Hankel operators on Hilbert spaces

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Abstract
In this paper we consider the Hankel operators from two points of view. On one hand the Hankel operator is induced by the coefficient sequence $a_0,a_1,a_2,...$ and operates on a Hilbert space $H^2(\beta)$ with $\Sigma_{n=0}^{\infty} \beta(n)^2 < \infty.$ In this situation we can find necessary conditions and sufficient conditions for the Hankel operator to be bounded. However, with compactness and Hilbert-Schmidt we can get only sufficient conditions. On the other hand we look at the Hankel operator $H_{f,\alpha}$ and little Hankel operator $h_{f,\alpha},$ with symbol function $f,$ that operates on a weighted Bergman space. In this case we can determine bounded, compact, Hilbert Schmidt, or trace class operators of the Hankel operator $H_{f}$ and $h_{f,\alpha}.$ We also give a good estimate of bounded norm of little Hankel operators with a particular symbol function $f = z\bar{g}$ where $g$ is in the Bloch space.

Keywords
Mathematics

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Hankel operators on Hilbert spaces

Wanpen, Pachara, Ph.D.
University of New Hampshire, 1993
HANKEL OPERATORS ON HILBERT SPACES

BY

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DISSERTATION

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4/20/93
Date
Dedication

To my Mother

Muoy Chaisuriya
Acknowledgments

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Foreword

There are at least two good reasons for studying Hankel operators. From a practical point of view Hankel operators play a major role in a variety of problems in physics, probability theory, information and control theory and several other fields. From a more abstract viewpoint Hankel operators constitute one of the most important classes of non-self adjoint operators and they are a fascinating example of the fruitful interplay between such topics as operator theory, function theory and the theory of Banach algebras.

The classical setting for Hankel operator is the Hardy space, in this space we have only one type of Hankel operator. There are several books dealing with Hankel operators on the Hardy space such as: [Power 80], [Partington 88], [Sarason 75], [GZ 58], [Nikol'skii 85], [Nehari 57], [Hartman 58] and others. The next setting for Hankel operators is on the usual Bergman space. In this space we have two types of Hankel operators which we will call the Hankel operator and the little Hankel operator. The first major progress into answering the boundedness and compactness questions of Hankel operators is in Axler's paper, [Axler 86]. The results of this paper shows that a Hankel operator is bounded if and only if its associated symbol function is in the Bloch space and compact if and only if its symbol function is in the little Bloch space.

Arazy, Fisher and Peetre, [AFP 88] studied Hankel operators with analytic symbol function on the weighted Bergman space their results about boundedness and compactness are parallel to those in [Axler 86]. They also considered the case of nonanalytic symbol functions. Zhu also studied the Hankel operator on the usual Bergman space with nonanalytic
symbol function. He is the first one to show the connection between size estimates of a Hankel operator and the mean oscillation of the symbol function in the Bergman metric. This idea is then generalized and developed systematically in [BCZ 87], and [BCZ 88]. In [Zhu 90] is given the characterization of those symbol functions for which the associated Hankel operator is bounded, compact and a member of a Schatten class.

In this paper we study the Hankel operator on the weighted Bergman space by generalized Zhu's techniques. In the case of boundedness and compactness we have results that parallel results in the usual Bergman space. However for the question of Schatten class membership we place some restriction on the weighted Bergman space.

The little Hankel on the usual Bergman space behaves like the Hankel operator on the Hardy space. Recently, Arazy, Fisher and Peetre, [AFP 88] gave the characterization of symbol function associated with a little Hankel operator on the weighted Bergman space. The conditions for boundedness, compactness and Schatten class membership are that the symbol functions are in the Bloch space, the little Bloch space and the Besov space respectively. In this paper we use a certain operator to characterize the boundedness, compactness and Schatten class membership of the little Hankel operators.

The paper is organized as follows:

Chapter 1 introduces the $H^2(\mathcal{B})$ space, the Hankel matrix and the Hankel operator in terms of a Hankel matrix related to an orthogonal basis. We give necessary conditions and sufficient conditions for an operator to be bounded and sufficient conditions for an operator to be Hilbert-Schmidt (which implies compact).

In the first section of chapter 2 we introduce the weighted Bergman space and some of its properties. In the next section we define the Hankel operator on the weighted Bergman space in terms of the Bergman projection and give three conditions on a symbol function that
imply the boundedness of the related Hankel operator. We also give a condition on a symbol function equivalent to the boundedness of two related Hankel operators. In section 3 we study the compactness of the Hankel operator. Some well-known definitions and properties are given which we use to find a necessary and sufficient condition for compactness of the operator. In section 4 we give a necessary and sufficient conditions for a Hankel operator to be a member of a Schatten class.

Chapter 3 presents the theory of little Hankel operators. We start by reviewing the development of the definition of this operator and give the definition of the little Hankel operator on the weighted Bergman space which we will use. We show that this operator is an integral operator and depends only on analytic part of its symbol function. We then introduce the important operator $V^*$. Some properties of $V^*$ are given and used to find a necessary and sufficient condition for the boundedness of the little Hankel operator. In section 2 of this chapter we introduce a certain finite rank little Hankel operator and give a theorem of compactness of this operator. Section 3 deals with Schatten class membership. A condition on a symbol functions is given which is equivalent to Schatten class membership for the associated operator. In each section of this chapter we relate our results to those in [AFP 88].

In Chapter 4 we consider the little Hankel operator with a particular symbol function, $z\bar{g}$ where $g$ is in the Bloch space. We show that the boundedness and compactness of the operator associated with $\bar{g}$ is equivalent to the boundedness and compactness of the operator associated with $z\bar{g}$. We estimate the bounded norm of the latter and show that this estimate generalizes that of Bonsall, [Bonsall 86].
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ABSTRACT

HANKEL OPERATORS ON HILBERT SPACES

by

Pachara Wanpen
University of New Hampshire, May, 1993

In this paper we consider the Hankel operators from two points of view. On one hand
the Hankel operator is induced by the coefficient sequence \( a_0, a_1, a_2, \ldots \) and operates on
a Hilbert space \( H^2(\beta) \) with \( \sum_{n=0}^{\infty} \beta(n)^2 < \infty \). In this situation we can find necessary
conditions and sufficient conditions for the Hankel operator to be bounded. However, with
compactness and Hilbert-Schmidt we can get only sufficient conditions. On the other hand
we look at the Hankel operator \( H_{f,\alpha} \) and little Hankel operator \( h_{f,\alpha} \), with symbol function
f, that operates on a weighted Bergman space. In this case we can determine bounded,
compact, Hilbert Schmidt, or trace class operators of the Hankel operator \( H_f \) and \( h_{f,\alpha} \).
We also give a good estimate of bounded norm of little Hankel operators with a particular
symbol function \( f = z\overline{g} \) where g is in the Bloch space.
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1

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Proposition 1.0.3 [Shields 74] An operator $A$ is an injective weighted shift on a Hilbert space $\mathcal{H}$ if and only if $A$ shifts some orthogonal basis, i.e., there exists an orthogonal basis $\{f_n : n = 0, 1, 2, \ldots\}$ of $\mathcal{H}$ such that $Af_n = f_{n+1}$, for $n = 0, 1, 2, \ldots$.

Definition 1.0.4 [Shields 74] Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$. Then $H^2(\beta)$ is the space of sequences $f = \{\hat{f}(n)\}$ such that

$$\sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2 < \infty.$$ 

We shall use the notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ whether or not the series converges for any (complex) values of $z$. These spaces are Hilbert spaces with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \beta(n)^2,$$

where $f = \{\hat{f}(n)\}$ and $g = \{\hat{g}(n)\}$, and the norm of $f$ is defined by

$$\|f\|_{H^2(\beta)}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2.$$ 

If $f_n(z) = z^n$, then $\{f_n\}$ is an orthogonal basis of $H^2(\beta)$, and $\|f_n\|_{H^2(\beta)} = \beta(n)$.

Example 1.0.5

1. When $\beta(n) = 1$ \ \forall n, $H^2(\beta)$ is the Hardy space.

2. When $\beta(n) = 1/\sqrt{n + 1}$ \ \forall n, $H^2(\beta)$ is the usual Bergman space.

3. When $\beta(n) = \sqrt{n}$ \ \forall n, $H^2(\beta)$ is the Fischer space.[Shields 74]

Definition 1.0.6 Consider the space $H^2(\beta)$ where the sequence $\left\{\frac{\beta(n+1)}{\beta(n)}\right\}$, is bounded.
Then $M_z$ is the linear transformation defined on $H^2(\beta)$ by

$$(M_z f)(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^{n+1}, \quad \text{where} \quad f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$ 

Since $(M_z f_n)(z) = z^{n+1} = f_{n+1}(z)$, $M_z$ shifts the orthogonal basis $\{f_n\}$ of $H^2(\beta)$.

**Proposition 1.0.7 [Shields 74]** The linear transformation $M_z$ is bounded and

$$\|M_z^n\| = \sup_k \left\{ \frac{\beta(k + n)}{\beta(k)} \right\}, \quad n = 0, 1, 2, \ldots .$$

**Proposition 1.0.8 [Shields 74]** The linear transformation $M_z$ on $H^2(\beta)$ is unitarily equivalent to an injective weighted shift operator (with the weight sequence $\{w_n\}$ given below).

Conversely, every injective weighted shift is unitarily equivalent to $M_z$ (acting on $H^2(\beta)$, for a suitable choice of $\beta$).

The relation between $\{w_n\}$ and $\{\beta(n)\}$ is given by the following equations

$$w_n = \frac{\beta(n + 1)}{\beta(n)} \quad \text{for all} \ n,$$

$$\beta(n) = w_0 w_1 \cdots w_{n-1}, \quad \text{for} \ n > 0, \ \text{and} \ \beta(0) = 1.$$

We will now relate Hankel operators to their matrix representations.

Suppose that $\{g_n : n = 0, 1, 2, \ldots \}$ is an orthogonal basis of a separable Hilbert space $\mathcal{H}$. Let $A \in \mathcal{B}(\mathcal{H})$. Each vector $Ag_n$ has an expansion $Ag_n = \sum_{m=0}^{\infty} a_{mn} g_m$, in which the coefficients are given by $a_{mn} = \frac{1}{\|g_n\|^2} \langle Ag_n, g_m \rangle$. In this way, we associate with each $A$ in
$B(\mathcal{H})$ a complex matrix $[a_{mn}]$ relative to the orthogonal basis $\{g_0, g_1, g_2, \ldots\}$. Since

$$
\frac{1}{\|g_m\|^2} \langle A^* g_n, g_m \rangle = \frac{1}{\|g_m\|^2} \langle g_n, A g_m \rangle = \frac{1}{\|g_m\|^2} \overline{\langle A g_m, g_n \rangle}
$$

$$
= \frac{\|g_n\|^2}{\|g_m\|^2} \delta_{nm},
$$

the matrix of $A^*$ has $\frac{\|g_n\|^2}{\|g_m\|^2} \delta_{nm}$ in the $(m, n)$ position. Suppose two operators $A, B$ of $B(\mathcal{H})$ have matrices $[a_{mn}]$ and $[b_{mn}]$, respectively, and let $C = AB$. Then

$$
\frac{1}{\|g_m\|^2} \langle C g_n, g_m \rangle = \frac{1}{\|g_m\|^2} \langle AB g_n, g_m \rangle = \frac{1}{\|g_m\|^2} \langle B g_n, A^* g_m \rangle,
$$

and Parseval's equation gives

$$
\frac{1}{\|g_m\|^2} \langle C g_n, g_m \rangle = \sum_{k=0}^{\infty} \frac{1}{\|g_m\|^2} \frac{1}{\|g_k\|^2} \langle B g_n, g_k \rangle \langle g_k, A^* g_m \rangle
$$

$$
= \sum_{k=0}^{\infty} \frac{1}{\|g_m\|^2} \langle A g_k, g_m \rangle \frac{1}{\|g_k\|^2} \langle B g_n, g_k \rangle
$$

$$
= \sum_{k=0}^{\infty} a_{mk} b_{kn}.
$$

Accordingly, the matrix $[c_{mn}]$ of $C = AB$ is given by

$$
c_{mn} = \sum_{k=0}^{\infty} a_{mk} b_{kn}.
$$

The algebraic relation between operators and matrices follows the pattern familiar in the finite-dimensional case, and $[a_{mn}]$ is the zero matrix only when $A = 0$.

Suppose $f = \sum_{n=0}^{\infty} b_n g_n$ and $Af = \sum_{m=0}^{\infty} c_m g_m$. Then

$$
c_m = \left\langle Af, \frac{g_m}{\|g_m\|^2} \right\rangle
$$
\[
\lim_{k \to \infty} \left\langle A \sum_{n=0}^{k} b_n g_n, \frac{g_m}{\|g_m\|^2} \right\rangle \\
= \lim_{k \to \infty} \sum_{n=0}^{k} \left\langle A g_n, \frac{g_m}{\|g_m\|^2} \right\rangle b_n \\
= \sum_{n=0}^{\infty} a_{mn} b_n.
\]

Thus
\[
\sum_{n=0}^{\infty} a_{mn} b_n \text{ converges.}
\]

By the above results the matrices, corresponding (through a fixed orthogonal basis) to bounded operators on a Hilbert space \( \mathcal{H} \), form an algebra relative to the usual sum, product, and scalar multiplication of matrices. Furthermore the mapping from bounded operators to the corresponding matrices is an isomorphism.

Hankel operators have been studied in several contexts, for example Hankel operators on the Hardy space and the Bergman space. In this chapter we want to study Hankel operators in a more general context, we will study Hankel operators on \( H^2(\beta) \) by looking at their matrix representations. Both the Hardy space and the Bergman space are \( H^2(\beta) \) spaces where \( \sum_{n=0}^{\infty} \beta(n)^2 \) does not converge. We will consider the spaces \( H^2(\beta) \) with the property that \( \sum_{n=0}^{\infty} \beta(n)^2 < \infty \).

A definition of Hankel operators in terms of matrix representations has been given in [Power 82] as follows:

Let \( \mathcal{H} \) be a separable Hilbert space with orthonormal basis \( \{e_0, e_1, e_2, \ldots\} \). An operator \( H \in \mathcal{B}(\mathcal{H}) \) is a Hankel operator provided that there exists a sequence of complex numbers \( \{a_n\} \) such that

\[
a_{n+m} = \langle He_m, e_n \rangle, \quad \text{for all } n, m \geq 0.
\]
Then the matrix of $H$ relative to the orthonormal basis $\{e_0, e_1, e_2, \ldots\}$ is

\[
\begin{pmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_1 & a_2 & a_3 & \cdots \\
a_2 & a_3 & a_4 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

We will call a matrix a Hankel matrix if it is of this form, and we say that $H$ is induced by the sequence $\{a_n\}$. Therefore another way to say $H \in \mathcal{B}(\mathcal{H})$ is a Hankel operator is that it has a Hankel matrix relative to some orthonormal basis.

Let $S$ be the unilateral shift; $Se_n = e_{n+1}$. If $H \in \mathcal{B}(\mathcal{H})$ is a Hankel operator, then

\[
\langle S^* H e_n, e_m \rangle = \langle He_n, Se_m \rangle \\
= \langle He_n, e_{m+1} \rangle \\
= a_{m+1+n} = \langle He_{n+1}, e_m \rangle \\
= \langle HSe_n, e_m \rangle \text{ for all } n, m \geq 0.
\]

Therefore another way to say $H \in \mathcal{B}(\mathcal{H})$ is a Hankel operator on $\mathcal{H}$ is that $S^* H = HS$. In this chapter we will define Hankel operators on separable Hilbert spaces with an orthogonal basis.

**Definition 1.0.9** Let $\mathcal{H}$ be a separable Hilbert space with orthogonal basis $\{g_n : n = 0, 1, 2, \ldots\}$. An operator $H \in \mathcal{B}(\mathcal{H})$ is called a Hankel operator provided there exists a sequence of complex numbers $\{a_n\}$ such that

\[
a_{i+j} = \frac{1}{\|g_i\|^2} \langle Hg_j, g_i \rangle \text{ for all } i, j \geq 0.
\]
Notice the matrix of a Hankel operator \( H \) relative to the orthogonal basis \( \{ g_n : n = 0, 1, 2, \ldots \} \) is a Hankel matrix

\[
\begin{pmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_1 & a_2 & a_3 & \cdots \\
a_2 & a_3 & a_4 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Therefore in this situation to say \( H \in \mathcal{B}(\mathcal{H}) \) is a Hankel operator is to say that it has a Hankel matrix relative to some orthogonal basis.

Let \( S \) be the operator defined on \( \mathcal{H} \) by \( S g_n = g_{n+1} \). Then for \( n, m \geq 0 \)

\[
\langle S^* H g_n, g_m \rangle = \langle H g_n, S g_m \rangle \\
= \langle H g_n, g_{m+1} \rangle \\
= \|g_{m+1}\|^2 a_{m+1+n} \\
= \frac{\|g_{m+1}\|^2}{\|g_m\|^2} \langle H S g_n, g_m \rangle.
\]

Therefore if \( \frac{\|g_{m+1}\|}{\|g_m\|} = 1 \) for all \( n \geq 0 \) \( S^* H = HS \), and so, if the basis of \( \mathcal{H} \) is normal, this definition of Hankel operator coincides with the one in [Power 82].

In the rest of this chapter we will consider only \( H^2(\beta) \) spaces satisfying the assumption that \( \sum_{n=0}^{\infty} \beta(n)^2 < \infty \), and \( H \) will be a Hankel operator acting on these spaces relative to the orthogonal basis \( \{ f_n : n = 0, 1, 2, \ldots \} \), defined by \( f_n(z) = z^n \) for \( n = 0, 1, 2, \ldots \).

**Proposition 1.0.10** Suppose \( H \) is a Hankel operator on the space \( H^2(\beta) \), and \( H \) is induced by a sequence of complex numbers \( \{ a_n \} \). Then for all complex sequences \( \{ b_n \} \) such that \( \sum_{n=0}^{\infty} |b_n|^2 \beta(n)^2 < \infty \), \( \sum_{n=0}^{\infty} a_{n+k} b_n \) converges for all \( k = 0, 1, 2, \ldots \). In particular
\[ \sum_{n=0}^{\infty} a_n b_n \text{ converges.} \]

**Proof.** The Proposition is an immediate consequence of the definition of matrix of an operator. \(\square\)

**Corollary 1.0.11** If \( H \) is as in Proposition 1.0.10, then the series \( \sum_{n=0}^{\infty} a_n \) is absolutely convergent.

**Proof.** Let \( \{b_n\} \) be a sequence defined by

\[
 b_n = \begin{cases} 
 \frac{|a_n|}{a_n} & \text{if } a_n \neq 0 \\
 0 & \text{if } a_n = 0.
\end{cases}
\]

Since \( \sum_{n=0}^{\infty} \beta(n)^2 < \infty \), therefore we can define \( g = \sum_{n=0}^{\infty} \beta(n)f_n \), \( g \in H^2(\beta) \)

\[
 \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \frac{|a_n|}{a_n} a_n
\]

which converges by Proposition 1.0.10. \(\square\)

**Corollary 1.0.12** If \( H \) is as in Proposition 1.0.10, then \( \left\{ \frac{a_n}{\beta(n)} \right\} \in l^2 \).

**Proof.** Let \( \{b_n\} \) be a sequence defined by \( b_n = \frac{a_n}{\beta(n)} \) where \( \{c_n\} \in l^2 \). Since \( \sum_{n=0}^{\infty} \left| \frac{a_n}{\beta(n)} \right|^2 \beta(n)^2 = \sum_{n=0}^{\infty} |c_n|^2 < \infty \), we can define \( g = \sum_{n=0}^{\infty} \frac{a_n}{\beta(n)} f_n \), then \( g \in H^2(\beta) \). By Proposition 1.0.10, \( \sum_{n=0}^{\infty} \frac{a_n}{\beta(n)} c_n \) converges for all \( k \). Using the fact that \( \{c_n\} \in l^2 \) and the principle of uniform boundedness, we have \( \left\{ \frac{a_n}{\beta(n)} \right\} \in l^2 \) for all \( k = 0, 1, 2, \ldots \). In particular \( \left\{ \frac{a_n}{\beta(n)} \right\} \in l^2 \). \(\square\)

**Theorem 1.0.13** Suppose \( \{a_n\} \) is a sequence such that \( \left\{ \frac{a_n}{\beta(n)} \right\} \) is in \( l^2 \) and \( \{|a_n|\} \) is monotonically decreasing. Then the Hankel matrix \( H \) induced by \( \{a_n\} \) is the matrix of a bounded operator.
Proof. Let \( g = \sum_{n=0}^{\infty} b_n f_n \in H^2(\beta) \). Then

\[
H g = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} a_{m+n} b_n \right) f_m.
\]

Thus

\[
\|Hg\|_{H^2(\beta)}^2 = \sum_{m=0}^{\infty} \left| \sum_{n=0}^{\infty} a_{m+n} b_n \right|^2 \beta(m)^2.
\]

Since \( \{a_n\} \) is monotone decreasing, we have

\[
\sum_{n=0}^{\infty} |a_{m+n} b_n| \leq \sum_{n=0}^{\infty} |a_n| |b_n| = \sum_{n=0}^{\infty} |a_n| \beta(n) |b_n| \beta(n).
\]

Use the fact that \( \left\{ \frac{a_n}{\beta(n)} \right\} \in l^2 \), \( g \in H^2(\beta) \) and apply Hölder's inequality to get

\[
\sum_{n=0}^{\infty} |a_{m+n} b_n| \leq \left( \sum_{n=0}^{\infty} \left| \frac{a_n}{\beta(n)} \right|^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} |b_n|^2 \beta(n)^2 \right)^{\frac{1}{2}}
\]

\[
= \left\| \left\{ \frac{a_n}{\beta(n)} \right\} \right\|_{l^2} \| g \|_{H^2(\beta)} < \infty.
\]

Therefore \( \sum_{n=0}^{\infty} a_{m+n} b_n \) is absolutely convergent. Now we have

\[
\|Hg\|_{H^2(\beta)}^2 \leq \left\| \left\{ \frac{a_n}{\beta(n)} \right\} \right\|_{l^2}^2 \| g \|_{H^2(\beta)}^2 \sum_{m=0}^{\infty} \beta(m)^2.
\]

Thus \( H \) is bounded and

\[
\|H\| \leq \left\| \left\{ \frac{a_n}{\beta(n)} \right\} \right\|_{l^2} \left( \sum_{m=0}^{\infty} \beta(m)^2 \right)^{\frac{1}{2}}.
\]
**Theorem 1.0.14** Suppose \( \{a_n\} \) is a sequence such that \( \{\beta_n\} \) is in \( l^2 \) and \( \{\beta(n)\} \) is monotonically decreasing. Then the Hankel matrix \( H \) induced by \( \{a_n\} \) is the matrix of a bounded operator.

**Proof.** Let \( g = \sum_{n=0}^{\infty} b_n f_n \in H^2(\beta) \). Then

\[
\|Hg\|_{H^2(\beta)}^2 = \sum_{m=0}^{\infty} \left| \sum_{n=0}^{\infty} a_{m+n} b_n \right|^2 \beta(m)^2.
\]

Consider

\[
\sum_{n=0}^{\infty} |a_{m+n} b_n| = \sum_{n=0}^{\infty} \left| \frac{a_{m+n}}{\beta(m+n)} b_n \right| \beta(m+n).
\]

Apply Hölder’s inequality and use the fact that \( \{\beta(n)\} \) is monotone decreasing we have

\[
\sum_{n=0}^{\infty} |a_{m+n} b_n| \leq \left( \sum_{n=0}^{\infty} \left| \frac{a_{m+n}}{\beta(m+n)} \right|^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} |b_n|^2 \beta(m+n)^2 \right)^{\frac{1}{2}} \leq \left( \sum_{n=0}^{\infty} \left| \frac{a_n}{\beta(n)} \right|^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} |b_n|^2 \beta(n)^2 \right)^{\frac{1}{2}}.
\]

Thus

\[
\|Hg\|_{H^2(\beta)}^2 \leq \left( \sum_{n=0}^{\infty} \left| \frac{a_n}{\beta(n)} \right|^2 \right) \left( \sum_{n=0}^{\infty} |b_n|^2 \beta(n)^2 \right) \sum_{m=0}^{\infty} \beta(m)^2
\]

\[
= \left\| \{ \frac{a_n}{\beta(n)} \} \right\|_{l^2}^2 \|g\|_{H^2(\beta)}^2 \sum_{m=0}^{\infty} \beta(m)^2.
\]

Hence \( H \) is bounded and

\[
\|H\| \leq \left\| \{ \frac{a_n}{\beta(n)} \} \right\|_{l^2} \left( \sum_{m=0}^{\infty} \beta(m)^2 \right)^{\frac{1}{2}}.
\]
From the above result we can estimate the size of a Hankel operator using the sequences \( \{a_n\} \) and \( \{\beta(n)\} \). If \( H \) is bounded, then

\[
\|H\| = \sup\{\|Hf\|_{H^2(\beta)} : f \in H^2(\beta), \|f\|_{H^2(\beta)} = 1\}.
\]

Consider

\[
f_0 \in H^2(\beta), \quad \|f_0\|_{H^2(\beta)} = \beta(0) = 1, \quad Hf_0 = \sum_{n=0}^{\infty} a_n f_n.
\]

Hence

\[
\|Hf_0\|_{H^2(\beta)}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 \leq \|H\|^2.
\]

Therefore

\[
\{a_n \beta(n)\} \in l^2 \quad \text{and} \quad \|H\| \geq \|\{a_n \beta(n)\}\|_{l^2}.
\]

From Theorems 1.0.13 and 1.0.14, we have

\[
\|H\| \leq \left\| \left\{ \frac{a_n}{\beta(n)} \right\} \right\|_{l^2} \left( \sum_{k=0}^{\infty} \beta(k)^2 \right)^{\frac{1}{2}}.
\]

We can estimate \( \|H\| \) by

\[
\|\{a_n \beta(n)\}\|_{l^2} \leq \|H\| \leq \left\| \left\{ \frac{a_n}{\beta(n)} \right\} \right\|_{l^2} \left( \sum_{k=0}^{\infty} \beta(k)^2 \right)^{\frac{1}{2}}.
\]

We have seen a set of sufficient conditions for Hankel operators to be bounded in the Theorems 1.0.13 and 1.0.14. Both theorems have the common conditions that \( \left\{ \frac{a_n}{\beta(n)} \right\} \in l^2 \) and \( \{\beta(n)\} \in l^2 \). Now we want to show in the next example that without the monotonicity of \( \{|a_n|\} \) or \( \{\beta(n)\} \), the Hankel operators induced by the sequence \( \{a_n\} \) may not be bounded.
Example 1.0.15 Let

\[ \beta(n) = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is odd} \\ \frac{1}{(n+1)^2} & \text{if } n \text{ is even} \end{cases} \]

\[ a_n = \begin{cases} \frac{1}{(n+1)^2} & \text{if } n \text{ is odd} \\ \frac{1}{(n+1)^2} & \text{if } n \text{ is even} \end{cases} \]

\[ b_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ (n+2) & \text{if } n \text{ is even}. \end{cases} \]

Then we have \( \left\{ \frac{a_n}{\beta(n)} \right\} \in l^2 \). \( \sum_{n=0}^{\infty} \beta(n)^2 < \infty \). Since

\[
\sum_{n=0}^{\infty} |b_n|^2 \beta(n)^2 = \sum_{n \text{ even}} |b_n|^2 \beta(n)^2 + \sum_{n \text{ odd}} |b_n|^2 \beta(n)^2 \\
= \sum_{n \text{ even}} (n+2)^2 \frac{1}{(n+1)^4} + \sum_{n \text{ odd}} \frac{1}{(n+1)^2} < \infty,
\]

we may define \( g \) in \( H^2(\beta) \) by \( g = \sum_{n=0}^{\infty} b_n f_n \). Consider

\[
\sum_{n=0}^{\infty} a_{n+1} b_n = \sum_{n \text{ even}} a_{n+1} b_n + \sum_{n \text{ odd}} a_{n+1} b_n \\
= \sum_{n \text{ even}} \frac{1}{(n+2)} + \sum_{n \text{ odd}} \frac{1}{(n+2)^2}.
\]

Therefore, \( \sum_{n=0}^{\infty} a_{n+1} b_n \) does not converge, and \( H \) is not bounded.

Theorem 1.0.16 Suppose \( \{a_n\} \) is a sequence such that \( \left\{ \frac{a_n}{\beta(n)} \right\} \) is in \( l^2 \) and \( \{|a_n|\} \) is monotonically decreasing. Then the Hankel matrix \( H \) induced by \( \{a_n\} \) is the matrix of a Hilbert-Schmidt operator.

Proof. By Theorem 1.0.13 we have \( H \) is a matrix of a bounded operator. Let \( \epsilon_n = \frac{a_n}{\beta(n)} \).
Then \( \{ e_n : n = 0, 1, 2, \ldots \} \) is an orthonormal basis of \( H^2(\beta) \). Consider

\[
\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} \| H e_n \|_{H^2(\beta)}^2 = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} \| H z^n \|_{H^2(\beta)}^2 = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} \sum_{m=0}^{\infty} a_{m+n} z^m \|_{H^2(\beta)}^2 = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} \sum_{m=0}^{\infty} |a_{m+n}|^2 \beta(m)^2.
\]

Use the fact that \( \{|a_n|\} \) is monotone decreasing and \( \sum_{m=0}^{\infty} \beta(m)^2 < \infty \) we have

\[
\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} \sum_{m=0}^{\infty} \beta(m)^2 = \left\| \left\{ \frac{a_n}{\beta(n)} \right\} \right\|_{l^2}^2 \sum_{m=0}^{\infty} \beta(m)^2 < \infty.
\]

Thus \( H \) is the matrix of a Hilbert-Schmidt operator.

\[ \square \]

**Corollary 1.0.17** Suppose \( \{ a_n \} \) is as in Theorem 1.0.16. Then the Hankel matrix \( H \) induced by \( \{ a_n \} \) is the matrix of a compact operator.

**Proof.** Use the fact that every Hilbert-Schmidt operator is a compact operator and then the result follows directly from Theorem 1.0.16.

\[ \square \]

**Theorem 1.0.18** Suppose \( \{ a_n \} \) is a sequence such that \( \left\{ \frac{a_n}{\beta(n)} \right\} \) is in \( l^2 \) and \( \{ \beta(n) \} \) is monotonically decreasing. Then the Hankel matrix \( H \) induced by \( \{ a_n \} \) is the matrix of a Hilbert-Schmidt operator.

