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DISTANCE FUNCTION CONSTRUCTIONS
IN TOPOLOGICAL SPACES

BY

Laurie J. Sawyer
A.B. Mount Holyoke College, 1982
M.S. University of New Hampshire, 1984

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

December, 1990

This dissertation has been examined and approved.

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November 26, 1990

Date

With love to my parents,

Don and Ellie Sawyer

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ABSTRACT

DISTANCE FUNCTION CONSTRUCTIONS IN TOPOLOGICAL SPACES

by

Laurie J. Sawyer
University of New Hampshire, December, 1990

This work investigates the use of distance function constructions in the study of semimetrizable spaces, especially as this relates to developable, K-semimetrizable and 1-continuously semimetrizable spaces.

A *distance function* for X is a nonnegative, symmetric, real-valued function d: $X \times X \to \mathbb{R}$ such that d(p,q) = 0 iff p = q. A distance function d is *developable* iff, when $d(x_n,p) \to 0$ and $d(y_n,p) \to 0$, then $d(x_n,y_n) \to 0$; and d is a *K-distance function* iff whenever $d(x_n,p) \to 0$, $d(y_n,q) \to 0$ and $d(x_n,y_n) \to 0$, then p = q.

A topological space $(X,\mathcal{T})$ is *semimetrizable* iff there is a distance function d for X such that, for every $A \subseteq X$, d-cl(A) = $\overline{A}^{\mathcal{T}}$. A topological space is *developable* (resp. *K-*, *1-continuously*) *semimetrizable* when d is a developable (resp. K-, 1-continuous) distance function.

First, we use our approach to prove the classical metrization theorems. Then, in searching for new results, we establish characterizations involving sequences of open covers and diagonal conditions.

Theorem. $(X,\mathcal{T})$ is Hausdorff and developable semimetrizable iff it is a wΔ-space with a $G_\delta{}^*$-diagonal.

Theorem. $(X,\mathcal{T})$ is K-developable semimetrizable iff it is a wΔ-space with a regular G_δ-diagonal.

We conclude our study with characterizations which are given in terms of neighborhood structures; $\{U_n(p): n\in\mathbb{N},\ p\in X\}$ is a *neighborhood structure* for $(X,\mathcal{T})$ iff $p\in U_n(p)\in\mathcal{T}$ and $U_{n+1}(p)\subseteq U_n(p)$, for every $n\in\mathbb{N}$. We characterize open, K- and developable semimetrizable spaces. For example,

Theorem. $(X,\mathcal{T})$ is developable semimetrizable iff there is a neighborhood structure $\{U_n(p): n\in\mathbb{N},\ p\in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{U_n(p): n\in\mathbb{N}\} = \{p\}$; and

(ii) if $x_n,\ p \in U_n(y_n)$ for some $y_n \in X$, then $x_n\to p$ (in $\mathcal{T}$).

In retrospect, we have found new characterizations or improved old characterizations of developable semimetrizable spaces and other more restricted kinds of developable spaces, while our study of 1-continuously semimetrizable spaces remains quite incomplete.

TABLE OF CONTENTS

OVERVIEW

The focus of this work is on the use of distance function constructions in topological spaces. The proofs will explicitly construct distance functions, as opposed to merely proving their existence.

A *distance function for a set* X is any nonnegative, real-valued function $d\colon X \times X \to \mathbb{R}$ such that $d(x,y) = d(y,x)$, and $d(x,y) = 0$ iff $x = y$, for every $x,y \in X$.

Distance functions are appealing because of their geometric nature. As such, they come equipped, topologically, with an intrinsic notion of convergence. Given a distance function d for a set X, a sequence $\{x_n\}$ is *d-convergent* provided that $d(x_n,p) \to 0$ for some $p \in X$. They also have a "neighborhood" structure. Namely, $S_d(x,\varepsilon) = \{y \in X: d(x,y) < \varepsilon\}$ is the *sphere of radius* ε *about* x.

In attempting to describe or characterize a topological space $(X,\mathcal{T})$, we often identify neighborhoods for each point. Then, the closed sets are those which contain their limit points. Thus, the question of how to describe the convergent sequences must also be answered. In dealing with these issues, we might make use of the concept of a "distance" for the set X.

In Chapter I we include a brief description of the various kinds of distance functions and some of their properties, and show that they can indeed be used to describe these topological structures.

Chapter II shows how our approach may be used to establish the classical metrization results.

Chapter III provides new proofs of old results, and establishes new results for developable semimetrizable and 1-continuously semimetrizable spaces.

Chapter IV characterizes the topological spaces under consideration in this work with neighborhood properties.

The reader is referred to Willard's General Topology [52] for definitions and standard topological notation which are not defined in this dissertation.

CHAPTER I

AN INTRODUCTION TO DISTANCE FUNCTIONS AND THEIR TOPOLOGIES

1. Kinds of Distance Functions

A *metric* is a distance function which satisfies the triangle inequality, that is, $d(x,y) \leq d(x,z) + d(z,y)$, for every $x,y,z \in X$.

A distance function d is *continuous* iff when $d(x_n,p) \to 0$ and $d(y_n,q) \to 0$, then $d(x_n,y_n) \to d(p,q)$; it is *1-continuous* iff, for any $q \in X$, when $d(x_n,p) \to 0$, then $d(x_n,q) \to d(p,q)$.

A distance function d is *developable* iff, when $d(x_n,p) \to 0$ and $d(y_n,p) \to 0$, then $d(x_n,y_n) \to 0$; it is *coherent* iff when $d(x_n,p) \to 0$ and $d(x_n,y_n) \to 0$, then $d(y_n,p) \to 0$.

A distance function d is a *K-distance function* iff whenever $d(x_n,p) \to 0$, $d(y_n,q) \to 0$, and $d(x_n,y_n) \to 0$, then $p = q$; it *has unique limits* iff when $d(x_n,p) \to 0$ and $d(x_n,q) \to 0$, then $p = q$.

I.1.1 Theorem. A metric is a continuous distance function. Any continuous distance function is a 1-continuous, developable, K-distance function. K-distance functions and 1-continuous distance functions always have unique limits.

These facts are summarized in the following diagram:

$$
\text{metric} \Rightarrow \text{continuous} \Rightarrow \begin{cases} \text{developable} \\ \text{1-continuous} \\ \text{K-distance function} \end{cases}
$$

This thesis will focus on developable distance functions, 1-continuous distance functions, and K-distance functions.

Given a distance function d for a set X, a sequence $\{x_n\}$ is *d-cauchy* iff for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for every $m, n \geq N$, $d(x_m, x_n) < \varepsilon$. One expects that d-convergent sequences are d-cauchy. However, this is not always the case. An immediate observation is the following theorem.

1.1.2 Theorem. If d is a distance function for X, the following are equivalent:

(i) d is developable;

(ii) for every $p \in X$, there are spheres, centered at p, of arbitrarily small diameter;

(iii) d-convergent sequences are d-cauchy.

The remainder of this section is devoted to constructing examples which will be used throughout this thesis. In each case, we define a distance function for the given set.

I.1.3 Example (The Single Sequence Space).

Let $X = \{a_n: n \in \mathbb{N}\} \cup \{0\}$, where $a_n = 1/3^n$, and define d as follows:

$$d(a_n,a_m) = 1,\ n \neq m;\ \text{and}$$
$$d(x,y) = |x - y|,\ \text{otherwise.}$$

The distance function d is a K-distance function which is not developable, since $\{a_n\}$ is a d-convergent sequence which is not d-cauchy. It is not 1-continuous since $d(a_n,0) \to 0$, but for any $m \in \mathbb{N}$, $d(a_n,a_m) \not\to d(0,a_m)$.

I.1.4 Example (The Double Sequence Space).

Let $X = A \cup B$, where $A = \{a_n: n \in \mathbb{N}\} \cup \{0\}$, $B = \{b_n: n \in \mathbb{N}\} \cup \{1\}$, $a_n = 1/3^n$, and $b_n = 1 + a_n$. Define d as follows:

$$d(a_n,b_n) = d(b_n,a_n) = 1/3^n;\ \text{and}$$
$$d(x,y) = |x - y|,\ \text{otherwise.}$$

Then, d is developable (each d-convergent sequence is d-cauchy) and 1-continuous. It is not a K-distance function because $d(a_n,0) \to 0$, $d(b_n,1) \to 0$, $d(a_n,b_n) \to 0$, but $0 \neq 1$. Thus, a 1-continuous, developable distance function need not be a continuous distance function nor a K-distance function.

I.1.5 Example (Galvin [20]). *Another double sequence space*

Let $X = A \cup B$, where $A = \{a_n: n \in \mathbb{N}\} \cup \{0\}$, $B = \{b_n: n \in \mathbb{N}\}$, $a_n = 1/3^n$, $b_n = -a_n$. Define d as follows:

$$d(x,y) = d(y,x) = \max\{2y, y-x\},\ \text{if } x < 0 < y;\ \text{and}$$

$d(x,y) = |x - y|$, otherwise.

A sequence $\{x_n\}$ is d-convergent iff it is eventually constant or $|x_n| \to 0$. Therefore, d is a developable K-distance function. However, d is not 1-continuous, since $d(b_n,0) \to 0$, but for any $k \in \mathbb{N}$, $d(b_n,a_k) \to 2a_k \neq d(0,a_k)$.

I.1.6 Definition. Two distance functions d_1 and d_2 *are equivalent* provided that $d_1(x_n,p) \to 0$ iff $d_2(x_n,p) \to 0$ for any sequence $\{x_n\}$ in X.

In both Examples I.1.3 and I.1.4 (the Single and Double Sequence Spaces), the usual Euclidean metric $\rho(x,y) = |x - y|$ is an equivalent distance function for X. This is not the case in the remaining examples of this section.

I.1.7 Example (Arhangel'skii [3]).
Let $X = [0,1]$, and $A \subseteq X$ be such that $A = \{1/3^n : n \in \mathbb{N}\} \cup \{0\}$.
Define d as follows:

$d(x,0) = d(0,x) = 1$ if $x \notin A$; and

$d(x,y) = |x-y|$, otherwise.

Then, d is a developable K-distance function that is not 1-continuous (for $1 \neq x \notin A$, we have $d(a_n,0) \to 0$ while $d(a_n,x) \not\to d(0,x)$).

I.1.8 Example (Shore-Uhland).

Let $X = (0,1)$ and define d as follows:

$$d(x,y) = \begin{cases} \min\{x,y\}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

A sequence $\{a_n\}$ is d-convergent iff it has at most one constant subsequence or it has a subsequence $\{a_{n_k}\}$ such that $|a_{n_k}| \to 0$. Therefore, d is a developable distance function for X. This distance function fails to have unique limits, since $\{1/3^n\}$ converges to any point in X; hence, d is neither 1-continuous nor a K-distance function.

