Graphical Representation of Function</td>
</tr>
<tr>
<td>Impact Of Technology On The Classroom, Research, and On Students’ Understandings Of The Graphical Representation Of Function</td>
</tr>
<tr>
<td>Types Of Graphing tasks used in the curriculum and in research</td>
</tr>
<tr>
<td>Research on Translating Between Representations Of Functions</td>
</tr>
<tr>
<td>Research on Student Understandings Of The Graphs Of Polynomial Functions</td>
</tr>
<tr>
<td>Graphs of Linear Functions</td>
</tr>
<tr>
<td>Graphs of Quadratic Functions</td>
</tr>
<tr>
<td>Graphs of Polynomial Functions of Degree Greater than Two</td>
</tr>
<tr>
<td>Section</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
</tr>
<tr>
<td>V. RESEARCH METHODOLOGY</td>
</tr>
<tr>
<td>THE NATURE OF QUALITATIVE RESEARCH</td>
</tr>
<tr>
<td>IMPLICATIONS OF THEORETICAL FRAMEWORK FOR RESEARCH METHODOLOGY</td>
</tr>
<tr>
<td>The Clinical Interview</td>
</tr>
<tr>
<td>Teaching Episodes and the Teaching Experiment</td>
</tr>
<tr>
<td>The Role of the Researcher</td>
</tr>
<tr>
<td>Design of the Current Study</td>
</tr>
<tr>
<td>PILOT STUDY AND DATA COLLECTION</td>
</tr>
<tr>
<td>Pilot Study</td>
</tr>
<tr>
<td>Phase I</td>
</tr>
<tr>
<td>Phase II</td>
</tr>
<tr>
<td>Phase III</td>
</tr>
<tr>
<td>Assessment of Teaching Episodes</td>
</tr>
<tr>
<td>DATA ANALYSIS</td>
</tr>
<tr>
<td>The Use of Case Studies</td>
</tr>
<tr>
<td>Data Reduction</td>
</tr>
<tr>
<td>SUMMARY</td>
</tr>
<tr>
<td>VI. CLASSROOM ENVIRONMENT</td>
</tr>
<tr>
<td>BACKGROUND</td>
</tr>
<tr>
<td>THE TEACHER INTERVIEW</td>
</tr>
<tr>
<td>Background</td>
</tr>
<tr>
<td>Algebra</td>
</tr>
<tr>
<td>Polynomial Functions</td>
</tr>
<tr>
<td>Functions</td>
</tr>
<tr>
<td>Graphing</td>
</tr>
<tr>
<td>Technology</td>
</tr>
<tr>
<td>Effect of Researcher's Presence On the Teacher</td>
</tr>
<tr>
<td>Summary</td>
</tr>
<tr>
<td>CLASSROOM INSTRUCTION</td>
</tr>
<tr>
<td>Modes Of Instruction</td>
</tr>
<tr>
<td>Instruction On Linear Functions</td>
</tr>
<tr>
<td>The Mid-Year Exam</td>
</tr>
<tr>
<td>Quadratic Equations And Functions</td>
</tr>
<tr>
<td>Instruction On Polynomial Functions Of Degree Greater Than Two</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Building Quadratic Functions</td>
<td>172</td>
</tr>
<tr>
<td>Working Backwards: Finding the Components of a Quadratic Function</td>
<td>176</td>
</tr>
<tr>
<td>Extending Activities to Cubic Polynomial Functions</td>
<td>181</td>
</tr>
<tr>
<td>Assessment of the Teaching Episodes: A Reevaluation of Lisa's Understanding</td>
<td>182</td>
</tr>
<tr>
<td>Summary</td>
<td>186</td>
</tr>
<tr>
<td>IX. BETH</td>
<td>190</td>
</tr>
<tr>
<td>Attitudes toward School and Mathematics</td>
<td>190</td>
</tr>
<tr>
<td>Definition of Mathematics</td>
<td>192</td>
</tr>
<tr>
<td>Knowledge of Functions and Graphs</td>
<td>193</td>
</tr>
<tr>
<td>Function Definition</td>
<td>193</td>
</tr>
<tr>
<td>Ability to Identify Functions</td>
<td>193</td>
</tr>
<tr>
<td>Describing Function Attributes</td>
<td>198</td>
</tr>
<tr>
<td>Translating to the Algebraic Representation</td>
<td>201</td>
</tr>
<tr>
<td>Understandings of Polynomial Functions</td>
<td>205</td>
</tr>
<tr>
<td>Connections between the Classes of Polynomial functions</td>
<td>205</td>
</tr>
<tr>
<td>Contributing/Inhibiting Factors to Making the Transition to Polynomials of Degree Greater than Two</td>
<td>208</td>
</tr>
<tr>
<td>Teaching Episodes: Forming Links Between the Classes of Polynomial Functions</td>
<td>212</td>
</tr>
<tr>
<td>Building Quadratic Functions</td>
<td>212</td>
</tr>
<tr>
<td>Working Backwards: Finding the Components of a Quadratic Function</td>
<td>218</td>
</tr>
<tr>
<td>Extending Activities to Cubic Polynomial Functions</td>
<td>221</td>
</tr>
<tr>
<td>Assessment of the Teaching Episodes: A Reevaluation of Beth's Understanding</td>
<td>223</td>
</tr>
<tr>
<td>Summary</td>
<td>227</td>
</tr>
<tr>
<td>VIII. Discussion and Conclusions</td>
<td>230</td>
</tr>
<tr>
<td>Making Connections: Building on Prior Understanding</td>
<td>230</td>
</tr>
<tr>
<td>Making the Transition to Polynomial Functions of Higher Degree:</td>
<td>230</td>
</tr>
<tr>
<td>Contributing/Inhibiting Factors</td>
<td>232</td>
</tr>
<tr>
<td>Activities Suggested by the Interviews to Make the Connections More Salient</td>
<td>234</td>
</tr>
<tr>
<td>Observed Changes in Understanding After the Teaching Episodes</td>
<td>235</td>
</tr>
<tr>
<td>Additional Activities Suggested by the Interviews that Would Promote Connections Between Classes</td>
<td>237</td>
</tr>
<tr>
<td>Consistencies in Understandings and Conceptual Development</td>
<td>238</td>
</tr>
<tr>
<td>Amongst Students</td>
<td>238</td>
</tr>
</tbody>
</table>
Verbal Descriptions of Graphs ................................................................. 239

The Influence of Technology on the Knowledge of Graphs .................. 240

Relation of Affective, Situational, and Contextual Factors to
Student Understanding ........................................................................... 242
  Affective Factors .............................................................................. 242
  Situational and Contextual Factors .................................................. 243

Implications for Instruction ................................................................. 250
  Fostering Connections .................................................................... 250
  Decreasing Emphasis on Algebraic Manipulations ....................... 252
  Fostering Communication ................................................................ 254
  Curriculum Materials ...................................................................... 255
  Implications for Teacher Education ............................................... 257

Implications for Research ................................................................... 258

Concluding Thoughts and Reflections ............................................... 259

Appendix A ....................................................................................... 262

Appendix B ....................................................................................... 266

Appendix C ....................................................................................... 271

Appendix D ....................................................................................... 274

Appendix E ....................................................................................... 276

Appendix F ....................................................................................... 284

Appendix G ....................................................................................... 297

References ....................................................................................... 302

xi
LIST OF FIGURES

1. Interpretations of the x-intercept ................................................................. 18
2. Concept definition controlling the concept image ....................................... 29
3. Reliance solely on concept image ................................................................. 29
4. Function block diagram .............................................................................. 39
5. Distance/time graph used by Kerslake ....................................................... 47
6. Graph showing motion of a car .................................................................... 48
7. Components of data analysis: Interactive model (Miles & Huberman) ....... 71
8. Letter coding of Graph #1 ......................................................................... 73
9. Task #1 (1-3) ............................................................................................ 106
10. Task #1 (4-6) .......................................................................................... 107
11. Task #1 (7-9) .......................................................................................... 108
12. Task #1 (10-12) ...................................................................................... 108
13. Task #1 (13) ............................................................................................ 109
14. Task #2 (1-3) ............................................................................................ 110
15. Task #2 (4-6) ............................................................................................ 112
16. Graph #1 - PC Emulation Software ......................................................... 117
17. Graph #2 - PC Emulation Software ............................................................. 121
18. Graph #3 - PC Emulation Software ............................................................. 123
19. Graph #1 - Working backwards (Mark) ..................................................... 132
20. Graph #2 - Working backwards (Mark) ..................................................... 132
21. Graph #3 - Working backwards (Mark) ..................................................... 134
22. Graph #4 - Working backwards (Mark) ......................................................... 135
23. Graph #1 - Working backwards (Lisa) .......................................................... 177
24. Graph #2 - Working backwards (Lisa) .......................................................... 179
25. Graph #3 - Working backwards (Lisa) .......................................................... 179
26. Graph #4 - Working backwards (Lisa) .......................................................... 180
27. Graph #1 - Working backwards, (Beth) ....................................................... 218
28. Graph #3 - Working backwards, (Beth) ....................................................... 219
29. Graph #3(b) - Working backwards, (Beth) ................................................... 220
30. Graph #4 - Working backwards, (Beth) ....................................................... 220
LIST OF TABLES

1. Coding of categories corresponding to research questions ........................................74
2. Subtopics corresponding to research questions ........................................................75
3. Distribution of questions on the midyear exam .........................................................90
4. Distribution of questions on polynomial functions test ...........................................99
5. Synopsis of student understanding before and after the teaching episodes ................237
ABSTRACT

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BY

Judith E. Curran
University of New Hampshire, December, 1995

Mathematics educators are realizing the impact that technology is having on the way mathematical functions can be represented and manipulated. The increased use of graphing technology in the classroom is paralleled by an increased emphasis on the role of the graphical representation of a function to solve problems. These changes together with a recognition of the significance and complexity of developing a rich understanding of the graphical representations of polynomial functions are the motivation behind this research.

The study was designed to explore students' conceptual understandings of the graphs of polynomial functions. Guided by a constructivist approach to conceptual change, the investigations were primarily directed towards determining how the students' ability to interpret the graphs of polynomial functions of degree greater than two depends and builds on their understandings of the graphs of linear and quadratic functions.

Based on three case studies of students within an Algebra II class in a large high school in Northern New England, the inquiry was qualitative in nature. Clinical interviews were used to probe into the students' developing
understandings of the graphs of linear, quadratic, and cubic functions. Based on findings from the clinical interviews, teaching episodes were designed in an attempt to enhance connections between the classes of polynomial functions. Data used in the analysis came from multiple sources: videotapes of the classroom instruction, collected student work, videotaped clinical interviews and teaching episodes, student journal entries, and field notes.

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The focus of the teaching episodes was on enhancing the connections between the classes of polynomial functions by building polynomial functions from products of linear expressions. This method did foster connections between the classes of polynomial functions, as well as connections between the algebraic and graphical representations.

Suggested modifications to the curriculum, implications for pedagogy, and avenues for future research are given.
CHAPTER I

INTRODUCTION

The Centrality of the Function Concept within the

K-12 Mathematics Curriculum

The Curriculum and Evaluation Standards for School Mathematics, published by the National Council of Teachers of Mathematics [NCTM] (1989) specify curricular goals for students of mathematics, grades K-12. To emphasize the importance of the study of functions in the curriculum, NCTM devotes an entire "standard" in both grades 5-8 and grades 9-12 to the function concept in the curriculum. Moreover, The Standards refer to the study of functions as a "central theme" and a "unifying idea" throughout K-12 mathematics and recommend that increased attention be given to the integration of functions across topics throughout the curriculum.

More recently, a statement prepared by the Board of Directors of the National Council of Teachers of Mathematics (NCTM, 1994) stated that a guiding principle in the effort to reconceptualize algebra in the high school curriculum is the following:

The primary role of algebra at the school level is to develop confidence and facility in using variables and functions to model numerical patterns and quantitative relations—both within pure mathematics and in a broad range of settings in which numerical data are important.

In addition, they state, "The use of graphing calculators and computers makes the focus on modeling and functions attractive and accessible for students . . . ". They
summarize their position by declaring that "refocusing algebra on modeling places functions and variables at center stage, where they ought to be."

**The Relevance of the Function Concept in Today's World**

The concept of function underlies many real-world phenomena. The function concept is foundational to situations in which one quantity affects change in another quantity. The computations involved in determining sales taxes, in converting units of measurement, in determining monetary change from transactions, are examples of functional situations encountered daily. As far back as 1922, Hedrick (1922), stated that:

> There can be no doubt at all of the value to all persons of any increase in their ability to see and to foresee the manner in which related quantities affect each other. (p. 165)

Of greater importance is the ability to extract patterns from these situations in which one quantity affects another. Fey (1990) states:

> For quantitative reasoning to yield results with greater power than unadorned number facts, it is essential that such reasoning be firmly rooted in general patterns of numbers and related computations. . . . Variables are not usually significant by themselves. In most realistic applications of algebra the fundamental reasoning task is not to find a value of x that satisfies one particular condition, but to analyze the relation between x and y "for all x." The most useful algebraic idea for thinking about relations of this sort is the concept of function. (p. 70)

The function concept is sure to maintain a prominent role in mathematics.

Currently, the notion of function is manifest throughout the curriculum. Eisenberg (1991) notes:

Functions are found everywhere in mathematics; the binary operations of ordinary addition and multiplication can be thought of as mappings from \( \mathbb{R} \times \mathbb{R} \) into \( \mathbb{R} \), the mechanics of solving the standard inequalities problems encountered in algebra and trigonometry can be thought of in a function format, as can many of the standard problems in differential and integral calculus. All techniques, which form a major
component of school and university mathematics courses, can be discussed from a functional approach. (p. 142)

Additional topics that embrace the concept of function are cited in *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989):

In geometry, functions relate sets of points to their images under motions such as flips, slides, and turns; and in probability, they relate events to their likelihoods. (p. 154)

In summary, during the reform climate of the 1990's, the emphasis has increasingly been placed on the function concept as a unifying principle and as a means of modeling real world phenomena. In a recent MAA publication entitled *The Concept of Function: Aspects of Epistemology and Pedagogy* (Harel & Dubinsky, 1992) the editors argued:

Function . . . is the single most important concept from kindergarten to graduate school and is critical throughout the full range of education. Arithmetic in the early grades, algebra in middle and high school, and transformational geometry in high school are all coming to be based on the idea of function. Increasingly, people involved in calculus reform are coming around to the view that a strong conception of function is an indispensable part of the background of any student who hopes to learn something about this subject. As mathematical education is being renewed and reformed throughout the world, we remain convinced that this movement requires that we learn more about the concept of function from both epistemological and pedagogical points of view. (p. vii)

**Overview of the Current Study**

The general class of functions called "polynomial functions" is comprised of functions of the form \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) where \( a_n \) is a non-zero real number, \( a_{n-1}, \ldots, a_1, a_0 \) are real numbers, and \( n \) is a nonnegative integer.

Linear functions (\( n=1 \)) and quadratic functions (\( n=2 \)) are considered special cases of this general class.
The present study was designed to investigate students' conceptual understandings of the graphs of polynomial functions. As will be seen in Chapter IV, research in the area of student understanding of the graphs of polynomial functions at the secondary or undergraduate level has concentrated mainly on the graphical interpretation of linear or quadratic functions, and very little has been done on student interpretation of the graphs of polynomial functions of degree greater than two. There is a need for research pertaining to student understanding of the graphs of polynomial functions of degree greater than two as these are a prerequisite for calculus and are used to model real-life situations.

In addition, there is a need for research that examines the connections that students make between the classes of polynomial functions, i.e. sets of polynomial functions distinguished by their degree. Chapter II is devoted to further examining the rationale behind this investigation, and to acknowledging research assumptions.

Conceptual understanding has been defined as "making one's own sense of knowledge" (West & Pines, 1985, p. 6). The learner constructs or builds his or her own understandings through experiences and interactions within the environment. Guided by a constructivist approach to conceptual change, which will be further described in Chapter III, the investigations in this study were primarily directed towards determining how the students' ability to interpret the graphs of polynomial functions of degree greater than two depends and builds on their understandings of the graphs of linear and quadratic functions.

The research methodology used in this study is described fully in Chapter V. The inquiry was designed to be qualitative in nature, predominantly based on three case studies (Chapter VII, VIII, IX) of students within an Algebra II class in a
large high school in Northern New England. The classroom environment, the modes of instruction, and the course structure will be described in detail in Chapter VI. Teaching episodes were developed in an attempt to enhance connections between classes of polynomial functions and to enrich the students' understandings of the graphs.

The data was triangulated from multiple sources in order to generate hypotheses in response to the research questions. These multiple sources include videotapes of the classroom instruction, collected student work, videotaped clinical interviews and teaching episodes, student journal entries, and field notes.

As the proposed study extends the research on student learning in the areas of functions and graphs, it has the potential to contribute to the mathematical education community's knowledge base on how students develop a conceptual understanding of the graphs of polynomial functions. Results from this investigation are sure to raise further questions about student understanding of functions and graphs, and hence, will motivate future research in this domain.

Information obtained from this study can serve to influence future classroom instruction in the area of polynomial functions and graphing as well as suggest the creation of classroom modules linked to knowledge obtained about the development of students' conceptual understandings of the graphs of polynomial functions. These changes in pedagogy and curriculum may, in turn, serve as a catalyst for further research in this domain.
Research Questions

The planning, investigations, and analysis of the conceptual understandings of a group of students relative to the graphs of polynomial functions were guided by the following research questions.

1. What connections do the students make between their study of the graphs of linear and quadratic functions and their study of the graphs of polynomial functions of degree greater than two?

2. What evidence suggests that the students' conceptual development during instruction on polynomial functions of degree greater than two builds on their understanding of the graphs of linear and quadratic functions and their general knowledge of functions and graphs?

3. What factors can be identified that contribute to/inhibit the student from making the transition from the graphical representations of linear and quadratic functions to those of higher degree?

4. Do observations of students suggest activities that would make the connections between classes of polynomial functions more salient?

5. In what ways is student understanding and conceptual development with respect to the graphs of polynomial functions of degree greater than two consistent amongst students?

6. What influence does technology have on student knowledge with respect to the graphs of polynomial functions of degree greater than two?

7. How do affective, situational, and contextual factors relate to the students' conceptual understanding of the graphs of polynomial functions? (This includes teacher attitudes and beliefs about the function concept, graphing, linear and quadratic functions, and polynomial functions.)
CHAPTER II

RATIONALE

*Conceptual Knowledge and Understanding*

**The Shift to Cognitive Psychology**

Since the mid-70's, educational researchers have seen a dramatic shift away from behavioristic theories of learning to cognitive psychology with its interest in observing learners in the process of learning. This shift was spurred on by the writings of Piaget and Vygotsky who deemed learners as active organizers of their learning experiences. Vygotsky (1962) identified two sources of knowledge in the individual: the knowledge acquired from social interaction in the environment and that acquired through formal instruction. Conceptual learning has been viewed as the integration of knowledge from these two sources (Cobb, Wood, & Yackel, 1990; West & Pines, 1985). Learners construct their own meanings from experiences within their environment. West and Pines (1985) have described conceptual understanding as "making one's own sense of knowledge" (p.2).

**Qualitative Research and Constructivism**

Research in a student's natural setting using qualitative methods is being increasingly recognized as the most appropriate approach to investigate students' conceptual understandings. Investigations seek to uncover the students
perceptions, reasoning, and purposes that they give to incoming information and seek to examine the ways in which the students apply or extend this information to new situations. Investigations may reveal patterns of thinking within the individual or across individuals. The application of these methods, like Piaget's clinical interview techniques (Hewson, 1985), has changed fundamental ways in which the learning process has been viewed: instead of students being regarded as passive recipients of knowledge, they are being seen as active participants in the construction of their own knowledge and instead of teachers being dispensers of knowledge, they are being viewed as facilitators of the students' knowledge construction.

Davis, Maher, and Noddings in the NCTM monograph entitled *Constructivist Views on the Teaching and Learning of Mathematics* (1990) write:

Learning mathematics requires construction, not passive reception, and to know mathematics requires constructive work with mathematical objects in a mathematical community. Each learner has a tool kit of conceptions and skills with which he or she must construct knowledge to solve problems presented by the environment. The role of the community—other learners and teacher—is to provide the setting, pose the challenges, and offer the support that will encourage mathematical construction. (pp. 2-3)

Constructivism is based on the belief that "knowledge does not reflect an 'objective' ontological reality, but exclusively an ordering and organization of a world constituted by our experience" (von Glasersfeld, 1984). Educators who regard learning in this way have been called constructivists. According to Davis, Maher, and Noddings (1990) constructivists agree that:

Mathematical learning involves the active manipulation of meanings, not just numbers and formulas. They reject the notion that mathematics is learned in a cumulative, linear fashion. Every stage of learning involves a search for meaning, and the acquisition of rote skills in no way ensures that learners will be able to use these skills intelligently in mathematical settings. (p. 187)
The Complexity of the Function Concept

The function concept, referred to as "the central theme" (NCTM, 1989) throughout the mathematics curriculum, has proven to be "one of the most difficult concepts to master in the learning sequence of school mathematics" (Eisenberg, 1991). The complexity of the concept can be attributed to the various representations of a function, the numerous subconcepts that make up the function concept, and the host of problem situations at varying levels of complexity and abstraction that are used to illustrate the concept. There is a growing consensus in the mathematics education community that, because of its complexity, students can only come to understand the concept of function slowly over several years (Monk, 1985).

Reevaluating the Way Functions are Used and Taught

Throughout the past decade, mathematics educators have been reevaluating the way functions are used and taught in the classroom. In secondary algebra courses, the typical path of instruction for the various representations of functions has been from algebraic expression, to ordered pairs, to graphs (Philipp, Martin & Richgels, 1993). Yerushalmy and Schwartz (1993) state:

It would seem that this sequential learning of the symbolic and graphical representations does not promote a tendency on the part of the learner to move between them. Each of the representations presents a separate symbol system for the learner with no mutual and constructive interaction. (p. 44)
Although the tabular and graphical representations of a function may not be as *computationally* useful as the symbolic form, these forms quickly convey a picture of the quantitative situation. An ability to interpret the tabular and graphical representations is fundamental to understanding the many tables, charts, and graphs used frequently to depict functional situations in newspapers and magazines. Fey (1990) writes:

> Graphic displays are becoming common and increasingly sophisticated. Thus it is important for mathematics students to become adept at interpreting graphic representations intelligently and to understand the connections among symbolic, graphic, and numerical forms of the same ideas. (p. 75)

It is important that research investigate the conceptual understandings of students as they try to understand the graphs of functions. With that information, the curriculum and teaching strategies could be modified to facilitate the development of a more meaningful and powerful understanding of the function concept. The following provides further justification for moving toward this goal.

**The Function Concept as a Foundation for Calculus**

For those students intending to major in one of the sciences at the college/university level, a strong conceptual foundation with respect to the function concept is helpful, if not essential. Academically, work with elementary functions forms a firm foundation for later work with limits, continuity, derivatives, area under a curve and other calculus concepts. More and more, people involved in calculus reform are suggesting that a strong conception of function is an indispensable part of the background of any student taking the course. The function concept has been said to be the "central underlying concept
in calculus" (Vinner, 1992, p. 195) and at the "very heart of calculus" (Thorpe, 1989, p. 12).

College mathematics teachers are nearly unanimous in their view that most students are blocked from ever understanding calculus by the inadequacies of their understanding of functions. (Monk, 1989, p. 10)

**Algebra Reform and the Impact of Technology**

Mathematics educators are realizing the impact technology is having on the way mathematical functions can be represented and manipulated. Advances in hand-held technology now make it possible to quickly generate graphs of functions, a task that was previously considered tedious by students. Dugdale (1993) writes:

> The easy manipulation of graphical representations allowed by current function-plotting tools has raised the possibility of visual representations of functions playing a more important role in mathematical reasoning, investigation, and argument. (p. 115)

This increased role of the use of the visual representation to solve problems is in contrast to the general dependence on algebraic manipulations that existed before graphing technology became available in the classroom, when the amount of time and effort needed to construct graphs outweighed the benefits of using the graphical representation (Romberg, Carpenter & Fennema, 1993). Hence, although algebra has usually been regarded as a language of symbols (Kieran, 1993), technology seems to be widening the view of what algebra is and how it should be taught. Heid (1995) writes:

> With the help of technology, the notion of function can be expanded from rules to compute output values from given input values to a dynamic study of the relationships between two or more quantities that vary. With this expansion of the concept, a functions approach to algebra becomes not only possible but actually appropriate. (p. 48)
With the new emphasis on the graphs of functional relationships brought about by the increased availability of graphing technology in the classroom, it is now appropriate for new research that investigates how students' understanding of the graphs of functions develops. Williams (1993) states:

Careful analysis of graphing tasks, graph comprehension tasks, translation tasks, and so forth, have yet to be synthesized in even a preliminary model of graph-related behaviors; a great deal of work remains to be done toward this end. (p. 331)

**The Place of Polynomial Functions in the Curriculum**

The focus of the present study is to investigate students' conceptual understandings of the graphs of a certain class of functions, polynomial functions. Polynomial functions are "ubiquitous in mathematics, and it is important that students be well grounded in them" (Eisenberg & Dreyfus, 1988, p. 113). The Standards (NCTM, 1989) state that college-intending students should develop an understanding of polynomial functions. An understanding of polynomial functions is necessary for a student's success in calculus and in differential equations, where textbook examples often assume an understanding of polynomial functions.

Linear functions are used to model any situation in which two quantities vary directly. In calculus, the derivative of a function is defined as the slope of the tangent line to the curve. It is assumed that calculus students have enough of an understanding of linear functions to be able to determine the tangent line to a curve. In understanding the derivative, an intuitive notion of the nature of the slope of a line is also helpful; i.e. if the derivative is positive, the slope of the tangent line is positive, and hence, the curve must be increasing. Linear
functions are also helpful background knowledge in graphing that involves asymptotic lines.

Quadratic functions are used to model various types of motion, area, distance between two points, sequences for which second differences are constant, applications in economics, and situations that involve a product of two linear expressions. Parabolic graphs are useful in calculus in exploring the concepts of the derivative and the integral.

Polynomial functions of degree greater than two are used in engineering designs, such as designing the hull of a sailboat, and in modeling volume and motion. They are also useful for exploring the core mathematical concepts of maxima, minima, points of inflection, end behavior, intercepts, and increasing/decreasing functions. All of these concepts can be exhibited within the context of the graphical representation of a single polynomial function of degree greater than two.

**The Merits of “Factoring”**

The structural character of the algebra curriculum that was evident at the beginning of the century is still largely in place today (Kieran, 1992). An emphasis of that curriculum has been on the factoring of polynomial functions. In describing the content of many first-year algebra courses, Kieran states:

After a brief period involving numerical substitution in both expressions and equations, the course generally continues with the properties of the different number systems, simplification of expressions, and the solving of equations through formal methods. The manipulation and factoring of polynomial and rational expressions of varying degrees of complexity soon become a regular feature. Interspersed among the various chapters are word problems, thinly disguised as “real-world” applications of whatever algebraic technique has just been learned. Students eventually encounter functions in their algebraic, tabular, and graphical representations. The functions
covered usually include linear, quadratic, cubic, exponential, logarithmic and trigonometric. (p. 395)

Solutions to problems dealing with real-world applications in algebra and calculus often require that an equation of the form \( f(x) = 0 \) be solved. By the Factor Theorem, finding the factors of a polynomial function, \( f(x) \), results in the determination of the solutions to the equation \( f(x) = 0 \). Defending the place of factoring in the secondary algebra curriculum, Philipp et al. (1993) state:

The search for solutions of equations has historical and current importance in a number of areas, including linear and abstract algebra, physics, and applied mathematics. The investigation of this problem has spurred development of a great deal of mathematics. (p. 262)

Finding the roots of a polynomial function of degree greater than two has traditionally been a trial and error process using the Rational Root Theorem and synthetic division. Now, however, graphing technology used in conjunction with these algebraic procedures can ease the search for solutions.

The use of graphs could even increase the intrinsic interest of factoring, as students explore the connections between graphical and algebraic representations of functions. Graphing utilities can help students discover the importance of factoring and enhance their interest in it by greatly reducing the emphasis on an algorithmic, manipulative approach, while revealing the significant information available from the factored form of an expression. (Philipp et al., 1993, p. 268)

When taken solely as a mechanical process, "factoring" has limited merits. Demana and Waits (1990) have developed many tasks that illustrate how the topic of factoring can be enhanced with the use of graphing calculators. Demana and Waits (1993) remark:

Limiting ourselves to algebraic techniques seriously restricts the types of equations students can solve. In the conventional curriculum, students solve linear equations, quadratic equations, easily factored contrived higher order polynomial equations, and other contrived equations whose form is special. However, using computer graphing,
students can easily solve very complex equations, even equations that do not admit an algebraic solution. (p. 32)

Technology, in the form of graphing software and graphing calculators, as will be seen in Chapter IV, can shed light on this mechanical process, highlighting the connections between the algebraic manipulations and a “picture” of the problem situation. Heid (1995) notes:

Whereas the ability to factor was a necessary tool in the pretechnological algebraic classroom, the ability to understand a factor is much more important in the technological algebraic world. (p. 25)

Investigations in the current study include examining student understandings of the linear factors of a polynomial function and seeking ways to enhance these understandings. It is an assumption of this research that in using technological tools, the ability to factor polynomial expressions remains an essential skill for an algebra student in solving problem applications. In addition, it is essential for an algebra student to develop an understanding of the relationships that a factor of a polynomial has with the graph of the function.

This assumption and others will be made explicit in the following section.

**Researcher’s Assumptions**

Given the rationale for this investigation of a group of students’ conceptual understandings relative to the graphs of polynomial functions, the reader should be made cognizant of the following beliefs and research assumptions embraced by the investigator:

1. An understanding of linear and quadratic functions lays a foundation for understanding polynomial functions of higher degree.

There are conceptual understandings that can and should be transferred from the study of the graphs of linear and quadratic functions to the study of the
graphs of polynomial functions of degree greater than two. For example, the zeroes of any polynomial function are represented graphically by points of the graph intersecting the independent axis. This notion, as a characterizing feature of all polynomial functions, is a common thread that should be woven, during instruction, across all classes of polynomial functions.

2. **Connections should be made between all classes of polynomial functions as determined by their degree.**

The NCTM Curriculum and Evaluation Standards (1989) highlight the importance of linking conceptual and procedural knowledge among the different topics in mathematics. In this way students are encouraged to see the patterns throughout mathematics, and to view mathematics as an integrated whole, rather than as a series of isolated topics. Instruction should focus on regarding the sundry classes of polynomial functions as variations of one another rather than as unrelated entities. The Standards state that “developing mathematics as an integrated whole also serves to increase the potential for retention and transfer of mathematical ideas.” (p. 149)

3. **Instruction on the graphs of polynomial functions of degree greater than two should build on the study of the graphs of linear and quadratic functions and other notions of function.**

This is a consequence of the first and second assumptions.

4. **The teaching and learning of the graphs of polynomial functions should also build on student intuitions about them.**

Mathematics researchers generally agree that developing an understanding of functions requires that teaching and learning build on the students’ intuitions about functions (Dreyfus & Eisenberg, 1982; Herscovics, 1989). Intuitions about
functions are senses about functions that arise predominantly from former experience and prior knowledge. The work of Piaget, Grize, Bang and Szeminska (1968) supports the existence of intuitions about functions in children as early as age 3½. He claims that children gain these intuitions from real-world phenomena. For instance, any cause and effect relationship, (i.e. when a child smashes a lump of clay), embodies the concept of function.

5. An adequate understanding of the graphs of polynomial functions includes a grasp of the relationship between the x-intercepts as seen on the graph, the zeros of the function, and the factors of the polynomial.

Given the algebraic form of a polynomial function, \( f(x) \), these "zeros" are the solutions to the equation, \( f(x)=0 \). Determining these values is what is often necessary to solve problem situations. By examining the graph of the polynomial function, the zeros appear as the x-intercepts; the points of intersection of the curve given by the function with the x-axis, \( y=0 \). Going the other way, given the graph of a polynomial function, these x-intercepts (or zeros) are a means to obtain the factors of the polynomial function, and thereby the algebraic representation of the function, a useful tool in generating other points on the curve.

The Relationships Between Roots, X-Intercepts, Zeros, and Factors

The diagram below shows the complexity of a meaningful understanding of the x-intercept as seen on the graph of a polynomial function. I developed this model to suggest nine different algebraic and graphical interpretations of the x-intercept that a student should be encouraged to make in order to fully develop their understanding of the x-intercept. These can also be viewed as connections between the algebraic and graphical representations of a function.
Figure 1: Interpretations of the x-intercept

The outer edges of the triangle involve predominantly symbolic or notational interpretations of the x-intercept, (c,0), while the interior arrows are interpretations of the x-intercept that are predominantly graphical. There are nine implications described below and illustrated in the model by an arrow labeled with a letter. Since interpretations of the x-intercepts can be said in a number of different ways, the assimilation of the implications is complicated. Rather than simplifying this model by the use of biconditionals, I decided to list each implication to reveal the variety and complexity of the interconnections.

**Graphical Interpretations**

- **Implication a:** An x-intercept as seen on the graph implies that the 2nd coordinate of that point is 0, i.e. the x-intercept is (c,0).

  Said differently:
An x-intercept as seen on the graph implies that the value of x is a zero of the function.

- **Implication b:** An x-intercept as seen on the graph implies that when c is substituted for x in the polynomial function, the function value becomes 0, i.e. 
  \[ P(c) = 0. \]
  Said differently:
  An x-intercept as seen on the graph implies that the value of x is a root of the function.

- **Implication c:** An x-intercept as seen on the graph implies that \((x-c)\) is a factor of the polynomial function \(P(x)\).

**Graphical Implications**

- **Implication d:** If \(c\) is a root of the polynomial, then \((c,0)\) is on the graph of the polynomial.

  Other ways of saying the same thing:
  
  If \(c\) is a root of the polynomial, then \(c\) is a zero of the polynomial.
  
  If \(P(c) = 0\), then \((c,0)\) is on the graph of the polynomial.
  
  If \(P(c) = 0\), then \(c\) is a zero of the polynomial.

- **Implication e:** If \((x-c)\) is a factor of the polynomial, then \((c,0)\) is on the graph of the polynomial.

  Said differently:
  
  If \((x-c)\) is a factor of the polynomial, then \(c\) is a zero of the polynomial.

**Notational Interpretations**

- **Implication f:** If \((c,0)\) is on the graph of the polynomial, then \((x-c)\) is a factor of the polynomial.
Said differently:
If $c$ is a zero of the polynomial, then $(x-c)$ is a factor of the polynomial.

- **Implication $g$:** If $(c,0)$ is on the graph of the polynomial, then $c$ is a root of the polynomial.

**Other ways of saying the same thing:**
If $(c,0)$ is on the graph of the polynomial, then $P(c)=0$.
If $c$ is a zero of the polynomial, then $c$ is a root of the polynomial.
If $c$ is a zero of the polynomial, then $P(c)=0$.

**Algebraic Interpretations**

- **Implication $h$:** If $c$ is a root of the polynomial, then $(x-c)$ is a factor of the polynomial.

  Said differently:
  If $P(c) = 0$, then $(x-c)$ is a factor of the polynomial.

- **Implication $i$:** If $(x-c)$ is a factor of the polynomial, then $c$ is a root of the polynomial.

  Said differently:
  If $(x-c)$ is a factor of the polynomial, then $P(c)=0$.

This study was designed to investigate only the interior implications of this triangle, i.e. the graphical interpretations $a$, $b$, and $c$. A student will be considered to have grasped an interpretation/implication if he/she gives evidence of understanding at least one of the statements listed for that interpretation/implication.
Summary

In this chapter, I have given a rationale behind this investigation into students' understandings of the graphs of polynomial functions. Advances in technology, the complexity of the function concept, and the importance of students having a rich understanding of the graphs of polynomial functions provided the impetus for the study. Looking ahead, the foundation for this study continues to be laid by discussing the theoretical framework that underlies the use of qualitative methods in the process of this inquiry (Chapter III), and reviewing the research previously done regarding student understanding of the graphs of polynomial functions (Chapter IV).
CHAPTER III

THEORETICAL FRAMEWORK

A theoretical framework can be viewed as a lens through which the researcher perceives the planning of the study, the collection of the data, and the analysis of that data. By describing a theoretical framework explicitly, the researcher is informing the reader of his or her frame of reference for the study. Only with this information, and by trying to view the study through this lens, can the reader make informed judgements as to the reasonableness of the researcher's analysis.

In this chapter, I describe the lens that was used in this study, a constructivist position on conceptual change.

The Theory Of Conceptual Change

The theoretical framework for the proposed study relies heavily on the theory of conceptual change as described by Strike and Posner (Posner, Strike, Hewson & Gertzog, 1982; Strike & Posner, 1985). This theory describes the interaction that takes place between an individual's experiences and his or her current conceptions and ideas. Derived from scientific philosophy (Kuhn, 1962/1970; Toulmin, 1972), conceptual change theory views learning as a type of inquiry in which students make judgements on the basis of available evidence (Posner et al., 1982). Students change their concepts in order to accommodate new information, apply old notions to new contexts, or establish connections.
between concepts. Hence, the student's conceptions are in a process of continual evolution.

Research on student learning indicates that a student's prior conceptual knowledge influences all aspects of how they process incoming information (Pintrich, Marx & Boyle, 1993). A student's prior conceptual knowledge forms a framework for understanding and interpreting new information and experiences and also forms a basis for judging the significance and validity of the information. The conceptual change model gives four conditions for conceptual change to occur: dissatisfaction with existing conceptions, intelligibility of the new conception (it must be minimally understood), plausibility of the new conception, and fruitfulness of the new conception (Posner et al., 1982).

Davis and Vinner (1986) have written about the conceptual conflict that occurs when a student is confronted with new information:

In general, learning a new idea does not obliterate an earlier idea. When faced with a question or task the student now has two ideas, and may retrieve the new one or may retrieve the old one. What is at stake is not the possession or non-possession of the new idea; but rather the selection (often unconscious) of which one to retrieve. Combinations of the two ideas are also possible, often with strikingly nonsensical results. (p. 284)

Though illustrative of the process of conceptual change, the previous statement by Davis and Vinner seems to oversimplify the learning process by suggesting that a student has only to choose between two ideas. In actuality, a new idea is pitted against the totality of the student's prior conceptual knowledge as well as sifted through personal, motivational, social and experiential elements. Moreover, conceptual change theory, which is modeled solely on cognition, does not seem to explain why some students do not activate or transfer appropriate prior conceptual knowledge to every new task.
Whereas conceptual change theory portrays learning as an entirely rational process, driven solely by logic, it is likely that "irrational", day-dependent, affective, situational, and contextual factors also contribute to how the student processes new information. Hence, the lens through which this study was viewed was enlarged beyond conceptual change to include other factors which may have an effect on the learning process. These factors are described in more detail below.

**The Role of Affective, Situational, and Contextual Factors on the Learning Process**

Students are individuals that engage with the subject matter in idiosyncratic ways. Students vary widely in abilities, study habits, and previous mathematics background. How they become engaged in the subject matter is partially a choice they make for themselves, and partially a result of their motivation on that day, a factor which in turn is affected by experiences, attitudes and beliefs about mathematics in general and about the particular mathematical topic being studied, self-efficacy and the classroom context. This suggests that motivation is situation and context specific, and hence the possibility that conceptual change will take place may vary daily. Situational factors include the classroom or interview environment, the attitudes and beliefs of the teacher, and the effects of the use of technology on learning.

In research, integrating affective factors with cognitive factors is not a new idea, as Piaget (1985) noted the inseparability of cognition and affect, suggesting that affective factors energized all cognitive activity. Though the influence of motivational or affective variables in the learning process was never actually
denied by Strike and Posner (1992); they later revised their original theory on conceptual change to include these factors. They state:

A wider range of factors needs to be taken into account in attempting to describe a learner's conceptual ecology. Motives and goals and the institutional and social sources of them need to be considered. The ideas of a conceptual ecology thus need to be larger than the epistemological factors suggested by the history and philosophy of science. (p. 162)

In summary, in this study, I postulate that the learner's beliefs, motivation, as well as the effects of contextual and situational factors, can either facilitate or hinder the potential for the development of conceptual understanding. These factors may be regarded as noise or interference to a study by some researchers, but the tenet in this study is that these factors are and should be an integral part of qualitative research. Though McLeod (1992) has pointed out that affective factors are often ignored by cognitive researchers, I maintain that a qualitative study's validity and reliability is seriously damaged by not considering and integrating these factors that are inextricably linked to the data. To withhold this information would give an inaccurate portrait to the reader of the students being investigated.

The following explains how viewing the conceptual change model of knowledge development through the perspective of constructivism ensures that these situational, contextual, and affective variables will be examined as the investigation of conceptual development unfolds.

*Constructivism and Conceptual Change*

The underlying framework for the current study is a constructivist position on conceptual change. This implies that the process of conceptual change is
affected by the active participation of the learner in the construction of his or her own knowledge.

"Constructivism", as described in the last chapter, has been characterized as both a cognitive position and a methodological perspective (Noddings, 1973). The emphasis on the actions of the learner from the constructivist point of view is what distinguishes this perspective from a purely cognitive theory such as conceptual change. While conceptual change theory focuses on the integration of an individual's conceptions with new information to form knowledge, constructivism focuses on the development of this knowledge through the student's activity with mathematical objects in a mathematical community (Davis, Maher & Noddings, 1990).

An outcome of a student's active involvement in the environment is the effect that the environment has on the student's beliefs, attitudes, self-efficacy and self-motivation. The contextual, affective, and social factors, play a significant role in the evolution of a student's mathematical understanding, as mentioned previously.

**The Individuality of the Learner**

Inherent in the statements about constructivism and conceptual change is that students are individuals "constructing" knowledge in a way that is unique to them and that a "right way" in which the students should be constructing that knowledge doesn't necessarily exist. In addition, the constructivist view implies that in trying to make sense of the world, two people with dissimilar existing knowledge structures can acquire different conceptions even when presented with the same information. It is likely, therefore, that students in the same classroom,
with the same teacher, and with the same curriculum materials, may construct
different understandings of a concept.

It has been shown that students generate strategies and conceptions
because they are useful (Chiu, Kessel, Lobato, Moschkovich & Munoz, 1993; Smith,
diSessa & Roschelle, (in press)). A student in interpreting new information
through their prior conceptions, is making their own sense of the information.

Goldenberg (1988) said:

As we first learn to read graphs, we interpret what we see in them
according to strategies that have been successful for us in other realms,
and we continue to use such strategies until our new experiences teach
us to do otherwise. (p. 142)

These statements about the individuality of the learner hold a caution for
researchers and teachers:

It is important for teachers to recognize that standard or textbook
strategies are not the only correct ones and that the student-generated
strategies can also be mathematically interesting and productive.
These alternative strategies need not be treated as “misconceptions” to
be rooted out, but rather as sensible constructions to be explored. . . .
The traditional instructional goal has often been to replace student
generated strategies with standard ones . . . in contrast, . . .
instruction should assist students in identifying both the power and
limitations of their own strategies and comparing their strategies with
the standard ones in terms of criteria and generalizability. (Chiu et
al., 1993, p. 190)

Therefore, the teacher/researcher should assist the student in determining
what strategies are interesting and productive and what strategies are limited and
unfruitful to pursue.

**Classifying Student Understandings of the Function Concept**

The sections above described the lens through which the planning of the
current study, the collection of data, and analysis of data were viewed. In an
attempt to analyze and interpret student understandings of the function concept,
several researchers have sought to classify the forms of understanding of the function concept that a student develops through their experiences within the environment. These ways of thinking about an understanding of the function concept are referred to quite often in the literature substantiating their existence as viable themes in the interpretation of students' understandings of functions. The terminology and ideas used by these researchers will sometimes be referred to in this study and are useful in distinguishing between student understandings of the graphs of polynomial functions. They are discussed below.

**Concept Image**

In describing student understanding of the function concept, Vinner (1989) defines a concept image to be “the set of all the mental pictures associated in the student’s mind with the concept name, together with all the properties characterizing them” (p. 356). The construction of this image is a result of the student's experience with examples and nonexamples explicating the concept. Vinner notes that a concept image can be spoken of only in relation to a specific individual, highlighting once again the individuality of the learner within the perspectives of constructivism and conceptual change.

Though teachers at the secondary or collegiate levels may assume that the concept definition controls the concept image that the student develops (Figure 2), Vinner deems this “naive” (1992, p. 198) and “wishful thinking” (1983, p. 295).

Vinner maintains that “very often the concept image is entirely shaped by some examples and it does not fit the concept definition.” He contends that in a problem situation a student will almost always evoke the concept image instead of the concept definition.
The naive pedagogical approach assumes also that when a mathematical task is presented to the student, he or she will consult the concept definition while working on the task . . . or sooner or later, the concept definition will be consulted. . . . However, because of the nature of spontaneous thought, many students will not consult at all their concept definition and will act as in [Figure 3]. (p. 198)


Relying heavily on the work of Piaget —Breidenbach, Dubinsky, Hawks, and Nichols (1992) have classified the various conceptions of functions that students have of the function concept. These conceptions are determined by the level of abstraction achieved and are referred to as action, process, and object conceptions of function. Breidenbach et al. caution that these are not to be thought of as "stages in development" but ways that students think about the function concept.
Many researchers in mathematics education interpret student understandings of functions from this perspective. (Cuoco, 1993; Even, 1990; Kieran, 1993; Moschkovich et al., 1993; Schwartz & Yerushalmy, 1992; Sfard, 1992).