**Proof.** By Theorem 1.0.14, we have \( H \) is the matrix of a bounded operator. From the proof of Theorem 1.0.16 we have

\[
\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} \sum_{m=0}^{\infty} \beta(m)^2 = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} \sum_{m=0}^{\infty} |a_{m+n}|^2 \beta(m)^2.
\]
Since \( \{\beta(n)\} \) is monotone decreasing we have

\[
\sum_{n=0}^{\infty} \| H \frac{z^n}{\beta(n)} \|_{H^2(\beta)}^2 \leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|a_{m+n}|^2}{\beta(m+n)^2} \beta(m)^2 \leq \left\| \left\{ \frac{a_n}{\beta(n)} \right\} \right\|_{l^2}^2 \sum_{m=0}^{\infty} \beta(m)^2 < \infty.
\]

Thus \( H \) is a matrix of a Hilbert-Schmidt operator.

\[ \square \]

**Corollary 1.0.19** Suppose \( \{a_n\} \) is as in Theorem 1.0.18. Then the Hankel matrix \( H \) induced by \( \{a_n\} \) is the matrix of a compact operator.

**Proof.** The proof of this corollary follows from the fact that every Hilbert-Schmidt operator is a compact operator and Theorem 1.0.18.

Among the open questions still remaining for Hankel operators on this space are the necessary conditions for compactness, Hilbert-Schmidt, characterizations of boundedness, and characterization of membership in the Schatten classes.
Chapter 2

Hankel Operators on Weighted Bergman spaces

In this chapter we consider Hankel Operators acting on weighted Bergman spaces. We apply techniques of Zhu used to study Hankel operators on the usual Bergman space. Necessary and sufficient conditions for these operators to be bounded, compact, and Schatten p-class are given.

2.1 Weighted Bergman spaces.

We begin this section by relating a weighted Bergman space to a subnormal injective weighted shift operator. Note that in the following operators are not necessarily bounded.

Definition 2.1.1 [Shields 74] An operator \( S \) on a Hilbert space \( \mathcal{H} \) is said to be subnormal if there is a Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) as a closed subspace, and a normal operator \( N \) on \( \mathcal{K} \) such that \( N \mathcal{H} \subseteq \mathcal{H} \) and \( S = N|_{\mathcal{H}} \).

Theorem 2.1.2 [Shields 74] Let \( \{e_0, e_1, \ldots\} \) be an orthonormal basis for \( \mathcal{H} \) and let \( S \) be a weighted shift relative to this basis with weight sequence \( \{\alpha_n\} \), with the property that \( \sup\{\alpha_n : n = 0, 1, \ldots\} = 1 \). Then the following statements are equivalent.
1. $S$ is subnormal of norm 1.

2. There is a probability measure $\nu$ on $[0,1]$ containing 1 in its support, such that for $n \geq 1$

$$\left(\alpha_2 \alpha_1 \cdots \alpha_{n-1}\right)^2 = \int_0^1 r^{2n} d\nu(r).$$

3. There is a probability measure $\nu$ on $[0,1]$ containing 1 in its support such that if $\rho$ is the measure defined on $\mathbb{C}$ by

$$d\rho(re^{i\theta}) = d\nu(r) \frac{d\theta}{2\pi},$$

then $S$ is unitarily equivalent to the subnormal injective unilateral shift $M_z$ on $H^2(\beta)$, where $\beta(n) = (\alpha_0 \alpha_1 \cdots \alpha_{n-1})$.

The correspondence between subnormal weighted shifts of norm 1 and probability measure on $[0,1]$ with 1 in their support, as described in (2), is bijective. The space $H^2(\beta)$ in this case is the span of the polynomials in $L^2(\rho)$.

Example 2.1.3

1. If $\nu$ is the unit point mass at 1, then the corresponding $H^2(\beta)$ space is the Hardy space that has sequence $\{\beta(n)\}$ such that $\beta(n) = 1$, for all $n$.

2. Let $\delta_0$ and $\delta_1$ be the unit point masses at 0 and 1 for any $\alpha > 0$. Define the measure $\nu = (\alpha \delta_0 + \delta_1)/(1 + \alpha)$. The corresponding $H^2(\beta)$ has sequence $\{\beta(n)\}$ such that $\beta(n) = (1 + \alpha)^{-\frac{1}{2}}$ for all $n$ [Conway 91].

3. If $d\nu(r) = 2rdr$ on $[0,1]$, then the corresponding $H^2(\beta)$ is the usual Bergman space that has measure $\rho = A$, where $dA = 2rdr \frac{d\theta}{2\pi}$. This space has a sequence $\{\beta(n)\}$ such that $\beta(n) = \frac{1}{\sqrt{n+1}}$ for all $n$. 
4. Let $\alpha > -1$, and $d\nu(r) = (\alpha + 1)(1 - r^2)^\alpha 2\pi r dr$. Then the corresponding $H^2(\beta)$ is a weighted Bergman space that has sequence $\{\beta(n)\}$ such that $\beta(n) = \left[\frac{\Gamma(\alpha + 2)\Gamma(n + 1)}{\Gamma(n + \alpha + 2)}\right]^{\frac{1}{2}}$.

Let $A_\alpha$ be the measure $\rho$ in this case.

From the above examples (3) and (4) we see that $dA_\alpha = (1 + \alpha)(1 - r^2)^\alpha dA$. We will use notation $L^2_\alpha(D, dA_\alpha)$ for this weighted Bergman space.

The following notations will be used often in this chapter.

Let $f$ be a measurable function on $D$

$$\|f\|_{L^2(D, dA_\alpha)} = \left\{ \int_D |f(z)|^2 dA_\alpha(z) \right\}^{\frac{1}{2}}$$

$L^2(D, dA_\alpha) = \{ f : \|f\|_{L^2(D, dA_\alpha)} < \infty \}$

$$\|f\|_\infty = \text{ess sup}\{ |f(z)| : z \in D \}$$

$L^\infty(D, dA_\alpha) = \{ f : D \to C, \|f\|_\infty < \infty \}$

$H^\infty(D, dA_\alpha) = \{ f : f \in L^\infty(D, dA_\alpha), f \text{ is analytic} \}$.

The essential supremum above is taken relative to any of the equivalent measure, $A_\alpha$, $\alpha > -1$.

**Proposition 2.1.4 [AFP 88]** The weighted Bergman space $L^2_\alpha(D, dA_\alpha)$ is a closed subspace of $L^2(D, dA_\alpha)$.

**Definition 2.1.5** A vector space $\mathcal{X}$ of functions on a set $S$, together with a norm on $\mathcal{X}$, is called a functional Hilbert space if for every $s \in S$ the evaluation functional at $s$ is
continuous, i.e. there is a constant $C_s > 0$ such that

$$|x(s)| \leq C_s \|x\|, \quad x \in X.$$  

**Proposition 2.1.6 [AFP 88]** The weighted Bergman space $L_2^\alpha(D, dA_\alpha)$ is a functional Hilbert space.

Since point evaluations are bounded linear functionals on the Hilbert space $L_2^\alpha(D, dA_\alpha)$, the Riesz Representation theorem gives us for each $z \in D$ there exists a unique function $K_z^\alpha$ such that $K_z^\alpha \in L_2^\alpha(D, dA_\alpha)$ with the property that for each $f \in L_2^\alpha(D, dA_\alpha)$

$$f(z) = \langle f, K_z^\alpha \rangle_{L_2^\alpha(D, dA_\alpha)} = \int_D f(w) \overline{K_z^\alpha}(w) dA_\alpha(w).$$

For convenience we will write $\overline{K_z^\alpha}(w) = K^{\alpha}(z, w)$ and call $K^{\alpha}(z, w)$ the reproducing kernel of $L_2^\alpha(D, dA_\alpha)$.

Since $L^2(D, dA_\alpha)$ is a Hilbert space and $L_2^\alpha(D, dA_\alpha)$ is a closed subspace of $L^2(D, dA_\alpha)$, there exists an orthogonal projection $P^\alpha$ on $L^2(D, dA_\alpha)$ with range $L_2^\alpha(D, dA_\alpha)$, such that for $f \in L^2(D, dA_\alpha)$, $z \in D$

$$P^\alpha f(z) = \langle P^\alpha f, K_z^\alpha \rangle_{L_2^\alpha(D, dA_\alpha)} = \langle f, K_z^\alpha \rangle_{L_2^\alpha(D, dA_\alpha)} = \int_D f(w) K^{\alpha}(z, w) dA_\alpha(w).$$

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Theorem 2.1.7 [Aronzajn 50] Suppose \( \{e_n\}_{n=1}^{\infty} \) is an orthonormal basis of \( L^2_a(D, dA_\alpha) \).

Then

\[
K^\alpha(z, w) = \sum_{n=1}^{\infty} e_n(z) \overline{e_n(w)}.
\]

In particular, \( K^\alpha(z, w) \) is independent of the choice of the orthonormal basis \( \{e_n\}_{n=1}^{\infty} \).

2.2 Hankel Operators on Weighted Bergman Spaces

In this section we will define one type of Hankel operator \( H_{f,\alpha} \) defined on the weighted Bergman space \( L^2_a(D, dA_\alpha) \) into \( L^2_a(D, dA_\alpha)^\perp \). Three conditions on \( f \) are given, each implying the boundedness of \( H_{f,\alpha} \). Finally, we give a condition on \( f \) equivalent to \( H_{f,\alpha} \) and \( H_{f,\alpha} \) being bounded.

The definition of Hankel operators on the usual Bergman space \( L^2_a(D, dA) \) has been given in [Axler 86]. They are defined as follows: Let \( P \) denote the orthogonal projection of \( L^2(D, dA) \) onto \( L^2_a(D, dA) \), so \( (1 - P) \) is the orthogonal projection of \( L^2(D, dA) \) onto \( L^2_a(D, dA)^\perp \). For \( f \in L^\infty(D, dA) \) the Hankel operator

\[
H_f : L^2_a(D, dA) \longrightarrow L^2_a(D, dA)^\perp \text{ is defined by } H_f g = (1 - P)(fg).
\]

Arazy, Fisher and Peetre [AFP 88] defined the Hankel operators on \( L^2_a(D, dA_\alpha) \) as follows:

\[
H_{f,\alpha} : L^2_a(D, dA_\alpha) \longrightarrow L^2_a(D, dA_\alpha)^\perp \text{ is defined by } H_{f,\alpha} g = (1 - P^\alpha)(fg).
\]

We will follow Axler and take for the definition of Hankel operator in this paper as following.
Definition 2.2.1 An operator $H_{f,\alpha}$ densely defined on $L^2_\alpha(D, dA_\alpha)$ into $L^2_\alpha(D, dA_\alpha)^\perp$ by

$$H_{f,\alpha}g = (1 - P^\alpha)fg$$

is called a Hankel operator, and we say that the Hankel operator $H_{f,\alpha}$ has symbol function $f$.

The definition of Hankel operator $H_{f,\alpha}$ in Definition 2.2.1 is the same as the Hankel operator $H_{f,\alpha}$ in [AFP 88].

In general $H_{f,\alpha}$ is unbounded, but if $f \in L^\infty(D, dA_\alpha)$, then

$$\|H_{f,\alpha}g\|_{L^2_\alpha(D, dA_\alpha)}^2 = \| (1 - P^\alpha)fg\|_{L^2_\alpha(D, dA_\alpha)}^2$$

$$\leq \int_D |f(w)g(w)|^2 dA_\alpha(w)$$

$$\leq \|f\|_\infty^2 \|g\|_{L^2_\alpha(D, dA_\alpha)}^2.$$

So an obvious sufficient condition for the Hankel operator $H_{f,\alpha}$ to be bounded is that its symbol function is in $L^\infty(D, dA_\alpha)$. For the second sufficient condition, we will need the following definitions and results from [Zhu 90].

1. For any $z \in D$, $\Phi_z : D \to D$ is defined by

$$\Phi_z(w) = \frac{z - w}{1 - \overline{z}w}, \quad w \in D.$$

2. The pseudo-hyperbolic distance $\rho(z, w)$ is defined on $D$ by

$$\rho(z, w) = |\phi_z(w)| = \frac{|z - w|}{|1 - \overline{z}w|}, \quad z, w \in D.$$
3. The hyperbolic (Bergman) metric $\beta(z, w)$ is defined on $D$ by

$$
\beta(z, w) = \frac{1}{2} \ln \frac{1 + \rho(z, w)}{1 - \rho(z, w)}, \quad z, w \in D.
$$

4. The hyperbolic disk $D(z, r)$ is defined by

$$
D(z, r) = \{ w \in D : \beta(z, w) < r \}.
$$

**Proposition 2.2.2 [Zhu 90]** Suppose $r > 0$.

1. If $\alpha > -1$, then there exists a constant $C > 0$ independent of $r$ and $\alpha$ such that

$$
\frac{1}{C} A_\alpha(D(z, r)) \leq A(D(z, r))^{1+\alpha} \leq C A_\alpha(D(z, r)),
$$

where $z \in D$.

2. There exists a constant $C > 0$ independent of $r$ such that for $z \in D$

$$
\frac{1}{C} A(D(z, r))^{\frac{1}{2}} \leq (1 - |w|^2) \leq CA(D(z, r))^{\frac{1}{2}}, \quad \text{when } w \in D(z, r).
$$

3. If $s = \tanh r = \frac{e^r - e^{-r}}{e^r + e^{-r}}$, then

$$
A(D(z, r)) = \frac{(1 - |z|^2)^2 s^2}{(1 - |z|^2 s^2)^2}, \quad z \in D.
$$

4. For any positive number $R$, there exists a constant $C > 0$ independent of $r$ such that

$$
\frac{1}{C} \leq \frac{A(D(z, r))}{A(D(w, r))} \leq C,
$$
for all $z$ and $w$ such that $\beta(z, w) \leq R$ and for all $0 < r \leq R$.

5. We also have

$$\int_{D} \int_{D} [\beta(u, v) + 1]^{2} dA(u) dA(v) < \infty.$$ 

6. If $p > 0$, then there exists a constant $C > 0$ independent of $r$ ($r \leq 1$) such that

$$(1 - |w|^2)^p \leq C(1 - |z|^2)^p,$$

for all $z \in D(w, 2r)$.

Proposition 2.2.3 [Rudin 80] Suppose $z \in D$, $c$ is a real, $t > -1$, and

$$I_{c,t}(z) = \int_{D} \frac{(1 - |w|^2)^t}{|1 - z\overline{w}|^{2 + 2t + c}} dA(w).$$

Then

1. If $c < 0$, then $I_{c,t}$ is bounded in $z$;

2. If $c > 0$, then $I_{c,t} \sim \frac{1}{|z|^{2t}}$ ($|z| \to 1^{-}$);

3. If $c = 0$, then $I_{c,t} \sim \log \frac{1}{|z|^{2t}}$ ($|z| \to 1^{-}$).

Theorem 2.2.4 [Schur 11] Suppose $(\mathcal{X}, \mu)$ is a measure space and $K$ is a nonnegative measurable function on $\mathcal{X} \times \mathcal{X}$, $T$ is the integral operator induced by $K$, i.e.

$$Tf(z) = \int_{\mathcal{X}} K(z, y)f(y)d\mu(y),$$

and $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there is a constant $C > 0$ and a positive measurable
function $h$ on $\mathcal{X}$ such that

$$\int_{\mathcal{X}} K(x, y)h(y)^r d\mu(y) \leq Ch(x)^q$$

for $\mu$–almost every $x \in \mathcal{X}$ and

$$\int_{\mathcal{X}} K(x, y)h(x)^r d\mu(x) \leq Ch(y)^p$$

for $\mu$–almost every $y \in \mathcal{X}$, then $T$ is bounded on $L^p(\mathcal{X}, d\mu)$ with norm less than or equal to $C$.

Let $f$ be a continuous function on $\mathcal{D}$ and $0 < r \leq 1$. Then the oscillation of $f$ in the Bergman metric disc is a function $O_r(f)$ defined on $\mathcal{D}$ by

$$O_r(f)(z) = \sup\{|f(z) - f(w)| : w \in D(z, r)\},$$

and

$$\|f\|_{BO_r} = \sup\{O_r(f)(z) : z \in \mathcal{D}\}.$$ 

Let $BO_r$ be the space of continuous functions on $\mathcal{D}$ with bounded oscillation, i.e.

$$BO_r = \{f : f \text{ is continuous on } \mathcal{D} \text{ and } \|f\|_{BO_r} < \infty\}.$$ 

$\|f\|_{BO_r}$ is a seminorm in $BO_r$.

**Proposition 2.2.5 [Zhu 90]** For any positive numbers $r$ and $s$, there is a constant $C > 0$
independent of \( r \) and \( s \) such that

\[
\frac{1}{C} \|f\|_{BO_r} \leq \|f\|_{BO_s} \leq C \|f\|_{BO_r}.
\]

for all continuous functions \( f \) on \( D \).

This proposition establishes that the space \( BO_r \) is independent of the choice of \( r \).

**Proposition 2.2.6 [Zhu 90]** If \( f \in BO_1 \), then

\[
|f(z) - f(w)| \leq \|f\|_{BO_1} \{\beta(z, w) + 1\}
\]

for all \( z, w \in D \).

**Proposition 2.2.7** If \( w \in D \) and \( t > 0 \), then \( (1 - |w|^2)^t \beta(0, w) \) is bounded.

**Proof.** Suppose \( w \in D \) and \( t > 0 \). Then

\[
(1 - |w|^2)^t \beta(0, w) = (1 - |w|^2)^t \left[ \frac{1}{2} \ln \frac{1 + \rho(0, w)}{1 - \rho(0, w)} \right]
\]

\[
= (1 - |w|^2)^t \left[ \frac{1}{2} \ln \frac{1 + |w|}{1 - |w|} \right]
\]

\[
= (1 - |w|^2)^t \frac{1}{2} \left[ \ln(1 + |w|) + \ln \frac{1}{1 - |w|} \right]
\]

\[
= \frac{1}{2} (1 - |w|^2)^t \ln(1 + |w|) + \frac{1}{2} (1 - |w|^2)^t \ln \frac{1}{1 - |w|}.
\]

The first summand is clearly bounded, and the second is bounded by virtue of the elementary fact \( \lim_{x \to 0} x^t \log x = 0 \). Therefore the proof is complete. \( \Box \)

Direct calculation gives us the following.
Proposition 2.2.8

1. For any $z \in D$, define $\Phi_z(w) = \frac{z-w}{1-z\overline{w}}$, $w \in D$. Then the real Jacobian determinant $J_{\Phi_z}(w)$ of $\Phi_z$ at $w$ is

$$J_{\Phi_z}(w) = \frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4}.$$

2. We also have

$$(1 - |\Phi_z(w)|^2)^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - z\overline{w}|^2},$$

and

$$(1 - |\Phi_z(w)|^2)^\alpha J_{\Phi_z}(w) = (1 - |w|^2)^\alpha |k_z^\alpha(w)|^2.$$
and

\[ k_\alpha(z, \Phi_z(w)) = \frac{(1 - \overline{w} z)^{2+\alpha}}{(1 - |z|^2)^{1+\frac{\alpha}{2}}} = \frac{1}{k_\alpha(z, w)}. \]

Proposition 2.2.10 Let \( g \) be in \( L^2(D, dA_\alpha) \), \( z \in D \). The operator \( T \) defined by

\[ Tg(z) = \int_D \beta(z, w)|K_\alpha(z, w)|g(w)dA_\alpha(w) \]

is bounded.

Proof. The operator \( T \) is an integral operator induced by \( \beta(z, w)|K_\alpha(z, w)| \). Apply Schur's theorem with \( p = q = 2 \) and \( h(z)^2 = (1 - |z|^2)^m \), where \( m \) will be specified later. In order to show that \( T \) is bounded, we need to show that, there exists a constant \( C > 0 \) such that

\[ \int_D \beta(z, w)|K_\alpha(z, w)|h(w)^2dA_\alpha(w) \leq Ch(z)^2. \]

Consider

\[ I = \int_D \beta(z, w)|K_\alpha(z, w)|h(w)^2dA_\alpha(w) \]

\[ = (\alpha + 1) \int_D \beta(z, w)|K_\alpha(z, w)|(1 - |w|^2)^{m+\alpha}dA(w). \]

Use the change of variable formula with \( w = \Phi_z(u) \) and use Propositions 2.2.8 and 2.2.9. This gives

\[ I = (\alpha + 1)h(z)^2 \int_D \frac{\beta(0, u)(1 - |u|^2)^\varepsilon(1 - |u|^2)^{m+\alpha}}{|1 - \overline{z}u|^{2+2m+\alpha}}dA(u) \]

for some fixed \( \varepsilon > 0 \). By Proposition 2.2.7, there exists a constant \( C_1 > 0 \) such that

\[ I \leq C_1 h(z)^2 \int_D \frac{(1 - |u|^2)^{m+\alpha-\varepsilon}}{|1 - \overline{z}u|^{2+2m+\alpha}}dA(u) \]
\[ C_1 h(z)^2 \int_D \frac{(1 - |u|^2)^{m+\alpha-\varepsilon}}{|1 - \overline{u} z^{2+(m+\alpha-\varepsilon)+(m+\varepsilon)}} dA(u). \]

Now we choose \( m \) so that \( m + \varepsilon < 0 \), and apply Proposition 2.2.3 to see that there exists a constant \( C > 0 \) such that \( I \leq C h(z)^2 \). This is what was to be proved. \( \Box \)

**Proposition 2.2.11** If \( g, z, \) and \( K^\alpha(z, w) \) are as in Proposition 2.2.10, then the operator \( T \) defined by

\[ T g(z) = \int_D |K^\alpha(z, w)| g(w) dA_\alpha(w) \]

is bounded.

**Proof.** The proof is similar to the proof of Proposition 2.2.10. \( \Box \)

We are now ready to show that we have a second condition on \( f \) which is sufficient to make \( H_{f,\alpha} \) bounded.

**Proposition 2.2.12** If \( f \in BO_1 \), then the Hankel operator \( H_{f,\alpha} \) defined on \( L^2_\alpha(D, dA_\alpha) \) is bounded. Moreover, there is a constant \( C > 0 \) such that

\[ ||H_{f,\alpha}|| \leq C ||f||_{BO_1}. \]

**Proof.** Let \( g \in L^2_\alpha(D, dA_\alpha) \) and \( z \in D \). Then

\[
(H_{f,\alpha} g)(z) = (1 - P^\alpha)(fg)(z)
\]

\[ = f(z)g(z) - P^\alpha(fg)(z) \]

\[ = f(z)(g, K^\alpha_z)_{L^2_\alpha(D, dA_\alpha)} - \langle P^\alpha(fg), K^\alpha_z \rangle_{L^2_\alpha(D, dA_\alpha)} \]

\[ = \int_D f(z)g(w)K^\alpha(z, w)dA_\alpha(w) - \int_D f(w)g(w)K^\alpha(z, w)dA_\alpha(w) \]

\[ = \int_D [f(z) - f(w)]g(w)K^\alpha(z, w)dA_\alpha(w). \]
Hence
\[ |(H_{f,\alpha}g)(z)| \leq \int_{D} |f(z) - f(w)||g(w)||K^{\alpha}(z, w)|dA_{\alpha}(w). \]

Since \( f \in BO_{1} \) Proposition 2.2.6 gives us
\[ |(H_{f,\alpha}g)(z)| \leq \|f\|_{BO_{1}} \int_{D} [\beta(z, w) + 1]|g(w)||K^{\alpha}(z, w)|dA_{\alpha}(w) \]
\[ = \|f\|_{BO_{1}} \left\{ \int_{D} \beta(z, w)|g(w)||K^{\alpha}(z, w)|dA_{\alpha}(w) + \int_{D} |g(w)||K^{\alpha}(z, w)|dA_{\alpha}(w) \right\}. \]

By Propositions 2.2.10 and 2.2.11
\[ \|(H_{f,\alpha}g)\|_{L^{2}(D, dA_{\alpha})} \leq \|f\|_{BO_{1}} \{ \|T_{1}\| + \|T_{2}\| \} \|g\|_{L^{2}(D, dA_{\alpha})}, \]

where \( T_{1} \) and \( T_{2} \) are the integral operators of Propositions 2.2.10 and 2.2.11. Therefore \( H_{f,\alpha} \) is bounded, and
\[ \|H_{f,\alpha}\| \leq C\|f\|_{BO_{1}} \text{ for some constant } C > 0. \]

We showed in the beginning of this section that if the symbol function \( f \) of a Hankel operator \( H_{f,\alpha} \) is in \( L^{\infty}(D, dA_{\alpha}) \), then \( H_{f,\alpha} \) is bounded. The next example shows that if \( f \) is in \( BO_{1} \), then \( f \) is not necessarily a bounded function.

**Example 2.2.13** Let \( f(z) = \beta(0, z) \). For \( w \in D \) such that \( \beta(z, w) \leq 1 \), consider
\[ |f(z) - f(w)| = |\beta(0, z) - \beta(0, w)| \]
\[ \leq \beta(0, z) + \beta(0, w) = \beta(z, w) \leq 1. \]
Therefore $f(z) = \beta(0, z)$ is in $BO_1$, but

$$f(z) = \frac{1}{2} \ln \frac{1 + |z|}{1 - |z|}$$

Thus $f(z)$ is not bounded in $D$.

For the third sufficient condition for a Hankel operator to be bounded, we need another definition.

1. Let $0 < r \leq 1$ and $f \in L^1(D, \, dA_\alpha)$. Then the average function $\hat{f}_{r, \alpha}$ is defined on $D$ by

$$\hat{f}_{r, \alpha}(z) = \frac{1}{A_\alpha(D(z, r))} \int_{D(z, r)} f(w) \, dA_\alpha(w).$$

2. Let

$$\|f\|_{BA_{r, \alpha}} = \sup \left\{ \left[ \|f\|^2_{r, \alpha}(z) \right]^{\frac{1}{2}} : z \in D \right\}.$$

Let $BA_{r, \alpha}$ be a space of functions $f$ with $\|f\|_{BA_{r, \alpha}} < \infty$. Then $\|f\|_{BA_{r, \alpha}}$ is a seminorm on $BA_{r, \alpha}$.

**Proposition 2.2.14 [CR 80]** There is a positive integer $N$ such that for any $r \leq 1$ there exists a sequence $\{\lambda_n\}$ in $D$ satisfying the following conditions

1. $D = \bigcup_{n=1}^{\infty} D(\lambda_n, r)$;

2. $D(\lambda_n, \frac{r}{4}) \cap D(\lambda_m, \frac{r}{4}) = \emptyset$ if $n \neq m$;

3. Any point in $D$ belongs to at most $N$ of the sets $D(\lambda_n, 2r)$.

**Proposition 2.2.15 [CR 80]** Let $r \leq 1$ and $\{D(\lambda_n, r)\}$ be a cover of $D$ satisfying the
conditions of the Proposition 2.2.14. For each $n$ we can find a measurable set $D_n$ with the following properties:

1. $D(\lambda_n, \frac{r}{n}) \subset D_n \subset D(\lambda_n, r)$, for all $n \geq 1$;
2. $D_n \cap D_m = \emptyset$ if $n \neq m$;
3. $\bigcup_{n=1}^{\infty} D_n = D$.

Proposition 2.2.16 [Zhu 90] There is a constant $C > 0$ such that

$$|f(z)|^p \leq \frac{C}{A(D(z,r))} \int_{D(z,r)} |f(w)|^p dA(w),$$

for all analytic function $f$, $z \in D$, $p > 0$ and $0 < r \leq 1$.

Proposition 2.2.17 Suppose $\alpha > -1$. Then

1. For any positive number $R$, there exists a constant $C > 0$ independent of $\alpha$ such that

$$\frac{1}{C} \leq \frac{A_\alpha(D(z,r))}{A_\alpha(D(w,r))} \leq C,$$

for all $\beta(z,w) \leq R$ and $0 < r \leq R$.

2. For $r > 0$, there exists a constant $C > 0$ depending on $r$ such that

$$A_\alpha(D(z,2r))^2 \leq CA_\alpha(D(z,r))A_\alpha(D(w,r)),$$

for all $z, w \in D$ such that $\beta(z,w) \leq r$.

Proof. (1) Suppose $0 < r \leq R$. From Proposition 2.2.2 (1) there are constants $C_1, C_2 > 0$
independent of $r$ and $\alpha$ such that

$$A_\alpha(D(z, r)) \leq C_1 A(D(z, r))^{1+\frac{\gamma}{2}}$$

and

$$A_\alpha(D(w, r)) \geq C_2 A(D(w, r))^{1+\frac{\gamma}{2}},$$

where $z, w \in D$. Therefore

$$\frac{A_\alpha(D(z, r))}{A_\alpha(D(w, r))} \leq \frac{C_1 A(D(z, r))^{1+\frac{\gamma}{2}}}{C_2 A(D(w, r))^{1+\frac{\gamma}{2}}}. $$

Since $\beta(z, w) \leq R$, and $0 < r \leq R$, Proposition 2.2.2 (4), implies that there exists a constant $C_3 > 0$ such that

$$\frac{A_\alpha(D(z, r))}{A_\alpha(D(w, r))} \leq \frac{C_1 C_3}{C_2} = C.$$ 

whenever $\beta(z, w) \leq R$.

Interchanging $w$ and $z$ we have

$$\frac{A_\alpha(D(z, r))}{A_\alpha(D(w, r))} \geq \frac{1}{C}.$$ 

Therefore (1) is done.

(2) Let $R > 0$, be such that $0 < 2r < R$. From (1) there exists a constant $C_1 > 0$ such that

$$A_\alpha(D(z, 2r))^2 \leq C_1 A_\alpha(D(w, 2r)) A_\alpha(D(z, 2r)).$$
for all \( \beta(z, w) \leq R \). Since for all \( z \in D \),

\[
0 < A_\alpha(D(z, r)) < A_\alpha(D(z, 2r)) < A_\alpha(D) = 1.
\]

Hence

\[
\frac{1}{A_\alpha(D(z, r))} A_\alpha(D(z, 2r)) < \frac{1}{A_\alpha(D(z, r))}.
\]

By the Archimedean property, there exists a natural number \( N \) depending on \( r \) such that

\[
A_\alpha(D(z, 2r)) < NA_\alpha(D(z, r)).
\]

Hence

\[
A_\alpha(D(z, 2r))^2 \leq C_1 N_1 N_2 A_\alpha(D(w, r)) A_\alpha(D(z, r)).
\]

Put \( C = C_1 N_1 N_2 > 0 \). Then \( C \) depends on \( r \) and the proof is complete. \( \square \)

**Proposition 2.2.18** If \( \alpha > -1 \), then there exists a constant \( C_\alpha > 0 \) such that

\[
|f(z)|^p \leq \frac{C_\alpha}{A_\alpha(D(z, r))} \int_{D(z, r)} |f(w)|^p dA_\alpha(w),
\]

for all analytic functions \( f \) on \( D \), \( z \in D \), \( 0 < r \leq 1 \) and \( p \neq 0 \).

