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Orbit-reflexivity

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ABSTRACT

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Suppose H is a separable, infinite dimensional Hilbert space and T and S are bounded linear transformations on H. Suppose that if $Sx \in \{x, Tx, T^2x, \ldots\}^-$ for every x implies that $S \in \{1, T, T^2, \ldots\}^{-SOT}$, then T is orbit-reflexive. Many operators are proven to be orbit-reflexive, including analytic Toeplitz operators and subnormal operators with cyclic vectors.

Suppose that if $Sx \in \{\lambda x : x \in H, \lambda \in \mathbb{C}\}^-$ for every x, implies that $S \in \{\lambda T^n : n \geq 0, \lambda \in \mathbb{C}\}^{-SOT}$, then T is $\mathbb{C}$-orbit-reflexive. Many operators are shown to be $\mathbb{C}$-orbit-reflexive. $\mathbb{C}$-orbit-reflexivity is shown to be the same as reflexivity for algebraic operators.

Chapter 1

Orbit-reflexivity

1.1 Introduction

In what follows H will mean a separable, infinite dimensional Hilbert space unless noted otherwise. B(H) will be the set of operators from H into H. Here 'operator' means a continuous, and therefore bounded, linear transformation from one Hilbert space into another, 'subspace' means a closed linear subset and 'nontrivial' means neither $\{0\}$ nor the whole space.

The invariant subspace problem is perhaps the most famous unsolved problem in operator theory. It asks whether every operator has at least one nontrivial invariant subspace. An operator with no nontrivial invariant subspace, if there is one, is called transitive. At the other extreme an operator with a lot of invariant subspaces is called reflexive. By 'a lot' we mean that it has so many invariant subspaces that the only other operators that leave every one invariant are those in the strongly closed algebra generated by 1 and the operator. There exist many reflexive operators; however there are also many non-reflexive operators.

An operator with no nontrivial invariant closed set is called orbit-transitive. An operator T with a lot of invariant closed sets is called orbit-reflexive. Here 'a lot' means that the only other operators that leave them all invariant are those in the strong closure of $\{1, T, T^2, \ldots\}$.

The set $\{1, T, T^2, \ldots\}$ is called the orbit of T and will be denoted by Orb(T). Similarly Orb$(T, x) = \{x, Tx, T^2x, \ldots\}$. Thus T is orbit-transitive if $Orb(T, x)$ is dense in H for every non-zero vector x in H. Also T is orbit-reflexive if $Sx \in \overline{Orb(T, x)}$ for every x in H, implies that S is in the strong closure of $Orb(T)$.

At this time a wide class of operators is known to be orbit-reflexive. On finite-dimensional spaces every operator is orbit-reflexive. But, as in the case of transitivity, there are no known orbit-transitive operators on a Hilbert space. On the other hand, no one has yet constructed an operator on a Hilbert space that is not orbit-reflexive.

Read [15] constructed an orbit-transitive operator on a Banach space; this operator, of course, also has no nontrivial invariant subspace.

Beauzamy [1] has studied the orbits of linear operators. Many interesting and important results on invariant subspaces have been obtained by Brown [2], Lomonosov [12], Pearcy [13] and Radjavi and Rosenthal [14].

If T is an operator, we define the $\mathbb{C}$-orbit of T, denoted by $\mathbb{C}orb(T)$, to be $\{\lambda T^n : \lambda \in \mathbb{C}, n \geq 0\}$. Also we define $\mathbb{C}orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \geq 0\}$ for each vector x. We call an operator T $\mathbb{C}$-orbit-reflexive if the only operators S satisfying $Sx \in \overline{\mathbb{C}orb(T, x)}$ for every x in H are operators in the strong closure of $\mathbb{C}orb(T)$.

In this paper we will expand the class of operators that are known to be orbit-reflexive and we will investigate some variations of the definition of orbit-reflexivity. In particular we give conditions for multipliers on functional Hilbert spaces to be orbit-reflexive. We show that many subnormal operators, including all cyclic ones, are orbit-reflexive. We give a characterization of $\mathbb{C}$-orbit-reflexive operators on a finite-dimensional space in terms of the Jordan canonical form.

1.2 Preliminaries

As usual, we define the norm on $B(H)$ by $\|T\| = \sup\{\|Tx\| : x \in H, \|x\| \leq 1\}$. This norm defines the norm topology on $B(H)$. We define the strong operator topology (SOT) by saying a net $\{T_i\}$ of operators converges in (SOT) to T if, for every x in H, $\|T_i x - Tx\| \to 0$. Similarly, we define the weak operator topology (WOT) by saying a net $\{T_i\}$ converges to T in (WOT) if, for every x and y in H, $(T_i x, y) \to (Tx, y)$. The adjoint T^* of an operator T is defined by $(T^*x, y) = (x, Ty)$ for all x and y in H. We say an operator T in $B(H)$ is normal if $T^*T = TT^*$. If $T \in B(H)$ and M is a closed linear subspace of H, we say M is an invariant subspace for T if $TM \subset M$. Also, the spectrum of an operator T, denoted by $\sigma(T)$, is the set of all complex numbers λ such that $T - \lambda$ is not invertible. The point-spectrum of T, denoted by $\sigma_p(T)$ is the set of eigenvalues of T. The first five results are due to Hadwin, Nordgren, Radjavi and Rosenthal [10].

Lemma 1 *Suppose that $S, T_1, T_2, \ldots \in B(H)$. If the set of vectors x in H for which $Sx \in \{T_1x, T_2x, \ldots\}$ is of the second category, then $S \in \{T_1, T_2, \ldots\}$.*

Proposition 1 *If $S, T \in B(H)$ and the set of vectors x for which $Sx \in \mathrm{Orb}(T, x)$ is of the second category, then $S \in \mathrm{Orb}(T)$.*

Proposition 2 *Suppose that N is a commmuting family of normal operators in B(H) and suppose that S is an operator such that $Sx \in \{Tx : T \in N\}^-$ for every x in H. Then $S \in N^{-SOT}$.*

Corollary 1 *Every normal operator is orbit-reflexive.*

Theorem 1 *Suppose that $T \in B(H)$. Then T is orbit-reflexive if any one of the following holds:*

1. *$Orb(T,x)$ is closed for every x in some non-empty open subset of H;*

2. *$Orb(T)^{-SOT}$ is countable and strongly compact;*

3. *there is a non-zero idempotent P in B(H) such that $PT = TP$ and $\|T^n x\| \to \infty$ for every non-zero x in* $\mathrm{ran}P$*;*

4. *there is a non-zero idempotent P in B(H) such that $PT = TP$ and $\sigma(T|\mathrm{ranP}) \subset \{z : |z| > 1\}$.*

5. *T is the direct sum of a normal operator A and an operator B such that $B^n \to 0$ in the weak operator topology;*

6. *$\|T\| \leq 1$;*

7. *T is algebraic;*

8. *T is compact;*

9. *$\sigma(T) \cap \{z : |z| = 1\} = \emptyset$;*

10. *T is a unilateral (forwards or backwards) or bilateral operator-weighted shift with commuting positive operator weights.*

1.3 Improvements

In this section we improve some parts of Theorem 1. We first improve parts (3) and (4) by weakening the hypotheses. Our first result is:

Theorem 2 *Suppose $T \in B(H)$. Then T is orbit-reflexive if either of the following cases holds:*

1. *there is a non-zero operator P in B(H) such that $PT = TP$ and $\|T^n x\| \to \infty$ for every non-zero x in* $\mathrm{ran}P$.

2. *there is a non-zero operator P in $B(H)$ such that $PT = TP$,* ran$P$ *is closed and* $\sigma(T|\ \mathrm{ran}P) \subset \{z : |z| > 1\}$.

Proof: Let $G = (\ker P)^c$, the complement of the kernel of P. Since P is nonzero, G is non-empty. Since $\ker P$ is closed, G is open. If $y \in G$, then $Py \neq 0$ so $||T^n Py|| \to \infty$. But $||T^n Py|| = ||PT^n y|| \leq ||P||||T^n y||$. Consequently, $||T^n y|| \to \infty$. This proves that Orb(T, y) is closed. Therefore T is orbit-reflexive by part (1) of Theorem 1. This concludes the proof of part (1).