Breidenbach et al. contend that a student needs to develop a "process conception" of function in order to have an adequate understanding of the function concept. The process conception is defined in terms of actions as follows:

An action is any repeatable physical or mental manipulation that transforms objects (e.g., numbers, geometric figures, sets) to obtain objects. When the total action can take place entirely in the mind of the subject, or just be imagined as taking place, without necessarily running through all of the specific steps, we say that the action has been interiorized to become a process. (p. 249)

An action, for example, would be when a student calculates the value of a function by "plugging in" a value for $x$. When the student is able to think of this as a complete activity, as linking the $x$ and $y$ values, then the student has developed a process conception. Cuoco (1993) has described the process conception as "a view that functions are procedural, machine-like entities that transform inputs to outputs in predictable ways." (p. 119) He clarifies the distinction between the action and process conceptions as follows:

Students who view functions as actions think of a function as a sequence of isolated calculations or manipulations, a loosely defined activity that is to be performed with objects. . . . Students who view functions as processes think of functions as dynamic (single-valued) transformations that can be composed with other transformations. (p. 120)

In describing the next level of abstraction, Breidenbach et al. (1992) state, "When it becomes possible for a process to be transformed by some action, then we say that it has been encapsulated to become an object." (p. 250) Moschkovich and her colleagues (1993) have characterized the object perspective as follows:
From the *object perspective*, a function or relation and any of its representations are thought of as entities—for example, algebraically as members of parametrized classes, or in the plane as graphs that, in colloquial language, are thought of as being "picked up whole" and rotated or translated. (p. 71)

To reiterate that these *levels* of abstraction are not to be considered as steps in development, both Kieran and Moschkovich refer to the "oscillation" between the levels as students develop an understanding of the concept of function.

The acquisition of structural conceptions by which expressions, equations, and functions are conceived as objects and are operated on as objects does not eliminate the continued need for the procedural conception ... both play important roles in mathematical activity. ... The challenge to classroom instruction is to ... develop the abilities to move back and forth between the procedural and structural conceptions and to see the advantages of being able to choose one perspective or the other depending on the task at hand. (Kieran, 1993)

Saying when a student actually "has" the object perspective is not a simple matter. It is not a yes/no kind of knowledge, but one of degrees, and the process of learning is not one of simple monotonic growth, but one that includes a fair amount of oscillation. ... Flexibility to move between perspectives is a hallmark of competence. (Moschkovich et al., 1993, p. 97)

Noting that students must make connections across representations, those that embrace this way of thinking about student understanding of functions realize that the three levels of abstraction can be differentially useful depending on the representation and the problem context but that coming to grips with all the perspectives is an essential part of learning about function and graphs (Even, 1990; Moschkovich, 1993; Schwartz & Yerushalmy, 1992; Sfard, 1992).

Both the process and object perspectives shed light on the behavior of functions, in every representation, but the perspectives are differentially useful, in that one perspective may be usefully involved in some problem contexts and not in others. In part, we argue that developing competency with linear relations means learning which perspectives and representations can be profitably employed in which contexts, and being able to select and move fluently among them to
achieve one's desired ends. . . . Connections between perspectives allow for the possibility of flexibly switching from viewing a line (or an equation) as an object that can be manipulated as a whole, to viewing a line (or an equation) as made up of individual points (ordered pairs).

(Moschkovich et al., 1993, p. 72)

Schwartz and Yerushalmy (1992) reported:

We see that the symbolic representation of function makes its process nature salient, while the graphical representation suppresses the process nature of the function and thus helps to make the functions more entity-like. A proper understanding of algebra requires that students be comfortable with both of these aspects of function. (p. 265)

When applying this model to the graphical representation of functions,

Sfard (1991) noted that:

Geometric ideas for which the unifying, static graphical representations appear to be more natural than any other, can probably be conceived structurally [as objects] even before full awareness of the alternative procedural [process] description has been achieved. (p. 10)

This is in contrast to an algebraic representation where “the majority of ideas originate in processes rather than in objects” (p. 11). Her comments raise questions as to the suitability of the action, process, object model when studying the graphical representations of functions. Kieran (1993) points out:

The traditional, nontechnology-supported teaching of graphical representations of functions has always emphasized a process approach to the graphical representation along with a process approach for the function’s algebraic and tabular representations: For example, take a linear equation; substitute values for x in the equation and set up a table of x-y values . . . The technology-supported projects described in this chapter have clearly shown that this route is not the one that has to be followed if we want to encourage students to learn to read the global features of graphs. We have choices now. (p. 232)

Summary

In this chapter, the theoretical framework, a constructivist position on conceptual change has been described. Combining conceptual change, as a theory
of learning, with the constructivist emphasis on the activity of the learner within
the environment, affective, situational, and contextual factors are integrated with
cognitive factors. Both contribute to how students process information and can
facilitate or hinder the potential for conceptual understanding.

Attempts at classifying student understanding of the function concept have
been made by several researchers. These ways of thinking about student
understanding were also discussed.
CHAPTER IV

BACKGROUND AND RELATED WORK

Qualitative studies are judged against a background of existent knowledge (Eisenhart and Howe, 1990).

For example, if the results of one study contradict those of another (or several others), then some sort of explanation of why this occurred is in order. This is where the familiar review of the literature comes into play. (p. 7)

The following review of literature serves as the background against which this study's value in contributing to the research on student understanding of the graphs of polynomial functions can be judged.

There is a growing body of research surrounding the concept of function being done by members of the mathematics, mathematics education, and cognitive psychology communities. The importance of the function concept in the curriculum, and the current emphasis being placed on it, has been highlighted via the publication of two recent volumes on the subject. The Concept of Function: Aspects of Epistemology and Pedagogy (Harel & Dubinsky, 1992) published by the Mathematical Association of America is a synthesis of theoretical analysis and empirical observations on the learning of functions. Integrating Research on the Graphical Representation of Functions (Romberg, Fennema & Carpenter, 1993) is concerned with integrating the research on teaching, learning, curriculum and assessment with respect to the domain. In addition to these two volumes, Leinhardt, Zaslavsky and Stein's (1990) review of research and theory related to
the teaching and learning of functions, graphs, and graphing is quite comprehensive. Together, these three references have helped tremendously in the formation of the following synopsis of the pertinent literature.

Following a brief historical analysis of the status of the function concept within mathematics and within the mathematics curriculum, this chapter gives an overview of the related research that focuses on students' understanding of the graphs of functions. Particular attention will be given to studies involving the graphs of polynomial functions (of any degree). Research related to student intuitions about polynomial functions, and affective, situational, and contextual factors affecting the teaching and learning of these functions appears to be absent from the literature.

Making The Function Concept Prominent: An Historical View

The Mathematical Development of the Function Concept

Though function-like activities and notions of dependence can be traced as far back as 2000 B.C. (Kline, 1972), functions were not studied as objects in their own right until the late 16th century. The concept of function originated when Galileo (1564-1642), in studying motion, searched for a way to describe his observations. The development of analytic geometry by Descartes (1637) also contributed to the development of function, as graphs were accepted as mathematical objects for the first time.

Over the next two centuries, Euler, the Bernoullis, and other mathematicians developed calculus to deal with physical problem situations.

For these mathematicians, a function was an analytic expression representing the relation between two variables with its graph having no corners; this is usually referred to as Euler's definition of function. (Malik, 1980, p. 490)
The modern definition of function was introduced by Dirichlet in the early 19th century to study more advanced mathematical concepts. This definition of function emphasized the correspondence between two variables: y is a function of x if for any value of x there is a rule which gives a unique value of y corresponding to x (Malik, 1980). Throughout the 19th century and into the 20th century, functions were viewed as either dependence relations or correspondences between variables (Kleiner, 1989).

As the concept of set was gaining recognition as the fundamental concept of mathematics in the first half of the 20th century, functions began to be described as correspondences between sets. In 1939, Bourbaki, proposed the following definition of a functional relation:

Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x in E, there exists a unique y in F which is in the given relation with x. (Kleiner, 1989, p. 299)

He defined a "function" to be the operation that associates the elements x in E with the elements y in F. By the end of the first half of this century, the Dirichlet-Bourbaki definition of function had become established as textbook terminology (Malik, 1980). Graphs of relations were defined as subsets of the Cartesian product of two sets.

The issue of what definition of function should be used is still not settled. Discussions in the past 30 years among mathematicians have attempted to replace the notion of function as a set of ordered pairs (based in set theory) to an association from one object to another object (based in category theory) (Kleiner, 1989). This indicates that the evolution of the function concept continues.
The Function Concept's Prominence in the K-12 Mathematics Curriculum

The desire to make the function concept central to the K-12 mathematics curriculum goes back approximately 100 years. In 1921, the National Committee on Mathematical Requirements of the Mathematical Association of America recommended that functional thinking be the unifying principle of secondary mathematics (Romberg et al., 1993). This recommendation was made in response to the German mathematician Klein (cited in Hamley, 1934, p. 52) who, in 1904, referred to functions as the "soul" of mathematics and stated that "an elementary treatment of the function . . . ought to be in the regular course of all types of high schools." It was during this time (1900-1920) that the topic of graphs entered the curriculum as a means of integrating geometry and algebra.

Despite these recommendations and others (Hedrick, 1922; Breslich, 1928) there was only a slight increase in emphasis in the teaching of functions during the first half of the century. Though secondary mathematics methods textbooks (Schorling, 1936; Butler & Wren, 1941) encouraged the teachers to make function the unifying principle, secondary school algebra textbooks provided only sparse and uneven implementation of the function concept (Breslich, 1928).

A new emphasis on set theory in school mathematics during the middle part of the 20th century caused significant changes in the teaching of functions. Recommendations made during the 50's placed the emphasis on a function as a set of ordered pairs, or a correspondence between elements of two sets, as opposed to earlier definitions that focused on the correspondence, rules, or graphical aspects of the function. Though this emphasis was challenged by many (Willoughby, 1967; Buck, 1970), an analysis of textbooks done by Cooney and
Wilson (1993) indicated that between 1958 and 1986 functions were consistently defined in terms of sets. An example is taken from the Algebra I series by Dolciani, Berman, Freilich (1962) which defined a function as a subset of relation: "A function is a relation which assigns to each element of the domain one and one element of the range" (pp. 438-439).

A significant increase in emphasis on the function concept did not occur until the 1960's when the School Mathematics Study Group (SMSG) produced a series of secondary textbooks that gave increased importance to the function concept. This was in response to statements such as those made by the Commission on Mathematics of the College Entrance Examination Board (CEEB, 1959). Recognizing the importance of the function concept, they recommended that a separate functions course, giving a unified treatment to the elementary functions, including polynomial, rational, logarithmic, exponential and trigonometric functions, replace the traditional advanced algebra course. A "unifying" treatment of the function concept, however, still eluded the secondary curriculum as textbooks continued to contain a separate chapter dealing with the function concept, usually following chapters on linear and quadratic equations (Cooney & Wilson, 1993). Not until pre-calculus mathematics was the function concept more evenly applied throughout the curriculum.

Although recommendations that the function concept take center stage in the curriculum have been around for about 100 years, and despite ongoing appeals from the National Council of Teachers of Mathematics (1989), the emphasis in grades K-12 on the function concept has just slightly increased and continues to take only a minor role until a student enters precalculus. Because of the heavy reliance of teachers on the textbook (Romberg & Carpenter, 1986) and
the hesitation of textbook writers to adopt a functional approach, the attempts to reorganize the curriculum around the function concept have been weak.

**The Complexity Of The Function Concept**

Feeding the resistance to put functions at center stage, is the recognition that the function concept is a complex one. Not only do students need to deal with subconcepts (i.e. domain, range, continuous and one-to-one functions), but also many different representations that are components of the function concept (i.e. graphs, formulas, tables, verbal descriptions). The complexity is compounded by the possible learning tasks and the number of different contexts and settings in which a functional situation could take place. The complexity of the function concept has been depicted by Eisenberg and Dreyfus (1982) through a "function-block diagram". (Figure 4)

![Function Block Diagram](image)

**Figure 4: Function Block Diagram**

In this diagram the x-axis contains the various representations of the function concept (ordered pairs, arrow diagrams, graphs, algebraic rule, table, etc.), the y-axis contains the various sub-concepts within the function concept (domain,
range, image, zeroes, maxima, etc.) and the z-axis contains the levels of abstraction or generalization (number of independent variables, properties of the domain and range, explicitly or implicitly defined functions, etc.)

Today, more importance is being placed on the development of a student's conception of function. It is now believed that to give students a stronger conceptual foundation, more abstract notions of the function concept should be delayed. Appealing for an early development of functions in Grades 1-6, NCTM maintains that students in grades 9-12 should continue the informal investigations into the function concept that were started in the earlier grades (NCTM, 1989).

Research On Student Views, Attitudes And Understandings Of The Graphical Representation Of Function

As "a picture is worth a thousand words", so a graph is a complex concentration of information. Though a graph can provide a wealth of information, some researchers conjecture that many students do not consider visualizations central to the nature of mathematics and, compared to the analytic representation, find them more difficult to understand. Balomenos, Ferrini-Mundy, and Dick (1988) concluded from their calculus research that:

Despite the calculus teacher's predilection for diagrams, our research indicates that students resist the use of geometric and spatial strategies in actually solving calculus problems. (p. 196)

Dreyfus and Eisenberg (1990) contend that one of the reasons that students are reluctant to visualize is that the elements of an analytic presentation are sequential and easier to absorb than the simultaneous elements of a diagrammatic presentation which are likely to be difficult to absorb, and interpret. In terms of thought processes, they suggest, therefore, that visual
processing requires a higher level of mental activity than analytic processing. Eisenberg (1992) maintains that although students are reluctant to think of functions visually, that “one of the components of a well developed sense for functions is the ability to tie together their graphical and analytical representations”.

The research shows that what a student sees in a graphical representation depends on his/her existing knowledge about functions and graphs (Schoenfeld, Smith & Arcavi, 1993; Larkin & Simon, 1987). Students see the graph through their current understanding of graphs of that type and through other experiences with graphs of that type. In other words, graphs are not transparent. Students don’t always see what we want or expect them to see, i.e. they see what their conceptual understanding at that time prepares them to see.

**Impact Of Technology On The Classroom, Research, And On Student’s Understandings Of The Graphical Representation Of Function**

Before hand-held graphing calculators and computers became readily available, the graphing of functions was regarded as a tedious and time-consuming process, and, therefore, often avoided in the classroom. The dependence on using algebraic techniques for finding roots, functions values, and equations, was a consequence of this situation and this technique prevailed despite the sense that graphical representations provided a visual representation of an algebraic expression, allowing the student another way of assigning meaning to the symbols. Now, with the aid of technology, the graphical representation of the function can be readily utilized in conjunction with the algebraic representation to perform these same tasks.
In recent years, there has been considerable support for a greater emphasis on the graphical representation of a function. Heid (1995) writes:

It is not as important for students to know how to sketch a graph by hand as it is for them to understand what the graph means and how it can be used to enhance their understanding of the function or relation the graph is describing. (p. 25)

Eisenberg (1992) has stated that the ability to visualize and interpret graphs is a major component of an understanding of functions. The widely held view is that the new emphasis on graphical representations will elucidate the function concept and make it easier to learn.

This interest in the graphical representation of functions has sparked a growth in the research base relative to the subject. The National Center for Research in Mathematical Sciences Education [NCRSME], brought together a group of scholars interested in research related to the graphical representation of functions. The result of this gathering was a book edited by Romberg, Fennema and Carpenter (1993) in which the authors examine the impact that a shift to graphical representations using technological tools has had on the content, learning, teaching, and assessment of functions. In bringing together mathematics education researchers with a common interest, this book represents a beginning point and a foundation for further research in this area.

Technology offers new methods to engage students in learning. Technology can change what is taught and how students develop a conceptual understanding of the subject matter. Research is necessary, however, to see in what ways technology interacts with student understanding.

Research has shown that technology can have both negative and positive effects on student understanding of functions and graphs. Confrey, Smith, Piliero
and Rizzuti (1991) remark on the use of the graphing software, *Function Probe* (Confrey, 1991/1992) which was used to analyze the development of the function concept in 23 senior precalculus students. They report:

Many of the students developed a strong qualitative sense of functional relationships that is not as easily developed when students examine hand drawn graphs. By quickly and efficiently graphing functions and presenting them to students as objects for examination, graphing software allows students to concentrate on global features of graphs, such as shape, direction, and location, and thus leads to understandings based on features which distinguish between classes of functions. (p. 50)

In contrast, Philipp, Martin, and Richgels (1993) remark on research results that report some of the negative effects of graphing software.

The ability of graphing technologies to improve the teaching and learning of mathematics is not all rosy, however, and in no area is this so clear as in studies of students' understanding of graphs (Clement, 1985; Goldenberg, 1987, 1988; Goldenberg & Kliman, 1988; Preece, 19893; Schoenfeld, 1990). Students possess a myriad of misconceptions about graphing, some of which are artifacts of the actual program being used. (p. 268)

Goldenberg (1988) reports from clinical studies that ambiguities in graphs can be caused in part by the interaction between the position and orientation of the graph and the shape of its window, and in part by the interaction between the scale of the graph and the scale of the window. He cautions:

It is tempting to assume that the world of shapes and visual gestalts is much more naturally interpretable than the world of algebraic symbols. . . . Most classroom observations suggest another possibility: namely that graphic representations of function are no more naturally interpretable than are the symbolic representations. They have their own rhetorical conventions that students must learn, and they contain ambiguities that are clarified only through the use of further specifications, such as those contained in the conventional algebraic notation. (p. 138)

Goldenberg points out the need for the "thoughtful" creation of graphing software. He reports that students may view a graph of a function $f(x) = ax^2 + bx +$
c more as a function of the constants a, b, and c, than as a function of x, if the graph is swept out from left to right without any student involvement except control over the function's parameters. Students have a tendency to view a, b, and c as the variables, so that attention is drawn to the graph itself as the output, rather than the output being the set of function values.

Simplistic software design or thoughtless use of computer graphing in classrooms may further obscure some of what we already find very difficult to teach. On the other hand, thoughtful design and use of graphing software presents new opportunities to focus on challenging and important mathematical issues that were always important to our students but that were never so accessible before (p. 135).

He notes that other research already attests that the concept of variable is difficult for students to learn, at least in static, inert media, and that the "unthinking use of such software may further obscure rather than clarify this difficult concept."

(p.152)

Types of Graphing Tasks Used in the Curriculum and in Research

According to Philipp and his colleagues (1993), most existing algebra textbooks today present graphs in stand-alone sections or chapters instead of integrating them with corresponding algebraic procedures. In addition, graphs are treated as ends in themselves.

Seldom are the students expected to use the graph to answer further questions about the function. Metaphorically, this approach to functions seems to be a half circle. One starts with the function and ends with its graph, never returning to the original function, or to the application that spawned the function. (p. 249)

Although this paints a discouraging picture of the progress being made in making functions and graphs more prominent in the curriculum, there are a few recently published textbooks that are making more radical strides away from the algebraic emphasis (Fey & Heid, 1995; Senk, Thompson & Viktora, 1990).
In the traditional algebra curriculum more tasks require the student to make local interpretations of graphs (e.g. the focus is on the points of the graph) rather than global interpretations of graphs (e.g. interpreting trends in the graph where the focus is on intervals or on the graph as a whole). This emphasis in the curriculum (Bell & Janvier, 1981; Clement, 1989; Kerslake, 1981) contributes to students viewing graphs as a collection of isolated points rather than as objects (Schoenfeld et al., 1993). There are also more graphing tasks that require a quantitative interpretation as opposed to a qualitative interpretation. A qualitative interpretation of a graph "requires looking at the entire graph (or part of it) and gaining meaning about the relationship between the two variables, and, in particular, their pattern of covariation" (Leinhardt et al., 1990). In a contextualized setting, Monk (1987) has found that students can answer "pointwise questions" fairly well, but have more difficulty with "across-time questions".

Efforts to improve students' understanding of graphs have emphasized the need to move beyond plotting and reading points to interpreting the global meaning of a graph and the functional relationship that it describes (Dugdale, 1993). Researchers at the Shell Centre for Mathematical Education at the University of Nottingham have been active in exploring students' perceptions of graphs and designing activities to develop qualitative understanding of graphs. Problems such as the bathtub problem, in which a graph is drawn to depict the height of the water in the bathtub while taking a bath, displayed by the computer program Eureka (Philipps, 1982) and graphs of electrical usage have been used to put graphs in the context of the students' everyday experiences. The use of "probeware" such as the microcomputer-based Labs [MBL] developed by the
Technical Education Research Center [TERC] and Texas Instruments provides a way that students can generate graphs related to experiments and activities using the probes. Barclay (1985) reported that students were more apt to answer questions about the graphs generated by these probes if the questions dealt with local interpretations of the graph than questions that concerned larger sections of the graph. Mokros and Tinker (1987) reported an improvement in students' ability to interpret and use graphs after using the probeware and a decline in the students' tendency to view graphs pictorially.

Leinhardt and her colleagues (1990) identify two types of student actions on graphing tasks, interpretation and construction. Interpretation is an action by which a student makes sense or gains meaning from a graph (or a portion of a graph) whereas construction is an action requiring the student to generate something new (e.g. graphing a function given the algebraic representation). Tasks used in the research on functions require one or both of these actions by the student. Leinhardt and her colleagues name four types of tasks that appear in the literature: prediction, classification, translation, and scaling.

A graphing task can be presented through a variety of problem contexts. Studies are usually based either on contextualized tasks (Bell & Janvier, 1981; Clement, 1989) or on abstract ones (Markovits, Eylon & Bruckheimer, 1986; Schoenfeld et al., 1993; Vinner, 1983). A few focus on both types of problem situations (Dreyfus & Eisenberg, 1982; Kerslake, 1981).

The studies that include contextualized tasks often are based on the assumption that it is easier for students to deal with problems that build on familiar situations (e.g. situations they either have experienced or are able to relate to in a meaningful way) than to deal with abstract situations. It is not clear, however, that a real-life kind of context always supports the learning process. For example, some situations have pictorial entailments to them that present the
Several studies have shown that time used as the independent variable aids student understanding (Janvier, 1981; Krabbendam, 1982). This seems to support the notion that tasks involving real-world situations are more easily understood by the student. However, these positive results are tempered by other work (on time and nontime graphs) that suggests that sometimes students are distracted by graphical features that correspond on a pictorial level with aspects of the situation (Kerslake, 1981; Stein & Leinhardt, 1989). Although, it has been shown that in a real world context graphs can provide a pictorial image of the function which is useful for interpretations, some students can interpret graphs too pictorially. An example of this is in a distance/time graph which, in Kerslake's study, looked to many children like journeys in a vertical plane with the object traveling "east", "north", etc... or "left", and then "right" (Figure 5). Kerslake hypothesized that children who are particularly strong visualisers have more trouble with this type of task.

![Distance/Time Graph](image)

**Figure 5: Distance/time graph used by Kerslake**

In another example, some students viewing a velocity time graph showing the motion of a car (Figure 6), thought that the car was moving up a hill.
Figure 6: Graph showing motion of a car

To combat this tendency to interpret graphs too pictorially, Goldenberg (1988) writes:

Multiple linked representations increase redundancy and thus can reduce ambiguities that might be present in any single representation. Said another way, each well-chosen representation conveys part of the meaning best; together, they should improve the fidelity of the whole message. (p. 136)

His statement indicates that the emphasis on the graphical representation is by no means an attempt to replace the use of the algebraic representation, but rather a means by which the algebraic representation can be illuminated to promote student understanding.

Research On Translating Between Representations Of Functions

Research done in the area of translating between representations of functions may offer insight into how students interpret the graphs of polynomial functions. In trying to make sense of the connections between the algebraic and graphical representations of linear functions, Moschkovich (1992) refers to the "conceptual asymmetry" between what is "perceptually salient" from the graph and the equation. That is, not all the information that is accessible from a graph of a linear function shows up in any one equation form.
For equations of the form $y=mx+b$, this conceptual asymmetry is evident in the x-intercept: while the x-intercept has status as a graphical variable, it does not as an algebraic variable in this form of the equation. (p. 131)

Algebraic symbolism is a means of communicating about the features of a graph that are embedded within a coordinate system. Hence, the algebraic representation of a polynomial function reveals useful information about the graph and vice-versa.

Much of the research that has been done in the area of translation between representations of functions involves the use of computers (Leinhardt et al., 1990). Computers are used because of the access to graphing software that permits multiple-linked representations that are believed to make the connections between representations salient. For instance, Goldenberg (1988) typified this optimism saying:

Common sense supports the notion that multiple representation will aid understanding. Used thoughtfully, multiple linked representations increase redundancy and thus can reduce ambiguities that might be present in any single representation. Said another way, each well-chosen representation conveys part of the meaning best; together, they should improve the fidelity of the whole message. Translation across multiple representations can help to reduce the isolation of each mathematical lesson and help to provide a more coherent and unified view of mathematical method and content. (p. 136)

But, through the use of clinical studies, Goldenberg detected the following:

Our research has also made it clear that graphing software that is developed along conventional lines—software that may seem perfectly adequate at first—can blur or obscure concepts of great importance... While potentially reducing ambiguity, multiple representation also presents a student with more places to look and is potentially complicating and distracting. (p. 136)

Schoenfeld et al. (1993) note that even with the use of technology, the mathematical connections between representations will not be obvious unless the student has adequate prior knowledge. In a detailed study of one student's
understanding of algebraic and graphical representations of linear functions using the computer program GRAPHER, the authors claimed that many of the student's difficulties arose because she had not made the "Cartesian connection". This consists of making Connection A: that the point \((x,y)\) is on the graph of \(y=f(x)\) only if \(y = f(x)\) and Connection B: that specific algebraic expressions have graphical identities (i.e. the student understands the structure of the plane).

Though making these connections may seem simple, their work reveals the complex knowledge structures that underlie these connections and, hence, the difficulties that students experience when translating between graphical and algebraic representations of functions.

**Research On Student Understandings Of The Graphs Of Polynomial Functions**

Most studies at the linear and quadratic level focus on the misconceptions and difficulties students have with graphs (Markovits et al., 1986; Vinner, 1983). These studies reveal that students have a tendency to revert toward features of linearity in a variety of situations, generally neglect or confuse the domain and range, and have more difficulty transferring from the graphical representation of a function to the algebraic representation than vice-versa.

The brevity of the following summary dealing with research among the different classes of polynomial functions indicates the lack of research that has been done in this specific area. In particular, the research on functions needs to be extended to include more studies concerned with the interpretation of the graphs of polynomial functions of degree greater than two and studies that explore the connections that students make between these classes of functions.
Graphs Of Linear Functions

The Functions Group at Berkeley has done extensive work in studying student understanding of linear functions using in-house software called Grapher (Schoenfeld, 1990). In observing the students, they noticed students developing their own strategies for determining the y-intercept of a graph instead of using the intended strategy of the curriculum (Chiu et al., 1993). In addition, this work reveals that students find the x-intercept every bit as salient perceptually as the y-intercept (Schoenfeld et al., 1993). In a related observation, they noted that students are often not even aware of some objects that were thought to be conspicuous, e.g. they focused on where intersecting lines entered and left the graphing window rather than the point of intersection. They explain (Moschkovich, Schoenfeld & Arcavi, 1993):

To sum up the moral of these examples in brief, what we perceive is shaped by what we know. People familiar with the domain know what to ignore and what to focus on, while newcomers to it do not. As a result, we see . . . differently than students do. What is obvious to us may not even be apparent to them. (p. 78)

In related work students saw the graph of \( y = x + b \) as a horizontal translation of \( b \) units of the graph of \( y = x \) (Chiu et al., 1993). Other research by Moschkovich (1992) noted a strong tendency to use the value of the x-intercept in the form \( y = mx + b \) for either \( m \) or \( b \). One cause of this phenomenon was attributed to the fact that a change in \( m \) results in a change in the x-intercept making it difficult to separate the two objects.

Graphs Of Quadratic Functions

Goldenberg (1988) found that in terms of quadratic functions, students had a well-defined notion that the "shape" or "pointiness" of the parabola was
controlled by the coefficient of $x^2$. Also, they knew that if the parabola was upside down, that coefficient was negative. The notion of "height" of the parabola was often lumped together with the constant term of the standard form of the quadratic.

Dreyfus and Halevi (1990/1991) used a computer-based learning environment with Grade 10 students to explore what affect the parameters $a$, $b$, and $c$ of a quadratic function $f(x) = ax^2 + bx + c$ have on the graph of the parabola. This teaching experiment was an attempt to establish connections between the algebraic and graphical representations and an intuitive familiarity with the set of all quadratic functions. The interactive manner of the learning process, the conceptual type of work and the immediate feedback the students got from the computer program is attributed to the students' increased depth of understanding in this domain.

**Graphs Of Polynomial Functions Of Degree Greater Than Two**

Dugdale, Wagner and Kibbey (1992) approached the graphs of polynomial functions of degree greater than two as sums of monomial graphs. As students explored this "monomial sums approach" to graphing they generated and shared various ideas about how the graph of the polynomial function related to the individual terms. Among the goals of their research were attempts to develop in their students a more intuitive and qualitative notion of why polynomial graphs behave as they do, and to emphasize graphical reasoning rather than rules to visualize the functional relationship. They report success in reaching these goals and noted that some students became quite good at recognizing the effects of individual terms of a polynomial function by the shape of its graph.
Summary

The literature summarized above has influenced the direction of the current study in both content and methodology. It serves to support the value of this study in extending the research on student understanding of the graphical representation of functions. It is evident from this review, from the complexity of the function concept, and from the rapid changes occurring in mathematics education reform and in technology, that researchers have only "scraped the tip of the iceberg" as far as the research that needs to be done in this area.
CHAPTER V

RESEARCH METHODOLOGY

The Nature Of Qualitative Research

As was indicated in Chapter II, research in a student's natural setting using qualitative methods is being increasingly recognized as the most appropriate approach to investigate students conceptual understandings. Qualitative data takes the forms of words rather than numbers. The focus is on "naturally occurring, ordinary events in natural settings" (Miles & Huberman, 1994, p. 10).

In the past ten years, qualitative inquiry has received greater acceptance as a valid research method. Having been used predominantly in the social sciences and in anthropology, the past decade has seen a shift to this method of inquiry by many other disciplines including mathematics education (Romberg, 1992). A strength of qualitative data is that it is embedded within a context and therefore is more vivid to the reader. Usually collected over a sustained period of time, the data holds a strong potential for revealing the complexity of a situation and for testing hypotheses.

There are still many concerns, however, with this relatively new method. A major concern is that there is no bank of explicit methods for doing qualitative research. This has sometimes caused a lack of confidence in qualitative research results. Glesne and Peshkin (1992) blame the lack of standardization on "the open, emergent nature" of qualitative research.

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Unlike quantitative inquiry, with its prespecified intent, qualitative inquiry is evolutionary, with a problem statement, a design, interview questions, and interpretation developing and changing along the way. ... We do not know of and thus do not provide clear criteria packaged into neat research steps. The openness sets the stage for discovery as well as for ambiguity that, particularly for the novice researcher, engenders a sometimes overwhelming sense of anxiety. (p.6)

In addition to the lack of a formalized methodology for doing qualitative inquiry, Miles and Huberman (1994) give the following list of concerns with the research paradigm:

Labor-intensiveness ... frequent data overload, the distinct possibility of researcher bias, the time demands of processing and coding data, the adequacy of sampling when only a few cases can be managed, the generalizability of findings, the credibility and quality of conclusions, and their utility in the world of policy and action. (p. 2)

They propose a series of 50 questions that researchers and readers can apply to a qualitative study to evaluate it's "goodness" (see p. 278-280). Many of these questions pertain to the level of description and detail given within the study in regards to the procedures used during research planning, data collection and data analysis. They recommend that research procedures be described "clearly enough so that others can understand them, reconstruct them, and subject them to scrutiny." (p. 281)

Glesne and Peshkin (1992) write that "time is a major factor in the acquisition of trustworthy data. Time at your research site, time spent interviewing, time to build sound relationships with respondents—all contribute to trustworthy data." (p. 146) In addition, they point out that the data becomes more trustworthy as it is triangulated from multiple sources, as you remain continually alert to your own biases and your own subjectivity, and as you realize the limitations of your study.
The concerns about the trustworthiness of qualitative inquiry, though seemingly vast, pale when set against the ability of this type of research to reveal the complexity of the subject matter by collecting the data over a sustained period of time and by accounting for the local context, rather than stripping it away. The data sheds light on the meanings that a student gives to the subject matter taking into account their attitudes, their social world, and their background.

**Implications of Theoretical Framework for Research Methodology**

**The Clinical Interview**

Working within the framework of a constructivist position on conceptual change, as described earlier, clinical interviews were used to investigate each student’s understandings. This method of inquiry has been described by Confrey (1981) as follows:

By clinical interviewing, I am referring to task-oriented, flexible interviews between a student and interviewer wherein the interviewer is expected to follow and pursue the student’s thinking, asking questions until the student’s reasons for response are understandable to the interviewer. (p. 6)

Confrey argues that it is as important to know why a student gave an answer as how that student gave an answer and hence clinical interviewing is an appropriate method for pursuing the “justifications” for student beliefs. Drawing upon the theory of conceptual change, she comments on the suitability of the clinical interview in investigating students’ understandings:

Concepts are not grasped in their entirety and thus do not entail fixed states, but include both progressive shifts in understanding and various processes necessary to accomplish those shifts. This resonates with the process-oriented, change-oriented view of clinical interviews. . . . Thus, in exploring a child’s understanding of a particular concept we are often led into other concepts whose meaning resounds in concert or in tension with a given one. In undertaking clinical
interviews, we find that configurations of concepts get considered, and their relative significance—not single, isolated concepts. (p. 11-12)

**Teaching Episodes and the Teaching Experiment**

Because of the constructivist emphasis on the activity of the learner, two other research techniques commonly used to investigate student's learning are teaching episodes and the teaching experiment (Cobb & Steffe, 1983; Steffe, 1984). When the researcher goes beyond observing and intentionally intervenes in knowledge construction, the interview is changed into a "teaching episode" (Steffe, 1984). It is more than a clinical interview, in that the teaching episode involves experimentation with the ways and means of influencing the student's knowledge development. Cobb and Steffe (1983) write:

Our emphasis on the researcher as teacher stems from our view that children's construction of mathematical knowledge is greatly influenced by the experience they gain through interaction with their teacher. (p. 83)

Primarily an exploratory tool, derived from Piaget's clinical interview techniques, teaching episodes also allow the researcher the opportunity to observe how the students do or construct mathematics, an essential aspect of learning that is not as readily observed in a clinical interview. The goal for the researcher/teacher is to "construct explanations of children's constructions" (Steffe, 1984).

A teaching experiment has been described to be a series of teaching episodes and individual interviews (Cobb & Steffe, 1983).

**The Role of the Researcher**

Norman (1993) characterizes the role of constructivist researchers as follows:
The basic assumption underlying the constructivist perspective is that students construct their mental structures (i.e., understandings) via their own idiosyncratic mental processes. Thus, the role of the researcher is to try to grasp the nature of the processes learners use to build their knowledge and the nature of the knowledge that learners possess. (p. 98)

To portray the "voice" of the student as authentically as possible, both clinical interviews and teaching episodes require "close listening" (Confrey & Smith, 1993) on the part of the researcher. The principles involved in close listening include (p. 5-6):

1. providing evidence in the students' words;
2. following the problem development carefully, trying to de-center from one's own perspective;
3. encouraging strongly autonomous expressions by students;
4. asking for clarification, rephrasing the statements in the student's language;
5. avoiding evaluative expressions except as they support articulation of method;
6. stepping out of the role of answer-giving;
7. checking that the student remains emotionally confident with the interview and involved in the course of problem solving;
8. allowing the student to identify errors and contradictions;
9. providing resources to allow the student a variety of routes to proceed; and
10. conducting interviews for a long enough duration to ensure that ample opportunity for expression is allowed.

**Design of the Current Study**

The purpose of this study was to investigate student understandings of the graphs of polynomial functions and in particular, the connections that students make graphically between the various classes of polynomial functions as distinguished by their degree. The following is an overview of the research components used to structure these investigations.
RESEARCH COMPONENTS

I. PILOT STUDY

ii. DATA COLLECTION

A. Phase One

i. Choose setting for study

ii. Initial contacts and observations

B. Phase Two

i. Videotaping of class instruction

ii. Selection of student participants

iii. Clinical Interviews

C. Phase Three

i. Teaching Episodes

ii. Assessment of Teaching Episodes

III. DATA ANALYSIS

A. Data Reduction

B. Data Display

C. Conclusions

The above overview also serves as an outline for describing the research methodology in the remainder of this chapter.

Pilot Study and Data Collection

Pilot Study

In preparation for the actual research study, pilot questions were administered to students in two different sections of a course entitled “Elementary Functions” at a state university, during the spring of 1992. Due to time
constraints, and the desire to elicit responses from a larger number of students, the questions and responses were written. They were given prior to the class instruction on polynomial functions of degree greater than two in an attempt to see if the questions were effective in probing students' intuitions about the graphs of these functions and in determining the nature of the connections the students made between the graphs of polynomial functions of degree greater than two and the graphs that they had already studied, i.e. graphs of quadratic functions, linear functions, and other elementary functions. The questions were revised based on the responses received in order to procure the information desired and the richness of response desired from the students in the actual study. The results of the pilot study were used for the evaluation of interview questions only.

Phase I

Setting. The study took place at a large secondary school in Southern New Hampshire. The school serves students from several communities with varying socio-economic structures. in operation since 1815, the maximum number of students in most classes is 30. Courses are divided into levels —level A designed to offer scholastic preparation for colleges and universities with a competitive admissions process, level B designed to offer both scholastic preparation for colleges and other post-secondary institutions, and general preparation in a wide variety of areas.

A level A Algebra II class was chosen at this school on the basis that the curriculum included the topics pertinent to this study, and the teacher had a number of years of experience teaching about functions and graphs. She also said
that she was working on increasing the incorporation of the graphing calculator in her classroom.

Beginning in October of the school year, I observed the class at least once a week in order for the students in the class and the teacher to start getting accustomed to my presence in the classroom. Field notes were taken to record initial observations of the classroom environment, including class format and types of interactions taking place. Once instruction on functions began, notes were gathered on how the function concept was introduced to the students and what balance existed among the different representations of function during the instruction. This information was used in the later data analysis, providing a background from which to interpret student dialogue. In addition, field notes recorded the time spent on topics, and time spent in various class formats, i.e. lecture, working in groups, etc. Worksheets that the teacher used to supplement the textbook were collected. This data was used to gain information on how the teacher was endeavoring to construct a foundation for the student's knowledge of the graphs of polynomial functions. This data combined with information recorded in a later interview with the teacher was used to make inferences about the teacher’s attitudes and beliefs about functions and her perspective on the importance of the various representations. Samples of student class work were collected during this first phase of the study.

Phase II

During the second phase of the study, which began in January, a video camera was used from the back of the classroom to record class work dealing with
quadratic functions in order to acquire a more complete account of the classroom instruction and discourse.

The traditional sequence of instruction, and the sequence that was adopted in this Algebra II class, was from linear functions to quadratic functions to polynomial functions. I believed that the graphs of quadratic functions would have the most effect on the student's ability to interpret the graphs of polynomial functions of degree greater than two because of the similar features of the graphs, i.e. maxima and minima, increasing as well as decreasing intervals.

Selection of student participants. At the end of January, the students were told more about the study that was taking place within their classroom and they were invited to volunteer to participate in the study, a commitment which would require about 5 hours of their time. Seven students volunteered. One student who wished to participate was denied the opportunity as he was taking a more advanced course simultaneously with the Algebra II class and was also involved with the mathematics team. I decided that this would distort the data relative to his conceptual development in the Algebra II class. Of the other six who volunteered, three participated only through the second interview, as I decided at that time to limit the number of case studies to three to make data analysis more manageable. From initial reviews of the first two interviews, three students were eliminated due to various characteristics that would have been interesting to explore but went beyond the scope of this study. One of these students was a mathematics team member and therefore had received instruction outside of the classroom that could have been considered an influence in her concept development. Another student revealed that he was dyslexic and I was concerned how this may have affected his ability to interpret graphs. The third
student had a very weak conceptual understanding of foundational function knowledge. The development of her knowledge with respect to the graphs of polynomial functions would be strongly affected by this insecure foundation.

Therefore, two females, Lisa and Beth, and one male, Mark, remained for the rest of the study. Class work, quizzes, and tests were collected from these students selected for the case studies. These were used to help assess each student's understanding of the graphs of polynomial functions.

**Clinical Interviews One and Two.** After consent documents were completed by the students and their guardians, I had two clinical interviews with each of the students. The interviews were conducted in an available empty classroom and videotaped. The purpose of these interviews was to obtain background information about the student and to investigate the student’s current understanding of various aspects of the function concept.

The general schedule of these interviews is given in Appendix B. Because of the “flexibility” required in clinical interviews and the variability in student responses, actual questions regarding the items in these tasks necessarily varied from those in the Appendix.

The tasks in the first clinical interview were developed to determine a baseline understanding of the function concept and of the graphical representations of functions for each student. With these responses serving as a baseline, comparisons and interpretations could be made with future responses as students changed their conceptions to accommodate new information. The tasks were designed by the researcher in an attempt to generate student responses that would be rich enough to sufficiently describe the student's current understandings of the graphs of polynomial functions. The graphs of linear and
quadratic functions were shown to the students as well as graphs of other elementary functions and non-functions. The tasks used in Interview #1 (Appendix E) were hand-written with thoughts of maintaining a casual, relaxed tone to the interview.

Tasks developed for Interview #2 were intended to be done with a personal computer. *PC-81 Emulation software* (Texas Instruments, Inc., 1991) was selected so that polynomial functions would be graphed quickly and efficiently for the student's observation yet be visible to the video camera. As this software was designed to resemble the student's hand-held TI-81 graphing calculator [Texas Instruments, Inc.], I decided that the student would more easily adjust to this software as compared to other types of graphing software. The student and I had access to a cursor that made pointing to sections or points of the graph convenient and made it possible to capture these activities on videotape.

Using this software, each student was presented with three different graphs in the following order, a polynomial function of degree three, a quadratic function, and a linear function. The main objective in showing the student these graphs was to observe the connections that the student made between the classes of polynomial functions. A secondary objective was to obtain a deeper perception of his/her current understandings of these graphs. The ordering of graphs was intentional, as I didn't want associations from cubic functions to linear/quadratic functions to be prompted or foremost in the student's minds by having just worked with the latter.

Tasks included describing various locations on each graph, as well as saying whether the location held any information relative to the graph's algebraic representation. In addition, students were asked to make some global
interpretations, discussing the general trend of the graph, the domain and range of the function, intervals on which the graph was increasing/decreasing, and intervals on which the function values were positive/negative. To repeat, the purpose behind these tasks was to evaluate student understanding of the graphs of polynomial functions and to see what connections the students made between the different classes of polynomial functions.

Following each interview, the student was asked to write in a journal their thoughts about the interview. These journal entries were a resource used to get a sense of student characteristics of an affective nature prominent during the interview (e.g. nervousness, fear, self-consciousness, etc...).

Phase III

Teaching episodes. The purpose of the clinical interviews was to probe the student's understanding of the graphs of polynomial functions to obtain a model of the student's current mathematical knowledge concerning them. Teaching episode tasks were made in light of the observations made in the clinical interviews. A goal of these teaching episodes was to present activities that would promote further connections between the classes of polynomial functions and develop the student's qualitative understanding of these graphs. Qualitative interpretation of graphs is an aspect of graphical interpretation considered underrepresented in the curriculum (Leinhardt et al., 1990, p. 11) and hence it is not surprising that researchers have found that students have difficulty in this area (Bell & Janvier, 1981).

As student interviews progressed in Phase II of this research study, the student's understandings of the graphs of polynomial functions was examined and
ways were considered in which this understanding could be enriched and extended. The assessment of their understanding is given in detail in Chapters VII - IX.

As evidenced by the clinical interviews in Phase II, students don't always make a viable connection between the zeros of a polynomial, \( f(x) \), the roots of the equation \( f(x)=0 \), the factors of the polynomial, and the intercepts of the graph. The hypothesis was that giving the students the opportunity to build polynomial functions by taking products of linear functions would make these connections more useful. Furthermore, the use of software that allowed the student to see the algebraic representation alongside the graphical representation would perhaps foster the formation of these connections.

Fortunately, the Educational Development Center in Newton, Massachusetts had already developed a suitable software tool to be utilized in the planned teaching episodes entitled \textit{The Function Supposer: Explorations in Algebra} (Educational Development Center, 1990). One program option in this computer environment is the construction of polynomial functions by taking products of linear functions. The software allows the user to see the graphs of the linear function components as well as the resulting polynomial function on the same graph. The algebraic representation can be shown beside these graphs.

Over the course of two teaching episodes, each student used \textit{The Function Supposer: Explorations in Algebra} to investigate the graph of the product of two linear functions. This idea was extended to products of three linear functions using worksheets that followed the format of the software. The teaching episodes had a general protocol (Appendix B) that was modified by the student's actions, constructions, and questions, as well as by information gained about their
understanding in the clinical interviews and previous teaching episodes. The ways in which these tasks affected the development of understandings with respect to the graphs of polynomial functions were explored.

Students were again asked to write in their journals following each teaching episode.

Assessment Of Teaching Episodes

Following the teaching episodes, student understanding of the graphs of polynomial functions was probed once again in clinical interviews using the PC-81 Emulation Software as in the second interview. Qualitative comparisons were made of student understandings as had been perceived by the researcher prior to, during, and after the teaching episodes. These comparisons led to an assessment of the teaching episode tasks.

Final journal entries were collected from the students.