I.1.9 Notation. When $X \subseteq \mathbb{R} \times \mathbb{R}$, we denote $x \in X$ by (x_1,x_2) and use $|x - y|$ to denote the usual Euclidean distance, $\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$, between x and y.

I.1.10 Example (The Split Disk Space).

Let $X = A \cup B$ where $A = \mathbb{R} \times \{0\}$, $B = \mathbb{R} \times (0,1)$ and define d as follows:

$$d(x,y) = d(y,x) = \begin{cases} |x - y| + 1, & \text{if } x,y \in A,\ x \neq y; \\ |x - y|, & \text{otherwise.} \end{cases}$$

The distance function d is a developable K-distance function. However, d is not 1-continuous since we can choose points $p,q \in A$, and a sequence $\{x_n\}$ in B such that $d(x_n,p) \to 0$, while $d(x_n,q) \not\to d(p,q)$.

I.1.11 Example (McAuley's Bow-Tie Space [41]).

Let $X = A \cup B$ where $A = \mathbb{R} \times \{0\}$, $B = \mathbb{R} \times (0,1)$. For any $x,y \in X$, define $\alpha(x,y)$ to be the smallest positive angle (expressed in radians) between the x-axis and the line joining x and y. Define d as follows:

$$d(x,y) = \begin{cases} |x-y|+\alpha(x,y), & \text{if } x_2 = 0 \text{ or } y_2 = 0; \\ |x-y|, & \text{otherwise.} \end{cases}$$

The distance function d is a K-distance function which is not developable (spheres at $p \in A$ have diameter at least $\pi/2$). Since we can choose points $p \in A$, $q \in B$, and a sequence $\{x_n\}$ in B such that $d(x_n,p) \to 0$, while $d(x_n,q) \not\to d(p,q)$, d is also not 1-continuous.

I.1.12 Example (Borges [7]).

Let $X = A \cup B$ where $A = \mathbb{P}_0 \times \{0\}$, $\mathbb{P}_0 = \mathbb{R} - \{x \in \mathbb{R}: x \in \mathbb{Q}$ or $x = q \pm \sqrt{2}/n$ for some $q \in \mathbb{Q}, n \in \mathbb{N}\}$, and $B = \mathbb{Q} \times \{\sqrt{2}/n: n \in \mathbb{N}\}$. For any $x,y \in X$, define $\alpha(x,y)$ as in Example I.1.11 and

$$d(x,y) = \begin{cases} |x-y|, & \text{if } \alpha(x,y) < \pi/4; \\ |x-y|+\alpha(x,y), & \text{if } \alpha(x,y) > \pi/4. \end{cases}$$

Notice that $\alpha(x,y) \neq \pi/4$ for every $x,y \in X$. The distance function d is a 1-continuous K-distance function which is not developable (spheres at $p \in \mathbb{P}_0$ have diameter at least $\pi/2$).

I.1.13 Example (Burke [8]).

Let $X = A \cup B$, where $A = A_0 \cup A_1$, $A_0 = \mathbb{P} \times \{0\}$, $A_1 = \mathbb{P} \times \{-1\}$, $B = \mathbb{Q} \times \{q \in \mathbb{Q}: 0 < q < 1\}$. For each $a \in A_0$, let $W_a = \{x \in X: x_2 \geq x_1 + a \text{ and } x_1 \geq a\}$. For each $a \in A_1$, let $W_a = \{(a,-1)\} \cup \{x \in X: x_2 \geq -x_1 + a \text{ and } x_1 \leq a\}$. Define a distance function d for X as follows:

$$\text{For each } a \in A,\ d(a,x) = d(x,a) = \begin{cases} x_2, & \text{if } x \in W_a; \\ 1, & \text{if } x \notin W_a; \end{cases}$$

and $d(x,y) = |x - y|$, otherwise.

The distance function d is developable since any d-convergent sequence is d-cauchy.

By using an argument similar to that given in Example I.1.11 (McAuley's Bow-Tie Space), it follows that d is not 1-continous.

To show that d is not a K-distance function, suppose $a \in \mathbb{Q}$. If we let $p = (a,0)$, $q = (a,-1)$, $x_n = (a + 1/2^n, 1/2^n)$ and $y_n = (a - 1/2^n, 1/2^n)$ for each $n \in \mathbb{N}$, we have $d(x_n,p) \to 0$, $d(y_n,q) \to 0$, and $d(x_n,y_n) \to 0$, while $p \neq q$.

I.1.14 Example (The Isbell-Mrówka Spaces [22; 51]).

Let $X = A \cup B$ where $B = \mathbb{N}$, and A is an infinite maximal family $\mathcal{R}$ of infinite, almost disjoint subsets of $\mathbb{N}$.

Define d as follows:

$$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y; \\ 1/2^x, & \text{if } x \in y \in \mathcal{R}; \\ |1/2^x - 1/2^y|, & \text{if } x,y \in \mathbb{N}; \\ 1, & \text{otherwise.} \end{cases}$$

For $p \in A$, $p \subsetneq \mathbb{N}$; if we denote $p = \{p_n\}$ with $p_n < p_{n+1}$, then $d(p_n,p) \to 0$. Essentially, for any $p \in A$, $\{p_n\}$ and its subsequences are the only nonconstant convergent sequences. Since any d-convergent sequence is d-cauchy, d is a developable distance function for X.

The distance function d is not 1-continuous. Consider $p \in A$, and $q \in \mathbb{N} - A$. Then $d(p_n,p) \to 0$, while $d(p_n,q) \to 1/2^q \neq 1 = d(p,q)$.

The distance function d is not a K-distance function. Consider $p,q \in A$, $p \neq q$. Then $d(p_n,p) \to 0$, $d(q_n,q) \to 0$, and $d(p_n,q_n) \to 0$, while $p \neq q$.

I.1.15 Example (Heath's V-space [27]).
Let $X = A \cup B$, where $A = \mathbb{R} \times \{0\}$ and $B = \mathbb{R} \times (0,1)$. For each $a \in A$, let $V_a = \{x \in X\colon x_1 = a + x_2 \text{ or } x_1 = a - x_2\}$ (i.e., a "V" with vertex at (a,0), sides with slopes of 1 and -1). Define d as follows:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \max(x_2,y_2), & \text{if } x,y \in V_a \text{ for some } a \in A; \\ 1, & \text{otherwise.} \end{cases}$$

A consideration of cases shows that d is a continuous distance function.

1.1.16 Example (The Niemytzki Space [22; **3K**]).

Let $X = A \cup B$ where $A = \{x \in \mathbb{R} \times \mathbb{R}: x_1^2 + x_2^2 = 1\}$ and $B = \{x \in \mathbb{R} \times \mathbb{R}: x_1^2 + x_2^2 < 1\}$. For any disk D, let $\alpha(D)$ be the usual Euclidean diameter. Define a distance function d for X as follows:

$$d(x,y) = d(y,x) = \inf\{\alpha(D): D \text{ is a disk with } x,y \in D \subseteq X\}.$$

Then d is a continuous distance function for X [37].

We conclude this section by recording some notation which will be used throughout this work.

1.1.17 Notation. For $A, B \subseteq X$, $x \in X$, and $\varepsilon > 0$, we use the following notation:

$d(x,A] = \inf\{d(x,y): y \in A\}$ is *the distance from* x *to* A;

$d[B,A] = \inf\{d(x,A]: x \in B\}$ is the *distance between* A *and* B;

$d\text{-cl}(B) = \{x \in X: d(x,B] = 0\}$ is the *d-closure* of B;

$\delta_d[A] = \sup\{d(a,b): a,b \in A\}$ is the *d-diameter* of A; and

$\mathcal{S}_d = \{S_d(x,\varepsilon): x \in X, \varepsilon > 0\}$ is the *set of spheres generated by* d.

2. Topological Connection

For any topological space $(X,\mathcal{T})$ we may consider its convergent sequences, that is, any sequence $\{x_n\}$ such that $x_n \to p$ (in $\mathcal{T}$) for some $p \in X$. For any distance function d, we have its d-convergent sequences, that is, any sequence $\{x_n\}$ such that $d(x_n,p) \to 0$ for some $p \in X$. This leads us to the question of when these two types of convergence are the same.

First, we note that any distance function d determines a topology for X, namely,

$\mathcal{T}_d = \{A \subseteq X$: when $a \in A$, then $S_d(a,\alpha) \subseteq A$ for some $\alpha\}$, which is called the *symmetric topology* for X [3].

I.2.1 Definition. A distance function d is a *symmetric for* $(X,\mathcal{T})$ iff $\mathcal{T} = \mathcal{T}_d$.

I.2.2 Remark. *Concerning convergence of sequences*

(i) For any distance function d, when $d(x_n,p) \to 0$, then $x_n \to p$ (in $\mathcal{T}_d$).

(ii) However, the converse may fail.

Let us consider the set $X = A \cup B$, where $A = \{a_n: n \in \mathbb{N}\} \cup \{0\}$, $B = \{b_n: n \in \mathbb{N}\}$, $a_n = 1/3^n$, $b_n = -1/3^n$ with the following distance function d for X:

$$d(a_m,b_n) = d(b_n,a_m) = 1/3^n;$$

$$d(0,b_n) = d(b_n,0) = 1; \text{ and}$$

$d(x,y) = |x - y|$, otherwise.

In this example, $b_n \to 0$ (in $\mathcal{T}_d$), but $d(b_n,0) \not\to 0$. The pathology of this example stems from the fact that for any $p \in X$, $d(b_n,p) \to 0$.

(iii) On the other hand, if d is a distance function with unique limits, then

$x_n \to p$ (in $\mathcal{T}_d$) iff $d(x_n,p) \to 0$.

<u>I.2.3 Remark</u>. We note that, for a metric d, the following are equivalent:

(i) d is a symmetric for $(X,\mathcal{T})$;

(ii) $\mathcal{S}_d = \{S_d(x,\varepsilon): x \in X, \varepsilon > 0\}$ is a base for $\mathcal{T}$;

(iii) for every $A \subseteq X$, $\overline{A}^{\mathcal{T}} = \{x \in X: d(x,A] = 0\} = d\text{-cl}(A)$, where $\overline{A}^{\mathcal{T}}$ is the topological closure.

We inquire about generalizing this result, and find that, for an arbitrary distance function d, (iii) $\Rightarrow$ (i); however, the converse may fail. In Arhangel'skii's Example (I.1.7), the d-closure is not a topological closure (since $d\text{-cl}(X-A) = (0,1] \neq [0,1] = d\text{-cl}[d\text{-cl}(X-A)]$).

Historically, distance functions whose d-closure is a topological closure have been called semi-metrics [53].