Let $T_1 = T|\mathrm{ran}P$. By hypothesis $0 \notin \sigma(T_1)$; i.e., T_1 is invertible. Now, $||T_1^n x|| \geq (\frac{1}{||T_1^{-1}||})^n ||x||$. If $||T_1^{-1}|| < 1$, then T is orbit-reflexive by part (1).

If $||\lambda|| \geq 1$, then $T_1 - 1/\lambda$ has an inverse. If we multiply that inverse by $(-1/\lambda)T_1$, we get an inverse for $T_1^{-1} - \lambda$. Therefore, elements of the spectrum of T_1^{-1} have absolute value less than 1. Since the spectrum is always closed, this implies that the spectral radius of T_1^{-1} is less than 1. (The spectral radius is the sup of the set of absolute values of elements of the spectrum.) Consequently $||T_1^{-1}||$ is less than 1. This completes the proof of part (2).

There is another result that is similar to parts (3) and (4) of Theorem 1.

Theorem 3 *Suppose $T \in B(H)$. Then T is orbit-reflexive if either of the following cases holds:*

1. *there is a non-zero operator P in $B(H)$ such that $PT = PTP$ and $||PT^n x|| \to \infty$ for every non-zero x in* ran$P$.

2. *there is a non-zero operator P in $B(H)$ such that $PT = PTP$,* ran$P$ *is closed and* $\sigma(PT|\mathrm{ran}P) \subset \{z : |z| > 1\}$.

Proof: Let $G = (\ker P)^c$. Since $\ker P$ is a closed set, G is open, and since P is non-zero, G is nonempty. Again we will use part (1) of Theorem 1. So, suppose $y \in G$. We need to show that $\mathrm{Orb}(T, y)$ is closed. To do that we need to show that the sequence $\{||T^n y||\}$ is unbounded. The norm $||PT^n Py||$ is unbounded by hypothesis, and $||PT^n Py|| = ||PT^n y||$. Since $||PT^n y|| \leq ||P|| \cdot ||T^n y||$ and P is non-zero, we get the desired conclusion. □

1.4 Functional Hilbert Spaces

Definition 1 *A* functional Hilbert space on a nonempty set *X is a Hilbert space H of complex valued functions on X such that:*

1. *if f and g are in H and if α and β are scalars, then $(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$ for all x in X, i.e. the evaluation functionals are linear on H,*
2. *to each x in X there corresponds a scalar γ_x, such that $|f(x)| \leq \gamma_x \|f\|$ for all f in H, i.e., the evaluation functionals on H are bounded,*
3. *for each x in X there is at least one f in H such that $f(x) \neq 0$, i.e., X has no null points. [8]*

Definition 2 *If H is a functional Hilbert space on X, then the linear functional $\hat{x}$ that maps $f \longrightarrow f(x)$ on H is bounded for each x in X. By the Riesz Representation Theorem, there is an element K_x of H such that $f(x) = < f, K_x >$ for all f. Here $<,>$ represents the inner product in H. The vector K_x is the* kernel function *for x.*

If $\varphi : X \to \mathbb{C}$ and $\varphi H \subset H$, where $\varphi H = \{\varphi f : f \in H\}$, then we call φ a multiplier of H, and we can define a map $M_\varphi : H \to H$ by $M_\varphi(f) = \varphi f$. We define $\|\varphi\|_\infty = \sup\{|\varphi(x)| : x \in X\}$.

Theorem 4 *Suppose X is a set and H is a functional Hilbert space on X, and φ is a multiplier of H. Then M_φ is bounded and $\|\varphi\| \leq \|M_\varphi\|$.*

Proof: We will use the Closed Graph Theorem. Suppose $f_i \to f$ and $M_\varphi f_i \to g$. To show: $M_\varphi f = g$.

For every linear functional $\hat{x}$, $\hat{x}(f_i) \to \hat{x}(f)$ and $\hat{x}(M_\varphi f_i) \to \hat{x}(g)$. This implies that for every x, $f_i(x) \to f(x)$ and $\varphi(x) f_i(x) \to g(x)$.

Multiplication is continuous in $\mathbb{C}$, so $f_i(x) \to f(x)$ implies that $\varphi(x)f_i(x) \to \varphi(x)f(x)$. Hence for every x, $g(x) = \varphi(x)f(x)$. Therefore $g = M_\varphi f$.

For every x in X, there exists a k_x in H such that $< f, k_x >= f(x)$. Now, $< \varphi k_x, k_x >= \varphi(x)k_x(x) = \varphi(x) < k_x, k_x >$. So,

$$\begin{aligned} |\varphi(x)|| < k_x, k_x > | &= | < \varphi k_x, k_x > | \\ &= | < M_\varphi k_x, k_x > | \\ &\leq ||M_\varphi k_x|| ||k_x|| \\ &\leq ||M_\varphi|| ||k_x||^2. \end{aligned}$$

For every x in X, $|\varphi(x)| ||k_x||^2 \leq ||M_\varphi|| ||k_x||^2$. As long as $||k_x|| \neq 0$, we get $||\varphi|| \leq ||M_\varphi||$.

In fact, for every x, $||k_x||^2 \neq 0$. Suppose to the contrary that $< k_x, k_x >= 0$ for some x. Then $k_x = 0$, and $< f, k_x >= 0$ for every f in H . This implies that $f(x) = 0$ for every f in H. This contradicts the fact that there are no null points. □

Theorem 5 *Suppose X is a nonempty set and H is a functional Hilbert space on X and φ is a multiplier of H. If $||M_\varphi|| = ||\varphi||$, then M_φ is orbit-reflexive.*

Proof: <u>*Case1*</u> for every x, $|\varphi(x)| \leq 1$. By Theorem 1, M_φ is orbit-reflexive because $||M_\varphi|| = ||\varphi|| \leq 1$.

<u>*Case2*</u> There is an x_0 in X such that $|\varphi(x_0)| > 1$. Since X has no null points, x_0 is not a null point. So,

$$(\ker \hat{x}_0)^c \neq \emptyset.$$

Since $\ker \hat{x}_0$ is closed, its complement, $(\ker \hat{x}_0)^c$, is open. Let $f \in (\ker \hat{x_0})^c$.

$$Orb(M_\varphi, f) = \{f, \varphi f, \varphi^2 f, \dots\}.$$

Take any subsequence $\{\varphi^{i_n} f\}$ in $Orb(M_\varphi, f)$.

$$\hat{x}_0(\varphi^{i_n} f) = \varphi^{i_n}(x_0) f(x_0) \to \infty.$$

Any convergent sequence of $Orb(M_\varphi, f)$ must be eventually constant. Therefore $Orb(M_\varphi, f)$ is closed and M_φ is orbit-reflexive by part (1) of Theorem 1. □

Let C denote the unit circle $\{z : |z| = 1\}$. Let $\mathcal{L}^2$ denote $\mathcal{L}^2(C, \mu)$, where μ is normalized Lebesgue measure on the circle; (i.e., $\mu(C) = 1$). For each integer n let e_n denote the function $e_n(z) = z^n$. Then $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonnormal basis for $\mathcal{L}^2$.

We next show how the preceding theorem can be applied. The space $\mathcal{H}^2$ is the subspace of $\mathcal{L}^2$ generated by $\{e_n : 0 \leq n < \infty\}$. If we let $\mathcal{L}^\infty$ denote the set of all multiplication operators on $\mathcal{L}^2$; then the space $\mathcal{H}^\infty$ is the subspace of $\mathcal{L}^\infty(C, \mu)$ defined by $\mathcal{H}^\infty = \mathcal{L}^\infty(C, \mu) \cap \mathcal{H}^2$. If $\phi \in \mathcal{H}^\infty$, then $M_\phi(\mathcal{H}^2) \subset \mathcal{H}^2$, since $\phi e_n \in \mathcal{H}^2$ for $n > 0$. It is well known that the functions in $\mathcal{H}^\infty$ are precisely the multipliers of the functional Hilbert space $\mathcal{H}^2$.