Data Analysis

The Use of Case Studies

Analysis of the data generated case studies of the three students who participated in the study. Driven by the aforementioned research questions, the case studies are indepth stories of the interviews and the teaching episodes. These stories provide information relative to each student's developing understanding of the graphs of polynomial functions as they progressed through the various phases of the study, through their actions, perceptions, and beliefs within the given circumstances and conditions. The use of case studies is
consistent with the researcher's belief in the individuality of the learner within
the conceptual change model and the theory of constructivism.

Though the use of case studies in research has generated criticism on the
grounds of reliability, validity and generalizability, Clarke (1991) contends:

The power of a reported case study lies in its ability to evoke
recognition, in the extent to which readers can identify the person,
event or institution described with similar elements of their personal
experience. (p. 8)

To facilitate this “recognition”, the data must be conveyed by the researcher in
sufficient detail. It is important not to strip the data from the context in which it
has been viewed. At the same time, a case study is of necessity a summary of the
collected data.

In presenting the findings of a case study in a form accessible to others
the researcher is obliged to select, summarize and simplify
observations already compressed into manageability and already
simplified by successive acts of categorization at the point of
observations and in subsequent description and analysis. The result is
a skeletal framework which presents only the essential structure of the
particular case, lacking flesh and substance. [Clarke, 1991, p. 8]

Hence, the researcher, using the research questions as a guide, has many
decisions regarding what should be considered the “essence” of the data.

Data Reduction

The term “thick description” has often been associated with case studies
(Geertz, 1973; Wolcott, 1990). Though the original data may take on this image,
as the data is sifted through decisions of significance, it takes on a “thinner” form,
one that conveys the “essence” of the data to the reader.

This data reduction should be considered part of the data analysis rather
than separate from it. It is a part of the analysis that “sharpens, sorts, focuses,
discards, and organizes data in such a way that 'final' conclusions can be drawn and verified." (Miles & Huberman, 1994, p. 11)

What is significant in the data and what inferences should be made from the selected data are subjective decisions by the researcher. Some would argue that the validity and reliability of the research is threatened by this subjectivity. Eisenhart and Howe (1990) state:

Especially relevant to qualitative research, is the researcher's own subjectivity (Peshkin, 1988). Peshkin has argued that subjectivity is the basis for the researcher's distinctive contribution, which comes from joining personal interpretations with the data that have been collected and analyzed. As with assumptions derived from the literature, subjectivities must be made explicit if they are to clarify, rather than obscure, research design and findings. (p. 7)

As was mentioned in the last chapter, as a defense against criticisms, researchers should report data in sufficient detail to give the reader a clear view of the setting, confirm their hypotheses by triangulating the data, and be self-critical to disclose their own biases.

From the start of data collection, and continuing on through the data reduction and data display, decisions are being made as to what the data means. The researcher looks for alternative frameworks (Toulmin, 1972) on the part of the student, patterns, connections, contradictions, and causality. A discussion of the meanings attributed to the data by the researcher for each case in this study is given in Chapters VII-IX.

Thompson (1993) commented on the types of factors that should be taken into consideration when doing a study and the risk involved if these factors are not included:

Our interpretation of students' performance must be conditioned by our knowledge that they are taught by teachers with their own images of what constitutes mathematics, and that both the learning and

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teaching of mathematics are conditioned by the cultures (school, ethnic, and national) in which they occur. . . . While ignoring this issue might simplify matters enormously for us as teachers and mathematics education researchers, we do so at the peril of losing generalizability and validity of our interpretations and conclusions. (p. 1-2)

**Analysis Procedure.** The data includes field notes (describing the classroom setting, interview setting, and classroom interactions), transcriptions and videotapes from both the interviews and the teaching episodes, videotapes of the teacher's instruction in the classroom, an audiotape of the teacher interview, student classwork, quizzes and tests, and student journal entries.

Data analysis begins at the start of data collection and continues on through data reduction and data display. During the nine months of data collection, remarks and reflections about the observations were made in the margins of the field notes or attached to the day's collected data. Glesne and Peshkin (1992) report that "data analysis done simultaneously with data collection enables you to focus and shape the study as it proceeds. . . . by getting your thoughts down as they occur, no matter how preliminary or in what form, you begin the analysis process." (p. 128-129) These first interpretations of the data are the beginnings of the categorizing, the synthesizing, and the searching for patterns that are part of interpreting the data.

Miles & Huberman depict these three types of analysis activity (Figure 7) as forming an interactive, cyclical process (p. 12). They write:

From the start of data collection, the qualitative analyst is beginning to decide what things mean—is noting regularities, patterns, explanations . . . The competent researcher holds these conclusions lightly, maintaining openness and skepticism, but the conclusions are still there, inchoate and vague at first, then increasingly explicit and grounded, to use the classic term of Glaser and Strauss (1967). (Miles & Huberman, 1994, p. 11)
As the data was being collected, audiotapes were made from the videotapes of the student interviews. These audiotapes were transcribed by a research assistant and then I checked the transcriptions for accuracy. By viewing the videotapes of each student again, I was able to add descriptions of facial expressions, and bodily movements to the audiotape transcriptions. Some portions of the tape were inaudible due to the student mumbling or speaking softly, or outside noise. For these portions of tape, I went back to the videotape and tried to read lips and/or body language to determine what was being said.

As the data accumulated in its various forms, it was organized by student and by date. The data was organized in this form to obtain a sequential view of the growth and flow in knowledge development for each student. A form was designed (Form 1, Appendix G) to summarize and organize information on the student's background, interests, and attitudes toward mathematics. The form also summarized the ways in which the student identified functions from non-functions and the features of curves that he/she used to describe the graphs of linear, quadratic, and cubic functions. I used these forms for a quick reference on
student traits in later data analysis. The forms were also helpful in determining any generalities a student made across classes of polynomial functions and helpful in making student comparisons. As another organizational tool, I designed a second form to keep track of student definitions for the terms "zero", "root", and "factor" given in the first interviews (Form 2, Appendix G).

When the student interview transcriptions were completed, I went on to view the videotapes taken of the classroom instruction. The sequence of topics over the nine month duration of the study were written down in chronological order (Appendix C). I carefully observed the teacher's use of mathematical language, writing down initial reflections about the various ways in which the language may have been interpreted and the use of words that may have conveyed alternate meanings to the students. These notes were later used to analyze the student's use of mathematical language. I also uncovered from these videotapes reactions, comments, and questions by the students in the class during instruction or group work. Frequently, however, student and teacher remarks became inaudible, especially when the teacher moved to the side of the room away from the microphone. Notes also indicate those topics for which the students used a graphing calculator as a tool in problem-solving or in investigation.

In another attempt to summarize and analyze the data, I designed a set of three forms to record student comments about each of the three graphs shown with the PC-Emulation Software during the second interview (Appendix G). Each form in this set used letter descriptors (A, B, C...) to label the intervals or points on the graph that the students were asked about (Figure 8).
Student comments corresponding to an interval or point were written on the form. These notes included comments describing the graph as well as comments that related to the particular interval's relationship or point's relationship to the algebraic representation for that graph.

The process of transcribing and examining the transcripts for accuracy and detail was repeated for the tapes of the teaching episodes and the tapes of the assessment of the teaching episodes. During data collection, I reread the transcripts of the interviews and teaching episodes for the purpose of coding dialogue that would contribute to the formation of conclusions relative to the research questions. The categories pulled from the research questions are listed in the table below.
<table>
<thead>
<tr>
<th>CODE #</th>
<th>TOPIC</th>
<th>RESEARCH QUESTION CORRESPONDENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Connections</td>
<td>1: Connections between study of the graphs of linear and quadratic functions and their study of the graphs of polynomial functions of degree greater than two</td>
</tr>
<tr>
<td>2</td>
<td>Building</td>
<td>2: Evidence suggesting that conceptual development builds on student understanding of previous knowledge</td>
</tr>
<tr>
<td>3</td>
<td>Transition</td>
<td>3: Factors that contribute to/inhibit student from making the transition from graphical representations of linear and quadratic polynomial functions to those of higher degree</td>
</tr>
<tr>
<td>4</td>
<td>Teaching Episodes</td>
<td>4: Observations that suggest activities to be used in teaching episodes</td>
</tr>
<tr>
<td>5</td>
<td>Cross-case</td>
<td>5: Similarities or differences between the student's conceptual development</td>
</tr>
<tr>
<td>6</td>
<td>Technology</td>
<td>6: Observed influence of technology on student knowledge</td>
</tr>
<tr>
<td>7</td>
<td>ASC Factors</td>
<td>7: Affective, situational or contextual factors that affect student conceptual understanding</td>
</tr>
</tbody>
</table>

Table 1: Coding of categories corresponding to research questions

The first three codes correspond to the research questions that pertain to the development of student knowledge. Dialogue marked “Code 4”, used prior to the time of the teaching episodes, corresponds to statements that hinted at a possible direction for the content of the teaching episodes. Code 5 was used when any strong similarities or differences were noted in student interpretation of the information. Codes 6 and 7 correspond to influencing factors on that knowledge development.

With the intent of writing up case studies, I sought evidence to support or refute generalizations and conclusions that were being formed relative to the research questions. As the transcripts were studied with the research questions in mind, I placed a code number in the margin of the transcript with a memo of how this dialogue contributed to making an assertion in the particular category.
This broke the data into "major code clumps". As Glesne and Peshkin (1992) have written:

Coding is a progressive process of sorting and defining and defining and sorting those scraps of collected data (i.e., observation notes, interview transcripts, memos, documents, and notes from relevant literature) that are applicable to our research purpose. By putting like-minded pieces together into data clumps, we create an organizational framework. It is progressive in that we first develop, out of the data, major code clumps by which to sort the data. Then we code the contents of each major code clump, thereby breaking down the major code into numerous subcodes. Eventually, we can place the various data clumps in a meaningful sequence that contributes to the chapters or sections of our manuscript. (p. 133)

Once the transcripts were coded as above, dialogue of the same code number was put together and then organized into subtopics that also correspond to the particular research question (Table 2). Code 4 is omitted from this table as its purpose was for the development of the teaching episodes and not for an analysis of student understanding or conceptual development.

<table>
<thead>
<tr>
<th>CODE</th>
<th>TOPIC</th>
<th>SUBTOPICS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Connections</td>
<td>i. Between linear functions and cubic functions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ii. Between quadratic functions and cubic functions</td>
</tr>
<tr>
<td>2</td>
<td>Building</td>
<td>i. Builds on general knowledge of functions and graphs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ii. Builds on knowledge of linear or quadratic functions</td>
</tr>
<tr>
<td>3</td>
<td>Transition</td>
<td>i. Factors that contribute to transition</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ii. Factors that inhibit transition</td>
</tr>
<tr>
<td>5</td>
<td>Cross-case</td>
<td>i. Similarities</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ii. Differences</td>
</tr>
<tr>
<td>6</td>
<td>Technology</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>ASC Factors</td>
<td>i. Affective</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ii. Situational</td>
</tr>
<tr>
<td></td>
<td></td>
<td>iii. Contextual</td>
</tr>
</tbody>
</table>

Table 2: Subtopics corresponding to research questions

After the transcripts were coded as above, the other data types were also coded with the same descriptors. The coded transcripts, journal entries,
classroom dialogue, the forms created in the preliminary analysis, and the completed worksheets, quizzes and tests, were all used as evidence and triangulated to support or disconfirm the assertions that were made.

Triangulation of data is used to support a finding by showing that independent observations agree with or, at least, do not contradict the finding. From triangulation "we may get corroboration, getting something like a confidence interval (Greene et al., 1989)." (Miles & Huberman, 1994, p. 267)

For the primary level of analysis, I analyzed the data corresponding to each student and then wrote a case study on each student. In writing these case studies, I am trying to provide the reader with a picture of the student's conceptual understanding with respect to the graphs of polynomial functions, with particular emphasis on the connections that the students made between the classes of polynomial functions. The account of the teaching episodes within the case studies describes my attempt to enhance these connections. The outcome of these teaching episodes was inferred from the final assessment given to the students.

In the second stage of analysis, the assertions made within the individual case studies were brought together to form responses to the research questions. These were organized by research question.

**Summary**

This chapter has included a description of the qualitative methods used in this study over the course of the data collection, data reduction, data analysis, and data display. While qualitative data takes on the form of words rather than
numbers, these words are interpretations and descriptions of the data that was viewed through the lens of the researcher.

The next chapter gives descriptions and interpretations of the data related to the setting of the study. It gives the reader a view of the classroom in which these students were learning about polynomial functions and a view of the teacher through a teacher interview and observations. A chronological account of the “units” taught during the students study of polynomial functions will also be given. This information will provide insight into later descriptions and interpretations of the students’ understanding.
CHAPTER VI

CLASSROOM ENVIRONMENT

Background

The selected Algebra II class met every day from 9:10 to 9:55 in the morning. The teacher, Mrs. Tucker, had a large, brightly lit classroom that contained six rows of student desks with five desks in each row. A bookcase containing old math team trophies stood next to her desk at one side of the classroom. At the other side of the room, school news, pictures, and posters were scattered on the bulletin boards. A large blackboard spread the entire width of the classroom, and Mrs. Tucker had placed a yellow sign in the middle of this that read "NO WHINING".

This particular class had 25 high school sophomores and juniors (11 male, and 14 female). The students were assigned a seat of their own choosing at the beginning of the year. There were a couple students in the class who belonged to the school math team, which Mrs. Tucker coached. With one exception, the students in the class were generally very well behaved, in that they were attentive to class instruction, willing to be diligent, and respectful of the teacher's authority.

The class used a 1989 edition of a text entitled Algebra II and Trigonometry (Dolciani, Graham, Swanson, & Sharro). The teacher used this text predominantly as a resource for homework exercises, departing frequently from the sequence of instruction presented in that text. She supplemented this
material with worksheets she had developed on her own and with materials from *The Language of Functions and Graphs* (Shell Center for Mathematics Education, 1985).

Mrs. Tucker communicated regularly with another teacher who taught another section of this algebra course. Both teachers had attended "The Functions Institute" at a state university two years beforehand, a project supported by a grant from the State Department of Education under the provisions of the Dwight D. Eisenhower Mathematics and Science Education Act. This week-long summer institute was designed to help teachers gain a deeper understanding of the function concept by exploring its centrality within the mathematics curriculum and by investigating how the graphing calculator could be used as a learning tool in their classrooms. Both teachers had been gradually incorporating some of these ideas and using the graphing calculator intermittently in their classes since attending the institute. Mrs. Tucker admitted that the other teacher was "a better trier [laugh]" and what the other teacher tried "gunho", Mrs. Tucker would "try a little . . . you know . . . work my way up to it". The school had purchased a classroom set of TI-81 calculators, and a TI-81 overhead calculator, that were available for their use. At the beginning of the year, however, Mrs. Tucker suggested to the algebra students that they consider buying a graphing calculator of their own. About a quarter of the students in the class had invested in one.
The Teacher Interview

Background

This was Mrs. Tucker's 20th year teaching mathematics at the high school level. A mother of two school-aged children, she had graduated from the state university in 1974 with a bachelor of science degree in mathematics. In an audiotaped interview, I asked her questions that would probe into her background, her attitudes about mathematics and teaching, her sense of autonomy in teaching, and in particular, her thoughts about teaching the function concept. I deemed these issues important, the teacher being a key part of the students' Algebra II learning environment. In particular, I was aware of the possible effect that the teacher's attitudes and beliefs about mathematics and the topics in the Algebra II curriculum could have on the student. I've included an overview of this interview here and some of the principal statements that were made by Mrs. Tucker in the interview as they may give the reader more of a portrait of the type of teacher the students had for Algebra II. In analyzing student data, these statements will be an added source of information in interpreting student understanding and attitudes toward graphing polynomial functions and the function concept in general.

Mrs. Tucker first became interested in mathematics in high school when she started receiving high grades in algebra class. At the end of her sophomore year in college, she switched from a degree in computers to a degree in mathematics with a minor in secondary education.

I asked her how "the reform movement in mathematics education" was affecting how she taught mathematics. She responded by saying that it had affected her the most in teaching Algebra II as she had been given more chances
to get ideas, like at the Functions Institute. The institute had encouraged her to
do more with mathematical modeling in her classes. She added that she agreed
with "The Standards" and that "the office" got the teachers to talk together about
them. She mentioned that she did some cooperative learning, but "not straight
cooperative", often getting the students to work in groups on worksheets.

She admitted that there were some things that she didn't like about the
book she used for her "Trig/Analyt" class, but because she didn't have time to
look or "put together her own", she hadn't done anything about it.

**Algebra**

Mrs. Tucker's teaching schedule had included an Algebra II class for about
8 years. I asked her if there were algebra topics that she liked teaching more than
others. She responded:

**MRS. TUCKER:** I don't think that specifically off the top of my head
anything . . . any that I like special more than the others . . .
Sometimes I tease them everytime we start a new chapter, I say "This is
my favorite chapter" . . . so, I don't think I have anything that I'd call
my favorite . . . I always tell the kids that I like graphing . . . and I've
always tried to do a lot of graphing.

Inverses was the only topic that she could think of that she didn't like to teach. "I
think it's mainly because I don't feel like I do a good job in it . . . so every year I
try a new lead in before I do inverses".

I asked Mrs. Tucker if she thought that the Algebra II content was
important for her students to know.

**MRS. TUCKER:** Anything we do in the advanced courses are so they can
take more advanced courses . . . you know, they're not going to walk out
of here and say well, I've had Algebra II and give me a $40,000 paying
job . . . I'm doing it so that . . . [it] will hopefully take them into a field
that they are interested in. They won't use it all . . . they'll use a little.
It's also good for their brain, practice, discipline, good thinking skills.
Polynomial Functions

Extending this last question a bit further, I asked if she thought learning about polynomial functions was important. She said that there are many applications of linear and quadratic functions around, "at least if you’re observant enough to see them", but "off the top of her head" she couldn’t think of any applications of polynomial functions of degree greater than two. She added, however, that she did know somebody who was an engineer who had told her that polynomial functions were used a lot in engineering.

I asked her if she thought that the students saw any connections between quadratics and polynomials of higher degree.

MRS. TUCKER: I don’t know. I try to tie them in—I say a lot of the things that work with quadratic equations work with polynomial functions—but I think the fact that they are using synthetic [division] to solve a polynomial, makes it seem different . . . but I don’t know how to avoid that . . . they can certainly use synthetic with quadratics . . . but by doing the quadratic formula . . . so I think they just . . . and that’s part of the reason why I didn’t split the test this year too. I wanted to see what keeping them together—because if I split it, then I’m making the split, you know, quadratics here and then . . . chung . . . polynomials . . . [laugh], I don’t know.

Mrs. Tucker seemed well aware of the separate treatment of quadratics and polynomial functions of degree greater than two in the curriculum she was using. She was also aware that, because the algorithms for solving equations of these types were different, the classes of polynomial functions appeared disunited to the students. Her comments indicate that she was trying to bring these topics closer together but didn’t know quite how to do it.

Functions

I asked Mrs. Tucker to tell me in which areas of mathematics she considered the function concept central. She responded:
MRS. TUCKER: I see it in everything. It's less in the Algebra I as it's taught now. I think because it's a hard concept... but it shows up a lot in Algebra II, and it shows up a lot in Trig, and Calculus. I would say like 80% of almost everything involves functions, right? Start right with lines and... what else do we do... lines, and then the basic functions, and then adding them... you know everything... it's in everything, it seems.

When Mrs. Tucker used the words "it shows up...", I was unsure whether "it" referred to function notation, or a broader relationship between variables. Looking back, I wish I asked her to clarify what she meant. Perhaps she was unsure too of what "it" meant, as she seemed to waiver in her thoughts through these statements with regard to how much "it showed up".

I asked her to talk a little bit about the "Function Institute" experience.

MRS. TUCKER: What the Functions Institute did was it took my idea of functions which was primarily graphing and then I think equations... it made me realize that, that it was more than that, it was the relationship rather than the graphs and equations... you know that there's a relationship between a couple of areas that may be defined by an equation, or may end up being shown by an equation, or shown by a picture, or shown by a graph... that it's the relationship itself that's a function.

I then inquired into her beliefs about whether the students found the function concept difficult to grasp.

MRS. TUCKER: Well, it's kind of abstract... I think... well, they find it abstract... it's one of those things that I don't think is all that abstract, but they seem to find abstract? Like when I ask them about functions at the beginning of the year—well, "it's when each x gets one y"—you know they need that little definition to keep track. It reminds me of percents... percents is like, you divide or you multiply, but the concept of percent being a part of a big whole and ratio seems to escape them.

She mentioned that over the years she had gradually increased the emphasis on the function concept in her teaching. "I try to really keep it as a constant thing now that's always there... that's always around."
**Graphing**

Mrs. Tucker saw graphing “as an aid... like a visual aid to help understand the relationships from an equation”.

MRS. TUCKER: So anyplace we're dealing with any kinds of functions, I would say graphing has to be there... and... and... back in the beginning [in October of the school year] when I started all the plain function stuff... I tried to give the graphs first... umm... to show that they have importance... you know, that they just don’t model the equation... that they also are legitimate... you know that they can actually show a relationship without an equation.

Reflecting on the current emphasis she gave to the function concept in her Algebra II class, she felt that it was adequate, considering that she had yet to squeeze many topics, such as exponentials, logarithms, and inverses, into the school year. It seemed that she saw these topics unrelated to the function concept. However, when she taught these topics later in the year, I observed that she often spoke of exponential functions, logarithmic functions, and inverse functions, as well as using function notation frequently on worksheets.

**Technology**

I then asked Mrs. Tucker more about her use of the graphing calculator in the classroom. She pointed out that this was the first year that she had used it “extensively”.

MRS. TUCKER: I had decided even at the beginning of the year, before you even came, that I was going to do a lot more of that. Last year, I think I brought it in once or twice and graphed some parabolas... I also brought in the computer that we have and graphed some there but I didn’t like that because we only had one and there was 25 people sitting around the computer, so I like them to be able to have their different calculators.

She went on to reflect on the use of calculators in mathematics teaching and learning.
MRS. TUCKER: There are times when any calculator bothers me because I feel like they use it too much . . . it irritates me when I see them doing $3^2$ and $8+5$ . . . the fact that so many of them use, or many of them use, the calculator as a crutch irritates me . . . but I think the good of being able to do types of things without getting bogged down outweighs it . . . you can tell them I said that [laugh] . . . I'm trying to convince myself . . . but, I think this year, because we have used them, they know the graphs are going to be better than they were. I remember doing, you know, synthetic substitution 20 times . . . 10 times . . . to get points to get a nice looking graph . . . it was such a pain in the neck and I always hated it, which took away from the graph. I know they can do synthetic [laugh] . . . the graphing calculator really, I think helps a lot . . . it helps me at the same time.

Mrs. Tucker explained why she had hesitated in using the graphing calculator in previous years.

MRS. TUCKER: I was afraid that I would make a bigger mess out of it . . . that it would take me long or I'd screw it up . . . I hate to . . . as a matter of fact . . . well, I don't like to screw up . . . I like giving the illusion that I can do all . . . figure out all . . . at least most of it. So I think I was afraid that I wouldn't . . . that I wouldn't be able to use it well . . . that I wouldn't . . . just the mechanics of it . . . that I wouldn't use it . . . you know, that I wouldn't have a good way to show it . . . and a good way to illustrate . . . bring it into their regular work. And . . . ah . . . I also was afraid that it would take a long time . . . which I don't have a lot of . . . so I guess that's the real reason . . . then . . . you know, once I did . . . once [the other teacher] tried a few things . . . I just knew it was time to do it.

Mrs. Tucker evidently was feeling pressure from her colleagues to use more technology in the classroom, as well as dealing with a fear of failure. Though she was still hesitant about using the graphing calculator, she had become more determined to work it into her classroom repertoire.

**Effect Of Researcher's Presence On The Teacher**

Mrs. Tucker claimed that since I had started joining the class early in the year, my presence had not made a "big difference" in what or how she taught the course.
MRS. TUCKER: I said I was already going to do all this stuff this year . . . so it kind of worked out . . . it's good! If you came in last year . . . um . . . I would have been in a panic. I had already decided I was going to do it . . . in a way . . . well, maybe in a way . . . you just ah . . . was like I couldn't back out. You know, panic, or say, "Well, maybe I'll do a little and try again next year".

Summary

The last five or so years of mathematics education reform had caused Mrs. Tucker to rethink the ways in which she taught mathematics, as in the areas of technology use, cooperative learning, and making connections. Though not "a trier", as she described it, after watching and listening to colleagues and attending the Function Institute, she had decided to inch off of her secure foundation, i.e. the ways in which she had always taught, and try a few new things. In a sense, the students in this Algebra II class were subjects of experimentation as she tried out these new practices in teaching.

Classroom instruction

This section will describe the instruction that the students in this algebra class received relative to functions, and in particular polynomial functions. The purpose of this study is not to analyze the classroom instruction, or any part of classroom discourse. The information below, however, is useful in analyzing the data collected later in the student interviews and the teaching episodes and sheds light on that discourse. References will be made to the content and modes of classroom instruction in the final discussion.

In addition, it should be noted that this description of the classroom instruction is of necessity a summary of that instruction, highlighting only the
segments which I, the researcher, sensed had the most effect on the students' understanding of the graphs of polynomial functions.

**Modes Of Instruction**

Mrs. Tucker did not have a set classroom routine, but varied her classroom repertoire from day to day. Students were allowed to use calculators freely in the classroom, and in most classes they spent at least part of the period working in groups of two to four on problems. These problems were then completed for homework. When Mrs. Tucker introduced new material to the class, her usual style was to ask questions that would lead into the new material, and then continue with this question/answer technique as she developed the new concept with the students. The students took notes on the material, sometimes asking questions to clarify the information. Mrs. Tucker guided the students by using several illustrative examples of typical problem situations before allowing the class to work on their own or in groups. When given the choice, most students in the class would choose to work with a partner or with a couple other students rather than work alone. The last 15 to 30 minutes of a class period was usually devoted to the students getting started or continuing their progress on worksheets that Mrs. Tucker had passed out. The problems on the worksheets often "stretched" the students' understanding, since a variety of problem types dealt with aspects of the concept that were new to the student. As an example, Mrs. Tucker used some of the Shell Centre materials (1985) to introduce graph interpretation and functions. Tasks included matching real-life situations to graphs, matching tables of values to graphs, or writing stories that could describe a graph. This unit culminated with the students designing a water tank of maximum volume out of a
given size sheet of material, making tables of values, and plotting graphs showing
the relationships between variables. Though Mrs. Tucker laid a foundation for
the concept, the students built upon this foundation by doing a variety of
problems.

A general itinerary of topics that was covered during the year is in
Appendix C. In addition to many interruptions to the school schedule due to
vacations or special school functions, the schedule was interrupted about six times
by "snow-days" throughout the winter months.

The first day that I attended this algebra class (October 1, 1993), Mrs.
Tucker used an overhead projector to project the answers to a worksheet on a
screen so that students could correct their own work. She walked around the room
helping the students with questions as they raised their hands. For the
remainder of the period, the students were allowed to move their desks together
in groups of two to four to work on another set of problems. Again, a few students
continued to work alone, but the majority of the class members chose to work with
nearby classmate(s).

**Instruction On Linear Functions**

After the unit on graph interpretation and identification and translation of
functions, the class moved on to study linear functions. Using the vertical line test
to differentiate functions from non-functions, the teacher asked "Are all lines
functions?". The students determined that all lines were functions except for
vertical lines because of the vertical line test. Mrs. Tucker pointed out that for
any equation that was a function, the variable y could be replaced with the
notation \( f(x) \), hence ordered pairs could be denoted \((x, f(x))\). The students practiced evaluating \( f(x) \) for different values of \( x \), given an equation.

Mrs. Tucker told the students that the slope of a line could be equated with a "rate of change", referring back to the graphical interpretation they had done before using the Shell Center materials (1985) that linked graphs to real-life situations. The students became familiar with linear equations in slope-intercept form and in standard form. At the end of class, Mrs. Tucker reflected that she should have used the terminology "linear function" rather than "linear equation" more than she had, more evidence that she was still trying to break away from some ingrained ways that she had taught, and spoken about, these topics in the past.

The next day, Mrs. Tucker used the overhead graphing calculator and the classroom set of graphing calculators to graph lines with the students. She was still a bit nervous about "messing up" but was determined to do it. The students "ooohed and aahed" when they entered the expression for \( y \), and pressed the "GRAPH" key. Mrs. Tucker had them change just the slope or the \( y \)-intercept of the expression and they discussed what effect those changes had on the graph of the line.

Mrs. Tucker continued by exploring graphs of the form \( y = ax^2 + b \). They exclaimed "It's a parabola!", recognizing this type of graph from the work they had done with graphical interpretation. For the next week, the students used the graphing calculators, while working in groups, doing worksheets that continued to investigate the effects of the various parameters on the graphs of \( y = a(x-c)^2 + d, y = a(x-c)^3 + d, y = \sqrt{x-c} + d, \) and \( y = ax - cl + d \). They were encouraged to look for patterns, rather than graphing every equation on the worksheets.
The Mid-Year Exam

After a test on functions that included their work with linear functions and translation of graphs, Mrs. Tucker did a unit on radicals that took the class through December to Christmas break. Following Christmas break, they began reviewing for their two hour mid-year exam. I include here what types of problems were on the mid-year exam, as this gives information about what Mrs. Tucker perceived as being important in the material covered during the first half of the year. There were 52 questions on the test, divided over the following areas (each question was placed in the area that I perceived as getting the major emphasis):

<table>
<thead>
<tr>
<th></th>
<th>Questions</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factoring of expressions</td>
<td>10</td>
<td>19.2%</td>
</tr>
<tr>
<td>Simplifying exponential expressions</td>
<td>3</td>
<td>5.8%</td>
</tr>
<tr>
<td>Complex numbers</td>
<td>2</td>
<td>3.8%</td>
</tr>
<tr>
<td>Simplifying radical expressions</td>
<td>11</td>
<td>21.2%</td>
</tr>
<tr>
<td>Solving equations or inequalities in one variable</td>
<td>15</td>
<td>28.8%</td>
</tr>
<tr>
<td>Functions (terminology, evaluation, lines, graphing)</td>
<td>11</td>
<td>21.2%</td>
</tr>
</tbody>
</table>

Table 3: Distribution of questions on the midyear exam

This distribution of questions seemed to reasonably correspond to the time and emphasis given to the different topics by Mrs. Tucker while I had observed the class.

Quadratic Equations And Functions

Following the mid-year exam, the class was taught the process of "completing the square" as a method to solve quadratic equations. Using a question/answer format, Mrs. Tucker then proceeded to derive the quadratic formula by completing the square of a quadratic equation in standard form. She pointed out other types of equations considered "in quadratic form" such as $x^4$ -
13x^2 + 36 = 0. She noted this equation had four solutions and wrote on the board:

**Theorem:** Any nth degree equation with complex coefficients has exactly n complex roots.

She clarified that "roots" meant "answers". They used the discriminant to discuss the nature of the roots of a quadratic equation. One student in the class complained that she didn't see why it was necessary to find the nature of the roots and not just the answers. Mrs. Tucker appeared a little stymied by this question. She answered by saying that sometimes you need to know if the roots are the kind of roots you want. This discussion of quadratic equations was devoid of any graphing, and concentrated solely on the symbolic manipulations necessary to solve the equations.

As Mrs. Tucker moved into quadratic functions, she clarified the difference between a quadratic function and a quadratic equation, emphasizing that the former has two variables and hence the students would be working with ordered pairs. She began with quadratic functions in the form $y = a(x-h)^2 + k$ which she described as "factored form" and "easy to graph form". She started with the equation $y=x^2$ which she referred to as "the basic parabola" and reviewed the work they had done translating this parabola. Using a table of values generated by $y=x^2$, the students studied the relationship between the coordinates of some of the points that were used to plot the graph of the basic parabola. They noted that a change of one unit from the origin (the vertex) in the x-direction produced a change of one unit in the y-direction, a change of two units from the origin in the x-direction produced a change of four units in the y-direction, etc..., i.e. the shape of the basic parabola was obtained by moving from the vertex "over one
and up one, over two and up four . . . ". This was a way for students to plot a more accurate graph by hand given any parabola that retained the shape of the basic parabola, for example, \( y = (x-2)^2 \) or \( y = x^2 - 1 \). This was extended to parabolas of the form \( y = ax^2 \). Members of the class noted that the equation \( y = 2x^2 \) produced a parabola that was \( \frac{1}{2} \) the width of the basic parabola, and then Mrs. Tucker added that instead of going over one and up one as in the "basic parabola", the graph went over one and up two, etc. . . Mrs. Tucker generalized this, saying, "a changes the jump for y". The students were encouraged to use the "relationship" between x and y to plot points on a parabola rather than always making tables of values. Students would check the graphs they determined by hand, using the graphing calculator.

The next day, graphing calculators were used in class to graph a quadratic function that was written on the board in the form \( y=f(x) \). The students were asked to "ZOOM" in on the vertex, and then the x-intercepts of the resultant parabola. Mrs. Tucker pointed out that "there are two x-intercepts". She asked the class "What does it mean to be an x-intercept?". When a few students responded, "Let \( y=0 \)", she answered, "Right . . . let \( y=0 \) . . . Look! [She replaced the variable y with 0 in the quadratic function written on the board.] . . . Lo and behold we have a quadratic equation." In showing the connection between these quadratic functions and quadratic equations, she may have been hoping that the students would link the roots of the quadratic equation with the x-intercepts, though this was not stated directly. The students also used the graphing calculator to "ZOOM" in on the y-intercept. Watching the values for x and y change in the window as they traced along the curve, they tried to get x close to 0.
The students continued to use the graphing calculators over the next couple of weeks as they worked in groups on quite a few modelling problems that dealt with quadratic functions. Most of the emphasis was placed on finding the quadratic function and then questions were answered by interpreting the graphs of these functions. Just a few questions dealt with interpreting the x-intercepts. Equal or greater emphasis was placed on interpreting the vertex, the y-intercept, and other isolated points.

Mrs. Tucker spent just one day on solving quadratic inequalities. Though emphasis was on the algebraic solution, the graphical interpretation of a quadratic inequality was shown.

**Instruction On Polynomial Functions Of Degree Greater Than Two**

The class then moved on to a unit on higher degree polynomial functions. Mrs. Tucker, in lecture style, introduced the material as an extension of quadratic functions saying that for quadratics they found the vertex, but for polynomials of higher degree they would be finding the "turning points". She illustrated what she meant by turning points by putting a graph of a polynomial function with three zeros on the board. She mentioned that like quadratic functions, for each function they would be finding x-intercepts and the y-intercept. She also explained, illustrating with graphs, that polynomial functions of the form \( y = x^n \) had one root while adding more terms to an expression of this type was what caused the graph to "go up and down". Some students asked questions [inaudible on the tape] and Mrs. Tucker took the time to answer each one.

After this introduction, Mrs. Tucker went on to explain, by example, the process of "synthetic substitution" as an easier means of evaluating a polynomial
function, $P(x)$, for various values of $x$. The students were attentive and enthused about using this method when they saw that it really was a shortcut to a lot of calculations. Mrs. Tucker pointed out that evaluating $P(0)$ returned the value of the $y$-intercept, "where the curve crosses the $y$-axis". She commented that to use the process of synthetic substitution for this calculation was unnecessary, since the value returned by this process was always equal to the constant of the polynomial expression.

Using the polynomial function, $P(x)=2x^4 + 3x^3 + 6x^2 + 12x - 8$, as an example, the students discovered that a value of $\frac{1}{2}$ returned $0$. Mrs. Tucker explained what this meant:

MRS. TUCKER: "This is one of the x-intercepts. We don't know how many it has yet. But we do know $\frac{1}{2}$ is an x-intercept. This $\frac{1}{2}$ is also called a 'zero of the function'. Do you know why?"

She proceeded to draw a W-shaped graph on the board and circled the x-intercepts, asking questions of the students to reveal that those x-intercepts had y-coordinates that equaled $0$ . . . "it makes the value of y zero". Some students asked questions during this lecture-based instruction. Their homework from the text involved practicing synthetic substitution. One student remarked at the end of class, "I think synthetic is fun!".

The following day, Mrs. Tucker worked through a long division problem, dividing a linear binomial into a cubic polynomial expression, and then did the same division using the process of synthetic substitution, and compared the two processes. This comparison was to bring out the notion that the linear divisor was a "factor" of the polynomial expression. Mrs. Tucker linked this idea of some linear expression $(x+4)$ being a "factor" of $P(x)=0$ to the graph of $P(x)=y$ that crossed over the x-axis at -4. Though Mrs. Tucker was trying to show the
connections between these processes, the reader should be reminded that, throughout this instruction, students were creating their own understanding of the material, which may have been vastly different from what Mrs. Tucker was trying to get across.

Throughout this instruction, Mrs. Tucker had primarily "told" the students how to solve the equations by synthetic substitution. During the next phase of the instruction, Mrs. Tucker used the classroom set of graphing calculators to allow the students to graph the polynomial functions in order to visually approximate the roots of the polynomial function. Using synthetic substitution, the students checked if the values of the observed x-intercepts were indeed roots ("answers"). Sensing some difficulty, Mrs. Tucker stopped the students.

MRS. TUCKER: So, we know the 1 works, so that means if I started factoring that thing, the first factor would be what? [A student responds.] Good, x-1... O.K., the other factor would be... [writes out and shows students how to determine the "depressed polynomial"].

The students proceeded to completely factor the polynomial expressions using the calculators in conjunction with synthetic substitution. Mrs. Tucker described this process as finding "all the x's that make it work" referring to values of x that returned a value of 0 in the process of synthetic substitution. By example, she also showed that "once they got down to a 2nd degree, they could use the quadratic formula" to finish solving the equation. A bit confused, students asked how the values for x found by using the quadratic formula could also be roots of the polynomial, an indication that the processes used for solving quadratic functions and synthetic division, used for polynomials of higher degree, lacked coherence in the students' minds. Mrs. Tucker went back and forth from the equation to the graph explaining why this was the case. There was also some
confusion as to what form the "answers" were to be in, i.e. factors or roots. She clarified that they needed to really pay attention to what was asked for in the given problem situation.

Mrs. Tucker reiterated that the "root is the value of x that makes y=0 . . . the answer, if you solve the equation". She also explained, by example, that some polynomial expressions will not factor completely over the real numbers into a product of linear factors, and that only the real roots will appear on the graph as x-intercepts.

MRS. TUCKER: Every real root is going to show up on your graph . . . you might have to search if I give you really bad coefficients . . . but every real root is going to show up on your graph. Where do the roots show up? How do you know that something's a root? [A few students respond.] Good...it hits the x-axis. How many times does this thing hit the x-axis? [A few students respond.] Is it hitting someplace we can't see it? [A few students answer.] Should I trace and go looking way farther over here? [A few students respond.] No, if you know this is supposed to be "W'ish" and you can see where the "W'ish" part down there is, right? So, it's not going to come back up and hit someplace else . . . 4th degrees don't do that . . . it goes on forever here . . . infinitely, so as your x is increasing right and decreasing left [uses hands to illustrate] the y's get bigger. So, what about the other two roots . . . they don't show up on the x-axis, so where are they? [A few students respond.] Imaginary, good . . . so the others will be imaginary.

In these classroom discussions, Mrs. Tucker seemed to presume that because some of the students were responding to her questions in what she considered an acceptable way, that the students in general were understanding what she was trying to get across.

After this class, Mrs. Tucker talked with me about the ways she had taught this material in the past. She noted that in prior classes she had given the students the rational root theorem before graphing the function, but preferred this way because she felt, through student responses, that they "made connections
better'. If the rational root theorem was used first, she felt that the students had a list of roots, but didn't visualize them on the graph. Again she seemed to be generalizing what she sensed was true for the few students who participated in class.

To be fair to Mrs. Tucker, I should add that besides class discussions and traditional quizzes and tests, she also acquired a sense of student progress by frequently walking around the classroom observing student work, while they worked in groups.

The following day she introduced the rational root theorem as an aid in solving polynomial equations with integral coefficients. She commented that they would need this theorem for solving an equation if they "didn't have a graph". I italicized "equations" above to emphasize for the reader the use of this word as opposed to functions. The students used this theorem in conjunction with the graphing calculator and synthetic substitution to find the roots. Mrs. Tucker referred to this theorem as another "way to pin down or narrow-down the guesses" and noted that it could also be used as a check against those values for x that appeared to be roots on the graphing calculator. Once again, she tried to clarify the distinction between the instructions "factor it" and "solve it" in response to a student's question.

MRS. TUCKER: If I say factor it, put these up here and give me all four factors [pointing to an example on the board], if I'm saying solve it, then I want the values of x... I want these things.

Mrs. Tucker sang the praises of the graphing calculator, telling the students that even in the year beforehand, the school did not have the graphing calculators to use as a tool to ease the process of finding roots. At that time, the students had
to find roots of polynomial functions using only the rational root theorem and synthetic substitution.

During the following week, the students worked in groups doing worksheets with a variety of problems on the roots and factors of polynomial equations and functions. In all these problems, the students were able to use the graphing calculator as an aide in determining the algebraic solutions. This was followed by a quiz on the material. The first problem on the quiz required that the students solve a polynomial equation of degree four. The next three problems involved interpreting the outcomes of synthetic substitution, and the last problem required that the students find a missing coefficient to a polynomial expression given that \((x+1)\) was a factor of that expression. The first problem was the only problem for which the graphing aspect of the graphing calculator was useful, as the other problems focused on the algorithmic use of synthetic substitution. This quiz was in partial preparation for a test that was to be given the following week.

The next type of problem that Mrs. Tucker spent a good portion of time on was "Given the roots, find the equation." The emphasis was on performing symbolic manipulation. She began by saying:

MRS. TUCKER: If that sounds familiar to you, it should, because we've already done all this, but we did it for quadratics, right? So, if I say the roots of a quadratic equation are 2 and -5, how would you find the equation? [A student responds.] Yeah! You just put 'em back in the binomials [writes on board] . . . \((x-2)\) times \((x+5)\), all right, you're gonna change the sign when you put 'em back in, set it equal to 0 and then do all the multiplying out . . . Is that O.K. for everybody?

She then asked:

MRS. TUCKER: What's our other way of getting the equation given the roots? [A student responds.] The "root rule", yes . . . What did the root rule say?
She reviewed the root rule by doing the same example as above and then comparing the results from the two methods. She commented, "So, it's really your choice . . . I don't know which you'd say is best." She did a few examples, claiming the root rule was probably easier if the roots were complex, the binomial method otherwise. She explained to the students that understanding these examples with quadratic equations was necessary in order to take the method further to 3rd and 4th degree polynomials.

She then did a problem with three roots, two of which were complex, the other an integer. She "threw" the integer into a binomial, and then used the root rule to get a quadratic factor with the other two complex roots. They checked the resultant equation with the graphing calculator. The next example had four roots, two that involved radicals, two that were complex.

On the last day of instruction relative to polynomial functions, Mrs. Tucker gave the students a problem that asked for the polynomial function determined by three zeros, 1, -2, and 3, and one other point on the graph, P(2)=8.

MRS. TUCKER: I don't want just the 3rd degree equation, I want the whole function . . . I have to use a little different terminology because functions don't technically have roots, those are the values of x that solve the equation, functions have zeros . . . they're, you know, technically the same place on the graph, just slightly different terminology.

Referring to the point, P(2)=8, she said:

MRS. TUCKER: Now, the reason I have to give this extra step is because of what those zeros represent [walks to other part of board and sketches some axes] . . . you must sketch them on your graph, you put a 1 here, you put a -2 here, you put a 3 here, right? . . . as the zeros of the function. Can you tell just from those where the function is? No, you can do a lot of guessing, you really can't know where it goes [she sketches various ways the curve could go] . . . but, if I tell you just one more point some place else, then I'll find one specific function. Now you'll be able to get what makes the up and down part [points to the graphs] and that's the coefficient of a, that I really haven't stressed
much with the polynomials but we did lots with the quadratics ... you
know, if $a = 2$, it's a lot thinner, if $a = \frac{1}{2}$, it's a lot wider, and that value
of $a$ affects the polynomial functions the same way.

Although, Mrs. Tucker was describing a connection here between quadratics and
polynomial functions of degree greater than two via the stretch factors, "$a"$, in the
same statement she was leaving quadratics out of the class of polynomial
functions.

Plugging the point $(2,8)$ into the function $y = a(x-1)(x+2)(x-3)$, the students
determined the value of $a$ algebraically and thus the "whole function". Mrs.
Tucker added, "Then you graph it to check yourself". They used graphing
calculators to see if the function actually did go through the given zeros and point.

After a three day weekend, the class spent a day doing more "practice
problems" to prepare for the following day's test (March 22nd). An analysis of the
test, revealed the following dispersion of problem types (separated by class of
function, actual test was not ordered in this way):

<table>
<thead>
<tr>
<th>Quadratics</th>
<th>(Total of 6 problems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solving quadratic equations.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Solving quadratic inequality.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Given quadratic function state the vertex.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Finding missing coefficient of quadratic equation.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Determining equation of quadratic given two points and the axis of symmetry.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Application problem finding maximum.</td>
<td>(1 problem)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Polynomial Functions of Higher Degree</th>
<th>(Total of 6 problems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Synthetic division to find quotient and remainder.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Solving equations of degree greater than two.</td>
<td>(3 problems)</td>
</tr>
<tr>
<td>Finding function given 4 zeros and another point.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Finding equation given 3 roots.</td>
<td>(1 problem)</td>
</tr>
<tr>
<td>Finding missing coefficient of 4th degree polynomial function.</td>
<td>(1 problem)</td>
</tr>
</tbody>
</table>

Table 3: Distribution of questions on polynomial functions test
The emphasis on this test was again on symbolic manipulation. There weren't any graphs on the test and no questions requiring the students to generate a graph on the test. They were able to use the graphing calculator as an aid or as a checking device when working on the test.