**Proof.** Since \( f \) is analytic on \( D \) and \( 0 < r \leq 1 \), Proposition 2.2.16 gives us that there exists a constant \( C_1 > 0 \) independent of \( f \) and \( r \) such that for \( z \in D \), \( p > 0 \)

\[
|f(z)|^p \leq \frac{C_1}{A(D(z, r))} \int_{D(z, r)} |f(w)|^p dA(w).
\]

Thus

\[
|f(z)|^p A(D(z, r))^2 \leq \frac{C_1 A(D(z, r))^2}{A(D(z, r))} \int_{D(z, r)} |f(w)|^p dA(w).
\]
By Proposition 2.2.2 (2) there exists a constant $C_2 > 0$ such that

$$|f(z)|^p A(D(z, r))^2 \leq \frac{C_1 C_2}{A(D(z, r))} \int_{D(z, r)} |f(w)|^p (1 - |w|^2)^\alpha dA(w).$$

Hence

$$|f(z)|^p \leq \frac{C_1 C_2}{(\alpha + 1) A(D(z, r))^{1+\frac{\alpha}{2}}} \int_{D(z, r)} |f(w)|^p dA_\alpha(w).$$

By Proposition 2.2.2 (1) there exists a constant $C_3 > 0$ such that

$$|f(z)|^p \leq \frac{C_1 C_2 C_3}{(\alpha + 1) A_\alpha(D(z, r))} \int_{D(z, r)} |f(w)|^p dA_\alpha(w).$$

Put $C = \frac{C_1 C_2 C_3}{(\alpha + 1)^2} > 0$. Then $C$ depends on $\alpha$ and the proof is complete. \qed

**Proposition 2.2.19** Suppose $\alpha > -1$, $0 < r \leq 1$ and $f \in BA_{r, \alpha}$. Then there exists a constant $C > 0$ depending on $\alpha$ and $r$ such that

$$\int_D |g(w)|^2 |f(w)|^2 dA_\alpha(w) \leq C \int_D |g(w)|^2 dA_\alpha(w),$$

for all $g \in L^2_0(D, dA_\alpha)$.

**Proof.** Let $0 < r \leq 1$ and $f \in BA_{r, \alpha}$. Proposition 2.2.14 gives us

$$\int_D |g(w)|^2 |f(w)|^2 dA_\alpha(w) = \sum_{n=1}^{\infty} \int_{D(\lambda_n r)} |g(w)|^2 |f(w)|^2 dA_\alpha(w),$$

where $g \in L^2_0(D, dA_\alpha)$. Thus

$$\int_D |g(w)|^2 |f(w)|^2 dA_\alpha(w) \leq \sum_{n=1}^{\infty} \sup_{\lambda_n r} \{ |g(z)|^2 : z \in D(\lambda_n r) \} \int_{D(\lambda_n r)} |f(w)|^2 dA_\alpha(w).$$
By Propositions 2.2.17 (1) and 2.2.18 with $g$ analytic and $p = 2$, there exists $C_\alpha > 0$, such that

$$\int_{D} |g(w)|^2 |f(w)|^2 dA_\alpha(w) \leq \sum_{n=1}^{\infty} \frac{C_\alpha}{A_\alpha(D(\lambda_n, r))} \int_{D(\lambda_n, 2r)} |g(w)|^2 dA_\alpha(w) \int_{D(\lambda_n, r)} |f(w)|^2 dA_\alpha(w).$$

Use the fact that $f \in \mathcal{B}A_{r, \alpha}$, to see that there exists a constant $C_1 > 0$ such that

$$\int_{D} |g(w)|^2 |f(w)|^2 dA_\alpha(w) \leq \sum_{n=1}^{\infty} C_\alpha C_1 \int_{D(\lambda_n, 2r)} |g(w)|^2 dA_\alpha(w).$$

By Proposition 2.2.14 (3), there exists a positive integer $N$ such that

$$\int_{D} |g(w)|^2 |f(w)|^2 dA_\alpha(w) \leq C_\alpha C_1 N \int_{D} |g(w)|^2 dA_\alpha(w).$$

Put $C = C_\alpha C_1 N > 0$. Then $C$ depends on $\alpha$ and $r$ and the proof is complete.

Now we are ready to prove the third sufficient condition for Hankel operators to be bounded.

**Proposition 2.2.20** If $f$ is in $\mathcal{B}A_{r, \alpha}$ then the Hankel operator $H_{f, \alpha}$ is bounded.

**Proof.** Let $g \in L^2_\alpha(D, dA_\alpha)$. Then

$$\|H_{f, \alpha}g\|_{L^2(D, dA_\alpha)} = \|(1 - P^\alpha)fg\|_{L^2(D, dA_\alpha)} \leq \|fg\|_{L^2(D, dA_\alpha)} \leq \left\{ \int_{D} |f(w)|^2 |g(w)|^2 dA_\alpha(w) \right\}.$$

Since $f \in \mathcal{B}A_{r, \alpha}$ and $g \in L^2_\alpha(D, dA_\alpha)$, by Proposition 2.2.19, there is a constant $C > 0$.
depending on \( r \) and \( \alpha \), such that

\[
\|H_{f,\alpha}g\|_{L^2(D, dA_{\alpha})}^2 \leq C \int_D |g(w)|^2 dA_{\alpha}(w) = C\|g\|_{L^2(D, dA_{\alpha})}^2.
\]

Thus \( H_{f,\alpha} \) is bounded. \( \square \)

**Definition 2.2.21**

1. Let \( f \) be a function in \( L^\infty(D, dA_{\alpha}) \). Define an operator \( T_{f,\alpha} \) on \( L^2(D, dA_{\alpha}) \) by

\[
T_{f,\alpha}g = P^\alpha(fg), \quad g \in L^2(D, dA_{\alpha}).
\]

\( T_{f,\alpha} \) is called the Toeplitz operator on \( L^2(D, dA_{\alpha}) \) with symbol function \( f \).

2. Let \( f \in L^1(D, dA_{\alpha}) \). The Berezin transform \( \tilde{f}_\alpha \) of \( f \) is defined on \( D \) by

\[
\tilde{f}_\alpha(z) = \int_D f(w)|k^\alpha_z(w)|^2 dA_{\alpha}(w).
\]

3. Let \( \mu \) be a finite complex Borel measure on \( D \). The Toeplitz operator \( T_{\mu,\alpha} \) on \( L^2(D, dA_{\alpha}) \) is defined by

\[
T_{\mu,\alpha}f(z) = \int_D K^\alpha(z, w)f(w)d\mu(w),
\]

where \( z \in D \).

4. Let \( \mu \) be a finite complex Borel measure on \( D \). The Berezin transform \( \tilde{\mu}_\alpha \) of a Toeplitz operator \( T_{\mu,\alpha} \) on \( L^2(D, dA_{\alpha}) \) is defined by

\[
\tilde{\mu}_\alpha(z) = \langle T_{\mu,\alpha}k^\alpha_z, k^\alpha_z \rangle_{L^2(D, dA_{\alpha})} = \int_D |k^\alpha_z(w)|^2 d\mu(w),
\]

\( \tilde{\mu}_\alpha \) is the Berezin transform of \( T_{\mu,\alpha} \).
where \( z \in D \).

**Proposition 2.2.22 [Zhu 90]** Suppose \( \alpha > -1 \), \( r > 0 \) and \( \varphi \) is a nonnegative function on \( D \). Then the following are equivalent:

1. \( T_{\varphi,\alpha} \) is bounded on \( L^2_\alpha(D, dA_\alpha) \);

2. \( \bar{\varphi}_\alpha(z) \) is bounded on \( D \);

3. \( \varphi_{r,\alpha}(z) \) is bounded on \( D \).

**Corollary 2.2.23** If \( f \) is in \( BA_{r,\alpha} \) then the Berezin transform of the function \( |f|^2 \) is bounded.

**Proof.** Since \( f \in BA_{r,\alpha} \), \( |f|^2 \) is bounded on \( D \). Put \( \varphi = |f|^2 \). Then \( \hat{\varphi}_{r,\alpha} = |f|^2_{r,\alpha} \) is bounded. By Proposition 2.2.22, \( \bar{\varphi}_\alpha \) is bounded and hence \( \bar{\varphi}_\alpha = |f|^2_\alpha \) is bounded on \( D \).

By Corollary 2.2.23, we can say that \( |f|^2_{r,\alpha} \) and \( |f|^2_\alpha \) are comparable, i.e. there exists a constant \( C > 0 \) such that

\[
\frac{1}{C} |f|^2_{r,\alpha} \leq |f|^2_\alpha \leq C |f|^2_{r,\alpha},
\]

for any \( r > 0 \). The above argument establishes that the space \( BA_{r,\alpha} \) is independent of \( r \).

We now define another function space that is independent of \( r \). Let

\[
\|f\|_{BA_\alpha} = \sup \{ |f|^2_\alpha(z) : z \in D \}
\]

and let

\[
BA_\alpha = \{ f : \|f\|_{BA_\alpha} < \infty \}.
\]
By Proposition 2.2.20 and Corollary 2.2.23 we can say that if $f$ is in $B A_{\alpha}$, then the Hankel operator $H_{f,\alpha}$ is bounded.

We have seen two conditions implying that the Hankel operator $H_{f,\alpha}$ is bounded. The first one is that the symbol function $f$ has to be in $B O_{r}$, for $0 < r \leq 1$ and the second is that the symbol function $f$ is in $B A_{\alpha}$. Now we are going to give a necessary and sufficient condition for the boundedness of the Hankel operator $H_{f,\alpha}$. We begin with a definition.

1. Let $f \in L^{2}(D, dA_{\alpha})$ and $r > 0$. Then the mean oscillation $MO_{r,\alpha}(f)$ of $f$ at $z$ in the Bergman metric $D(z, r)$ is defined by

$$MO_{r,\alpha}(f)(z) = \left[ \frac{1}{A_{\alpha}(D(z, r))} \int_{D(z, r)} |f(w) - \hat{f}_{r,\alpha}(z)|^2 dA_{\alpha}(w) \right]^{\frac{1}{2}}.$$

Let $BMO_{r,\alpha}$ be the space of functions on $D$ with

$$\|f\|_{BMO_{r,\alpha}} = \sup\{ MO_{r,\alpha}(f)(z) : z \in D \} < \infty.$$

2. Let $f \in L^{2}(D, dA_{\alpha})$. Then $MO_{\alpha}(f)$ is defined on $D$ by

$$MO_{\alpha}(f)(z) = \left[ |\hat{f}_{\alpha}(z)|^2 - |\tilde{f}_{\alpha}(z)|^2 \right]^{\frac{1}{2}}, \quad z \in D,$$

and $\|f\|_{BMO_{\alpha}} = \sup\{ MO_{\alpha}(f)(z) : z \in D \}$. Let $BMO_{\theta,\alpha}$ be the space of functions $f$ with $\|f\|_{BMO_{\theta,\alpha}} < \infty$.

Proposition 2.2.24 Let $f \in L^{2}(D, dA_{\alpha}), z \in D, r > 0$. Then

1. $MO_{r,\alpha}(f)(z) = \left[ |\hat{f}_{r,\alpha}(z)|^2 - |\tilde{f}_{r,\alpha}(z)|^2 \right]^{\frac{1}{2}}$;

2. $MO_{r,\alpha}(f)(z) = \left[ \frac{1}{2A_{\alpha}(D(z, r))} \int_{D(z, r)} \int_{D(z, r)} |f(u) - f(v)|^2 dA_{\alpha}(u) dA_{\alpha}(v) \right]^{\frac{1}{2}}$;
3. There exists a constant $C > 0$, independent of $z$, $r$ and $f$ such that

$$M_{O r, \alpha}(f)(z) \leq C M_{O 2r, \alpha}(f)(z).$$

**Proof.** (1) From the definition of $M_{O r, \alpha}(f)(z)$ we have

$$[M_{O r, \alpha}(f)(z)]^2 = \frac{1}{A_\alpha(D(z, r))} \int_{D(z, r)} |f(w) - \tilde{f}_{r, \alpha}(z)|^2 dA_\alpha(w).$$

Expand the integrand and integrate each term separately to obtain

$$[M_{O r, \alpha}(f)(z)]^2 = |\tilde{f}_{r, \alpha}(z)|^2 - |\tilde{f}_{r, \alpha}(z)|^2.$$

(2) Consider

$$I = \frac{1}{2A_\alpha(D(z, r))^2} \int_{D(z, r)} \int_{D(z, r)} |f(u) - f(v)|^2 dA_\alpha(u) dA_\alpha(v).$$

Expand the term inside the integrals and integrate each term separately. We have

$$I = \frac{1}{2} \left[ |\tilde{f}_{r, \alpha}(z)| - \tilde{f}_{r, \alpha}(z) \tilde{f}_{r, \alpha}(z) - \tilde{f}_{r, \alpha}(z) \tilde{f}_{r, \alpha}(z) + |\tilde{f}_{r, \alpha}(z)|^2 \right]$$

$$= |\tilde{f}_{r, \alpha}(z)|^2 - |\tilde{f}_{r, \alpha}(z)|^2.$$

By the first part, (2) is proved.

(3) By using (2) and the fact that $D(z, r) \subset D(z, 2r)$ we have

$$M_{O r, \alpha}(f)(z) \leq \left[ \frac{1}{2A_\alpha(D(z, r))^2} \int_{D(z, 2r)} \int_{D(z, 2r)} |f(u) - f(v)|^2 dA_\alpha(u) dA_\alpha(v) \right]^\frac{1}{2}.$$
From Proposition 2.2.17 (2), there exists a constant \( C > 0 \) such that

\[
A_\alpha(D(z, r))^2 \geq \frac{1}{C} A_\alpha(D(z, 2r))^2.
\]

Therefore

\[
\frac{1}{2A_\alpha(D(z, r))^2} \leq \frac{C}{2A_\alpha(D(z, 2r))^2}.
\]

Thus

\[
MO_{r, \alpha}(f)(z) \leq \left[ \frac{C}{2A_\alpha(D(z, 2r))^2} \int_{D(z, 2r)} \int_{D(z, 2r)} |f(u) - f(v)|^2 dA_\alpha(u) dA_\alpha(v) \right]^{\frac{1}{2}} = CMO_{2r, \alpha}(f)(z).
\]

The last equality follows from (2).

\[
\square
\]

**Proposition 2.2.25** Suppose \( f \in L^2(D, dA_\alpha) \) and \( z \in D \). Then

\[
MO_\alpha(f)(z)^2 = \frac{1}{2} \int_D \int_D |f(u) - f(v)|^2 [k_\alpha^2(u)]^2 [k_\alpha^2(v)]^2 dA_\alpha(u) dA_\alpha(v).
\]

**Proof.** The proof is similar to that of Proposition 2.2.24 (2).

\[
\square
\]

**Proposition 2.2.26** Suppose \( r > 0 \) and \( f \) is in \( BMO_{2r, \alpha} \). Then

1. The function \( \tilde{f}_{r, \alpha} \) is in \( BO_r \) and there exists a constant \( C > 0 \) independent of \( r \) and \( f \) such that

\[
\|\tilde{f}_{r, \alpha}\|_{BO_r} \leq C\|f\|_{BMO_{2r, \alpha}}.
\]

2. The function \( f - \tilde{f}_{r, \alpha} \) is in \( BA_{r, \alpha} \), and there exists a constant \( C > 0 \) independent of \( r \) and \( f \) such that

\[
\|f - \tilde{f}_{r, \alpha}\|_{BA_{r, \alpha}} \leq C\|f\|_{BMO_{2r, \alpha}}.
\]
Proof. (1) Let $z, w \in D$, with $\beta(z, w) \leq r$. Consider

$$
\tilde{f}_{r, \alpha}(z) - \tilde{f}_{r, \alpha}(w) = \frac{1}{A_{\alpha}(D(z, r))} \int_{D(z, r)} f(u) dA_{\alpha}(u) - \frac{1}{A_{\alpha}(D(w, r))} \int_{D(w, r)} f(v) dA_{\alpha}(v)
$$

$$
= \frac{1}{A_{\alpha}(D(z, r)) A_{\alpha}(D(w, r))} \int_{D(w, r)} \int_{D(z, r)} f(u) dA_{\alpha}(u) dA_{\alpha}(v) - \frac{1}{A_{\alpha}(D(z, r)) A_{\alpha}(D(w, r))} \int_{D(w, r)} \int_{D(z, r)} f(v) dA_{\alpha}(u) dA_{\alpha}(v).
$$

Using Fubini's theorem and applying Hölder's inequality twice, we have

$$
|\tilde{f}_{r, \alpha}(z) - \tilde{f}_{r, \alpha}(w)| \leq \frac{1}{A_{\alpha}(D(z, r))^{\frac{1}{2}} A_{\alpha}(D(w, r))^{\frac{1}{2}}} \left( \int_{D(w, r)} \int_{D(z, r)} |f(u) - f(v)|^2 dA_{\alpha}(u) dA_{\alpha}(v) \right)^{\frac{1}{2}}.
$$

Hence

$$
|\tilde{f}_{r, \alpha}(z) - \tilde{f}_{r, \alpha}(w)|^2 \leq \frac{1}{A_{\alpha}(D(z, r)) A_{\alpha}(D(w, r))} \int_{D(w, r)} \int_{D(z, r)} |f(u) - f(v)|^2 dA_{\alpha}(u) dA_{\alpha}(v).
$$

Since $\beta(z, w) \leq r$, $D(w, r) \subseteq D(z, 2r)$. By Proposition 2.2.17 (2), there exists a constant $C > 0$ such that

$$
|\tilde{f}_{r, \alpha}(z) - \tilde{f}_{r, \alpha}(w)|^2 \leq \frac{C}{A_{\alpha}(D(z, 2r))^2} \int_{D(z, 2r)} \int_{D(z, 2r)} |f(u) - f(v)|^2 dA_{\alpha}(u) dA_{\alpha}(v).
$$

By Proposition 2.2.24 (2), we have

$$
|\tilde{f}_{r, \alpha}(z) - \tilde{f}_{r, \alpha}(w)| \leq C \MO_{2r, \alpha}(f)(z) < \infty.
$$

Therefore $\tilde{f}_{r, \alpha} \in BO_r$ and $\|f\|_{BO_r} \leq C\|f\|_{\MO_{2r, \alpha}}$. 

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(2) Put \( g(z) = f(z) - \hat{f}_{r,\alpha}(z), \) for \( z \in \mathbf{D}. \) Consider

\[
\left[ |g|^{2} r_{\alpha}(z) \right]^{\frac{1}{2}} = \left[ \frac{1}{A_{\alpha}(D(z, r))} \int_{D(z, r)} |f(w) - \hat{f}_{r,\alpha}(w)|^{2} dA_{\alpha}(w) \right]^{\frac{1}{2}}
\]

\[
= \left[ \frac{1}{A_{\alpha}(D(z, r))} \int_{D(z, r)} |f(w) - \hat{f}_{r,\alpha}(z) + \hat{f}_{r,\alpha}(z) - \hat{f}_{r,\alpha}(w)|^{2} dA_{\alpha}(w) \right]^{\frac{1}{2}}.
\]

Apply Minkowski’s inequality to get

\[
\left[ |g|^{2} r_{\alpha}(z) \right]^{\frac{1}{2}} \leq \frac{1}{A_{\alpha}(D(z, r))} \left\{ \int_{D(z, r)} |f(w) - \hat{f}_{r,\alpha}(z)|^{2} dA_{\alpha}(w) \right\}^{\frac{1}{2}}
\]

\[
+ \frac{1}{A_{\alpha}(D(z, r))} \left\{ \int_{D(z, r)} |\hat{f}_{r,\alpha}(z) - \hat{f}_{r,\alpha}(w)|^{2} dA_{\alpha}(w) \right\}^{\frac{1}{2}}.
\]

By the definition of \( MO_{r,\alpha}(f) \) and the result from (1) there exists a constant \( C > 0 \) such that

\[
\left[ |g|^{2} r_{\alpha}(z) \right]^{\frac{1}{2}} \leq MO_{r,\alpha}(f) + CMO_{2r,\alpha}(f)(z).
\]

Since \( MO_{r,\alpha}(f) \) is comparable to \( MO_{2r,\alpha}(f) \), there exists a constant \( C_1 > 0 \) such that

\[
\|f - \hat{f}_{r,\alpha}\|_{BA_{\alpha}} \leq (C_1 + C)MO_{2r,\alpha}(f)(z) < \infty.
\]

The last inequality follows from the fact that \( f \) is in \( BMO_{2r,\alpha}. \) Therefore the proof is complete. \( \square \)

**Proposition 2.2.27**

1. \( \int_{\mathbf{D}} \int_{\mathbf{D}}[3(u, v) + 1]^{2} dA_{\alpha}(u)dA_{\alpha}(v) \leq \infty; \)

2. \( \inf \{ |k_{\alpha}(w)|^{2} : w \in D(z, r) \} = \left[ \frac{(1-|z|\beta^{2})}{(1-|z|\beta^{2})} \right]^{2+\alpha}; \)

3. \( \sup \{ |k_{\alpha}(w)|^{2} : w \in D(z, r) \} = \left[ \frac{(1+|z|\beta^{2})}{(1-|z|\beta^{2})} \right]^{2+\alpha}, \)
where \(0 < r \leq 1\), and \(s = \frac{1 - r}{1 + r} = \tanh r \in (0, 1)\).

**Proof.** Let 
\[
I = \int_D \int_D [\beta(u, v) + 1]^2 dA_\alpha(u) dA_\alpha(v).
\]
Then
\[
I = (\alpha + 1)^2 \int_D \int_D [\beta(u, v) + 1]^2 (1 - |u|^2)^\alpha (1 - |v|^2)^\alpha dA(u) dA(v)
\leq (\alpha + 1)^2 \sup \{ (1 - |u|^2)^\alpha : u \in D \}^2 \int_D \int_D [\beta(u, v) + 1]^2 dA(u) dA(v)
= (\alpha + 1)^2 \int_D \int_D [\beta(u, v) + 1]^2 dA(u) dA(v).
\]
The double integral converges by Proposition 2.2.2 (5) and therefore (1) is proved.

(2) Let \(0 < r \leq 1\), and \(\tanh r \in (0, 1)\). Then
\[
\inf \{ |k_\alpha^\circ(w)|^2 : w \in D(z, r) \} = \inf \left\{ |k_\alpha^\circ(\Phi_z(w))|^2 : w \in D(0, r) \right\}
= \inf \left\{ \left| 1 - \frac{2w}{(1 - |z|^2)^{1+\alpha}} + 2 \right|^2 : w \in D(0, r) \right\}
= \left[ \frac{(1 - |z|^2)^{1+\alpha}}{(1 - |w|^2)^{1+\alpha}} \right].
\]
The last equality follows from the fact that \(D(0, r)\) is a Euclidean disc with Euclidean center 0 and radius \(s\) [Garnett 81].

(3) The proof is similar to (2). \(\square\)

**Proposition 2.2.28** Suppose \(r > 0\) and \(f \in L^2(D, dA_\alpha)\). Then the following are equivalent:

1. \(f \in BMO_{2r, \alpha}\);

2. \(f = f_1 + f_2\) with \(f_1 \in BO_1\) and \(f_2 \in BA_\alpha\);

3. \(f \in BMO_{0, \alpha}\).
Proof. (1) ⇒ (2)

Suppose \( f \in BMO_{2r,\alpha} \). Put \( f = (f - \hat{f}_{r,\alpha}) + \hat{f}_{r,\alpha} \). By Proposition 2.2.26, \( f - \hat{f}_{r,\alpha} \in BA_{r,\alpha} \) and \( \hat{f}_{r,\alpha} \in BO_r \). Let \( f_1 = \hat{f}_{r,\alpha} \) and \( f_2 = f - \hat{f}_{r,\alpha} \). Then (1) ⇒ (2) is done.

(2) ⇒ (3)

Suppose \( f = f_1 + f_2 \), where \( f_1 \in BO_1 \) and \( f_2 \in BA_\alpha \). Then for \( z \in D \), Proposition 2.2.25 gives us

\[
MO_\alpha(f_1)(z)^2 = \frac{1}{2} \int_D \int_D |f_1(u) - f_1(v)|^2 |k_1^\alpha(u)|^2 |k_1^\alpha(v)|^2 dA_\alpha(u)dA_\alpha(v).
\]

Since \( f_1 \in BO_1 \), by applying Proposition 2.2.6 we have

\[
MO_\alpha(f_1)(z)^2 \leq \frac{1}{2} \|f_1\|_{BO_1}^2 \int_D \int_D [\beta(u, v) + 1]^2 |k_1^\alpha(u)|^2 |k_1^\alpha(v)|^2 dA_\alpha(u)dA_\alpha(v).
\]

By the change of variables \( u = \Phi_2(s) \) and \( v = \Phi_2(t) \), we have

\[
MO_\alpha(f_1)(z)^2 \leq \frac{1}{2} \|f_1\|_{BO_1}^2 \int_D \int_D [\beta(s, t) + 1]^2 dA_\alpha(s)dA_\alpha(t).
\]

Apply Proposition 2.2.27 (1) to see that there exists a constant \( C > 0 \) such that

\[
MO_\alpha(f_1)(z) \leq C\|f_1\|_{BO_1}.
\]

Thus

\[
\|f_1\|_{BMO_{\theta,\alpha}} \leq C\|f_1\|_{BO_1} < \infty.
\]

For \( f_2 \in BA_\alpha \), since

\[
\|f_2\|_{BA_\alpha} = \sup \left\{ \left[ \int_D [f_2(z)]^\frac{1}{\alpha} \right]^{\frac{\alpha}{2}} : z \in D \right\}.
\]
and

\[ MO_\alpha(f_2)(z) = \left[ |\overline{f_2}_\alpha^2(z)| - |\overline{f_\alpha}(z)|^2 \right]^{\frac{1}{2}} \leq \left[ |f_\alpha^2(z)| \right]^{\frac{1}{2}}. \]

Hence

\[ \sup \{ MO_\alpha(f_2)(z) : z \in D \} \leq \sup \left\{ \left[ |\overline{f_2}_\alpha^2(z)| \right]^{\frac{1}{2}} : z \in D \right\}. \]

Therefore

\[ \|f_2\|_{BMO_\alpha} \leq \|f_\alpha\|_{BA_\alpha} < \infty. \]

It follows that

\[ \|f\|_{BMO_\alpha} = \|f_1 + f_2\|_{BMO_\alpha} \leq \|f_1\|_{BMO_\alpha} + \|f_2\|_{BMO_\alpha} < \infty. \]

Hence we have (2) \Rightarrow (3).

(3) \Rightarrow (1)

Suppose \( f \in BMO_\alpha \). Then by Proposition 2.2.25,

\[
[M_\alpha(f)(z)]^2 = \frac{1}{2} \int_D \int_D |f(u) - f(v)|^2 |k_\alpha(u)|^2 |k_\alpha(v)|^2 dA_\alpha(u) dA_\alpha(v) \\
\geq \frac{1}{2} \left[ \int_{D(z,2r)} \int_{D(z,2r)} |f(u) - f(v)|^2 |k_\alpha(u)|^2 |k_\alpha(v)|^2 dA_\alpha(u) dA_\alpha(v) \right] \\
\geq \frac{1}{2} \left[ \inf \left\{ |k_\alpha(w)|^2 : w \in D(z,2r) \right\} \right]^2 \\
\int_{D(z,2r)} \int_{D(z,2r)} |f(u) - f(v)|^2 dA_\alpha(u) dA_\alpha(v). 
\]

By Proposition 2.2.27 (2), we have

\[
[M_\alpha(f)(z)]^2 \geq \frac{1}{2} \left[ \frac{(1 - |z|s)^2}{(1 - |z|^2)} \right]^{4+2\alpha} \int_{D(z,2r)} \int_{D(z,2r)} |f(u) - f(v)|^2 dA_\alpha(u) dA_\alpha(v). 
\]

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By Proposition 2.2.2 (1) and (3), there exists a constant $C_1 > 0$ such that

$$C_1 \left[ \frac{(1-|z|^2)s}{(1-|z|^2)^2} \right]^{2+\alpha} \leq A_\alpha(D(z,r))$$

and hence

$$C_1 \left[ \frac{s}{(1 + |z|s)} \right]^{2+\alpha} \left[ \frac{1-|z|^2}{1-|z|s} \right]^{2+\alpha} \leq A_\alpha(D(z,r)).$$

It follows that there exists a constant $C_2 > 0$ such that

$$C_2 \left[ \frac{1-|z|^2}{1-|z|s} \right]^{2+\alpha} \leq A_\alpha(D(z,r)).$$

Thus

$$\left[ \frac{1-|z|s}{1-|z|^2} \right]^{4+2\alpha} \geq \frac{C_2}{A_\alpha(D(z,r))^2}.$$  

So, we have

$$[MO_\alpha(f)(z)]^2 \geq \frac{C_2(1-|z|s)^{4+2\alpha}}{2A_\alpha(D(z,r))^2} \int_{D(z,2\epsilon)} \int_{D(z,2r)} |f(u) - f(v)|^2 dA_\alpha(u)dA_\alpha(v)$$

$$\geq \frac{C}{2A_\alpha(D(z,2r))^2} \int_{D(z,2\epsilon)} \int_{D(z,2r)} |f(u) - f(v)|^2 dA_\alpha(u)dA_\alpha(v),$$

where $C = C_2(1-s)^{4+2\alpha} > 0$. Proposition 2.2.24 (2) gives us

$$[MO_\alpha(f)(z)]^2 \geq C[MO_{2\epsilon,\alpha}(f)]^2.$$  

Thus $(3) \Rightarrow (1)$ is proved. \qed
Proposition 2.2.29 Suppose $f \in L^2(D, dA_\alpha)$ and $z \in D$. Then

$$MO_\alpha(f)(z) = \left[ |f \Phi_\alpha(z)|^2 - |\tilde{f}_\alpha(z)|^2 \right]^{\frac{1}{2}} = \|f \circ \Phi_z - \tilde{f}_\alpha(z)\|_{L^2(D, dA_\alpha)}.$$ 

Proof. Consider

$$\|f \circ \Phi_z - \tilde{f}_\alpha(z)\|^2_{L^2(D, dA_\alpha)} = \int_D [f(\Phi_z(w)) - \tilde{f}_\alpha(z)]^2 dA_\alpha(w).$$

Expand the integrand and integrate each of the four terms separately to obtain

$$\|f \circ \Phi_z - \tilde{f}_\alpha(z)\|^2_{L^2(D, dA_\alpha)} = \int_D [f(\Phi_z(w))]^2 dA_\alpha(w) - \tilde{f}_\alpha(z) \int_D f(\Phi_z(w)) dA_\alpha(w)$$

$$- \tilde{f}_\alpha(z) \int_D f(\Phi_z(w)) dA_\alpha(w) + |\tilde{f}_\alpha(z)|^2.$$

Use the change of variable formula with $w = \Phi_z(u)$ to obtain

$$\|f \circ \Phi_z - \tilde{f}_\alpha(z)\|^2_{L^2(D, dA_\alpha)} = \int_D [f(u)]^2 |k^\alpha_z(u)|^2 dA_\alpha(u) - \tilde{f}_\alpha(z) \int_D f(u) |k^\alpha_z(u)|^2 dA_\alpha(u)$$

$$- \tilde{f}_\alpha(z) \int_D f(u) |k^\alpha_z(u)|^2 dA_\alpha(u) + |\tilde{f}_\alpha(z)|^2$$

$$= |f|^2_{L^2}(z) - \tilde{f}_\alpha(z) \tilde{f}_\alpha(z) - \tilde{f}_\alpha(z) \tilde{f}_\alpha(z) + |\tilde{f}_\alpha(z)|^2$$

$$= |f|^2_{L^2}(z) - |\tilde{f}_\alpha(z)|^2.$$

□

Proposition 2.2.30 For any $z \in D$, let $U_z$ be the operator defined on $L^2(D, dA_\alpha)$ by

$$U_z f = (f \circ \Phi_z) k^\alpha_z.$$ Then

1. $U_z = U^*_z$;

2. $U_z U^*_z = 1$;
3. \( U_z L_a^2(D, dA_\alpha) \subseteq L_a^2(D, dA_\alpha) \);

4. \( U_z L_a^2(D, dA_\alpha)^ \perp \subseteq L_a^2(D, dA_\alpha)^ \perp \);

5. \( U_z(f \circ \Phi_z) = f k_z^\alpha \).