<u>I.2.4 Definition</u>. A distance function d is a <u>*semimetric for*</u> $(X,\mathcal{T})$ iff for every $A \subseteq X$, $d\text{-cl}(A) = \overline{A}^{\mathcal{T}}$.

A topological space is <u>*semimetrizable*</u> iff there is a semimetric for $(X,\mathcal{T})$.

Note that, if d is a semimetric for $(X,\mathcal{T})$, then $\mathcal{T} = \mathcal{T}_d$. When $\mathcal{T}_d \subset \mathcal{T}$, we say that d *is a* *distance function on* $(X,\mathcal{T})$.

I.2.5 Theorem. For any distance function d, the following are equivalent:

(i) d is a semimetric for $(X,\mathcal{T})$;

(ii) for every $x \in X$, $\{S_d(x,\varepsilon): \varepsilon > 0\}$ is a neighborhood base for x in $\mathcal{T}$; and

(iii) $(X,\mathcal{T})$ is first countable and $x_n \to p$ (in $\mathcal{T}$) iff $d(x_n,p) \to 0$.

I.2.6 Remark. *Symmetrizable vs. semimetrizable spaces*

If $(X,\mathcal{T})$ is semimetrizable, then it is symmetrizable. For Hausdorff spaces, the converse holds if $(X,\mathcal{T})$ is first countable.

I.2.7 Remarks. *When $\mathcal{S}_d$ is a base for $\mathcal{T}$*

(i) In the Single Sequence Space (Example I.1.3), the set of spheres generated by d, $\mathcal{S}_d = \{S_d(x,\varepsilon): x \in X, \varepsilon > 0\}$ is not a base for a topology (since the spheres are not necessarily open, e.g., $S_d(1/3,1/2)$ is not open).

(ii) If d is a semimetric for $(X,\mathcal{T})$ and $\mathcal{S}_d \subsetneq \mathcal{T}$, then $\mathcal{S}_d$ is a base for $(X,\mathcal{T})$. Surprisingly, the converse may fail.

Consider Galvin's Example (I.1.5). The distance function d determines a semimetric topology $\mathcal{T}_1$, while the set of spheres generated by d is a base for another topology $\mathcal{T}_2$. Both the sequence $\{b_n\}$ and the sequence $\{a_n\}$ converge to 0 in $\mathcal{T}_1$. However, the sequence $\{b_n\}$ does not converge to 0 in $\mathcal{T}_2$. Note that A is in $\mathcal{T}_2$ but not in $\mathcal{T}_1$.

With the exception of the Single Sequence Space (Example I.1.3), Arhangel'skii's Example (I.1.7), and Galvin's Example (I.1.5), all of our examples have distance functions whose spheres form a local base.

(iii) If d is a symmetric for $\mathcal{T}$ and $\mathcal{S}_d \subseteq \mathcal{T}$, then d is a semimetric for $(X,\mathcal{T})$ and $\mathcal{S}_d$ is a base for $\mathcal{T}$.

(iv) If $x \in S_d(p,\alpha) \Rightarrow S_d(x,\delta) \subseteq S_d(p,\alpha)$ for some $\delta > 0$, then $\mathcal{S}_d$ is a base for $\mathcal{T}_d$ and d is a semimetric for $\mathcal{T}_d$.

If d is 1-continuous, then d is a semimetric for $\mathcal{T}_d$ such that $\{S_d(p,\varepsilon): \varepsilon > 0\}$ is a local base for p in $(X,\mathcal{T})$.

I.2.8 Remark [28], [35]. *Countability conditions in Semimetrizable spaces*

If $(X,\mathcal{T})$ is a semimetrizable space, then

(i) $(X,\mathcal{T})$ is a first countable T_1-space;

(ii) $(X,\mathcal{T})$ is Lindelöf iff it is $\aleph_1$-compact; and

(iii) if $(X,\mathcal{T})$ is Lindelöf, then it is separable.

I.2.9 Remark. *Compactness conditions in Semimetrizable spaces*

(i) If $(X,\mathcal{T})$ is a semimetrizable space, then $(X,\mathcal{T})$ is compact if and only if it is sequentially compact.

(ii) If $(X,\mathcal{T})$ is semimetrizable, compact space, then it is metrizable.

I.2.10 Remark. *Separation in Semimetrizable spaces*

(i) Semimetrizable spaces need not be Hausdorff, e.g., the Shore-Uhland Example (I.1.8).

(ii) Hausdorff semimetrizable spaces need not be regular, e.g., the Split Disk Space, (Example I.1.10).

(iii) Tychonoff semimetrizable spaces need not be normal, e.g., Borges' Example (I.1.12), Heath's V-space (Example I.1.3), The Isbell-Mrówka Spaces (Example I.1.14), and the Niemytzki Space (Example I.1.16).

A topological space is *continuously semimetrizable* iff there is a semimetric d for $(X,\mathcal{T})$ which is continuous; in this case we call d a continuous semimetric for $(X,\mathcal{T})$. Developable semimetrizable and 1-continuously semimetrizable spaces are defined similarly. A topological space is *K-semimetrizable* iff there is a semimetric d for $(X,\mathcal{T})$ which is a K-distance function. In this case d is called a *K-semimetric* for $(X,\mathcal{T})$.

I.2.11 Remark. *Concerning developable semimetrics*

(i) In developable semimetrizable spaces, second countable, Lindelöf, and $\aleph_1$-compact are equivalent [28], [35].

(ii) Hausdorff developable semimetrizable spaces need not be regular (the Split Disk Space, Example I.1.10).

(iii) If a topological space is Hausdorff, developable semimetrizable, and paracompact, then it is metrizable [4].

1.2.12 Theorem. *Concerning 1-continuous semimetrics*

If d is a 1-continuous distance function, then $(X,\mathcal{T}_d)$ is a completely regular Hausdorff space such that $\mathcal{S}_d$ is a base for $\mathcal{T}_d$. Thus, d is a semimetric for $(X,\mathcal{T}_d)$.

1.2.13 Theorem. *Concerning K-semimetrics*

If d is a semimetric for $(X,\mathcal{T})$, then,

(i) if d is a K-distance function, then $(X,\mathcal{T})$ is Hausdorff;

(ii) d is a K-distance function iff d separates disjoint compact sets in $\mathcal{T}$ (i.e., for disjoint compact sets $A,B \subseteq X$, $d[A,B] > 0$).

Proof. (i) Suppose d is a semimetric for $(X,\mathcal{T})$ and that X is not Hausdorff. Choose distinct points $x,y \in X$ which cannot be separated by disjoint open sets. Since $(X,\mathcal{T})$ is first countable, there are decreasing neighborhood bases, respectively, $\{U_n(x): n\in \mathbb{N}\}$ for x, and $\{U_n(y): n\in \mathbb{N}\}$ for y. Now choose $z_n \in U_n(x) \cap U_n(y)$ for every $n \in \mathbb{N}$. Thus $z_n \to x$ and $z_n \to y$. Since d is a semimetric for $\mathcal{T}$, $d(z_n,x) \to 0$, $d(z_n,y) \to 0$, and $d(z_n,z_n) \to 0$, while $x \neq y$. Therefore d is not a K-distance function.

(ii) First, suppose that d is a K-semimetric for $(X,\mathcal{T})$. Suppose that A and B are compact subsets of X such that $d[A,B]=0$. Thus, we may choose sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that $d(x_n,y_n) \to 0$. Since A and B are compact subsets of a semimetric space, they are also sequentially compact. Thus, we may choose subsequences $\{a_n\}$ of $\{x_n\}$ in A and $\{b_n\}$ of $\{y_n\}$ in B such that $a_n \to a \in A$, $b_n \to b \in B$, and $d(a_n,b_n) \to 0$. It follows that $d(a_n,a) \to 0$, $d(b_n,b) \to 0$, and $d(a_n,b_n) \to 0$. Since d is a K-distance function, a = b and, hence, $A \cap B \neq \emptyset$.

Conversely suppose that d is not a K-distance function. Then there are sequences $\{x_n\}$, $\{y_n\}$ in X such that $d(x_n,p) \to 0$, $d(y_n,q) \to 0$, and $d(x_n,y_n) \to 0$ for some $p,q \in X$, and $p \neq q$. It follows that $A = [\{x_n: n \in \mathbb{N}\} - (\{y_n: n \in \mathbb{N}\} \cup \{q\})] \cup \{p\}$ and $B = [\{y_n: n \in \mathbb{N}\} \cup \{q\}] - \{p\}$ are disjoint compact sets with $d[A,B] = 0$, i.e., d does not separate disjoint compact sets in $\mathcal{T}$.

3. Historical Remarks

The history of distance functions is as extensive as the history of general topology itself. At its inception, topological activity centered around the construction of distance functions. Given a particular type of distance function, it was not uncommon for the researcher to try to find an equivalent metric. In trying to find an appropriate theory of limit points, which would ultimately describe the closed sets of a topology, this approach made sense because of a metric's relationship to Euclidean geometry. We begin this section by sketching the progression of the development of this theory.

In his 1906 thesis Fréchet [15] initiated the study of distance functions. Given a *voisinage*, which is a distance function d such that for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $d(p,x) < \varepsilon$ and $d(x,y) < \varepsilon$ implies that $d(p,y) < \delta(\varepsilon)$, he asked the question of whether or not there was an equivalent metric. At this time, he called a metric an *écart*. This type of distance function was appealing because it had a geometric, as opposed to a topological property. In 1917 Chittenden [10] showed that, for any voisinage, there is an equivalent metric. In 1918, Pitcher and Chittenden [46] introduced the idea of a *local écart*, which is a distance function d such that for every $\varepsilon > 0$ and every $p \in X$, there is $\delta > 0$ such that, when $d(p,x) < \delta$ and $d(x,y) < \delta$, then $d(p,y) < \varepsilon$. They proved that, if the given topological space was compact, for any local écart, there was an equivalent metric. Whether or not the condition of being compact was necessary was left as an open question. In 1927 Niemytzki [45] showed, by establishing the existence of an equivalent metric, that for any local écart, there is always an equivalent metric. He gave two proofs, one based on the work of Chittenden and the other on the work of Alexandroff and Urysohn [2]. We present the second approach in the next chapter, but instead of giving an existence proof, the desired metric is explicitly constructed.

As the study of distance functions continued, it became clear that the following two problems needed to be resolved: (i) for limit points generated by arbitrary distance functions, a limit point of the set of limit points of a set A was not necessarily a limit point of A; and (ii) d-convergent sequences need not be d-cauchy.

Pitcher and Chittenden introduced developable and coherent distance functions and showed that, for coherent distance functions, a limit of a sequence of limit points of a set A is itself a limit point of A. As it was noted in Chapter I, it is the developable distance functions for which d-convergent sequences are necessarily d-cauchy.