**Summary**

The thrust of this chapter was to create a portrait of the classroom environment in which the three student participants in the study were taught. It summarizes for the reader observations from that classroom as seen through the lens of the researcher via sketches from the teacher interview and the classroom instruction. Though the teacher was trying to take steps away from the traditional classroom where the teacher lectured, the students listened, and all work was done by hand, this class still exhibited these characteristics a lot of the time. Though the teacher was trying to incorporate the graphing calculator into the classes, an emphasis on symbol manipulation came through most of the lectures. The chapter serves as a background through which the reader can judge the validity of future assertions about student understanding that will be made as this discussion of the research continues.
While observing the class, Mark, a junior, had always appeared to me to be pretty self-confident. Thin and of average height, he kept his hair parted in the middle, and, like many of the high school males, wore an earring in one ear. When students had the option of working in groups, he normally worked with about three or four other males at the back of the room. He seemed to be a leader in this group, often giving direction as they solved problems together.

Mark appeared a little apprehensive coming into the first interview, lacking the air of confidence that I usually perceived. Sitting down with his arms wrapped across his chest, he followed me carefully with his eyes as I adjusted the video camera.

**Attitudes Toward School and Mathematics**

In the first interview, I asked Mark about his background in school, particularly with mathematics, in an effort to glean from his comments a sense of his attitudes toward the same.

When I asked Mark what was his favorite part of the school day, he responded, "um...the end? [laugh]". I went on to ask whether he had a favorite subject:

MARK: I kinda like Drama... that's fun... doing pretty well in Chemistry, too... so that's not bad.
... and a least favorite subject:

MARK: Algebra's right up there . . . [laugh] . . . I don't like Algebra too much.

Pursuing this a bit further, I asked if he generally disliked mathematics:

MARK: Oh . . . I liked Geometry, I didn't like Algebra I.
JC: Was that the teacher or the subject?
MARK: I think it's kinda the subject . . . it's a lot harder to understand than Geometry.

In later conversations he clarified what he meant by this last statement. When asked to explain what Algebra is, he replied:


This was an interesting distinction for Mark to make and I was drawn to observe how this need to "see it" would be reflected in future investigations into polynomial functions. Mark's feeling of success in Geometry was reinforced by the grades he received in that course, "high B's . . . I think I might have had an A in there, but don't quote me on that." In Algebra I, Mark had received a grade of B for each of the first three quarters, and then received a C to close out the year. At the time of this interview, he had received a grade of B for each of the first two quarters of Algebra II and had received a grade of 78 on his midyear exam.

Mark said that he had lived in Europe for a period of time before freshman year in high school and when he came back to the States "started getting B's and C's" in mathematics courses "instead of B's and A's". When I reminded him that he must have done "O.K." since he was currently in a pretty high level Algebra class, he responded, "Yeah, I can get it . . . it just takes me a few minutes." I took this to mean that Mark did not find algebra very hard. In general, throughout the study, I did find that Mark was quick to grasp new ideas as compared to his peers.
From these few statements about his studies, my impression was that Mark enjoyed a subject if it was "fun", not "hard" for him, or if he was doing "pretty well" in it. This attitude toward his studies was reinforced when I asked Mark about his goals after school. He replied, "I'd like to do something with commercial art or the movies or something kind of fun".

Mark laughed when I asked him about how much time he spent on homework per night. "I don't usually do a lot of homework... maybe... maybe a half hour... depending on how many reports I have coming up."

Mark did not own his own graphing calculator but reported, "I like them." He mentioned that he would have to get one soon as he was planning to take "Trig and Analyt" next year. He thought that purchasing one would be necessary after talking to students who were taking the course this year. He mentioned that he had three computers at home, "two PC's and a Compaq" and that he used them for playing games and doing reports.

**Definition of Mathematics**

While Mark was reflecting on his background in mathematics, I asked him, "What would you say mathematics is?". With his arms still held tightly across his chest, he responded:

MARK: What it is? [laugh] um... using a whole bunch of numbers... like somewhat logically... most of the time it's logically [laugh]... to get answers for whatever you want... like... if you're looking for something... like... I don't know, it's hard to explain... that's a hard question!
JC: Yeah... first reactions, I guess, when you think about mathematics.
MARK: Hard. [laugh]
Though my impressions from Mark's statements above were that mathematics came fairly easy to him, it was interesting that he considered mathematics "hard".

Knowledge of Functions and Graphs

Function Definition

Mark was asked in the first interview to describe what a function is, in his own words. He smiled, sighed, and then paused, before saying:

MARK: It's the . . . the domain and range, and . . . I forget which one's which, but, its like a one-to-one ratio or . . . No, actually, I can do it better with the graph. It's like, when you have the x-axis and if you can go straight up and there's only one point or no points? That's a function . . . it's like . . . if there's no more than one y-value to each x-value, that's it.

Mark uncrossed his arms and used his hands to help describe the Cartesian plane. He started by describing the function concept with a mention of domain and range, but finding these terms a bit confused, resorted to a graph and what appeared to be an application of the vertical line test. His voice was confident as he moved his hands up, down, and sideways to describe the x-axis and the vertical line up and down. Once he did this, he was able to return to the notion of a "one-to-one ratio" that he had tried to start with, and say with more confidence that there could be "no more than one y-value to each x-value". After he was able to visualize the idea, he seemed more confident in describing it verbally. This substantiates his earlier statement that he liked to "see it" which I understood as being able to visualize the concept pictorially.
Ability to Identify Functions

When handed Task #1 (Figures 9-13), Mark was asked to identify which of the thirteen items were functions and to explain his reasoning. The following summary of his responses is organized by the types of representations presented in this task: sets of points and tables of values, graphs, and equations.

Mark used his definition "no more than one y-value to each x-value" as criteria to decide if a set of points, or a table of values was a function but had developed alternate versions of this definition which are italicized below.

1. {(1,2), (3,4), (0,6)}
   2. x  | y
       |   0 0
       | 2 4
       | 3 6
       | 4 7

Figure 9: Task #1 (1-3)

For the first item, (Figure 9, #1), a set of three points, Mark asked, "Is that a line or is that just points?" Perhaps in asking this question he was reminded of situations in which he had a small number of points, yet connected those points to obtain an entire line. When I clarified that it was a set of points, he answered, "O.K., that's a function." In clarifying his reasoning, he said, "because there's . . . let me see . . . there's . . . all the y values are different". He quickly added that the second item (Figure 9, #2), a table of values consisting of four points, was also a function for the same reason. Using this alternate version of the definition of function, which I believe he thought equivalent to his first definition, caused Mark some problems when he arrived at items #5 and #6.
He glanced quickly at the fifth item (Figure 10, #5) and stated, "I think that's a function too." According to his definition, "all the y-values are different", his evaluation of this item was correct. I asked him to look at the item again, and then he responded, "Oh, no, it's not, because there's 2 x's . . . no, there's one x with 2 y's . . . O.K.". I believe at this point he had realized on his own that "all the y-values are different" was not enough of a justification to say that the item was a function, as he didn't use this definition again throughout the study. Like the other students I interviewed, Mark often interchanged the x and y variables, but he was usually very quick to discover his own error and correct it.

For the sixth item (Figure 10, #6), Mark quickly said, "That's not a function", using the last definition of function that he used for #5. On #13 (Figure 13), Mark returned to his earlier statement used when asked for the definition of function, "and that's a function . . . cause there is only one y to each x."
Mark seemed to use the vertical line test to decide if graphs were functions, as for the first graph (Figure 9, #3), when he explained, "That's a function because there's only one point in each... only one y". He glanced up from the paper, to make sure that I was understanding what he was trying to say. I affirmed his response with a nod of my head. For the remainder of the graphs, Mark revealed the alternate ways in which he thought about this test. Again, I believe he saw these alternate ways as equivalent. For the next graph (Figure 10, #4), Mark commented, "Yeah... that's a function because it doesn't, like, cross itself and come back". For the graph in #8 (Figure 11), he commented, "That's not a function because there's two". He used his finger to show a vertical line going through two points of the parabola in #8. He switched back and forth between these various statements when looking at the remaining graphs.
Items 7, 9 and 11 were equations. Mark hesitated and wrinkled his forehead when he looked at the equation in item #7 (Figure 11), saying, “I think that’s a function.” He glanced up to see if I approved. I smiled, and he continued, “Yeah . . . because it’s just a basic parabola . . . well not basic, but, it’s just a parabola.” He recognized that the graph of this equation would be a parabola, but not a basic parabola as had been described in class. I noted that he again visualized the graph before making his decision, indicated by referring back to item #8 (Figure 11). He reasoned, “It doesn’t like go sideways like that one”, . . . showing with his finger once more that it was not a function by drawing a vertical line crossing two points.

For item 9 (Figure 11), Mark again hesitated and said, “That’s a function because, it’s another parabola and they don’t cross.” When asked why he hesitated, he responded, “Well, it’s . . . it’s just that it takes me a second to graph number 9.” This statement gave more credence to Mark’s assertion that he preferred to “see it”, as he used the visual representation rather than the algebraic representation to make a decision.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
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<tr>
<td>1</td>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
<td>2</td>
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</tbody>
</table>

Figure 13: Task #1 (13)

For the linear equation given in item #11 (Figure 12), Mark stated:

MARK: . . . and for #11, “um . . . that’s a function . . . cause the graph probably goes something like . . . that . . . [He draws a line showing its direction with his finger.] . . . and there’s only one y to each x . . . it doesn’t go straight up and down.”
This statement was an indication of his understanding that all non-vertical lines were functions.

In summary, Mark was able to distinguish functions from non-functions and as the task progressed became more confident with the tools he was using to do this. As the theory of conceptual change suggests, at times he altered his existing definition of function to accommodate the examples he was given. This was most clearly seen when he realized that his definition "all the y-values are different" used for identifying functions was not adequate. I believe that in being given the opportunity to express verbally his thinking, he was having to make decisions about the rationality of his thinking.

Describing Function Attributes

For Task #2, I told Mark, "This is kind of a game that we're going to play next." I handed him a “little book of graphs”, the contents of which are in Figures 14 and 15 below, and explained that he should describe each graph well enough to me so that I could “draw it”. The purpose of this exercise was to note which attributes of the graphs stood out most for Mark and also to investigate his use of terminology.

Knowing that it was probably obvious to Mark that I already knew what was in this little book, I asked him to pretend that I didn’t. I also mentioned that he could assume that I knew some terminology about functions and graphs.

Mark thought that I was asking him to give the equation of the graph, suggesting that he viewed an equation as a type of description of a graph. I asked him to describe the graph first without using the equation and that we would come back to talking about the equations of the graphs.
For the first graph (Figure 14, #1), which was a line with a positive slope, Mark's focus went to the y-intercept. He continued by giving the x-intercept and two more points before he said I could connect all the points with a line. 

MARK: O.K., go on the y-axis, put a point at 1, and on the x-axis put a point at -3 . . . and let's see . . . I'm trying to figure out another point . . . and put another point at 3,2 . . . and . . . and at -2,-6 . . . yeah, no, hang on . . . sorry . . . -6,-2 . . . and you connect those with a line . . . I'm assuming this goes on forever. 

For the second graph (Figure 14, #2), a parabola, his attention again went to the y-intercept, which, in this case, was also the vertex of the parabola. 

MARK: O.K . . . -2 on the y-axis and draw a parabola opening up . . . and it's . . . let's see . . . whoa . . . that's funky . . . um . . . [laugh] . . . man, forget that parabola bit . . . put a point at 2 and -3 on the x-axis . . . and draw a curve . . . like . . . connecting the point on the y-axis and the 2 points on the x-axis. 

Mark's method of determining other points on the parabola was a technique the teacher had shown them in class. The students had been shown that when sketching basic parabolas if they begin at the vertex, they could find other points on the parabola by following the pattern: over one, up one; over two, up four; over three, up nine; etc., and then using symmetry to sketch or find points on the opposite half of the parabola. Mark, it seemed, had concluded that all parabolas
followed this pattern, and hence his comment about "forget the parabola bit". The graph he was viewing did not match his understanding of what a parabola was, so he referred to it as a "curve" instead.

For the third graph (Figure 14, #3), also a parabola, Mark noted the vertex/x-intercept first and then used two pairs of symmetric points to describe its shape.

MARK: Go to 4 on the x-axis and put a point . . . and then put a point at 3,-1 and 5,-1 . . . and 2,-4 and 6,-4 and connect them to make a parabola.

Though this parabola didn't classify as a basic parabola, Mark didn't get bothered by it this time.

Though Mark had not used the words vertex or x-intercepts, he was familiar with the terminology as is seen in the following dialogue.

JC: Do you have a name for this point here?
MARK: Vertex.
JC: O.K., and how about those points here . . . back on number 2?
MARK: X-intercepts.

Figure 15: Task #2 (4-6)

Mark's description of the fourth graph (Figure 15, #4), was based on terminology that had been used by Mrs. Tucker in class. He stated, "O.K., . . . um . . . just draw a basic swivel going through the origin". Having observed the class
during the time the teacher talked about “swivels”, I knew what he meant by this word. Mrs. Tucker used this terminology to describe the shape of the graph f(x) = x³ and at other times to describe the shape of 4th degree polynomial functions. As far as I could determine, any graph that was not parabolic or linear, yet was the graph of a polynomial function, “swiveled”. I began to like this terminology as it seemed to reveal the dynamic aspect of the polynomial function more than the terms “parabola”, “line”, or “cubic”.

For the fifth graph (Figure 15, #5), Mark started on the y-axis naming the “open and closed dots”, and then named two additional points for each section of the graph.

For the sixth graph (Figure 15, #6), Mark started with a “point at 2 on the x-axis” and then continued:

MARK: Draw a graph of an absolute value from there like um ... a curve that’s ... um ... let’s see ... this is a hard one ... pretend you’re drawing half a parabola but only draw the half of a sideways parabola, but only draw the half of ... that’s like ... on the top side of the x-axis.
JC: O.K., so it goes up like this?
MARK: No ... um ... a sideways one. So if you held that paper sideways it would be opening down? Yeah, like that.

The class had done worksheets translating the graphs of the functions y=lx| and y=√x, and I believe Mark was confusing these two functions.

In summary, it was seen from this task, that Mark recognized shapes of some basic functions and used the names given in class, i.e. “swivel”, “parabola”, and “line”, to describe the graphs. Other than these names he used points to describe and position the graph, often giving more points than was actually necessary, as in the case of a line. When the graph was a parabola, his attention seemed drawn to the vertex first, and then to the intercepts.
Translating to the Algebraic Representation

Having finished describing the graphs to me, I asked Mark to say what he could about the equations of the same graphs.

For the first graph (Figure 14, #1), the line, Mark determined the equation by first finding the y-intercept, then the slope, and then plugging these values into the form \( y = mx + b \).

\[
\text{MARK: O.K... that's } y = \frac{1}{2}x + 1?
\]

\[
\text{JC: O.K., how'd you get that?}
\]

\[
\text{MARK: Well, the y-intercept is 1 and it seems to... be like... it's... when you go up 2 for the } x \text{ you go up 1 for the } y \text{? So, I guess it's } \frac{1}{2}
\]

Instead of using the word "slope", Mark described the process he used to find it.

For the second graph (Figure 14, #2), Mark guessed:

\[
\text{MARK: Well, I'd say it's probably } \frac{1}{2}x^2 - 2... \text{ that's a guess, though, I just got that because I know the basic parabola would go... oh, no, O.K... the basic parabola would be thinner because you go over one and up 4, since you go over 2 and up 4, I mean... and this one seems to go over 2 and up 2.}
\]

Mark was slightly confused about the coefficient of the squared term. His technique for finding this coefficient was to compare this graph to the relative changes in \( x \) and \( y \) for the basic parabola, a technique used often in class. This technique seemed to foster the connections between linear and quadratic equations in that for both equations the leading coefficient is a rate of change and the constant term is the y-intercept.

Mark confidently gave the equation for the graph of #3 (Figure 14), “\( y = (x - 4)^2 \)”, probably due to the fact that his algebra class had spent quite a few days on translations of parabolas. He also responded quickly with an equation for the fourth graph (Figure 15, #4), “\( y = x^3 \)”. I was a bit surprised when he gave the equation for #5 almost as readily.
MARK: \( y = \ldots x+2 \ldots \) greater than \ldots with the restriction greater than or equal to 0 \ldots \( x \) is greater than or equal to 0 \ldots and \ldots \( y \) is equal to \( x-2 \), restriction, \( x \) is less than 0?

He was also able to give the equation for the last graph, “\( y=\sqrt{x-2} \)” without hesitation.

**Journal entry.** Mark’s journal entry after this interview was:

I thought the questions you asked at the beginning were strange, but I understand. I thought it was easier to give you the equations of the graphs than to describe them to you. That’s it.

Though Mark indicated previously that he was more visual, this journal entry reveals that he had a harder time describing a graph visually than describing it by an equation. This was evident in the interview. Putting this together with his statement that the questions at the beginning were “strange”, and with his disuse of terminology such as “slope”, “x-intercept”, or “y-intercept”, I sense that he had not had many experiences in verbalizing the visual representation before. For the time I observed his class, I can’t recall the students ever being asked to do this.

**UndenUmdina* of Polynomial Functions**

The purpose of the following activity was to obtain a deeper look into Mark’s current understandings of the graphs of polynomial functions. As indicated in Chapter V, the students were shown three graphs with **PC-81 Emulation software** (Texas Instruments, Inc., 1991): a cubic polynomial function, a quadratic function, and a linear function. For each graph, I moved the cursor slowly along the curve from left to right, stopping at particular intervals or points and asking Mark to comment on the location, pressing also for the points connection to the algebraic representation. If I wanted a student to talk more
about a location, having the student think and communicate about the algebraic representation often brought out connections that the student was making between the graph and the graphs of other classes of polynomial functions. At the time of this interview, Mark had just finished the unit in class on polynomial functions of degree two and higher. This interview followed the test on that unit for which he received an 83%.

The discussion of these interviews is organized around the first three research questions:

1. What connections do the students make between their study of the graphs of linear and quadratic functions and their study of the graphs of polynomial functions of degree greater than two?
2. What evidence suggests that the students' conceptual development during instruction on polynomial functions of degree greater than two builds on their understanding of the graphs of linear and quadratic functions and their general knowledge of functions and graphs?
3. What factors can be identified that contribute to/inhibit the student from making the transition from the graphical representations of linear and quadratic functions to those of higher degree?

The following section headings correspond to the focus of these research questions: "Connections Between the Classes of Polynomial Functions" (questions 1 and 2), and "Contributing/Inhibiting Factors to Making the Transition to Polynomials of Degree Greater Than Two" (question 3).

As these interviews progressed, I thought about appropriate activities to use in the forthcoming teaching episodes that could enhance the students' understandings of the graphs of polynomial functions. The activities were to be
selected based on the understandings I was perceiving through the interviews (research question #4).

**Connections Between the Classes of Polynomial Functions**

Mark was shown a graph of a cubic polynomial function (Figure 16). Mark identified the graph as a third degree polynomial function.

![Graph #1 - PC Emulation Software](image)

**Figure 16: Graph #1 - PC Emulation Software**

As I moved the cursor up to the first turning point on the curve, Mark said:

MARK: That's the apex, I think, or the vertex ... you know ... one of those "ex" ones [laugh] ... that's where the y's start to go down again? ... the x's are still going up.

When I inquired whether that point on the curve had anything to do with the equation, he responded:

MARK: um ... you know, if it was like a parabola, you could plug in the ... oh, let me see ... yeah, you could plug in the numbers in the ... a times x-h squared + k ... you could plug them in there and that would help you get the equation.
Mark was viewing this turning point as a vertex which reminded him of the form \( y = a(x-h)^2 + k \) which he had used in determining the equations of parabolas when the vertex was known.

As we continued tracing along the curve, and I stopped the cursor on the y-axis, Mark offered:

**MARK:** That's probably... like in a parabola... that would be like what you add on at the end... or if it was a line you'd add that on... but, again, I don't know what this graph... or what the equation is.

I noted that he did not use the term "y-intercept" until I asked him if he had a name for that point. From his comments in the first interview with other graphs, I believe that he viewed the constant term of a polynomial function as a result of a translation of the curve "up" or "down", rather than the result of evaluating the polynomial for \( x=0 \). Because the class had done a lot of work at the beginning of the year involving translation of graphs, this was not surprising.

In trying to answer my questions about the points on the cubic, Mark kept referring back to his knowledge of quadratic and linear functions, providing evidence that he viewed this graph through prior experiences with the graphs of quadratic and linear functions.

Following this graph, we moved along the curves of a quadratic function and a linear function in the same way. The graphs were purposely shown to Mark in this order because I didn't want the discussion about the attributes of the linear and quadratic graphs to trigger responses regarding the cubic. Mark viewed the quadratic function as a translation of the "basic" parabola, \( y = x^2 \), though noted that you could find the equation of the quadratic by plugging the vertex into the form \( y = a(x-h)^2 + k \). For the linear function, Mark determined the
equation using the slope and the y-intercept. The y-intercept, which he said is what "you add on at the end", seemed to be the prominent linkage for Mark between the three classes of polynomial functions.

**Contributing/Inhibiting Factors to Making the Transition to Polynomials of Degree Greater than Two**

As noted above, Mark made several connections between the graph of the cubic function and the graphs of a quadratic and linear function, i.e. linking "turning points" with vertices, and y-intercepts with the constant in each polynomial function. Tracing along the curve of the cubic, I had stopped on the first x-intercept and asked Mark if that point was useful in determining the equation of the graph to see what connections Mark made between this graph and other graphs or with the algebraic representation.

MARK: I don't really know how the equation for this goes... something like ... x^3 + x^2 + x and something else... I don't know.

JC: So, you don't know if that point would have anything to do with that... it wouldn't help you to find that equation?

MARK: I couldn't find that equation.

Later, I asked him to summarize his sense of what the equation would be. He responded:

MARK: Something like y equals x^3 + x^2 + x + 1.

JC: O.K. [I point to the x-intercepts.] Do these intercepts have anything to do with that equation?

MARK: I'd guess that those are like... well, I'll take a guess and say that's the um... coefficient of the x terms... but I don't know, that's a guess.

When Mark took his test on this unit, one of the questions stated, "A fourth degree polynomial equation has zeros of -2, 1, 3, and 4. If P(-1) = 20, find P(2)."

To solve this problem, Mark had written on his test paper, "a(x+2)(x-1)(x-3)(x-4) = P(x)". It appears he understood the relationship between zeros and factors in an
algebraic sense, i.e. could "plug" the roots into the factors, but had not thought about pulling them right off of the graph and doing the same thing, as the previous quote shows. Even though he had just completed the unit dealing with the factor theorem, he didn't associate the x-intercept as seen on the graph with a linear factor of the equation. This could be due to the greater emphasis in class on problems involving algebraic manipulations as the one on the test rather than problems requiring graphical interpretation. It occurred to me that the emphasis was on the algebraic interpretation perhaps because Mrs. Tucker found it easier to make up a worksheet with algebraic problems as opposed to having to draw graphs. This is one of those questions which in hindsight would have been good to ask Mrs. Tucker, but these thoughts occurred to me long after the actual study when the questions would have been inappropriate.

When Mark did not suggest any link between the x-intercepts as seen on the graph of the cubic polynomial function with quadratic and linear functions or with the algebraic representation of the function, I introduced the terminology of factor into our discussion by asking Mark, "Could you tell me what a factor of that equation would be by looking at the graph?"

MARK: Um . . . let's see . . . 1/2?
JC: Where are you getting that from?
MARK: That's about where the x-coordinate is . . . the x-intercept.
JC: How many factors do you think there would be?
MARK: 3 . . . there's x^3 . . . so, you get three factors.
JC: O.K., what would be the other factors?
MARK: Probably about 1/2 and almost -2?

Mark combined information from the graph (the values of the x-intercepts), with information from the leading term, x^3, to arrive at a response.

As the cursor moved to the x-axis, on the graph of the quadratic function (Figure 17), Mark noted that the point was the "x-intercept", but again did not
think that this point had anything to do with the algebraic representation of the function. He did however, note a connection between the vertex and the algebraic representation. Our conversation follows:

**Figure 17: Graph #2 - PC Emulation Software**

MARK: That's... the x-intercept again.
JC: Does that have anything to do with the equation?... would it help you find the equation?
MARK: Would it help me find it?... no.

For the point on the y-axis, Mark said:

MARK: That's the y-intercept and the vertex.
JC: O.K... and does that have anything to do with the equation?
MARK: Yeah, the vertex is... where you plug in to that... I forget what it is called... the a times x-h squared + k... you plug in the vertex for the h and k.

Moving to the second x-intercept, Mark modified the answer he had given for the first x-intercept. I believe he made these comments as a result of our earlier conversation about "factors" when we were looking at the graph of the cubic polynomial function.

JC: Does that have anything to do with the equation?
MARK: Um... I think they're factors.
JC: O.K.
MARK: So you put 'em in and you get 0.
JC: What did you do?
MARK: You plug them into the equation... you get 0... is that right?

Mark seemed to be identifying the x-intercept as a value that made the function 0. I asked him what he thought the equation for this parabola was.

MARK: Um... y = x^2 - 2?
JC: O.K., and how are you getting that?
MARK: It looks like its stretch factor is one... [points to screen] it kind of goes up one and over one.
JC: O.K.
MARK: ... and I just subtracted 2 to bring it down?

To obtain the equation, rather than using the form y=a(x-h)^2 + k, as he had alluded to when talking about the vertex, he viewed this parabola as a translation of the basic parabola.

When shown the graph of the line (Figure 18), I asked Mark if the x-intercept as seen on the graph had anything to do with the equation of the line.

MARK: I forget if they have factors or not. I think so. I think that's a factor of it...

Since Mark previously equated "factor" with "x-intercept", it appeared that he tried to also equate the x-intercept of a linear function with "factor". He was unsure of his response because he couldn't remember experiences for which linear equations had factors. This is another instance where he made a connection between the classes of polynomial functions. He suggested that this point had something to do with the equation of the line, but didn't say how they were related.
Mark went on to get the equation of the line by using the slope and y-intercept.

**Summary**

The diagram entitled "Interpretations of the x-intercept" that was more fully described in Chapter II is shown again below. From the comments Mark made above, I would classify Mark's conceptions of the following connections as:

**Acquired:** b - Mark understood that an x-intercept as seen on the graph makes the function value equal to 0.

**Tentative:** a - It is not clear whether Mark understood that if c is an x-intercept, that the coordinates of the x-intercept are (c,0).

**Missing:** c - Mark did not make a connection between the x-intercept, c, and the factor (x-c) of the polynomial.
Mark equated "factor" with the value of the x-intercept, and held on to this perception throughout the entire study. For Mark, the x-intercepts were the points that made \( y = 0 \) when they were put into the function. He did not link these x-intercepts to the linear factors of the function for any of the classes of polynomial functions.

One of the assumptions of this study (Chapter II) is that it is essential for an algebra student to develop an understanding of the relationships (note the plural here) that a factor of a polynomial has with the graph of the function. I believe Mark's understanding of polynomial functions of degree greater than two would have more effectively built on his understanding of the graphs of quadratic and linear functions if he had understood the link between the x-intercepts as seen on the graph and the linear factors of the polynomial expression.

**Journal entry.** Mark's entry in his journal after the 2nd interview was:
I made the stupidest mistake - the x-intercepts/factors of the equation do help me get the equation for instance if they are 2, 4 you 
\((x+2)(x+4)=0 \Rightarrow x^2 + 6x + 8\). I feel so stupid.

In reflecting about our session together, he was able to connect the x-intercepts with the linear expression. He still referred to the x-intercepts as the factors rather than the linear expressions. I noted that, in his example, he equated the product of linear factors with 0. This was not a surprise, however, as most of the factoring that had been done in class was for the purpose of solving equations.

**Teaching Episodes: Forming Links Between the Classes of Polynomial Functions**

In light of the observations made in the clinical interviews, the teaching episodes were designed to strengthen connections between the classes of polynomial functions by building polynomial functions from products of linear functions. Using software that allowed the student to see the algebraic representation of the functions alongside of the graphical representation would perhaps foster the formation of connections between the classes of polynomial functions. It was felt, in particular, that these activities would make the connections between the x-intercept and the factors of the polynomial more salient.

In the teaching episodes, linear functions were viewed as the building blocks of all classes of polynomial functions. These activities, therefore, were meant to expand on the student’s current understandings of linear functions. As the theory of conceptual change suggests, students change their present conceptions to accommodate new information. It was hoped that this “new” way of looking at polynomial functions would enrich and broaden their current conceptions.
Since the processing of the new information is influenced by the totality of the student's prior conceptual knowledge as well as personal, social and contextual elements, I expected each student to process and accommodate the information in different ways. The affect of the teaching episodes on the student's conceptions was, therefore, unclear.

**Building Quadratic Functions**

Using the *Function Supposer: Explorations in Algebra* software (Appendix E), I asked Mark to type in any two linear functions, \( f(x) \) and \( g(x) \), keeping them fairly simple. He typed in the equations \( f(x) = \frac{1}{2}x + 3 \) and \( g(x) = -x - 2 \).

I then asked, "If you were to take the product of these two linear expressions . . . what would you get for a quadratic?"

**MARK:** Oh! Do you want me to like type in the equation . . . or . . .

**JC:** No. Can you . . . do you have any feel for what the graph of the product of those two would be?

**MARK:** Oh, it would be a parabola.

**JC:** How do you know that?

**MARK:** Cause it'd be . . . you'd have \( x^2 \).

**JC:** O.K., and do you know anything about the parabola . . . anything at all?

**MARK:** Eeee . . . it would be . . . um . . . it would be wide, well, it wouldn't be a parabola . . . well, it'd be like a parabola but wider.

Mark revealed again here his notion that the only parabolas were *basic* parabolas.

When asked about the product, Mark focused on the degree of the parabola rather than any specific points. As indicated in the clinical interviews, his focus with linear functions was on the \( y \)-intercepts and the slope, but these attributes of the lines did not seem to enter into his determination of what the parabola would look like. As is indicated by the following, Mark was focusing on algebraic manipulations rather than on the attributes of the graphs.

**JC:** O.K . . . why do you sense it would be wider?
MARK: [He points to the expression for f(x).] Cause you have \( V_i \) there so when you take that out you'd get \( x + \frac{V_2}{2} \ldots \) multiply that \ldots multiply it by \( V_2 \ldots \) um \ldots I gotta do all the multiplication in my head \ldots \) um \ldots let's see \ldots it'll open down.

JC: How do you know that?

MARK: Cause that's a \(-x\) and when you multiply that through \ldots and I don't know \ldots it's not going to be on the origin \ldots I'm not sure where it would be \ldots I'd need a pencil and paper to figure that out.

Mark knew the product of two linear expressions would be quadratic but in order to know anything more about the graph of the parabola, he thought that he would have to multiply the linear expressions out and then change that product to the form \( y = a(x-h)^2 + k \) to determine its position.

When the parabola was graphed on the same axes as the lines, the x-intercepts on the graph were not visually prominent; he did not focus on their placement until told to do so. Finally, with surprise, he realized that the lines had the same x-intercepts as the parabola.

After doing a few more examples, Mark tried to equate the form \( a(x-h)^2 + k \) to the factored form \((x-c_1)(x-c_2), c_1, \text{ and } c_2 \) being the x-intercepts. He felt something was missing still in the latter form. He was concerned about the y-intercept, whether it would automatically be given by the product of linear factors or if it would have to be added on at the end, like the \( k \) in the first form. We discussed why, in the product of linear factors, everything was there except for, perhaps, a "stretch factor".

I showed Mark how the software could now express a new function \( h(x) \) as the product of these linear expressions. It could show all three graphs separately and then together on one set of coordinate axes, and color code them so they could be distinguished easily. I asked Mark if the resultant parabola was what he expected.
MARK: Um . . . I think so. Is that wider?
JC: Well, yeah, probably.
MARK: Yeah, pretty much.

I asked Mark to tell me more about the parabola's position.

JC: Anything you note about where that parabola is?
MARK: Well, it moved on the x-axis . . . I didn't know it was going to do that, but I guess I kind of really figured it out when I did the problem . . . I figured it'd be on the y-axis, but . . .
JC: O.K., what about the x-axis?
MARK: It moved on it . . . it moved sideways.
JC: O.K., do you notice anything about where it . . . where it went?
MARK: Oh, its near the . . . the vertex is kind of close to the . . .
JC: Uh, huh . . . O.K. And how about where it crosses the x-axis?
MARK: [He looks at the graph.] . . . -2 and -6 . . . is that right? . . . all right.
JC: How do you think . . . why do you think that?
MARK: No, that wouldn't be right . . . [He looks at the product of the linear expressions.] Yeah, cause if you take the negative out of that and then you have x + 2 and that would make it . . . um . . . make the . . . ah . . . factor 2 . . . -2 . . . and you take the V2 out of that you get a 6 . . . so you make it a -6.
JC: All right, and how does that correspond to the linear expression?
MARK: Oh . . . yeah, that's right where they cross . . . huh . . . huh . . . huh . . .

Mark was surprised that the parabola went through the x-axis at the same points as the lines did. His attention was drawn more to the global position of the graph relative to its first position, i.e. that it had moved left or right and was no longer centered on the y-axis. He noted also that the vertex of the parabola was close to the intersection of the two lines. Not until I asked about how the parabola's x-intercepts corresponded to the x-intercepts of the lines, did he actually realize they were the same.

We repeated this activity and Mark entered two new functions for f(x) and g(x). This time he let f(x) = 2x + 1 and g(x) = V4x - 1. I again asked Mark to predict what the parabola would look like.
MARK: So it will open up and it is going to go right through there at -1 [He corrected this later] and through there at 4.
JC: 4... how do you know that it's 4?
MARK: [Points to linear expression for f(x)] Cause you take that out and get 4 there.

The computer confirmed Mark's prediction.

I went on to explore with Mark a way to determine where the graph would cross the y-axis. When he multiplied the linear expressions, the result was a quadratic expression in standard form that ended with a constant. The constant term for the product of the particular functions he had entered was -1.

JC: O.K., now, what has that to do with the graph?... the -1.
MARK: Well, that's where it crosses here but... I don't know if that will be right for all of them.
JC: O.K., but if the... the y-intercept, how do you define that? What's another way of thinking about it?
MARK: Oh... that's where it crosses the y-axis so that's where x is zero, so if you plug in...
JC: Right, if you put zero in...
MARK: Yeah... it would always be the constant.

In the interviews it had seemed that Mark knew the constant term was the y-intercept. I believe, as mentioned previously, that he viewed the constant term's purpose as a shift up or down, which allowed him to find the position of the curve on the y-axis. This was the first time that Mark indicated an understanding of the connection between the constant and the y-intercept, i.e., that if c is the y-intercept, then \( f(0) = c \).

I hoped that Mark would see the graphical side of the relationship between the constants of the linear expressions and the constant of the product. Observing where the linear functions crossed the y-axis and then taking the product of these intercepts would give the y-intercept of the graph obtained by taking the product of the linear expressions. So the y-intercept of the resultant graph could be found by just focusing on the graphs of the linear functions as opposed to focusing on
the linear expressions. I hypothesized that if Mark worked in both representations to find the y-intercept, he would build connections between the two representations as well as develop a stronger understanding of the y-intercept.

To enhance the relationship between the lines and the parabola even further, I took a strip of paper and put the edge of the paper on the leftmost x-intercept, covering the portions of the graphs to the right of that x-intercept. I asked Mark whether the y-values for each of the portions of the graphs that were still showing were positive or negative. This was to allow Mark to see that for each value of x, the sign of the y-value for the parabola would always have the same sign as the product of the y-values of the lines.

JC: O.K. So, ... again, we're thinking of this as the product of these two, so where this is negative this is negative, when you multiply these two together you get the positive part of the parabola.
MARK: Oh. Yeah, O.K. I see. Yeah, so like right there one part of one is negative so you get a negative parabola.
JC: Yeah ... I sliced it right at the x-intercept there.
MARK: Yeah. O.K., so if you cut it right where they [points to screen]. . . yeah, right there, it's positive cause both of these are negative, and if you slice it like right there, that's negative cause one's positive and one's negative ... and yeah . . .

We continued looking at the portions of graphs between the intercepts and beyond the last intercept to see that this was the case for all values of x.

Before we ended this session together, I tried to talk with Mark about his understanding of the words "factor" and "root". I explained why the term "factor", in the context of polynomial functions, customarily referred to a linear expression rather than an x-intercept. I felt it necessary to clarify the meaning of these terms, as I was using them so frequently.
Mark was then asked to type in functions for \( f(x) \) and \( g(x) \) once more. This time he entered \( f(x) = 2x - 4 \) and \( g(x) = \sqrt{3x} \) and was able to predict the position of the graph on the axes accurately.

**Working Backwards: Finding the Components of a Quadratic Function**

Mark's prior experiences included factoring quadratic expressions into linear factors, which can be thought of as breaking the quadratic into its building blocks or components. The counterpart to this process, graphically, is to break the graph of a quadratic function into its components, the lines. To illuminate factoring in a visual way, I asked Mark to work backwards, i.e., given a parabola, to try to find lines that could possibly represent its linear factors. Since the computer software did not allow for this option, I had sketched parabolas on worksheets in a layout similar to the layout of the *Function Supposer* software (Appendix E).

Without my saying a word, Mark knew when handed the sheet what he was going to be asked to do. "Sketch the lines that make those?" Handing him a pencil, I commented, "I'll let you start with a pencil." He laughed and went to work taking about 20 seconds to sketch in two lines (Figure 21).
I asked him to explain why he had sketched those particular lines.

MARK: It crosses there because that's where they both are negative... so it's up there? Right there one changes to positive, so... it's negative cause when you multiply a positive by a negative... and then it crosses right there cause that is when the other one changes to positive so it crosses back up and goes positive.

We color-coded the lines he had drawn and moved on to the next parabola (Figure 22). Mark took about 12 seconds this time sketching in the lines, again using the switches in sign of the parabola as a guide. I noted that for this graph, however, that the y-intercepts of the lines did not correspond to the y-intercept of the parabola.

JC: Now how about the y-intercept in that one?
MARK: Ooh . . .
JC: Ooh . . . [laugh]
MARK: Eee . . . um . . .
JC: Do you want to change that?
MARK: Yeah.

Mark struggled a bit as he readjusted the lines.

MARK: [He hits the paper with his fist.] Messed up again!
JC: That's why I gave you a pencil.
MARK: . . . $1 \frac{1}{2}$ . . . that's $1 \frac{1}{2}$ . . . that's better . . . O.K.
JC: . . . and then you multiply that by . . .
MARK: Yeah, you multiply that by the -2 and you come out to -3 which is the y-intercept.

I asked if he thought whether those were the only two lines that would work or if he thought he could come up with other lines that would also work.

MARK: You could come up with other lines . . . you could come up with like . . . 3 . . . let me see . . . yeah . . . you'd go through 3 in that one and then . . . this one a -1 . . . or -3 and 1 . . . you can come up with just about any combination.

At first thought, one might imply from what Mark was saying that a quadratic expression is not factored uniquely into linear expressions. This factorization into linear expressions, however, is only unique up to unit factors. Unit factors can be split over the linear expressions to give an infinite number of combinations of lines. The only limitations, as Mark pointed out, are that the product of the y-intercepts of the lines be equal to the value of the y-intercept for the parabola and that the roots remain the same.

Journal entry. Mark wrote in his journal after this first teaching episode:

I found our last meeting very interesting. No one has ever explained that to me that way. I can really see why you get the parabola you get when you multiply lines together. Knowing why really helps, I thought I understood before but now I can really see it. Thank you, it was very interesting.

At our next meeting, we reviewed our work on the computer from the first teaching episode by doing one more example of generating a parabola by entering
the equations of two linear functions and then taking the product. It had been
over a month since I had spent time with Mark, but Mark seemed to remember the
ideas that we had discussed the previous time and easily predicted the position of
the parabola from the graphs of the two given lines.

We looked at a couple more graphs of parabolas where I again asked Mark
to sketch a pair of lines representing the linear factors of the quadratic function. I
warned him that he might find these a bit harder than the last ones he did, in
case he started to feel uncomfortable finding the lines.

I asked Mark to tell me what he was thinking as he worked on the first
problem (Figure 23).

![Figure 23: Graph #3 - Working Backwards (Mark)]

MARK: This one has to be . . . one has to go through 0 . . . and one has
to go through . . . (sighs and puts chin in hand) . . . um . . . well
[pause] . . . I don't know . . . let me see . . . let's try this . . . that's not
very straight but I guess you'll know what I mean . . . um . . . [long
pause] . . . would it go through like there? . . . [Mark drew another line
going through a point on the x-axis different from the vertex.]
JC: O.K . . . with this one though you get another . . . wouldn't this
parabola go through that intercept also?
MARK: Yeah, that's what I couldn't figure out . . . but I had to . . . one
had to be 0 . . . Oh, hang on . . . let's see 0 and . . . ah . . . Yeah, it has
to be 0 and a y of 1, doesn't it?
Mark spent a period of time thinking that the y-intercept of the parabola was 0, and hence looking for a pair of lines so that the product of y-intercepts would be 0.

JC: Yeah, but what's the y-intercept of the parabola?
MARK: Oh, its up there. . . Oh, the x. . . Oh, O.K. [He erases his work.]

Mark struggled with the positioning on the x-axis of the second line.

I offered a hint asking Mark if the lines had to be different.

MARK: I was just thinking that. . . cause they. . . let me see. . . what. . . they both have to go through there. . . [He points to the vertex of the parabola.]

I reminded Mark that he could also think about the changes in sign of the parabola to help determine the lines.

MARK: Yeah, but um. . . if you have. . . well, they both go through there. . . and one's here and one's here, then they both are positive here, so. . .
JC: O.K.
MARK: This would be positive, they'd be both. . . both be negative here, so that would be positive. . .
JC: O.K. Good. . . uh-huh. . .
MARK: So, if it goes through the 2 then it's the same line, but if it goes through the 1 it's two different ones. . . it's 1 and 4.

Finally feeling success with that task, Mark glanced ahead at the next parabola on the sheet (Figure 24).

Figure 24: Graph #4 - Working Backwards (Mark)
MARK: Oh, yeah? [laugh]
JC: [laugh] . . . first reactions?
MARK: I don't know how I am going to get 'em to like multiply to be . . .
for everything it would be above the line . . . so that . . . I don't know
how to explain it [laugh] . . . O.K . . . um . . . would it have to pass
through the vertex? . . . be two lines like passing through the vertex?
Would that work?

I pointed out to Mark that the lines he was thinking about would eventually pass
trough the x-axis. Seeing that Mark was giving up, I referred back to the
previous problems.

JC: Let's talk about this one again [pointing to example we had done at
the beginning of session on the computer]. We talked about these
being factors of that expression . . . So, if you knew an x-intercept, you
could find one of the factors. All right? So, up here [referring now to
the previous problem on the sheet] for this one you did . . . what would
be the factors of that expression?
MARK: 3 . . . + or -3?
JC: Um . . . a little more than that . . . it's not + or -3 . . . a factor would
be that whole expression? . . . it's a linear expression . . . that makes
up the . . . it's part of the quadratic.
MARK: Oh, yeah . . . so it would be like . . . um . . . -2/3 x + 2 . . .
would that be it? . . . or . . .
JC: Where are you getting that?
MARK: The slope of that one is -2/3 cause you go down 2 and up 3 . . .
JC: I see.
MARK: . . . and then its intercept is 2 . . . so . . . and if you then
square that you get the parabola.

I noticed that Mark was using the form mx+b to get the linear expression rather
than the factor theorem and therefore did not seem to be making a connection
between the linear expression and the x-intercept. I felt that it was necessary that
he make this connection in order to understand the current problem. Since there
were no x-intercepts for the current parabola, there would be no linear factors, i.e.
the quadratic expression describing this parabola does not factor over the real
numbers. Departing from the problem for a few minutes, I decided to discuss the
relationship between the equation of a line and its x-intercept. I drew a line on
the piece of paper going through one on the x-axis and two on the y-axis. I then asked Mark:

JC: O.K., could you find the equation of that line by just looking at the x-intercept?
MARK: No.
JC: I'm just trying to think of how to put this to you... now you're used to y=mx+b.
MARK: Yeah.

Asking Mark to give me the slope and y-intercept, I wrote down the equation of the line in slope-intercept form next to its graph (y=-2x+1).

JC: You're used to the slope and the y-intercept. What I'm saying is instead of the y-intercept, you could use the x-intercept too... if you pulled out the -2 on this [points to screen]... pull it out of the whole expression, you get x...
MARK: Oh, yeah... if you pull out the 2?
JC: Pull out the -2.
MARK: -2?... then you get x + V2
JC: O.K. Now, what that is... this is the slope of that line still... but this here is the x-intercept, where the line crossed the x-axis... and how do I know that?
MARK: Oh, is that kind of like the... ah... let's see... y=a times x... no, that's a parabola.

Mark made an immediate link to the form he had used to find the equation of a parabola: y = a(x-h)^2 +k. He pulled back once he remembered that that form was associated with parabolas. This reaction shows his tendency to regard linear and quadratic functions as unconnected entities. I decided to build on this connection that he had begun to make.

JC: It's exactly the same thing.
MARK: Without a k, because it doesn't have a vertex.
JC: That's right... so, another way of thinking of this is m(x-h) if you want... so here your x-intercept still would be h... it's -V2. All right, so for this particular line, you could write the equations like -2(x-1)... and that would be about the same as if you were thinking about the y-intercept... it would be -2x+2... -2 slope.
After talking about this association between the forms of the equation, I worried a bit that Mark would start confusing x-intercepts with the vertex. I moved back to the problem that had caused this whole digression.