**Proof.** (1) Let \( f, g \in L^2(D, dA_\alpha) \). Then

\[
\langle U_z^* f, g \rangle_{L^2(D, dA_\alpha)} = \langle f, U_z g \rangle_{L^2(D, dA_\alpha)} = \langle f, (g \circ \Phi_z) k_z^\alpha \rangle_{L^2(D, dA_\alpha)} = \int_D f(w) \bar{g}(\Phi_z(w)) k_z^\alpha(w) dA_\alpha(w).
\]

Use the change of variable formula with \( w = \Phi_z(u) \) to obtain

\[
\langle U_z^* f, g \rangle_{L^2(D, dA_\alpha)} = \int_D (f \circ \Phi_z)(u) \bar{g}(u) k_z^\alpha(u) dA_\alpha(u) = \langle (f \circ \Phi_z) k_z^\alpha, g \rangle_{L^2(D, dA_\alpha)}.
\]

Therefore (1) is done.

(2) Let \( f, g \in L^2(D, dA_\alpha) \). Then

\[
\langle U_z U_z^* f, g \rangle_{L^2(D, dA_\alpha)} = \langle U_z^* f, U_z^* g \rangle_{L^2(D, dA_\alpha)} = \langle (f \circ \Phi_z) k_z^\alpha, (g \circ \Phi_z) k_z^\alpha \rangle_{L^2(D, dA_\alpha)} = \int_D (f \circ \Phi_z)(w) k_z^\alpha(w) (g \circ \Phi_z)(w) \bar{k_z^\alpha(w)} dA_\alpha(w).
\]

Use the change of variable formula with \( w = \Phi_z(u) \) to obtain

\[
\langle U_z U_z^* f, g \rangle_{L^2(D, dA_\alpha)} = \int_D f(u) \bar{g}(u) dA_\alpha(u).
\]
which proves (2).

(3) Part (3) follows directly from the definition of \( U_z \) and the boundedness of \( k_2^\alpha \).

(4) By part (1), \( U_z \) is self-adjoint. Every invariant subspace of a self-adjoint operator is reducing. The assertion follows from part (3).

(5) If \( f \in L^2(D, dA_\alpha) \), then \( f \circ \Phi_z \in L^2(D, dA_\alpha) \), and

\[
U_z(f \circ \Phi_z) = (f \circ \Phi_z) \circ \Phi_z k_2^\alpha = f k_2^\alpha.
\]

\[\square\]

**Proposition 2.2.31** Suppose \( f \in L^2(D, dA_\alpha) \) and \( z \in D \). Then

1. \( \tilde{f}_\alpha(z) = P^\alpha(f \circ \Phi_z)(0) \);
2. \( P^\alpha(P^\alpha(f \circ \Phi_z)) = P^\alpha(f \circ \Phi_z)(0) \);
3. \( \|H_{f',\alpha} k_2^\alpha\|_{L^2(D, dA_\alpha)} = \|(1 - P^\alpha)(f \circ \Phi_z)\|_{L^2(D, dA_\alpha)} \);
4. \( \|H_{f',\alpha} k_2^\alpha\|_{L^2(D, dA_\alpha)} = \|f \circ \Phi_z - P^\alpha(f \circ \Phi_z)\|_{L^2(D, dA_\alpha)} \).

**Proof.** Suppose \( f \in L^2(D, dA_\alpha) \) and \( z \in D \). Then

\[
\tilde{f}_\alpha(z) = \int_D f(w)|k_2^\alpha(w)|^2 dA_\alpha(w).
\]

Use the change of variable formula with \( w = \Phi_z(u) \), and use \( K^\alpha(0, u) = 1 \), to obtain

\[
\tilde{f}_\alpha(z) = \int_D f(\Phi_z(u))K^\alpha(0, u)dA_\alpha(u)
= \langle f \circ \Phi_z, K_0^\alpha \rangle_{L^2(D, dA_\alpha)}
\]
\[ = \langle f \circ \Phi_z, P^\alpha K_0^\alpha \rangle_{L^2(D, dA_a)} \]
\[ = \langle P^\alpha(f \circ \Phi_z), K_0^\alpha \rangle_{L^2(D, dA_a)} \]
\[ = P^\alpha(f \circ \Phi_z)(0). \]

(2) Let \( g = P^\alpha(f \circ \Phi_z) \). Then \( g \in L^2(D, dA_a) \). Since 0 is orthogonal to \( L^2(D, dA_a) \) and \( P^\alpha 1 = 1 \), \( \overline{P^\alpha g} = g(0) \). In addition, \( P^\alpha 1 = 1 \) implies

\[ \overline{g(0)} = \langle 1, \overline{f \circ \Phi_z} \rangle_{L^2(D, dA_a)} \]
\[ = \langle f \circ \Phi_z, 1 \rangle_{L^2(D, dA_a)} \]
\[ = P^\alpha(f \circ \Phi_z)(0). \]

It follows that

\[ P^\alpha P^\alpha(f \circ \Phi_z) = P^\alpha g = P^\alpha(f \circ \Phi_z)(0). \]

(3) By definition,

\[ \| H_{f, \alpha} k_2^\alpha \|_{L^2(D, dA_a)} = \|(1 - P^\alpha)f k_2^\alpha \|_{L^2(D, dA_a)}. \]

By Proposition 2.2.30 (5), we have

\[ \| H_{f, \alpha} k_2^\alpha \|_{L^2(D, dA_a)} = \|(1 - P^\alpha)U_z(f \circ \Phi_z)\|. \]

By Proposition 2.2.30 (3) and (4), \( U_z \) commutes with \( 1 - P^\alpha \), and \( \|U_z\| = 1 \). Then

\[ \| H_{f, \alpha} k_2^\alpha \|_{L^2(D, dA_a)} = \|U_z(1 - P^\alpha)(f \circ \Phi_z)\|_{L^2(D, dA_a)} \]
\[ = \|(1 - P^\alpha)(f \circ \Phi_z)\|_{L^2(D, dA_a)}. \]
(4) The proof of (4) is similar to that of (3).

\begin{theorem}
A function \( f \) is in \( BMO_{\partial, \alpha} \) if and only if both \( H_{f, \alpha} \) and \( H_{f, \alpha} \) are bounded.

Moreover, there is a constant \( C > 0 \) such that

\[
\frac{1}{C} \| f \|_{BMO_{\partial, \alpha}} \leq \| H_{f, \alpha} \| + \| H_{f, \alpha} \| \leq C \| f \|_{BMO_{\partial, \alpha}}
\]

for all \( f \in BMO_{\partial, \alpha} \).

\end{theorem}

\begin{proof}
Suppose \( H_{f, \alpha} \) and \( H_{f, \alpha} \) are bounded. By Propositions 2.2.29 and 2.2.31, we have

\[
MO_\alpha(f)(z) = \| f \circ \Phi_z - \bar{f}_\alpha(z) \|_{L^2(\mathbb{D}, dA_\alpha)}
\]

\[
= \| f \circ \Phi_z - P^\alpha(\bar{f}_\alpha)(0) \|_{L^2(\mathbb{D}, dA_\alpha)}
\]

\[
= \| f \circ \Phi_z - P^\alpha(f \circ \Phi_z) + P^\alpha(f \circ \Phi_z) - P^\alpha(\bar{f}_\alpha)(0) \|_{L^2(\mathbb{D}, dA_\alpha)}
\]

\[
\leq \| f \circ \Phi_z - P^\alpha(f \circ \Phi_z) \|_{L^2(\mathbb{D}, dA_\alpha)} + \| P^\alpha(f \circ \Phi_z) - P^\alpha(\bar{f}_\alpha)(0) \|_{L^2(\mathbb{D}, dA_\alpha)}
\]

It follows from Proposition 2.2.31 (2) that

\[
MO_\alpha(f)(z) \leq \|(1 - P^\alpha)(f \circ \Phi_z)\|_{L^2(\mathbb{D}, dA_\alpha)} + \|P^\alpha(f \circ \Phi_z) - P^\alpha(\overline{P^\alpha(\bar{f}_\alpha)})\|_{L^2(\mathbb{D}, dA_\alpha)}
\]

\[
= \|(1 - P^\alpha)(f \circ \Phi_z)\|_{L^2(\mathbb{D}, dA_\alpha)} + \|P^\alpha[(f \circ \Phi_z) - (\overline{P^\alpha(\bar{f}_\alpha)})]\|_{L^2(\mathbb{D}, dA_\alpha)}
\]

\[
\leq \| H_{f, \alpha} k^\alpha_{\partial} \|_{L^2(\mathbb{D}, dA_\alpha)} + \| f \circ \Phi_z - (\overline{P^\alpha(\bar{f}_\alpha)}) \|_{L^2(\mathbb{D}, dA_\alpha)}
\]

\[
\leq \| H_{f, \alpha} k^\alpha_{\partial} \|_{L^2(\mathbb{D}, dA_\alpha)} + \| H_{f, \alpha} k^\alpha_{\partial} \|_{L^2(\mathbb{D}, dA_\alpha)}
\]

\[
\leq \| H_{f, \alpha} \| + \| H_{f, \alpha} \|.
\]

\end{proof}
Therefore $f$ is in $BMO_{\Theta,\alpha}$, and

$$
\|H_{f,\alpha}\| + \|H_{f,\alpha}\| \geq \|f\|_{BMO_{\Theta,\alpha}}.
$$

Conversely, if $f$ is in $BMO_{\Theta,\alpha}$, then by Proposition 2.2.28 $f = f_1 + f_2$ where $f_1 \in BO_1$ and $f_2 \in BA_\alpha$. Let $g \in L^2_\alpha(D, dA_\alpha)$. Then

$$
\|H_{f,\alpha}g\|_{L^2(D, dA_\alpha)} = \|H_{f_1 + f_2,\alpha}g\|_{L^2(D, dA_\alpha)}
$$

$$
= \|(1 - P^{\alpha})(f_1 + f_2)g\|_{L^2(D, dA_\alpha)}
$$

$$
\leq \|(1 - P^{\alpha})f_1 g\|_{L^2(D, dA_\alpha)} + \|(1 - P^{\alpha})f_2 g\|_{L^2(D, dA_\alpha)}
$$

$$
= \|H_{f_1,\alpha}g\|_{L^2(D, dA_\alpha)} + \|H_{f_2,\alpha}g\|_{L^2(D, dA_\alpha)}
$$

$$
\leq (\|H_{f_1,\alpha}\| + \|H_{f_2,\alpha}\|)\|g\|_{L^2_\alpha(D, dA_\alpha)}.
$$

Since $f_1 \in BO_1$, Proposition 2.2.12 implies $H_{f_1,\alpha}$ is bounded. Since $f_2 \in BA_\alpha$, Proposition 2.2.20 implies $H_{f_2,\alpha}$ is bounded. Consequently, $H_{f,\alpha}$ is bounded.

By the definition of $BMO_{\Theta,\alpha}$ we see that if $f$ is in $BMO_{\Theta,\alpha}$, then $\overline{f}$ is also in $BMO_{\Theta,\alpha}$. Therefore $H_{f,\alpha}$ is also bounded.

Thus we see that the boundedness properties of Hankel operators on the weighted Bergman space $L^2_\alpha(D, dA_\alpha)$ parallel those of Hankel operators on the usual Bergman space.

### 2.3 Compact Hankel Operators on Weighted Bergman Spaces

In this section we will characterize the symbol functions $f$ that make bounded Hankel operators $H_{f,\alpha}$ and $H_{f,\alpha}$ compact. We have seen in the previous section that if $f$ is in $BMO_{\Theta,\alpha}$ then the Hankel operators $H_{f,\alpha}$ and $H_{f,\alpha}$ are both bounded. We begin by outlining some
well known properties of compact operators on a separable Hilbert space $\mathcal{H}$.

**Theorem 2.3.1** Let $T$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

1. $T$ is compact;
2. $T^*T$ is compact;
3. $(T^*T)^\frac{1}{2}$ is compact.

**Proposition 2.3.2** Suppose $\{e_n\}$ and $\{\sigma_n\}$ are orthonormal sets in a Hilbert space $\mathcal{H}$ and $\{\lambda_n\}$ is a sequence of complex numbers tending to 0. Let $T$ be the linear operator on $\mathcal{H}$ defined by

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \sigma_n, \quad x \in \mathcal{H}. $$

Then $T$ is compact.

**Proposition 2.3.3** If $T$ is a self-adjoint compact operator on $\mathcal{H}$, then there exists a sequence of real numbers $\{\lambda_n\}$ tending to 0 and there exists an orthonormal set $\{e_n\}$ in $\mathcal{H}$ such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n $$

for all $x \in \mathcal{H}$.

By using Propositions 2.3.1 and 2.3.3 we can get the converse of Proposition 2.3.2 as follows: Suppose $T$ is compact, but not necessarily self-adjoint. Then we can consider $(T^*T)^\frac{1}{2}$ which is compact and self-adjoint. By Proposition 2.3.3 there is an orthonormal set $\{e_n\}$ in $\mathcal{H}$ and sequence of real numbers $\{\lambda_n\}$, where $\lambda_n \to 0$ as $n \to \infty$ such that

$$(T^*T)^\frac{1}{2}x = \sum_n \lambda_n \langle x, e_n \rangle e_n, \quad x \in \mathcal{H}. $$
By the polar decomposition of the bounded operator $T$, there is a partial isometry $V$ such that

$$ T = V(T^*T)^{\frac{1}{2}}. $$

Now let $\sigma_n = V e_n$ for each $n$. Then $\{\sigma_n\}$ is an orthonormal set and for each $x \in \mathcal{H}$

$$ Tx = V(T^*T)^{\frac{1}{2}}x = V \left( \sum_n \lambda_n(x, e_n) e_n \right) = \sum_n \lambda_n(x, e_n) \sigma_n. $$

Thus the converse of Proposition 2.3.2 is proved. The above formula for a compact operator $T$ will be called the canonical decomposition. The sequence $\{\lambda_n\}$ may be arranged to be non-increasing and is called the sequence of singular values of $T$.

**Definition 2.3.4** Let $\mathcal{H}$ be a separable Hilbert space. An operator $T \in B(\mathcal{H})$ is called a Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n : n = 0, 1, 2, \ldots\}$ of $\mathcal{H}$ such that $\sum_{n=0}^{\infty} \|Te_n\|^2 < \infty$.

We will call the finite value $\sum_{n=0}^{\infty} \|Te_n\|^2$ the Hilbert-Schmidt norm of $T$ and denote it by $\text{trace}(T^*T)$. It is a well known result that if an operator $T \in B(\mathcal{H})$ is a Hilbert-Schmidt operator, then $T$ is compact. The set of Hilbert-Schmidt operators is denoted by $S_2$.

The compactness of Hankel operators is related to certain Mobius invariant spaces that we will introduce here.
Definition 2.3.5 The Möbius group $G$ consists of all one-to-one analytic functions that map the open unit disc $D$ onto itself. Each $\phi \in G$ has the form

$$\phi(z) = \lambda \Phi_a(z),$$

where

$$\Phi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad |a| < 1 \quad \text{and} \quad |\lambda| = 1.$$ 

Definition 2.3.6 The Bloch space $B$ consists of those analytic functions $f$ on $D$ for which

$$\sup \left\{ |f'(z)|(1 - |z|^2) : |z| < 1 \right\} = \rho_B(f)$$

is finite.

Definition 2.3.7 [AFP 85] [RT 79] Let $X$ be a linear space of analytic functions on the open unit disc $D$ with a seminorm $\rho$. $X$ is Möbius invariant if

1. $X$ is a subset of the Bloch space $B$ and

$$\rho_B(f) \leq C \rho(f),$$

for all $f \in X$, where $C$ is a positive constant independent of $f$;

2. The seminorm $\rho$ is complete in $X$;

3. For each $\phi \in G$ and each $f \in X$, the composite function $f \circ \phi$ is in $X$ and $\rho(f \circ \phi) \leq K \rho(f)$ for all $f \in X$, where $K$ is a constant independent of $\phi$ and $f$. If $K = 1$, so that

$$\rho(f \circ \phi) = \rho(f)$$

for all $f \in X$ and all $\phi \in G$, then we say that $X$ is strictly Möbius invariant.
4. For each $f \in \mathcal{X}$, the mapping $\phi \mapsto f \circ \phi$ is continuous from $G$ into $\mathcal{X}$.

**Example 2.3.8** There are many examples of Mobius invariant spaces [AFP 85]. We will mention two here.

1. The little Bloch space $B_0$ of $D$ is the closed subspace of the Bloch space $B$ consisting of functions $f$ with $(1 - |z|^2)f'(z) \to 0$ as $|z| \to 1$.

2. For $1 < p < \infty$, the Besov space $B_p$ of $D$ is defined to be the space of analytic functions $f$ in $D$ such that

$$
\|f\|_{B_p} = \left[ \int_D (1 - |z|^2)^p |f'(z)|^p d\lambda(z) \right]^{\frac{1}{p}} < \infty,
$$

where

$$
d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}
$$

is the Mobius invariant measure on $D$. And the seminorm $\| \|_{B_p}$ is complete on $B_p$.

Moreover, $\| \|_{B_p}$ is Mobius invariant in the sense that $\|f \circ \phi\|_{B_p} = \|f\|_{B_p}$ for all $f \in B_p$ and $\phi \in$ Mobius group $G$. The space $B_p$ becomes a Banach space with the norm

$$
\|f\| = |f(0)| + \|f\|_{B_p}.
$$

Note that for $p = 2, B_2 = D$ is the classical Dirichlet space.

**Theorem 2.3.9** [AFP 88] Let $f$ be an analytic function on $D$. Then the Hankel operator induced by $f$ is a Hilbert-Schmidt operator on $L^2_\alpha(D, dA_\alpha)$ if and only if $f$ belongs to the Dirichlet space $D$. Moreover, the association $f \mapsto H_{f,\alpha}$ is an isometry,

$$
\|f\|_D = \|H_{f,\alpha}\|_{S_2}
$$
and
\[(f \cdot g)_D = (H_{f, \alpha}, H_{f, \alpha})_{s_2}.
\]

The above theorem establishes the following result.

**Corollary 2.3.10** If \(f\) is a polynomial on \(D\), then \(H_{f, \alpha}\) is compact.

**Proof.** If \(f\) is a polynomial in \(D\), then \(\|f\|_D < \infty\). The desired result follows directly from Theorem 2.3.9.

**Proposition 2.3.11** [AFP 88] Let \(T\) be a linear operator on \(L^2_\alpha(D, dA_\alpha)\) which is either trace class or nonnegative. Then

\[
\text{trace}(T) = \int_D \langle Tk_\alpha^0, k_\alpha^0 \rangle_{L^2_\alpha(D, dA_\alpha)} d\lambda(z).
\]

**Proposition 2.3.12** Let \(f \in L^2_\alpha(D, dA_\alpha)\) and \(z \in D\). Then

1. \(P^{\alpha}(f k_\alpha^0) = \tilde{f}(z)k_\alpha^0\);
2. \(\tilde{f}_\alpha(z) = f(z)\);
3. \(\|H_{f, \alpha} k_\alpha^0\|_{L^2_\alpha(D, dA_\alpha)}^2 = |\tilde{f}_\alpha(z)|^2 - |\tilde{f}(z)|^2\).

**Proof.** (1) Let \(g\) be a polynomial. Then

\[
(P^{\alpha}(f k_\alpha^0), g)_{L^2_\alpha(D, dA_\alpha)} = (\tilde{f} k_\alpha^0, g)_{L^2_\alpha(D, dA_\alpha)}
= \frac{1}{\|K_\alpha^0\|} (K_\alpha^0 f g)_{L^2_\alpha(D, dA_\alpha)}
= \frac{1}{\|K_\alpha^0\|} \tilde{f}(z)g(z)
= (\tilde{f}(z) k_\alpha^0, g)_{L^2_\alpha(D, dA_\alpha)}.
\]

Use the fact that the polynomials are dense in \(L^2_\alpha(D, dA_\alpha)\) to obtain the result.
(2) From the definition of $\tilde{f}_\alpha(z)$, we have

$$\tilde{f}_\alpha(z) = \int D f(w)|k_\alpha^\circ(w)|^2 dA_\alpha(w).$$

Expand $|k_\alpha^\circ(w)|^2$ and change the integral to the inner product in $L^2(D, dA_\alpha)$ to obtain

$$\tilde{f}_\alpha(z) = \langle k_\alpha^\circ, \tilde{f}_\alpha^\circ \rangle_{L^2(D, dA_\alpha)}$$

$$= \langle k_\alpha^\circ, P^\circ(\tilde{f}_\alpha^\circ) \rangle_{L^2(D, dA_\alpha)}.$$

Using part(1) and the fact that $k_\alpha^\circ$ is a unit vector in $L^2(D, dA_\alpha)$, we have

$$\tilde{f}_\alpha(z) = \langle k_\alpha^\circ, f(z)k_\alpha^\circ \rangle_{L^2(D, dA_\alpha)} = f(z).$$

(3) The definition of Hankel operator gives us

$$\|H_{f, \alpha}k_\alpha^\circ\|^2_{L^2(D, dA_\alpha)} = \|(1 - P^\circ)\tilde{f}_\alpha^\circ\|^2_{L^2(D, dA_\alpha)}.$$

By (1) and the definition of norm in $L^2(D, dA_\alpha)$, we have

$$\|H_{f, \alpha}k_\alpha^\circ\|^2_{L^2(D, dA_\alpha)} = \|\tilde{f}_\alpha^\circ - f(z)k_\alpha^\circ\|^2_{L^2(D, dA_\alpha)}$$

$$= \int_D |k_\alpha^\circ(w)|^2 |\tilde{f}(w) - f(z)|^2 dA_\alpha(w).$$

Expand the integrand and integrate each term separately to obtain

$$\|H_{f, \alpha}k_\alpha^\circ\|^2_{L^2(D, dA_\alpha)} = |\tilde{f}|^2_\alpha(z) - \tilde{f}(z)\tilde{f}_\alpha(z) - f(z)\tilde{f}_\alpha(z) + |f(z)|^2.$$
The result from (2) gives us

\[ \| H_{f, \alpha} k_2^2 \|_{L^2(D, dA_\alpha)} = |f|_{\alpha}(z) - |f_0(z)|^2. \]

\[ \square \]

**Proposition 2.3.13** For any \( f \in L^2(D, dA_\alpha) \), we have

\[
\text{trace}(H_{f, \alpha}^* H_{f, \alpha}) = \int_D \left( |f|_{\alpha}(z) - |f_0(z)|^2 \right) d\lambda(z)
\]

\[ = \frac{1}{2} \int_D \int_D \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^{4+2\alpha}} dA_\alpha(z) dA_\alpha(w). \]

**Proof.** From Proposition 2.3.11 we have

\[
\text{trace}(H_{f, \alpha}^* H_{f, \alpha}) = \int_D \| H_{f, \alpha} k_2^2 \|_{L^2(D, dA_\alpha)}^2 d\lambda(z)
\]

and Proposition 2.3.12 (3) gives us the first equality. Next we consider

\[
\int_D \int_D \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^{4+2\alpha}} dA_\alpha(z) dA_\alpha(w).
\]

Expand the numerator of integrand and integrate each term separately to get

\[
\int_D \int_D \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^{4+2\alpha}} dA_\alpha(z) dA_\alpha(w) = I_1 + I_2 + I_3 + I_4,
\]

where

\[
I_1 = \int_D \int_D \frac{|f(z)|^2}{|1 - z\overline{w}|^{4+2\alpha}} dA_\alpha(z) dA_\alpha(w)
\]

\[
I_2 = \int_D \int_D \frac{|f(w)|^2}{|1 - z\overline{w}|^{4+2\alpha}} dA_\alpha(z) dA_\alpha(w)
\]
\[ I_3 = \int_D \int_D \frac{f(w) \bar{f}(z)}{|1 - z \bar{w}|^{4+2\alpha}} \, dA_\alpha(z) \, dA_\alpha(w), \]

and

\[ I_4 = \int_D \int_D \frac{f(z) \bar{f}(w)}{|1 - z \bar{w}|^{4+2\alpha}} \, dA_\alpha(z) \, dA_\alpha(w). \]

Use the fact that \(|K_\alpha^\alpha(z)|^2 = \frac{1}{|1 - z \bar{w}|^{4+2\alpha}}\) and \(||K_\alpha^\alpha||^2 \, dA_\alpha(w) = d\lambda(w)\) to obtain

\[ I_1 = I_2 = \int_D |\bar{f}_\alpha(z)|^2 \, d\lambda(z), \]

and

\[ I_3 = I_4 = \int_D |\bar{f}_\alpha(z)|^2 \, d\lambda(z). \]

Hence

\[ \int_D \int_D \frac{|f(z) - f(w)|^2}{|1 - z \bar{w}|^{4+2\alpha}} \, dA_\alpha(z) \, dA_\alpha(w) = 2 \int_D \left( |\bar{f}_\alpha(z)|^2 - |\bar{f}_\alpha(z)|^2 \right) \, d\lambda(z). \]

Therefore the proof is complete.

Before we give a necessary and sufficient condition for the compactness of the Hankel operator \(H_{f, \alpha}\), we need the following definitions.

1. Let \(VMO_{\beta, \alpha}\) be the closed subspace of \(BMO_{\beta, \alpha}\) consisting of functions \(f\) with

\[ \lim_{|z| \to 1^-} \left( |\bar{f}_\alpha(z)|^2 - |\bar{f}_\alpha(z)|^2 \right) = 0. \]

2. Let \(VO_1\) be the closed subspace of \(BO_1\) consisting of functions \(f\) with

\[ \lim_{|z| \to 1^-} \text{Or}(f)(z) = 0. \]
3. Let $VA_\alpha$ be the closed subspace of $BA_\alpha$ consisting of functions $f$ with

$$\lim_{|z| \to 1^-} \widehat{|f|^2}_\alpha(z) = 0.$$

**Theorem 2.3.14** Suppose $r > 0$ and $f \in BMO_{\theta,\alpha}$. Then the following are equivalent

1. $f \in VMO_{\theta,\alpha}$;

2. $\lim_{|z| \to 1^-} \left( |\widehat{f_r^2}_r,\alpha(z)| - |\widehat{f}_r,\alpha(z)|^2 \right) = 0$;

3. $f = f_1 + f_2$ where $f_1 \in VO_1$, and $f_2 \in VA_\alpha$.

**Proof.** (1) $\Rightarrow$ (2)

Suppose $f \in VMO_{\theta,\alpha}$. By Propositions 2.2.24 (1) and 2.2.28 there exists a constant $C > 0$ such that

$$\lim_{|z| \to 1^-} MO_{r,\alpha}(f)(z)^2 = \lim_{|z| \to 1^-} \left( |\widehat{f_r^2}_r,\alpha(z)| - |\widehat{f}_r,\alpha(z)|^2 \right) \leq C \lim_{|z| \to 1^-} \left( |\widehat{f_r^2}_r,\alpha(z)| - |\widehat{f}_r,\alpha(z)|^2 \right) = 0.$$

Therefore (1) $\Rightarrow$ (2) is done.

(2) $\Rightarrow$ (3) follows from the result in Proposition 2.2.28.

(3) $\Rightarrow$ (1)

Consider

$$\|f\|_{BMO_{\theta,\alpha}} = \|f_1 + f_2\|_{BMO_{\theta,\alpha}} \leq \|f_1\|_{BMO_{\theta,\alpha}} + \|f_2\|_{BMO_{\theta,\alpha}}.$$
By the proof in Proposition 2.2.28 there exist constants $C_1, C_2 > 0$ such that

$$\|f\|_{BMO_{\partial,\alpha}} \leq C_1\|f_1\|_{BO_1} + C_2\|f_2\|_{BA_\alpha}.$$ 

Take the limit as $|z| \to 1^-$ and the desired result follows from the fact that $f_1 \in VO_1$ and $f_2 \in VA_\alpha$. 

\[\square\]

**Proposition 2.3.15 [Zhu 90]** The subspace $VO_1$ of $BO_1$ is generated by $C(\overline{D})$ in $BO_1$.

**Proposition 2.3.16** The subspace $VA_\alpha$ is generated by the functions in $BA_\alpha$ with compact support in the open disc $D$.