The study of metrics culminated with a result by Frink in 1937 which we discuss in more detail in chapter II.

Fréchet's introduction of abstract spaces with a topological structure marked the beginning of the evolution of the notion of a topological space. In the 1920's, the open set was one of the most important topological considerations. The definition of a topological space which is generally accepted today was formulated (in terms of a closure operator) by Kuratowski in 1922 [14]. After the notion of a topological space was established, the question of what topological spaces were generated from metrics arose. We discuss the earliest solutions to this question, which involve sequences of open covers, in chapter II. This work of Alexandroff and Urysohn (1923) and of Bing, Smirnov, and Nagata (1951) motivated the "Moore School" (developable topological spaces) which we discuss in chapter III. In chapter IV we discuss neighborhood characterizations, as motivated by the "Jones School", including the work of McAuley, Heath, and Hodel.

CHAPTER II

EXPLICIT METRIZATION

This chapter provides an historical basis for this work. We highlight distance function constructions and provide direct proofs of the most famous metrization theorems.

1. Constructing Metrics from other Distance Functions

Historically, these results illustrate the earliest of the metrization investigations. We note an early result of Pitcher and Chittenden and suggest that it is a precursor to the powerful result of Frink.

II.1.1 Theorem (Pitcher-Chittenden's Theorem [46]). If d is a coherent distance function for X, then there is an equivalent coherent developable distance function for X.

Proof. Suppose that d is a coherent distance function for X. Then the distance function d_1 such that

$$d_1(x,y) = \inf \{d(x,z) + d(z,y): z \in X\}$$

is an equivalent coherent developable distance function for X.

It follows immediately from the definitions that a distance function is coherent iff it is a local écart. Since any local écart is equivalent to a metric, the above theorem is really about metrizable spaces. This leads us to a Frink's result. We give a proof which invokes the explicit construction of the desired metric.

__II.1.2 Frink's Theorem__ (1937; [17]). If d is a distance function for X such that

(*) for any $a,b,z \in X$, it is not possible that both
$d(a,z) < \frac{1}{2}d(a,b)$ and $d(b,z) < \frac{1}{2}d(a,b)$,

then there is a metric ρ which is equivalent to d; in fact, the distance function ρ such that

$$\rho(x,y) = \inf\{d(x,z_1)+d(z_1,z_2)+ \ldots +d(z_n,y): z_1,z_2, \ldots z_n \in X \text{ for some } n \in \mathbb{N}\}$$

is such a metric.

__Proof__. Suppose that d is a distance function for X which satisfies (*). We use (*) to prove:

__Frink's Claim__. For any $z_1,z_2, \ldots ,z_n$ in X,

(**) $d(a,b) \leq 2d(a,z_1) + 4d(z_1,z_2)+ \ldots + 4d(z_{n-1},z_n) + 2d(z_n,b)$.

__Proof__. The proof is given by induction on n.

(a) For $n = 1$, the property is obvious from (*) since either $\frac{1}{2}d(a,b) \leq d(a,z_1)$ or $\frac{1}{2}d(a,b) \leq d(b,z_1)$.

(b) Next, assume that the property holds for fewer than n z's and consider $z_1, z_2, \ldots, z_n$.

If either $\frac{1}{2}d(a,b) \le d(a,z_1)$ or $\frac{1}{2}d(a,b) \le d(b,z_n)$, then the property (**) again holds. Otherwise, $d(a, z_1) < \frac{1}{2}d(a,b)$ and $d(b,z_n) < \frac{1}{2}d(a,b)$. From the second of these and (*) it follows that $\frac{1}{2}d(a,b) \le d(a, z_n)$.

Thus, there is a $k^* \in \mathbb{N}$ such that $1 \le k^* < n$ and $d(a,z_{k^*}) < \frac{1}{2}d(a,b) \le d(a,z_{k^*+1})$.

From the first inequality and (*) it follows that:

$$\begin{aligned}\tfrac{1}{2}d(a,b) &\le d(z_{k^*},b)\\ &\le 2d(z_{k^*}, z_{k^*+1}) + 4d(z_{k^*+1},z_{k^*+2}) + \ldots +\\ &\quad 4d(z_{n-1},z_n) + 2d(z_n,b)\end{aligned}$$

and from the second:

$$\begin{aligned}\tfrac{1}{2}d(a,b) &\le d(a,z_{k^*+1})\\ &\le 2d(a,z_1)+4d(z_1,z_2)+ \ldots +4d(z_{k^*-1},z_{k^*})+2d(z_{k^*},z_{k^*+1}).\end{aligned}$$

Thus,

$$d(a,b) \le 2d(a,z_1)+4d(z_1,z_2)+ \ldots +4d(z_{n-1},z_n)+2d(z_n,b).$$

Next, consider the distance function ρ for X such that:

$$\rho(x,y) = \inf\{d(x,z_1) + d(z_1,z_2) + \ldots + d(z_n,y) : z_1,z_2, \ldots ,z_n \in X \text{ for some } n \in \mathbb{N}\}.$$

It follows, by Frink's claim, that $\rho \le d \le 4\rho$. Therefore ρ is a metric such that $d(x_n,p) \to 0$ iff $\rho(x_n,p) \to 0$. Hence ρ is equivalent to d.

II.1.3 Remark. Frink's original theorem assumed that the distance function d was such that, when $d(x,z) < \varepsilon$ and $d(z,y) < \varepsilon$, then $d(x,y) < 2\varepsilon$. Our version of Frink's result uses a formally weaker condition which we call $(*)$.

2. The Classical Metrization Theorems

This section begins with an alternative construction technique which involves the use of a countable family of pseudometrics. We use this approach to prove several of the early metrization theorems.

II.2.1 Theorem. For any first countable Hausdorff space $(X,\mathcal{T})$, if $\{d_k: k \in \mathbb{N}\}$ is a countable family of pseudometrics for X such that, for any sequence $\{x_n\}$ in X,

(***) $x_n \to p$ (in $\mathcal{T}$) if and only if, for each k, $d_k(x_n,p) \to 0$;

then $\rho = \Sigma \min\{d_k, 1/2^k\}$ is a metric for $(X,\mathcal{T})$.

Proof. Clearly, ρ is a pseudometric and $x_n \to p$ (in $\mathcal{T}$) if and only if $\rho(x_n,p) \to 0$. Since $(X,\mathcal{T})$ is first countable, ρ is a pseudometric for $(X,\mathcal{T})$. Finally, ρ is a metric because $(X,\mathcal{T})$ is Hausdorff.

II.2.2 The Urysohn-Tychonoff Theorem (1925; [50], [51]). A second countable, regular Hausdorff space is metrizable.

Proof. Suppose that $\mathcal{B}$ is a countable base for the regular, Hausdorff topology $\mathcal{T}$ for X. Then $\mathcal{T}$ is first countable, and since $\mathcal{T}$ is regular and Lindelöf, it follows ([14], [52, p. 111]) that it is also normal.

Next, index the countable set $\Lambda = \{(V,U): V,U \in \mathcal{B}$ with $\overline{V} \subseteq U\}$ with the positive integers k. For each k, use Urysohn's Lemma [52, p. 102] to obtain a continuous function $f_k: X \to [0,1]$ such that:

$$f_k(x) = \begin{cases} 0, & \text{if } x \in V_k; \\ 1, & \text{if } x \notin U_k. \end{cases}$$

and use f_k to define a pseudometric d_k for X as follows:

$$d_k(x,y) = | f_k(x) - f_k(y) |.$$

Since each d_k is continuous, it follows that, when $x_n \to p$ (in $\mathcal{T}$), then $d_k(x_n,p) \to 0$. Conversely, suppose that, for each k, $d_k(x_n,p) \to 0$. If $p \in G \in \mathcal{T}$, then since $(X,\mathcal{T})$ is regular, there are basic open sets V and U in $\mathcal{B}$ such that $p \in V \subseteq \overline{V} \subseteq U \subseteq G$. Thus, $(V,U) = (V_k,U_k)$ for some k.

It follows that the sphere $S_{d_k}(p,1) \subseteq G$ so that, from our supposition, x_n is eventually in G, and, therefore, $x_n \to p$ (in $\mathcal{T}$). We have now shown that $x_n \to p$ (in $\mathcal{T}$) if and only if, for each k, $d_k(x_n,p) \to 0$, i.e., that (***) holds.

Since $(X,\mathcal{T})$ is first countable, Hausdorff, and satisfies (***), $\rho = \Sigma \min\{d_k, 1/2^k\}$ is a metric for $\mathcal{T}$.

II.2.3 The Nagata, Smirnov, or Bing Theorem (1951; [4], [44], [49]). A regular, Hausdorff space with a σ-locally finite base is metrizable.

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \mathbb{N}\}$, the countable union of locally finite families $\mathcal{B}_n$, be a base for the regular Hausdorff topology $\mathcal{T}$ for X. Since $\mathcal{B}$ is a countable union of locally finite families, it follows that $\mathcal{T}$ is first countable. $\mathcal{T}$ is paracompact, since we have a regular T_1-space such that each open cover has an open σ-locally finite refinement [52, p.146] and therefore it is also normal.

Next, index the countable set $\mathbb{N} \times \mathbb{N}$ with the positive integers k. For each (m_k, n_k) in $\mathbb{N} \times \mathbb{N}$, if $U \in \mathcal{B}_{n_k}$ and $V_{m_k} = \cup\{V : V \in \mathcal{B}_{m_k} \text{ with } \overline{V} \subsetneq U\}$, then we may use Urysohn's Lemma to obtain a continuous function $f_{k,U}: X \rightarrow [0,1]$, such that:

$$f_{k,U}(x) = \begin{cases} 0, & \text{if } x \in V_{m_k}; \\ 1, & \text{if } x \notin U. \end{cases}$$

Note that, because locally finite families are always closure-preserving, $V_{m_k} = \cup\{\overline{V} : V \in \mathcal{B}_{m_k} \text{ with } \overline{V} \subsetneq U\} \subsetneq U$.

Next, for each k, define the pseudometric d_k for X such that:

$$d_k(x,y) = \sum\{ \mid f_{k,U}(x) - f_{k,U}(y) \mid : U \in \mathcal{B}_{n_k} \}.$$

Note that, due to the fact that we have a σ-locally finite base, for any $x,y \in X$, there are only finitely many nonzero terms in this sum.

Since each d_k is continuous, it follows that, when $x_n \to p$ (in $\mathcal{T}$), then $d_k(x_n,p) \to 0$. Conversely, suppose that, for each k, $d_k(x_n,p) \to 0$. If $p \in G \in \mathcal{T}$, then since $\mathcal{T}$ is regular, there are basic open sets V and U in $\mathcal{B}$ such that $p \in V \subsetneq \overline{V} \subsetneq U \subsetneq G$. Therefore, $V \in \mathcal{B}_{m_k}$ and $U \in \mathcal{B}_{n_k}$ for some k.