An operator T on $\mathcal{H}^2$ is an *an analytic Toeplitz operator* if there exists a $\phi \in \mathcal{H}^\infty$ such that $T = M_\phi | \mathcal{H}^2$. If $\phi \in \mathcal{H}^\infty$, then the norm of the analytic Toeplitz operator corresponding to ϕ is $||\phi||$. These facts can be found in Douglas [7]

Corollary 2 *(To Theorem 5) Every analytic Toeplitz operator is orbit-reflexive.*

Proof: The corollary now follows immediately from the theorem. □

1.5 Direct Sums

In this section we prove some partial results about direct sums and orbit-reflexivity.

Theorem 6 *Suppose $T, A \in B(H)$. If T is orbit-reflexive and invertible and A is nilpotent, then $T \oplus A$ is orbit-reflexive.*

Proof: Let $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$, $\overline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $T \oplus A = \begin{pmatrix} T & 0 \\ 0 & A \end{pmatrix}$. Suppose for every x in H, $S\overline{x} \in \overline{Orb(T \oplus A, \overline{x})}$.

Since A is nilpotent there exists $k > 1$ such that $A^{k-1} \neq 0$ and $A^k = 0$. By letting $y = 0$, we get $S_3 = 0$; by letting $x = 0$, we get $S_2 = 0$. Consequently, for every x in H,

$$\begin{pmatrix} S_1 x \\ S_4 y \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} Tx \\ Ay \end{pmatrix}, \begin{pmatrix} T^2 x \\ A^2 y \end{pmatrix}, \ldots, \begin{pmatrix} T^{k-1} x \\ A^{k-1} y \end{pmatrix}, \begin{pmatrix} T^k x \\ 0 \end{pmatrix}, \ldots \right\}^- .$$

By letting $y = 0$, we get that for every x in H,

$$S_1 x \in \{x, Tx, T^2, \ldots\}^- .$$

Since T is orbit-reflexive, $S_1 \in Orb(T)^-$.

By letting $x = 0$ we get for every y in H,

$$S_4 y \in \{y, Ay, A^2 y, \ldots, A^{k-1} y, 0\}^- .$$

Now Lemma 1 and the fact that the closure of any finite set in H is itself imply

$$S_4 \in \{0, 1, A, A^2, \ldots, A^{k-1}\}.$$

Case 1: $S_4 = 0$. Then for every $\overline{x}$,

$$\begin{pmatrix} S_1x \\ 0 \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} Tx \\ Ay \end{pmatrix}, \ldots, \begin{pmatrix} T^{k-1}x \\ A^{k-1}y \end{pmatrix}, \begin{pmatrix} T^kx \\ 0 \end{pmatrix}, \ldots \right\}^- .$$

Since $A^{k-1} \neq 0$, there is a y such that none of $y, Ay, \ldots, A^{k-1}y$ are zero. Hence for every x in H, $S_1x \in \{T^kx, T^{k+1}x, \ldots\}^-$. Applying T^{-k} to S_1x says that for every x in H, $T^{-k}S_1x \in \{x, Tx, T^2x, \ldots\}^-$. The orbit-reflexivity of T puts $T^{-k}S_1$ in Orb(T). Now applying T^k gives

$$S_1 \in \{T^k, T^{k+1}, T^{k+2}, \ldots\}^- .$$

Hence,

$$S_1 \oplus 0 \in \{T^k \oplus 0, T^{k+1} \oplus 0, \ldots\}^- .$$

Case 2: $S_4 = 1$. For every x and y in H,

$$\begin{pmatrix} S_1x \\ y \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} Tx \\ Ay \end{pmatrix}, \ldots, \begin{pmatrix} T^{k-1}x \\ A^{k-1}y \end{pmatrix}, \begin{pmatrix} T^kx \\ 0 \end{pmatrix}, \ldots \right\}^- .$$

The nilpotency of A implies that there is some non-zero y such that no two of the first $k-1$ powers of A applied to y are equal. So for every x, $S_1x = x$. Therefore $S_1 = 1$. Clearly $1 \oplus 1 \in Orb(T \oplus A)^-$.

Case 3: For some i_0, $1 \leq i_0 < k$, $S_4 = A^{i_0}$. So for every $\overline{x}$,

$$\begin{pmatrix} S_1x \\ A^iy \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} Tx \\ Ay \end{pmatrix}, \ldots, \begin{pmatrix} T^{k-1}x \\ A^{k-1}y \end{pmatrix}, \begin{pmatrix} T^kx \\ 0 \end{pmatrix}, \ldots \right\}^- .$$

Again no two of the A^iy can be equal. So $S_1 = T^i$. And $T^i \oplus A^i \in Orb(T \oplus A)^-$. □

Definition 3 *Let $H^{(n)}$ be the direct sum of n copies of H and $S^{(n)}$ and $T^{(n)}$ be the direct sums of n copies of S and T respectively.*

The following is a generalization of a result in Radjavi and Rosenthal [14] .

Theorem 7 *Suppose for every n and every $\overline{x}$ in H^n, $S^{(n)}\overline{x} \in \overline{Orb(T^{(n)}, \overline{x})}$, then $S \in \overline{Orb(T)}$.*

Proof: If $\varepsilon > 0$ and $\{x_1, \ldots, x_k\}$, then we need an m such that $T^m \in Orb(T)$ and $\|T^m x_i - Sx_i\| < \varepsilon$ for every i, $1 \leq i \leq k$. Let $\overline{x} = (x_1, \ldots, x_k)$ be an element of $H^{(n)}$. So $S^{(k)}\overline{x} \in \overline{Orb(T^{(k)}, \overline{x})}$. This implies that there is an m such that $\|(T^{(k)})^m\overline{x} - S^{(k)}\overline{x}\| < \varepsilon$. □

The following result shows, for certain operators, it is enough to check if $0 \in \overline{Orb(T)}$ in order to prove that T is orbit-reflexive.

Theorem 8 *Assume $\bigcap_{n=1}^{\infty} \overline{ranT^n} = \{0\}$. Suppose that for all x, $Sx \in \overline{Orb(T, x)}$. Then $S \in Orb(T) \bigcup \{0\}$. In particular, if either $0 \in \overline{Orb(T)}$ or $0 \in \overline{Orb(T, x)}$ for some $x \in H$, then T is orbit-reflexive.*

Proof: If $Sx \in \overline{Orb(T, x)}$, then either $Sx \in Orb(T, x)$ or there is a sequence $\{n_i\}$ such that $T^{n_i} \to Sx$, where $n_1 < n_2 < n_3 < \ldots$. If $T^{n_i}x \to Sx$, then for every n, there is a sequence in $ranT^n$ that converges to Sx (the part of the sequence $\{T^{n_i}\}$ such that $n_i \geq n$). So for every n, $Sx \in \overline{ranT^n}$. So $Sx \in \bigcap_{n=1}^{\infty} \overline{ranT^n}$. Therefore $Sx = 0$.

In either case $Sx \in \{0, x, Tx, T^2x, T^3x, \ldots\}$. That is, for every x, $Sx \in \{0x, 1x, Tx, T^2x, T^3x, \ldots\}$. So, by Lemma 1, $S \in \{0, 1, T, T^2, T^3, \ldots\}$. □

Corollary 3 *Suppose $|r| \geq 1$, then* $T = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} \oplus \ldots$ *is orbit reflexive.*

Proof: Suppose $Sx \in \overline{Orb(T,x)}$ for all x. Since T is a direct sum of nilpotents we know that if $\{T^{n_i}x\}$ converges, it converges to 0. Therefore $Sx \in \{0, x, Tx, T^2x, \ldots\}$ for all x. By Lemma 1, $S \in \{0, 1, T, T^2, \ldots\}$. Let $x_0 = \begin{pmatrix} 0 \\ 1/r \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 1/r^2 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/r^3 \end{pmatrix} \oplus \ldots$. Then $\|x_0\|^2 = \sum_{n=1}^{\infty} (1/r)^{2n} = \frac{\frac{1}{r^2}}{1-\frac{1}{r^2}} = \frac{1}{r^2-1}$. This implies $\|x_0\| = \sqrt{\frac{1}{r^2-1}} \neq 0$. Hence $Tx_0 =$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1/r \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 1/r \\ 0 \end{pmatrix} \oplus \ldots,$$