JC: O.K., now why am I saying all that? I'm not trying to confuse you, really... so that's the idea down here...
MARK: Yeah, so I can find the x-intercept of that, but it's imaginary so it would have to be like $i$ something... something $i$.
JC: O.K., that's what I wanted you to get at there. There aren't any real roots here... so...
MARK: Yeah.
JC: So you can't draw the lines.
MARK: There are no lines that make that parabola?

Going beyond even terminology that I had ever used or heard before, I suggested that perhaps we could think of *imaginary lines*, but there were no *real lines* we could find that made up the parabola. I found myself inventing this terminology to make my point.

**Extending Activities to Cubic Polynomial Functions**

The software I used up to this point was incapable of extending these ideas beyond polynomial functions of degree two. Desiring to extend these ideas to polynomials of degree three, I designed a worksheet in the same format as the *Function Supposer* window, but with boxes for three linear functions and their product (Appendix E). Handing Mark this worksheet, I asked him to write down three linear functions and sketch their graphs in the appropriate boxes. When Mark was finished entering the linear functions separately, he color-coded them and redrew them together on the same axes in the larger box. I had him write down the product of the linear expressions before he tried to predict what the curve would look like. Mark writes in the three linear functions.

MARK: Oh, O.K... um... it would be... $x^3$. 

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JC: You don't have to multiply it out... just write it as a product here of linear expressions.

As he sketched the curve, he described what he was doing.

MARK: So, see if we can find this... um... first it's going to go through there. It's going to come up like from... no... it's going to... yeah, O.K., it's going to come from there... and here it's gonna go down and here it will come back up.

JC: O.K., how did you know that?
MARK: Cause, right... like back here all the lines are negative, so they come up... they'll all... these two... when you multiply 'em it will be a positive and then you multiply them by a negative, you get a negative. And here, two are... did I do that right?... yeah, here one is positive and two are negative so it comes out positive. Here two are positive, one is negative, so you come out negative, and here they're all positive... so...

He had extended our work of building quadratic functions to cubics by using the x-intercepts and the changes in sign of the lines to determine the curve.

JC: Good. O.K. um... how about the y-intercept? Did that work out?
MARK: The y-intercept is... 0.
JC: O.K... and again as an extension of this, it would come out to be the product of these three y-intercepts... when you multiply all these things together.
MARK: Yeah, 0 times those two.

The period ended and I asked Mark to do another example for homework as well as write in his journal.

Journal entry. Mark's journal entry at the end of the second teaching episode was:

I kind of expected that we were going to do 3rd degree equations, but it was still interesting. I can kind of see what other equations would make other things e.g. [sic] x+y=8 (x) x-y=2 (=) x^2 - y^2 = 16 (→) x^2/16 - y^2/16 = 1 [Parentheses were added for clarity]. Is that right? I think so.

I was intrigued that he had tried to extend these ideas further, and that he had made the claim that he did, even though the claim is mathematically incorrect. He had departed from what we had done in the teaching episodes by
multiplying equations here, as opposed to expressions, suggesting that what he knew to be a hyperbola could be broken down into two objects (it is not clear whether he knew these were also lines) of the form \( x \pm y = k \).

**Assessment of the Teaching Episodes: A Reevaluation of Mark's Understanding**

I began the final interview with Mark by asking him to explain what he did on his take-home worksheet on “Building polynomials” (Appendix E). Pointing to the intercepts, he explained how he got the y-intercept of the resultant curve first.

MARK: O.K., ah... the y-intercept's at 6, because that's at 1 and that's at -3... -1 and that's -3, and that's 2, and then multiplying that out makes 6.

He then explained how he found the positions of the x-intercepts by factoring the leading coefficient out of the linear expressions.

MARK: Ah... the first x-intercept is right there... and I don't know. the... um... let's see... so that'd be... 4?... and if you took it out of that one it would be 6, so it's 4 and 6. But I drew that one wrong... cause that one's gonna come up here. Yeah, that one would be 6, too? Yeah, this one would be 6. So it comes down and crosses at 4 and goes back up and touches the axis... the x-axis at 6, and comes down... doesn't really go through it.

JC: O.K., how do you know that it just touches?
MARK: Because both of these go through there... so, it's 0, but it never gets positive?

Though very procedural, his conversation indicates a good understanding of the relationship between the lines and the curve. After this, we did one more example of building a second degree polynomial on the computer. I did this to make sure that Mark was still able to predict the curve of a product of linear expressions of this type.
Looking at the first graph (Figure 25), and stopping on the x-axis, I asked Mark what was "going on with the curve or the function". Mark changed his answer quite a bit from the previous interview.

![Figure 25: Graph #1 - PC Emulation Software](image)

MARK: Then you take that and you put $x \ldots$ what is it ... I don't know ... about 1.8? ... so you could go $x - 1.8$ and multiply it by these to get the equation.

JC: Multiply it by those, what do you mean?

MARK: Multiply it by whatever that is, [he points to the other intercepts] ... so I guess $V_2$ and $1 \frac{1}{2}$

JC: O.K.

MARK: So, you'd have $x + V_2$ and ...

JC: O.K., what are those things called that we're talking about.

MARK: Factors? ... x-intercepts, too.

Though, he now connected the x-intercepts to the equation, he was still a bit "fuzzy" with the distinction between the terms "factor" and "x-intercept". (As we moved along the curve, Mark referred to another x-intercept as an "x-intercept slash factor".)

MARK: Was that right? ... or did you change these like ... if that's - 1.8 ... would it change it to an $x + 1.8$? ... or ...

JC: Yeah.

MARK: O.K. I wasn't sure if you had to change the signs ...
JC: Yeah, it's $x$ minus ... because that's the place where $y$ is 0 so it would be $x$ minus whatever that is ... call it $h$.
MARK: Yeah.
JC: And when you plug in the $x$-value, $y$ would be 0.

Mark agreed with my brief explanation, but just by the shaking of his head, so I remained unconvinced that he really was making the connections between the factors of the equation and the fact that the $x$-intercepts were the zeros of the polynomial. We also discussed briefly whether the product of these three factors would definitely be the equation of the polynomial.

MARK: Let me see ... no ... well, maybe [laugh].
JC: Yeah, it could be.
MARK: Yeah, it ... you might have something added on to the end, maybe?
JC: um ... or a constant out front.
MARK: Oh, yeah ... or a stretch factor, yeah, that's right.

In hindsight, I should have added at this time he could find the exact equation by substituting the coordinates of another point on the curve.

I continued to trace along the curve and then stopped at a turning point.

Mark referred to this point as "the vertex of one of the peaks". At the $y$-intercept, Mark said:

MARK: Ummm ... that's ... like when you have the ... ah ... that's this times that times that ... all the $x$-intercepts combined ... when you take out the stretch factor.

Mark had noticed a relationship between the $x$-intercepts and the $y$-intercept, as the product of the constant terms in the linear expressions. Though this way of getting the $y$-intercept would give the right value in some cases, it would not work all the time.

After we finished tracing along the cubic, I changed to graph #2 (Figure 26). I asked Mark how this curve differed from the cubic.
MARK: Well, there's only two x-intercepts and so it's only got two factors... unless like it turns up there... or... no, it wouldn't... let me see... it's got two x-intercepts, so it's got two factors... but it has no imaginary factors like the... if the... swivel had a ah... had only two it would have one imaginary? It's a second degree function... and the other one's a third.

I began to trace along the curve and stopped at the x-axis. Unlike the previous interview, Mark could give some information about this point with respect to the equation of the parabola.

MARK: Um... that would be a factor and you go x minus whatever that number is times x minus whatever the other number is and... to get the equation.

JC: So, if I asked you to find the equation of this parabola... suppose you didn't know what it was... um... what would be your way of finding it?

MARK: [He grabs a piece of paper and writes down his thinking.] Um... I'd take... I don't know... it looks like 1.5?... for the factors, + or -... so I'd go (x - 1.5) times (x + 1.5), and multiply them out.

Mark used the x-intercepts to find the equation rather than a translation of the basic parabola as he had in the previous interview. I sensed that by this time he knew that I was focusing on these x-intercepts and the "factors" of the quadratic,
so that was probably what I wanted to hear. I asked Mark if he remembered how
he had found the equation the previous time.

      MARK: I went to the vertex, and did the ... a(x-h) + k [sic]
    JC: You went to the vertex, and that gave you the equation ... and
    that does ... the vertex will give you the equation ... um ... but now
    you're going to the x-intercepts.
    MARK: Yeah. Well, I still kind of like the h and the k one ... it's a little
easier ... but ... for me, I don't know ... if I know ... the vertex.

He referred back to the way he had done it this time.

      MARK: When you do that ... um ... does that automatically give you
      like where the vertex is? ... or ... 

I sensed Mark was trying to see the connection between the various forms of the
quadratic equation. On a piece of paper, I showed Mark how you'd still have to
multiply the factors to get the quadratic in standard form and then go through the
process of “completing the square” to get the quadratic in the form a(x-h)^2 + k.

After showing him this procedure, Mark said:

      MARK: But, ah ... what I mean is when you take the factors and you
      multiply them out ... does that automatically give you that? ... 
      probably ... I just didn't know if you had to like subtract anything.

Now it seemed that Mark seemed to be comparing the two forms and seeing a
product of linear factors in both forms, but noticing the “+k” at the end of one of
them. He clarified what he was trying to say.

      MARK: O.K., um ... if you take that, and says there's a 1.5, you go x -
      1.5 times x + 1.5, and you get the answer for that ... once you get that
      answer, will you have to subtract 2 to make it move down? ... or will it
      automatically do that?
    JC: The only thing that you don't have by just multiplying these two
    together is how wide it is ... you don't have the stretch factor out
    there.
    MARK: Oh ... oh, yeah ... so you have to be an a in front of it, yeah.

I illustrated with an example to show Mark that once the product of linear factors
was multiplied by the “stretch factor” it was equivalent to the other form.
For the y-intercept, Mark returned to the idea of it being the constant at the end of the quadratic expression: "You subtract it off the end of whatever you have for an equation . . . you usually add it on . . .". He was referring to the y-intercept in this case being negative.

Moving ahead to the graph of the line (Figure 27), I asked Mark how he would find its equation. His method of determining it had not changed from the previous interview.

![Figure 27: Graph #3 - PC Emulation Software](image)

MARK: I would go . . . find the slopes, so it'd be . . . let me see . . . these are wider, so it'd be 1 . . . and then minus 2 . . . so it'd be x - 2.

I asked if he could find the equation using the x-intercept instead.

MARK: um . . . x-2? . . . let me see . . . yeah, cause it's just like the parabola, it's a factor, so you'd go, x minus the factor, like you do for all the other ones . . . so it's x-2.

Even though he again blurred the terms "x-intercept" and "factor", he was able to connect this task to the way he got the equation of the parabola.

Probing his understanding further, I wrote on a piece of paper the slope-intercept form of the equation for a line, \( y = mx + b \).
JC: Is there a way we can rewrite this so it's the slope, x-intercept form... do you think?

MARK: x-intercept form... rewrite this one like in terms of mx and b?... um... let me see... [laugh]. You could have like... could you do... just switch it around?... x=my+b or my-b or something?... or do you want it with y equals?

JC: Yeah, I want it with y equals.

MARK: Um... well, as soon as you take the slope out, you get the x-intercept, like, as soon... as soon as you like factor out the m from the b... but I don't know how to write that, like... oh, yeah, it's... um... divided... b over m.

JC: Yeah. So, you have y=m(x+b/m) and um... what is b/m?

MARK: b/m is the x-intercept... oh, the negative x-intercept.

JC: Yeah, right... and that's the slope, so that's like slope, x-intercept form.

MARK: Yeah.

We did this again on paper with another linear equation.

After analyzing Mark's responses in this last interview, I was able to make the following conclusions about Mark's interpretations of the x-intercept as shown in the model (Figure 28).

Acquired: a - There was no evidence of change in this interpretation of the x-intercept.

Tentative: a, c - There was no change evident for interpretation a, but for interpretation c, there was a slight change. I described the change as tentative because I am unsure whether the understanding was solely procedural or if there was an understanding of the concepts involved.

**Journal entry.** Mark wrote in his journal after our last session together:

Our last meeting was weird. I didn't know that what I was learning during our other meeting affected that but I guess it did. Good luck on your dissertation [sic]!

Is that spelled right?
Summary

I described Mark as confident. He enjoyed subjects that were "easy" or "fun", and he indicated that though he generally disliked mathematics because it was "hard", he could "get it after a few minutes". A "B" student, he preferred things that you could "see" and algebra did not fit into that category. He could see geometry and admitted that he had enjoyed that more. On the whole, he viewed mathematics as symbolic, "mostly" logical, and a way to get answers to problems.

Mark readily separated functions from non-functions using the vertical line test or by thinking in terms of a "one-to-one ratio". He was adept at picturing the graphs of linear and quadratic functions given the algebraic representation. Unlike the other students, Mark seemed to have the perception that all parabolas were "basic" parabolas. He sometimes reversed the x and y coordinates, but quickly corrected himself. He equated "factor" with x-intercept, and held on to this perception throughout the entire study.

Describing graphs was not a task he was used to, he struggled with this and failed to bring up commonly used terminology such as "x-intercept" or "slope". He noted in his journal entry that it was easier to give equations then describe them.

Mark tried to draw the connection between the vertex of a parabola and the turning point of the cubic, linking the coordinates of the turning point to the form $y = a(x-h)^2 + k$. In addition, Mark made the observation that the y-intercept was the constant term, "added on at the end", for all classes of polynomial functions. He seemed to view the constant term as a result of a translation of the curve "up" or "down". For Mark, the x-intercepts were the points that made $y = 0$ when they
were put into the equation. He did not relate the x-intercepts to the factors of the equation for any of the classes of polynomial functions.

Mark knew the product of two linear expressions would be quadratic but in order to know anything more about the graph of the parabola, he thought that he would have to multiply the linear expressions out and then change that product to the form \( y = a(x-h)^2 + k \) to determine its position. He viewed parabolas in a global fashion, focusing on the position relative to the axes. The x-intercepts were not prominent, and he did not focus on their placement until told to do so. Finally, with surprise, he realized that the lines had the same x-intercepts as the parabola.

After the teaching episodes, Mark was able to give the factors of the cubic and quadratic polynomial functions by seeing the x-intercepts on the graph. It was by taking the product of these factors that he would obtain the equation of the polynomial function for both classes. He also referred to the y-intercept as the product of the x-intercepts.

Mark did not change his method of obtaining the equation for a linear function. Though, he still used \( y = mx + b \), he was able to describe how this could be written in slope, x-intercept form and show how this form could be obtained from the graph. The teaching episodes shifted part of Mark’s attention to the x-intercepts on these curves.
CHAPTER VIII

LISA

Lisa, a junior in the Algebra II class, was the socialite of the group. She loved to talk and laugh. When the students were allowed to work together on problems, she was quick to move the desks together and chat. She waved her hands around a lot when she spoke, always with a lot of enthusiasm. Tall and attractive, her long blond hair was often stylishly braided and her general appearance very neat.

Attitudes Toward School and Mathematics

Lisa told me in the first interview that in addition to Algebra II, she was taking Chemistry, English, Spanish, and History. She admitted that she liked school.

JC: What do you like most about it or least about it?
LISA: um... Well, [laugh] least about it I like math! [laugh] I don't like... I don't know... I've always had problems with math, but English and like science... that's what I like the best and after school activities I like a lot... the math... I've never been good in math.

I asked her why she liked the English and science best.

LISA: I like to write. Then in science, I like the physical... like... I like Chemistry better than I like Biology... more like the... experiments and stuff.

Lisa had not felt a lot of success in mathematics evidenced by her statement that "I've never been good in math" and indicated it was the subject she liked least. I found out that these feelings were primarily focused on the mathematics
she had had since seventh grade when I asked her to talk about her experiences with "math" in elementary school.

LISA: Well, I was always... I was good in math except for fractions... until I got in like seventh grade, we didn't have a math teacher, so... well, we had a math teacher but she didn't... she... was... um... old?... and our class was really bad... we had made the teacher before leave... the teacher the year before.

JC: Oh, swell. [laugh]

LISA: [laugh] So we had this other teacher this year and she just was like... basically... we had homework and stuff, but we didn't have to do it unless we needed the... well, she never like corrected it or anything, so if you could figure out how to do it for the test and you could still get A's, so that's basically what I did... I never... I don't think I ever learned like the basic ways how to do some things... so I think that's kinda screwed me up [laugh]... and then eighth grade we had a good teacher... but he taught us like mostly what I know about math now... I learned from eighth grade... freshman year.

JC: What did you learn in eighth grade? Do you remember what you learned?

LISA: We did some pre-algebra... like mostly it was pre-algebra stuff... and he tried to kinda teach us what we didn't learn the year before [laugh].

Lisa, therefore, seemed to attribute a lot of her current difficulty with mathematics on the teachers she had in the sixth and seventh grades. She thought that there were gaps in her learning because of these teachers and thought those gaps caused her difficulty now. I imagine that this belief left Lisa with a fatalistic attitude that understanding certain portions of mathematics, perhaps algebra, was now out of her own control, i.e. the damage had been done.

Later, when I asked Lisa to describe mathematics, she responded:

LISA: [laugh]... what mathematics is?... hard [laugh]... um... confusing... well, I just... I don't like it because there's one set way to do everything [laugh]... if you make one mistake it's too... I don't know... specific... um... not creative enough, I think... bland... probably what mathematics is to me. [laugh]

Lisa's idea that "there's one set way to do everything" also seemed to be a cause of her dislike of mathematics. I noticed throughout the study that her fear of
making a mistake often seemed to keep her from persevering and from thinking clearly. I believe this fear also contributed to her sense that she was unsuccessful at mathematics.

Lisa went on to talk about Algebra I, which she had received "A's and B's" in.

Lisa: um... Algebra I, I did O.K. in... I had Mr. Smith, and he's a good teacher, but I don't know... I didn't learn from him that well, cause he's kinda... um... he's really smart and he expects you to know... kinda... [laugh] he used to yell at us, "You know this! Don't be stupid!" [laugh]... and a lot of us didn't... like, even now... he taught us some different methods, so she's [Mrs. Tucker] like... like Ryan's method... we'll be factoring and I always use Ryan's method and everyone else is like "who?"... "what is that"?... me and Doug are the only ones who know.

Out of curiosity, I asked Lisa to explain this "Ryan's method" before we continued, as I had never heard of it either.

When Lisa was asked about her experiences in Geometry, it was evident that she preferred geometry over algebra.

Lisa: I like proofs. I don't know why I liked it, but... once I got back to doing some algebra... like algebra... I said "Oh, my God".

Lisa said that she spent about two hours per night on homework, but tried to get a lot done in study halls as she ran track after school and also worked.

I asked how her parents were in math and whether they helped her with it.

Lisa: Well, my Dad helps me more than my Mom... my Mom's an English major and she always hated math [laugh], so I think I'm more like her... so she'll help me on my English, but math, my Dad knows... probably, is better at figuring... he's better at that... so... he'll help me with my Algebra if I want.

I wondered to what degree her mother had influenced Lisa's dislike of mathematics and had swayed Lisa into believing that she had "never been good" at it.
Lisa did not have a graphing calculator of her own and did not have a computer at home. She did enjoy using the graphing calculator in school, however.

**LISA:** They help me a lot . . . seeing them . . . I don't know . . . I can . . . it's easier to visualize, definitely, when I can see it on the calculator in front of me.

I asked Lisa if she had any plans after graduation from high school.

**LISA:** I'm going to college . . . I don't know for what . . . probably something in science, but I don't know what . . . I have no clue.

**Definition of Mathematics**

I asked Lisa to define "mathematics".

**LISA:** Adding and subtracting [laugh] . . . multiplying . . . um . . . I don't even think of Geometry as math as much as I do Algebra.

Her view of mathematics was apparently doing manipulations on symbols. Lisa was focusing on the "action" or "doing" aspect of mathematics. An "action" conception of mathematics, as described in Chapter III, emphasizes transformations rather than entire processes. Mathematics is viewed in a step-by-step manner, as a recipe.

I asked Lisa to describe Algebra. Her response again indicated a focus on the manipulations or action aspect of Algebra.

**LISA:** Like x . . . finding x and factoring, like a lot of ah . . . numbers . . . it's a lot of mental figuring.

I continued by asking about the aspects of Algebra she liked or did not like.

**LISA:** Like, I like word problems . . . I don't know just that . . . I don't know . . . other stuff.
**JC:** How about graphing? You've been doing a lot of graphing in Algebra.
**LISA:** [laugh] I don't know . . . I don't like graphing. [laugh]
**JC:** That's interesting because you like Geometry . . . do you see any similarities between . . .
Lisa: Ah, no... I... the Geometry... like in Geometry when we did graphing I was... it always confused me... the functions. I'm just starting now to understand functions... like I understand what we're doing now more than...

In working with Lisa, I knew that these attitudes towards graphing, functions, algebra, and mathematics in general, affected her self-perception of what she was capable of doing and would be a hindering force in getting her to be open to learning more in this area.

**Knowledge of Functions and Graphs**

I asked Lisa some questions that would investigate her understanding of function notation. Showing Lisa the graph of a cubic polynomial function, f(x), I asked her to point to the part or parts of the graph where f(x) was greater than 0.

![Graph](image)

**Figure 28: Graph #1, PC-Emulation Software**

Lisa: [Points to portions of graph.] Well, from like here and here, up here and then here... so... it would be if like you have like an and here and then or this. [She smiles with satisfaction in her response.]

She was also able to identify the section of the graph where x was bigger than 0.

When asked to point to where f(x) was equal to 2, she responded:
LISA: Hmmm . . . wait a minute . . . right here . . . f(x) is y . . . actually like here . . . I think.
JC: O.K., is there any other place where f(x) is equal to 2?
LISA: um . . . yeah . . . here and here . . . I think . . . I'll go with that.

She was able to answer these questions without a lot of hesitation. Because of her responses to these questions, I felt her understanding of function notation was adequate.

**Function Definition**

At the time of this interview, the class had just finished the unit on polynomial functions of degree greater than two, which were most often denoted by P(x). Within that context, I asked Lisa if she could tell me, in her own words, what a function *is*.

LISA: No. [laugh]
JC: Think about it for a second and see if you can explain to me what a function is.
LISA: Oh, boy . . . um . . . all I can think of is P of x [laugh] . . . um, like the function is . . . doesn't it equal y? P of x is y . . . and that's always confusing to me because how can x be y? It's not logical to me.
JC: What do you see when someone says the word “function”? What comes to your mind?
LISA: P of x [laugh], that's honestly what comes into my mind.

Lisa really didn't want to think about functions, and this was an example of when I felt her fear of being wrong seemed to affect her ability to think clearly about the question.

**Ability to Identify Functions**

Lisa was given Task #1 (Figures 29-33) and was asked to identify which of the thirteen items were functions. Focusing her attention on a task, and away from my questioning, she was able to verbalize her thinking about the function
concept more clearly. In looking at the first item on the sheet (Figure 29, #1), she commented:

LUISA: um, Let's see... I think that's a function because... um... there's one point... oh, wait a second... I remember what's a function... there's, um... wait a second... one... one... one y for every x?... something like that [laugh]... it's coming back... uh... all I know, if you draw a line, it won't hit two points [laugh]... it will only hit one.

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Figure 29: Task #1 (1-3)

She claimed that the second item (Figure 29, #2) was a function, because she assumed it was a line, and the graph (Figure 29, #3) was "definitely a function" because "regular straight lines won't double back". She claimed that the graph in #4 (Figure 30) was a function, for the same reason. Like Mark, she seemed to have alternate ways of viewing the vertical line test; envisioning lines that will "only hit one" point or having curves that "won't double back". At this point, I noticed that Lisa seemed to find it easier to visually identify functions in graphical form.
There was about a fifteen second pause before Lisa gave any response to the table of values in item #5 (Figure 30). Her response indicated her confusion in applying her "one y for every x" rule to this set of points.

**Lisa:** Number 5 is a function because you know... it's like a curved line, but I don't think there's any points where if you drew a line, it would intersect both.

There was a similar long pause before giving an answer for the set of points in #6 (Figure 30). Again, Lisa did not recognize that the last two points in this set of ordered pairs were on the same vertical line graphically. I believe she was focusing on the constant difference in the y-coordinates as she responded:

**Lisa:** That's a function, too, because it's straight.
**JC:** If you had a set of points like that... is there any way it wouldn't be a function or would every set of points be a function?
**Lisa:** Well, if you drew a line and it connected... oh, boy... I don't remember [laugh]... if you drew... I think if you drew a line and they went like number 8... number 8, I know, is definitely not a function... cause if you draw it, it would intersect...
Lisa's first comment on the equation in #7 (Figure 31) was:

LISA: I have no clue... I would have to do it... it would take me a
long time.

Explaining, Lisa points out her need to get this equation in the form
\[ y = a(x-h)^2 + k \]

before concluding whether or not it was a function.

LISA: I have to put it in the other form.
JC: What other form?
LISA: The a times x - h + ...
JC: So you know what that is?
LISA: That's... um... a function... I think.
JC: Any particular type?
LISA: Linear?... Quadratic?... Quadratic, no... yeah... I think...
I don't know [laugh]... I'm so confused. I hate functions!... like my
worst part! I think on the final... the one I did... on the midterm,
that was like the part I did worse on... functions.

I am sure that I was contributing to Lisa's "function anxiety" by asking her these
questions. Lisa, in this dialogue, was not clear of the accepted association between
the degree of the polynomial and the name given to the function. This is
evidenced again by her comments for the ninth and twelfth items. In #9 (Figure
31), she claimed that the quadratic equation was a function, because "It's a line, I
think", and in #12 (Figure 32), the graph was a function because "It's kinda like a
swivel... like a 4th degree quadratic... something like that".

Lisa went on to say that the graph in #8 (Figure 31) was not a function
because "for every y there are two x's... I think... or is it the other way
around?* She could visualize what she was saying, but was confused about the algebraic interpretation in terms of x and y.

\[ f(x) = 5x - 3 \]

Figure 32: Task #1 (10-12)

For item 10 (Figure 32), Lisa responded without hesitation that the graph was a function because “It's a parabola.” For #11 (Figure 32), Lisa identified the equation as a function, because it was a line and also identified the table of values in #13 (Figure 32) as a function because “That's a line”.

JC: How do you know that's a line?
LISA: Well, cause they have the same like . . . going up at the same increments . . . the same, um . . . what's the word I'm looking for? . . . I can picture the graph too . . .

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Figure 33: Task #1 (13)

Like the other students I interviewed, Lisa sometimes groped for the appropriate terminology. She knew what she wanted to say, because she could picture it, but
didn't know how to put it into words. At least in part, this could be attributed to nervousness and/or not having to verbalize the concept on a regular basis in class.

**Describing Function Attributes**

Lisa was handed the "little book of graphs" that comprised Task #2. She was asked to describe each graph in sufficient enough detail to guide me to "draw it" correctly.

![Figure 34: Task #2 (1-3)](image)

For the first graph (Figure 34, #1), Lisa gave me the coordinates of the y-intercept, followed by the coordinates of the x-intercept, and then asked me to "draw a straight line between them . . . and go both ways".

To describe the second graph (Figure 34, #2), she declared, "It's a parabola", and began looking for the coordinates of the vertex. Because the graph was a little off-center, she remarked that she couldn't " . . . tell where the vertex is" and she proceeded to give the coordinates of the two x-intercepts and the y-intercept instead.

For the parabola in #3 (Figure 34), she noted that "it bounces off the . . . touches the x-axis" and then gave the position of the "x-intercept", which appeared to be the spot where the graph "bounced off". After giving the value of
the y-intercept, she showed me with her hands how the parabola was "negative". This characterization of an inverted parabola as "negative" is used by Beth also in the next chapter.

Lisa referred to the fourth graph (Figure 35, #4) as "a swivel... with a point at the origin". Feeling unable to describe it any better, she said "I'll come back to this one". After she finished describing the other graphs, she did return to this graph, again using her hands to illustrate:

LISA: It starts like at the top of the y... kind of... oh, wait a second... this could be a way... let's see... at the point (3,5) is kind of where it starts and it kind of swivels down through the origin and... kind of keeps going.

Looking at the fifth graph (Figure 35, #5), she said:

LISA: "Oh, boy, I remember doing these... but I don't remember what they are called. Oh! I think that's when we did the and and or thing? I'm not really... no, that was the number line... these are different...".

Looking at this graph she made some connection to the work she had done in graphing inequalities on the number line. Rather than continuing to explore and glean from these observed connections, she jumped back to the coordinate axes as though deciding that the two were unconnected concepts. She gave the
coordinates of the y-intercept referring to the point on the graph as a "closed circle" and then said "It's kind of diagonal".

LISA: And then there's another line where the . . . um . . . point does not include . . . the starting point does not include the . . . does not include the y-intercept at 2 . . . -2, I mean . . . and that one goes a down diagonal.

Using her hands, she described item #6 (Figure 35) as a graph that starts at an x-intercept of 2 and "kind of gently slopes up and to the right".

LISA: . . . looks kinda like a parabola, but . . . um . . . it looks like it's missing the negative half, so it could be for like a word problem . . . when you need only the positive value.

Her dialogue indicated that she was making connections to other work she had done. She seemed to make sense of these last graphs by connecting them to other experiences she had with other types of graphs. As the theory of conceptual change suggests, Lisa was interpreting new tasks through existing information received from prior experiences.

Translating to the Algebraic Representation

While still looking at the "little book of graphs", I asked Lisa if she could determine the equation of the line in #1 (Figure 36).

LISA: um . . . It would be . . . um . . . oh, slope . . . all that stuff . . . um . . . you find the slope . . . which is like what . . . -1 over 3, I think . . . and then you have your . . . it's a linear function . . . so, I don't remember . . . I hate those . . . I'll go on to the next one.

She was only able to partially give the equation. As in this instance, when Lisa got confused or "didn't remember", she would often pull away from the problem as quick as possible without trying to deal with it.

When asked about the equation for the parabola in #2 (Figure 36), she said:
LISA: It is like a third degree... isn't it? [She laughs.]... a third degree function... quadratic kinda... I think.
JC: O.K.
LISA: Like the a times x-h...

Figure 36: Task #2 (1-3)

Speaking about the equation of the inverted parabola in #3 (Figure 36), she remarked:

LISA: The a is negative so that it flips it over... [She uses her hands to illustrate] and then... with the same like equation [as #2]... and like the, um... yeah... it has a regular stretch factor it seems... like no stretch factor and stuff... the regular basic parabola... seems just like the -a and moved over to the right... so, +h has a... is that what it is?... I think... yeah... no k.
JC: You said that's third degree?
LISA: I think so...

In this dialogue, Lisa referred to the "basic parabola", as "a regular parabola".

Another detail to note is that Lisa referred to a stretch factor of 1 as having "no" stretch factor.

Moving on to discuss the equation of the "swivel" in #4 (Figure 37), Lisa said:

LISA: ... "swivels" are... 4th and 5th?... swivels are 4th degree?
I'm not sure...

This was all the information she offered about the equation. For the graphs in #5, and #6 (Figure 37), she didn't add anymore about the equation than she had when describing the graphs (refer to last section).
As stated previously, Lisa was quite confused with the terminology used for the varying degrees of the polynomial functions. Though it seemed she was able to picture the graph, she didn't connect the accepted terminology with its graph.

**Journal Entry.** At the end of this first interview with Lisa, she wrote in her journal:

I felt a little nervous and was afraid to give the wrong answer. I felt more and more comfortable as we went along, but I was still apprehensive [sic]. The camera didn't bother me, but my fear of saying something really ignorant did.

Her journal entry confirmed what I had expected, her nervousness and fear of making a mistake interfered with her ability to think clearly. In addition, it gives credence to her previously stated perception of mathematics - that "there's one set way to do everything".

**Understandings of Polynomial Functions**

The second interview was designed to obtain a deeper look into Lisa's current understandings of the graphs of polynomial functions. *PC-Emulation Software* (Texas Instruments, Inc., 1991) was used in the second interview to show the graphs of linear, quadratic, and cubic polynomial functions. This
activity followed a class test on the unit on polynomial functions for which Lisa received a 74%.

As was done for Mark's case study, the discussion of this interview is organized around the first three research questions:

1. What connections do the students make between their study of the graphs of linear and quadratic functions and their study of the graphs of polynomial functions of degree greater than two?
2. What evidence suggests that the students' conceptual development during instruction on polynomial functions of degree greater than two builds on their understanding of the graphs of linear and quadratic functions and their general knowledge of functions and graphs?
3. What factors can be identified that contribute to/inhibit the student from making the transition from the graphical representations of linear and quadratic functions to those of higher degree?

The section headings correspond to the focus of these research questions: "Connections Between the Classes of Polynomial Functions" (questions 1 and 2), and "Contributing/Inhibiting Factors to Making the Transition to Polynomials of Degree Greater than Two" (question 3).

Additionally, as this interview progressed, I considered appropriate activities to use in the forthcoming teaching episodes that could enhance the students' understandings of the graphs of polynomial functions based on the understandings I was perceiving through the interviews (research question 4).

Connections Between the Classes of Polynomial Functions
Starting with the graph of the cubic polynomial function (Figure 38), which Lisa noted was the graph of “a cubic”, I traced along the curve beginning at the leftmost part of the graph, and asked Lisa what she could say about various locations on the curve.

![Graph #1, PC-Emulation Software](image)

Figure 38: Graph #1, PC-Emulation Software

At the first turning point, Lisa noted,

Lisa: Well, we've hit a turning point and um... it's not really a vertex, it's not a parabola... it's a turning point! I remember that! [laugh]...
and... um... it's... ah... (1,-3).

Lisa often confused the signs of coordinates, but usually recognized her own error as she seems to have done when plugging this vertex into the equation expressed in the following dialogue.

JC: Does that have anything to do with the equation?
Lisa: This is um... when... oh, well, like equations... do you mean like the standard equation or the working equation?
JC: What's the difference?
Lisa: Well, one is the a times x minus h plus k and then the other one like the... quadratic one... actually you know... this is like a is um... it would be like a times x+1 then +3... well, in parentheses, +3... I think, something like that... for the vertex.
It was interesting that she had classified the “working equation” \( y = a(x-h)^2 + k \) as a third degree polynomial in a previous dialogue, but referred to the “standard equation”, \( y = ax^2 + bx + c \), as the quadratic one. Though we were dealing with a cubic she was using these 2nd degree forms. It was unclear whether she considered these two forms of a quadratic equation to be equivalent. She continued to use the “working equation” when we got to the other turning point and also referred to it as a “kind of a vertex”.

Lisa seemed to view the graphs of the linear and quadratic functions as isolated entities, and disconnected from the other graphs of polynomial functions. The only connection that I observed Lisa making between the three types of graphs was that for each curve the value of the x-intercept made \( y = 0 \) in the function’s algebraic representation and the y-intercept made \( x = 0 \) in the algebraic representation. I would characterize Lisa’s understanding of the x and y intercepts as an action understanding. She understood the action of taking the value of the intercept and plugging it into the equation to return 0 for the other variable. When I moved to the x-intercept of the line and asked about it, Lisa described the action of finding the x-intercept given the equation.

LISA: The x-intercept, \( y \) is 0... um... in the equation... well, you just figure it out. If you want to find the x-intercept... you just put 0 in for \( y \) and you can figure it out.

Likewise, when I stopped at the x-intercept while tracing along the graph of the cubic, she commented:

LISA: Well, it would be the x-intercept, so, the \( y \) is 0 and um... the x is still negative... I don’t know like what else [laugh].

Arriving on the y-axis, Lisa said, “Well, y-intercept and... the x is 0...”. I received basically the same responses moving along the curve of the quadratic
function. With all the students I interviewed, speaking of the y-intercept as where x = 0 seemed very well rehearsed. However, unlike Mark, Lisa did not seem to connect the value of the y-intercept to the constant in the equation.

**Contributing/Inhibiting Factors to Making the Transition to Polynomials of Degree Greater than Two**

Lisa's transition to cubic polynomial functions, I believe, was hindered by the lack of connections that Lisa made between the graphs of cubic functions and the work done with linear and quadratic functions.

Stopping at an x-intercept on the graph of the cubic, I asked:

**JC:** Does that have anything to do with the equation?
**USA:** Yeah, it does, but I don't know what.

She added at the next x-intercept:

**LISA:** I'm trying to remember . . . all I can think of is that synthetic stuff and that has nothing to do with it . . . really, it has something to do with it, but not really . . . that much.

Looking back at the classwork in this unit, I was reminded that the students were given very few activities for homework problems and on quizzes and tests in which they were asked to obtain the linear factors of the polynomial from the graph when compared to the number of problems in which they found linear factors given the roots algebraically.

Investigating her understanding of the word *factor*, I asked:

**JC:** Do you know what the . . . what some *factors* of this equation would look like?
**LISA:** um . . . probably . . . about √3, a little less than √2, I don't know . . . then like 1⅔ maybe . . . that's just a little over 1⅓.

As Mark had done, Lisa equated the x-intercepts, or the roots, with the word *factor*. Being very suggestive, I directed her attention to the x-intercepts of
the graph and I asked if she had other names for those points beside "x-intercepts".

Lisa: Those are zeros... those are also... they're factors and they're zeros of the function.
JC: How about the word roots? Do you ever use the word roots?
Lisa: Yeah, those are... those are what the... um... the basic equation... they make the equation... true... the answers.

It was not clear from this answer if she also connected roots, or "the answers", with the x-intercepts.

When I put the graph of the quadratic function in the window (Figure 39), Lisa reacted with relief, "It's a parabola!", glad to get away from my questioning on the cubic.

![Graph](image)

**Figure 39: Graph #2, PC-Emulation Software**

JC: You like that graph? [laugh]
Lisa: I like it better than the last one. [laugh]
JC: What type of equation do you have for this graph-I should say, function?
Lisa: [Hesitantly] It's a quadratic function.
JC: So would this point... that point on the x-axis [pointing to an x-intercept]... have anything to do with that?
Lisa: Yeah, um [nervous laugh]... I'm trying to think of what... but I don't remember.
Again, feeling frustrated, she sort of pushed this question away with an "I don't remember". Moving to the vertex of the parabola, which appeared to Lisa to be close to, but not the same as, the y-intercept, I asked if that point had anything to do with the equation.

LISA: Yes. Well, the quadratic equation thing . . . the $h$ is . . . wait . . . the vertex, something . . . yeah, it does . . . um . . . If you're given the equation you can figure out what the exact vertex is . . . the $b/a$ and stuff . . . it would be $b/a$ and the $y$-value would be $b^2-4ac/4a$ . . . would be the $y$-value to figure the vertex out.

Her understanding seemed dependent on remembering how to do these actions. Lisa seemed to flip back and forth between what she had called the "working equation" and the "standard equation". She now remembered that the vertex was related to both of these algebraic forms, but I still wondered whether she realized that these algebraic forms were equivalent. Again, she was focusing on the manipulations necessary in solving this problem, what you had to do in order to "figure" it out, i.e. the actions.

I asked Lisa if she knew what the factors of the quadratic equation would be.

LISA: Factors. Um . . . It would be -2 . . . try -2 and +2 . . . you'd try them both . . . I don't know if they'd work . . . actually wait, wait . . . [points to tick marks on x-axis] . . . I'm thinking that was a line in there [laugh] . . . probably 1 and -1 . . . and I don't know . . . go with $\sqrt{3}$ and probably $+1\frac{1}{3}$, so they'll all be the same.

Again, she seemed to equate the term factor with the x-intercepts. She then added, however, that you would have to "try them both" to see if they worked. I assumed, perhaps incorrectly, that she was referring to the standard form of the equation. This is another instance when I wished that I had inquired more deeply into her thought processes.
Lisa was getting pretty frustrated with this whole interview. When I changed the graph to a linear function (Figure 40), Lisa exclaimed, “Ah! A linear function!”, now glad to be rid of the quadratic function.

![Graph #3](image)

**Figure 40: Graph #3 - PC-Emulation Software**

JC: O.K., which do you like the best of the three?
USA: Probably the linear function [laugh].

Tracing along the curve from the left, I stopped at the y-intercept.

JC: Does that have anything to do with the equation?
USA: Not really, well . . . I don't know.
JC: If I asked you to find the equation of that could you?
USA: Um . . . Probably . . . I think so . . . if I can remember. I think so. If I had time . . . if I remember how . . . [pause] y = . . . y = x + 2 . . . I don't know . . . Yeah, I think that's . . . something like that.
JC: How are you getting that?
USA: Actually . . . y = x-2 [laugh] . . . well, cause, like, um . . . when x is like 2, y is 0 and when x is . . . yeah, it's x-2 [laugh] . . . when y . . . I figured it out just by doing that one. [She waves her hand at the graph.]

Lisa had not been comfortable using the slope-intercept form, \( y = mx+b \), to find the equation through any of the interviews. She hadn't used it in this instance either, as far as I could tell, but justified the equation from the x-intercept.
Below is the diagram entitled "Interpretations of the x-intercept" that was more fully described in Chapter II. As indicated in that chapter, this study was designed to investigate only interpretations a, b, and c. From the comments Lisa made above, I would classify Lisa's conceptions of these interpretations as:

Acquired: a, b - Lisa indicated that an x-intercept as seen on the graph makes the function value equal to 0 and in testing the suitability of the equation of the linear function y = x - 2 had used both coordinates of the x-intercept, (2,0).

Missing: c - Lisa did not give evidence that she made the connection between the x-intercepts, as seen on the graph, and the linear factor(s) of the polynomial function.

Figure 41: Interpretations of the x-intercept

*Journal entry.* At the end of this second interview, Lisa wrote in her journal:

I still felt nervous, but it was better than the last interview. I was confused as to exactly how you wanted me to describe the graphs.
As for Mark, I sensed that this was the first time Lisa had been asked to "describe" a graph. Both students felt awkward verbalizing these descriptions, probably thinking that I was looking for more than they were saying or for a particular response.

**Teaching Episodes: Forming Links Between the Classes of Polynomial Functions**

As discussed previously, the purpose of the teaching episodes was to strengthen connections between the classes of polynomial functions by building polynomial functions from products of linear functions. Made in the light of the preceding clinical interviews, the primary objective of this activity was to foster the connections between the x-intercept, as seen on the graph, and the factors of the polynomial, across the classes of polynomial functions. For the sake of comparison of the effectiveness of these teaching episodes on the students, the same activities were used for Lisa and Mark even though their understandings of the graphs of polynomial functions as gleaned from the first interviews were different.

**Building Quadratic Functions**

Using *The Function Supposer: Explorations in Algebra*, I asked Lisa to type in two, simple, linear functions. These two functions, \( f(x) = x - 2 \) and \( g(x) = x \), were graphed in the small boxes at the right of the screen (Appendix E). The graphs were also shown simultaneously in a larger window, above their algebraic representation.

JC: What do you notice about those lines?
Lisa: They're parallel.
JC: Parallel . . . and what else?
LISA: They have the same slope ... they're both positive slopes ... I don't know ...
JC: O.K., good. And what are other characteristics ... you can talk about each one separately if you want.
LISA: Well, g(x) is closer to the origin.
JC: And how about f(x)?
LISA: It's ... um ... two off the ... It's just 2 over, x-wise, positive actually ... not negative.

This explanation of the movement of the linear function \( f(x) = x-2 \) as a horizontal translation along the x-axis, rather than a vertical translation along the y-axis, substantiates the research on student understanding of linear functions done by the Berkeley Group (Moschkovich et al., 1993). Their research similarly indicates that students view a change in the y-intercept as a movement of the line along the x-axis rather than a movement along the y-axis. Probing deeper, I asked:

JC: How do you know that's 2?
LISA: Because of the um ... value for ... what is it, \( h \). Wait a minute ... I'm thinking ... I don't know.

Lisa was using the letter \( h \) to represent this translation of a linear function, corresponding to it's use when using the "working form" of a quadratic function, \( y = a(x-h)^2 + k \), in graphing a parabola. In this form \( h \) does represent a horizontal translation of the parabola. This was an interesting connection between the algebraic representations of linear and quadratic polynomial functions not observed in our initial interviews.

Continuing, I asked if she could predict what the graph of the product of the linear expressions for \( f(x) \) and \( g(x) \) would look like.

LISA: It would be ... um ... it would be a parabola, wouldn't it? ... I think?
JC: Why would you guess that?
LISA: Because of ... you'd have a ... um ... a squared term and then, just a regular x ... I think it would be a parabola.
JC: O.K., all right, do you know anything about that parabola? ... what it would look like?
Lisa determined that the graph would be a parabola by multiplying the linear expressions. When we graphed the product, I asked Lisa what she noticed about the parabola.

Lisa: Well, um... you can do the negative... you can do the ... 2, it brings it down, it brings the vertex down... and over... gives it the... um... gives it the 2 because of the stretch factor before the x... moves it over and down. Other than that, I know it's a stretch factor... looks like a stretch factor... it's just not the same... I don't know.

Concentrating her attention on the vertex and the "stretch factor" for the parabola, it was not immediately obvious to Lisa that the graphs had the same x-intercepts.

Judy: So, what else do you note about the parabola in relationship to the other two lines?
Lisa: The other two lines? Well, it goes through both of them.
Judy: It what?
Lisa: It goes... it intersects both of them.
Judy: O.K., where?
Lisa: Um... x at the origin... at... um... where both the lines go through the... um... at the x-intercepts of both the lines, the parabola also intersects.

It took a good nudge in that direction for her to recognize that the parabola had the same x-intercepts as the lines. By comparing the algebraic representations of the linear functions to that of the quadratic function, and by discussing how the product of two binomials could equal 0, we talked about why this was the case.

I decided to work through another example with her. This time she typed in the functions \( f(x) = x + 4 \) and \( g(x) = -x \). I again asked if she could predict what the graph of the product would look like.