It follows that the sphere $S_{d_k}(p,1) \subsetneq G$ so that, from our supposition, x_n is eventually in G, and, therefore, $x_n \to p$ (in $\mathcal{T}$). We have now shown that $x_n \to p$ (in $\mathcal{T}$) if and only if, for each k, $d_k(x_n,p) \to 0$, i.e., that $(***)$ holds.

Since $(X,\mathcal{T})$ is first countable, Hausdorff, and $(***)$ holds, $\rho = \Sigma\min\{d_k, 1/2^k\}$ is a metric for $\mathcal{T}$.

<u>II.2.4 Remark</u>. In this proof, we used paracompactness to imply normality. However, it can be shown directly that a regular space with a σ-locally finite base is always normal.

We conclude this section with the following well known, but non-classical, result.

<u>II.2.5 Theorem</u>. Any compact Hausdorff space whose diagonal is a G_δ-set in $X \times X$ is metrizable.

<u>Proof</u>. Suppose that $(X,\mathcal{T})$ is a compact Hausdorff space whose diagonal is a G_δ-set in $X \times X$. Since $X \times X$ is normal, the

diagonal (a closed G_δ-set) is the zero set of a real-valued, continuous function $F: X \times X \to [0,1]$. We may also assume that $F(x,y) = F(y,x)$.

For each $x \in X$, let $U_n(x) = \{y \in X: F(x,y) < 1/2^n\}$. Then $\{U_n(x): n \in \mathbb{N}\}$ is a countable local base for x. It follows that $(X,\mathcal{T})$ is separable with countable dense set $A = \{a_k: k \in \mathbb{N}\}$. Hence, for each k, $d_k(x,y) = |F(a_k,x) - F(a_k,y))|$ is a continuous pseudometric for $(X,\mathcal{T})$ so that $\rho = \Sigma\min\{d_k,1/2^k\}$ is a continuous metric on X (because $F(x,a) = F(y,a)$ for each $a \in A$ iff $x = y$). Since compact topologies are minimal among Hausdorff topologies, ρ determines the topology $\mathcal{T}$.

__II.2.6 Remark__. Since any semimetrizable space $(X,\mathcal{T})$ has a diagonal which is a G_δ-set in $X \times X$, Theorem II.2.5 generalizes Remark I.2.9(ii) which notes that a compact semimetrizable space is metrizable.

3. Constructing Metrics from a Family of Covers

We conclude this chapter with a proof of the first metrization theorem which was established by P.S. Alexandroff and Paul Urysohn in 1923. They raised the question of which topological spaces were generated from metrics and, in their solution, use a countable family of covers to construct the metric.

II.3.1 The Alexandroff-Urysohn Metrization Theorem (1923; [2]). If there is a sequence $\{\mathcal{G}_n\}$ of open covers of X such that:

(i) $\{\mathcal{G}_n\}$ is a *regular sequence* (i.e., when $A \cap B \neq \varnothing$ for $A,B \in \mathcal{G}_n$, then $A \cup B \subsetneq C$, for some $C \in \mathcal{G}_{n-1}$);

(ii) $\{\mathcal{G}_n\}$ is a *G_δ-diagonal sequence* (i.e., for $x \neq y$, there is an n such that $x \notin st(y,\mathcal{G}_n) = \cup\{A \in \mathcal{G}_n : y \in A\}$); and

(iii) $\{\mathcal{G}_n\}$ is a *$w\Delta$-sequence* (i.e., when $a_n \in st(p,\mathcal{G}_n)$ for some p, then the sequence $\{a_n\}$ clusters in X),

then X is metrizable.

Proof. Let $\{\mathcal{G}_n\}$ be a sequence of open covers of X satisfying (i)-(iii) as above, and let $\mathcal{T}$ denote the topology for X. Consider the distance function d for X such that:

$$d(x,y) = \begin{cases} 1/2^k, \text{ where } k = \min\{n \in \mathbb{N} : x \notin st(y,\mathcal{G}_n)\}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Clearly $d(x,y) = d(y,x)$. Because $\{\mathcal{G}_n\}$ is a G_δ-diagonal sequence, $d(x,y) = 0$ iff $x = y$.

Because $\{\mathcal{G}_n\}$ is a regular sequence, we have,

$(*)$ for any $a,b,z \in X$, it is not possible that both $d(a,z) < \frac{1}{2}d(a,b)$ and $d(b,z) < \frac{1}{2}d(a,b)$.

Thus, as in Frink's Theorem II.1.2, there is an equivalent metric d^* such that:

$$d^*(x,y) = \inf\{d(x,z_1) + \Sigma\, d(z_i,z_{i+1}) + d(z_n,y):\ z_1,z_2,\dots, z_n \in X\}$$

and $d(x_n,p) \to 0$ iff $d^*(x_n,p) \to 0$. Note that the metric d^* determines a topology $\mathcal{T}^*$ for X which has as its base the set of all spheres $\{S_{d^*}(p,\varepsilon): p \in X,\ \varepsilon > 0\}$.

Since $S_d(p,1/2^n) = st(p,\mathcal{G}_n)$, which is open in $(X,\mathcal{T})$, it follows that $\mathcal{T}^* \subseteq \mathcal{T}$ and, hence, $\overline{A}^{\mathcal{T}} \subseteq \overline{A}^{\mathcal{T}^*}$. To show that the reverse inclusion also holds, suppose that $p \in \overline{A}^{\mathcal{T}^*}$. Then, since $d(p,A] = 0$, for each $n \in \mathbb{N}$, there is $a_n \in A$ such that $a_n \in S_d(p,1/2^n) = st(p,\mathcal{G}_n)$. Thus, since $\{\mathcal{G}_n\}$ is a $w\Delta$- sequence, the sequence $\{a_n\}$ clusters to some point $q \in X$. For $A_o = \{a_n: n \in \mathbb{N}\}$, it follows that $q \in \overline{A_o}^{\mathcal{T}} \subseteq \overline{A_o}^{\mathcal{T}^*}$ and, therefore, that there must be a subsequence $\{a_{k_n}\}$ of $\{a_n\}$ such that $d^*(a_{k_n},q) \to 0$. But, $d^*(a_{k_n},p) \to 0$ so that $p = q$ and $p \in \overline{A}^{\mathcal{T}}$. Thus it follows that $\overline{A}^{\mathcal{T}} = \overline{A}^{\mathcal{T}^*}$; hence, $\mathcal{T} = \mathcal{T}^*$ so that X is metrizable.

II.3.2 Remarks. *Concerning the Alexandroff-Urysohn Metrization Theorem*

(i) The ideas of both a regular sequence and a G_δ-diagonal sequence were introduced by Alexandroff and Urysohn in [2] and are discussed in [34]. The "G_δ-diagonal" terminology is well chosen. It is easy to show [9] that a topological space has a G_δ-diagonal sequence iff its diagonal Δ_X is a G_δ-set in the product space $X \times X$.

The original version of this theorem included a condition, which they said made $\{\mathcal{G}_n\}$ *complete*, i.e., for any $p \in X$, if, for each $n \in \mathbb{N}$, $p \in G_n \in \mathcal{G}_n$, then $\{G_n\colon n \in \mathbb{N}\}$ is a local base for p. The condition of being complete is equivalent to a condition, which R.L. Moore [42] said made $\{\mathcal{G}_n\}$ a *development*, i.e., for any $p \in X$, $\{st(p,\mathcal{G}_n)\colon n \in \mathbb{N}\}$ is a local base for p. If a topological space has a development, it is called *developable*. A *Moore space* is a regular Hausdorff space which has a development.

The notion of a $w\Delta$-sequence was introduced by Borges [6], and is a condition which is weaker than the condition of being complete.

(ii) The converse of Theorem II.3.1 also holds. If d is a metric for $(X,\mathcal{T})$ and $\mathcal{G}_n = \{S_d(x,1/2^n)\colon x \in X\}$, then $\{\mathcal{G}_n\}$ is a sequence of open covers which is a regular sequence, a G_δ-diagonal sequence, and a $w\Delta$-sequence.

CHAPTER III

EXPLICIT SEMIMETRIZATION

1. Developable Semimetrizable Spaces

In the study of developable semimetrizable spaces, our aim is to find conditions which characterize Hausdorff developable, K-developable and 1-continuous-developable semimetrizable spaces. We begin with several definitions which are used in these characterizations. Recall that a topological space $(X,\mathcal{T})$ has a *G_δ-diagonal* iff there is a *G_δ-diagonal sequence* for X; that is,

there is a sequence $\{\mathcal{G}_n\}$ of open covers of X such that
$\{p\} = \cap\{st(p,\mathcal{G}_n): n \in \mathbb{N}\}$.

III.1.1 Definition. A topological space $(X,\mathcal{T})$ has a *$G_\delta{}^*$-diagonal* [31] iff there is a *$G_\delta{}^*$-diagonal sequence* for X; that is,

there is a sequence $\{\mathcal{G}_n\}$ of open covers of X such that
$\{p\} = \cap\{\overline{st(p,\mathcal{G}_n)}: n \in \mathbb{N}\}$.

$(X,\mathcal{T})$ has a *regular G_δ-diagonal* [54] iff the diagonal of X, $\Delta_X = \{(x,x): x \in X\}$, is a countable intersection of regular closed sets; that is, there is a sequence $\{V_n\}$ of open sets in $X \times X$ such that

$$\Delta_X = \cap\{V_n: n \in \mathbb{N}\} = \cap\{\overline{V}_n: n \in \mathbb{N}\}.$$

$(X,\mathcal{T})$ has a ***zero set diagonal*** iff its diagonal is a zero set in the product $X \times X$.

III.1.2 Remark. *Concerning G_δ-diagonals*

For any topological space $(X,\mathcal{T})$,

(i) $(X,\mathcal{T})$ has a zero set diagonal iff there is a continuous distance function on X;

(ii) if $(X,\mathcal{T})$ has a zero set diagonal, then it also has a regular G_δ-diagonal;

(iii) if $(X,\mathcal{T})$ has a regular G_δ-diagonal, then it has a G_δ^*-diagonal [31];

(iv) if $(X,\mathcal{T})$ has a G_δ^*-diagonal, then it is Hausdorff; and

(v) if $(X,\mathcal{T})$ is semimetrizable, then it has a G_δ-diagonal.

III.1.3 Definition [6]. A topological space $(X,\mathcal{T})$ is a ***wΔ-space*** iff it has a wΔ-sequence (see Theorem II.3.1).

Hodel [31, Theorem 2.5] used sequences of open covers to prove that every Hausdorff wΔ-space with a G_δ^*-diagonal has a development (as in II.3.2(ii)). It is known that a topological space $(X,\mathcal{T})$ is developable semimetrizable iff it is a T_1-space and has a development (see IV.3.2). Our theorem, which includes Hodel's Theorem, is established with distance function constructions.