$T^2x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1/r \\ 0 \\ 0 \end{pmatrix} \oplus \ldots$. For every n, $\|T^n x_0\| \geq 1$. So $0 \notin \overline{Orb(T, x_0)}$. Therefore $S \neq 0$. This means $S \in \{1, T, T^2, \ldots\}$; which implies that $S \in \overline{Orb(T)}$. Therefore T is orbit reflexive. □

Theorem 9 *If T is an orbit-reflexive operator, then $T \oplus 1$ is orbit-reflexive.*

Proof: Suppose $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \in Orb\left(T \oplus 1, \begin{pmatrix} x \\ y \end{pmatrix}\right)^-$ for every $\begin{pmatrix} x \\ y \end{pmatrix}$. Let $y = 0$. We get $S_3 = 0$ and $S_1 x \in Orb(T,x)^-$. Since T is orbit-reflexive $S_1 \in Orb(T)^-$. Let $x = 0$. Then $S_2 = 0$ and $S_4 = 1$. So $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = S_1 \oplus 1$. So

$S \in Orb(T \oplus 1)^{-}$. □

Lemma 2 *Suppose* $T \in B(H)$, $T = A \oplus B$ *and* $A = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$. *Also suppose that either* α *or* β *has norm larger than 1. Then* T *is orbit-reflexive.*

Proof: Case 1: $|\beta| > 1$. Let $R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus (0)$.

$$\overline{x} \in \ker R \Rightarrow \overline{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \oplus (z),$$

for some x and z.

$$(\ker R)^c = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \oplus (z) : y \neq 0 \right\}.$$

Note: $(\ker R)^c$ is open and nonempty.

Claim: Orb(T,$\overline{x}$) is closed.

Note that $\|T^n\overline{x}\| \geq |\beta^n y|$ because $\beta^n y$ is the second component of the first summand of $T^n\overline{x}$. The sequence $|\beta^n y|$ approaches infinity, so the only convergent sequences in $Orb(T, \overline{x})$ are the eventually constant ones. Therefore T is orbit-reflexive by Theorem 1.

Case 2: $|\alpha| > 1$. If $\alpha = \beta$ then we are back in case 1. If $\gamma = 0$, then $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and we can use virtuallly the same argument as in case 1. If $\gamma \neq 0$, then let $S_1 = \begin{pmatrix} 1 & 1 \\ 0 & \frac{\beta-\alpha}{\gamma} \end{pmatrix}$ and $S_1^{-1} = \begin{pmatrix} 1 & \frac{-\gamma}{\beta-\alpha} \\ 0 & \frac{\gamma}{\beta-\alpha} \end{pmatrix}$.

Now let

$$W = S_1 \oplus I$$

$$W^{-1}TW = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \oplus B.$$

Now we have the case where $\gamma = 0$. The operator $W^{-1}TW$ is orbit-reflexive and by similarity T is also. □

Lemma 3 *Suppose* $T = A \oplus B$, $A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$, $|\alpha| = 1$ *and* $\beta \neq 0$. *Then* T *is orbit-reflexive.*

Proof: The powers of A look like this:

$$A^n = \begin{pmatrix} \alpha^n & n\alpha^{n-1}\beta \\ 0 & \alpha^n \end{pmatrix}.$$

Let $\overline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \oplus (z)$. The set $G = \{\overline{x} \in H : y \neq 0\}$ is a nonempty open subset of H.

Suppose $\overline{x} \in G$. Then $A^n\overline{x} = \begin{pmatrix} \alpha^n x + n\alpha^{n-1}\beta y \\ \alpha^n y \end{pmatrix}$.

The sequence of norms, $||T^n\overline{x}||$, approaches infinity because the first component of $||A^n x||$ does. So, by Theorem 1, T is orbit-reflexive. □

1.6 Purely Subnormal Operators

In section 1.2 we cited the result that all normal operators are orbit-reflexive. Subnormal operators constitute a somewhat larger class of operators. Our attempts to prove that all subnormal operators are orbit-reflexive have not completely succeeded. Below we will show that not only are all purely subnormal operators with a cyclic vector orbit-reflexive, but any analytic function of such an operator is orbit-reflexive.

Definition 4 *An operator A on a Hilbert space H is* subnormal *if there exists a normal operator B on a Hilbert space K such that H is a subspace of K, the subspace H is invariant under the operator B, and the restriction of B to H coincides with A.*

Definition 5 *A subnormal operator is* purely subnormal *if it has no normal direct summands.*

Definition 6 *An operator T on a Hilbert space H has a* cyclic vector x *if the set of all vectors of the form $p(T)x$, where p varies over all polynomials, is dense in H.*

Definition 7 *If T is an operator then the* point spectrum $\sigma_p(T)$, *is defined by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some non-zero vector } x\}.$$

I.e., the point spectrum of T is the set of eigenvalues of T.

The main tool in our results is a powerful theorem of J. Thomson [4] .

Theorem 10 *If T is a pure subnormal operator with a cyclic vector, then* $\sup\{|\lambda| : \lambda \in \sigma_p(T^*)\} = \|T\|$.

Definition 8 *If $\lambda \in \mathbb{C}$ and $T \in B(H)$, then λ is a* bounded point evaluation *for T if* $\overline{\lambda} \in \sigma_p(T)$.

Lemma 4 *(Conway [5]) If $T \in B(H)$, T is subnormal and T has a cyclic vector, then the bounded point evaluations are dense in the spectrum of T.*

Definition 9 *If T is an operator, then the* spectral radius *of T, is defined by*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Theorem 11 *Suppose T is a subnormal operator with a cyclic vector and the function f is analytic in a neighborhood of $\sigma(T)$. Then $f(T)$ is orbit-reflexive.*

Proof. If $||f(T)|| \leq 1$, then $f(T)$ is orbit-reflexive by Theorem 1.

Suppose $||f(T)|| > 1$. The spectral radius of $f(T)$ is equal to $||f(T)||$. Using the spectral mapping theorem (twice) and the previous lemma, we can show there is some element, $\overline{\lambda_0}$, of $\sigma_p(f(T)^*)$, with absolute value greater than one. The closed linear subspace generated by λ_0 is invariant for $f(T)$. Using the projection onto that subspace as P, the result follows from Theorem 3. □

1.7 Some Power Bounded Operators

If M is a subspace of H, then P_M denotes the projection onto M.

Theorem 12 *Suppose T is an operator on a Hilbert space H and $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ are invariant subspaces for T. Suppose also that $\overline{\cup M_k} = H$ and that* $\sup \|T^n\| < \infty$ *(i.e., T is power bounded). If for every k, $T|_{M_k}$ is orbit-reflexive, then T is orbit-reflexive.*

Proof: First a quick fact. For each x in H, $P_{M_i}x \to x$. If it did not , then no sequence in H could converge to x because, by definition, $P_{M_i}x$ is the element of M_i that is closest to x.

Suppose for every x in H, $Sx \in \overline{Orb(T,x)}$. To prove that T is orbit-reflexive we must show that $S \in \overline{Orb(T)}^{SOT}$. To do that we must show that every strong neighborhood of S intersects $Orb(T)$. So, suppose that $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset H$. There is an i_0 such that for each $1 \leq j \leq n$,

$$\|x_j - P_{M_{i_0}} x_j\| < \varepsilon.$$

The subspace M_{i_0} will play an important role in this proof. To ease the notation a little let $M_0 = M_{i_0}$.

If $x \in M_0$, then $S|_{M_0}x \in \overline{Orb(T|_{M_0}, x)}$. Since $T|_{M_0}$ is orbit-reflexive $S|_{M_0} \in \overline{Orb(T)}^{SOT}$. So for the set $\{x_1, \ldots, x_n\}$, there is a N such that for every j

$$\| \; T^N|_{M_0}(P_{M_0}x_j) \; - \; S|_{M_0}(P_{M_0}x_j) \; \| < \varepsilon.$$

Using the fact that T is power bounded and writing x_j as $x_j + P_{M_0}x_j - P_{M_0}x_j$ we can show that $\|Sx_j - T^N x_j\| < \varepsilon$. Actually we need to go back and use bounds smaller than the given ε. The computations are straight forward. □

1.8 Direct Integrals

Direct integrals can be viewed as "continuous direct sums." So it is natural to try to prove that direct integrals analogous to the direct sums in section 1.4 are orbit-reflexive. Suppose (Ω, μ) is a finite measure space and H is a separable Hilbert space.