Lisa: This time it will go down... it will be that way.
Judy: Why? How did you know that?
Lisa: Because you'll have a negative for your squared term, so... that's it.
JC: O.K., good... and what else do you think will happen with that parabola?
USA: Well, it probably will intersect down there. [She points to the x-axis.]
JC: Why do you think that?
USA: Cause it's the same thing we did the last time [laugh].

With Lisa's laugh, she knew that this certainly wasn't a very strong justification, but an easy way out. On viewing the graph of the parabola, she found that her prediction was correct.

At this time, I took a little strip of paper and covered the portions of the graph to the right of the leftmost x-intercept to illustrate, as I had done with all the students, how the product of the "signs" of the linear functions corresponded to the "sign" for that portion of the parabola above them. This again was to illustrate the connections and relationships between the classes of functions. Lisa was a bit confused about where the y-values were negative and where they were positive, although she had been able to find where \( f(x) > 0 \) on the cubic during the first interview. Like Beth, in the next chapter, finding positive values of the function seemed to be clearer when the function notation, \( f(x) \), was used in place of \( y \).

The third time that Lisa typed in linear functions, she let \( f(x) = -3x \) and she let \( g(x) = -x + 6 \). She again characterized the resultant parabola as "positive".

LISA: It should be positive this time... and it'll be, let me see here, and it'll be narrower, I think... cause it will have... instead of having 1 as a stretch factor, it will have a 3.
JC: O.K.
LISA: It should be over too.
JC: Why do you think it will be over?
LISA: Because of the... 6.
JC: Can you tell me where it will be?
LISA: It will be... let me see... I don't know... it will be in the middle though. [Laugh.]
JC: O.K., from what we did before, where will it be?
LISA: It will be... um... somewhere down here like that.
She was focusing on the global characteristics of shape and the position of the graph relative to the axes rather than on any specific points. The graph of this parabola required that we ZOOM out to see the vertex, as its second coordinate was -18.

The next time we built a quadratic function from two linear functions, I asked Lisa to type in two linear functions neither of which would go through the origin. She typed in \( f(x) = 2x + 3 \) and \( g(x) = -3x - 2 \). This time when asked to predict what the parabola would look like, she did talk about the positioning of the \( x \)-intercepts after making note of its stretch factor and that it opened down.

**Lisa:** It will be . . . um . . . it will go like that [shows with her hands]. It will be narrow, because the stretch factor will be like . . . 6, so it would be narrow and it will go up here and intersect in the 2 points [points to the points where the lines cross the \( x \)-axis].

In doing these examples and others that are not included in this summary, Lisa leaned more to the algebraic representation of the quadratic function in order to determine the position of the parabola than she did to the graphical representations of the lines.

In other examples, we investigated how the \( y \)-intercept of the parabola was related to the \( y \)-intercepts of the linear functions.

**Working Backwards: Finding the Components of a Quadratic Function**

The intent of the next component of this teaching episode was to strengthen Lisa's understanding of this concept of building quadratic functions from linear functions by working backwards, i.e. starting with a parabola and trying to find a pair of lines such that the product of the linear expressions could possibly give the quadratic expression given by the parabola. As the software was limited to taking the product of two linear functions only, I extended the idea to a
product of three linear functions using paper and an assortment of colored pens (Appendix E).

Handing Lisa a graph of a parabola (Figure 42), I asked her to try to come up with two lines so that the product of the corresponding linear expressions would give the expression corresponding to the quadratic function. Even though she only had a graph, she interpreted what I said as a request for the algebraic representation of the two lines. When this became evident, I rephrased the instructions.

JC: You don't have to give me the equations . . . but more sketch them.
LISA: Oh, just sketch them? Oh. That's easier than making lines.

As she worked, I asked her to tell me what she was thinking.

LISA: I'm trying to think of the . . . when you told me the . . . like the parabola is positive over here, so I'm thinking the y's have to be positive for both of the lines. [She erased the lines she had sketched in and sketched the lines shown in Figure 42.]

![Figure 42: Working backwards, Graph #1 (Lisa)](image)

When she had finished, I noted that the lines seemed parallel and asked Lisa if she thought that those were the only two lines that would have worked.

LISA: No. I think . . . um . . . I could like change the slopes.
Lisa claimed that the second graph I handed her was more difficult, but said confidently after she had done a bit of erasing and was done, “That works.” The lines she had drawn satisfied the conditions, i.e. they went through the x-intercepts of the parabola and the product of the signs of the function values of the lines corresponded to the signs of the function values of the parabola. I asked if she thought about the values of the y-intercepts.

Lisa: Um ... No. [Laugh] I didn’t.
JC: Remember what we talked about for that?
Lisa: Yeah ... the y-intercept of this ... is it multiplied by the y-intercept of that? ... and that gives you ... I think ... or added ... I don’t know.

Again Lisa seemed to care only about knowing what actions to perform rather than having a conceptual understanding of the relationship between the points on the graph. We reviewed why the y-intercept of the parabola was the product of the y-intercepts for the lines. She shifted the lines a bit so that the product of the y-intercepts came out to be -3. However, in making these shifts, one of the lines no longer went through the x-intercept (Figure 43).
During our next time together, we did another example of building a quadratic function from linear functions. For review purposes, we then went on to another example of working backwards from a parabola (Figure 44). This parabola was a bit different from the others we had done, as it only had one zero. I asked Lisa to tell me what she was thinking as she tried to find the component lines of the parabola.

Lisa: I'm thinking that the whole parabola's positive, so the lines have to be positive. And then there... I guess they're about 2, so...
JC: Do they both have to be positive?
Lisa: Or they can both be negative.
Lisa referred to the lines as "positive" which I interpreted, from the context, that their y-values were all positive. Lisa sketched in a couple of horizontal lines, $y = 1$ and $y = 2$. I asked her to multiply the expressions together and in doing so she realized that the resultant expression was not quadratic. Offering her another hint, I suggested that the lines she chose really didn't have to be different. When she still didn't make any progress, I prompted her to draw a couple lines through the same x-intercept. We then checked the lines against the parabola by comparing their "signs" and by comparing the y-intercepts.

The last parabola I showed Lisa (Figure 45) didn't have any zeroes. I asked Lisa what this indicated about the equation. Taking what I thought to be just a stab in the dark, Lisa responded:

LISA: It doesn't have that middle term?
JC: But I can have like $x^2 - 4 \ldots$ that does intersect at 2 and -2.
LISA: That's right \ldots I have no idea.

We discussed the meaning behind a graph having no x-intercepts, i.e. that the quadratic expression could not be factored into linear expressions (with real coefficients), i.e. had no real roots. Lisa commented, "I kind of understand." She did not express much confidence in this statement.

![Figure 45: Working Backwards - Graph #4 (Lisa)](image)
Extending Activities to Cubic Polynomial Functions

In the last teaching episode, we extended the process of building polynomial functions from linear functions to polynomial functions of degree three. Modeling a worksheet after the software that was used in the first teaching episode (Appendix E), Lisa was asked to write down three linear functions and graph each of them in the small boxes on the right of the worksheet. She predicted that the graph of the product of the three linear expressions would be a "swivel". She wrote down three linear functions, all three of them with a slope of one, and I asked her to predict the position of this swivel. She drew a graph that "swiveled" through just two of the x-intercepts, and when asked, said she "didn't know why" she had avoided the other one. To check her understanding, I asked how the product of the three expressions could equal 0. She answered, "Then either one's zero, or all . . . two are zero, or all three are zero". I added that when the value of x was equal to the value of the x-intercept in the linear expression that the value of the linear expression became 0. She adjusted the swivel to go through all three x-intercepts and then checked the graph, by my request, to see if the "signs" and the y-intercepts all corresponded.

When we finished with this example, Lisa commented:

Lisa: Actually, this makes more sense to me than the way we learned it in class.

I handed her another worksheet, and asked her to build another cubic polynomial from linear functions for the next interview. At the next meeting, when she showed me what she had done, she remarked that she couldn't remember why the
swivel went through the same x-intercepts as the lines. She had the actions down, but did not have a meaningful understanding of the concepts behind the actions.

**Assessment of the Teaching Episodes: A Reevaluation of Lisa's Understanding**

The primary purpose of this last interview was to detect any changes in Lisa's conceptual understandings of the graphs of polynomial functions after the teaching episodes. The same graphs that were used in the second interview, using the PC-Emulation software, were used again for this comparison of understandings.

I traced along the first graph (Figure 46) from left to right. As I started moving up the curve, I asked Lisa what was going on in that particular interval.

LISA: Well, there's three lines and they... um... have to be... the y-value has to turn out to be negative when they're all multiplied together right there.

![Figure 46: Graph #1, PC-Emulation Software](image)

Continuing the movement of the cursor up to the x-axis, Lisa added:
LISA: Well, one of the lines goes through that point and you told me it had something to do with it, and I still don't remember from the last time we did it two seconds ago [embarrassed laugh].

At the turning point, Lisa laughed as she found herself only commenting, "They're nice", when asked specifically about the function value at that point on the curve. Moving to the y-intercept, Lisa remarked, "The terms at the end multiplied equal that number."

Her comments about this curve had changed from the previous interview, accommodating many of the ideas developed in the teaching episodes. I recognized that the reason she perhaps chose to use the ideas of the teaching episodes regarding the points on the graphs was because this was the perspective she thought I wanted to hear. It was also the last perspective through which she had done these tasks. Therefore, it was the most prominent perspective in her mind. I was also aware that though she was able to picture the lines that made up the cubic function, her understanding of why the lines shared the same x-intercepts as the cubic function was very weak. This was not surprising considering her level of engagement in the interviews. She did not seem to expend a lot of effort in trying to understand the concepts.

I asked Lisa if she knew what the factors of the equation would be.

LISA: Well, it would be x+2... would be a factor. And then, that's supposed to be, what? ... like 1/2? I don't know ... x - 1/2 and then x - 1 1/2.

Her understanding of what a factor of the equation would be had changed from being synonymous with the value of the x-intercept. She noted how the y-intercept was a product of the constant terms of the linear expressions.

Showing Lisa the graph of a parabola (Figure 47), I again traced along the curve starting at the leftmost point in the window. She again chose to view this
graph through what we had done in the teaching episodes. As I started moving
down along the curve, she commented:

LISA: The lines that make it are . . . the two lines that make it, make
positive y-values right there.
JC: How many lines do you have?
LISA: Two.

When I moved the cursor to the x-intercept, she said:

LISA: Um . . . Um . . . [laugh]. One of the lines goes through there . . .
y is 0.
JC: Do you know anything about the equation of that line?
LISA: No. I probably do somewhere in my memory . . . but I don't
remember right now.

With this remark, Lisa had chosen not to think about it anymore. Feeling more
comfortable as the cursor was placed on the y-axis, she noted:

LISA: I like this one . . . the two terms multiplied equal the y-
intercept.

Moving along this curve, Lisa never referred to the vertex or the "stretch-factor" as
in the first interviews. When I moved to the second x-intercept, Lisa moved
uncomfortably in her chair. "Grabbing for straws" and frustrated that she
couldn't say more, she coyly said, "I think that's a y. [Laugh.] There's always more x-intercepts".

looking at the graph of the line (Figure 48), I asked Lisa to find the equation of the line.

![Figure 48: Graph #3, PC-Emulation Software](image)

Lisa: O.K., ... um ... y= ... um ... x ... minus 2 ... no, x ...

minus 2, yeah, x-2.

JC: O.K., how did you get that?

Lisa: Um, well ... the slope I got cause it's just a 1 ... and then the y-intercept is -2.

JC: How did you get the slope?

Lisa: Well, you can tell it's 1 cause it's ... for every ... you go up 2 over 2 ... so, visually.

In previous interviews, Lisa had been unable to determine the equation of a line.

This was the first time that Lisa was able to find the slope of the line and the y-intercept of the line and apply these numbers to the form y=mx+b. It wasn't clear whether this change was a result of classwork or of our conversations. Moving the cursor to the x-intercept, I asked:

JC: O.K., so you didn't use that point at all to get the equation.

Lisa: I didn't ... someone else might [laugh].
JC: Well, you’ve been trained to use the y-intercept, so that’s why... but, would that help you find... could you use that [referring to x-intercept] to find the equation?

LISA: Yup! [laugh] You would put... um... O.K... let me think... we just did this... you put +2... O.K... wait a minute... took out that number... trying to remember... I don’t remember now.

Unable to remember the manipulations, she gave up. Even with more probing, Lisa was unable to see what connection this x-intercept had to the equation of the line.

At the end of this interview, Lisa remarked, “I still wish it was English though... it’s a little easier”.

Summary

Lisa liked mathematics the least of all her subjects. She claimed that she always had problems with it and had never been good at it. She seemed to blame her current problems with math on math teachers in her past. Admitting that she enjoyed doing the proofs in Geometry, and that she did like the word problems in Algebra, she described mathematics as generally hard and confusing and noted that there was one set way to do everything. This perception of mathematics may have been behind her fear of giving a wrong answer in the interviews. In our meetings together, this pressure of giving a right answer seemed to be stressful and inhibit her from thinking clearly.

Lisa mentioned her struggle with functions many times throughout the interviews. A lack of success in this area of mathematics had left her with what I have called “function anxiety”. When frustrated, she’d say, “I hate functions” or “This is my worst part - functions”. She used the “one y for every x” relationship and the vertical line test to identify functions though she had trouble applying the relationship to sets of points and tables of values. She identified the graphs of
lines, parabolas, and "swivels" as functions, but was unsure of the degree of the
equation for these graphs. The graph of a parabola was "3rd degree, quadratic
kinda" and "swivels" were 4th or 5th degree.

Lisa seemed to focus on the manipulations in algebra, describing it as
finding x and factoring. Given the equation $S(t) = 3t^2 - 3t + 1$, she said that she
would have to put it in the form $y = a(x-h)^2 + k$ to know if it was a function. Given
the graph of a parabola, she focused on the vertex and the stretch factor, "a", with
the intent of using the form $y = a(x-h)^2 + k$, which she referred to as the "working
equation", to find the equation. She had memorized (incorrectly) that the
coordinates of the vertex corresponded to $(b/a, (b^2-4ac)/(4a))$ if given the
"standard equation", $y=ax^2 + bx +c$, which was the "quadratic one". Like Mark, she
had a hard time viewing these two different forms of the quadratic equation as
synonymous.

As Mark had done, Lisa used the word "factor" synonymously with x-
intercept. Looking at the x-intercept on the graph of the cubic, she stated that for
the x-intercept, "y=0", and she recalled that the "synthetic stuff" had something to
do with this point and finding the equation, but "not really . . . that much".

When given the graph of a line, Lisa's attention went to the x and y-
intercepts. She could not remember what the y-intercept had to do with the
equation for the line. She knew that you could get the x-intercept from the
equation by putting 0 in for y and "figuring it out". She was unable to get the
equation of the line, but knew that it involved finding "the slope, and all that
stuff".
In terms of connections drawn between the classes of polynomial functions, Lisa described a turning point on the graph of the cubic polynomial as "not really a vertex". However, she thought that the form \( y = a(x-h)^2 + k \) might still have something to do with this point. Later in the teaching episodes, Lisa used the form \( y = x - h \) in lieu of \( y = mx + b \) to express the equation of a line.

When building quadratic functions from linear functions, Lisa was able to predict the "wideness" of the parabola and say whether it was inverted or not by multiplying the leading coefficients of the linear expressions. When shown the graph of the product, she compared the resultant graph to that of the "basic parabola" noting that the vertex was "down and over". She took on a global view of the graph. When asked about the relationship between the parabola and the lines, she noticed that the parabola crossed through the lines, but didn't notice without a lot of prodding, that the lines and the parabola had the same x-intercepts.

In continuing to work with Lisa through the teaching episodes, I believe she learned the mechanics of the placement of the parabola, but never really understood why the lines and the parabola crossed the x-axis at the same points. When I asked why they shared the same x-intercepts she responded with a laugh, "Cause it's the same thing we did last time." However, following these teaching episodes, she commented, "Actually this makes more sense to me than the way we learned it in class."

Lisa's understandings of the graphs after the teaching episodes were based on the work we had done in the teaching episodes. Except for the linear function, for which she determined the equation by finding the slope and the y-intercept, she talked about the curves as being the product of lines but again said she really
didn't know why the graphs hit the x-axis at the points they did. She knew that the y-intercept would be the product of the "terms at the end" of the linear expressions. She was also able to give the linear factors of the cubic polynomial function without any hesitation.
CHAPTER IX

Beth

Beth was a sophomore in an Algebra II class full of juniors. She had moved to New Hampshire from Nebraska before her freshman year. She kept to herself most of the time, usually choosing to work alone rather than work in groups during classwork. She appeared quiet and timid, but after I spent some time with her she relaxed and became quite talkative. Though naturally attractive, she seemed less concerned about her appearance and impressing others than most of the high school girls. Her straight blond hair fell to her shoulders in a carefree style and her clothing always looked relaxed and comfortable. She had a slight frame, and smiled a lot, though she rarely laughed. During our first interview, Beth appeared a bit nervous, looking curiously at the video camera, and rocking back and forth in her chair as she spoke.

Attitudes Toward School and Mathematics

We met during Beth’s lunch period, a time when she normally went to the library and worked on homework or talked to her friends. She said that her day at school began by “wandering around and talking (for hours) to her volleyball coach.”

BETH: Then there’s Gym and English ... boring ... and then go to Algebra ... usually, I feel pretty good cause I understand and [Mrs. Tucker], I like the way she teaches, so ... and then ... I go through that ... Geometry is fourth and that’s pretty easy ... the book we have is very blunt ... simple ... then there’s Biology which is fun ... we watch movies and I love labs ... they’re the best ... and then German. I didn’t take a foreign language last year, so I’m kind of catching up ... oh, and then 8th period, I have Basic Business which has tests about everyday ... and so that is my school day.
Not only did she find Geometry class easy, but later she noted it was the favorite part of her day.

BETH: I like Geometry cause [the teacher] is wicked funny. He hates our book because it's pretty stupid, so he makes fun of it all day ... it's got like the stupidest things ... like, oh, there are lots of story problems about Sherlock Homey which is Sherlock Holmes cousin ... O.K ... [said sarcastically].

Beth talked about what she remembered from elementary school mathematics:

BETH: We always had to do those flashcards and then I remember on T.V. they had, on the information channel? ... they had the ... multiply quickly in your head? That thing works so well ... just from what they do ... like O.K., 5x27, and you could do it, like cool.

JC: Anything else you remember about elementary school?

BETH: I remember the first time my teacher threw in a letter and we were kind of like, "You can't do that!" ... cause we were just used to 1+1, 1x1 ... so, very confusing.

It became more evident in the later interviews that Beth liked the "tricks" used in mathematics to remember rules.

In middle school, Beth spoke about taking a national mathematics exam and the day the lights went out:

BETH: I remember we took that in eighth grade and, like, that was kind of hard. I always remember the calculators, they're all solar and the lights went out one day in school and we couldn't do our math [she smiles]. It was horrible! That was the worst day!

She was again indicating that mathematics class had been the favorite part of her day and she hated to miss it.

Beth had a course in pre-Algebra in the 8th grade and took Algebra I during her freshman year. It was her favorite course during freshman year. She admitted that mathematics always came easier for her than other subjects. The following quote suggests a level of self-confidence in doing mathematics.

BETH: I had it last period so you have all your English classes and everything and you're all frazzled through that and get to Algebra and you're like, "Yes, I can do this!" ... I'm better at math.
Beth thought that she spent about two hours on homework each night. I asked what most of this time was spent on.

BETH: um ... Usually it's German or ... Algebra, I usually finish in class and Geometry, I finish in class, and then everything else ... it's kind of collective ... bring every book home ... kinda study ... go over your notes for the whole day, that you took, and then do some of the homework ... and I'd throw this book aside and go to that one and take that one back and ... and just everything.

JC: That sounds like a good study habit...going over your notes. Where did you learn that?

BETH: Yeah. Seventh grade reading class. "If you don't review your notes within 24 hours, you are going to forget it", so ... I do that.

Beth did have a graphing calculator at home that she often used in doing her homework. She also had a computer at home that she used solely for word processing.

I asked Beth what she wanted to do after high school.

BETH: I'd like to join the Peace Corps for a year or two and ... but I'd also like to go to ... um ... I'd like to work at NASA. So, just kinda ... or Lockheed. I'd really like to go to ... um ... to Colorado to the Air Force Academy and then just kinda go through that, cause that kinda gets you right into it ... the aeronautical stuff ... but I'd like to take a year off to relax ... and just "experience" [she smiles].

Though Beth was considering quite a few options, her statement depicts her sometimes introverted/sometimes extroverted nature.

**Definition of Mathematics**

I then asked Beth to describe mathematics in her own words.

BETH: It's ... I don't know ... numbers? ... letters? ... things equaling out ... it's ... gosh ... it's expressions that ... like you can make things work ... and you have reasons for it ... like rules that you can see ... like how they equal out ... not like English how it's ... it just happens that way ... this way you can explain it ... how it happens.

I noted Beth's tendency to focus on the symbolism by her use of the phrases: "expressions", "you can make things work", "rules that you can see", and "how they equal out".

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Probing further, I asked Beth to describe Algebra. Corroborating my theory that she focused on symbolism, she responded:

BETH: Algebra is . . . I guess . . . how the numbers work? . . . and how you can fit little letters in or variables to find out what they are . . . and . . . just . . . I guess that's good enough.

In future conversations with Beth, as will be noted in this case study, Beth continued to portray mathematics as a language of symbols.

**Knowledge of Functions and Graphs**

**Function Definition**

Though Beth remembered studying about functions in other years, she said that this year was the first time she understood them at all. I asked her to tell me, in her own words, what a function is.

BETH: Function . . . let's see . . . actually, I think I almost remember the definition, but . . . forget that [waves her hands]. Function would be when you have your little equation and you . . . it's how . . . when you can get whatever your variable is and you have a set answer for it, according to the numbers or variables you have.

J.C: So what do you picture in your mind when I say "function"?

BETH: One little person running through a little map and getting the answer that their genes or whatever would make them get . . . they have no choice.

Again, the emphasis in her definition seemed to be on a function as an equation. I also noted that she seemed to focus on the whole functional process; the input, the rule, and the output.

**Ability to Identify Functions**

I handed Beth Task #1 (Figures 49-53) and asked her to tell me which of the thirteen items were functions.

BETH: Hmm . . . O.K., I guess one is a function because you have your x, which would give you your letter . . . any number . . . and y which is x used this way.
She continued:

**BETH**: O.K., number two is the same as number one because it's just written differently... so you have your x's gave you certain y's also.

She seemed to be focusing again on the input/output relationship between x and y in deciding if these first two items were functions.

![Table and Graph](image)

**Figure 49: Task #1 (1-3)**

**BETH**: [Looking ahead at the following graphs.] I don't know any of these things [in a shrill voice].

**JC**: (laugh) That's O.K.

**BETH**: ... and number three? It's not a function... maybe not... I don't know... [Looks up from paper] Could we just say I don't know?

**JC**: Sure you can.

**BETH**: [Says quickly.] I don't know.

**JC**: What is that... if it's not a function?

**BETH**: A line... Actually, I guess... for a function you can only have one answer, right?... maybe... for one thing?

**JC**: What do you mean?

**BETH**: For every x there is one y?

This correspondence between the x and y coordinates started coming more into focus for Beth as she saw the graphical representation. In prior statements, she seemed to be looking more for an equation to determine if a table of values was a function. With this notion of the correspondence between coordinates coming back she looked back at the previous items she had done.

**BETH**: Oh, no... The first two are just points, and... cause you can get any other number... or you can get y's... with any x's I guess... cause they're just points... they could be anything.
She now seemed to be viewing items one and two as sets of ordered pairs, and interpreting the points as only a subset of a larger set which wasn't shown. She continued on to the graphs in the next items with more certainty.

**BETH:** O.K, three, yes. Four, yes, because... yeah, O.K... For every x there is one y. So, because that's horizontal... it'd be... sure.

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</tbody>
</table>

**Figure 50: Task #1 (4-6)**

**BETH:** Number five? No!... because for every x there's one and only one y? So, there's two fours in this... two different answers.

**JC:** O.K., how is that different from number two?

**BETH:** um... There's two x's... um... two of the x's are the same?

**JC:** So you said this one isn't a function [pointing to number two].

**BETH:** Oh, dread! I don't know. Well, this one... I don't know.

As evidenced by this dialogue, Beth viewed the y-value as an "answer" (the output) even under this notion of function as a correspondence. She was now starting to experience a lot of confusion as she verbally interchanged the positions of the x and y coordinates. With relief she looked ahead to number seven (Figure 51).

**BETH:** Number seven... yay!

**JC:** [laugh] Number seven... yay... why? Why is that a yay?

**BETH:** It's an equation!

She seemed to be a lot happier with this representation, supporting my sense that she was more comfortable with equations than with tables or graphs. Beth went on to say coyly that this was a function because "math says it is". The notation told her it was a function. She added:

**BETH:** That's the function thingie.

**JC:** Function thingie... what's...?
BETH: It's the function ... the S of t ... which would make it a function of t.

7. $S(t) = 3t^2 - 3t + 1$  8.  9. $y = 2x^2 + 1$

Figure 51: Task #1 (7-9)

BETH: [With confidence.] Number eight would be negative! ... Boo ... cause there's two y's for every x ... and number nine is also a function because y is really $f(x)$. I learned that this year ... that's why I was so confused before ... I didn't understand how it worked.

Beth suggested again that equations were functions if they had the function notation, and since $y$ was just another name for $f(x)$, #9 (Figure 51) would be considered a function. Beth, like Lisa, acknowledged that "$f(x)$ was just another name for $y$" was not something that readily made sense, but it took awhile before she could understand that.

10.  11. $f(x) = 5x - 3$  12.

Figure 52: Task #1 (10-12)

The following statements seem to indicate that Beth accepted items as functions according to former experiences with items of that type.

BETH: O.K., number ten is a parabola ... I can't even spell it, and so that's a function. This one [looking at number eleven] can be a function because it's $f(x)$, I guess ... and this one's a swivel [referring to number twelve], so that's a function too ... yeah ... yeah, it is.
Moving on to item thirteen, Beth asked if the table of values continued.

BETH: Are we assuming that this kinda stops right there . . . does it? . . . stop right there?
JC: Yes.

As I mentioned when Mark asked this same question, I felt that this comment reflected prior experiences of being asked to graph a continuous curve given a set of only a few points.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 53: Task #1 (13)

BETH: O.K., well then that's . . . can be a function . . . because it doesn't have any double y's.

She then went back and clarified her thinking on the first two items, understanding now that these sets of points did "stop". Beth had a sense of humor which was now coming out more and more as she became more relaxed in our discussion. As will be seen in many pieces of dialogue, Beth often humanized mathematical objects.

BETH: O.K., so back to one and two! They can be functions if they want to . . . because if they are just little points . . . they can be functions.

She added:

BETH: Oh! that's another thing I remember from eighth grade . . . ah . . . ninth grade Algebra. We had . . . um . . . for functions we had "every women can have one husband but all the husbands can have as many wives as they wanted" . . . or something.
Despite this "trick" for remembering which sets of points were functions, she continued to interchange the x and y-coordinates throughout the entire study and had trouble identifying sets of points or tables of values as functions.

**Describing Function Attributes**

Handing Beth the "little book of graphs" to begin Task #2, I asked her to give me enough information about each graph so that I could "draw it". Beth asked:

BETH: Can I give you points?
JC: Anything you want. You can actually tell me if you recognize it as a specific shape... you can tell me what it is.
BETH: O.K. [Referring to the first item.] This is a line. O.K., there's a point (0,1) ... and another point that is approximately -3, 1 ... 0 ... (-3,0) ... and there's a slope through there.

![Figure 54: Task #2 (1-3)](image)

I assumed that she was trying to say there was a slanted line through the points.

Moving on to the second graph (Figure 54), she asserted:

BETH: This is a parabolaish... yeah... sure. Yeah. (laugh) O.K., it will be a parabola. It's kind of off center, but, that's O.K. We have a point at (0,2) and -3 ... or (0,-3) ... O.K., hold on ... (0,2) ... (0,-3). ... no [annoyed]... if it's ... like... oh gosh. O.K., the x is right ... so it'd be ... wait a minute then the last point would be wrong... that one was (0,1), right? ... the one before? O.K., this is going to be (-3,0) ... yeah.

Though she was still interchanging the positions of the x and y coordinates, I sensed that Beth was trying to give me the coordinates of the x-intercepts. Unlike
the other students in the study, Beth's focus had gone to the x-intercepts first, as opposed to the vertex. She then continued by giving the coordinates of the vertex.

BETH: I'm very confused and I wish I wasn't. O.K. . . . and then . . . ah . . . the vertex we'll put a point . . . a negative point . . . a (-1/2, -2) . . . oh, no . . . oh, yes, oh, yes, O.K., never mind . . . and then . . . oh, my gosh . . . the slope is . . . the x . . . a is . . . O.K., hold on, I'm getting it . . . I'm doing fine . . . mmm . . . 2, yeah, 2. [She counts up from the vertex over to the point 1 unit farther to the right on the graph with her fingers.]
JC: How do you know that?
BETH: Because fractions make it smaller and . . . regular . . . or integers make it larger . . . and it's positive.

She was commenting on the effect of the value of a, the leading coefficient, on the shape of the parabola. She had referred to a as "the slope", indicating now that she connected the coefficient "a", the size of the parabola, with "m" which indicated the "steepness" of lines. I noted also how she thought of a point with negative coefficients as a "negative point".

For the third graph, Beth commented:

BETH: (laugh) . . . all right, this one's negative . . . and the . . . it's upside down.

She was again using her own terminology to describe the graphs. As she had described lines and points as either positive or negative according to the sign of the leading coefficient and coordinates respectively, she also described parabolas as positive or negative according to the sign of the leading coefficient. She went on to give the x-intercept (the vertex) and the y-intercept of the graph. She then asked:

BETH: . . . do parabolas have slopes?
JC: That's a good question.
BETH: But, you won't answer?
JC: Do you think they have slopes?
BETH: They don't actually, but they kind of do . . . they have the . . . after you exit the vertex area they kind of . . . O.K., never mind . . . it's \( x^2 \) . . . so this one's . . . the letter in front of \( x \) would be -1 . . . so it's upside down.
I found it interesting that Beth thought the parabola might have a slope after you exit the vertex area. I interpreted this to mean that there wasn't a slope at the vertex. I am not sure if she was saying that a slope didn't exist here or simply that it was 0. I believe this question of whether parabolas have slopes was spurred on by linking the leading coefficients of linear and quadratic functions. It seemed as if Beth was looking for connections between the two graphs.

Beth then gave two more points, (3,-1) and (5,-1), to describe the shape of the parabola as I drew it on my paper.

Beth acted relieved when she saw the next graph (Figure 55, #4). She identified the graph as a "swivel", adopting, as Mark had done, the terminology used in class. When looking at the graph more closely, however, she was bothered that the points weren't in the pattern she thought they should be.

BETH: Wonderful... this is a swivel... this one's x^3, I think... O.K., nevermind... we'll just go on... there is a point at (0,0), (1,1), (-1,1) ... (-1,-1)... that's what I meant... I'm sorry... and then... it swivels through... but you can't do that, O.K... (2,3)?... is an approximate point... O.K., can you swivel through that?... Do you need another one on the bottom? O.K., we'll just flip that one around... (-2,-3).

JC: O.K... you seemed to know that this... at least you said... that this was y=x^3, is that right?
BETH: Mmmm... 2... O.K., hold on... 2, 8, 4... oh, 1, 1, 2, 2, 8... no, that does not appear to be correct... well...

By plugging in the coordinates of points, she determined that this was not the graph of y=x^3. This activity gave evidence that Beth seemed to have a good understanding of what Moschkovich, Schoenfeld, and Arcavi (1993) call the Cartesian Connection, i.e. that a point is on a graph G if and only if its coordinates satisfy the equation of G.
For the fifth graph (Figure 55), Beth began by giving the open and closed y-intercepts, and one other point to determine the upper half of the graph. For the lower part of the graph, using terminology similar to the type she used before, she said, "Use the same slope and progress it from that open dot . . . Excellent". For the last graph, Beth said:

BETH: O.K., this is a square root . . . right? . . . point at (2,0) . . . I feel like the guy from the Navy who goes "15 knots" (laugh) . . . and then . . . [makes arc with hands to the right] . . . whoosh . . .

**Translating to the Algebraic Representation**

I asked Beth if there was an equation for the graph of the line in item one (Figure 56). She continued to produce her own terminology in response to this question.

BETH: Yeah. There probably . . . there is one, and it would be a *positive* one . . . or negative . . . positive . . . yeah, it's positive.
JC: What do you mean it's positive?
BETH: The x is positive.
JC: The x is positive . . . why?
BETH: Because negative ones go the other way . . . or . . . Oh! How about this? The slope is positive . . . scratch that other part. The slope is positive because it increases as it goes to the right.
I asked if she knew anything else about the equation. She again interchanged x and y in her response.

BETH: um..The x and y intercepts. The y-intercept is the constant... is it? I don't remember that part, and... ooh... the slope is the number in front of the x, as in ax + b... so... but, I always mess up the slope equation thing... I don't know... either the x is on top or the y is on top, but I don't know which. So, that's all I know about that guy.

Beth continued to "humanize" graphs and equations as in this last statement. I noted her use of "a" rather than "m" in the slope-intercept form of the line. It was interesting that she used the variable "a" for both lines and later with parabolas, although in class, the slope of a line had almost always been denoted by m. She seemed to connect the leading coefficients of these two classes of polynomial functions in some way.

Beth determined the equation of the parabolas by plugging the coordinates of the vertex into the form $y = a(x-h)^2 + k$.

When asked about the equation for the third item in Task #2 (Figure 56), she noted:

BETH: Negative something... oh, that one's got a half inside... or something? No, that makes that a 2... Oh, my... O.K. you have 4... +4 on the outside... it's fine on the inside and it's negative 1... so, $-x^2 + 4$. 

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She was using the formula $y = a(x-h)^2 + k$, hence the "inside"/"outside" terminology referring to the parentheses. She switched the positions of the coordinates of the vertex, however, in using this formula. She then added, "... and those are all equal to zero". She was indicating that the algebraic representation would be written in the form $0=a(x-h)^2 + k$. I attributed this statement to the emphasis in class on solving quadratic equations of the form $f(x)=0$ rather than being asked to find the zeros of functions in the form $y=f(x)$.

When asking about the equation for the graph of the cubic, I asked:

JC: Could that be ... um ... a first degree equation?
BETH: No, then it would be a line, wouldn't it?
JC: O.K... How about ... could that be a second degree equation?
BETH: It would have to be a parabola. So, it'd be a third ... I just don't know what kind ... it'd be a positive ...

Her answers here indicated that she had an understanding of the relationship between the degree of the polynomial and the shape of the curve, at least for linear and quadratic functions. Heid (1995) argues for the importance of student's being able to recognize the relationship between the shape of a curve and its symbolic form, saying that this skill is important in recognizing the appropriate equation to use in situations involving mathematical modeling.

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**Figure 57: Task #2 (4-6)**

When asked if she knew the equation for the graph in item five (Figure 57, #5), Beth said:
BETH: No... oh, yes, actually... No, actually, I changed my mind again. Oh, it's something about... there's no numbers in between... x's... goodness... well, the... O.K... is x domain or range?... can you say?
JC: x is the domain.
BETH: O.K., so, the domain is equal to all reals but the range is... greater than -2... or 2?... less than... or greater than or equal to 2 and less than or equal to -2.
JC: You hold the paper sideways to see that?
BETH: [Mrs. Tucker] said, if you wanted to, you could, because its kind of like the y... or the x... cause I forget everything unless there's a nice little trick to it.
JC: Is there an equation though for that... number 5?
BETH: Sure, I don't know it though.

I wish I could say why Beth kept interchanging the roles of x and y. She did this much more than any of the other students I interviewed. Beth admitted here the need to have a "trick" to remember the mathematics, but again, in this case, the "trick" didn't seem to be helping.

Going on to the last graph in the booklet (Figure 57, #6):

BETH: It's a square... or a square root... the square root of x + 2.
JC: Square root of x+... 2 on the outside?
BETH: On the outside, yeah.
JC: On the outside... what is that equal to, anything?
BETH: Zero.

It's not clear whether she just accepted my statement about the 2 being on the "outside" of the parentheses without giving it any more thought, or if she really thought that was the case. Regretfully, I did not probe any deeper at the time. She continued to believe that the equation for this graph was of the form f(x)=0.

Journal entry. After the first interview, Beth wrote the following in her journal:

Goodness, I was so nervous when you started asking questions. On some questions I felt I knew the answer, but had no way to explain it. I was totally flustered by some of the graphs, but I guess afraid I'd give a wrong answer.

I was aware that all the students would experience some anxiety in being interviewed and that this would affect their ability to think clearly. This "anxiety factor" should be considered in weighing the students' understanding as
described in these case studies. Sometimes their statements may not give a clear picture of their understanding because of this anxiety.

Beth noted her own inability to explain some ideas. All of the students I interviewed seemed to be a bit frustrated trying to express their answers verbally. This was not really surprising due to a lack of classroom experience in having to talk about the mathematics involved in a problem situation.

**Understandings of Polynomial Functions**

This section, as for Mark and Lisa, is organized around the first three research questions. Again, the conversations in this section took place after Beth's test on the "unit" on polynomial functions for which she received an 85%.

**Connections Between the Classes of Polynomial Functions**

Some connections between the classes of polynomial functions that Beth made can be extracted from the above discussion. For example, Beth saw commonalities between the algebraic representations of linear and quadratic functions in their leading coefficients. This was evidenced both in referring to the slope of a line as \( a \) instead of \( m \) and in referring to the stretch factor of the polynomial as "the slope". In addition, if the leading coefficient was negative, she seemed to note that it affected the orientation of these graphs similarly, i.e. "it would go the other way". When reacting to the graph of the cubic function, for example, (Figure 58), she noted:

**BETH:** If it was negative it would go the other way [shows with hands].
**JC:** If what was negative?
**BETH:** The number before \( x \).

The leading coefficient, in Beth's terminology, made the curve "positive or negative". Beth used this terminology across the classes of polynomial functions that we examined.
Beth thought the *PC-81 Emulation Software* was "cool". When I put the graph of the cubic polynomial function (Figure 58) in the window, she called it "a swivel... It's an $x^3$!"

Starting at the leftmost part of the graph and tracing along the curve, I asked her to tell me "what was going on with the graph" at various locations along the way. When I stopped at the x-intercept, she interchanged the values of the x and y coordinates as she had done so often in the previous interview.

**BETH:** um... x is equal to 0... the... ah... x-intercept... be a zero of the function.

I asked her to explain what she meant by a zero of the function.

**BETH:** It's when the... O.K., it's when the... when you enter x for the... when the function is equal to 0... whatever number you have for x.

She seemed to be viewing the zero of a function from an algorithmic perspective, i.e. plugging in the value of x in the algebraic representation to find the corresponding y value, rather than from a graphical perspective, i.e. in terms of the the second coordinate of the point being 0.

As I moved to the minimum value on the parabola (Figure 59), Beth noted:
BETH: That's the bottom.
JC: Which you call? . . .
BETH: The vertex?

![Graph #2, PC-Emulation Software](image)

She referred to the relative maximum on the cubic, however, as a "turning point" which she described as where "the y's start to go back down but the x's are still increasing". When I asked Beth if the turning points revealed any information about the equation, Beth commented in a hesitating manner:

BETH: The number of turning points it has tells you what kind of... whether it's x squared or x cubed or x to the fourth... you have one less curve then there is... number, exponent... number?

With this statement, Beth was stating a principle that she thought applied across the classes of polynomial functions. In a later interview, I talked with Beth further about the relationship between the degree of the polynomial and the number of turning points. I was curious whether having her recall the graph of $y=x^3$ would change her theory. She remembered what the graph of $y=x^3$ looked like, describing it in the following way:

BETH: Goes right through the little... it goes zhoooom [She sketched the graph with her finger in the window going from left to right].
JC: How many turning points does that have?
BETH: It has two itsy-bitsy ones... only they don't like...
Her interpretation of the graph of $y = x^3$ fit reasonably into her rule regarding the relationship between the number of turning points and the degree of the polynomial. There was no need for her to change this conception.

Moving along the curve to the y-axis, Beth spoke of the the y-intercept as where $y$ equaled 0 as she had previously done in another interview. I believe she was interchanging the roles of $x$ and $y$ again at this point.

**Contributing/Inhibiting Factors to Making the Transition to Polynomials of Degree Greater than Two**

Beth was asked questions that probed into her understanding of function notation and some related terminology. With some degree of hesitation, she was able to point on the graph to intervals where the function, $f(x)$, was greater than 0 and to intervals where $x$ was greater than 0. When asked, she also clearly expressed these intervals in words. She was able to identify the points on the curve where $f(x)=2$ and she was also able to cite the "domain" and "range" of the function. Hence, her interpretation of functional notation in a graphical sense seemed quite good, and though she had a lot of trouble distinguishing $x$ from $y$ on a graph, she had little trouble distinguishing $x$ from $f(x)$. I was also realizing that if we were moving along a curve, she had less difficulty distinguishing $x$ from $y$, whereas for a static position, i.e. a point, she usually had a hard time. The dynamic aspect of moving up, down, left, or right along the curve seemed to help her distinguish the $x$-coordinate from the $y$-coordinate.

BETH: Oh, the $x$-intercept ... ooooh ... is it? I always thought it was the $y$ ...  
JC: That is the $y$-intercept right there.  
BETH: Oh, than why was the $x$ zero? Is that how it goes? 
BETH: Oh, O.K ... I won't look at the numbers anymore, it's confusing.
Beth was referring to the numerical values of the x and y coordinates which are shown at the bottom of the window as the cursor moves. I now understood where at least some of Beth's confusion over the values was coming from.

In the interviews with Beth, she did not suggest that there was a link in the properties of the x-intercepts or the y-intercept across the classes of polynomial functions.

When I asked about the x-intercepts when moving along the cubic, she commented:

*BETH:* Those ... you can use them and they'll give you your equation.

*JC:* How?

*BETH:* I don't know, that's the one I got wrong [referring to the test, she smiles as she says this]... um ... *if it's just an equation,* you can use those as ... like ... hmmm ... Oh! you can ... Oh, my God ... oh, you can do x minus whatever number your y is there ... your x is ... no, O.K., wait ... there's something you can do ... O.K., you have equal to zero at the end ... ummm ... use all those numbers x minus that number [points to x-intercept on graph] then another thing with x minus that number and then fill in your little x's and ... oh, no ... don't fill in the x's ... put an a out front and multiply ... and then you have an equation.

This process of using the "Factor Theorem" seemed quite mechanical for her, and I felt that in being mechanical she didn't really understand the meanings or purposes behind the process. I decided to ask if she could give me a "factor" of the equation while we were exploring the graph of the cubic.

*JC:* Do you know what a factor of this ... the equation here would be?

*BETH:* Is that like a zero?

*JC:* It has something to do with a zero.

*BETH:* O.K. ... factor.

*JC:* A factor of the equation.

*BETH:* So like the function of 2 would be equal to 20 ... is that a factor maybe?

*JC:* No.

*BETH:* No? O.K., well then, I don't know.

I wished later that I probed into what she was thinking when she said f(2)=20 had something to do with a factor.

As with the cubic polynomial function, Beth made no mention of the usefulness of the x-intercepts in finding the equation of a parabola and
surprisingly, she made no mention of the usefulness of the vertex. I was
surprised because this was the aspect of a parabola stressed in class. Tracing
along the curve of the parabola, I asked her about the x-intercepts specifically.

J: O.K., how about the... um... how about the x-intercept, does
that have anything to do with the equation?
B: No.
J: And the other x-intercept... that wouldn't either?
B: Right.

After we finished tracing along the parabola, I asked Beth if there were any parts
of the curve that would help her to find the equation.

B: The y-intercept.
J: O.K., how does that help you to find the equation?
B: Cause... well, maybe... well... because a regular equation .
.. or just \(x^2\) would be... the y-intercept would be at zero and this
one's at -2, so... that makes the constant negative.

She had viewed this parabola as a translation of the basic parabola \(y=x^2\) and
determined the equation in that way. This was the first time she seemed to pay
any attention to the y-intercept as useful in determining the equation.

When asked how she would determine the equation for the line (Figure 60),
she responded:

Figure 60: Graph #3, PC-Emulation Software
BETH: You could find the... like, find your slope... which is... well, you'd have to find two points and then... subtract the... something... either \( x_1 - x_2 \) over \( y_1 - y_2 \) or \( y_2 - y_1 \)...
I get it backwards, so I'm not sure which letters it is any more... and that would be your coefficient of \( x \) and then your constant would be the \( y \)-intercept.

JC: All right, does the \( x \)-intercept have anything to do with the equation?

BETH: No.

Beth remembered the type of manipulations necessary to find the slope, but couldn't remember them exactly. This was another indication of an emphasis in her learning on the symbolism and manipulations, i.e. the rules in finding the slope, and less emphasis on a conceptual understanding, i.e. understanding the meaning of slope.

To summarize Beth's understanding, I will refer to the model of interpretations of the \( x \)-intercept given in Chapter II and shown again in Figure 61. The reader is reminded that this study was designed to investigate understandings for implications a, b, and c in this model. From the interviews, I would classify Beth's understandings of these interpretations as:

**Acquired:** b -Beth repeatedly spoke of the \( x \)-intercept as the value that when entered into the function would make the function value zero.

**Tentative:** a -Because of Beth's repeated interchanging of coordinates, I have listed this interpretation as tentative. Her understanding of the value of the coordinates for the intercepts was not robust.