III.1.4 Theorem. A topological space is Hausdorff and developable semimetrizable iff it is a wΔ-space with a G_δ^*-diagonal.

Proof. Suppose that d is a developable semimetric for the Hausdorff space $(X,\mathcal{T})$. For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{G \in \mathcal{T}: \delta_d[G] < 1/2^n\}$. Since d is a semimetric for $(X,\mathcal{T})$, the set of spheres $\{S_d(p,\varepsilon): \varepsilon > 0\}$ is a neighborhood base for p in $(X,\mathcal{T})$. Since d is developable, there are spheres centered at p of arbitrarily small diameter. Therefore, for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is an open cover of X.

Next, we show that $st(p,\mathcal{G}_n) \subseteq S_d(p,1/2^n)$. Suppose $x \in st(p,\mathcal{G}_n)$. Then, there is a $G \in \mathcal{T}$ such that $x,p \in G$ and $\delta_d[G] < 1/2^n$. Therefore, $d(x,p) < 1/2^n$ and $x \in S_d(p,1/2^n)$. It follows that $\{st(p,\mathcal{G}_n): n \in \mathbb{N}\}$ is a local base for p in $(X,\mathcal{T}_d)$.

Hence, if for every $n \in \mathbb{N}$, $x_n \in st(p,\mathcal{G}_n)$, then $d(x_n,p) < 1/2^n$ so that $x_n \to p$; thus, $\{\mathcal{G}_n\}$ is a wΔ-sequence for $(X,\mathcal{T})$.

Since $(X,\mathcal{T})$ is Hausdorff, if $q \neq p$, then there is an $n_q \in \mathbb{N}$ such that $q \notin \overline{st(p,\mathcal{G}_{n_q})}$. Since we can find such an n_q for any $q \neq p$, it follows that $\{p\} = \cap\{\overline{st(p,\mathcal{G}_n)}: n \in \mathbb{N}\}$ and, therefore, that $\{\mathcal{G}_n\}$ is a G_δ^*-diagonal sequence.

Conversely, suppose that $\{\mathcal{W}_n\}$ is a wΔ-sequence and that $\{\mathcal{G}_n\}$ is a sequence of open covers of X such that $\{p\} = \cap\{\overline{st(p,\mathcal{G}_n)}: n \in N\}$ for each $p \in X$. We may also assume that, for each $n \in \mathbb{N}$, $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$, and $\mathcal{W}_{n+1}$ refines $\mathcal{W}_n$. If, for each $n \in \mathbb{N}$, we let $\mathcal{U}_n = \{W \cap G: W \in \mathcal{W}_n, G \in \mathcal{G}_n\}$, we have a new sequence $\{\mathcal{U}_n\}$ of open covers such that $\mathcal{U}_n$ refines both

$\mathcal{W}_n$ and $\mathcal{G}_n$. The sequence $\{\mathcal{U}_n\}$ is also a wΔ-sequence. Since $(X,\mathcal{T})$ has a G_δ^*-diagonal, it is Hausdorff.

Define a distance function d for X as follows:

$$d(x,y) = \begin{cases} 1/2^n, \text{ where } n = \min\{k \in \mathbb{N} : x \notin st(y,\mathcal{U}_k)\}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Then $S_d(p,1/2^n) = st(p,\mathcal{U}_n)$, and therefore $\mathcal{T}_d \subsetneq \mathcal{T}$. To prove that d is a semimetric for $(X,\mathcal{T})$, we show that $\mathcal{T} \subsetneq \mathcal{T}_d$, i.e., if $p \in G \in \mathcal{T}$, then $\{S_d(p,1/2^n): n \in \mathbb{N}\} \subsetneq G$ for some $n \in \mathbb{N}$.

If not, then there is an open set $G \in \mathcal{T}$ and a sequence $\{x_n\}$ such that $p \in G$, $x_n \in S_d(p,1/2^n)$, while $x_n \notin G$. But $\{\mathcal{U}_n\}$ is a wΔ-sequence, so the sequence $\{x_n\}$ has a cluster point q; but, $q \neq p$. Since $\{\mathcal{U}_n\}$ refines $\{\mathcal{G}_n\}$, $\{p\} = \cap\{\overline{st(p,\mathcal{U}_n)}: n \in N\}$. It follows that there is an open set $U \in \mathcal{T}$ and an $n \in \mathbb{N}$ such that $q \in U$ and $U \cap st(p,\mathcal{U}_n) = \emptyset$. Hence $x_n \notin U$ for every $n \in \mathbb{N}$, which contradicts that q is a cluster point of $\{x_n\}$.

Since each open set U in $\mathcal{U}_n$ has d-diameter less than $1/2^n$, we conclude that d is developable.

<u>III.1.5 Theorem</u>. A topological space is K-developable semimetrizable iff it is a wΔ-space with a regular G_δ-diagonal.

<u>Proof</u>. Suppose that d is a K-developable semimetric for $(X,\mathcal{T})$. For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{G \in \mathcal{T}: \delta_d[G] < 1/2^n\}$. As in the proof of the preceding theorem, $\{\mathcal{G}_n\}$ is a wΔ-sequence, and $\{S_d(p,\varepsilon): \varepsilon > 0\}$ is a neighborhood base for p in $(X,\mathcal{T})$.

To show that $(X,\mathcal{T})$ has a regular G_δ-diagonal, let $U_n = \cup\{G \times G: G \in \mathcal{G}_n\}$. Clearly $\Delta_X \subsetneq \cap\{\overline{U}_n: n \in \mathbb{N}\}$. We will show that $\Delta_X = \cap\{\overline{U}_n: n \in \mathbb{N}\}$.

Suppose $(p,q) \in \cap\{\overline{U}_n: n \in \mathbb{N}\}$. Then for each n, there is $(x_n,y_n) \in G \times G$, for some $G \in \mathcal{G}_n$, such that $(x_n,y_n) \in S_d(p,1/2^n) \times S_d(q,1/2^n)$. It follows that, for every n, $x_n,y_n \in G$ where $\delta_d[G] < 1/2^n$. Therefore $d(x_n,p) \to 0$, $d(y_n,q) \to 0$, and $d(x_n,y_n) \to 0$. Since d is a K-distance function, $p = q$, and $\Delta_X = \cap\{\overline{U}_n: n \in \mathbb{N}\}$.

To prove the converse, suppose that the topological space $(X,\mathcal{T})$ is a wΔ-space with a regular G_δ-diagonal. Then there is a wΔ-sequence, $\{\mathcal{W}_n\}$, for X, and a decreasing sequence of open sets, $\{U_n\}$, in $X \times X$ such that $\Delta_X = \cap\{U_n: n \in \mathbb{N}\} = \cap\{\overline{U}_n: n \in \mathbb{N}\}$. We may also assume that, for each $n \in \mathbb{N}$, $\mathcal{W}_{n+1}$ refines $\mathcal{W}_n$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{G \in \mathcal{T}: G \times G \subsetneq U_n\}$, and note that $\mathcal{U}_n$ is an open cover of X. If, for each $n \in \mathbb{N}$, we let $\mathcal{G}_n = \{W \cap U: W \in \mathcal{W}_n, U \in \mathcal{U}_n\}$, we have a new sequence $\{\mathcal{G}_n\}$ of open covers such that $\mathcal{G}_n$ refines both $\mathcal{W}_n$ and $\mathcal{U}_n$. The sequence $\{\mathcal{G}_n\}$ is also a wΔ-sequence. Define a distance function d for X as follows:

$$d(x,y) = \begin{cases} 1/2^n, \text{ where } n = \min\{k \in \mathbb{N}: x \notin st(y,\mathcal{G}_k)\}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

By using a procedure which is similar to that of the proof of the preceding theorem, it follows that d is a developable semimetric for $(X,\mathcal{T})$.

To show that d is a K-distance function suppose that $d(x_n,p) \to 0$, $d(y_n,q) \to 0$, and $d(x_n,y_n) \to 0$. We may assume that $d(x_n,y_n)$, $d(x_n,p)$, $d(y_n,q) < 1/2^n$ for every n. Since $x_n \in S_d(p,1/2^n)$, $y_n \in S_d(q,1/2^n)$, and $(x_m,y_m) \in U_m \subset U_k$ for $m \geq k$, it follows that $(p,q) \in \overline{U_k}$ for every k (every open set G about (p,q) intersects U_k). Therefore, $(p,q) \in \cap\{\overline{U_n}: n \in \mathbb{N}\} = \Delta_X$, which implies that $p = q$. Thus, d is a K-distance function.

<u>**III.1.6 Corollary**</u>. A developable semimetrizable space is K-developable semimetrizable iff it has a regular G_δ-diagonal.

<u>**Proof**</u>. A developable semimetrizable space is a $w\Delta$-space.

<u>**III.1.7 Theorem**</u>. A topological space $(X,\mathcal{T})$ is developable semimetrizable and subcontinuously semimetrizable (that is, there is a continuous distance function on X) iff it is a $w\Delta$-space with a zero set diagonal.

<u>**Proof**</u>. Suppose that $(X,\mathcal{T})$ is developable semimetrizable and that there is a continuous distance function on X. Then $(X,\mathcal{T})$ is a $w\Delta$-space (since it is developable semimetrizable) with a zero set diagonal (see Remark III.1.2).

Conversely, suppose that $(X,\mathcal{T})$ is a $w\Delta$-space with a zero set diagonal. Then $(X,\mathcal{T})$ has a G_δ^*-diagonal; hence is a Hausdorff space. It follows that $(X,\mathcal{T})$ is developable semimetrizable (Theorem III.1.4). Since $(X,\mathcal{T})$ has a zero set diagonal, there is a continuous distance function on X.

Now we use the results of this section to investigate some of the examples from Chapter I.

III.1.8 Remark. *Burke's Example*

The distance function given in Burke's Example (I.1.13) is a developable distance functon for X such that $\mathcal{S}_d \subsetneq \mathcal{T}_d$. Therefore d is a developable semimetric for the Hausdorff space $(X,\mathcal{T}_d)$. Thus, by Theorem III.1.4, $(X,\mathcal{T}_d)$ is a $w\Delta$-space with a $G_\delta{}^*$-diagonal.

Burke has shown that no semimetric for $(X,\mathcal{T})$ is a K-distance function [8]; hence, $(X,\mathcal{T})$ is not K-developable semimetrizable. Since $(X,\mathcal{T})$ is a $w\Delta$-space, it follows (from Theorem III.1.5) that Burke's Example does not have a regular G_δ-diagonal.