Definition 10 *Define*

$$L^2(\mu, H) = \{f \mid f : \Omega \to H,\ f \text{ is norm measurable, and } \int_\Omega \|f(\omega)\|^2 \, d\mu(\omega) < \infty\}.$$

and

$$L^\infty(\mu, B(H)) = \{\phi : \Omega \to B(H) \mid \phi \text{ is SOT-measurable and the function } [\omega \mapsto \|\phi(\omega)\|] \in L^\infty(\mu)\}.$$

The elements of $L^\infty(\mu, B(H))$ *are called* direct integrals.

Theorem 13 *Suppose* $H = \mathbb{C}^2$ *and* (Ω, μ) *is a measure space. Suppose also that* $T \in L^\infty(\mu, B(H))$ *and* $T = \begin{pmatrix} a(\omega) & b(\omega) \\ 0 & d(\omega) \end{pmatrix}$. *If* $d(\omega)$ *has absolute value greater than 1 on a set of positive measure, then* T *is orbit-reflexive.*

Proof. Suppose that $t > 1$ and $|d(\omega)| > t$ for all $\omega \in S$ where $S \subset \Omega$ and $\mu(S) > 0$. Define a new function from Ω into $B(H)$ by

$$R(\omega) = \begin{pmatrix} 0 & 0 \\ 0 & f(\omega) \end{pmatrix} \text{ where } f(\omega) = \begin{cases} 0 & \omega \notin S \\ 1 & \omega \in S \end{cases}$$

Notice that $R \in L^\infty(\mu, B(H))$. Let $\varphi(\omega) = \begin{pmatrix} \varphi_1(\omega) \\ \varphi_2(\omega) \end{pmatrix}$ be an arbitrary element of $L^2(\mu, H)$. Then $\ker R = \{\varphi \in L^2(\mu, H) \mid \varphi_2(\omega) = 0 \text{ a.e. on } S\}$. The set $\ker R$ is closed, so $(\ker R)^c$ is open and nonempty.

If $\varphi \in (\ker R)^c$, then $Orb(T, \varphi)$ is closed. This is true because

$$\|T^n(\varphi)\|^2 \geq \int_S t^n |\varphi_2(\omega)|^2 d\mu$$

and the second quantity approaches infinity. So T is orbit-reflexive by Theorem 1. □

Theorem 14 *Suppose $T \in L^\infty(\mu, B(H))$ and $A \subset \Omega$ with $\mu(A) > 0$. Suppose that a.e. on Ω,*

$$T(\omega) = \begin{pmatrix} \alpha(\omega) & \beta(\omega) \\ 0 & \alpha(\omega) \end{pmatrix},$$

where $|\alpha(\omega)| = 1$ and $\beta(\omega) \neq 0$. Then T is orbit-reflexive.

Proof: Define $R \in L^\infty(\mu, B(H))$ by

$$R(\omega) = \begin{pmatrix} 0 & 0 \\ 0 & f(\omega) \end{pmatrix}.$$

Where

$$f(\omega) = \begin{cases} 0 & if \ \ \omega \notin A \\ 1 & if \ \ \omega \in A \end{cases}$$

$(\ker R(\omega))^c$ is a nonempty open set.

$$\varphi \in L^2(\mu, H), \quad \varphi(\omega) = \begin{pmatrix} \varphi_1(\omega) \\ \varphi_2(\omega) \end{pmatrix}.$$

$$\varphi \in (\ker R(\omega))^c \Leftrightarrow \varphi_2(\omega) \neq 0 \quad a.e. \ on \ A.$$

Suppose $\varphi \in (\ker R(\omega))^c$. Look at $Orb(T, \varphi)$.

$$T^n(\omega) = \begin{pmatrix} \alpha^n(\omega) & n\alpha^{n-1}(\omega)\beta(\omega) \\ 0 & \alpha^n(\omega) \end{pmatrix}.$$

$$T^n(\omega)\varphi(\omega) = \begin{pmatrix} \alpha^n(\omega)\varphi_1(\omega) + n\alpha^{n-1}(\omega)\beta(\omega)\varphi_2(\omega) \\ \alpha^n(\omega)\varphi_2(\omega) \end{pmatrix}.$$

For almost all ω in A, the first component approaches infinity ($n \to \infty$, $|\alpha^{n-1}(\omega)| = 1$, $\beta(\omega) \neq 0$, $\varphi_2(\omega) \neq 0$). So Orb(T,φ) is closed for all $\varphi \in (\ker R(\omega))^c$. So by Theorem 1, T is orbit-reflexive. □

Chapter 2

C-Orbit-reflexivity

2.1 Preliminaries

In this chapter we define and investigate $\mathbb{C}$-orbit-reflexive operators. If T is an operator in $B(H)$ and $x \in H$, we define

$$\mathbb{C}orb(T) = \{\lambda T^n : n \geq 0,\ \lambda \in \mathbb{C}\} \text{ and}$$

$$\mathbb{C}orb(T, x) = \{\lambda T^n x : n \geq 0,\ \lambda \in \mathbb{C}\}.$$

Definition 11 *An operator T is* $\mathbb{C}$-orbit-reflexive *if the only operators S such that $Sx \in \overline{\mathbb{C}orb(T,x)}$ for every vector x are the operators in $\overline{\mathbb{C}orb(T)}^{SOT}$.*

Theorem 15 *Suppose that A, B and T are operators and T is invertible. If A is $\mathbb{C}$-orbit-reflexive and $B = T^{-1}AT$, then B is $\mathbb{C}$-orbit-reflexive.*

Proof: Suppose $x \in H$ and $Sx \in \overline{\mathbb{C}orb(B,x)}$. Then $Sx \in \overline{\mathbb{C}orb(T^{-1}AT, x)}$. By continuity and linearity

$$TSx \in \{\lambda A^n Tx : n \geq 0,\ \lambda \in \mathbb{C}\}^-.$$

For every vector x in H,

$$TST^{-1}x \in \{\lambda A^n TT^{-1}x : \ n \geq 0, \ \lambda \in \mathbb{C}\}^-.$$

So for every x, $TST^{-1}x \in \overline{\mathbb{C}orb(A,x)}$. Since A is $\mathbb{C}$-orbit-reflexive, $TST^{-1} \in \overline{\mathbb{C}orb(A)}^{SOT}$. This implies that $S \in \overline{\mathbb{C}orb(T^{-1}AT)}^{SOT}$, equivalently $S \in \overline{\mathbb{C}orb(B)}^{SOT}$. Therefore B is $\mathbb{C}$-orbit-reflexive. □

Theorem 16 *Any normal operator is $\mathbb{C}$-orbit-reflexive.*

Proof: If T is a normal operator, then $\mathbb{C}orb(T)$ is a commuting family of normal operators. Suppose $Sx \in \overline{\mathbb{C}orb(T,x)}$ for every x. By Proposition 2, $S \in \overline{\mathbb{C}orb(T)}^{SOT}$. Therefore T is $\mathbb{C}$-orbit-reflexive. □

2.2 Finite Dimensional Operators

The following theorem due to Deddens and Fillmore [6] will be very useful in the rest of this chapter.