**Proof.** Suppose $f \in VA_\alpha$. Then define $f_r(z)$ by

$$f_r(z) = \begin{cases} 
  f(z) & \text{if } |z| \leq r \\
  0 & \text{if } |z| > r,
\end{cases}$$

where $z \in D$ and $r \in (0, 1)$. It is clear that $f_r$ is in $BA_\alpha$ and has compact support in $D$.

Given $\epsilon > 0$, $f \in VA_\alpha$ implies that there exists $\delta \in (0, 1)$ such that

$$|f|^2_\alpha(z) = \int_D |f(w)|^2 |k_\alpha^\alpha(w)|^2 dA_\alpha(w) < \epsilon$$

for all $\delta < |z| < 1$. Now if $\delta < |z| < 1$, then

$$\int_D |(f - f_r)(w)|^2 |k_\alpha^\alpha(w)|^2 dA_\alpha(w) \leq \int_D |f(w)|^2 |k_\alpha^\alpha(w)|^2 dA_\alpha(w) < \epsilon.$$
For $|z| \leq \delta$

$$\int_{\mathcal{D}} |(f - f_r)(w)|^2 |k_\alpha^2(w)|^2 dA_\alpha(w) \leq \frac{1}{(1 - \delta)^{4 + 2\alpha}} \int_{r < |z| < 1} |f(w)|^2 dA_\alpha(w).$$

Take the limit as $r \to 1^-$ and use the fact that $f$ is in $L^2(\mathcal{D}, dA_\alpha)$ to obtain

$$\limsup_{r \to 1^-} \|f - f_r\|_{BA_\alpha} < \epsilon$$

for any $\epsilon > 0$. Therefore the proof is complete. \hfill \Box

**Corollary 2.3.17** The closed subspace $VMO_{\theta, \alpha}$ of $BMO_{\theta, \alpha}$ is generated by $C(\overline{\mathcal{D}})$ and the functions in $BA_\alpha$ with compact support.

**Proof.** Suppose $f \in VMO_{\theta, \alpha}$. Then Theorem 2.3.14 gives us $f = f_1 + f_2$ where $f_1 \in VO_1$ and $f_2 \in VA_\alpha$. Propositions 2.3.15 and 2.3.16 give the desired result. \hfill \Box

The next proposition describes a relationship between the Toeplitz operator (see Definition 2.2.21) and the Hankel operator on the space $L^2_\alpha(\mathcal{D}, dA_\alpha)$.

**Proposition 2.3.18** [Axler 86][Zhu 90]

1. For all $f$ and $g$ in $BMO_{\theta, \alpha}$

$$H_{f, \alpha}^* H_{g, \alpha} = T_{f_2, \alpha} - T_{f, \alpha} T_{g, \alpha}.$$

2. For all $f \in BMO_{\theta, \alpha}$ and $g \in B$,

$$H_{f, \alpha}^* H_{f, \alpha} = T_{f, \alpha} H_{f, \alpha}^* H_{f, \alpha} T_{g, \alpha}.$$
Theorem 2.3.19 [Zhu 90] Suppose $\alpha > -1, r > 0$, and $\mu$ is a positive Borel measure on $D$. Then the following are equivalent:

1. $T_{\mu,\alpha}$ is compact on $L^2(D, dA_\alpha)$;
2. $\tilde{\mu}_\alpha(z) \to 0$ as $|z| \to 1^-$;
3. $\tilde{\mu}_{r,\alpha}(z) \to 0$ as $|z| \to 1^-$.

Corollary 2.3.20 [Zhu 90] If $\alpha > -1, r > 0$, and $\varphi$ is a nonnegative function on $D$, then the following are equivalent:

1. $T_{\varphi,\alpha}$ is compact on $L^2_\alpha(D, dA_\alpha)$;
2. $\tilde{\varphi}_\alpha(z) \to 0$ as $|z| \to 1^-$;
3. $\tilde{\varphi}_{r,\alpha}(z) \to 0$ as $|z| \to 1^-$.

Proposition 2.3.21 For all $f \in C(\overline{D}), H_{f,\alpha}$ is compact.

Proof. Let $\{f_n\} \subset C(\overline{D})$ be such that $f_n \to f$. For $g \in L^2_\alpha(D, dA_\alpha)$

$$
\|(H_{f_n,\alpha} - H_{f,\alpha})g\|_{L^2(D, dA_\alpha)} = \|((1 - P^\alpha)(f_n - f))g\|_{L^2(D, dA_\alpha)} \\
\leq \|f_n - f\|_\infty \|g\|_{L^2_\alpha(D, dA_\alpha)}.
$$

Hence

$$
\|H_{f_n,\alpha} - H_{f,\alpha}\| \leq \|f_n - f\|_\infty.
$$

Take the limit as $n \to \infty$. Then $\|H_{f_n,\alpha} - H_{f,\alpha}\| \to 0$. Therefore the mapping

$$
\Psi : f \mapsto H_{f,\alpha}.
$$

$$
\Psi : C(\overline{D}) \to B(L^2_\alpha(D, dA_\alpha), L^2_\alpha(D, dA_\alpha))
$$
is norm continuous. By Proposition 2.3.18 (2) we have

$$H_{z^n z^m, \alpha} H_{z^n z^m, \alpha} = T_{z^n \alpha} H_{z^n z^m, \alpha} H_{z^n z^m, \alpha} T_{z^n \alpha}$$

where $n, m \geq 0$. By Corollary 2.3.10, $H_{z^n z^m, \alpha} H_{z^n z^m, \alpha}$ is compact, which implies the compactness of $H_{z^n z^m, \alpha}$. Since $C(\overline{D})$ is generated by functions of the form $z^n z^m$ for $m, n \geq 0$, the Stone-Weirestrass approximation theorem implies that $H_{f, \alpha}$ is compact for all $f \in C(\overline{D})$.

\[ \square \]

**Proposition 2.3.22** If $f \in BMO_{\partial, \alpha}$ has compact support in $D$, then $H_{f, \alpha}$ is compact.

**Proof.** From Proposition 2.3.18 (1), for all $f \in BMO_{\partial, \alpha}$

$$H_{f, \alpha}^* H_{f, \alpha} = T_{|f|^2, \alpha} - T_{f, \alpha} T_{f, \alpha} \leq T_{|f|^2, \alpha}.$$

By Corollary 2.3.20, $T_{|f|^2, \alpha}$ is compact, which implies $H_{f, \alpha}$ is compact. Therefore the proof is complete. \[ \square \]

**Theorem 2.3.23** If $f \in BMO_{\partial, \alpha}$, then $H_{f, \alpha}$ and $H_{f, \alpha}$ are both compact if and only if $f \in VMO_{\partial, \alpha}$.

**Proof.** Suppose $H_{f, \alpha}$ and $H_{f, \alpha}$ are compact. Since $k_z^\alpha \to 0$ weakly in $L^2(D, dA_\alpha)$ as $|z| \to 1^-$,

$$\|H_{f, \alpha} k_z^\alpha\|_{L^2(D, dA_\alpha)} \to 0 \quad \text{and} \quad \|H_{f, \alpha} k_z^\alpha\|_{L^2(D, dA_\alpha)} \to 0.$$

By the proof of Theorem 2.2.32

$$MO_\alpha(f)(z) \leq \|H_{f, \alpha} k_z^\alpha\|_{L^2(D, dA_\alpha)} + \|H_{f, \alpha} k_z^\alpha\|_{L^2(D, dA_\alpha)}.$$
Take the limit as $|z| \to 1^-$. Then we have $MO_\alpha(f)(z) \to 0$. Therefore $f \in VMO_{\beta, \alpha}$.

Conversely, let $f \in VMO_{\beta, \alpha}$. By Theorem 2.3.14, $f = f_1 + f_2$ with $f_1 \in VO_1$ and $f_2 \in VA_\alpha$. Propositions 2.3.15 and 2.3.16 imply compactness of $H_{f_1, \alpha}$ and $H_{f_2, \alpha}$. Thus

$$H_{f, \alpha} = H_{f_1 + f_2, \alpha} = H_{f_1, \alpha} + H_{f_2, \alpha}$$

is compact. \hfill \Box

In this section we have seen that the compact Hankel operators on the weighted Bergman space $L^2_\alpha(Dd\lambda_\alpha)$ for $\alpha > -1$ have properties parallel to the compact Hankel operators on usual Bergman space $L^2(D, dA)$.

2.4 Schatten Class Hankel Operators on Weighted Bergman Spaces

In this section we will characterize the symbol functions $f$ that make the bounded Hankel operators $H_{f, \alpha}$ and $H_{f_2, \alpha}$ members of the Schatten classes. We are now going to outline definitions and some well known properties of Schatten class operators on a separable Hilbert space. For references on this material see [Douglas 72], [DS 58], [GK 69], [Ringrose 71] and [Simon 79].

**Definition 2.4.1** Let $T$ be a compact operator on a separable Hilbert space $\mathcal{H}$ with the canonical decomposition

$$Tx = \sum_n \lambda_n(x, e_n)\sigma_n,$$

where $x \in \mathcal{H}$, $\{e_n\}$, $\{\sigma_n\}$ are orthonormal sets in $\mathcal{H}$ and $\lambda_n$ is the $n$-th singular value of $T$. Given $0 < p < \infty$, we define the Schatten $p$-class of $\mathcal{H}$, denoted by $S_p(\mathcal{H})$ or $S_p$, to be
the space of all compact operators $T$ on $\mathcal{H}$ with singular value sequence $\{\lambda_n\} \in l^p$. 

If $1 \leq p < \infty$, then $S_p$ is a Banach space with the norm 

$$
\|T\|_p = \left[ \sum_n |\lambda_n|^p \right]^{\frac{1}{p}}.
$$

Note that $S_1$ is called the trace class of $\mathcal{H}$, and $S_2$ is called the Hilbert -Schmidt class.

**Theorem 2.4.2** If $T$ is a compact operator on $\mathcal{H}$ with singular values $\{\lambda_n\}$, then 

$$
\sum_n |\lambda_n|^2 = \sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n,m=1}^{\infty} |(Te_n, e_m)|^2,
$$

for any orthonormal basis $\{e_n\}$ of $\mathcal{H}$.

**Theorem 2.4.3** If $T$ is a positive compact operator on $\mathcal{H}$ with the canonical decomposition 

$$
Tx = \sum_n \lambda_n(x, e_n)e_n, \ x \in \mathcal{H},
$$

then 

$$
\sum_n \lambda_n = \sum_{k=1}^{\infty} (T\sigma_k, \sigma_k),
$$

for any orthonormal basis $\{\sigma_k\}$ of $\mathcal{H}$.

**Theorem 2.4.4** If $T$ is in $S_1$, then the series $\sum_{k=1}^{\infty} (Te_k, e_k)$ converges absolutely for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$ and the sum is independent of the choice of the orthonormal basis. We call this value the trace of $T$ and denote it by trace$(T)$.

**Proposition 2.4.5** Suppose $T$ is a positive compact operator on $\mathcal{H}$ and $p > 0$. Then $T \in S_p$ if and only if $T^p \in S_1$. Moreover $\|T\|^p_p = \|T^p\|_1$.
Theorem 2.4.6 If $T$ is a compact operator on $\mathcal{H}$ and $p \geq 1$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{\frac{p}{2}} \in S_1$ if and only if $T^*T \in S_1^{\frac{p}{2}}$. Moreover,

$$
||T||_p = \left(\sum_{n} |\langle Te_n, e_n \rangle|^p\right)^{\frac{1}{p}} = ||(T^*T)|^p||_1 = ||T^*T||_{\frac{p}{2}}.
$$

Theorem 2.4.7 Suppose $T$ is a compact operator on $\mathcal{H}$ and $p \geq 1$. Then $T$ is in $S_p$ if and only if $\sum_{n} |\langle Te_n, e_n \rangle|^p < \infty$ for all orthonormal sets $\{e_n\}$. Moreover,

$$
||T||_p = \sup \left\{ \left(\sum_{n} |\langle Te_n, e_n \rangle|^p\right)^{\frac{1}{p}} : \{e_n\} \text{ orthonormal} \right\}.
$$

Theorem 2.4.8 Suppose $T$ is a compact operator on $\mathcal{H}$ and $p \geq 1$. Then $T$ is in $S_p$ if and only if $\sum_{n} |\langle Te_n, \sigma_n \rangle|^p < \infty$ for all orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$. Moreover,

$$
||T||_p = \sup \left\{ \left(\sum_{n} |\langle Te_n, \sigma_n \rangle|^p\right)^{\frac{1}{p}} : \{e_n\} \text{ and } \{\sigma_n\} \text{ are orthonormal} \right\}.
$$

Theorem 2.4.9 Suppose $T$ is a compact operator on $\mathcal{H}$ and $p \geq 2$. Then $T$ is in $S_p$ if and only if $\sum_{n} \|Te_n\|^p < \infty$ for all orthonormal sets $\{e_n\}$ in $\mathcal{H}$. Moreover,

$$
||T||_p = \sup \left\{ \left(\sum_{n} \|Te_n\|^p\right)^{\frac{1}{p}} : \{e_n\} \text{ orthonormal} \right\}.
$$

Proposition 2.4.10 [AFP 88] Suppose $T$ is a positive operator on a Hilbert space $\mathcal{H}$ and $x$ is a unit vector in $\mathcal{H}$. Then

1. $\langle T^p x, x \rangle \geq \langle Tx, x \rangle^p$ for all $p \geq 1$;

2. $\langle T^p x, x \rangle \leq \langle Tx, x \rangle^p$ for all $0 < p \leq 1$. 

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Theorem 2.4.11 [Zhu 90] Suppose $\mu$ is a finite positive Borel measure on $D$, $1 \leq p \leq \infty$ and $r > 0$. Then the following are equivalent:

1. $T_{\mu,\alpha}$ is in $S_p$;

2. $\bar{\mu}_\alpha$ is in $L^p(D, d\lambda)$;

3. $\bar{\mu}_{r,\alpha}$ is in $L^p(D, d\lambda)$.

Corollary 2.4.12 [Zhu 90] Suppose $\varphi$ is a nonnegative function on $D$, $1 \leq p < \infty$ and $r > 0$. Then the following are equivalent:

1. $T_{\varphi,\alpha}$ is in $S_p$;

2. $\bar{\varphi}_\alpha(z)$ is in $L^p(D, d\lambda)$;

3. $\bar{\varphi}_{r,\alpha}$ is in $L^p(D, d\lambda)$.

Let $\gamma(t)$ be a smooth curve in $D$ and $s(t)$ be the arclength of $\gamma(t)$ in the Bergman metric. Then

$$\frac{ds}{dt} = \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2}.$$ 

Let $P_{\gamma(t)}$ be the rank one orthogonal projection from $L^2_\alpha(D, dA_{\alpha})$ onto the one dimensional subspace of $L^2_\alpha(D, dA_{\alpha})$ spanned by $k^\alpha_{\gamma(t)}$. Then we have the following proposition.

Proposition 2.4.13 For any smooth curve $\gamma(t)$ in $D$

$$\frac{ds}{dt} = \frac{1}{\sqrt{2 + \alpha}} \|(1 - P_{\gamma(t)})(\frac{d}{dt} k^\alpha_{\gamma(t)})\|_{L^2(D, dA_{\alpha})}.$$ 

Proof. We have

$$k^\alpha_{\gamma(t)}(z) = \frac{(1 - |\gamma(t)|^2)^{1 + \frac{\alpha}{2}}}{[1 - \gamma(t)z]^{2 + \alpha}},$$

for any $t \in [0, 1]$ and $z \in D$. 

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Differentiate $k_{\gamma(t)}^\alpha(z)$ with respect to $t$, we have
\[
\frac{d}{dt}(k_{\gamma(t)}^\alpha)(z) = \frac{(2 + \alpha)(1 - |\gamma(t)|^2)^{1 + \frac{\alpha}{2}} z \gamma'(t)}{[1 - \overline{\gamma(t)}z]^{3 + \alpha}} - \frac{(1 + \frac{\alpha}{2})(1 - |\gamma(t)|^2)^{\frac{\alpha}{2}} (\gamma'(t)\overline{\gamma(t)} + \gamma(t)\overline{\gamma'(t)})}{[1 - \overline{\gamma(t)}z]^{2 + \alpha}}.
\]

Since $\frac{d}{dt}k_{\gamma(t)}^\alpha \in L^2_\alpha(D, dA_\alpha)$ and the subspace spanned by $k_{\gamma(t)}^\alpha$ is included in $L^2_\alpha(D, dA_\alpha)$, we have
\[
P_{\gamma(t)}\left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right)(z) = \left\langle P_{\gamma(t)}\left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right), K_z^\alpha\right\rangle_{L^2_\alpha(D, dA_\alpha)} = \left\langle \left\langle \left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right), K_{\gamma(t)}^\alpha\right\rangle_{L^2_\alpha(D, dA_\alpha)} k_{\gamma(t)}^\alpha(t), K_z^\alpha\right\rangle_{L^2_\alpha(D, dA_\alpha)} = \frac{1}{\|K_{\gamma(t)}^\alpha\|} \left\langle \left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right), K_{\gamma(t)}^\alpha\right\rangle_{L^2_\alpha(D, dA_\alpha)} (k_{\gamma(t)}^\alpha(t), K_z^\alpha)_{L^2_\alpha(D, dA_\alpha)}.
\]

The reproducing property of the kernel function of $L^2_\alpha(D, dA_\alpha)$ gives us
\[
P_{\gamma(t)}\left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right)(z) = \frac{1}{\|K_{\gamma(t)}^\alpha\|} \left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right)(\gamma(t)) k_{\gamma(t)}^\alpha(z).
\]

Substituting in the above formulas of $\left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right)(\gamma(t))$ and $k_{\gamma(t)}^\alpha(z)$, we have
\[
P_{\gamma(t)}\left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right)(z) = \frac{(2 + \alpha)(1 - |\gamma(t)|^2)^{1 + \frac{\alpha}{2}} z \gamma'(t)}{[1 - \overline{\gamma(t)}z]^{3 + \alpha}} - \frac{(2 + \alpha)(1 - |\gamma(t)|^2)^{\frac{\alpha}{2}} (\gamma(t)\overline{\gamma'(t)})}{[1 - \overline{\gamma(t)}z]^{2 + \alpha}}.
\]

Therefore
\[
\left(1 - P_{\gamma(t)}\right)\left(\frac{d}{dt}k_{\gamma(t)}^\alpha\right)(z) = \frac{(2 + \alpha)(1 - |\gamma(t)|^2)^{1 + \frac{\alpha}{2}} z \gamma'(t)}{[1 - \overline{\gamma(t)}z]^{3 + \alpha}} - \frac{(2 + \alpha)(1 - |\gamma(t)|^2)^{\frac{\alpha}{2}} (\gamma(t)\overline{\gamma'(t)})}{[1 - \overline{\gamma(t)}z]^{2 + \alpha}}.
\]
Combine the terms and use the formula of $k_{\gamma(t)}^\alpha(z)$ to get

\[
(1 - P_{\gamma(t)}) \left( \frac{d}{dt} k_{\gamma(t)}^\alpha \right)(z) = \frac{(2 + \alpha)\gamma'(t)(z - \gamma(t))}{(1 - |\gamma(t)|^2)(1 - \overline{\gamma(t)}z)} k_{\gamma(t)}^\alpha(z).
\]

Thus

\[
\| (1 - P_{\gamma(t)}) \frac{d}{dt} k_{\gamma(t)}^\alpha \|_{L^2(D, dA_\alpha)}^2 = \frac{(2 + \alpha)^2 |\gamma'(t)|^2}{(1 - |\gamma(t)|^2)^2} \int_D \left| \frac{z - \gamma(t)}{1 - \overline{\gamma(t)}z} \right|^2 |k_{\gamma(t)}^\alpha(z)|^2 dA_\alpha(z).
\]

Using the change of variable formula with $z = \Phi_{\gamma(t)}(w)$, we have

\[
\| (1 - P_{\gamma(t)}) \frac{d}{dt} k_{\gamma(t)}^\alpha \|_{L^2(D, dA_\alpha)}^2 = \frac{(2 + \alpha)^2 |\gamma'(t)|^2}{(1 - |\gamma(t)|^2)^2} \int_D |w|^2 dA_\alpha(w).
\]

Direct calculation of the integral gives us

\[
\| (1 - P_{\gamma(t)}) \frac{d}{dt} k_{\gamma(t)}^\alpha \|_{L^2(D, dA_\alpha)}^2 = \frac{(2 + \alpha)|\gamma'(t)|^2}{(1 - |\gamma(t)|^2)^2}.
\]

Hence

\[
\| (1 - P_{\gamma(t)}) \frac{d}{dt} k_{\gamma(t)}^\alpha \|_{L^2(D, dA_\alpha)} = \frac{(2 + \alpha)^{\frac{1}{2}} |\gamma'(t)|}{1 - |\gamma(t)|^2} \frac{ds}{dt} = (2 + \alpha)^{\frac{1}{2}} \frac{ds}{dt}.
\]

Therefore the proof is complete. \(\Box\)

**Proposition 2.4.14** Let $\gamma : [0, 1] \rightarrow D$ be a smooth curve. Then

1. $\text{Re} \left( \frac{d}{dt} k_{\gamma(t)}^\alpha, k_{\gamma(t)}^\alpha \right) = 0$;
2. If $P_{\gamma(t)}$ is the orthogonal projection (that was defined as above), then

$$\text{Re} \left[ P_{\gamma(t)} \left( \frac{d}{dt} \kappa^\alpha_{\gamma(t)} \right)(w) \kappa^\alpha_{\gamma(t)}(w) \right] = 0;$$

3. $\| (f - \tilde{f}_\alpha(\gamma(t))) \kappa^\alpha_{\gamma(t)} \|_{L^2(D, dA_\omega)}^2 = [M_{\alpha}(f)(\gamma(t))]^2.$

**Proof.** (1) Since $k^\alpha_{\gamma(t)}$ is a unit vector in $L^2(D, dA_\omega)$, we have

$$\int_{D} |k^\alpha_{\gamma(t)}(w)|^2 dA_\omega(w) = 1.$$

Differentiating both sides with respect to $t$, we have

$$2 \left[ \text{Re} \int_{D} \left( \frac{d}{dt} k^\alpha_{\gamma(t)}(w) \overline{k^\alpha_{\gamma(t)}(w)} \right) dA_\omega(w) \right] = 0.$$

Thus the definition of inner product in $L^2(D, dA_\omega)$ gives us

$$2 \text{Re} \left< \frac{d}{dt} k^\alpha_{\gamma(t)}, k^\alpha_{\gamma(t)} \right>_{L^2(D, dA_\omega)} = 0.$$

Therefore (1) is done.

(2) For $w \in D$, the definition of $P_{\gamma(t)}$ gives us

$$P_{\gamma(t)} \left( \frac{d}{dt} k^\alpha_{\gamma(t)} \right)(w) = \left< P_{\gamma(t)} \frac{d}{dt} k^\alpha_{\gamma(t)}, k^\alpha_{\gamma(t)} \right>_{L^2(D, dA_\omega)} k^\alpha_{\gamma(t)}(w)$$

$$= \left< \frac{d}{dt} k^\alpha_{\gamma(t)}, k^\alpha_{\gamma(t)} \right>_{L^2(D, dA_\omega)} k^\alpha_{\gamma(t)}(w).$$

Multiplying both sides with $\overline{k^\alpha_{\gamma(t)}(w)}$ and comparing the real parts, we have

$$\text{Re} \left[ \left( P_{\gamma(t)} \left( \frac{d}{dt} k^\alpha_{\gamma(t)} \right)(w) \right) \overline{k^\alpha_{\gamma(t)}(w)} \right] = |k^\alpha_{\gamma(t)}(w)|^2 \text{Re} \left< \frac{d}{dt} k^\alpha_{\gamma(t)}, k^\alpha_{\gamma(t)} \right>_{L^2(D, dA_\omega)}.$$
By (1), the proof of (2) is complete.

(3) From the definition of norm in $L^2(D, dA_\alpha)$ we have

$$|||f - \tilde{f}_\alpha(\gamma(t))||_{L^2(D, dA_\alpha)}^2 = \int_D |f(w) - \tilde{f}_\alpha(\gamma(t))|^2 |k^\alpha_{\gamma(t)}(w)|^2 dA_\alpha(w).$$

Expand the integrand and integrate term by term separately to obtain

$$|||f - \tilde{f}_\alpha(\gamma(t))||_{L^2(D, dA_\alpha)}^2 = [\tilde{f}_\alpha(\gamma(t)) - |\tilde{f}_\alpha(\gamma(t))|^2$$

$$= [MO_\alpha(f)(\gamma(t))]^2.$$ 

\[ \square \]

**Proposition 2.4.15** If $\gamma : [0, 1] \to D$ is a smooth curve and $f \in BMO_{\theta, \alpha}$, then

$$\left| \left( \frac{d}{dt} \tilde{f}_\alpha \right)(\gamma(t)) \right| = 2\sqrt{2 + \alpha MO_\alpha(f)(\gamma(t))} \frac{ds}{dt}.$$

**Proof.** From the definition, we have

$$\tilde{f}_\alpha(\gamma(t)) = \int_D f(w) |k^\alpha_{\gamma(t)}(w)|^2 dA_\alpha(w).$$

Differentiating both sides with respect to $t$, we have

$$\left( \frac{d}{dt} \tilde{f}_\alpha \right)(\gamma(t)) = 2 \int_D f(w) Re \left[ \left( \frac{d}{dt} k^\alpha_{\gamma(t)} \right)(w)(\overline{k^\alpha_{\gamma(t)}(w)}) \right] dA_\alpha(w).$$

By Proposition 2.4.14 (2) we have

$$R = 2\tilde{f}_\alpha(\gamma(t)) Re \left[ P_{\gamma(t)} \left( \frac{d}{dt} k^\alpha_{\gamma(t)} \right)(w)(\overline{k^\alpha_{\gamma(t)}(w)}) \right] = 0.$$
Since $R = 0$, subtract $R$ from the inside of the integral to get

$$
\left( \frac{d}{dt} \tilde{f}_\alpha \right) (\gamma(t)) = 2 \int_D \left[ f(w) - \tilde{f}_\alpha(\gamma(t)) \right] \Re \left[ (1 - P_{\gamma(t)} \left( \frac{d}{dt} k_{\gamma(t)}^\alpha \right)(w)(\overline{k_{\gamma(t)}^\alpha(w)}) \right] dA_\alpha(w).
$$

Thus

$$
\left| \left( \frac{d}{dt} \tilde{f}_\alpha \right) (\gamma(t)) \right| \leq 2 \int_D |f(w) - \tilde{f}_\alpha(\gamma(t))||k_{\gamma(t)}^\alpha(w)| \left| (1 - P_{\gamma(t)} \left( \frac{d}{dt} k_{\gamma(t)}^\alpha \right)(w) \right| dA_\alpha(w).
$$

Apply Hölder's inequality to obtain

$$
\left| \left( \frac{d}{dt} \tilde{f}_\alpha \right) (\gamma(t)) \right| \leq 2 \left\| (f - \tilde{f}_\alpha(\gamma(t)))k_{\gamma(t)}^\alpha \right\|_{L^2(D,dA_\alpha)} \left\| (1 - P_{\gamma(t)} \left( \frac{d}{dt} k_{\gamma(t)}^\alpha \right)) \right\|_{L^2(D,dA_\alpha)}.
$$

The Propositions 2.4.14 (3) and 2.4.13 give us

$$
\left| \left( \frac{d}{dt} \tilde{f}_\alpha \right) (\gamma(t)) \right| \leq 2\sqrt{2 + \alpha |MO_\alpha(f)(\gamma(t))| \frac{ds}{dt}}.
$$

□

**Proposition 2.4.16** Let $z$ be a real number such that $z > 1$. Then there exists a constant $C_z > 0$ such that for $z \in D$

$$
|z|^x \int_0^1 \frac{dt}{(1 - t|z|)^{x + \frac{1}{2}}} \leq \frac{C_z}{(1 - |z|)^{\frac{z-1}{2}}}.
$$

**Proof.** Let

$$
I = |z|^x \int_0^1 \frac{dt}{(1 - t|z|)^{x + \frac{1}{2}}}
$$
Since $x > 1$, $\frac{\xi}{2} + \frac{1}{2} > 1$, direct calculation of the integral gives us

$$I = \left\| \frac{z^{x-1}}{\frac{\xi}{2} - \frac{1}{2}} \left[ 1 - (1 - |z|)^{\frac{\xi}{2} - \frac{1}{2}} \right] \right\|.$$

Use the fact that $z \in D$, and $x > 1$ to obtain

$$I \leq \frac{1}{(\frac{\xi-1}{2})(1 - |z|)^{\frac{\xi-1}{2}}}.$$

Put $C_x = \frac{2}{x-1}$. Then $C_x > 0$ and

$$I \leq \frac{C_x}{(1 - |z|)^{\frac{\xi-1}{2}}}.$$

\[ \square \]

**Proposition 2.4.17** Suppose $\alpha > -1$ and $f \in BMO_{\alpha, \alpha}$. Then there exists a constant $C_\alpha > 0$ independent of $f$ such that

$$\int_D |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(0)|dA_\alpha(z) \leq C_\alpha \int_D |MO_\alpha(f)(z)| \frac{dA_\alpha(z)}{|z|}.$$

**Proof.** Fix any $z \in D$ and let $\gamma(t) = tz, (0 \leq t \leq 1)$ be the geodesic in the Bergman metric from 0 to $z$. By Proposition 2.4.15 and $\frac{dz}{dt} = \frac{\frac{dz}{dt}}{1 - |tz|^2}$, we have

$$\left| \frac{d}{dt} \tilde{f}_\alpha(tz) \right| \leq 2\sqrt{2 + \alpha} MO_\alpha(f)(tz) \frac{|(tz)'|}{1 - |tz|^2}$$

$$= 2\sqrt{2 + \alpha} MO_\alpha(f)(tz) \frac{|z|}{1 - |tz|^2}.$$
Therefore the Fundamental Theorem of Calculus gives us

\[ |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(0)| \leq 2\sqrt{2 + \alpha} \int_0^1 \frac{(MO_\alpha(f)(tz))|z|}{1 - |tz|^2} \, dt. \]

Thus

\[ \int_\mathcal{D} |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(0)| dA_\alpha(z) \leq 2\sqrt{2 + \alpha} \int_\mathcal{D} \int_0^1 \frac{(MO_\alpha(f)(tz))|z|}{1 - |tz|^2} \, dt \, dA_\alpha(z). \]