**Missing:** c -Though Beth did speak of an \( x \)-intercept, c, as linked to the expression, \( x-c \), she saw this expression only as part of the equation, \( f(x)=0 \), rather than as part of the function, \( y=f(x) \).
Figure 61: Interpretations of the x-intercept

**Journal entry.** Following this second interview with Beth, she wrote in her journal:

I was much more comfortable today. Whenever you ask if I know anymore I feel like I've forgotten everything I am supposed to know. Algebra II has taught me everything mostly you've asked about. I learned some of it in Algebra I, but I never understood it until this year. X- and y-intercepts, actually anything with the x and y lines confuses me. I forget which line is which and then mess it all up. All of the interviews have helped me sort out the information in my head.

**Teaching Episodes: Forming Links Between the Classes of Polynomial Functions**

At our next meeting, the software entitled *The Function Supposer: Explorations in Algebra* was used as an intervention in Beth's knowledge construction through a teaching episode (Appendix B).

**Building Quadratic Functions**

As I had done with Mark and Lisa, I asked Beth to type in a couple of simple, linear functions, \( f(x) \) and \( g(x) \). Beth typed in a constant function for \( f(x) \).
We talked about the graph of that function, and then I mentioned that, for the purposes of what we were going to do, I'd like her to start with a linear function that had a slope different from 0.

The two linear functions she typed in were \( f(x) = \frac{1}{4} x - 3 \) and \( g(x) = -x + 4 \). After they were graphed in the windows, I asked her to talk about the differences between the two lines graphically. She noted that \( f(x) \) had a positive slope, and \( g(x) \) had a negative slope and that the y-intercepts also differed in sign. I "ZOOMed" out on the graphs so that Beth could also see where the lines intersected.

**JC:** O.K., how about where those lines cross?

**BETH:** That's where the . . . number for the . . . the numbers are the same . . . where the values of \( x \) are equal . . . or make the y's equal.

**JC:** O.K., is it the same value for \( x \), too?

**BETH:** No . . . yes . . . ummm . . . no . . . can't be.

I asked Beth to think about what the graph of the product of those two linear expressions would look like.

**BETH:** A regular parabola.

**JC:** Why?

**BETH:** Because, then you'd have an \( x^2 \)?

She had multiplied the corresponding linear expressions in her head. In asking Beth what that parabola would look like, she responded:

**BETH:** Probably . . . maybe it would kind of look like the graph right now, would it?

She pointed to the graph, indicating that the vertex of the parabola might be near the intersection of the lines. She sketched with her finger what it would look like. She knew it was upside down, and thought it would be "kind of skinny".

**JC:** Why do you think it's skinny?

**BETH:** Cause you have a lot of fractions . . . you have \( \frac{1}{4} \) . . . and . . . fractions make it skinny! . . . so it'd be . . . [shows with her hand] . . . Mmmmyowww.

I asked if she could tell where the graph of the parabola would intersect the \( x \)-axis.
BETH: At . . . no, I don't. Do you want me to guess?
JC: Sure . . . go ahead and guess.
BETH: How about -3 and 4 . . . +4?
JC: Why do you guess that?
BETH: Cause they're right there. [She points to the y-intercepts of the lines.]
JC: Oh, O.K. Well, what are those on those graphs?
BETH: Those are the y-intercepts so . . . maybe, no . . . well, then I can't do it . . . I don't know.

At this point, I decided to let Beth see the graph of the parabola.

JC: O.K., so what do you notice about that parabola now? . . . you were right, it is upside down . . . inverted.
BETH: Inverted . . . O.K., it's . . . Oh, it has the same x-intercepts as the other one did.

Unlike Mark and Lisa, Beth noticed readily that the x-intercepts of the parabola were the same as the x-intercepts of the lines. We discussed the algebraic reasons why the linear factors of the quadratic function had the same x-intercepts as the linear functions.

After clearing the window, I asked Beth to type in other linear functions for f(x) and g(x). This time, she was able to pinpoint the position of the resultant parabola on the x-axis.

When we did it for a third time, Beth entered f(x)=5 and g(x)= .3x - 4 for the linear functions. This time I let her keep the constant function. The graph of g(x) went outside of the window before it crossed the x-axis. I asked about this graph.

JC: Will it go through the x-axis at all?
BETH: Yeah, eventually.
JC: Where?
BETH: Probably like over here. [She points to the right of the window.]
JC: Why?
BETH: Cause it's got a very slim slope, so it will take a long time.

Her use of the word "slim" here seemed similar to the word "skinny" she had used for the shape of parabolas determined by the leading coefficient, again humanizing these graphs. Knowing she was probably just extending the line
visually, I was curious if she could determine the actual value of the x-intercept algebraically.

JC: O.K., so can you tell just from the algebra where it crossed?
BETH: Yeah... no... well, you could do it. Want me to?... I have to think about lots of stuff... I'm getting better with my numbers though, I know, this one is (-1, 0) there, cause we've been graphing a lot.
JC: O.K., tell me what you are doing.
BETH: I'm trying to remember which one you want zero for... you want y to be zero... so... divided by .3 that will be... a number.
JC: A number. [laugh]
BETH: O.K., so that would be... 40/3... be over there.

Beth was trying to remember here whether the x-intercept implied that x was 0 or that y was 0. It didn't seem that she was really thinking about the position of the x-intercept on the graph to remember this, but trying to recall a memorized fact about the x-intercept. I "ZOOMed" out so that we could see the x-intercept.

I asked her to tell me what the product of the two linear functions would be, knowing that this time the product would be a linear function since f(x) was a constant function.

BETH: Well, it's going to have the one intercept where this one would be... because the other ones did... so... it should... it should just be another line, shouldn't it?
JC: Right... what is the slope of it though?
BETH: So, ... ooh... point... it'd be 1.5, so it would be more vertical than the other ones... yeah!

She seemed to accept the fact that the graph of the product went through the same x-intercept as the original line just from the pattern she had observed from the other examples. She multiplied the algebraic expressions out to determine the slope of the line. In observing the graph of the product, I commented:

JC: It does have that same intercept... it's steeper, like you said... um... and what do you think this 5 had to do with it... changing... it just made the slope steeper?
BETH: And it moved the y-intercept.
JC: O.K., do you see any relationship between the y-intercepts of [f(x) and g(x)]... the first two and the last one?
BETH: Well, 5 times -4 is -20.
She made this observation about the relationship between the y-intercepts by comparing the graphical representations rather than studying the algebraic representations.

After clearing the window, Beth entered the linear functions \( f(x) = -\frac{4}{5}x \) and \( g(x) = \frac{7}{44}x - 5 \). (Beth was definitely not afraid to explore the concept with a wide variety of linear functions.) She again predicted the position of the parabola, saying that it passed through the x-intercepts of the lines and was inverted. When the parabola was actually graphed, her predictions were confirmed.

JC: Is it what you expected to happen?  
BETH: It is very much so.

At our next meeting, a couple days later, Beth did one more example of building a quadratic function from linear functions using the same software. Beth entered \( f(x) = x + 3 \) and \( g(x) = .5x - 3 \).

JC: O.K., so can you tell me what the quadratic function will look like... the parabola will look like?  
BETH: O.K. It will cross at... right there... at that intercept and at that intercept... and it will be "not" inverted but just regular?  
JC: Again, how do you know that?  
BETH: Because the coefficient of \( x \) is positive and both of them are, so it will be slightly thin...  
JC: How do you know that?  
BETH: The \( \frac{1}{2} \)... maybe 1... it would be one or the other... it will be slightly wider.  
JC: O.K., how about the y-intercept of the parabola? Do you know what that is?  
BETH: The y-intercept?... no.  
JC: You don't?  
BETH: Maybe... I get -9.  
JC: O.K., how are you getting that?  
BETH: if both the x's were 0... let's say if x was 0, it would be -9.

There are a couple comments to make concerning this dialogue. First, throughout the dialogue Beth continued to provide her own version of the terminology to describe the graphs. This parabola was not "inverted", so she described it as "regular" and it was "slightly thin" which she changed to "slightly wider".
Secondly, she determined the y-intercept by letting \( x = 0 \) in each of the expressions. This method of determining the y-intercept seemed to be the most prominent definition for her. Recognizing the constant as the y-intercept did not seem as prominent.

From this point, we graphed the quadratic function on the screen, and after adjusting the scale of the axes so that all the intercepts were visible in the window, the graph confirmed her predictions. I asked Beth to identify portions of the graph for which the y-values were positive.

Beth: O.K., I'm thinking... I'm thinking, oh, geez... O.K., I don't know... um... like this half down?... so, zhoom... [She waves her hand over the part of the graph below the y-axis.]

JC: All this part, your y-values are less than 0, cause you are below the x-axis?

Beth: Oh! O.K., then this way. [She points to one side of the parabola above the x-axis.]

JC: Right, and also over there... [pointing to the other side]

Beth: Yes.

It was interesting that Beth had no trouble answering a similar question in the second interview when I asked where \( f(x) \) was greater than 0. Her confusion seemed restricted to questions about y and x.

I placed a vertical strip of paper along the leftmost x-intercept, to hide the parts of the graph to the right of the intercept. I then asked whether the lines to the left of that intercept were above the x-axis or below the x-axis.

Beth: Below?

JC: So they are both... for that part of the graph, they are both...

Beth: Negative!

JC: Negative. So, you have a negative times a negative and you get the positive.

Taking another strip of paper, I hid all parts of the graphs except for the parts between the two x-intercepts. We compared the "signs" of the lines to that of the parabola. We then compared the "signs" of the graphs beyond the second intercept. The suggestion was that observing the "signs" could also be useful in determining the position of the parabola.
**Working Backwards: Finding the Components of a Quadratic Function**

Reversing this procedure of predicting the parabola given two lines, I handed Beth a piece of paper with a graph of a parabola on it and asked her to try to sketch two lines whose product could possibly produce the parabola.

After Beth sketched the lines (Figure 62), I asked:

JC: Can you tell me why you picked those lines?
BETH: Cause, I've got the intercepts to make them work nicely... at the double negative [referring to the "signs" of the lines], positive [referring to the "sign" of the parabola]... and then there's the negative and one positive, and then cross the x-axis again it goes positive.

![Figure 62: Graph #1 - Working backwards (Beth)](image)

Beth did seem to have a good understanding of how the "signs" of the lines gave the "sign" of the parabola.

JC: Great. I think that is real good. I like it a lot. I think you are right. I wonder if those are the only two lines that would give you that.
BETH: I doubt it... there's probably plenty, you think? You could go that way, couldn't you? [She sketches two other lines passing through the same x-intercepts, but both with negative slopes.]
JC: Possibly, I don't see why not...
BETH: You can do a lot of fun things.

The y-intercept for the parabola was 0, and so was one of the x-intercepts.

Giving her another parabola, I gave her the same directions. She drew two lines, but then admitted that she forgot to take into consideration the y-intercept.
After discussing again what the product of the y-intercepts of the lines needed to be, Beth changed one of the lines to correspond to the y-intercept of the parabola.

The next couple of parabolas (Figures 63-65) that I gave to Beth were a bit more difficult.

BETH: Oh, my gosh. [Pause.] I'm confused.

![Graph #3 - Working backwards (Beth)](image)

BETH: Can you give any helpful hints?
JC: Well, can you tell me what one line would be, maybe? Well, that's confusing you, cause ... let me throw this hint at you. The lines don't have to be different.
BETH: Oh, yeah. So ... you just throw them right through the middle? Would that do it? [Pause.]
JC: Well, the lines could have the same x-intercept.
BETH: Yeah, right ... throw one down ... but, if I put one like that and another one like that, would that do it? [She draws one line with a positive slope and the other with a negative slope going through the intercept.]  
JC: You can try.
BETH: I don't knmoowww. I'm confused ... I don't even know what I'm confused about ... O.K. I understand they have the same x-intercept ... or they can ...

To keep her from getting too frustrated with this problem, and for the sake of time, I decided to give her another hint.
JC: Does the y-intercept help you?
JC: The parabola has a y-intercept of...
BETH: 2?
JC: 2. So the two lines would... their y-intercepts would what?
BETH: Equal 2?
JC: When you...
BETH: Multiplied?
JC: ...multiplied them out... so what... give me...
BETH: -2... O.K., so... Oh, Yeah! [Erases.] Like that.

Figure 64: Graph #3(b) - Working backwards (Beth)

Beth still seemed a bit dissatisfied with these lines, but I thought that perhaps the next parabola would shed some light on this confusion. I shared this with Beth as I handed her the next parabola (Figure 65).

Figure 65: Graph #4 - Working backwards (Beth)

BETH: [Laugh.] I know this one doesn’t have any x-intercepts... so, they’re both up here [referring to the lines]. Yay! So, let’s see...
JC: What kind of line... is there a line that won’t intersect that axis?
BETH: A horizontal line . . . [She sketches in \( f(x)=4 \) and \( g(x)=1 \).]

The choice of these lines was very logical for her. They satisfied the conditions for the \( y \)-intercept, the \( x \)-intercepts (there were none), and the "signs". We talked about the equations of the lines she had drawn. She seemed disappointed that the product of these two expressions was another constant and not a parabola.

JC: Yeah. So, I've kind of thrown in something that . . .

BETH: You cheated!

JC: Yeah. This one can't be broken down into two linear functions because it doesn't cross the axis.

We continued to talk about what this meant in terms of the algebraic representations, i.e. that there were no real roots, and hence it couldn't be factored except into complex expressions. I then compared this graph to the parabola we had done before that had just one root. Despite encouragement, Beth remained unconvinced about the arrangement of lines she had given for that third example.

**Extending Activities to Cubic Polynomial Functions**

The next component of this teaching episode involved building cubic polynomial functions from linear functions. This was done on paper as the software could not show four different functions on the graph simultaneously.

The layout of the paper was very similar to what was seen on the screen when we built quadratic functions (Appendix E).

Beth wrote the equation for three different linear functions and graphed each one on separate axes and then in a larger box on the same axes. She color-coded the lines as the software had done.

For the first example, she chose the three linear functions: \( f(x)=x + 1 \), \( g(x)=2x - 1 \), and \( h(x)=x \). I then asked her what she would get if she multiplied the expressions.

BETH: A cube.
JC: A cubic. O.K., and these things are called what . . . of that cubic expression?
BETH: Factors!

Although, in the previous interview, Beth was unsure what was meant by a factor, she was now confidently identifying the linear expressions as factors. I asked her to sketch the graph of the cubic using the three linear functions as guides. At first she didn’t understand how to go about doing this and she started multiplying the linear expressions out to get the cubic expression. I believe from this point she intended to use algebraic techniques that she had learned in class to graph the cubic. I stopped her and talked about this task as an extension of the prior tasks where we built the quadratic functions from linear functions.

BETH: I understand better now. O.K. You can go . . . rrump . . .
JC: Yeah, that’s the idea. But, what are you thinking when you say that?
BETH: um . . . It crosses that x-intercept, that one, and that one.
JC: O.K., and what about the y-intercept . . . what do you think about the y-intercept?
BETH: There’s only one. Oh, it would be at the origin.
JC: Why?
BETH: Because, that’s . . . one of the . . . factors . . . I don’t know.

I sensed again that she was just following the pattern from previous examples, and not really understanding why the cubic went through the same x-intercepts as the lines. We again used the strip of paper to observe how the “signs” of the three linear factors combined to give the “sign” of the cubic. The graph she had drawn was reasonable, but she really had not considered these signs in getting the graph.

I gave Beth another piece of paper so that she could do the same thing for homework, build a cubic from three linear functions. At our next meeting together, she talked about what she had done to get the cubic from the three linear functions.

BETH: O.K., I have the same x-intercepts going . . .
JC: O.K.
BETH: And the y-intercept would be . . . these three numbers multiplied together . . .
JC: And what's that?
BETH: 9?
JC: Yeah.
BETH: And since it was negative it went that way instead of that way.

Beth was referring to the product of the leading coefficients of the linear factors when she said "negative". The direction of the graph had been found with linear functions and quadratic functions in this way. When looking at the graph of the cubic again, she was unsure that "negative" caused the graph to weave through the x-intercepts as she had drawn it. She had not looked at sections of the graph divided by the x-intercepts to determine the changes in sign for the cubic.

I asked Beth what type of a function was generated by multiplying three linear expressions. When she responded, "quadratic", I countered that quadratics were of degree 2, and she corrected the response to "a cubic". She added that in the following journal entry she had said "quadratic" when she meant to say "cubic".

**Journal entry.**

The last two meetings were very fun. The graphing functions computer was a great trick. I never really thought of quadratic functions that way and I found the idea very interesting. I suppose I knew the information, but it was brought to my attention in an interesting new way.

Though I felt that Beth's understanding of the graphs of polynomial functions had been slightly enriched through the teaching episodes, I was a bit disturbed that she viewed our work as another "trick". I didn't want to give her just more rules to learn, or another trick to memorize, but to promote understanding of these graphs by building connections between them through the teaching episodes.

**Assessment of the Teaching Episodes: A Reevaluation of Beth's Understanding**

The following interview was meant to determine changes in Beth's understanding of the graphs of polynomial functions from the first interviews.
Having finished the teaching episodes with Beth, I traced along the graph of a cubic polynomial function using the PC-81 Emulation Software as I had done in the previous clinical interviews. I asked her to talk about the various parts of the graph as I moved along the curve.

Her answers did not change from those of the previous interview, until I asked her what parts of the curve provided information about the equation.

JC: If you didn’t know what this equation was, at all, is there a way you could find that equation by looking at various parts of the graph?
BETH: You take your x-intercepts and do x minus the first one . . . and then do that for each of the intercepts and then multiply them together and you have a lovely . . .
JC: So, those are . . . what are those called?
BETH: Um, factors?

Looking at my questionning later, I wished that I had used the word function instead of equation here with Beth. I was left unsure as to whether she understood that the factor theorem could be used in finding functions of the form \( y = f(x) \) as well as in finding equations of the form \( f(x) = 0 \). I clarified that there possibly could be a stretch factor, a constant “out front”, also affecting the curve. I asked about the y-intercept.

BETH: That would be . . . actually the x-intercepts multiplied together, right?
JC: Um . . .
BETH: No?
JC: Ah, not the x-intercepts.
BETH: Oh! The y-intercepts multiplied together.
JC: Of what?
BETH: Of the . . . linear equations? . . . of each of the factors?

Beth had repeated many of the ideas we had used in the teaching episodes. She seemed to be making more connections between the roots of the function, the zeros of the function, and the factors of the polynomial.

Putting the graph of a quadratic function in the window, I asked what parts of the parabola Beth would use to find the equation.

BETH: um . . . x-intercepts and the slope?
JC: x-intercepts and the slope. What do you mean by the slope?
BETH: The . . . which would make it the factor . . . the coefficient of \( x \)
... how fast it goes wide? How wide it is?

It was interesting that Beth still used the word "slope" to describe the leading coefficient of the quadratic equation. By "factor", I believe she was referring to what had been called the "stretch factor". This additional use of the word factor may have caused confusion as to its meaning as a linear expression. We talked in more depth about the linear factors of this quadratic function and about the value of the y-intercept.

JC: Let's say this intercept [pointing to an x-intercept] was ... um ... 3, and this one's 5. O.K.? So, what would be the equation? Can you come up with it?
BETH: x-3 times x-5.
JC: O.K., and what would make the y-intercept?
BETH: The ... 15?

We talked about the stretch factor as the finishing touch to the equation. I reminded Beth that in the previous interviews she had used the vertex to determine the equation. I assured her that both ways were valid and that either the vertex or the intercepts could be used to find the equation.

The last graph that I showed Beth was the graph of a linear function. I asked how she would determine the equation of the line.

BETH: Take your ... y-intercept down at the end of your equation and then find your slope.
JC: O.K.
BETH: Rise over run!
JC: So, you're going back to y=mx+b?
BETH: Yeah.

Beth had developed confidence, somewhere along the way, in determining the equation of the line using the form y=mx+b. During the first interview, she had been unsure of how to find the slope. When asked if she could get the equation of the line using the x-intercept, Beth didn't know how but thought that you probably could. Her use of the x-intercept across classes of polynomial functions did not include linear functions.
Zero of polynomial:
(c,0) is on the graph

Root of Polynomial:
P(c)=0

Figure 66: Interpretations of the x-intercept

Looking again at the model above, I would classify Beth's interpretations of the x-intercept following the teaching episodes as:

Acquired:  b - There was no indication of any change in this already acquired interpretation.

Tentative: a, c - Beth still interchanged the coordinates of the intercepts, so I feel interpretation "a" remained a tentative interpretation. Although Beth could use the Factor Theorem to find the factors of a polynomial, I was unclear whether she viewed it as only useful in determining an equation of the form f(x)=0, or if she also viewed it as useful in determining the algebraic representation of the function, y=f(x).

Journal entry. Beth wrote in her journal after these teaching episodes:

I had lots of fun with this project. I did really bad on my last test [she got an 85%] because I wasn't concentrating at all and now I know a lot better. I still loved the computer programs and wish we could use them in class more often.
Summary

Although Beth at first appeared quite shy, she was quite talkative once she got to know you and exhibited a dry sense of humor. She kept to herself in this algebra class, however, choosing to work alone rather than work in groups. She was confident in her ability to do mathematics, and enjoyed mathematics. Viewing mathematics as rule-based, and oriented around finding the answer, she said that she was more apt to remember the rules if there was a nice little trick attached to them.

Her initial perception of function was an equation, but when given a graph, remembered the vertical line test, which she interpreted as "you can have only one answer". She was able to see on a graph that there was "1 y for each x" but was not able to recognize this from a set of points or a table of values.

She was relieved when she saw function notation, taking that in itself to be enough of a guarantee that the equation was indeed a function. Beth had a great deal of difficulty knowing x from y and the ordering of the coordinates in the ordered pairs. What confused her is that for an x-intercept, y=0, and for a y-intercept, x=0. She struggled with this throughout the study. It affected a lot of her work, from finding the slope to describing intercepts.

Beth often groped for words, inventing terminology. Like the other students, describing graphs felt new to her. In describing a line, Beth gave two points and then said, "There's a slope through there." If the leading coefficient was positive, she labeled the equation "positive", if the leading coefficient was negative, the equation was "negative". She commented in her journal, that though she often felt she knew the answer, she "had no way to explain it". Later, in another journal entry, she reported that the interviews were helping her sort out the information. Having her talk about the graphs was strengthening her understanding and causing her to deal with some conflicting conceptual ideas.
Beth was able to express the equation of the cubic as “a” times the product of linear factors determined by the x-intercepts. She did not use the term “factor” however, she had no idea what this term meant in relation to this graph. She knew that the number of turning points gave the degree of the polynomial, and “a” determined whether the curve was “positive” or “negative”. More comfortable with the graph of a quadratic function, Beth determined its equation by shifting the basic parabola $y=x^2$ using the form $y = a(x-h)^2 + k$. When asked if the x-intercepts had anything to do with the equation, she answered, “No”.

She used the slope/y-intercept form of the equation, to determine the equation of a line. She was unsure of the correct way to determine the slope because of her confusion with the orientation of x and y. She did not think that the x-intercept could be used to determine a linear equation.

Whenever asked to give an equation for the graph of a polynomial function in factored form, she expressed the equation in the form $f(x) = 0$. This was due to usually seeing polynomials in this form in the context of solving equations. She had seldom seen the function in factored form, so that the product of the factors was equal to $f(x)$ or $y$.

Beth linked the equations of the various classes of polynomial functions together by often using the same coefficients for the variables, i.e. instead of $y=mx+b$, she would say $y=ax+b$ likening this form to the form of a parabola, $y = a(x-h)^2 + k$. She also referred to “a” in this latter form as the “slope of the parabola”. This was a departure from the notation and terminology used in class, revealing her desire to draw connections between these topics.

During the teaching episodes, Beth predicted the graph of the product of the linear expressions would be a “regular parabola”. She described it as skinnier and “negative” due to the negative, fractional value for “a”. Not clear as to where...
the graph would intersect the x-axis, she showed surprise when she saw that the
parabola had the same x-intercepts as the lines.

She didn't naturally extend the idea of building polynomial functions to
polynomial functions of third degree. When asked about the product of three
linear expressions, she began to revert to algebraic techniques to graph the cubic,
intending to multiply the three linear expressions and then use the rational root
theorem, and synthetic division to graph it. She held on to the sign of “a” to
determine the orientation of the graph through the intercepts, rather than linking
the signs of the functional values of the lines to the signs of the parabola.

A change in her understanding after the teaching episodes was observed
when she was able to give the factors for both the cubic and quadratic polynomial
functions when asked about finding their equations. She referred again to the
“slope” of the parabola as the value of “a” that would be put in front of the factors.
She claimed, as Mark had done, that the value of the y-intercept was equal to the
product of the x-intercepts. In addition to these changes, her ability to determine
the equation of the line, using the form y=mx+b, had improved, since she was now
confident in finding the slope of the line.
CHAPTER X

DISCUSSION AND CONCLUSIONS

Through the case studies of Mark, Lisa, and Beth, I have presented a view of each student’s understandings of the graphs of polynomial functions. In this chapter, I pull together data from the study to provide information in response to the research questions. The titles of the section headings indicate their relation to the seven research questions listed in Chapter 1.

Before closing remarks and reflections, recommendations suggested by the current study are cited for instruction and future research.

Making Connections: Building on Prior Understanding

The three students in this study often linked their understandings of the graph of the cubic function with the graphs of the linear and quadratic functions in statements that they made in the interviews and teaching episodes. These links between the classes took the form of observed similarities in the attributes of the graphs or in observed similarities concerning the effect of the equation’s coefficients on the graph. This section pulls together portions of the case studies that refer to the connections that the students made between their study of the graphs of polynomial functions of degree greater than two and their study of the graphs of linear and quadratic functions.

Perhaps the most obvious way in which two of the students linked their understandings of the graph of the cubic to the graph of a quadratic function was
in viewing the turning point of the cubic "as a kind of vertex". Mark indicated that "if it was like a parabola . . . you could plug in the numbers in the \(a(x-h)^2 + k\). . . and that would help you get the equation". Lisa also made statements of this type.

Lisa and Beth often replaced the coefficients, \(m\) and \(b\), in the slope-intercept form with \(a\) and/or \(h\) respectively, linking the coefficients of the algebraic form used to graph a line, \(y = mx+b\), with the algebraic form used to graph a parabola, \(y=a(x-h)^2 + k\). Furthermore, Beth referred to the "stretch factor", \(a\), of the quadratic equation as the "slope of the parabola" and then later questioned, "Do parabolas have slopes?". These students noted a similarity between the leading coefficients, \(m\) and \(a\) respectively, in that both coefficients affected the shape of the graph in terms of "wideness" or "slope" as well as in direction, i.e. as stated by Beth, the sign of the leading coefficients made the curve "positive or negative" or, said differently, made the curve "go the other way".

Other connections that were made by these students between the classes of polynomial functions are actually inherent to the graphs of all functions. As an example, all functions have the property that the 2nd coordinate of the x-intercept is 0 and the first coordinate of the y-intercept is 0. This property is a connection between Cartesian graphs in general and not solely a tie between the classes of polynomial functions. When Mark referred to the value of the y-intercept in the algebraic representation of lines and parabolas as "what you add on at the end", he was making a link to his general knowledge of function and graphs rather than a link to his knowledge of linear and quadratic functions in particular.
Understanding that all graphs could be physically translated, Mark and Beth viewed the value of the y-intercept as a shift up or down.

Additionally, the students in this study retained some "blurred conceptions" throughout their study of polynomial functions that were based on prior understanding of functions and graphs. Monk (1992) describes a "blurred concept" as a concept that is available, "but not at all robust . . . sometimes they are fused, conflated, or exchanged." (p. 176). A prominent blurred conception of this type was the orientation of the Cartesian coordinates, x and y. Beth, in particular, reversed the positions of the x and y coordinates in ordered pairs, in defining "function", and in giving a definition for "slope".

The connections that the students made provide evidence that the students' conceptual understandings of the graphs of polynomial functions was built on their understandings of the graphs of linear and quadratic functions and their general knowledge of functions and graphs.

Making the Transition to Polynomial Functions of Higher Degree: Contributing/Inhibiting Factors

Although Mrs. Tucker tried to introduce the classes of polynomial functions as extensions of one another, instructional factors can be identified that worked together to inhibit the students from making more connections than they did between the classes: the notation used (using P(x) for polynomial functions of degree greater than two), the emphasis on certain attributes of the graphs (for example, the vertex with parabolas), the emphasis on particular forms of the functions (for example, y=a(x-h)^2+k for quadratic functions), the techniques used to solve equations (factoring quadratic equations vs. synthetic division), the organization of class activities, and an emphasis on manipulations rather than
understanding. (These inhibiting factors will be talked about in greater detail later in the section dealing with situational and contextual factors.)

Mrs. Tucker was aware of at least some of these problems, evidenced by a comment in her interview:

MRS. TUCKER: I try to tie them in— I say a lot of the things that work with quadratic equations work with polynomial functions. But I think the fact that they are using synthetic to solve a polynomial, makes it seem different . . . but I don't know how to avoid that. They can certainly use synthetic with quadratics . . . but by doing the quadratic formula . . . so I think they just . . . and that's part of the reason why I didn't split the test this year too. I wanted to see what keeping them together . . . because if I split it, then I'm making the split . . . you know, quadratics here and then . . . polynomials . . . [laugh]. I don't know.

In addition to trying to tie the classes of polynomial functions together, Mrs. Tucker was now trying to find ways to tie the graphical and algebraic representations together. As an example, she decided to teach the students to graph polynomial functions of degree greater than two by using the graphing calculator in conjunction with the Rational Root Theorem. In previous years, she had reserved the Rational Root Theorem for solving polynomial equations algebraically. She told me in the interview that she preferred this new way because the students "made connections better" when they generated a list of possible roots using the Rational Root Theorem and were also able to see them on the graph.

Although the instructor attempted to make some of the connections between the graphical and algebraic representations more evident, the students maintained weak conceptions of the links between them. As evidence, the students did not link the x-intercepts as seen on the graph with the factors of the algebraic representation for any of the classes of polynomial functions. Although
Mark could form factors if given a list of roots to determine the equation, he couldn’t form factors when shown the x-intercepts on the graph. A possible explanation for this inability is that the classroom emphasis of finding factors had been on algebraic manipulations.

A factor that seemed to block the students from making connections between the graphical and algebraic representations of the quadratic functions was that, depending on the task at hand, different forms of the quadratic equation were used. Lisa called \( y = a(x-h)^2 + k \) the “working equation”, which she classified as a third degree polynomial, and \( y = ax^2 + bx + c \) the “standard form”, which was the “quadratic one”. In general, the students did not seem to view the two forms of the quadratic equation as synonymous, i.e. different forms for the same graph. This, perhaps, was due to the fact that the former was expressed as equal to “\( y \)” and used primarily for constructing or interpreting graphs, and the latter was usually expressed as equal to “\( 0 \)” and used to algebraically find the roots. As a result, though students could graph the parabola using the form \( y = a(x-h)^2 + k \), they did not connect the x-intercepts on that graph with any form of the quadratic equation.

**Activities Suggested by the Interviews to Make the Connections More Sallent**

I thought that the method of building polynomials from linear expressions used in the teaching episodes might be one way not only to foster connections between the classes of polynomial functions, but also to foster connections between the graphical and algebraic representations of these functions. In particular, the graphical interpretations of the x-intercept, i.e. the relation
between the zeros of a polynomial $f(x)$, the roots of the equation $f(x)=0$, the factors of the polynomial, and the intercepts of the graph, would become more evident across classes.

The following paragraphs indicate the changes made in the students' understandings as a result of these teaching episodes by giving a synopsis of the students' understandings relative to the graphical interpretation of the $x$-intercept before and after the teaching episodes.

**Observed Changes in Understanding After the Teaching Episodes**

The model given in Chapter II, that portrays the various interpretations of the $x$-intercept has been repeated below. The description of implications "a", "b", and "c", are repeated below the model. These were the interpretations of the $x$-intercept investigated in this particular study.

![Figure 67: Interpretations of the $x$-intercept](image)
Implication a: An x-intercept as seen on the graph implies that the 2nd coordinate of that point is 0, i.e. the x-intercept is \((c,0)\).

Said differently:
An x-intercept as seen on the graph implies that the value of \(x\) is a zero of the function.

Implication b: An x-intercept as seen on the graph implies that when \(c\) is substituted for \(x\) in the polynomial function, the function value becomes 0, i.e. \(P(c) = 0\).

Said differently:
An x-intercept as seen on the graph implies that the value of \(x\) is a root of the function.

Implication c: An x-intercept as seen on the graph implies that \((x-c)\) is a factor of the polynomial function \(P(x)\).

The table that follows gives a synopsis of the students' understandings of implications a, b, and c before and after the teaching episodes. The terms "Acquired", "Tentative", and "Missing" are used to describe the degree to which the student has grasped the concept, or in the words of Monk (1992), the degree to which these concepts appear "robust". I equate a "tentative" conception with Monk's notion of a "blurred" conception.

The activities used in the teaching episodes were primarily directed towards making connections between the classes of polynomial functions by graphically focusing on the linear factors of these functions. Though this activity did strengthen the students conception of a factor of a polynomial expression in relation to the graph, this was the only implication from the model that showed significant strengthening. As indicated in this table, the understandings for
interpretations "a" and "b" did not noticeably change for any of the students.

Interpretation "c" was affected by the teaching episodes for two of the students.

<table>
<thead>
<tr>
<th>Implication &quot;a&quot;</th>
<th>Implication &quot;b&quot;</th>
<th>Implication &quot;c&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before</td>
<td>After</td>
<td>Before</td>
</tr>
<tr>
<td>MARK</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>LISA</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>BETH</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

KEY: A = Acquired, T = Tentative, M = Missing

TABLE 4: Synopsis of student understandings before and after the teaching episodes

The table also indicates that student understanding and the conceptual development with respect to the interpretations of the x-intercept was inconsistent amongst the students, i.e. these students developed this knowledge in idiosyncratic ways.

Additional Activities Suggested by the Interviews that Would Promote Connections Between Classes

Due to time constraints in this study, the activity used in the teaching episodes was the only activity I asked the students to participate in. There are certainly other activities that might strengthen the other implications as well. In thinking about what these other activities would be, I came up with the following list of suggestions for a "reformed study of polynomial functions" that I hypothesize would enhance the connections between the classes of polynomial functions and between representations.
1. A graph's component parts (lines) should be shown whenever an equation is broken into its component parts (factors).

2. Make the rational root theorem an aid in solving quadratic equations as well as polynomial equations of degree greater than two.

3. Place more emphasis on the roots, rather than on the vertex and y-intercept of the graphical representation of a parabola.

4. Given two roots, $c_1$ and $c_2$, exploring the family of parabolas in the form $y = a(x - c_1)(x - c_2)$.

5. Given a root, $c_1$, explore the family of lines going through that root.

6. Use the slope/x-intercept form to obtain the equation of a line.

7. As another method of determining the x-intercept of a linear function, have students factor out the $m$ from the form $y = mx + b$.

These suggestions have an underlying theme - that linear and quadratic functions should be explicitly considered as first and second degree polynomial equations respectively, and as such constitute examples for more general polynomials.

Graphing technologies should be used as much as possible as they allow the user to see different representations simultaneously and they make it easier to obtain and compare graphs.

**Consistencies in Understandings and Conceptual Development Amongst Students**

There were quite a few similarities between students in their interpretation of the graphs of polynomial functions. At least in part, this is due to the fact that they were all influenced by the same teacher and had pretty much the same experiences in class. These experiences contributed to the way they approached tasks. When shown a "table of values", for example, and asked if that
table was a function, the three students questioned whether the table values were part of a larger set of values. I attributed this to their experiences with prior exercises involving tables in which they developed a sense of the shape of the graph by putting only a few ordered pairs in a table.

**Graphical Attributes That Received the Focus of Attention**

In the first interviews, all three students focused on the same portions of the graphs. For lines, this was the slope and the y-intercept. For parabolas, it was the vertex and the "wideness" of the curve. For cubics, it was the overall shape and relative position on the axes.

In addition, the students all leaned to the same algebraic forms of the polynomial functions when asked to determine an equation from a graph. For linear functions, this was the form $y=mx+b$. For quadratic functions, it was the form $y = a(x-h)^2 + k$. For polynomial functions of higher degree, the form was

$$P(x)=a_nx^n + an_{-1}x^{n-1} + an_{-2}x^{n-2} +...+ a_1x^1 + a_0.$$  This is not surprising as these forms were the focus of classroom instruction.

**Verbal Descriptions of Graphs**

All three students said they found describing graphs difficult. Rather than using the customary terminology associated with the graphs in the classroom, they often tried to describe a graph with more familiar vocabulary or with words pulled from mathematics but used in a different sense. Beth, for instance, would describe a line as "positive" if the slope was positive and she would describe a parabola as "positive" if the leading coefficient in the algebraic representation...
was positive. Mark avoided using terminology such as "x-intercept", by saying, for example,"Put a point at 2 on the x-axis". In addition, instead of trying to describe the slope of a line, he would give about 4 points on the line to give the idea. He noted in his journal entry that it was easier to give equations then describe them. Though the students had heard the mathematical terms used many times in class, the students were not accustomed to using this terminology in discussion. The students usually understood what the terms meant from hearing them in class, but these terms had not yet become a part of their routine vocabulary. Being more comfortable with mathematical terminology would probably have made the interviews easier for the students, but I don't believe that use of standard terminology necessarily implies a higher level of understanding of the mathematics.

_The Influence of Technology on the Knowledge of Graphs_

All three students that I interviewed commented that they enjoyed using the graphing calculator. Though none of the students had one of their own, they all wished that they did. The ability to quickly generate graphs was very appealing.

Though the teacher was a little nervous about being able to effectively instruct the students on the calculator's use, she seemed to relax as she allowed its use more and more in the class. As mentioned earlier, she recognized the benefits of the graphing calculator in promoting connections between the graphical and algebraic representations. The time consuming process of plotting points to generate graphs is eliminated by using the graphing calculator, so
exposing the students to both algebraic and graphical representations simultaneously is much less of a problem.

Despite the benefits of graphing technology, this study corroborates research discussed in Chapter IV that has shown that what a student sees in a graphical representation depends on his/her existing knowledge about functions and graphs (Schoenfeld, Smith & Arcavi, 1993; Larkin & Simon, 1987). As explained in that chapter, students see a graph through their current understanding of graphs of that type. Students don't always see what we expect them to see, i.e. they see what their conceptual understanding at that time prepares them to see.

The statements from the research above were exemplified in the current study by the students' attention being consistently drawn to certain attributes of a graph. From a pointwise perspective, the students focused on the portions of the graph according to the focus that they had experienced in class. This will be further discussed in the next section. Viewing the graph from a global perspective, all three students described the graph of a cubic polynomial functions as a "swivel", the graph of y=x² as a "basic parabola", and other parabolas as translations and/or stretchings of the "basic parabola".

The students' understandings of the symbolic conventions of the Cartesian plane also played a role in their understandings of a graph. In interpreting portions of the graphs, the students found themselves having to fit what they saw with the notational conventions they understood. Sometimes their explanation of a graph did not mesh with their understanding of the notation, which made them confused and unsure of themselves, as when they interchanged the positions of the x and y coordinates. Beth repeatedly said the x-coordinate was 0 at an x-
intercept, and the y-coordinate was 0 at a y-intercept. Function notation caused problems for Lisa. When thinking about the meaning of $P(x) = y$, she commented, "that's always confusing to me, because how can $x$ be $y"."

Goldenberg (1988) has said:

Graphic representations . . . have their own rhetorical conventions that students must learn, and they contain ambiguities that are clarified only through the use of further specifications, such as those contained in the conventional algebraic notation. (p. 138)

Hence, a graph is interpreted using the algebraic conventions, and thus if a student has a weak understanding of the algebraic conventions he/she will be hindered from understanding the graph.

**Relation of Affective, Situational, and Contextual Factors to Student Understandings**

The following sections discuss the role that affective variables, and the role that situational and contextual factors, seemed to play in this study in the development of student understanding, and the ability of the students to make connections. Classroom assessment, the teacher's use of terminology, and classroom emphases are the situational and contextual factors having the greatest perceived impact on their understanding.

**Affective Factors**

As stated in Chapter III, a student's personality, motivation, and attitudes play an important role in the degree to which the student will become engaged in the subject matter and whether conceptual change will take place.

It is not surprising that the measure of success that the students in this study had experienced in mathematics in earlier years contributed to their current enjoyment of mathematics and their amount of self-confidence in doing
mathematics. I found that both Mark and Beth, who indicated previous positive experiences with mathematics, enjoyed doing mathematics and were fairly confident approaching new tasks, while Lisa, coming from many negative experiences in mathematics, did not enjoy doing mathematics and did not have a lot of confidence when approaching new tasks.

The three students in this study differed in their personalities, their degree of "function anxiety", their attitudes toward mathematics, and their perceptions of what mathematics and algebra is. Their uneasiness about giving a "wrong" answer differed between students, Lisa probably being the most worried about appearing "ignorant". This disquietude sometimes affected their ability to think clearly in the interviews.

Each student seemed to be attracted to different aspects of mathematics in general, and different representations of a function in particular. The students seemed more comfortable when these representations were salient when given a task. For example, Mark liked "to see it", he seemed to favor the graphical representation of the function. When confused, he often resorted to the graph to sort out his thinking. Beth seemed to be more comfortable with an equation and a lot less comfortable with graphs and tables. Her focus when giving the definition of function was on a rule, a process by which x-values were assigned to y-values. Lisa expressed her hatred of functions in general, and added in particular "I don't like graphing!".

**Situational and Contextual Factors**

Caught in the midst of the reform movement in mathematics education, Mrs. Tucker was trying to adapt to the rapid changes taking place. Having taught
Algebra II for eight years, she was now submitting to the pressure of using technology in her classroom and of putting a greater emphasis on functions and graphing throughout the curriculum she used. She agreed with these calls for reform, thinking that they were ultimately for the betterment of the students' understanding. At the same time, however, she found that it required considerable effort to change some of the deep-seated methods she had acquired over the years of teaching the material. After one class, in which she tried to generate discussion about the interpretation of graphs that modeled real-life situations, she remarked, "I always think that one of the reasons I went into Math is so I wouldn't have to do discussion!" She made this comment in reference to calls for increased verbal expression of mathematical language in the classroom.

Mrs. Tucker chose to emphasize functions and graphing over the course of the year. In so doing, she became conscious of changing certain terminology she used in class. After one class in which instruction focused on the graphs of linear functions, she admitted that she should have used the word "function" a lot more, instead of "equation", when talking to the students. The graphing calculator was used often by the students as an exploratory tool.

*Classroom assessment.* Though the classroom instruction seemed to reflect an emphasis on functions and graphing, the assessment that Mrs. Tucker used did not (Appendix F). The focus of her tests and quizzes was predominantly on algebraic manipulations and procedures. A student with an ability to perform these manipulations could have received a decent grade on these quizzes and tests while maintaining a weak conceptual understanding of the subject matter. As an example, to solve a polynomial equation of degree greater than two, the three students in the study were quite proficient at using the Rational Root
Theorem and synthetic substitution. The students were given the following problem on their test: A fourth degree polynomial equation has zeros of -2, 1, 3, and 4. If \( P(-1) = 20 \), find \( P(2) \). All three students in the study were able to express the polynomial as the product of factors, \( a(x+2)(x-1)(x-3)(x-4) = P(x) \). However, in evaluating this expression for \( x = -1 \), two of the three students chose to multiply the four factors together and use synthetic substitution in some way to solve the problem. Both students got bogged down with these manipulations and were unable to finish the problem. Though they were proficient with synthetic substitution, they lacked a clear understanding of when it was useful.

Mrs. Tucker seemed to infer by her choice of items on these instruments, what she felt was important for the students to understand about the subject matter. Though this may not correspond to her belief of what was important for them to understand, the emphasis on procedural skills on the quizzes and tests may have affected what the students sensed was important to her. The students, responding to experience and their own "survival" instinct, would focus on procedural skills in studying for quizzes and tests.

Though Mrs. Tucker had made some changes in emphasizing functions and graphing, in using technology, and in allowing students to work together in her classroom, these changes were not reflected in her assessment practices. Hence, I would not regard these changes as deep changes, inferring by this that deep changes would affect all aspects of instruction including assessment.

**Terminology.** There are quite a few instances in which the students adopted the language of the teacher in talking about functions and graphs. As mentioned previously, all three students described the graph of a cubic
polynomial function as a "swivel", referred to the graph of $y=x^2$ as a "basic parabola", and equated the word "roots" with "answers" as she had done in class.

The use of terminology in the classroom seemed to have an effect on some of the students' perceptions of the subject matter. Some terminology she used may have caused confusion in the students' understandings of concepts. For instance, she used "factored form" when speaking of a quadratic function in the form $y = a(x-h)^2 + k$. This may have inhibited students from identifying a "factor", $(x-c)$ where $c$ is a root, of a polynomial function of degree other than two. The terminology "stretch factor" used with the graphs of parabolas and cubics and equated with the leading coefficient of the polynomial, might also have been a deterrent in understanding the meaning of the word "factor" when used to describe a linear expression.

Looking at the terminology used from a conceptual change point of view, in trying to "make sense" of the subject matter, the students piece their past experiences together, to obtain a concept image of what a "factor" is. In attempts to form a mental image of what a factor is, a student who refers to $y = a(x-h)^2 + k$ as "factored form" when the terms of this form are added together, or who refers to a coefficient as a "stretch factor", might experience trouble fitting in the piece that describes a linear expression as a factor. The incongruity may contribute to their inability to obtain a clear understanding of what a "factor" is. As indicated in the case studies, two of the three students in this study equated "factor" with an x-intercept, the other student claimed to have "no idea" of what was meant by "factor".
Mrs. Tucker had also referred to the form, \( y = a(x-h)^2 + k \), as "easy to graph form". Referring to the form in this way, highlighted, for the students, its usefulness in graphing quadratic functions. The students perception of this form as "easy" could have made it a first resort in any problem situation requiring the graph of a quadratic function. I would argue that the form which is the most expedient, i.e. the "easiest" to use, however, depends on the problem situation. For example, when the students were asked in the teaching episodes, to explain what the graph of the product of two linear expressions would look like, all three students started to multiply the linear expressions together to get the standard form of the function. Mark commented that he would have to change the standard form into the form \( y = a(x-h)^2 + k \) to know what the graph looked like. None of the students could determine the position on the axes from the linear factors, though this might have been easier.