Theorem 17 *Let Q be a nilpotent linear transformation in a finite-dimensional vector space V, and let $Q = \sum_{i=1}^{k} \oplus Q_i$ be a decomposition into cyclic parts on subspaces of dimensions $n_1 \geq n_2 \geq \ldots \geq n_k$. Then Q is reflexive if and only if $n_1 = n_2$, or $n_1 = n_2 + 1$, or $dim V = 1$.*

Theorem 18 *Suppose $x_0 \in H$, $\mathcal{S} \subset B(H)$, and the evaluation map from $\mathcal{S}$ to $\mathcal{S}x_0$ is a SOT homeomorphism. Also suppose that $[\forall x \in H, \ Tx \in \overline{\mathcal{S}x}] \Rightarrow T \in \mathcal{S}^{-SOT}$. If $\mathcal{S}_0 \subset \mathcal{S}$, then $[\forall x \in H, \ Tx \in \overline{\mathcal{S}_0 x}] \Rightarrow T \in \mathcal{S}_0^{-SOT}$.*

Proof: Suppose $\mathcal{S}_0 \subset \mathcal{S}$ and $Tx \in \overline{\mathcal{S}_0 x}$, $\forall x$. Then $Tx_0 \in \overline{\mathcal{S}_0 x}$. So $Tx_0 \in \overline{\mathcal{S}_0 x_0}$. So there exists a sequence $\{S_{0i}\}$ in $\mathcal{S}_0$ such that $S_{0i}x_0 \to Tx_0$. Therefore, since the evaluation map is a homeomorphism, $S_{0i} \to T$ (SOT). So $T \in \mathcal{S}_0^{-SOT}$. □

Theorem 19 *Suppose T is a nilpotent operator on a finite dimensional space. Then T is $\mathbb{C}$-orbit-reflexive if and only if T is reflexive.*

Proof:

Assume that T is a nilpotent operator on a finite dimensional space and that T is reflexive. Let $\mathcal{S}$ be the set of polynomials in T. Suppose $Rx \in \overline{\mathcal{S}x}$, $\forall x$. Then R leaves invariant every T invariant subspace. (Suppose $TM \subset M$. If $x \in M$, then $Rx \in \overline{\mathcal{S}x}$ and $\mathcal{S}x$ is a subset of M.) Since T is reflexive, $R \in \mathcal{S}^{-SOT}$. Since T is nilpotent there is an n such that $T^n = 0$ and $T^{n-1} \neq 0$. Choose $x_0 \in H$ such that $T^{n-1} \neq 0$. The evaluation map at x_0 from $\mathcal{S}$ to $\mathcal{S}x_0$ is clearly onto. It is also one-to-one because T is linear and nilpotent. If $S_i \to S$ (SOT) in $\mathcal{S}$, then certainly $S_i x_0 \to Sx_0$. Suppose $S_i x_0 \to Sx_0$. By using the linearity and nilpotency of T it can be shown that each sequence of coefficients converges to the corresponding coefficient of S. Thus $S_i \to S$ (SOT). This proves that the evaluation map at x_0 is a SOT homeomorphism. Now let $\mathcal{S}_0 = \mathbb{C}orb(T)$. The previous lemma implies that T is in $\mathbb{C}orb(T)$.

To prove the other direction suppose T is not reflexive. Let $T = \sum_{i=1}^{k} \oplus T_i$ be a decomposition of T into cyclic parts on subspaces of dimensions $n_1 \geq n_2 \geq \ldots \geq n_k$. By Theorem 17,$n_1 \geq n_2+2$. Let $S = \begin{pmatrix} 0 & \ldots & 0 & 1 & 1 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & & \ldots & 0 \end{pmatrix} \oplus (0)$, where the first component acts on the same space as T_1. Let $\overline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n_1} \end{pmatrix} \oplus (0)$. Then $S\overline{x} = \begin{pmatrix} x_{n_1-1} + x_{n_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus (0)$.

If $x_{n_1} \neq 0$, then $S\overline{x} \in \mathbb{C}\{T^{n_1-1}\overline{x}\}$. If $x_{n_1} = 0$, then $S\overline{x} \in \mathbb{C}\{T^{n_1-2}\overline{x}\}$.

In either case $Sx \in \overline{\mathbb{C}orb(T,x)}$. Clearly $S \notin \overline{\mathbb{C}orb(T)}$. Therefore T is not $\mathbb{C}$-orbit-reflexive. □

Theorem 20 *Suppose H is a finite dimensional Hilbert space and $T \in B(H)$. Let $\delta =$ $\max\{|\lambda| : \lambda$ is an eigenvalue for $T\}$. Consider all the Jordan blocks that correspond to eigenvalues with absolute value equal to δ. Let T_1 be the largest of these blocks and let T_2 be the second largest. Then T is $\mathbb{C}$-orbit-reflexive if and only if the size of T_1 is at most one greater than the size of T_2. (If T has just one block T_1, then T is $\mathbb{C}$-orbit-reflexive if and only if T_1 is 1 by 1.)*

Proof: Suppose T is $\mathbb{C}$-orbit-reflexive and T_1 and T_2 differ in size by at least 2. Suppose T_1 is a k by k matrix. Let $S = C \oplus D$ be an element of $B(H)$. Let C be the k by k matrix with zeroes everywhere except in the last two spots of the first row, where there are ones. Specifically, $C = (c_{ij})$ where $c_{1k-1} = 1$, $c_{1k} = 1$ and $c_{ij} = 0$ for all other choices of i and j. Let $D = 0$.

Since we are trying to show that T is not $\mathbb{C}$-orbit-reflexive, it is sufficient to consider the case where T_1 is the k by k matrix with ones on the main diagonal and the first super diagonal and zeroes everywhere else and all the other blocks have a main diagonal element with absolute value less than or equal to one. So,

$$T_1^n = \begin{pmatrix} 1 & n & \binom{n}{2} & \cdots & \binom{n}{k-2} & \binom{n}{k-1} \\ 0 & 1 & n & \binom{n}{2} & \cdots & \binom{n}{k-2} \\ \vdots & & & & & \vdots \\ 0 & 0 & & \cdots & & 1 \end{pmatrix},$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

The diagonals of T_1^n are constant. The second to the last super diagonal of C is not constant. So, $C \notin \overline{\mathbb{C}orb(T_1)}$. Therefore $S \notin \overline{\mathbb{C}orb(T)}$.

To prove that T is not $\mathbb{C}$-orbit-reflexive, we must now show that

$$S\overline{x} \in \overline{\mathbb{C}orb(T, \overline{x})} \text{ for every } \overline{x} \in H.$$

Let

$$\overline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{pmatrix} \oplus y.$$

$$S\overline{x} = \begin{pmatrix} x_{k-1} + x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus (0).$$

Case 1: $x_k = 0$. As n approaches infinity each element of

$$\frac{(k-2)!}{n^{k-2}} T^n \overline{x}$$

will approach 0, except for the first element which will approach x_{k-1}. The last statement is true because for any constant k,

$$\frac{n!}{(n-k+2)!n^{k-2}}$$

will approach 1.

Case 2: $x_k \neq 0$.

By a similar argument $\frac{(k-1)!}{n^{k-1}} T^n \overline{x}$ will approach $S\overline{x}$.

Either way $S\overline{x} \in \overline{\mathbb{C}orb(T, \overline{x})}$.

To prove the other direction suppose $S\overline{x} \in \overline{\mathbb{C}orb(T, \overline{x})}$ for all $\overline{x} \in H$. To show that T is $\mathbb{C}$-orbit-reflexive we must show that $S \in \overline{\mathbb{C}orb(T)}^{SOT}$. Let $T = T_1 \oplus T_2 \oplus T_3$ where T_1 is the m by m matrix described in the proof of the other direction and T_2 is the following p by p matrix, where $p = m$ or $p = m - 1$, $\begin{pmatrix} e^{i\theta} & 1 & 0 & 0 & \dots & 0 \\ 0 & e^{i\theta} & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & e^{i\theta} \end{pmatrix}$ and T_3 consists of the remaining Jordan blocks. Let $S = A \oplus B \oplus C$ where A is a m by m matrix and B is a p by p matrix. Our immediate goal is to show that each of the diagonals of S is constant.

$$T^n = \begin{pmatrix} 1 & n & \binom{n}{2} & \cdots & \binom{n}{m-1} \\ 0 & 1 & n & \cdots & \binom{n}{m-2} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & e^{in\theta 0} & ne^{in\theta} & \cdots & \binom{n}{p-1} \\ 0 & 1 & e^{in\theta} & \cdots & \binom{n}{p-2} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \bigoplus T_3^n.$$

Here $p = m$ or $p = m - 1$. Since $S\overline{x} \in \overline{\mathbb{C}orb(T,x)}$ for all x, it is fairly easy to show that all the elements of S that are below the main diagonal are 0. So all the subdiagonals are constant.