By Fubini's theorem and the change of variable \( tz = w \), we have

\[ \int_\mathcal{D} |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(0)| dA_\alpha(z) \leq 2\sqrt{2 + \alpha(\alpha + 1)} \int_0^1 \int_\mathcal{D} \frac{1}{t^3(1 - |w|^2)} \left( 1 - \left| \frac{w}{t} \right|^2 \right)^\alpha dA(w) \, dt. \]

Fubini's theorem gives us

\[ \int_\mathcal{D} |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(0)| dA_\alpha(z) \leq 2\sqrt{2 + \alpha(\alpha + 1)} \int_\mathcal{D} \frac{[MO_\alpha(f)(w)]|w|}{(1 - |w|^2)} \int_0^1 \frac{1}{t^3} \left( 1 - \left| \frac{w}{t} \right|^2 \right)^\alpha \, dt \, dA(w). \]

Calculation of the second integral gives us

\[ \int_\mathcal{D} |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(0)| dA_\alpha(z) \leq 2\sqrt{2 + \alpha} \int_\mathcal{D} \frac{MO_\alpha(f)(w)(1 - |w|^2)^\alpha}{|w|} dA(w) \]

\[ = \frac{2\sqrt{2 + \alpha}}{\alpha + 1} \int_\mathcal{D} MO_\alpha(f)(w) \frac{dA_\alpha(w)}{|w|}. \]

Put \( C_\alpha = \frac{2\sqrt{2 + \alpha}}{\alpha + 1} \). Then \( C_\alpha > 0 \) and the proof is complete. \( \square \)
Proposition 2.4.18 Suppose $\alpha > -\frac{1}{2}, 1 < p < \infty$ and $f \in BMO_{\alpha, \alpha}$. Then there exists a constant $C_{\alpha, p} > 0$ independent of $f$ such that

$$\int_D |f_\alpha(z) - \tilde{f}_\alpha(0)|^p dA_\alpha(z) \leq C_{\alpha, p} \int_D |MO_\alpha(f)(z)|^p \frac{dA_\alpha(z)}{|z|}.$$ 

Proof. Let $1 < p < \infty$ and let $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. From the previous proof and using the fact that $|tz| < 1$, we have

$$|f_\alpha(z) - \tilde{f}_\alpha(0)| \leq 2\sqrt{2 + \alpha} \int_0^1 MO_\alpha(f)(tz) \frac{|z|}{1 - |tz|} dt.$$ 

Hölder's inequality gives us

$$|f_\alpha(z) - \tilde{f}_\alpha(0)| \leq 2\sqrt{2 + \alpha} \int_0^1 \frac{MO_\alpha(f)(tz)}{(1 - |tz|)^{1/p}} \frac{|z|}{(1 - |z|)^{1/2} + \frac{1}{2}} dt.$$ 

$$\leq 2\sqrt{2 + \alpha} \left( \int_0^1 \frac{|MO_\alpha(f)(tz)|^p}{(1 - t|z|)^{1/2}} dt \right)^{1/2} \left( \int_0^1 \frac{|z|^q dt}{(1 - |z|)^{1/2 + \frac{q}{2}}} \right)^{1/2}.$$ 

Apply Proposition 2.4.16 to the second integral. Then there exists a constant $C_q > 0$ such that

$$|f_\alpha(z) - \tilde{f}_\alpha(0)| \leq 2\sqrt{2 + \alpha} \frac{C_q}{(1 - |z|)^{1/2 - \frac{q}{2}}} \left( \int_0^1 \frac{|MO_\alpha(f)(tz)|^p}{(1 - |tz|)^{1/2}} dt \right)^{1/2}.$$ 

Hence

$$|f_\alpha(z) - \tilde{f}_\alpha(0)|^p \leq (2\sqrt{2 + \alpha} C_q)^p \frac{1}{(1 - |z|)^{1/2 - \frac{q}{2}}} \int_0^1 \frac{|MO_\alpha(f)(tz)|^p}{(1 - |tz|)^{1/2}} dt.$$ 

Fubini's theorem gives us

$$\int_D |f_\alpha(z) - \tilde{f}_\alpha(0)|^p dA_\alpha(z) \leq (2\sqrt{2 + \alpha} C_q)^p \int_0^1 \int_D \frac{|MO_\alpha(f)(tz)|^p}{(1 - |z|)^{1/2} (1 - t|z|)^{1/2}} dA_\alpha(z) dt.$$
Use the change of variable $tz = w$ to get

$$\int_{D} |\tilde{f}_{\alpha}(z) - \tilde{f}_{\alpha}(0)|^p dA_{\alpha}(z) \leq (\alpha + 1)(2\sqrt{2} + \alpha C_{q})^p \int_{0}^{1} \int_{D} \chi_{D}(w) [MO_{\alpha}(f)(w)]^p (1 - |w|^2)^{\alpha} \left(1 - \frac{|w|^2}{t^2}\right)^{\frac{\alpha}{2}} dA(w) dt.$$ 

Fubini's theorem gives us

$$\int_{D} |\tilde{f}_{\alpha}(z) - \tilde{f}_{\alpha}(0)|^p dA_{\alpha}(z) \leq 2^{\alpha}(\alpha + 1)(2\sqrt{2} + \alpha C_{q})^p \int_{D} \frac{[MO_{\alpha}(f)(w)]^p}{(1 - |w|)^{\frac{\alpha}{2}}} \int_{|w|}^{1} \frac{1}{t^2} \left(1 - \frac{|w|}{t}\right)^{\frac{\alpha}{2}} dtdA(w).$$

Integrate the second integral to obtain

$$\int_{D} |\tilde{f}_{\alpha}(z) - \tilde{f}_{\alpha}(0)|^p dA_{\alpha}(z) \leq \frac{2^{\alpha}(2\sqrt{2} + \alpha C_{q})^p}{(\alpha + \frac{1}{2})} \int_{D} [MO_{\alpha}(f)(w)]^p dA_{\alpha}(w).$$

Put $C_{\alpha,p} = \frac{2^{\alpha}(2\sqrt{2} + \alpha C_{q})^p}{(\alpha + \frac{1}{2})}$. Then $C_{\alpha,p} > 0$ and the proof is complete. \(\square\)

**Proposition 2.4.19** Let $p \geq 2$ and $f \in BMO_{S_{\alpha}}$. If $H_{f,\alpha}$ and $H_{f,\alpha}$ are both in $S_{p}$, then $MO_{\alpha}(f)(z) = (|f|_{\alpha}(z) - |\tilde{f}_{\alpha}(z)|)^{\frac{1}{2}}$ is in $L^p(D, d\lambda)$.

**Proof.** Suppose $H_{f,\alpha}$ and $H_{f,\alpha}$ are in $S_{p}$. When $H_{f,\alpha}$ is in $S_{p}$, Theorem 2.4.6 implies that $(H_{f,\alpha}^* H_{f,\alpha})^p$ is in $S_{1}$, and Proposition 2.3.11 gives us that

$$\text{trace}(H_{f,\alpha}^* H_{f,\alpha})^p = \int_{D} \langle (H_{f,\alpha}^* H_{f,\alpha})^p k_{z}^{\alpha}, k_{z}^{\alpha} \rangle_{L^2(D, dA_{\alpha})} d\lambda(z) < \infty.$$ 

Since $p \geq 2$ and $k_{z}^{\alpha}$ is a unit vector in $L^2_{\alpha}(D, dA_{\alpha})$, Proposition 2.4.10 gives us

$$\int_{D} \langle (H_{f,\alpha}^* H_{f,\alpha}) k_{z}^{\alpha}, k_{z}^{\alpha} \rangle_{L^2(D, dA_{\alpha})} d\lambda(z) \leq \text{trace}(H_{f,\alpha}^* H_{f,\alpha})^p < \infty.$$
Thus

$$\int_{D} \| H_{f,\alpha} k_\alpha \|_{L^p(D,dA_\alpha)}^p d\lambda(z) < \infty.$$

Similarly, when $H_{f,\alpha}$ is in $S_p$ we have

$$\int_{D} \| H_{f,\alpha} k_\alpha \|_{L^p(D,dA_\alpha)}^p d\lambda(z) < \infty.$$

Since $f \in BMO_{\beta,\alpha}$, by the proof of Theorem 2.2.32 for $z \in D$, we have

$$MO_\alpha(f)(z) \leq \| H_{f,\alpha} k_\alpha \|_{L^1(D,dA_\alpha)} + \| H_{f,\alpha} k_\alpha \|_{L^2(D,dA_\alpha)}.$$

Hence

$$\left( \int_{D} [MO_\alpha(f)(z)]^p d\lambda(z) \right)^{\frac{1}{p}} \leq \left( \int_{D} \left( \| H_{f,\alpha} k_\alpha \|_{L^p(D,dA_\alpha)} + \| H_{f,\alpha} k_\alpha \|_{L^2(D,dA_\alpha)} \right)^p d\lambda(z) \right)^{\frac{1}{p}}.$$

By applying Minkowski's inequality, we have

$$\left( \int_{D} [MO_\alpha(f)(z)]^p d\lambda(z) \right)^{\frac{1}{p}} \leq \left( \int_{D} \left( \| H_{f,\alpha} k_\alpha \|_{L^p(D,dA_\alpha)} \right)^p d\lambda(z) \right)^{\frac{1}{p}}$$

$$+ \left( \int_{D} \left( \| H_{f,\alpha} k_\alpha \|_{L^2(D,dA_\alpha)} \right)^p d\lambda(z) \right)^{\frac{1}{p}} < \infty.$$

Therefore the proof is complete. \qed

In order to show the converse of Proposition 2.4.19 we need the method of complex interpolation that was motivated by the Riesz-Thorin interpolation theorem. The complex interpolation method was mainly developed by Calderon [Calderon 64]. For further references and historical remarks see [Peetre 85], [BL 76] or [Zhu 90]. See Zhu for the following definitions and theorems 2.4.20 to 2.4.23.
1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. We say that $\mathcal{X}$ and $\mathcal{Y}$ are compatible provided there exists a Hausdorff topological linear space $\mathcal{Z}$ containing both of them.

2. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be compatible Banach spaces. A Banach space $\mathcal{X}$ is called an intermediate space between $\mathcal{X}_1$ and $\mathcal{X}_2$ if

$$\mathcal{X}_1 \cap \mathcal{X}_2 \subset \mathcal{X} \subset \mathcal{X}_1 + \mathcal{X}_2$$

with continuous inclusions.

3. An intermediate space $\mathcal{X}$ between $\mathcal{X}_1$ and $\mathcal{X}_2$ is called interpolation space provided any linear mapping on $\mathcal{X}_1 + \mathcal{X}_2$ which is bounded from $\mathcal{X}_1$ into $\mathcal{X}_1$ and bounded from $\mathcal{X}_2$ into $\mathcal{X}_2$ is also bounded from $\mathcal{X}$ into $\mathcal{X}$.

Let $S$ be the open strip in the complex plane consisting of points with real parts between 0 and 1. Let $\overline{S}$ be the corresponding closed strip. Then we have the following theorem:

**Theorem 2.4.20** Let $f(z)$ be a bounded continuous function on $\overline{S}$ which is analytic in the open strip $S$. For any $\theta \in [0, 1]$, let

$$M_\theta = \sup \{|f(\theta + iy)| : |y| < \infty\}.$$

Then

$$M_\theta \leq M_0^{1-\theta} M_1^\theta.$$

Let $\mathcal{X}$ be a Banach space. We say that a function $f : S \rightarrow \mathcal{X}$ is analytic if for any bounded linear functional $\zeta$ on $\mathcal{X}$, the composite function $\zeta \circ f$ is analytic in the usual sense.

Now we are ready to introduce the complex method of interpolation.
Suppose $\mathcal{X}_0$ and $\mathcal{X}_1$ are compatible Banach spaces. Let $F(\mathcal{X}_0, \mathcal{X}_1)$ be the space of all functions $f$ from $\mathcal{F}$ into $\mathcal{X}_0 + \mathcal{X}_1$ with the following properties:

1. $f$ is bounded and continuous on $\mathcal{F}$ and $f$ is analytic in $\mathcal{S}$;

2. $y \mapsto f(k + iy), \ (k = 0, 1)$ are continuous from the real line into $\mathcal{X}_k$.

Then $F(\mathcal{X}_0, \mathcal{X}_1)$ is a vector space. $F(\mathcal{X}_0, \mathcal{X}_1)$ with the norm

$$\|f\|_F = \max\{ \sup_x \|f(iz)\|_{\mathcal{X}_0}, \sup_x \|f(1 + iz)\|_{\mathcal{X}_1} \}$$

becomes a Banach space. Given $0 \leq \theta \leq 1$, let $\mathcal{X}_\theta$ be the space of vectors $v$ in $\mathcal{X}_0 + \mathcal{X}_1$ such that $v = f(\theta)$ for some $f$ in $F(\mathcal{X}_0, \mathcal{X}_1)$. Let

$$\|v\|_\theta = \inf\{ \|f\|_F : v = f(\theta) \}.$$

Then $\|v\|_\theta$ is a norm in $\mathcal{X}_\theta$. With this construction we have the following results.

**Theorem 2.4.21** If $\mathcal{X}_0$ and $\mathcal{X}_1$ are compatible Banach spaces and $0 < \theta < 1$, then

1. $\mathcal{X}_\theta$ is a Banach space;

2. $\mathcal{X}_\theta$ is an interpolation space between $\mathcal{X}_0$ and $\mathcal{X}_1$;

3. the construction of $\mathcal{X}_\theta$ is functional in the sense that if $\mathcal{X}_0, \mathcal{X}_1$ and $\mathcal{Y}_0, \mathcal{Y}_1$ are compatible pairs of Banach spaces and $T : \mathcal{X}_0 + \mathcal{X}_1 \rightarrow \mathcal{Y}_0 + \mathcal{Y}_1$ is a bounded linear mapping such that $T$ maps $\mathcal{X}_k$ boundedly into $\mathcal{Y}_k$, $(k = 0, 1)$, then $T$ maps $\mathcal{X}_\theta$ boundedly into $\mathcal{Y}_\theta$ for all $0 < \theta < 1$.

Note that we will write $[\mathcal{X}_0, \mathcal{X}_1]_\theta = \mathcal{X}_\theta$. 

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**Theorem 2.4.22** Suppose $\mu$ is a positive Borel measure on a locally compact topological space $\mathcal{X}$ and $L^p = L^p(\mathcal{X}, d\mu)$, for $1 \leq p \leq \infty$. Then

$$[L^{p_0}, L^{p_1}]_\theta = L^p,$$

(with equal norms) for all $1 \leq p_0 < p_1 \leq \infty$ and $\theta \in (0, 1)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

**Theorem 2.4.23** For $1 \leq p < \infty$, let $S_p$ be the Schatten p-class of a Hilbert space and let $S_\infty$ be the space of all bounded linear operators on the Hilbert space. Then we have

$$[S_{p_0}, S_{p_1}]_\theta = S_p,$$

(with equal norms) for all $1 \leq p_0 < p_1 \leq \infty$ and all $\theta \in (0, 1)$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

The next two propositions have been proven in the case of the usual Bergman space in [Zhu 90]. The proof for the weighted Bergman space is similar to the one in [Zhu 90].

**Proposition 2.4.24** Suppose $A_G$ is the operator on $L^2(\mathcal{D}, dA_\alpha)$ defined by

$$A_G f(z) = \int_{\mathcal{D}} G(z, w) K_\alpha(z, w) f(w) dA_\alpha(w),$$

where $G(z, w)$ is a measurable function on $\mathcal{D} \times \mathcal{D}$ and $K_\alpha(z, w)$ is the Bergman kernel of $\mathcal{D}$. If

$$\int_{\mathcal{D}} \int_{\mathcal{D}} |G(z, w)|^2 |K_\alpha(z, w)|^2 dA_\alpha(z) dA_\alpha(w) < \infty,$$

then $A_G$ is in $S_2$ of $L^2(\mathcal{D}, dA_\alpha)$.  

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Proof. It is a standard that a square integrable kernel induces a Hilbert-Schmidt integral operator.

\[ \text{Proposition 2.4.25} \]

Suppose \( p \geq 2 \) and \( A_G \) is the operator on \( L^2(D, dA_\alpha) \) defined by

\[ A_G f(z) = \int_D G(z, w) K^\alpha(z, w) f(w) dA_\alpha(w), \]

where \( G(z, w) \) is a measurable function on \( D \times D \) and \( K^\alpha(z, w) \) is the Bergman kernel of \( D \). If

\[ \int_D \int_D |G(z, w)|^p |K^\alpha(z, w)|^2 dA_\alpha(z) dA_\alpha(w) < \infty, \]

then \( A_G \) is in \( S_p \) of \( L^2(D, dA_\alpha) \).

Proof. Let \( \Omega = D \times D \) and \( d\mu(z) = |K^\alpha(z, w)|^2 dA_\alpha(z) dA_\alpha(w) \). If \( G(z, w) \in L^{\infty}(\Omega, d\mu) \), then for all \( f \in L^2(D, dA_\alpha) \)

\[ |A_G f(z)| \leq \|G\|_\infty \int_D |K^\alpha(z, w)| |f(w)| dA_\alpha(w). \]

Thus

\[ \|A_G f\|_{L^2(D, dA_\alpha)} \leq \|G\|_\infty \|T\| \|f\|_{L^2(D, dA_\alpha)}, \]

where \( T \) is the operator in Proposition 2.2.11. Therefore \( A_G \) is a bounded operator on \( L^2(D, dA_\alpha) \).

Consider the mapping \( F : L^2(\Omega, d\mu) + L^\infty(\Omega, d\mu) \to B(L^2(D, dA_\alpha)) \) defined by \( F(G) = A_G \), where \( B(L^2(D, dA_\alpha)) = S_\infty(L^2(D, dA_\alpha)) \) is the full algebra of all bounded linear operators on \( L^2(D, dA_\alpha) \). It follows from Proposition 2.4.24 that \( F : L^2(\Omega, d\mu) \to S_2 \) and with the above argument we have \( F : L^\infty(\Omega, d\mu) \to S_\infty \). Hence \( F \) is a bounded linear mapping.
By Theorems 2.4.22 and 2.4.23 we have

\[ [L^2, L^\infty]_\theta = L^p, \text{ and } [S_2, S_\infty]_\theta = S_p, \]

where \( \theta \in (0, 1) \) and

\[
\frac{1}{p} = \frac{1 - \theta}{2} + \frac{\theta}{\infty} = \frac{1 - \theta}{2}.
\]

Therefore \( 2 \leq p < \infty \).

Thus Theorem 2.4.21 implies that \( F \) maps \( L^p(\Omega, d\mu) \) boundedly into \( S_p \) for all \( 2 \leq p < \infty \).

\( \square \)

**Proposition 2.4.26** Suppose \( \alpha > -1 \). Then

\[
\int_D \frac{|k_\theta^\alpha(z)|^2}{|\Phi_\omega(z)|} dA_\alpha(z) = 1.
\]

**Proof.** Let

\[
I = \int_D \frac{|k_\theta^\alpha(z)|^2}{|\Phi_\omega(z)|} dA_\alpha(z).
\]

Then use the change of variable formula with \( z = \Phi_\omega(u) \), we have

\[
I = \int_D \frac{dA_\alpha(u)}{|u|}
= \frac{(\alpha + 1)}{\pi} \int_0^{2\pi} \int_0^1 (1 - r^2)^\alpha dr d\theta
= (\alpha + 1)B(1, \alpha + 1),
\]
where $B$ is the beta function. The relationship between the beta and gamma functions gives us

$$I = (\alpha + 1) \frac{\Gamma(1)\Gamma(\alpha + 1)}{\Gamma(\alpha + 2)} = 1.$$ 

\[\square\]

**Proposition 2.4.27** Suppose $p \geq 2, \alpha > -\frac{1}{2}, f \in BMO_{B_\alpha} \text{ and } MO_\alpha(f) \text{ is in } L^p(D, d\lambda).$

Then

$$\left[ \int_D |\tilde{f}_\alpha(\Phi_z(0)) - \tilde{f}_\alpha(\Phi_z(u))|^2 dA_\alpha(u) \right]^{\frac{1}{2}}$$

is in $L^p(D, d\lambda)$.

**Proof.** Let $p > 2$. Consider

$$I = \int_D \left[ \int_D |\tilde{f}_\alpha(\Phi_z(0)) - \tilde{f}_\alpha(\Phi_z(u))|^2 dA_\alpha(u) \right]^{\frac{1}{2}} d\lambda(z).$$

Let $s = \frac{p}{2}$ and $t = \frac{p}{p-2}$. Then $\frac{1}{s} + \frac{1}{t} = 1$. By an application of Hölder's inequality to the inside integral with the conjugate pair $s$ and $t$, we have

$$I \leq \int_D \int_D |\tilde{f}_\alpha(\Phi_z(0)) - \tilde{f}_\alpha(\Phi_z(u))|^p dA_\alpha(u) d\lambda(z).$$

Use the facts that $\tilde{f}_\alpha \circ \Phi_z = (f \circ \Phi_z)_\alpha, \ MO_\alpha(f \circ \Phi_z)(u) = MO_\alpha(f)(\Phi_z(u))$ and Proposition 2.4.18 to obtain

$$I \leq C_{\alpha, p} \int_D \int_D [MO_\alpha(f)(\Phi_z(u))]^p \frac{dA_\alpha(u)}{|u|} d\lambda(z).$$

Use the change of variable formula with $u = \Phi_z(w)$ to obtain

$$I \leq C_{\alpha, p} \int_D \int_D [MO_\alpha(f)(w)]^p \frac{|k_\alpha(z)|^2}{|\Phi_w(z)|} d\lambda(w) dA_\alpha(z).$$
Fubini's theorem and Proposition 2.4.26 imply that $I < \infty$.

In the case of $p = 2$, we don't need to apply the H"older inequality. The rest of the proof will be similar to the case of $p > 2$. Hence the proposition is complete. \hfill \square

**Proposition 2.4.28** Suppose $p \geq 2, \alpha > -\frac{1}{2}$, $f \in BMO_{0,\alpha}$ and $MO_{\alpha}(f)(z)$ is in $L^p(D, d\lambda(z))$.

Then

1. $H_{(\tilde{f}_\alpha)_{\alpha}}$ is in $S_p$;
2. $H_{(f - \tilde{f}_\alpha)_{\alpha}}$ is in $S_p$.

**Proof.** (1) Let $g \in L^2(D, dA_\alpha)$ and $z \in D$. Then

$$H_{(\tilde{f}_\alpha)_{\alpha}} g(z) = (1 - P^\alpha)(\tilde{f}_\alpha g)(z)$$

$$= \tilde{f}_\alpha(z)g(z) - P^\alpha(\tilde{f}_\alpha g)(z)$$

$$= \tilde{f}_\alpha(z)g(z) - \langle \tilde{f}_\alpha g, K^\alpha_z \rangle$$

$$= \tilde{f}_\alpha(z) \int_D [\tilde{f}_\alpha(z) - \tilde{f}_\alpha(w)]g(w)K^\alpha(z, w)dA_\alpha(w).$$

By Proposition 2.4.25, to prove that $H_{(\tilde{f}_\alpha)_{\alpha}} \in S_p$ it suffices to show that

$$I = \int_D \int_D |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(w)|^2|K^\alpha(z, w)|^2 dA_\alpha(w)dA_\alpha(z) < \infty.$$

Since $d\lambda(z) = \|K^\alpha_z\|^2 dA_\alpha(z)$, we have

$$I = \int_D \int_D |\tilde{f}_\alpha(z) - \tilde{f}_\alpha(w)|^2|k^\alpha_z(w)|^2 dA_\alpha(w)d\lambda(z).$$
Use the change of variable formula with \( w = \Phi_z(u) \) and the facts that \( \Phi_z(0) = z \), \( \tilde{f}_\alpha \circ \Phi_z = (f \circ \Phi_z)_\alpha \) to obtain

\[
I = \int_D \int_D |\tilde{f}_\alpha(\Phi_z(0)) - \tilde{f}_\alpha(\Phi_z(u))|^p dA_\alpha(u) d\lambda(z)
\]

\[
= \int_D \int_D |(f \circ \Phi_z)_\alpha(0) - (f \circ \Phi_z)_\alpha(u)|^p dA_\alpha(u) d\lambda(z).
\]

Apply Proposition 2.4.18 to the inside integral and use the invariance of the Berezin transform to see that there exists a constant \( C_{\alpha,p} > 0 \) such that

\[
I \leq C_{\alpha,p} \int_D \int_D [MO_\alpha(f \circ \Phi_z)(u)]^p \frac{dA_\alpha(u)}{|u|} d\lambda(z).
\]

A simple calculation gives us that

\[
MO_\alpha(f \circ \Phi_z)(u) = MO_\alpha(f)(\Phi_z(u)).
\]

Thus

\[
I \leq C_{\alpha,p} \int_D \int_D [MO_\alpha(f)(\Phi_z(u))]^p \frac{dA_\alpha(u)}{|u|} d\lambda(z).
\]

Changing the variable with \( u = \Phi_z(w) \), using \( d\lambda = \|A_\alpha^w\|^2 dA_\alpha(z) \) and using Fubini’s theorem, we obtain

\[
I \leq C_{\alpha,p} \int_D [MO_\alpha(f)(w)]^p \left( \int_D \frac{|k_\alpha^w(z)|^2}{|\Phi_w(z)|} dA_\alpha(z) \right) d\lambda(w).
\]

By Proposition 2.4.26, we have

\[
I \leq C_{\alpha,p} \int_D [MO_\alpha(f)(w)]^p d\lambda(w).
\]
Now use the fact that $MO_{\alpha}(f) \in L^p(D, d\lambda_{\alpha})$ to obtain $I < \infty$. This is what was to be proved.

(2) Let $g = f - \bar{f}_{\alpha}$. Since $f \in BMO_{\beta, \alpha}$, Propositions 2.2.28 and 2.2.6 give us that $g \in BMO_{\beta, \alpha}$. By Proposition 2.3.18, we have

$$H_{g, \alpha}^* H_{g, \alpha} = T_{|g|^2, \alpha} - T_{\beta, \alpha} T_{\beta, \alpha} \leq T_{|g|^2, \alpha}.$$

By Theorem 2.4.6, to show $H_{g, \alpha} \in S_p$ it suffices to show $H_{g, \alpha}^* H_{g, \alpha} \in S_{\frac{p}{2}}$. By the relationship between Hankel operators and Toeplitz operators as above it is enough to show that $T_{|g|^2, \alpha}$ is in $S_{\frac{p}{2}}$.

Consider

$$\left[|g|^2_{\alpha}(z)\right]^\frac{1}{2} = \left[\int_D |f(w) - \bar{f}_{\alpha}(w)|^2|k_{\alpha}^2(w)|^2 dA_{\alpha}(w)\right]^\frac{1}{2}$$

$$= \left[\int_D |f(w) - \bar{f}_{\alpha}(z) + \bar{f}_{\alpha}(z) - \bar{f}_{\alpha}(w)|^2|k_{\alpha}^2(w)|^2 dA_{\alpha}(w)\right]^\frac{1}{2}.$$

The triangle and Minkowski’s inequalities give us

$$\left[|g|^2_{\alpha}(z)\right]^\frac{1}{2} \leq \left[\int_D |k_{\alpha}^2(w)|^2|f(w) - \bar{f}_{\alpha}(z)|^2 dA_{\alpha}(w)\right]^\frac{1}{2}$$

$$+ \left[\int_D |k_{\alpha}^2(w)|^2|\bar{f}_{\alpha}(z) - \bar{f}_{\alpha}(w)|^2 dA_{\alpha}(w)\right]^\frac{1}{2}$$

$$= I_1 + I_2,$$

where

$$I_1 = \left[\int_D |k_{\alpha}^2(w)|^2|f(w) - \bar{f}_{\alpha}(z)|^2 dA_{\alpha}(w)\right]^\frac{1}{2}.$$
and
\[
I_2 = \left[ \int_D |k^\alpha_z(w)|^2 |\vec{f}_\alpha(z) - \vec{f}_\alpha(w)|^2 dA_\alpha(w) \right]^{\frac{1}{2}}.
\]

Consider
\[
I_1 = \left[ \int_D |k^\alpha_z(w)|^2 |f(w) - \vec{f}_\alpha(z)|^2 dA_\alpha(w) \right]^{\frac{1}{2}}.
\]

Expanding the integrand and integrating each term separately to obtain
\[
I_1 = \left[ |f|^2_\alpha(z) - |\vec{f}_\alpha(z)|^2 \right]^{\frac{1}{2}} = MO_\alpha(f)(z).
\]

Now consider
\[
I_2 = \left[ \int_D |k^\alpha_z(w)|^2 |\vec{f}_\alpha(z) - \vec{f}_\alpha(w)|^2 dA_\alpha(w) \right]^{\frac{1}{2}}.
\]

Using the change of variable formula with \( w = \Phi_z(u) \), we have
\[
I_2 = \left[ \int_D |\vec{f}_\alpha(\Phi_z(u)) - \vec{f}_\alpha(\Phi_z(0))|^2 dA_\alpha(u) \right]^{\frac{1}{2}}.
\]

Thus
\[
\left[ |g|^2_\alpha(z) \right]^{\frac{1}{2}} \leq MO_\alpha(f)(z) + \left[ \int_D |\vec{f}_\alpha(\Phi_z(0)) - \vec{f}_\alpha(\Phi_z(u))|^2 dA_\alpha(u) \right]^{\frac{1}{2}}.
\]

Now using the fact that \( MO_\alpha(f) \in L^p(D, d\lambda) \) and Proposition 2.4.27, we have \( |g|^2_\alpha \in L^p(D, d\lambda) \) or \( |g|^2_\alpha \in L^\infty(D, d\lambda) \). Since \( 1 \leq \frac{p}{2} < \infty \), \( r > 0 \) and \( |g|^2 \) is a nonnegative function on \( D \), it follows from Corollary 2.4.12 that the operator \( T_{g^2} \) is in the Schatten class \( S_{\frac{p}{2}} \). Consequently \( H_{g, \alpha} \in S_p \). Therefore (2) is proved.

\[\square\]

**Proposition 2.4.29** Let \( p \geq 2 \), \( \alpha > -\frac{1}{2} \) and \( f \in BMO_{\theta, \alpha} \). If \( MO_\alpha(f)(z) = (|f|^2_\alpha(z) - |\vec{f}_\alpha(z)|^2)^{\frac{1}{2}} \) is in \( L^p(D, d\lambda) \), then \( H_{f, \alpha} \) and \( H_{f, \alpha} \) both are in \( S_p \).
Proof. Suppose \( MO_\alpha(f) \in L^p(D, dm) \). Put \( f = \tilde{f}_\alpha + f - \tilde{f}_\alpha \). Consider

\[
\|H_{f,\alpha}\|_{S_p} = \|H_{(f + f - \tilde{f}_\alpha),\alpha}\|_{S_p}
= \|H_{(f + f - \tilde{f}_\alpha),\alpha}\|_{S_p} + \|H_{(f - \tilde{f}_\alpha),\alpha}\|_{S_p} < \infty.
\]

The last inequality follows from Proposition 2.4.28. Therefore \( H_{f,\alpha} \) is in \( S_p \). Similarly, we also have \( H_{f,\alpha} \) is in \( S_p \). \( \square \)

Propositions 2.4.19 and 2.4.29 establish the following theorem.