III.1.9 Remark. *Borges' Example*

By Theorem I.2.11, no semimetric for Borges' Example (I.1.12) is developable (since it is Lindelöf, but not second countable). Since this example is subcontinuously semimetrizable (the usual Euclidean metric is continuous on X), but not developable semimetrizable, it follows from Theorem III.1.7 that Borges' Example is not a $w\Delta$-space.

Similar remarks hold for McAuley's Bow-Tie Space (Example I.1.11).

Before investigating the properties of the Isbell-Mrówka spaces (Example I.1.14), we note the following theorem of McArthur.

III.1.10 Theorem (McArthur [40]). If $(X,\mathcal{T})$ is a pseudocompact, completely regular, Hausdorff space with a regular G_δ-diagonal, then it is metrizable.

III.1.11 Theorem. The Isbell-Mrówka space $\psi_{\mathcal{R}}$ is developable semimetrizable and K-semimetrizable, but it is not K-developable semimetrizable.

Proof. In considering Example I.1.14, note that the Isbell-Mrówka space $\psi_{\mathcal{R}} = \mathbb{N} \cup \mathcal{R}$ has the topology which has the following properties: (i) each $p \in \mathcal{R}$ has $\{U_k(p): k \in \mathbb{N}\}$ as a local base, where $U_k(p) = \{p\} \cup \{n \in p: k \leq n\}$; and (ii) each $n \in \mathbb{N}$ has $\{n\}$ as a local base [22]. The following distance function, which is given in Example I.1.14,

$$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y; \\ 1/2^x, & \text{if } x \in y \in \mathcal{R}; \\ |1/2^x - 1/2^y|, & \text{if } x,y \in \mathbb{N}; \\ 1, & \text{otherwise} \end{cases}$$

is a developable semimetric for $\psi_{\mathcal{R}}$.

By modifying this distance function as follows, we obtain a K-semimetric for $\psi_{\mathcal{R}}$:

$$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y; \\ 1/2^x, & \text{if } x \in y \in \mathcal{R}; \\ 1, & \text{otherwise.} \end{cases}$$

By Theorem I.1.1.iii, this distance function d is not developable since each $p \in \mathcal{R}$, viewed as an increasing sequence $\{p_n\}$ in $\mathbb{N} \subsetneq \mathbb{N} \cup \mathcal{R}$, is d-convergent, but is not d-cauchy.

Next, we show that $\psi_{\mathcal{R}}$ is not K-developable semimetrizable. Because $\psi_{\mathcal{R}}$ is not normal, it is also not metrizable. Since it is a pseudocompact, completely regular, Hausdorff space, it follows from McArthur's Theorem III.1.10 that $\psi_{\mathcal{R}}$ must not have a regular G_δ-diagonal. Therefore, by Theorem III.1.5 it is not K-developable semimetrizable.

2. 1-Continuously Semimetrizable Spaces

Our study of 1-continuously semimetrizable spaces focuses on finding characterizations for K-1-continuously semimetrizable spaces and developable-1-continuously semimetrizable spaces. In this case we seek characterizations in the context of diagonal properties and covering conditions.

III.2.1 Remark. *Concerning G_δ^*-diagonals*

(i) If $(X,\mathcal{T})$ is 1-continuously semimetrizable, it has a G_δ^*-diagonal.

(ii) If $(X,\mathcal{T})$ is continuously semimetrizable, it has a zero set diagonal.

III.2.2 Theorem. If there is a 1-continuous distance function d on the separable space $(X,\mathcal{T})$ such that $\mathcal{T}_d \subsetneq \mathcal{T}$, then $(X,\mathcal{T})$ is submetrizable, that is, there is a metric $\rho: X \times X \to \mathbb{R}$ such that $\mathcal{T}_\rho \subsetneq \mathcal{T}$.

Proof. Suppose that the topological space $(X,\mathcal{T})$ is separable with countable dense subset $A = \{a_n : n \in \mathbb{N}\}$, and that d is a 1-continuous distance function on X such that $\mathcal{T}_d \subsetneq \mathcal{T}$. For each $n \in \mathbb{N}$, define a distance function $\rho_n: X \times X \to \mathbb{R}$ such that $\rho_n(x,y) = \min\{1/2^n, |d(x,a_n)-d(y,a_n)|\}$. Since each ρ_n is a pseudometric, the distance function $\rho: X \times X \to \mathbb{R}$ such that $\rho(x,y) = \Sigma\{\rho_n(x,y): n \in \mathbb{N}\}$ is also a pseudometric.

If $\rho(x,y) = 0$, then $|d(x,a_n) - d(y,a_n)| = 0$, and, therefore $d(x,a_n) = d(y,a_n)$ for every $n \in \mathbb{N}$. Since A is dense, there is a sequence $\{a_{n_k}\}$, which is a subset of A, such that $d(a_{n_k},x) \to 0$. But d is 1-continuous, so that $d(a_{n_k},y) \to d(x,y)$. Since $d(a_{n_k},y) = d(a_{n_k},x)$ and $d(a_{n_k},x) \to 0$, it follows that $d(x,y) = 0$. Therefore $x = y$, and ρ is a metric.

Furthermore, if $d(x_n,p) \to 0$, then $\rho(x_n,p) \to 0$ so that $\mathcal{T}_\rho \subsetneq \mathcal{T}_d \subsetneq \mathcal{T}$.

__III.2.3 Corollary__. $\psi_{\mathcal{R}}$ is not 1-continuously semimetrizable.

__Proof__. Since $\psi_{\mathcal{R}}$ does not have a regular G_δ-diagonal (as noted in the proof of Theorem III.1.11), there are no continuous distance functions (Remark III.1.2), hence, no metrics on $\psi_{\mathcal{R}}$. But $\psi_{\mathcal{R}}$ is separable, since $\mathbb{N}$ is a countable dense subset. It follows from Theorem III.2.2 that there are no 1-continuous distance functions on $\psi_{\mathcal{R}}$.

__III.2.4 Corollary__. If $(X,\mathcal{T})$ is a separable 1-continuously semimetrizable topological space, then it is also K-1-continuously semimetrizable, that is, there is a K-semimetric for $(X,\mathcal{T})$ which is also 1-continuous.

__Proof__. Suppose that d is a 1-continuous semimetric for the separable space $(X,\mathcal{T})$. By Theorem III.2.2 there is a metric ρ

on X. Clearly, ρ is also a K-distance function. Because ρ is a K-distance function, and both d and ρ are 1-continuous, the distance function d + ρ is a K-distance function which is also 1-continuous. Since $\mathcal{T}_\rho \subsetneq \mathcal{T}_d$, it follows that $d(x_n,p) \to 0$ iff $(d + \rho)(x_n,p) \to 0$. Therefore d + ρ is a K-1-continuous semimetric for $(X,\mathcal{T})$.

III.2.5 Remark. *Borges' Example Revisited*

Borges has shown that his example (I.1.12) is 1-continuously semimetrizable in [7]. From Corollary III.2.4, since $(X,\mathcal{T})$ is separable, $(X,\mathcal{T})$ is K-1-continuously semimetrizable. In fact, the distance function d for X which we describe in Chapter I is a K-1-continuous semimetric for $(X,\mathcal{T})$.

III.2.6 Remark. *Burke's Example Revisited*

It follows from Corollary III.2.4 that Burke's Example (I.1.13) is not 1-continuously semimetrizable, since it is separable but no semimetric for $(X,\mathcal{T})$ is a K-distance function (Remark III.1.8).

III.2.7 Remark. *A K-developable semimetrizable, separable space with a zero set diagonal need not be 1-continuously semimetrizable*

The distance function given in Example I.1.10 (the Split Disk Space) is a developable distance function for X such that $\mathcal{S}_d \subsetneq \mathcal{T}_d$. Therefore, d is a developable semimetric for $\mathcal{T}_d$.

We denote the usual Euclidean metric by **e**, and observe that $\mathbf{e} \le d$. Then $S_d(p,\varepsilon) \subsetneq S_{\mathbf{e}}(p,\varepsilon)$ for every $p \in X$, $\varepsilon > 0$; hence

$\mathcal{S}_d \subsetneq \mathcal{S}_{\mathbf{e}}$. If $d(x_n, p) \to 0$, $d(y_n, q) \to 0$, and $d(x_n, y_n) \to 0$, then $\mathbf{e}(x_n, p) \to 0$, $\mathbf{e}(y_n, q) \to 0$, and $\mathbf{e}(x_n, y_n) \to 0$. But $\mathbf{e}$ is a metric, so $p = q$. Therefore, d is also a K-distance function.

Thus, the Split Disk Space is K-developable semimetrizable, separable, and has a zero set diagonal, but is still not 1-continuously semimetrizable (since it is not completely regular).

<u>**III.2.8 Remark**</u>. *Concerning the wΔ-Space Problem*

The distance function given in the Shore-Uhland Example (I.1.8) is a developable distance function for X such that $\mathcal{S}_d \subsetneq \mathcal{T}_d$. Therefore, d is a developable semimetric for $\mathcal{T}_d$.

$(X, \mathcal{T}_d)$ is not Hausdorff; hence, it is also not K-semimetrizable.

It is a wΔ-space with a G_δ-diagonal since it is developable semimetrizable. However, it does not have a G_δ^*-diagonal since it is not Hausdorff.

The Shore-Uhland Example is of interest because of its relationship to the <u>*wΔ-space problem*</u>: must every wΔ-space with a G_δ-diagonal be developable?

For Hausdorff spaces, because of Theorem III.1.4, this becomes: must every wΔ-space with a G_δ-diagonal have a G_δ^*-diagonal?

The Shore-Uhland Example is a counterexample to this conjecture since it is a wΔ-space with a G_δ-diagonal, but does not have a G_δ^*-diagonal.

III.2.9 Remark. *Concerning the Normal Moore Space Conjecture*

Borges' Example I.1.12 is normal because it is regular and Lindelöf. Since it is not developable semimetrizable, it is also not continuously semimetrizable. Thus we have an example of a normal 1-continuously semimetrizable space which is not continuously semimetrizable.

Borges' Example is of interest because of its relationship to the *normal Moore space conjecture*, which states that every normal Moore space is metrizable [35]. Borges' Example shows, without the use of extra set theoretic axioms, that a parallel result for 1-continuously semimetrizable spaces does not hold.

We now note that only two of our examples are continuously semimetrizable. In each case the distance function described in Chapter I is a continuous semimetric for $(X,\mathcal{T})$.

III.2.10 Theorem. Heath's V-space (Example I.1.15) and the Niemytzki Space (Example I.1.16) are continuously semimetrizable.

We now consider any semimetrizable space which has the strongest of our diagonal properties, that is, has a zero set diagonal.

III.2.11 Theorem. If $(X,\mathcal{T})$ is a semimetrizable space with a zero set diagonal, then $(X,\mathcal{T})$ is K-semimetrizable.