Let $A = (a_{ij})$ and $B = (b_{ij})$.

<u>case 1</u>: $a_{11} \neq 0$.

Claim 1: In A the main diagonal is constant.

We will show that for every $l \in \mathrm{N}$, $a_{ll} = a_{11}$. The proof is by induction. Suppose $a_{kk} = a_{11}$. We need to show that $a_{k+1,k+1} = a_{11}$.

Let e_k be the m-tuple (or p-tuple) with a 1 in the kth component and zeroes everywhere else.

$$A \oplus B(e_k \oplus e_1) = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \begin{pmatrix} b_{11} \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix}.$$

$$(T_1 \oplus T_2)^n(e_k \oplus e_1) = \begin{pmatrix} \binom{n}{k-1} \\ \binom{n}{k-2} \\ \vdots \\ \binom{n}{1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \begin{pmatrix} e^{in\theta} \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Since $S(e_k \oplus e_1 \oplus 0) \in \overline{Corb(T, e_k \oplus e_1 \oplus 0)}$, there is a sequence $\{\alpha_r\}$ in $\mathbb{C}$, such that $\alpha_r \to a_{kk}$ and $\alpha_r e^{in_r\theta} \to b_{11}$, where each n_r is an integer greater than or equal to 0. Therefore $|b_{11}| = |a_{kk}|$.

Now,

$$A \oplus B(e_{k+1} \oplus e_1) = \begin{pmatrix} a_{1k+1} \\ a_{2k+1} \\ \vdots \\ a_{kk+1} \\ a_{k+1k+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \begin{pmatrix} b_{11} \\ 0 \\ 0 \\ \\ \vdots \\ \\ 0 \\ 0 \end{pmatrix},$$

and

$$(T_1 \oplus T_2)^n(e_{k+1} \oplus e_1) = \begin{pmatrix} \binom{n}{k} \\ \binom{n}{k-1} \\ \vdots \\ n \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \begin{pmatrix} e^{in\theta} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Since $(A \oplus B)(e_{k+1} \oplus e_1) \in \overline{\mathbb{C}orb((T_1 \oplus T_2), e_{k+1} \oplus e_1)}$, there is a sequence α_r in $\mathbb{C}$ such that $\alpha_r \to a_{k+1k+1}$, and $\alpha_r n_r \to a_{kk+1}$ and $\alpha_r e^{in_r\theta} \to b_{11}$. Consequently,

$$|b_{11}| = |a_{k+1,k+1}|.$$

Note that $\alpha_r \not\to 0$. If it did $b_{11} = 0$ and so $a_{kk} = 0$, which it is not.

Also,

$$a_{kk+1} = \gamma_1 a_{k+1k+1},$$

where γ_1 is an integer greater than or equal to zero.

Now we look at the image of $e_k + e_{k+1}$ under A and T_1^n.

$$A(e_k+e_{k+1}) = \begin{pmatrix} a_{1k} + a_{1k+1} \\ \vdots \\ a_{kk-1} + a_{k-1k+1} \\ a_{kk} + a_{kk+1} \\ a_{k+1k+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } T_1^n(e_k+e_{k+1}) = \begin{pmatrix} \binom{n}{k-1} + \binom{n}{k} \\ \binom{n}{k-2} + \binom{n}{k-1} \\ \vdots \\ n+1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now we get a sequence $\{\alpha_r\}$ in $\mathbb{C}$, such that $\alpha_r \to a_{k+1k+1}$ and $\alpha_r(n_r+1) \to a_{kk}+a_{kk+1}$. Since a_{k+1k+1} is not equal to zero and $\{\alpha_r\}$ does not converge to zero, the sequence $\{n_r+1\}$ must converge. Using some of the previous identities and the fact that any convergent sequence of natural numbers is eventually constant, we get

$$(n_0+1)a_{k+1k+1} = a_{kk} + \gamma_1 a_{k+1k+1}$$

$$a_{kk} = (n_0 + 1 - \gamma_1).$$

Hence a_{kk} is an integer multiple of a_{k+1k+1}. Since they both have the same absolute value, either $a_{k+1k+1} = a_{kk}$ or $a_{k+1k+1} = -a_{kk}$. The second equation is impossible.

Assume $a_{k+1k+1} = -a_{kk}$.

$$A(2e_k + 3e_{k+1}) = \begin{pmatrix} \vdots \\ 2a_{kk} + 3a_{kk+1} \\ 3a_{k+1k+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } T_1^n(2e_k + 3e_{k+1}) = \begin{pmatrix} \vdots \\ 2+3n \\ 3 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

There is a sequence $\{\alpha_r\}$ in $\mathbb{C}$ such that $3\alpha_r \to 3a_{k+1k+1}$ and $\alpha_r(2+3n_r) \to 2a_{kk} + 3(\gamma_1 a_{k+1k+1})$. Now,

$$-a_{kk}(3n_1 + 2) = 2a_{kk} + 3\gamma_1(-a_{kk}).$$

$$-3n_1 - 2 = 2 - 3\gamma_1.$$

$$n_1 = \gamma_1 - 4/3.$$

But n_1 is an integer and γ_1 is a natural number. This contradicts the last equation. The proof of claim 1 is now complete.

Claim 2: All the other diagonals of A are constant.

Proof: The proof is by induction. Assume the first $t-1$ diagonals are constant. To show: $a_{q,q+t} = a_{q+1,q+t+1}$ for all q.

$$Ae_{q+t} = \begin{pmatrix} a_{1,q+t} \\ \vdots \\ a_{q,q+t} \\ a_{q+1,q+t} \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } T_1^n e_{q+t} = \begin{pmatrix} \binom{n}{q+t-1} \\ \vdots \\ \binom{n}{t} \\ \binom{n}{t-1} \\ \vdots \\ n \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This implies there is a seqence $\{\alpha_r\}$ in $\mathbb{C}$ such that

$$\alpha_r \to a_{11},$$

$$\alpha_r \binom{n_r}{t} \to a_{q,q+t} \text{ and}$$

$$\alpha_r \binom{n_r}{t-1} \to a_{q+1,q+t}.$$

$$Ae_{q+t+1} = \begin{pmatrix} a_{1,q+t+1} \\ \vdots \\ a_{q,q+t+1} \\ a_{q+1,q+t+1} \\ \vdots \\ a_{q+t+1,q+t+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ and } T_1^n e_{q+t+1} = \begin{pmatrix} \binom{n}{q+t} \\ \binom{n}{q+t-1} \\ \vdots \\ \binom{n}{t+1} \\ \binom{n}{t} \\ \binom{n}{t-1} \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

There must be a sequence $\{\alpha_s\}$ in $\mathbb{C}$ such that,

$$\alpha_s \to a_{11},$$

$$\alpha_s \binom{n_s}{t} \to a_{qq+1,q+t+1} \text{ and}$$

$$\alpha_s \begin{pmatrix} n_s \\ t-1 \end{pmatrix} \to a_{q+2,q+t+1}.$$

The sequence $\left\{ \begin{pmatrix} n_r \\ t \end{pmatrix} \right\}$ must converge because a_{11} is not zero. The sequence $\{n_r\}$ must be eventually constant. Consequently,

$$a_{q,q+t} = a_{11} \begin{pmatrix} n_{r_0} \\ t \end{pmatrix}$$

and

$$a_{q+1,q+t} = a_{11} \begin{pmatrix} n_{r_0} \\ t-1 \end{pmatrix}.$$

Similarly $a_{q+1,q+t+1} = a_{11} \begin{pmatrix} n_{s_0} \\ t \end{pmatrix}$ and $a_{q+2,q+t+1} = a_{11} \begin{pmatrix} n_{s_0} \\ t-1 \end{pmatrix}$. Now $a_{q+1,q+t}$ and $a_{q+2,q+t+1}$ are on the $t-1$ diagonal and are therefore equal. So $n_{r_0} = n_{s_0}$ and $a_{q,q+t} = a_{q+1,q+t+1}$. This finishes the proof of claim 2.