**Theorem 2.4.30** Let \( p \geq 2, \alpha > -\frac{1}{2} \) and \( f \in BMO_{\beta, \alpha} \). Then \( H_{f, \alpha} \) and \( H_{\tilde{f}, \alpha} \) are both in \( S_p \) if and only if \( MO_\alpha(f)(z) = (|f|^2_{\alpha}(z) - |f_\alpha(z)|^2)^{\frac{1}{2}} \) is in \( L^p(D, dm) \).

Note that by carefully examining the proof of Propositions 2.4.19, 2.4.28 and 2.4.29, we will see that there is a constant \( C > 0 \) (depending on \( \alpha \) and \( p \)) such that

\[
C^{-1}\|MO_\alpha(f)\|_{L^p(D, dm)} \leq \|H_{f,\alpha}\|_{S_p} + \|H_{\tilde{f},\alpha}\|_{S_p} \leq \|MO_\alpha(f)\|_{L^p(D, dm)}.
\]

We have seen in this section that the Schatten p classes Hankel operators on the weighted Bergman space \( L^2_\alpha(D, dA_\alpha) \) for \( p \geq 2 \) and \( \alpha > -\frac{1}{2} \) have properties parallel to the Schatten p-classes Hankel operators on the usual Bergman space \( L^2(D, dA) \). The case \( -1 < \alpha \leq -\frac{1}{2} \) is an open question.
Chapter 3

Little Hankel Operators on

Weighted Bergman spaces

In this chapter we will study the so-called little Hankel operators on the weighted Bergman space $L^2_\omega(D, dA_\omega)$. We will characterize the symbol functions which make these operators bounded, compact, or Schatten p-class. The little Hankel operators have been considered briefly by Coifman, Rochberg and Weiss [CRW 76], and by Axler, Conway and McDonald [ACM 82] and more fully by Peetre [Peetre 82]. Among other papers dealing with the little Hankel operators are [JPR 87], [AFP 88], [Zhu 88], [Zhu 90] and [Zhu 91].

We are now going to define the little Hankel operator. First we review the development of this operator.

Power, [Power 82], has defined the Hankel operators on the Hardy space as follows. Let $L^2$ be the space of Lesbegue square integrable functions on the unit circle, $\{z^n : n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2$. Let $H^2$ be the Hardy space, i.e. the span of $\{z^n : n = 0, 1, 2, \ldots\}$. Define the unitary operator $J$ on $L^2$ by $Jf(z) = f(\bar{z})$. Let $P$ be the orthogonal projection of $L^2$ onto $H^2$, and for $\varphi \in L^\infty$, let $M_\varphi$ be the operator of multiplication by $\varphi$ on $L^2$. Then the Hankel operator is $h_\varphi = PJM_\varphi|_{H^2}$. Observe that the matrix of $h_\varphi$ relative to the basis $\{z^n\}$ is a Hankel matrix as defined in Chapter 1.
Bonsall [Bonsall 86] defines the little Hankel operator on the usual Bergman space $L^2_\alpha(\mathbb{D}, dA)$ as follows: Given $f \in L^2(\mathbb{D}, dA)$, let $h_\varphi$ denote the operator in $L^2_\alpha(\mathbb{D}, dA)$ with domain $H^\infty$ defined by
\[ h_\varphi f = PJ(\varphi f), \quad f \in H^\infty, \]
where $J$ is the self-adjoint unitary operator on $L^2(\mathbb{D}, dA)$ given by $(Jf)(z) = f(\bar{z})$ and $P$ is the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L^2_\alpha(\mathbb{D}, dA)$. The operator $h_\varphi$ is a Hankel operator in $L^2_\alpha(\mathbb{D}, dA)$.

Arazy, Fisher and Peetre [AFP 88], defined the little Hankel operators on the weighted Bergman space $L^2(\mathbb{D}, dA_\alpha)$ as follows: The little Hankel operator with symbol $f$ is defined by
\[ h_{f,\alpha}(u) = \overline{P^\alpha f} \overline{P^\alpha u}, \quad u \in L^2(\mathbb{D}, dA_\alpha), \]
where $P^\alpha$ is the orthogonal projection of $L^2(\mathbb{D}, dA_\alpha)$ onto $L^2_\alpha(\mathbb{D}, dA_\alpha)$ and $\overline{P^\alpha}$ is the orthogonal projection of $L^2(\mathbb{D}, dA_\alpha)$ onto $\overline{L^2_\alpha(\mathbb{D}, dA_\alpha)}$, the space of complex conjugates of elements of $L^2_\alpha(\mathbb{D}, dA_\alpha)$.

In this paper we will define the little Hankel operators as follows:

**Definition 3.0.1** Let $f \in L^2(\mathbb{D}, dA_\alpha)$. The little Hankel operator $h_{f,\alpha}$ with symbol $f$ is the operator densely defined on $L^2_\alpha(\mathbb{D}, dA_\alpha)$ into $\overline{L^2_\alpha(\mathbb{D}, dA_\alpha)}$ by
\[ h_{f,\alpha}g = \overline{P^\alpha}(fg) \]
for $g \in L^2_\alpha(\mathbb{D}, dA_\alpha)$.

Note that the little Hankel operator $h_{f,\alpha}$ in definition 3.0.1 is the same as $h_{f,\alpha}$ in [AFP 88].
Let $P_0$ be the rank one projection (onto the constants). Then it is easy to see that $P_0 - P_0 \leq 1 - P^\alpha$. (We have seen the Hankel operators $H_{f,\alpha}$ and $h_{f,\alpha}$ defined by $H_{f,\alpha}g = (1 - P^\alpha)(fg)$, and $h_{f,\alpha}g = P_0(\omega g)$, it is because $P_0 - P_0 \leq 1 - P^\alpha$, that we call $h_{f,\alpha}$ the little Hankel operator.) The above also makes it clear that $h_{f,\alpha}$ is bounded (or compact) if $H_{f,\alpha}$ is. One advantage of working with the little Hankel operator $h_{f,\alpha}$ is that it only depends on the analytic part of $f$, in other words we always have $h_{f,\alpha} = h_{P_0 f,\alpha}$.

3.1 Bounded Little Hankel Operators on Weighted Bergman Spaces

In this section we will determine the properties of the symbol function $f$ that make the little Hankel operator $h_{f,\alpha}$ bounded.

We will start with an obvious property.

Proposition 3.1.1 If $f$ is in $L^\infty(D, dA_\alpha)$, then $h_{f,\alpha}$ and $h_{f,\alpha}$ are bounded.

Proof. Suppose $f$ is in $L^\infty(D, dA_\alpha)$. Then for all $g \in L^2(D, dA_\alpha)$

$$h_{f,\alpha}g = P_0(\omega g).$$

Thus

$$\|h_{f,\alpha}g\|_{L^2(D, dA_\alpha)} \leq \|fg\|_{L^2(D, dA_\alpha)} \leq \|f\|_\infty \|g\|_{L^2(D, dA_\alpha)}.$$

Therefore $\|h_{f,\alpha}\| \leq \|f\|_\infty$. Hence $h_{f,\alpha}$ is bounded. Similarly we have $h_{f,\alpha}$ is bounded. □
Proposition 3.1.2 1. For all $g$ in $L^2(D, dA_\alpha)$ and $z \in D$, we have

$$P_\alpha g(z) = \int_D K_\alpha(z, w)g(w)dA_\alpha(w).$$

2. Let $f \in L^2(D, dA_\alpha)$. Then for all $g$ in $H^\infty(D, dA_\alpha)$, we have

$$h_{f, \alpha}g(z) = \int_D K_\alpha(w, z)f(w)g(w)dA_\alpha(w)$$

$$= \int_D \frac{f(w)g(w)}{(1 - \bar{z}w)^{2+\alpha}}dA_\alpha(w).$$

Proof. (1) Let $g \in L^2(D, dA_\alpha)$. Then $g = P_\alpha g + h$, where $h \in L^2_0(D, dA_\alpha)$ and $h(0) = 0$. Thus $\overline{g} = (P_\alpha g) + \bar{h}$, where $P_\alpha g \in L^2_0(D, dA_\alpha)$, and $\bar{h} \perp L^2_0(D, dA_\alpha)$. Therefore

$$\overline{(P_\alpha g)(z)} = \langle P_\alpha g, K_\alpha^\alpha \rangle_{L^2_0(D, dA_\alpha)}$$

$$= \langle g, K_\alpha^\alpha \rangle_{L^2(D, dA_\alpha)}.$$

Hence

$$P_\alpha g(z) = \langle K_\alpha^\alpha, \overline{g} \rangle_{L^2(D, dA_\alpha)}$$

$$= \int_D K_\alpha(w, z)g(w)dA_\alpha(w)$$

$$= \int_D \frac{f(w)g(w)}{(1 - \bar{z}w)^{2+\alpha}}dA_\alpha(w).$$

(2) Let $g$ be in $H^\infty(D, dA_\alpha), z \in D$. Then we have

$$h_{f, \alpha}g(z) = P_\alpha(fg)(z).$$
Proposition 3.1.2 and the fact that $\overline{K^\alpha(z, w)} = K^\alpha(w, z)$ gives us

\[
h_{f, \alpha} g(z) = \int_D K^\alpha(w, z) f(w) g(w) dA_\alpha(w)
\]

\[
= \int_D \frac{f(w) g(w)}{(1 - \overline{z} w)^{2+\alpha}} dA_\alpha(w).
\]

Use the fact that $H^\infty(D, dA_\alpha)$ is dense in $L^2_\alpha(D, dA_\alpha)$ to obtain the desired result.

\[\square\]

Proposition 3.1.2 shows that the little Hankel operator $h_{f, \alpha}$ is an integral operator.

**Proposition 3.1.3** If $f \in L^2(D, dA_\alpha)$, then $h_{f, \alpha} = h_{(P_\alpha f), \alpha}$ in the sense that

\[
h_{f, \alpha} g = h_{(P_\alpha f), \alpha} g \quad \text{for all } g \in H^\infty(D, dA_\alpha).
\]

**Proof.** Suppose $u \in L^2_\alpha(D, dA_\alpha)$. Then for all $g \in H^\infty(D, dA_\alpha)$

\[
\langle h_{f, \alpha} g, u \rangle_{L^2(D, dA_\alpha)} = \langle P_\alpha(f g), u \rangle_{L^2(D, dA_\alpha)}
\]

\[
= \langle f g, u \rangle_{L^2(D, dA_\alpha)}
\]

\[
= \langle \bar{u} g, f \rangle_{L^2(D, dA_\alpha)}.
\]

Since $\bar{u} g \in L^2_\alpha(D, dA_\alpha)$, we have

\[
\langle h_{f, \alpha} g, u \rangle_{L^2(D, dA_\alpha)} = \langle \bar{u} g, P_\alpha f \rangle_{L^2(D, dA_\alpha)}
\]

\[
= \langle P_\alpha f g, u \rangle_{L^2(D, dA_\alpha)}.
\]
The definition of $h_{\mathbb{P}^\alpha f, \alpha}$ gives us

$$
(h_{f, \alpha}g, u)_{L^2(D, dA_\alpha)} = (h_{(\mathbb{P}^\alpha f), \alpha}g, u)_{L^2(D, dA_\alpha)},
$$

for all $u$ in $L^2_\alpha(D, dA_\alpha)$. Thus the proof is complete. □

Proposition 3.1.3 establishes that the little Hankel operator $h_{f, \alpha}$ depends only on the analytic part of symbol function $f$.

**Proposition 3.1.4 [Rudin 80],[Zhu 90]** Suppose $\alpha > -1$ and $z \in D$. Define the operator $P_\alpha$ on $L^1(D, dA_\alpha)$ by

$$
P_\alpha f(z) = (\alpha + 1) \int_D \frac{(1 - |w|^2)^\alpha f(w)}{(1 - z\bar{w})^{2+\alpha}} dA(w).
$$

Then $P_\alpha f = f$ for all analytic $f$ in $L^1(D, dA_\alpha)$.

**Proposition 3.1.5** Suppose $\alpha > -1$ and $z \in D$. Define the operator $V$ on $L^2(D, dA_\alpha)$ by

$$
V f(z) = (3 + 2\alpha) \int_D \frac{(1 - |w|^2)^{2+\alpha} f(w)}{(1 - z\bar{w})^{4+2\alpha}} dA_\alpha(w).
$$

Then

1. $V f = f$ for all analytic function $f$ on $L^2(D, dA_\alpha)$;
2. $V$ is a bounded operator on $L^2(D, dA_\alpha)$.

**Proof.** (1) Put $\beta = 2 + 2\alpha > 0$. Then we have

$$
V f(z) = (\beta + 1) \int_D \frac{(1 - |w|^2)^\beta f(w)}{(1 - z\bar{w})^{2+\beta}} dA(w).
$$
Since $f \in L^2(D, dA_\alpha)$ it follows that $f \in L^1(D, dA_\alpha)$. By Proposition 3.1.4 with $\alpha = \beta > 0$ yields $Vf = P_\beta f = f$, for all analytic function $f \in L^2(D, dA_\alpha)$.

(2) Put

$$G^\alpha(z, w) = \frac{(3 + 2\alpha)(1 - |w|^2)^{2+\alpha}}{(1 - z\bar{w})^{4+2\alpha}}.$$ 

Then

$$Vf(z) = \int_D G^\alpha(z, w)f(w)dA_\alpha(w).$$

The operator $V$ is an integral operator induced by $G^\alpha(z, w)$. Apply Schur's theorem with $h^2(z) = (1 - |z|^2)^m$, where $m < 0$. In order to show that $V$ is bounded on $L^2(D, dA_\alpha)$, we need to show that, there exist constants $C_1$ and $C_2 > 0$ such that

$$I_1 = \int_D |G^\alpha(z, w)|h^2(w)dA_\alpha(w) \leq C_1 h^2(z),$$

and

$$I_2 = \int_D |G^\alpha(z, w)|h^2(z)dA_\alpha(z) \leq C_2 h^2(w).$$

Consider

$$I_1 = (3 + 2\alpha)(1 + \alpha) \int_D \frac{(1 - |w|^2)^{2+2\alpha+m}}{|1 - z\bar{w}|^{4+2\alpha}}dA(w).$$

Applying Proposition 2.2.3 with $m < 0$, there exists a constant $C_1 > 0$ such that

$$I_1 \leq C_1 h^2(z).$$

Consider

$$I_2 = (3 + 2\alpha)(1 + \alpha)(1 - |w|^2)^{2+\alpha} \int_D \frac{(1 - |z|^2)^{m+\alpha}}{|1 - z\bar{w}|^{4+2\alpha}}dA(z).$$
Since \( m < 0, \ 2 + \alpha - m > 0 \). By Proposition 2.2.3 there exists a constant \( C_2 > 0 \) such that

\[
I_2 \leq C_2 h^2(w).
\]

Thus the proof is complete. \( \Box \)

Note that Proposition 3.1.5 gives us that the adjoint of \( V, V^* \), is also a bounded operator on \( L^2(D, dA_\alpha) \) and

\[
V^* g(z) = (3 + 2\alpha)(1 - |z|^2)^{2+\alpha} \int_D \frac{g(w)}{(1 - \bar{w}z)^{4+2\alpha}} dA_\alpha(w).
\]

**Proposition 3.1.6** Let \( V \) be the operator as in Proposition 3.1.5 and \( z \in D \). Then

\[
V^* f(z) = (3 + 2\alpha)\langle \overline{k^2_z}, h f, \alpha k^0_x \rangle_{L^2(D, dA_\alpha)},
\]

for all \( f \in L^2(D, dA_\alpha) \).

**Proof.** Consider

\[
\langle \overline{k^2_z}, h f, \alpha k^0_x \rangle_{L^2(D, dA_\alpha)} = \langle \overline{k^2_z}, Pf^\alpha \rangle_{L^2(D, dA_\alpha)}.
\]

Since \( P^\alpha \) is a projection onto \( L^2_0(D, dA_\alpha) \), and \( \overline{k^2_z} \in \overline{L^2_0(D, dA_\alpha)} \), we have

\[
\langle \overline{k^2_z}, h f, \alpha k^0_x \rangle_{L^2(D, dA_\alpha)} = \langle f, (k^2_z)^2 \rangle_{L^2(D, dA_\alpha)}
\]

\[
= (1 - |z|^2)^{2+\alpha} \int_D \frac{f(w)}{(1 - \bar{w}z)^{4+2\alpha}} dA_\alpha(w).
\]

Thus the remark under Proposition 3.1.5 gives us the desired result. \( \Box \)
Proposition 3.1.7 Let $V$ be the operator as in Proposition 3.1.5. Then

$$V^* P^o = V^*,$$

in the sense that for all $f \in L^2(D, dA_o)$ and $z \in D$

$$V^* P^o f(z) = V^* f(z).$$

Proof. Let $f \in L^2(D, dA_o)$ and $z \in D$. Since $P^o f \in L^2(D, dA_o)$, Proposition 3.1.6 gives us

$$V^* (P^o f(z)) = (3 + 2\alpha) \langle \overline{k_z^o}, h_{(P^o f)_o}, k_z^o \rangle_{L^2(D, dA_o)}.$$

Since $k_z^o \in H^\infty(D, dA_o)$, Proposition 3.1.3 gives us

$$V^* (P^o (f(z))) = (3 + 2\alpha) \langle \overline{k_z^o}, h_{f_o}, k_z^o \rangle_{L^2(D, dA_o)}.$$

Proposition 3.1.6 gives us the desired result. $\square$

Proposition 3.1.8 Let $V$ be the operator as in Proposition 3.1.5. Then

$$P^o V^* f = P^o f,$$

for all $f \in L^2(D, dA_o)$.

Proof. Let $z \in D$. It suffices to show that $P^o V^* f(z) = P^o f(z)$. Since $(P^o V^*) f$ is in $L^2(D, dA_o)$, the property of kernel function in $L^2(D, dA_o)$ gives us

$$P^o V^* f(z) = \langle P^o V^* f, K_z^o \rangle_{L^2(D, dA_o)}.$$
\[ = \langle V^* f, K^\alpha_z \rangle_{L^2(D, dA_\alpha)} \]
\[ = \langle f, V K^\alpha_z \rangle_{L^2(D, dA_\alpha)}. \]

Since \( K^\alpha_z \in L^2(D, dA_\alpha) \), Proposition 3.1.5 gives us
\[ P^\alpha V^* f(z) = \langle f, K^\alpha_z \rangle_{L^2(D, dA_\alpha)} \]
\[ = \langle P^\alpha f, K^\alpha_z \rangle \]
\[ = P^\alpha f(z). \]

\[ \square \]

**Theorem 3.1.9** Suppose \( f \in L^2(D, dA_\alpha) \). Then \( h_{f,\alpha} \) is bounded if and only if \( V^* f(z) \) is bounded in \( D \).

**Proof.** Suppose \( f \in L^2(D, dA_\alpha), z \in D \) and \( h_{f,\alpha} \) is bounded. Consider
\[ V^* f(z) = (3 + 2\alpha)(k^\alpha_z, h_{f,\alpha} k^\alpha_z)_{L^2(D, dA_\alpha)}. \]

Apply the Cauchy Schwarz inequality, using the fact that \( h_{f,\alpha} \) is bounded, and \( k^\alpha_z \) is a unit vector in \( L^2(D, dA_\alpha) \) to get
\[ |V^* f(z)| \leq (3 + 2\alpha)\|k^\alpha_z\|_{L^2(D, dA_\alpha)}\|h_{f,\alpha}\|\|k^\alpha_z\|_{L^2(D, dA_\alpha)} \]
\[ = (3 + 2\alpha)\|h_{f,\alpha}\|. \]

Therefore \( V^* f(z) \) is bounded for all \( z \in D \).

Conversely, suppose \( V^* f(z) \) is bounded for all \( z \in D \). Consider \( h_{f,\alpha} : H^\infty(D, dA_\alpha) \rightarrow \)
By propositions 3.1.3 and 3.1.8, we have

\[ h_{f,\alpha} = h_{(P \ast f),\alpha} = h_{(P \ast Vf),\alpha}. \]

Again by Proposition 3.1.3, we have \( h_{f,\alpha} = h_{(V^* f),\alpha} \). Use the fact that \( V^* f \in L^\infty(D, dA_\alpha) \) and apply Proposition 3.1.1 to get \( h_{f,\alpha} \) is bounded and \( \|h_{f,\alpha}\| \leq \|V^* f\|_\infty. \)

\[ \Box \]

**Corollary 3.1.10** The little Hankel operator \( h_{f,\alpha} = h_{V^* f,\alpha} \) for all \( f \in L^2(D, dA_\alpha) \). It follows that \( h_{f,\alpha} \) is bounded if and only if \( h_{f,\alpha} = h_{g,\alpha} \) for some \( g \in L^\infty(D, dA_\alpha) \).

**Proof.** The proof of the corollary follows from Propositions 3.1.3, 3.1.8 and Theorem 3.1.9.

\[ \Box \]

**Proposition 3.1.11** [JPR 87] The operator \( V^* \) is an embedding from \( B \) into \( L^\infty(D, dA_\alpha) \).

The next theorem shows that the symbol function \( f \) in the Bloch space will make the operator \( h_{f,\alpha} \) bounded. The proof of the theorem in [AFP 88] referred to [Bonsall 86]. We will use the result of this section to prove it.

**Theorem 3.1.12** [AFP 88] The little Hankel operator \( h_{f,\alpha} \) is bounded if and only if \( f \) is in the Bloch space \( B \).

**Proof.** Suppose \( h_{f,\alpha} \) is bounded. Since the operator \( h_{f,\alpha} \) depends only on analytic part of \( f \), we can assume \( f \in L^2_0(D, dA_\alpha) \). By Proposition 3.1.4, we have

\[ f(z) = P_{1+2\alpha} f(z) = 2 \int_D \frac{(1 - |w|^2)^{1+\alpha} f(w)}{(1 - \bar{z}w)^{3+2\alpha}} dA_\alpha(w). \]

Differentiate both sides with respect to \( z \) to obtain

\[ f'(z) = 2(3 + 2\alpha) \int_D \frac{(1 - |w|^2)^{1+\alpha} \bar{w} f(w)}{(1 - \bar{z}w)^{4+2\alpha}} dA_\alpha(w). \]
Thus

\[ |f'(z)| \leq 2 \left| (3 + 2\alpha) \int_D \frac{f(w)}{(1 - \frac{w}{z})^{2\alpha+2}} dA_\alpha(w) \right| \]

\[ = \frac{2}{(1 - |z|^2)^{2+\alpha}} \left| (3 + 2\alpha)(1 - |z|^2)^{2+\alpha} \int_D \frac{f(w)}{(1 - \frac{w}{z})^{2\alpha+2}} dA_\alpha(w) \right| \]

\[ = \frac{2}{(1 - |z|^2)^{2+\alpha}} |V^* f(z)|. \]

Therefore

\[ \frac{(1 - |z|^2)^{1+\alpha}}{2} (1 - |z|^2)|f'(z)| \leq |V^* f(z)|. \]

Since \( h_{f,\alpha} \) is bounded, Theorem 3.1.9 gives us \( V^* f \in L^\infty(D, dA_\alpha) \). Thus \( f \) is in the Bloch space.

Conversely, suppose \( f \in B \). By Proposition 3.1.11, we have \( V^* f \in L^\infty(D, dA_\alpha) \) and Theorem 3.1.9 gives us that \( h_{f,\alpha} \) is bounded. \( \square \)

### 3.2 Compact Little Hankel Operators on Weighted Bergman Spaces

We have seen in the previous section that the boundedness of the little Hankel operator \( h_{f,\alpha} \) on the weighted Bergman space \( L^2_\alpha(D, dA_\alpha) \) depends on the boundedness of \( V^* f(z) \) on \( D \), where \( f \in L^2(D, dA_\alpha) \), and \( V^* \) is the adjoint of the operator \( V \) as defined in Proposition 3.1.5. In this section we will characterize compactness of \( h_{f,\alpha} \) in terms of the function \( V^* f \).

We begin this section with a certain finite rank little Hankel operator.

**Proposition 3.2.1** Suppose \( n, m \geq 0 \). Then \( h_{z^n \bar{z}^m,\alpha} \) is a finite rank operator.
Proof. Consider
\[
\left\langle h_{z^n} z^m \frac{z^i}{\|z^i\|}, \frac{z^j}{\|z^j\|} \right\rangle_{L^2(D, dA_\alpha)}
= \frac{1}{\|z^j\| \|z^i\|} \left\langle \overline{P(\alpha) z^n z^{m+j}}, z^i \right\rangle_{L^2(D, dA_\alpha)}
= \frac{1}{\|z^j\| \|z^i\|} \left\langle z^{m+i+j}, z^n \right\rangle_{L^2(D, dA_\alpha)}.
\]

Thus
\[
\left\langle h_{z^n} z^m \frac{z^i}{\|z^i\|}, \frac{z^j}{\|z^j\|} \right\rangle_{L^2(D, dA_\alpha)} = \begin{cases} 
0 & \text{if } n - m \neq i + j \\
\frac{\|z^n\|_{L^2(D, dA_\alpha)}^2}{\|z^j\|_{L^2(D, dA_\alpha)}^2 \|z^i\|_{L^2(D, dA_\alpha)}^2} & \text{if } n - m = i + j
\end{cases} \quad (3.1)
\]

Hence the matrix representation of the operator $h_{z^n} z^m \alpha$ relative to orthonormal basis $\{\frac{z^j}{\|z^j\|} : j = 0, 1, 2, \ldots\}$ for $L^2(D, dA_\alpha)$ and $\{\frac{z^i}{\|z^i\|} : i = 0, 1, 2, \ldots\}$ for $L^2(D, dA_\alpha)$ has finitely many nonzero terms. Therefore $h_{z^n} z^m \alpha$ is a finite rank operator. \qed

Theorem 3.2.2 Suppose $f \in L^2(D, dA_\alpha)$ and $V$ is the operator as in Proposition 3.1.5. Then the following are equivalent:

1. $h_{f, \alpha}$ is compact;
2. $V^* f$ is in $C_0(D)$;
3. $V^* f$ is in $C(\overline{D})$.

Proof. (1) $\Rightarrow$ (2)

Suppose $h_{f, \alpha}$ is compact. Since $(1 - |z|^2)^{2+\alpha}$ is continuous and $K^{\alpha}(z, w)$ is analytic on $z$, it follows that $V^* f(z)$ is continuous for all $z \in D$. By Proposition 3.1.6, we have
\[
V^* f(z) = (3 + 2\alpha)(k_{\alpha}^0, h_{f, \alpha} k_{\alpha}^0)_{L^2(D, dA_\alpha)}.
\]
Apply the Cauchy Schwarz inequality and use the fact that $k^2_\alpha$ is a unit vector in $L^2(D, dA_\alpha)$ to obtain

$$V^*f(z) \leq (3 + 2\alpha)\|h_{f,\alpha}k^2_\alpha\|_{L^2(D, dA_\alpha)}.$$ 

Take the limit as $|z| \to 1^-$, use the fact that $h_{f,\alpha}$ is compact and $k^2_\alpha \to 0$ weakly as $|z| \to 1^-$ to obtain

$$\lim_{|z| \to 1^-} |V^*f(z)| = 0.$$ 

Hence (1) $\Rightarrow$ (2) is done.

(2) $\Rightarrow$ (3) 

Obvious.

(3) $\Rightarrow$ (1)

Let $S = \text{span}(\{z^n z^m : n, m \geq 0\})$. By the Stone Weierstrass Approximation theorem we have $\overline{S} = C(\overline{D})$. It follows from Corollary 3.1.10 that in order to prove (3) $\Rightarrow$ (1) it suffices to show that for all $f \in C(\overline{D})$, $h_{f,\alpha}$ is compact.

Suppose $f \in C(\overline{D})$. If $f \in S$, then Proposition 3.2.1 gives us that $h_{f,\alpha}$ is compact. If $f \in \overline{S}$, then there exists a sequence $\{f_n\}$ in $S$ such that $f_n \to f$ as $n \to \infty$. Since the mapping $\Psi : f \mapsto h_{f,\alpha}$ is norm continuous, we have

$$\|h_{f_n,\alpha} - h_{f,\alpha}\| \to 0 \text{ as } n \to \infty.$$ 

Therefore $h_{f,\alpha}$ is compact. Hence (3) $\Rightarrow$ (1) is done. \qed

**Proposition 3.2.3 [JPR 87]** The operator $V^*$ is an embedding from $\mathcal{B}_0$ into $C_0(D)$.

**Theorem 3.2.4 [AFP 88]** The little Hankel operator $h_{f,\alpha}$ is compact if and only if $f$ is in $\mathcal{B}_0$, the little Bloch space.
Proof. Suppose $h_{f,\alpha}$ is compact. By Theorem 3.1.12, we have $f \in \mathcal{B}$. From the proof in Theorem 3.1.12, we will see that there exists a constant $C > 0$ such that

$$|V^*f(z)| \geq C(1 - |z|^2)|f'(z)|.$$  

Thus

$$\lim_{|z| \to 1^-} |V^*f(z)| \geq C \lim_{|z| \to 1^-} (1 - |z|^2)f'(z).$$

Since $h_{f,\alpha}$ is compact, Theorem 3.2.2 gives us

$$C \lim_{|z| \to 1^-} (1 - |z|^2)f'(z) \leq 0.$$  

We always have

$$C \lim_{|z| \to 1^-} (1 - |z|^2)f'(z) \geq 0.$$  

Therefore

$$\lim_{|z| \to 1^-} (1 - |z|^2)f'(z) = 0,$$

that implies $f \in \mathcal{B}_0$.

Conversely, suppose $f \in \mathcal{B}_0$. By Proposition 3.2.3, we have $V^*f \in C_0(D)$. Therefore $h_{f,\alpha}$ is compact by Theorem 3.2.2.