There has been discussion in other research as to the progression between perspectives in the conceptual development of the function concept, i.e. process conception to an object conception or vice versa (Moschkovich, Schoenfeld, Arcavi, 1993; Kieran, 1993; Schwartz & Yerushalmy, 1992) but it is generally agreed that students should eventually be comfortable with both perspectives and that different representations will make different perspectives salient. I am suggesting here that the terminology may also affect from which perspective a student views a representation.

As an example, Mrs. Tucker used the word "swivel" to describe polynomial functions of degree three. Not only did this word seem to leave an impression about the shape of the curve to the students, the word also gave a dynamic quality
to the graph. In describing these graphs, Beth would say “it swivels through” or after giving points on the curve say “can you swivel through that?”. Lisa remarked when looking at a swivel, “It kind of swivels down through the origin and . . . kind of . . . keeps going”. On the other hand, the word used for the graph of a quadratic function, “parabola”, seems to leave more of a static impression.

Together these words may leave the impression that a parabola is an object and a “swivel” is a process. The impression of a parabola as an object is then reinforced when a parabola is picked up and moved under a translation. A parabola described as “wide” or “thin”, also suggests that the graph has an object quality rather than a process quality, as these adjectives are most often used to describe objects rather than processes.

**Classroom Emphases.** The emphasis that Mrs. Tucker placed on the attributes of the graphs, for each class of polynomial function, contributed to the students being drawn to these attributes when given a graph. For linear functions, the classroom emphasis was on the y-intercept and the slope. These were the attributes that were generally used to determine the equation of the line. For quadratic functions, the emphasis was placed primarily on the vertex and the “wideness” of the parabola and the position of the parabola was viewed as a variation or translation of the “basic parabola” y=x^2. For polynomial functions of degree greater than two, the emphasis was placed graphically on the x-intercepts and the number of “turning points”.

Likewise, the teacher’s choice of a particular algebraic form for each class of polynomial function had an influence on the student’s choice of form. For linear functions, the form used most often was y=mx+b. For quadratic functions, the
form used most often was \( y = a(x-h)^2 + k \) except when solving equations. When solving equations, standard form was used: \( 0 = ax^2 + bx + c \).

For polynomial functions of degree greater than two, the form used was
\[
P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0
\]
The function was written in this form for just about all algorithmic processes. As can be seen by the test on polynomial functions in Appendix F, the students were seldom, if ever, asked to graph a polynomial in this form. However, they were often asked to solve a polynomial equation in the form \( 0 = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0 \) and in the process would use their graphing calculators to graph the function \( y = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0 \) as a tool in determining the roots.

This study also shows that giving students repeated exercises of the same type can restrict the students’ perspective and understanding of a problem situation. As an example, Beth began to think that the algebraic representations of polynomial functions were all written as \( 0 = f(x) \). I attributed this to her being given so many factoring problems in this form. In addition, the process of factoring was most often done in the context of algebraically solving a polynomial equation. None of the students that I interviewed applied this process to the context of finding the algebraic representation of a polynomial function given the graph of a polynomial function or to using factors to sketch the graph of the function.
Implications for Instruction

As the urgent need for algebra reform in the mathematics curriculum is now more popularly recognized and as mathematics educators begin making larger strides to change what algebra is taught and how it is taught, there are implications from this study that should be considered.

Fostering Connections

The Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) highlight the importance of linking conceptual and procedural knowledge among the different topics in mathematics. A major thrust of the Standards is that instruction should encourage students to see the patterns throughout mathematics, and to view mathematics as an integrated whole, rather than as a series of isolated topics. The document suggests that new concepts be introduced, where possible, as extensions of familiar mathematics. The authors claim that "developing mathematics as an integrated whole also serves to increase the potential for retention and transfer of mathematical ideas." (p. 149)

The assumptions that I held as I began this study were given in Chapter II. The focus of these assumptions was on the connections that should be made with respect to polynomial functions. Two of these assumptions were:

1. Linear and quadratic functions are a foundation for further study of polynomial functions of higher degree.
2. Connections should be made between all classes of polynomial functions as determined by their degree.

After doing this research, I still hold these assumptions. I believe that this research has reinforced these presuppositions and that students do build their
conceptions of polynomial functions of degree greater than two on their prior understanding of linear and quadratic functions.

According to the theory of conceptual change, learners try to give meaning to bits of information by fitting new information into their existing knowledge structures. This study has shown that instruction should focus on regarding the sundry classes of polynomial functions as variations of one another rather than as unrelated entities. As evidenced in this study, connections are hindered when the various classes of polynomial functions are treated as separate units, and when the focus on the graphs and equations for these various classes are unrelated. The focus in instruction has emphasized the "parts", i.e. linear functions, quadratic functions, . . . , rather than "the whole" in terms of polynomial functions. Hiebert and Lefevre (1986) write:

Conceptual knowledge is . . . knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information . . . . A unit [of information] . . . is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information. (p. 3-4)

The emphasis in instruction needs to be on the commonalities between classes, and the themes that are transferred from one class to another. This gives a sense of wholeness to the content, a gestalt from which to view all polynomial functions.

Combining research from the physical and social sciences, Caine and Caine (1994) argue that students learn by "patternning", which they describe as the meaningful organization and categorization of information. They write:

The brain is designed to perceive and generate patterns, and it resists having meaningless patterns imposed on it. "Meaningless" patterns are isolated pieces of information unrelated to what makes sense to a student. When the brain's natural capacity to integrate information is acknowledged and involved in teaching, vast amounts of initially
unrelated or seemingly random information and activities can be presented and assimilated. (p. 89)

Therefore, the primary implication of this study is that educators need to build connections between the classes of polynomial functions. In general, educators need to acknowledge that instruction should build on students prior conceptions as much as possible. Instruction should build on what students already know, relating information to previous information, to assist the students in making order out of the chaos of seemingly unrelated pieces of information.

The technique of building polynomial functions from linear functions used in this study is perhaps just one of many ways that the connections between the classes of polynomial functions can be fostered. It was shown that a glue that holds the classes of polynomial functions together is the Fundamental Theorem of Algebra; every polynomial function can be expressed as a product of linear factors. Linear and quadratic functions can be thought of as the "building blocks" or components of all other classes of polynomial functions.

**Decreasing Emphasis on Algebraic Manipulations**

This study also highlights the emphasis that has been placed on algebraic manipulations in the traditional curriculum. Heid (1995) writes:

> Currently, in spite of the best efforts of their teachers, many students come away from their algebra experience with a fragile understanding of the concepts and procedures with which they have been working. In particular, they come to view symbolic representations as entities to be manipulated rather than as meaningful representations in themselves. (p. 123)

This study corroborates Heid's statement. The emphasis on algebraic manipulations often resulted in the sacrifice of a conceptual understanding of the concept.
In concluding statements from their research on student understanding of linear functions, Moschkovich et al. (Romberg et al., 1993) declare that the purpose of teaching should be to enhance connections and understanding rather than procedural competence. Research tells us that students seem to be learning techniques at the expense of understanding the larger picture (Eisenberg & Dreyfus, 1992).

Though acknowledging the usefulness of symbolism in algebra, Heid (1995) predicts three ways in which symbolic work will differ in the future:

- There will be a greater concentration on learning the meaning of symbols as representations.
- Instead of focusing only on the application of rules for producing equivalent forms, work with equivalent forms will center on understanding the meaning of equivalence.
- Work in solving equations will draw on new ways of viewing the properties of equality and will involve greater attention to interpreting the solutions produced by computer-algebra systems. (p. 123)

The current study suggests that to pursue a meaningful understanding of polynomial functions, graphing needs to be more heavily integrated with the algebraic manipulations. As much as possible, students should be interpreting and visualizing algebraic results through the graphical representation. Visualizing the algebraic representation gives more meaning to the symbols.

Eisenberg (1992) states:

One of the main components of having a well developed sense for functions is the ability to tie together their graphical and analytical representations. (p. 154)

Currently, the emphasis with quadratic functions is finding the roots by means of factoring or the quadratic formula. Textbooks begin with a polynomial function in standard form and treat its factors as a byproduct of this form rather than vice
versa. Since the emphasis is on finding the roots for the algebraic manipulations, a proportionate emphasis needs to be placed on the graphical representation of these manipulations. A sufficient accumulation of experiences with these emphases will enhance the connections between the algebraic and the graphical representations of the functions. As the Standards (NCTM, 1989) have stated:

The connections between algebra and geometry are among the most important in high school mathematics...finding a zero of a function in algebra corresponds to determining an x-intercept of the graph of the function"... Many problems traditionally solved using algebra can now be solved efficiently and in more general cases using the geometric representation and computer-graphing techniques. This approach removes algebraic manipulative skill as a prerequisite, thereby allowing all students to address interesting problems and explore important mathematical ideas. (p. 147-148)

The availability of technology in the classroom has made it possible to more easily exploit these connections between the classes of polynomial functions by putting a greater emphasis on the graphs and thereby melding the algebraic and geometric components of these functions together.

A few examples of how technology can be used have been given in this study. Not only did the use of technology make the graphs of polynomial functions easy to explore, but it highlighted the connections between the algebraic and geometric concepts by often having dual representations visible at one time.

Fostering Communication

Another implication of this study is that students need to be given more opportunities to verbally communicate their understanding of polynomial functions. Describing graphs was not a task Mark was used to; he struggled with this and failed to bring up useful terminology such as "x-intercept" or "slope". He noted in his journal entry that it was easier to give equations than describe them.
Beth searched for words, often inventing terminology. Having to describe the graphs was a task that was new to her. She commented in her journal, that though she often felt she knew the answer, she "had no way to explain it*. Later, in another journal entry, she reported that the interviews were helping her sort out the information. Having her talk about the graphs was strengthening her understanding and making her deal with conceptual conflict. Caine and Caine (1994) write:

We acquire deeper insights in part as we find clearer ways to talk about and describe what we experience and what it means to us. It is not just a matter of getting the right answer. As we talk about a subject or skill in complex and appropriate ways—and that includes making jokes and playing games—we actually begin to feel better about the subject and master it. That is why the everyday use of relevant terms and the appropriate use of language should be incorporated in every course from the beginning. (p. 131)

To promote communication in the classroom, curriculum materials need to model this emphasis. Creating tasks that require oral and written communication will aid the teacher in giving student communication a larger role in their own classrooms. In addition, teacher preservice and inservice workshops on communication can convey to the teachers the benefits of having students talk and write about the subject matter regularly.

**Curriculum Materials**

If connections are to be emphasized in the curriculum, much work needs to be done in developing tasks, and teaching modules that take seriously the formation of these connections. Earlier in this chapter, I listed some suggestions for a modified curriculum that were indicated by the results of this study. Hopefully, curriculum writers will be encouraged to make such
modifications/additions to the curriculum so that students' understandings of the
graphs of polynomial functions can be enhanced.

The ability to build connections between numeric and graphical
representations can start long before a student gets into high school algebra.
Demana, Schoen, and Waits (1993) analyzed the content on graphing in typical
mathematics textbooks prior to high school algebra and argued that many
common student difficulties with graphing could be understood in light of the
inadequacy of this content. In each of the grades 1-6, they found that only 1-2%
of the mathematics textbook pages contained graphing content and that most of
this material was contained in special sections labeled “enrichment” or “problem-
solving”. In addition, they state that students in grades 1-6 have almost no
experience constructing a graph of any kind. In grades 7-8, graphing content was
found in only 3% of the pages.

Textbooks at the secondary level should also take more radical strides away
from an emphasis on solving equations algebraically toward a more graphical
approach. Textbooks are a prime influence in what goes on in the classroom for a
majority of teachers. This means that either teachers should receive the training
and the time to rely more heavily on their own creativity and intuitions in
selecting appropriate materials for their classes, or teachers should make
demands for curriculum materials that take greater strides away from an
emphasis on algebraic manipulations to an emphasis on graphical interpretation.
Dugdale (1993) writes:

The availability of function-plotting software has raised the possibility
of visual representations of algebraic functions playing a more
important role in mathematical reasoning, but new instructional
models are needed to encourage graphical reasoning. (p. 101)
Implications for Teacher Education

Most mathematics teachers today were educated at a time when technology did not exist in every classroom, so that their experiences with learning algebra focused on symbolic manipulation. Their pre-service training did not promote the graphical emphasis that is advocated today. For the most part, the topics in algebra were broken into isolated units devoid of unifying connections. Their teaching reflects these experiences. As a result of this study, I would recommend that teachers be given opportunities to explore the connections in algebra, be given time to think about their own learning and understanding, have experiences in which they can discover on their own how communication fosters understanding, and see how technology can be used effectively in the classroom. These topics should be addressed in preservice education, at inservice workshops, and at conferences.

The methods of teaching about the classes of polynomial functions are deeply entrenched. Though "there is a mathematically accepted way to think about the subject matter" (Moschovitz, 1992, p. 129), promoting meaningful learning often means that educators must look beyond these entrenched ideas, and come to grips with the complexity involved in the many facets of an understanding of polynomial functions. This study indicates that there are many connections between the classes of polynomial functions and between the graphical and algebraic representations of a function that students should develop in order to form a powerful understanding of polynomial functions. The preceding sections point to classroom practices that currently hinder the formation of these connections.
Teachers need to reflect on their own practices and, using knowledge on how students learn, work towards practices that foster the connections between classes and between representations. Knowledge of student conceptions needs to form the basis for designing instruction. Fennema and Franke (1992) point out that there is not much information available on how students learn most topics in the school curriculum. After reviewing research on the relationship between teacher knowledge and practice, Fennema and Franke conclude that teachers need to develop a knowledge of how students learn particular topics for the reason that this knowledge will influence classroom instruction in a positive way. The results of this study provide information on how students develop an understanding of the graphical representations of polynomial functions. As the next section indicates, additional research is needed to further explore this understanding and the many other facets of an understanding of polynomial functions.

**Implications For Research**

The current study has extended the research that investigates the development of student knowledge in the area of the graphs of polynomial functions and has the potential to add to the mathematics community's knowledge of how students develop a conceptual understanding of the graphs of polynomial functions. This research may be replicated and additional teaching episodes could be designed that would continue the move towards enhancing connections among the classes of polynomial functions.

Earlier in this chapter, suggestions were made for a modified curriculum. Each of these suggestions for improvement in instruction needs to be tried in the classroom and research needs to be done to determine if they, in fact, promote
understanding. In addition, in order to stimulate change in teaching and in the curriculum, more research is needed which investigates the effects of a graphical emphasis and the effects of the use of technology on algebra learning. That research, in turn, will hopefully result in the creation of additional modules to be tried in the classroom and also in teaching experiments that will continue to broaden the research on polynomial functions.

This research focused only on a small piece of an understanding of the graphs of polynomial functions. Results of the teaching episodes in this study indicate that connections were enhanced in just one area of the students' understandings of the x-intercept. The positive results, however, in this one area, suggest that there are teaching methods and modes of exploration that may pull a student to a more robust understanding of the concepts involved in other areas. It is the responsibility of researchers and teacher-researchers to explore the possibilities.

As this study investigated implications a, b, and c of the model portraying interpretations of the x-intercepts, research could likewise be directed toward investigating the other six implications in that model. How knowledge of these implications can be enhanced and how this knowledge is built on prior understanding would be avenues to investigate.

The clinical interviews and the teaching episodes in this study revealed that much can be learned from listening to a student communicate their understanding of a concept. I would encourage additional qualitative research on topics in the algebra curriculum that would reveal additional information about students thinking and understanding. The journal entries in this study also gave the students time to reflect on what and how they learned. Not only do these
activities give the students the opportunity to acknowledge what they know, determine what concepts are blurry, and realize what they don't understand, it allows the teacher/researcher a view into the student's understandings that is much richer than the view received from traditional quizzes and tests. Research that makes a stronger statement for communication in the classroom would be valuable. Classroom experiments are necessary that explore effective ways in which a teacher can increase student communication.

**Concluding Thoughts and Reflections**

The complexity of analyzing the data obtained in this study was compounded by the complexity of the subject matter which was being investigated—"functions". Though it was possible to get a sense of the students' current understandings of the graphs of polynomial functions, the research could only look at a very small part of the complex puzzle that contributes to the students' understandings of polynomial functions.

Following my investigations in this study, I would not change my initial assumptions concerning the connections that should be made between the classes of polynomial functions. I have increased my own efforts in the classes I teach to promote connections between topics and to listen to the students' understandings. I am not saying these are easy tasks. I, too, am working against 15 years of ingrained teaching experiences, and a society that still often views traditional methods as the way it should be done. I am also working against the limited period of time I have with my students, as well as having to search for materials that promote these ideals.
A variety of circumstantial aspects affect a student's understandings. Each student's understandings in this study were uniquely affected by the experiences within the environment in which they were placed. These experiences were determined by factors largely beyond their control; in particular, a long history of mathematicians and mathematics educators who have determined what and how they are to learn and what should be emphasized, the makers of their curriculum, and their teachers. As each student's understandings are a unique product of many years of mathematical experiences, we are obligated to get to know our students individually so that we obtain a sense of the foundation of conceptual understandings we are building upon.

Educational practices continue to evolve as our culture changes. Recently, with the advent of technology, the changes are fast-moving and widespread. We are often charting unexplored territory as we investigate what educational practices make sense for today's world. I would like to think that this study will have some effect on the evolution of mathematics education practices. As researchers disseminate information concerning their investigations about student understandings, recommendations can then be made to develop an appropriate mathematics education for today's youth.
APPENDIX A

CONSENT FORMS
Informed Consent Form

Purpose: The purpose of this research is to study the development in student
knowledge of the graphs of polynomial functions.

Description: Participation in this study means that you will have approximately
3 interviews. In the first interview, you will be asked questions about your
mathematics background, your attitudes toward mathematics, and your beliefs
about mathematics. In later interviews, you will be asked to discuss your
understanding of the graphs of polynomial functions. These interviews will be
videotaped. Classwork, quizzes, and tests will be collected to gain further
information about your understanding. A post-study questionnaire will ask you
to give your opinions about participation in the study.

Please read the following statements and respond as to whether or
not you are willing to participate:

1. I understand that the use of human subjects in this research has been
approved by the UNH Institutional Review Board for the Protection of Human
Subjects in Research.
2. I understand the purpose of this research project and the scope of activities in
which I am being asked to participate during the course of the remainder of
3. I understand that anonymity of the school and the participants will be fully
maintained and that the videotapes acquired in this study will be strictly for
the use of data collection and will not be available for public viewing at any
time.
4. I understand that my consent to participate in this research is entirely
voluntary, and that my refusal to participate will involve no prejudice or
penalty in regards to my future endeavors in this class.
5. I further understand that if I consent to participate, I may discontinue my
participation at any time without prejudice or penalty.
6. I understand that I will not be provided financial incentive for my
participation by the University of New Hampshire. I also confirm that no
coercion of any other kind was used in seeking my participation in this
research project.
7. I understand that any information gained about me as a result of my
participation in this study will be provided to me at the conclusion of my
involvement in this research project by my request or by my guardian(s)
request.

I have read and fully understand what my participation in this
project entails.

- I, _______________, consent/agree to participate in this research project.
- I, _______________, refuse/do not agree to participate in this research

project.

________________________________________  __________________________
Signature of subject                     Date

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GUARDIAN Informed Consent Form

Your son/daughter has been asked to participate in a research study to be conducted by a Ph.D. candidate from the University of New Hampshire in cooperation with [school].

Purpose: The purpose of this research is to study the development in student knowledge of the graphs of polynomial functions.

Description: Participants in this study will have approximately 3 interviews. In the first interview, they will be asked questions concerning their mathematics background, their attitudes toward mathematics, and their beliefs about mathematics. In later interviews, they will be asked to talk about their understanding of the graphs of polynomial functions. These interviews will be videotaped. Classwork, quizzes, and tests will be collected to gain further information about the participants' understanding. A post-study questionnaire will ask their opinions about participation in the study.

Please read the following statements and respond as to whether or not you are willing to allow your son/daughter to participate:

1. I understand that the use of human subjects in this research has been approved by the UNH Institutional Review Board for the Protection of Human Subjects in Research.

2. I understand the purpose of this research project and the scope of activities in which my son/daughter is being asked to participate during the course of the remainder of the school year 1993-1994.

3. I understand that anonymity of the school and the participants will be fully maintained and that the videotapes acquired in this study will be strictly for the use of data collection and will not be available for public viewing at any time.

4. I understand that consent to participate in this research is entirely voluntary, and that refusal to participate will involve no prejudice or penalty in regards to his/her future endeavors in the class.

5. I further understand that if I give my consent for him/her to participate, I may withdraw my consent at any time without prejudice or penalty.

6. I understand that she/he will not be provided financial incentive for participation in this study by the University of New Hampshire and that no coercion of any other kind was used in seeking his/her participation in this research project.

I have read and fully understand what participation in this project entails.

1. I give my permission for __________________ to participate in this research project.

2. I prefer that __________________ does not participate in this research project.

__________________________________________
Signature of Guardian  Date

For further questions about this study, you may contact the project director, Mrs. Judy Curran, at [phone number].
Teacher Informed Consent form

Mrs. Judith Curran, a UNH doctoral candidate in Mathematics Education wishes to do a research study within your 3rd period Algebra II class. She has chosen this class because the students will be learning the topics pertinent to her study, and because, as a teacher, you have a number of years experience in teaching about functions and graphs.

Purpose: The purpose of this research is to study, within a classroom situation, the development in student knowledge of the graphs of polynomial functions.

Description: The researcher will be attending the class and videotaping class instruction on polynomial functions. Of the students willing to participate in the study, 3 or 4 will be selected to have approximately three interviews with the researcher during their free time. The researcher will be asking them questions concerning their mathematics background, their attitudes toward mathematics, their beliefs about mathematics and concerning their understanding of the graphs of polynomial functions. Examples of classwork, quizzes, and tests will be collected from these students.

As a teacher's attitudes toward mathematics, beliefs about mathematics, and style of teaching affect the students and contribute to their understanding of mathematics, you will be asked to answer a few questions relative to your teaching experience, and your attitudes and beliefs about mathematics (especially functions and graphing). Responses to these questions may be audiotaped, but only used to further analyze student understanding.

The data will be used to compile case studies of student understanding with respect to the graphs of polynomial functions for inclusion in the researcher's doctoral dissertation.

Please read the following statements and respond as to whether or not you are willing for this research project to take place within your classroom.

1. I understand the purpose and scope of this research project that is to take place during the remainder of the school year 1993-1994 in my classroom.
2. I understand that anonymity of the school and the participants will be fully maintained and that the videotapes acquired in this study will be strictly for the use of data collection and will not be available for public viewing at any time.
3. I understand that my consent is required in order for this study to take place within my classroom, that my consent is entirely voluntary, and that no coercion was used in obtaining that consent.
4. I understand that I will receive no monetary compensation for my participation in this research project.

I have read and fully understand what allowing this research project to take place in my classroom entails.

_ I, ______________, give my permission for this research project to take place within my classroom.

_ I, ______________, do not give my permission for this research project to take place within my classroom.

_________________________  ______________________
Signature of Teacher       Date
APPENDIX B

INTERVIEW AND TEACHING EPISODE FORMATS
INTERVIEW I

PURPOSE: To find out more about the student in terms of attitudes and beliefs about school, mathematics, algebra II. To investigate students current understanding of functions and graphing.

I. Attitudes toward school:

-Tell me about a typical day for you at school.
-What part of school do you like the most? least?
-Do you have a favorite subject? What? Why?
-On an average, how much time do you think you spend on homework a night?
-Any thoughts on what you'd like to do after high school?

II. Attitudes toward mathematics/background:

-Tell me what you remember about learning mathematics growing up? (start at elementary school)
(get at enjoyment and whether they find it difficult)
-How did you do in Algebra I? Geometry?
-Can you describe what mathematics is? (Explore)
-What is algebra?
-Do you own a graphing calculator? If so, how often do you use it and what do you use it for? Do you use it for anything else besides math?...(Same questions for computer.)

III. Functions:

- Can you tell me what a function is, in your own words? (Explore)
- (Give Task #1 to student). Would you look at this sheet, and tell me what is/isn't a function and why?

IV. Graphs of various functions:

-(Give Task #2 to student). We're going to play a little game. Here is a book of graphs. Tell me about each graph so that I will draw the same graph in my notebook. You can assume I know a lot of terminology and try to pretend that I don't know what the graphs are.
-Go back through the graphs, and see if you can tell me the equation for the graph.
INTERVIEW II

PURPOSE: To investigate what students see in the graph of a cubic polynomial function and compare that to what they see in the graph of a quadratic function and a linear function.

I'm going to put a graph on the monitor, and I'll be asking you some questions about it.

(SHOW GRAPH OF CUBIC POLYNOMIAL USING PC-81 EMULATOR)

1. Get initial reactions to the graph.

2. I'm going to move the cursor to the leftmost part of this graph, and then I'll gradually move it along the curve. (Point to different places/points on the graph.) I'd like you to say what you can about this location on the graph.

Examples:
   a) How would you describe the graph between these two points? (Trace the interval between 2 points with pointer)
   b) What can you say about the 2nd-coordinates of the points as you go from here to here?
   c) Does the point/place give any information about the algebraic equation for the graph? What?

3. What conclusions can you make about the algebraic equation for this graph? (What parts of the curve tell you about the equation.)

4. What other ideas/thoughts about the graph or its' equation do you have? (If the student doesn't mention "factor", ask if they know what a factor of the equation would be.) Domain, Range??

-Do same for graph of a quadratic function, and then graph of a linear function.

-Put a couple of the functions on the axes at the same time. Have the student discuss similarities/differences between the curves.

For Journal: Think about your interview today and write down your thoughts. I would be interested in anything from the atmosphere during the interview, your feelings of comfortableness/uncomfortableness before/during/after the interview, and reflections on the questions that were asked.
Teacher Interview

I. Background questions:
- How long have you been teaching high school?
- How long have you been teaching Algebra II?
- Has Algebra II changed since you’ve been teaching it? How?
- You’ve also been math team coach, for how many years?
- What made you become a math teacher?

II. Beliefs/attitudes about mathematics:
- There is major reform going on now in mathematics education. How has this affected what and how you teach?
- Do you question what mathematics should be taught? In what way?
- Do you question how it should be taught?
- Do you have any input into curriculum decisions and/or textbook choices for the courses you teach? Do you like the textbook that you use in Algebra II?
- Are there topics that you like teaching best? least?

III. Function questions:
- You attended a Function Institute at UNH? When?
- Can you talk a little about experience and how it has affected what and how you teach?
- What’s the first thing that comes to mind when I say function?
- In what areas of mathematics do you consider the function concept central?
- Do you think the function concept is an easy or difficult one for students to grasp? How was it for you?
- Has your teaching changed with respect to the function concept over the years? How?
- What emphasis would you say should be given to the function concept? How does that compare to the emphasis you currently give?

IV. Graphing:
- Graphing...how do you see the emphasis on it and how it’s taught changing?
- Graphing calculator...you’re trying to use it more...could you talk talk more on this? What are pros and cons of it?

V. Researcher’s presence:
- Has having me in the class affected how and what you teach?
Teaching Episode #1

Introduce students to the Function Supposer.

1. At main menu, have the student put a linear function into $F(x)$, and then a different linear function (with a different slope) into $G(x)$. Have the student talk about their observations and compare the two graphs as they do this.

2. Ask the student what they think the product of these two linear functions would look like. (They can use paper here, if they wish.)

3. Have the computer take the product of these linear functions. Discuss with the student the characteristics of the parabola and how these characteristics are derived from the linear factors...the roots, places where $H(x)>0$ and where $H(x)<0$, y-intercept. Allow the student to make their own observations and talk about their own thinking.

4. Repeat (1-3), as time is available. When the student seems to understand the relationship between the linear factors and the product, graph a parabola and see if the student can go in the opposite direction (find the linear factors graphically).
APPENDIX C

ITINERARY OF TOPICS
Itinerary of Topics (October, 1993 - May, 1994)

<table>
<thead>
<tr>
<th>DATE (Week of...)</th>
<th>TOPIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 4</td>
<td>Factoring</td>
</tr>
<tr>
<td>October 11</td>
<td>Word problems</td>
</tr>
<tr>
<td>November 1</td>
<td>Interpretation of graphs</td>
</tr>
<tr>
<td>November 8</td>
<td>Making water tanks and graphing</td>
</tr>
<tr>
<td>November 15</td>
<td>*Function notation, vertical line test, domain and range, linear functions</td>
</tr>
<tr>
<td>November 22</td>
<td>*Translation of quadratic functions and other functions and continuing to do worksheets on linear functions</td>
</tr>
<tr>
<td>December 6</td>
<td><em>FUNCTIONS TEST</em>, begin radicals</td>
</tr>
<tr>
<td>December 13</td>
<td>Radicals continued...</td>
</tr>
<tr>
<td>December 20</td>
<td>finishing radicals, begin review for midyear exam</td>
</tr>
</tbody>
</table>

CHRISTMAS BREAK

MID-YEAR EXAMS

<table>
<thead>
<tr>
<th>DATE</th>
<th>TOPIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 24</td>
<td>Solving quadratic equations algebraically (completing the square)</td>
</tr>
<tr>
<td>January 31</td>
<td>Derive quadratic formula, nature of roots</td>
</tr>
<tr>
<td>February 7</td>
<td>*Graphing quadratic functions</td>
</tr>
<tr>
<td>February 14</td>
<td>*Graphing quadratic functions continued..., modeling</td>
</tr>
</tbody>
</table>

VACATION WEEK

<table>
<thead>
<tr>
<th>DATE</th>
<th>TOPIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 28</td>
<td>*Modeling continued...</td>
</tr>
<tr>
<td>March 7</td>
<td>*Quadratic inequalities, began polynomial functions of degree greater than two</td>
</tr>
<tr>
<td>March 14</td>
<td>*Polynomial functions of degree greater than two continued...</td>
</tr>
<tr>
<td>March 21</td>
<td>POLYNOMIAL FUNCTIONS TEST, exponential form</td>
</tr>
<tr>
<td>March 28</td>
<td>Exponential form continued..., inverse functions</td>
</tr>
<tr>
<td>April 4</td>
<td>Inverse functions continued..., logarithms</td>
</tr>
<tr>
<td>April 11</td>
<td>Logarithms continued..., modeling</td>
</tr>
<tr>
<td>April 18</td>
<td>EXPONENTS and LOGS TEST, *exponential growth and decay</td>
</tr>
</tbody>
</table>

VACATION WEEK
May 2  Conics (lines)
May 9  Conics continued...(circles)
May 16 Conics continued...(parabolas)
May 23 Conics continued...(ellipses, hyperbolas)
May 30 Conics continued...(hyperbolas), CONICS TEST
June 6  Geometric sequences
June 13 Review for Final Exam, Last day of classes (the 14th)

*Teacher used classroom set of graphing calculators in this unit
APPENDIX D

STUDENT GRADES
# STUDENT GRADES

<table>
<thead>
<tr>
<th>Class</th>
<th>Class Avg (X)</th>
<th>σ(μ)</th>
<th>Mark</th>
<th>Beth</th>
<th>Lisa</th>
</tr>
</thead>
<tbody>
<tr>
<td>FUNCTIONS TEST</td>
<td>90.19</td>
<td>7.5</td>
<td>92</td>
<td>94</td>
<td>82</td>
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<tr>
<td>MIDYEAR EXAM</td>
<td>79.93</td>
<td>14.3</td>
<td>78</td>
<td>98</td>
<td>74</td>
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<td>POLYNOMIAL TEST</td>
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<td>14.0</td>
<td>83</td>
<td>85</td>
<td>74</td>
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<td>QUARTER I</td>
<td>85.82</td>
<td>6.8</td>
<td>86</td>
<td>92</td>
<td>81</td>
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<tr>
<td>QUARTER II</td>
<td>83.67</td>
<td>9.9</td>
<td>88</td>
<td>92</td>
<td>82</td>
</tr>
<tr>
<td>QUARTER III</td>
<td>84.24</td>
<td>13.1</td>
<td>82</td>
<td>92</td>
<td>80</td>
</tr>
</tbody>
</table>
TASK 1

identify the following as functions or non-functions.

1. \{(1,2), (3,4), (0,6)\}  
2. \[\begin{array}{c|c} x & y \\ \hline 0 & 0 \\ 2 & 4 \\ 3 & 6 \\ 4 & 7 \end{array}\]  
3.  
4.  
5. \[\begin{array}{c|c} m & n \\ \hline -1 & 0 \\ 3 & 5 \\ 4 & 4 \\ 4 & 8 \end{array}\]  
6. \{(-1,0), (-1,2), (-3,4), (-3,6)\}  
7. \(S(t) = 3t^2 - 3t + 1\)  
8.  
9. \(y = 2x^2 + 1\)  
10.  
11. \(f(x) = 5x - 3\)  
12.  

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<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
TASK 2

Describe each graph in such a way that the investigator can draw it. Can you determine the equations for these graphs?

1.  

2.  

3.  

4.  

5.  

6.
PC-81 EMULATION SOFTWARE GRAPHS

A cubic polynomial function

A quadratic function
A linear function
WORKING BACKWARDS

Sketch the graphs of two linear functions that could possibly be the components of the quadratic function.

#1.  

#2.  

#3.  

#4.
**Building Polynomials**

<table>
<thead>
<tr>
<th>f(x) =</th>
<th>g(x) =</th>
<th>h(x) =</th>
<th>k(x) =</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>-6</td>
<td>-8</td>
<td>-2</td>
</tr>
<tr>
<td>-4</td>
<td>-6</td>
<td>-8</td>
<td>-6</td>
</tr>
<tr>
<td>-3</td>
<td>-6</td>
<td>-8</td>
<td>-6</td>
</tr>
</tbody>
</table>

f(x) 2

\[ g(x) \]

\[ h(x) = \]

\[ k(x) = \]
11/19/93: Functions quiz

I. State whether or not each graph represents a function.

II. State the Domain & Range of each graph (in set form!)

III. 9. What is the definition of function that I gave you?

10. $f(x) = 8x-2$. Find $f(3)$.

11. $f(x) = 2x+5$. Find $x$ if $f(x) = 21$.

IV. Read each statement carefully. Tell me what the independent variable should be.

12. the mass of a person is related to his or her height.

13. the diameter of a pizza is related to the price you pay.

14. the number of letters in the corner mailbox is related to the time of day.

15. your efficiency at studying algebra is related to how late at night is it.

V. 16. If I were to give you an equation and ask if it were a function, how could you tell?

17. What is the slope of the line between $(5,-3)$ and $(-1,2)$?

18. What is the slope of the line $y = -2x + 6$
12/6/93: Functions Test

I. For each of the equations listed, choose the graph that best describes it and place the letter of that graph to the left of the equation.

1. \( y = x^2 - 3 \)
2. \( f(x) = -x^2 + 3 \)
3. \( y = x^2 + 3 \)
4. \( f(x) = (x-3)^2 \)
5. \( f(x) = x^2 \)
6. \( y = -x^2 - 3 \)
7. \( y = (x+3)^2 \)
8. \( y = (x-3)^2 - 3 \)

II. 1. Find the equation of the linear function in which \( f(0) = 5 \) and \( f(3) = -3 \)

2. \( f(x) = 2kx + 7 \), \( f(3) = -5 \). Find \( f(-2) \)

3. Find \( k \) so that the slope of the line between \( (0,-5) \) and \( (5,k) \) is \( 3/7 \).

4. Find \( k \) so that the line through \( (-3,1) \) and \( (6,k) \) is \( \perp \) to the line \( 3x + 2y = 5 \)

5. \( f(x) = 2x - 3 \)  \( g(x) = x^2 + 2 \)
   a. find \( f(2) \)
   b. find \( g(1) \)
   c. find \( g(f(2)) \)
   d. find \( f(g(3)) \)

6. What is the domain and range of \( y = x^2 - 4 \)

7. What is the domain and range of \( y = |x-2| \)

8. What is the domain and range of \( y = x^3 + 2 \)
9. What is the domain and range of \( y = \sqrt{x} - 2 \)

10. What is the domain and range of \( y = -x^2 \)

III. For each of the equations listed, choose the graph that best describes it and place the letter of that graph next to the equation.

- 1. \( y = x - 2 \)
- 2. \( y = \lvert x \rvert + 2 \)
- 3. \( y = -x - 2 \)
- 4. \( y = \lvert x + 2 \rvert \)
- 5. \( y = \sqrt{x} - 2 \)
- 6. \( y = -\sqrt{x} \)
- 7. \( y = \sqrt{2 - x} \)
- 8. \( y = \sqrt{x + 2} \)

IV. State which variable is the independent variable

1. the weight of a bag of chips and its cost are related.

2. the distance you are from a reading lamp and the amount of light it shines on your book are related.

3. the smaller the boxes are, the more can be loaded on the van.
V. Give an example of:

1. an equation of a horizontal line ____________________________
2. an equation of a line with slope 2/3 __________________________
3. an equation of a vertical line _________________________________
4. an equation whose graph is a parabola _________________________
5. an equation whose graph is a "swivel" _________________________
6. an equation that represents a function _________________________
7. an equation that is not a function _____________________________

VI. 1. Write $4x + 3y = 8$ in slope intercept form

2. Write $y = -\sqrt{4} \times - \frac{1}{4}$ in standard form
1/19/94: Midyear exam

For each of the following equations, make a table of at least 5 ordered pairs and an accurate graph.

1. \[ y = x^2 - 3 \]
2. \[ y = (x-2)^2 \]
3. \[ y = |x| - 2 \]
4. \[ y = \sqrt{x} + 4 \]

5. Find the equation of the linear function for which \( f(0)=4 \) and \( f(2) = 5 \).

6. \( f(x) = 2kx + 7 \), \( f(2)=6 \) find \( f(3) \).

7. Find \( k \) so that the line through \((-2,1)\) and \((5,k)\) is \( \perp \) to the line \( 3x - 2y - 5 = 0 \).

8. What is the domain and range of \( y = x^2 - 4 \)?

9. What is the domain and range of \( y = |x - 2| \)?

10. Write \( 4x + 3y = 8 \) in slope-intercept form.

11. Which of the variables is the independent variable: the smaller the boxes are, the more can be loaded in the van.

In #12-17, Factor over the Integers

12. \[ 49 - 16a^2 \]
13. \[ 8 - 27x^3 \]
14. \[ 6x^2 - 11x - 10 \]
15. \[ x^4 + 3x^2 - 4xy - 4y^2 \]
16. \[ 4 - x^2 - 4xy - 4y^2 \]
17. \( ax - 5x - 5y + ay \)

\textit{Simplify}

18. \((2x^4y^7)(3x^4y^5)\)

19. \( \frac{(2a^3b^{-3})^5}{8a^3b^{-5}} \)

20. \( \frac{2^x \cdot 8^{x+y}}{4^{x+y}} \)

\textit{Solve over } \mathbb{R}

21. \( 5 = 2x(4x - 3) \)

22. \( 75x^2 - 30x + 3 = 0 \)

23. \((x + 5)(x - 1) \geq 0\)

24. \((x + 3)^2(x-1) \geq 0\)

25. \( x^2 = 36 \)

26. \( x^7 = 10 \)

27. \( \sqrt{6x + 4x} = x + 2 \)

28. \( |2x + 7| \leq 15 \)

\textit{Simplify over } \mathbb{R}

36. \( \sqrt[6]{81a^6} \)

37. \( \sqrt[6]{2a^6b^{12}} \)

38. \( 4\sqrt{5a^3} + 2a\sqrt{45a} \)

39. \( \frac{3}{\sqrt{5} - 2} \)
Simplify over C

43. \( \sqrt{-75} \)

44. \( 2\sqrt{-2} \cdot \sqrt{-8} \)

45. \( \frac{2\sqrt{8}}{\sqrt{48}} \)

Factor over \( \mathbb{R} \)

49. \( x^2 - 50 \)

50. \( x^2 - 4\sqrt{5}x - 15 \)

Factor over \( C \)

51. \( x^2 + 25 \)

52. \( x^2 - 4ix + 12 \)

46. \( (-3 + 5i) - (4 - 6i) \)

47. \( \frac{4}{2 + 3i} \)

48. \( 3\sqrt{48} - 3\sqrt{3} \)
2/2/94 Quiz on solving quadratic equations

I. Solve by completing the square

1. \(x^2 + 12x + 4 = 0\)  
2. \(2x^2 = 6x - 5\)

II. Solve using the Quadratic Formula

3. \(2(x+2)^2 = 15\)  
4. \(2x^2 + \sqrt{5}x = 5\)

5. \(x^2 = ix + 6 = 0\)

Bonus: Derive the quadratic formula from the general quadratic equation \(ax^2 + bx + c = 0\)
2/10/94: Quiz on quadratic equations

1. Solve by completing the square:
   \[ 2x^2 - 3x + 6 = 0 \]

2. Solve by quadratic formula
   \[ 5x^2 = 6x + 2 \]

3. Find D and state whether there are 2 distinct real roots, 2 equal real roots or 2 imaginary conjugate roots:
   \[ x^2 - \frac{3}{2}x = 10 \]

4. Find k so that \[ x^2 + 3x + k - 2 = 0 \] has 2 imaginary conjugate roots.

5. Solve: \[ x^4 - 7x^2 - 18 = 0 \]

6. Solve: \[ x^3 - 27 = 0 \]

7. The roots of a quadratic equation are \( 5 \pm 3\sqrt{2} \). Find the equation.

8. The roots of a quadratic equation are \( 1/3 \pm 4/3 \). Find the equation.

**Bonus:** Solve for \((x, y)\):
\[ 3x - 2y = 11 \]
\[ x + 4y = 13 \]
2/18/94: Quiz on graphing quadratic functions

1. Graph: \( y = -2(x-1)^2 + 3 \)

2. Graph: \( y = \frac{1}{2} x^2 - 2 \)

3. Graph: \( y = (x - 4)^2 \)

4. Graph: \( y = x^2 + 6x + 4 \)

5. Find the x and y intercepts of \( y = 2(x - 1)^2 - 2 \)

6. Find the equation of the parabola with \( x=3 \) as its axis of symmetry and containing the points (2,5) and (5,-1)

7. Find the equation of the quadratic function containing the points (-4,2) (0,2) and (-2,6)

8. A rectangular garden is to be enclosed on three sides by fencing. The fourth side is to be against the house. What is the largest area that can be enclosed by 40 m of fencing?
3/16/94: Quiz on polynomial functions of degree greater than two.

1. \( 2x^4 + 3x^3 + 6x^2 + 12x - 8 = 0 \). Solve.

2. Use synthetic division to find \( Q(x) \) and \( R \): \( (3x^3 - 4x^2 + 10x - 8) + (3x - 1) \)

3. Use synthetic division to find \( k \) given that when \( x^3 - 2x^2 + kx + 6 \) is divided by \( x - 2 \), the remainder is 10.

4. \( P(x) = x^3 + 2ix^2 + 4i \). Find \( P(2i) \) using synthetic substitution.

5. Find \( m \) so that \((x+1)\) is a factor of \( x^{48} + 2x^{18} + mx^5 - 2 \).
3/22/94: Test on Polynomial functions

1. Solve: \( 2x^2 = 6x - 5 \)

2. Solve: \( x^6 - 7x^3 - 8 = 0 \)

3. Find \( k \) so that \( 2x^2 - 10x + 5k = 0 \) has 2 imaginary conjugate roots.

4. A parabola has \( x=3 \) as its axis of symmetry. The points \((2,5)\) and \((5,-1)\) are on its graph. Find the equation.

5. Use synthetic division to find the quotient and remainder: \((2x^3 + x^2 - 6x - 8) + (2x + 1)\)

6. Solve: \( x^2 + 6x + 6 \geq 0 \)

7. State the vertex: \( y = 2x^2 - 8x + 7 \)

8. Write the third degree equation whose roots are \(-3, 2 \pm \sqrt{6}\)

9. Solve: \( x^4 - 8x^3 + 24x^2 - 32x + 216 = 0 \)

10. \( P(x) = 3x^4 - 5x^2 + kx + 25 \) Find \( k \) so that \( P(2) = P(-4) \)

11. A rectangular pen is to be built along a wall from 50 m of fencing. Find the value of \( x \) that will give the maximum area.

12. Solve: \( 3x^4 - 16x^3 + 8x^2 + 20x - 7 = 0 \)

13. A fourth degree polynomial equation has zeros of -2, 1, 3, and 4. If \( P(-1) = 20 \), find \( P(2) \).
FORM 1

NAME____________________

FAVORITE SUBJECT:

LEAST FAVORITE SUBJECT:

PART OF MATH LIKED MOST:

PART OF MATH LIKED LEAST:

AFTER HIGH SCHOOL PLANS:

PARENTS:

MATHEMATICS IS....

A FUNCTION IS....

TOOLS USED TO IDENTIFY FUNCTIONS:

SET OF POINTS:

LINE:

OTHER:

GRAPHING CALCULATOR:

DESCRIBING GRAPHS:

LINE
FORM 2

DEFINITIONS:

ZERO OF A FUNCTION:

ROOT:

FACTOR:
FORM 3

GRAPH #1

A:
  LISA:
  MARK:
  BETH:

B:
  LISA:
  MARK:
  BETH:

C:
  LISA:
  MARK:
  BETH:


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TI-81 Graphing Calculator. Lubbock, TX: Texas Instruments, Inc.