Claim 3: The diagonals of B are all constant. The proof is by induction. The main diagonal of B is constant. Assume that the first $t-1$ diagonals are constant and that $b_{i,j}$ and $b_{i+1,j+1}$ are on the t diagonal. To show: $b_{i,j} = b_{i+1,j+1}$for all i and j. Evaluate $A \oplus B$ and $(T_1 \oplus T_2)^n$ at $e_1 \oplus e_j$ and at $e_1 \oplus e_{j+1}$. An argument very similar to that in the proof of claim 2 implies that $b_{i,j} = b_{i+1,j+1}$.

Claim 4: The diagonals of the matrices that comprise C are all constant. The proof is virtually identical to the proof of claim 3.

case 2: $a_{11} = 0$. As in the proof of claim 1, we can show that $b_{11} = 0$ and $a_{k,k} = 0$. Therefore the main diagonal of A consists of zeroes. It is not hard to show the main diagonal of B consists of zeroes. Now we can treat the first super diagonals of A and B as the main diagonals and argue as in case 1. To prove that the first non-zero diagonal of A is constant we need to use the first non-zero element of the first row of B. If $m = p - 1$, we may run out of non-zero elements of B. That would only happen if A has only one non-zero element: A_{mm}. In that case the diagonal has only one element and is therefore constant.

Therefore all diagonals of the all the matrices of the Jordan canonical form of S are constant. So S commutes with T. So $S \in AlgLat(T) \cap (T)' = \{p(T) : p \text{ is a polynomial }\}$. From this we can prove that $S \in \overline{\mathbb{C}orb(T)}$. □

An algebraic operator T is an operator for which there is a nonzero polynomial p such that $p(T) = 0$. Every algebraic linear transformation on a complex vector space has a Jordan cononical form, with possibly infinitely many Jordan blocks. We can extend our finite-dimensional result to arbitrary algebraic operators.

Theorem 21 *Suppose T is an algebraic operator. Then T is $\mathbb{C}$-orbit-refleive if and only if T is reflexive.*

Proof: Suppose T is an algebraic operator and T is reflexive. Suppose for every $x \in H$, $Sx \in \overline{\mathbb{C}orb(T,x)}$. Let $P(T) = \{q(T) : \quad q \text{ is a polynomial }\}$.

Since T is algebraic, H is an algebraic direct sum of subspaces that are invariant under T. More precisely, there exist eigenvalues $e_i \in H$ such that $H = \sum_{i\in I}^{\oplus} H_i$, where $H_i = \{P(T)e_i\}$. Also $T = \sum_{i\in I}^{\oplus} T_i$ where $T_i = T|_{H_i}$.

To show that $S \in \overline{\mathbb{C}orb(T)}^{SOT}$, we must prove that every strong neighborhood of S intersects $\mathbb{C}orb(T)$. Suppose $\{f_1, \ldots, f_m\} \subset H$ and $\varepsilon > 0$. Let $M = P(T)f_1 + P(T)f_2 + \ldots + P(T)f_m + H_1 + H_2$. The subspace M is finite dimensional, therefore M is closed. Clearly M is invariant for T. For every x, $Sx \in \overline{P(T)x}$; so $S \in AlgLat\ T$. Therefore $S(M) \subset M$. Now for every $x \in M$, $(S|_M)x \in \mathbb{C}orb((T|_M), x)$. The eigenvalues of $T|_M$ are all eigenvalues of T, so the decomposition of $T|_M$ is part of the decomposition of T. Since H_1, $H_2 \subset M$, the two largest blocks of $T|_M$ differ by at most 1. By the previous result $T|_M$ is $\mathbb{C}$-orbit-reflexive. Therefore $S|_M \in \overline{\mathbb{C}orb(T|M)}^{SOT}$. Since $f_1, \ldots, f_m \in M$, there exist $\lambda \in \mathbb{C}$ and $k \geq 0$ such that $\|S|_M f_i - \lambda(T|_M)^k f_i\| < \varepsilon$ for $1 \leq i \leq m$. So $\|Sf_i - \lambda T^k f_i\| < \varepsilon$ for $1 \leq i \leq m$.

To prove the opposite direction we can use the same argument as in the finite dimensional case. □

Chapter 3

Hereditarily Orbit-reflexive Operators

Suppose $\mathcal{S} \subset B(H)$ and $\mathcal{S}$ is a multiplicative semigroup. We define $\mathcal{S}$ to be *orbit-reflexive* if the following implication is true:

$$Sx \in \overline{\mathcal{S}x}, \text{ for all } x \in H \quad \Rightarrow S \in \overline{\mathcal{S}}^{SOT}.$$

Definition 12 *$\mathcal{S}$ is* hereditarily-orbit-reflexive *if every subsemigroup $\mathcal{S}_0$ of $\mathcal{S}$ is orbit-reflexive.*

Definition 13 *An operator T is* hereditarily-orbit-reflexive *if $Orb(T)$ is hereditarily-orbit-reflexive.*

Theorem 22 *Any normal operator T is hereditarily-orbit-reflexive.*

Proof: It must be shown that $Orb(T)$ is hereditarily-orbit-reflexive. So suppose $\mathcal{S}_0$ is a subsemigroup of $Orb(T)$. Suppose $Sx \in \overline{\mathcal{S}_0 x}$ for all x. $\mathcal{S}_0$ is a commuting family of normal operators. So, by proposition 2, $S \in \overline{\mathcal{S}_0}^{SOT}$. □

Theorem 23 *If $Orb(T, x)$ is closed for every x in a nonempty open set, then T is hereditarily-orbit-reflexive.*

Proof: Suppose $Sx \in \overline{\mathcal{S}x}$ for every x. For every x, $\mathcal{S}x \subset Orb(T,x)$ and for every x in the given non-empty open set $\overline{\mathcal{S}x} \subset Orb(T,x)$. So by Lemma 1 $S \in \overline{\mathcal{S}}^{SOT}$. □

Theorem 24 *If $\overline{Orb(T)}^{SOT}$ is countable and strongly compact, then T is hereditarily-orbit-reflexive.*

Proof: Suppose $\mathcal{S}$ is a subsemigroup of $Orb(T)$. Suppose $Rx \in \overline{\mathcal{S}x}$, for every x. For any particular x_0 there is a sequence $\{S_i\}$ in $\mathcal{S}$ such that $S_i x_0 \to Rx_0$. The sequence $\{S_i\}$ is in $\overline{Orb(T)}^{SOT}$ which is strongly compact, so there is a subsequence $\{S_{ij}\}$ such that $S_{ij} \to S_0$ (SOT). Since each S_{ij} is also in $\mathcal{S}$, $S_0 \in \overline{\mathcal{S}}^{SOT}$. So for every x, $Rx \in \overline{\mathcal{S}}^{SOT}x$. Since $\overline{Orb(T)}^{SOT}$ is countable, so is $\overline{\mathcal{S}}^{SOT}$. By Proposition 1, $R \in \overline{\mathcal{S}}^{SOT}$. □

Theorem 25 *Suppose $T \in B(H)$ and there is a nonzero operator $p \in B(H)$ such that $TP = PT$ and $\|T^n x\| \to \infty$ for every nonzero x in the range of P.*

Proof: Suppose $x \in (\ker P)^c$, then $Px \neq 0$ and $Px \in ranP$. So,

$$\|PT^n(x)\| = \|T^n P(x)\| \to \infty,$$

and so

$$\|T^n x\| \to \infty.$$

So $Orb(T,x)$ is closed for a non-empty open subset, $(\ker P)^c$. So by Theorem 23, T is hereditarily-orbit-reflexive. □

Theorem 26 *Suppose $T^n \to A$ (SOT), then T is hereditarily-orbit-reflexive.*

Proof: Suppose $\mathcal{S}$ is a subsemigroup of $Orb(T)$. Note that $\overline{\mathcal{S}}^{SOT} = \mathcal{S} \cup \{A\}$. Suppose that $Sx \in \overline{\mathcal{S}x}$, for every x.

$$\overline{\mathcal{S}x} = \mathcal{S}x \cup \{Ax\},$$

which is a countable set. By Proposition 1, $S \in \mathcal{S} \cup \{A\}$. Hence $S \in \overline{\mathcal{S}}^{SOT}$. Therefore T is herditarily-orbit-reflexive. □

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