3.3 Schatten Class Little Hankel Operators on Weighted Bergman Spaces

In this section we will characterize membership of $h_{f,\alpha}$ in a Schatten $p$-Class, where $1 \leq p \leq \infty$, in terms of the functions $V^*f$.  

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Theorem 3.3.1 Suppose $f \in L^2(D, dA_\alpha)$ and $1 \leq p < \infty$. Then $h_{f, \alpha} \in S_p$ of $L^2(D, dA_\alpha)$ if and only if $V^* f$ is in $L^p(D, d\lambda)$, where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^{\alpha}}$.

Proof. Suppose $1 \leq p < \infty$ and $h_{f, \alpha} \in S_p$ of $L^2(D, dA_\alpha)$. Then by Proposition 2.4.6, we have $(h_{f, \alpha}^*, h_{f, \alpha})^\frac{p}{2} \in S_1$ of $L^2(D, dA_\alpha)$. Now apply Proposition 2.3.11 to obtain

$$\text{trace}(h_{f, \alpha}^*, h_{f, \alpha})^\frac{p}{2} = \int_D \langle (h_{f, \alpha}^*, h_{f, \alpha})^\frac{p}{2} k_\alpha^o, k_\alpha^o \rangle_{L^2(D, dA_\alpha)} d\lambda(z) < \infty.$$ 

Since $\frac{p}{2} \geq 1$, Proposition 2.4.10 gives us

$$\langle (h_{f, \alpha}^*, h_{f, \alpha})^\frac{p}{2} k_\alpha^o, k_\alpha^o \rangle_{L^2(D, dA_\alpha)} \geq \langle (h_{f, \alpha}^*, h_{f, \alpha}) k_\alpha^o, k_\alpha^o \rangle_{L^2(D, dA_\alpha)}^\frac{p}{2} \geq \|h_{f, \alpha} k_\alpha^o\|_{L^2(D, dA_\alpha)}^p.$$ 

Thus

$$\int_D \|h_{f, \alpha} k_\alpha^o\|_{L^2(D, dA_\alpha)}^p d\lambda(z) \leq \int_D \langle (h_{f, \alpha}^*, h_{f, \alpha})^\frac{p}{2} k_\alpha^o, k_\alpha^o \rangle_{L^2(D, dA_\alpha)} d\lambda(z) < \infty.$$ 

Consider

$$V^* f(z) = (3 + 2\alpha) \langle \overline{k_\alpha^o}, h_{f, \alpha} k_\alpha^o \rangle_{L^2(D, dA_\alpha)}.$$ 

Apply the Cauchy Schwarz inequality and use the fact that $k_\alpha^o$ is a unit vector in $L^2(D, dA_\alpha)$ to obtain

$$|V^* f(z)| \leq (3 + 2\alpha) \|h_{f, \alpha} k_\alpha^o\|_{L^2(D, dA_\alpha)}.$$ 

Thus

$$|V^* f(z)|^p \leq (3 + 2\alpha)^p \|h_{f, \alpha} k_\alpha^o\|_{L^2(D, dA_\alpha)}^p.$$
Therefore

\[
\int_{\mathcal{D}} |V^*f(z)|^p d\lambda(z) \leq (3 + 2\alpha)^p \int_{\mathcal{D}} \|h_{f, \alpha}k_\omega\|^p_{L^2(\mathcal{D}, dA_\alpha)} d\lambda(z) < \infty.
\]

Hence \( V^*f \) is in \( L^p(\mathcal{D}, d\lambda) \).

Conversely, suppose \( V^*f \) is in \( L^p(\mathcal{D}, d\lambda) \). By Propositions 3.1.3 and 3.1.8, we have

\[
h_{(V^*f), \alpha} = h_{(P\alpha V^*f), \alpha} = h_{(P\alpha f), \alpha} = h_{f, \alpha}.
\]

Thus if we can show that \( h_{\varphi, \alpha} \) is in \( S_p \) of \( L^2(\mathcal{D}, dA_\alpha) \) for all \( \varphi \) is in \( L^p(\mathcal{D}, d\lambda) \), then the proof is complete.

Let \( \varphi \in L^p(\mathcal{D}, d\lambda) \). By Proposition 3.1.2 (2), we have for all \( g \in L^2_\alpha(\mathcal{D}, dA_\alpha) \) and \( z \in \mathcal{D} \),

\[
h_{\varphi, \alpha} g(z) = \int_{\mathcal{D}} K^\alpha(w, z)\varphi(w)g(w)dA_\alpha(w).
\]

By Proposition 2.4.25, to prove \( h_{\varphi, \alpha} \in S_p \) of \( L^2(\mathcal{D}, dA_\alpha) \) it suffices to show that

\[
I = \int_{\mathcal{D}} \int_{\mathcal{D}} |\varphi(w)|^p |K^\alpha(w, z)|^2 dA_\alpha(w)dA_\alpha(z) < \infty.
\]

Apply Fubini's theorem, using \( d\lambda(w) = \|K^\alpha_\omega\|^2 dA_\alpha(w) \) and the fact that \( k_\omega^\alpha \) is a unit vector in \( L^2_\alpha(\mathcal{D}, dA_\alpha) \) to obtain

\[
I = \int_{\mathcal{D}} |\varphi(w)|^p \int_{\mathcal{D}} \frac{|K^\alpha(w, z)|^2}{\|K^\alpha_\omega\|^2_{L^2(\mathcal{D}, dA_\alpha)}} dA_\alpha(z)d\lambda(w)
\]

\[
= \int_{\mathcal{D}} |\varphi(w)|^p \int_{\mathcal{D}} |k_\omega^\alpha|^2 dA_\alpha(z)d\lambda(w)
\]
\[ = \int_{\mathcal{D}} |\varphi(w)|^p d\lambda(w) < \infty. \]

The last inequality follows from the fact that $\varphi \in L^p(\mathcal{D}, d\lambda)$. \qed

**Corollary 3.3.2** Suppose $p \geq 1$ and $f \in L^2(\mathcal{D}, dA_\alpha)$. Then $h_{f,\alpha}$ is in $S_p$ if and only if $h_{f,\alpha} = h_{g,\alpha}$, for some $g \in L^p(\mathcal{D}, d\lambda)$.

**Proof.** The result follows directly from the theorem. \qed

**Proposition 3.3.3 [JPR 87]** The operator $V^*$ is an embedding from $\mathcal{B}_p$ into $L^p(\mathcal{D}, d\lambda)$ for all $1 \leq p < \infty$.

**Theorem 3.3.4 [AFP 88]** Let $1 \leq p < \infty$. The operator $h_{f,\alpha}$ is in $S_p$ of $L^2(\mathcal{D}, dA_\alpha)$ if and only if $f$ is in $\mathcal{B}_p$, the Besov space.

**Proof.** The proof of the Theorem follows from Proposition 3.3.3 and Theorem 3.3.2. \qed
Chapter 4

Good Estimates for Norms of

Little Hankel Operators on

Weighted Bergman spaces

In this chapter we will characterize the little Hankel operators $h_{z\beta,\alpha}$ on the weighted Bergman space $L^2_\alpha(D, dA_\alpha)$, where $g$ is in $L^2_\alpha(D, dA_\alpha)$ and $z \in D$. The main objective is to get a good estimate of the bounded norm of the operator $h_{z\beta,\alpha}$. The next proposition was proved by Bonsall [Bonsall 86] in the case of the usual Bergman space; We will give an alternate proof using results from section 3.1.

Proposition 4.0.1 Suppose $g \in L^2_\alpha(D, dA_\alpha)$. If $g$ is in the Bloch space, then $h_{z\beta,\alpha}$ is bounded.

Proof. Suppose $g \in \mathcal{B}$ and $z \in D$. Then we have $h_{z\beta,\alpha} = h_{\beta,\alpha}M_z$. By Proposition 3.1.12, we know that $h_{\beta,\alpha}$ is bounded. If we can show $M_z$ is bounded, then the proof is done. Since $L^2_\alpha(D, dA_\alpha)$ is an $H^2(\beta)$ space with $\beta(n)^2 = \|z^n\|^2 = \frac{n!\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}$, $w_n = \frac{\beta(n+1)}{\beta(n)} = \sqrt{\frac{n+1}{n+\alpha+2}}$. Thus $\{w_n\}$ is monotonically increasing with limit 1, so $M_z$ is bounded by Proposition 1.0.7. Therefore $h_{z\beta,\alpha}$ is bounded.

\[\Box\]
Suppose $h_{zg}$ is bounded. By the argument in the proof of above theorem, we have $M_z = SD$, where $S$ is the (unweighted) unilateral shift and $D$ is the diagonal operator with weight sequence $\{w_n\}$. Let $U = D^{-1}S^*$. We have $h_{\bar{g},a} = h_{zg,a}U + \bar{g} \otimes \epsilon_0$, and hence the boundedness of $h_{zg,a}$ implies that of $h_{\bar{g},a}$. This also shows that $h_{zg,a}$ is compact if and only if $h_{\bar{g},a}$ is compact.

4.1 A Good Estimate for the Bounded Norms of Little Hankel Operators on Weighted Bergman Spaces

In the last section we have seen that the operator $h_{(zg),a}$, where $g$ is in the Bloch space, is bounded. In this section we will show that if $g \in L_2^b(D, dA_\alpha)$ and the operator $h_{(zg),a}$ is bounded then $g$ is in the Bloch space. Moreover, we also get a good estimate of (bounded) the norm of this operator.

Bonsall, [Bonsall 86], has estimated this norm in the usual Bergman space $L_2^b(D, dA)$. He estimates the norm of $h_{zg}$ as follows:

$$\frac{\pi}{8} ||g||_a \leq ||h_{zg}|| \leq ||g||_a.$$ 

He also showed that the constant $\frac{\pi}{8}$ is best possible.

For $f \in L_2^b(D, dA_\alpha)$, and $g \in B$ we will write

$$[f,g] = \int_D f(w)(1 - |w|^2)g(w)|dA_\alpha(w).$$
Thus if \( f \in L^2_a(\mathbb{D}, dA) \) we will have

\[
[f, g] = \langle f, (1 - |z|^2)g' \rangle_{L^2(\mathbb{D}, dA)}.
\]

In this section we will need the definitions and certain properties of the beta and gamma functions which we will summarize here for convenience. For reference one can see any advanced calculus book.

**Definition 4.1.1**

1. The beta function \( B(x, y) \) is defined in \( \{(x, y) : x > 0, y > 0\} \) by

\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.
\]

2. The gamma function \( \Gamma(x) \) is defined in \( \{x : x > 0\} \) by

\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.
\]

**Proposition 4.1.2**

1. If \( x > 0 \), then we have

\[
\Gamma(x + 1) = x\Gamma(x).
\]

2. If \( k \) is any nonnegative integer, then

\[
\Gamma(k + 1) = k!.
\]

3. \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \).
4. For $x > 0$, and $k$ is nonnegative integer, we have

$$\Gamma(x) = \frac{\Gamma(x + k + 1)}{x(x + 1)(x + 2)\ldots(x + k)}.$$

5. For $x, y > 0$, we have

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

**Proposition 4.1.3 (Stirling’s Formula)** Suppose $c$ is a real number and $t > -1$. Then

$$\frac{\Gamma(n + \frac{1}{2}(2 + t + c))^2}{n!\Gamma(n + t + 2)} \sim n^{c-1}, \quad (n \to \infty)$$

**Proposition 4.1.4** Suppose $g \in L^2_\alpha(D, dA_\alpha)$. If $u$ is a polynomial and $g \in B$, then

$$[u, g] = (\alpha + 1)\langle zu, g \rangle_{L^2(D, dA_\alpha)}.$$

**Proof.** It suffices to show that for $m \geq 0$

$$[z^m, g] = (\alpha + 1)\langle zz^m, g \rangle_{L^2(D, dA_\alpha)}.$$

Let $g = \sum_{n=0}^{\infty} a_n z^n = \lim_{k \to \infty} \sum_{n=0}^{k} a_n z^n$. Consider

$$\langle zz^m, g \rangle_{L^2(D, dA_\alpha)} = \left \langle z^{m+1}, \lim_{k \to \infty} \sum_{n=0}^{k} a_n z^n \right \rangle_{L^2(D, dA_\alpha)}$$

$$= \lim_{k \to \infty} \sum_{n=0}^{k} \frac{a_n}{n!} (z^{m+1}, z^n)_{L^2(D, dA_\alpha)}$$

$$= \frac{a_{m+1}}{a_{m+1}} ||z^{m+1}||^2.$$
Substitute the value of \(\|z^{n+1}\|^2\) to get

\[
\langle zz^m, g \rangle_{L^2(D, dA_\alpha)} = \overline{a}_{m+1} \frac{\Gamma(\alpha + 2)\Gamma(m + 2)}{\Gamma(m + \alpha + 3)}.
\]

Since \(z^m \in L^2_d(D, dA_\alpha)\) and \(g \in B\),

\[
[z^m, g] = \left\langle z^m, (1 - |z|^2) \lim_{k \to \infty} \sum_{n=1}^{k} n \overline{a}_n z^{n-1} \right\rangle_{L^2(D, dA_\alpha)}
\]

\[
= \lim_{k \to \infty} \sum_{n=1}^{k} n \overline{a}_n \langle z^m, (1 - z\overline{z})z^{n-1} \rangle_{L^2(D, dA_\alpha)}
\]

\[
= \lim_{k \to \infty} \sum_{n=1}^{k} n \overline{a}_n \left\{ \langle z^m, z^{n-1} \rangle_{L^2(D, dA_\alpha)} - \langle z^{m+1}, z^n \rangle_{L^2(D, dA_\alpha)} \right\}
\]

\[
= (m + 1)\overline{a}_{m+1} \left\{ \|z^m\|_{L^2(D, dA_\alpha)}^2 - \|z^{m+1}\|_{L^2(D, dA_\alpha)}^2 \right\}.
\]

Substitute the values of \(\|z^m\|_{L^2(D, dA_\alpha)}\), and \(\|z^{m+1}\|_{L^2(D, dA_\alpha)}\) to get

\[
[z^m, g] = (m + 1)\overline{a}_{m+1} \left\{ \frac{\Gamma(\alpha + 2)\Gamma(m + 1)}{\Gamma(m + \alpha + 2)} - \frac{\Gamma(\alpha + 2)\Gamma(m + 2)}{\Gamma(m + \alpha + 3)} \right\}.
\]

Combining the terms and using \(z\Gamma(z) = \Gamma(z + 1)\), we have

\[
[z^m, g] = (\alpha + 1)\overline{a}_{m+1} \frac{\Gamma(\alpha + 2)\Gamma(m + 2)}{\Gamma(m + \alpha + 3)}
\]

\[
= (\alpha + 1) \langle zz^m, g \rangle_{L^2(D, dA_\alpha)}.
\]

The last equality follows from the previous proof. Hence the proof is complete.
Proposition 4.1.5 Suppose \( \alpha > -1 \). Then the sequence \( \{a_n\} \) where

\[
a_n = \left( \frac{\frac{3 + \alpha}{2}}{n} \right)^2 \frac{n!}{\Gamma(n + \alpha + 2)}, \quad n = 0, 1, 2, \ldots
\]

is an increasing sequence.

Proof. Expand the binomial coefficient to get

\[
a_n = \frac{\Gamma\left(\frac{3 + \alpha}{2} + n\right)^2}{n! \Gamma\left(\frac{3 + \alpha}{2}\right)^2 \Gamma(n + \alpha + 2)}, \quad n = 0, 1, 2, \ldots
\]

Thus

\[
a_{n+1} = \frac{\Gamma\left(\frac{3 + \alpha}{2} + n + 1\right)^2}{(n + 1)! \Gamma\left(\frac{3 + \alpha}{2}\right)^2 \Gamma(n + \alpha + 3)}.
\]

We may use properties of the Gamma function to get

\[
a_{n+1} = a_n \left\{ \frac{(\frac{3 + \alpha}{2} + n)^2}{(n + 1)(n + \alpha + 2)} \right\}.
\]

Hence

\[
a_{n+1} - a_n = a_n \left\{ \frac{(\frac{3 + \alpha}{2} + n)^2}{(n + 1)(n + \alpha + 2)} - 1 \right\}.
\]

Since \( a_n \) is positive, if we can show that

\[
\left( \frac{3 + \alpha}{2} + n \right)^2 \geq (n + 1)(n + \alpha + 2),
\]

or

\[
\left( \frac{3 + \alpha}{2} + n \right) \geq \sqrt{n + 1} \sqrt{n + \alpha + 2},
\]

then the proof is complete.
The left hand side is the arithmetic mean of \((n + 1)\) and \((n + \alpha + 2)\) and the right hand side is the geometric mean of \((n + 1)\), and \((n + \alpha + 2)\). Use the fact that the arithmetic mean is always greater than or equal to the geometric mean to get the desired result.

\[ \square \]

**Proposition 4.1.6** Suppose \(\{a_n\}\) is a sequence as in Proposition 4.1.5. Then

\[
\lim_{n \to \infty} a_n = \frac{1}{\Gamma \left( \frac{3 + 2\alpha}{2} \right)^2}.
\]

**Proof.** Consider

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\Gamma \left( \frac{3 + \alpha}{2} + n \right)^2}{n! \Gamma \left( \frac{3 + \alpha}{2} \right)^2 \Gamma (n + \alpha + 2)}
= \frac{1}{\Gamma \left( \frac{3 + \alpha}{2} \right)^2} \lim_{n \to \infty} \frac{\Gamma \left( \frac{3 + \alpha}{2} + n \right)^2}{n! \Gamma (n + \alpha + 2)}.
\]

By using Stirling's formula, we have the last limit equal to 1. Thus

\[
\lim_{n \to \infty} a_n = \frac{1}{\Gamma \left( \frac{3 + \alpha}{2} \right)^2}.
\]

\[ \square \]

**Proposition 4.1.7** Suppose \(\alpha > -1, \ w \in \mathbb{D}\). Then

\[
\int_{\mathbb{D}} \frac{1}{|1 - wz|^{3+\alpha}} dA_\alpha(z) \leq \frac{\Gamma (\alpha + 2)}{\Gamma \left( \frac{3 + \alpha}{2} \right)^2 (1 - |w|^2)}.
\]

**Proof.** Consider

\[
(1 - wre^{-i\theta})^{-(3+\alpha)} = \left\{ (1 - wre^{-i\theta})^{\frac{3+\alpha}{2}} \right\}^2.
\]
Applying the binomial theorem to the inside term, we have

\[
(1 - wr^{-i\theta})^{-(3+\alpha)} = \left\{ \sum_{n=0}^{\infty} (-1)^n \left( \frac{3+\alpha}{n} \right) w^n r^n e^{-in\theta} \right\}^2.
\]

Thus

\[
\frac{1}{|1 - wr^{-i\theta}|^{3+\alpha}} = \left| \sum_{n=0}^{\infty} \left( \frac{3+\alpha}{n} \right) w^n r^n e^{-in\theta} \right|^2.
\]

Let

\[
I = \int_D \frac{1}{|1 - wz^{3+\alpha}|} dA_\alpha(z),
\]

\[z = re^{i\theta}\] and change the integral into polar coordinate form.

\[
I = \frac{(\alpha + 1)}{2\pi} \int_0^1 \int_0^{2\pi} \left| \sum_{n=0}^{\infty} \left( \frac{3+\alpha}{n} \right) w^n r^n e^{-in\theta} \right|^2 (1 - r^2)^\alpha \theta^2 r dr d\theta.
\]

Using the fact that the inside integral is a norm of function in \(L^2(d\theta)\), we have

\[
I = 2(\alpha + 1) \int_0^1 (1 - r^2)^\alpha \sum_{n=0}^{\infty} \left( \frac{3+\alpha}{n} \right)^2 |w|^{2n} r^{2n} dr.
\]

By the definition of the beta function, we have

\[
I = (\alpha + 1) \sum_{n=0}^{\infty} \left( \frac{3+\alpha}{n} \right)^2 \frac{1}{\Gamma(n+1, \alpha + 1)} |w|^{2n}.
\]
Now the relationship between beta and gamma functions gives us

\[
I = (\alpha + 1) \sum_{n=0}^{\infty} \left( \frac{\Gamma(3+\alpha)}{2\pi} \right)^2 \frac{n!\Gamma(\alpha + 1)|w|^{2n}}{\Gamma(n + \alpha + 2)}
= \Gamma(\alpha + 2) \sum_{n=0}^{\infty} a_n |w|^{2n},
\]

where \(a_n\) is a sequence as in Proposition 4.1.5.

Finally the Propositions 4.1.5 and 4.1.6 give us

\[
I \leq \frac{\Gamma(2 + \alpha)}{(1 - |w|^2)\Gamma(\frac{3+\alpha}{2})^2}.
\]

The proof is complete. \(\square\)

**Proposition 4.1.8** Suppose \(z, w \in D, \alpha > 1\) and

\[
b_w(z) = \frac{(2 + \alpha)(1 - |w|^2)}{(1 - z\overline{w})^{3+\alpha}}.
\]

Then

1. \(b_w(z) \in H^\infty(D, dA_\alpha)\);

2. \(\|b_w\|_{L^1_{\alpha}(D, dA_\alpha)} \leq \frac{\Gamma(3+\alpha)}{\Gamma(\frac{3+\alpha}{2})^2}\).

**Proof.** (1) is obvious.

(2) The norm in \(L^1_{\alpha}(D, dA_\alpha)\) gives us

\[
\|b_w\|_{L^1_{\alpha}(D, dA_\alpha)} = (2 + \alpha)(1 - |w|^2) \int_D \frac{1}{|1 - z\overline{w}|^{3+\alpha}} dA_\alpha(z).
\]
Apply Proposition 4.1.7 to obtain
\[ |b_w|_{L_1(D,dA_\alpha)} \leq \frac{\Gamma(3 + \alpha)}{\Gamma(\frac{3+\alpha}{2})^2}. \]

\[ \square \]

**Proposition 4.1.9** Suppose \( g \in L_0^2(D,dA_\alpha) \), and \( h_{(z\bar{z}),\alpha} \) is bounded. Then \( g \in \mathcal{B} \) and
\[ \frac{\Gamma(\frac{3+\alpha}{2})^2}{\Gamma(3 + \alpha)} \|g\|_{\mathcal{B}} \leq \|h_{(z\bar{z}),\alpha}\|. \]

**Proof.** Since \( g \in L_0^2(D,dA_\alpha) \) we have
\[
g(w) = \langle g, K_w^0 \rangle_{L_1^2(D,dA_\alpha)} = \int_D \frac{g(z)}{(1 - w\bar{z})^{2+\alpha}} dA_\alpha(z).
\]
Differentiating both sides with respect to \( w \), we have
\[ g'(w) = (2 + \alpha) \int_D \frac{\bar{z}g(z)}{(1 - w\bar{z})^{3+\alpha}} dA_\alpha(z). \]
Thus
\[ \overline{g'(w)} = (2 + \alpha) \int_D \frac{\bar{z}\bar{g}(z)}{(1 - \bar{w}z)^{3+\alpha}} dA_\alpha(z). \]
Therefore
\[
(1 - |w|^2)g'(w) = (2 + \alpha) \int_D z\bar{g}(z) \frac{(1 - |w|^2)}{(1 - w\bar{z})^{3+\alpha}} dA_\alpha(z)
= \int_D z\bar{g}(z)h_{\omega}(z) dA_\alpha(z),
\]
where $b_w(z)$ is as in Proposition 4.1.8.

Substituting the value of $c_w$ in the above integral, we have

$$(1 - |w|^2)g'(w) = \int_D \bar{z}g(z)c_w(z)^2 dA_\alpha(z).$$

The definition of inner product in $L^2(D,dA_\alpha)$ gives us

$$(1 - |w|^2)g'(w) = \langle w\bar{g}c_w, \overline{c_w} \rangle_{L^2(D,dA_\alpha)}.$$

Use the fact that $\overline{c_w} \in \overline{L^2(D,dA_\alpha)}$ to get

$$(1 - |w|^2)g'(w) = \langle P_\alpha(w\bar{g}c_w, \overline{c_w}) \rangle_{L^2(D,dA_\alpha)}$$

$$= \langle h_{\langle w\bar{g}, \alpha \rangle}c_w, \overline{c_w} \rangle_{L^2(D,dA_\alpha)}.$$ .

Apply the Cauchy Schwarz inequality to get

$$(1 - |w|^2)|g'(w)| \leq \|h_{\langle w\bar{g}, \alpha \rangle}c_w\|_{L^2(D,dA_\alpha)}\|\overline{c_w}\|_{L^2(D,dA_\alpha)}.$$ Using the fact that $h_{\langle w\bar{g}, \alpha \rangle}$ is bounded, we have

$$(1 - |w|^2)|g'(w)| \leq \|h_{\langle w\bar{g}, \alpha \rangle}\|_{L^2(D,dA_\alpha)}\|c_w\|_{L^2(D,dA_\alpha)}\|\overline{c_w}\|_{L^2(D,dA_\alpha)}$$

$$= \|h_{\langle w\bar{g}, \alpha \rangle}\|_{L^2(D,dA_\alpha)}\|b_w\|_{L^2(D,dA_\alpha)}.$$ Thus

$$\|g\|_\theta \leq \|h_{\langle w\bar{g}, \alpha \rangle}\|_{L^2(D,dA_\alpha)}\|b_w\|_{L^2(D,dA_\alpha)}.$$
Therefore \( g \in \mathcal{B} \), and Proposition 4.1.8 gives us

\[
\frac{\Gamma\left(\frac{3+\alpha}{2}\right)^2}{\Gamma(3+\alpha)} \|g\|_B \leq \|h_{(\omega g)}\|_A.
\]

\[\square\]

**Proposition 4.1.10** Suppose \( g \in L^2_\alpha(D,dA_\alpha) \) and \( g \in \mathcal{B} \). Then

\[
\|h_{(\omega g)}\|_A \leq \frac{1}{(\alpha + 1)} \|g\|_B.
\]

**Proof.** Let \( u, v \) be polynomials, and \( \|u\|_{L^2_\alpha(D,dA_\alpha)} = \|v\|_{L^2_\alpha(D,dA_\alpha)} = 1 \). Consider

\[
\langle h_{(\omega g)}u, \bar{v} \rangle_{L^2(D,dA_\alpha)} = \langle z \bar{g}u, \bar{v} \rangle_{L^2(D,dA_\alpha)} = \langle zuv, g \rangle_{L^2(D,dA_\alpha)}.
\]

Since \( uv \) is a polynomial, Proposition 4.1.4 implies that

\[
\langle h_{(\omega g)}u, \bar{v} \rangle_{L^2(D,dA_\alpha)} = \frac{1}{(\alpha + 1)} [uv, g].
\]

Writing \([uv, g]\) in the integral form, we have

\[
\langle h_{(\omega g)}u, \bar{v} \rangle_{L^2(D,dA_\alpha)} = \frac{1}{\alpha + 1} \int_D u(w)v(w)(1 - |w|^2)g'(w)dA_\alpha(w).
\]

Hence

\[
|\langle h_{(\omega g)}u, \bar{v} \rangle_{L^2(D,dA_\alpha)}| \leq \frac{1}{(\alpha + 1)} \|g\|_B \int_D |u(w)v(w)|dA_\alpha(w).
\]
Apply Hölder's inequality and use the fact that \( \|v\|_{L^2_\alpha(D,dA_\alpha)} = \|u\|_{L^2_\alpha(D,dA_\alpha)} = 1 \) to get

\[
|\langle (h_{(zg)}\cdot u, \bar{v} \rangle_{L^2(D,dA_\alpha)} | \leq \frac{1}{\alpha + 1} \|g\|_B.
\]

Since polynomials are dense in \( L^2_\alpha(D,dA_\alpha) \), the desired result follows.

\[\square\]

**Theorem 4.1.11.** Suppose \( g \in L^2_\alpha(D,dA_\alpha) \). Then \( h_{(zg)}\cdot \) is bounded on \( L^2_\alpha(D,dA_\alpha) \) if and only if \( g \) is in the Bloch space. Furthermore

\[
\frac{\Gamma(\frac{3+\alpha}{2})^2}{(2+\alpha)\Gamma(\alpha + 2)} \|g\|_B \leq \|h_{(zg)}\cdot \| \leq \frac{1}{\alpha + 1} \|g\|_B.
\]

**Proof.** The results follow directly from Propositions 4.1.9, 4.1.10 and 4.1.11.

\[\square\]

If we apply Theorem 4.1.12 to the case of \( \alpha = 0 \), then we have an estimate of \( h_{zg} \) on the usual Bergman space as follows.

\[
\frac{\Gamma(\frac{3}{2})^2}{2\Gamma(2)} \|g\|_B \leq \|h_{zg}\| \leq \|g\|_B.
\]

Thus

\[
\frac{(\frac{1}{2})^2 \Gamma(\frac{1}{2})^2}{2} \|g\|_B \leq \|h_{zg}\| \leq \|g\|_B.
\]

Now use the fact that \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \) to obtain

\[
\frac{\pi}{8} \|g\|_B \leq \|h_{zg}\| \leq \|g\|_B.
\]

Therefore Theorem 4.1.11 implies the estimate of [Bonsall 86]. The constant \( \frac{\pi}{8} \) is the best possible for the case of usual Bergman space. Whether the constant \( \frac{\Gamma(\frac{3+\alpha}{2})^2}{(2+\alpha)\Gamma(\alpha + 2)} \) is the best
possible for the case of the weighted Bergman space is an open question.

Bonsall, [Bonsall 86], also has good estimates of the Hilbert-Schmidt norm \( \| h_{2g} \|_{S_2} \) and the trace class norm \( \| h_{2g} \|_{S_1} \) on the usual Bergman space \( L^2_0(D,dA) \) as follows.

\[
6^{-\frac{1}{2}} \| g' \|_{L^2_0(D,dA)} \leq \| h_{2g} \|_{S_2} \leq 2^{-1} \| g'' \|_{L^2_0(D,dA)},
\]

where \( g \in L^2_0(D,dA) \), and \( g' \in L^2_0(D,dA) \).

\[
6^{-1} \| g'' \|_{L^1_0(D,dA)} \leq \| h_{2g} \|_{S_1} \leq 2^{-1} \| g''' \|_{L^1_0(D,dA)},
\]

where \( g \in L^2_0(D,dA) \), \( g'' \in L^1_0(D,dA) \). In the case of the weighted Bergman space \( L^2_0(D,dA_\alpha) \) the estimates of \( \| h_{(2g)_\alpha} \|_{S_2} \) and \( \| h_{(2g)_\alpha} \|_{S_1} \) are open questions.
Bibliography