Proof. Suppose that d is a semimetric for the topological space $(X,\mathcal{T})$, and that $f: X \times X \to [0,1]$ is a continuous function such that $Z_f = \Delta_X$. Let d_1 be the distance function for X such that $d_1(x,y) = \min\{f(x,y),f(y,x)\}$. Then d_1 is a continuous distance function, and $d + d_1$ is a K-semimetric for $(X,\mathcal{T})$.

Since X is subcontinuously semimetrizable iff it has a zero set diagonal, we have the following corollary to Theorem III.2.11.

III.2.12 Corollary. A semimetrizable topological space which is also subcontinuously semimetrizable is K-semimetrizable.

III.2.13 Remark. *McAuley's Bow-Tie Space Revisited*
Since $(X,\mathcal{T})$ has a zero set diagonal ($\mathbf{e}$ is a continuous distance function on X, since $\mathbf{e} \leq d$), it follows from Theorem III.2.11 that $(X,\mathcal{T})$ is K-semimetrizable. In fact, d is a K-semimetric for $(X,\mathcal{T})$.

CHAPTER IV

NEIGHBORHOOD CHARACTERIZATIONS

1. Neighborhood Structures

In this chapter our study seeks to establish neighborhood characterizations for the topological spaces under consideration in this work. Historically, Fréchet [16] initiated this study, and Hausdorff [24] recorded and added to the study.

For our work, we introduce a definition.

IV.1.1 Definition. A collection of sets $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a *neighborhood structure* for $(X,\mathcal{T})$ iff $p \in U_n(p) \in \mathcal{T}$ and $U_{n+1}(p) \subsetneq U_n(p)$, for every $n \in \mathbb{N}$.

We observe that:

IV.1.2 Theorem. If $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ such that:

for every sequence $\{x_n\}$, whenever $x_n \in U_n(p)$ for every $n \in \mathbb{N}$, then $x_n \to p$ (in $\mathcal{T}$),

then $\{U_n(p)\}$ is a local base for p in $(X,\mathcal{T})$.

IV.1.3 Corollary. A topological space $(X,\mathcal{T})$ is first countable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that if $x_n \in U_n(p)$ for every $n \in \mathbb{N}$, then $x_n \to p$ (in $\mathcal{T}$).

Throughout this chapter we seek to characterize topological properties with theorems analogous to this one.

2. Semimetrizable Spaces

We begin with a characterization of semimetrizable spaces and proceed to find characterizations for more restricted semimetrizable spaces.

IV.2.1 Theorem. A topological space $(X,\mathcal{T})$ is semimetrizable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{U_n(p): n \in \mathbb{N}\} = \{p\}$;

(ii) if $x_n \in U_n(p)$, then $x_n \to p$ (in $\mathcal{T}$); and

(iii) if $p \in U_n(x_n)$, then $x_n \to p$ (in $\mathcal{T}$).

Proof. Suppose d is a semimetric for $(X,\mathcal{T})$. For $n \in \mathbb{N}$, $p \in X$, let $U_n(p) = \text{int}_{\mathcal{T}} S_d(p,1/2^n)$. Then $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ with properties (i) - (iii).

Conversely, suppose that $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ with properties (i) - (iii). Define a distance function d for X as follows:

$$d(p,q) = 1/2^n, \text{ where } n = \min\{k: p \notin U_k(q) \text{ and } q \notin U_k(p)\}.$$

Since $U_n(p) \subseteq S_d(p,1/2^n)$ and $\{U_n(p): n \in \mathbb{N}\}$ is a local base for p in $(X,\mathcal{T})$, it follows that $(X,\mathcal{T})$ is first countable and $x_n \to p$ (in $\mathcal{T}$) iff $d(x_n,p) \to 0$. Thus, d is a semimetric for $(X,\mathcal{T})$.

<u>IV.2.2 Remark</u>. *Factorization theorems*

This theorem illustrates the attempts of the "Jones School" to create theorems that "factor" topological properties; see McAuley [41], and Heath [26]. Our theorem and its proof is suggested by the work of Heath.

To further illustrate this factorization, we note that:

(a) A topological space $(X,\mathcal{T})$ is a T_1-space if there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ with IV.2.1(i), that is, $\cap\{U_n(p): n \in \mathbb{N}\} = \{p\}$.

We have already noted:

(b) A topological space $(X,\mathcal{T})$ is first countable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ with IV.2.1(ii), that is, if $x_n \in U_n(p)$ for every $n \in \mathbb{N}$, then $x_n \to p$ (in $\mathcal{T}$).

Following Hodel [31], we define:

(c) A topological space $(X,\mathcal{T})$ is <u>*semistratifiable*</u> iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ with IV.2.1(iii), that is, if $p \in U_n(x_n)$, then $x_n \to p$ (in $\mathcal{T}$).

Thus we have "factored" the concept of semimetrizable in our Theorem IV.2.3 as follows:

A topological space $(X,\mathcal{T})$ is semimetrizable iff it is

(a) a T_1-space that is

(b) first countable and

(c) semistratifiable.

Finally, note that the notion of semistratifiability was introduced by Michael as a derivative of stratifiable spaces [5]. Stratifiable is Borges' terminology for Ceder's "M_3-spaces". Semistratifiable spaces are studied by Heath's student, Geoffrey Creede, in [13].

<u>IV.2.4 Corollary</u>. *Other factorization theorems*

(a) A topological space $(X,\mathcal{T})$ is a first countable T_1-space iff there are neighborhood structures for $(X,\mathcal{T})$ with IV.2.1(i) and (ii).

(b) A topological space $(X,\mathcal{T})$ is a first countable semistratifiable space iff there are neighborhood structures for $(X,\mathcal{T})$ with IV.2.1(ii) and (iii).

(c) A topological space $(X,\mathcal{T})$ is a semistratifiable T_1-space if there are neighborhood structures for $(X,\mathcal{T})$ with IV.2.1(i) and (iii).

<u>Proof</u>. We illustrate the nature of the proofs by proving (a). The others follow similarly. Suppose $(X,\mathcal{T})$ is a first countable T_1-space. For each $p \in X$, let $\{U_n(p): n \in \mathbb{N}\}$ be a local base for p in $(X,\mathcal{T})$ with $U_{n+1}(p) \subsetneq U_n(p)$ for every $n \in \mathbb{N}$. Then $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ with IV.2.1(i) and (ii).

Conversely suppose there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ with IV.2.1(i), and a neighborhood structure $\{V_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ with IV.2.1(ii). For each $n \in \mathbb{N}$ and $p \in X$, let $G_n(p) = U_n(p) \cap V_n(p)$.

Then $\{G_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ with IV.2.1(i) and (ii). Therefore $(X,\mathcal{T})$ is a first countable T_1-space.

Our next theorems strengthen the separation property for semimetrizable spaces and continue to provide factorization theorems.

<u>IV.2.5 Theorem</u>. A topological space $(X,\mathcal{T})$ is Hausdorff and semimetrizable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{\overline{U_n(p)}: n \in \mathbb{N}\} = \{p\}$;

(ii) if $x_n \in U_n(p)$, then $x_n \to p$ (in $\mathcal{T}$); and

(iii) if $p \in U_n(x_n)$, then $x_n \to p$ (in $\mathcal{T}$).

<u>Proof</u>. Note first that $(X,\mathcal{T})$ is Hausdorff iff $p \neq q$ implies that there is a $G \in \mathcal{T}$ such that $p \in G$ and $q \notin \overline{G}$. Now the proof follows as in Theorem IV.2.1.

<u>IV.2.6 Theorem</u>. A topological space $(X,\mathcal{T})$ is regular and semimetrizable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{U_n(p): n \in \mathbb{N}\} = \{p\}$;

(i') for every $n \in \mathbb{N}$, there is an $m \geq n$ such that $\overline{U_m(p)} \subsetneq U_n(p)$;

(ii) if $x_n \in U_n(p)$, then $x_n \to p$ (in $\mathcal{T}$); and

(iii) if $p \in U_n(x_n)$, then $x_n \to p$ (in $\mathcal{T}$).

Proof. The useful local characterization here is that $(X,\mathcal{T})$ is regular iff $p \in G \in \mathcal{T}$ implies there is an $H \in \mathcal{T}$ such that $p \in H \subseteq \overline{H} \subseteq G$. From this fact, the proofs follow easily, as in Theorem IV.2.1.

We now turn our attention to the characterization of K-semimetrizable spaces. This requires a strengthening of condition (i).

IV.2.7 Theorem. A topological space $(X,\mathcal{T})$ is K-semimetrizable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) For disjoint compact $A,B \subseteq X$, there is an $n \in \mathbb{N}$ such that $U_n[A] \cap B = \varnothing$, where $U_n[A] = \cup\{U_n(a): a \in A\}$;

(ii) if $x_n \in U_n(p)$, then $x_n \to p$ (in $\mathcal{T}$); and

(iii) if $p \in U_n(x_n)$, then $x_n \to p$ (in $\mathcal{T}$).

Proof. Suppose d is a K-semimetric for $(X,\mathcal{T})$. For each $n \in \mathbb{N}$, $p \in X$, let $U_n(p) = \mathrm{int}_{\mathcal{T}} S_d(p,1/2^n)$. Next suppose that $A,B \subseteq X$ are disjoint and compact. Since d is a K-semimetric for $(X,\mathcal{T})$, $d[A,B] > 0$, say $d[A,B] > 1/2^n$. Thus $S_d[A,1/2^n) \cap B = \varnothing$ (otherwise if $x \in S_d[A,1/2^n) \cap B$, then $d(x,a) < 1/2^n$ for some $a \in A$ where $x \in B$, which implies that $d[A,B] < 1/2^n$, a contradiction). It follows that $U_n[A] = \cup\{U_n(a): a \in A\} = \cup\{\mathrm{int}_{\mathcal{T}} S_d(a,1/2^n): a \in A\} \subseteq S_d[A,1/2^n)$ and therefore $U_n[A] \cap B = \varnothing$. Thus (i) - (iii) hold.

Conversely suppose there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ with properties (i) - (iii). Define a distance function d as follows:

$$d(p,q) = 1/2^n, \text{ where } n = \min\{k: p \notin U_k(q) \text{ and } q \notin U_k(p)\}.$$

Then d is a semimetric for $(X,\mathcal{T})$. To show that d is a K-distance function, suppose $A \cap B = \emptyset$. By (i), there are $n_a, n_b \in \mathbb{N}$ such that $U_{n_a}[A] \cap B = \emptyset$ and $A \cap U_{n_b}[B] = \emptyset$. Let $n = \max\{n_a, n_b\}$. Then $U_n[A] \cap B = \emptyset$ and $A \cap U_n[B] = \emptyset$. If $a \in A$ and $b \in B$, then $a \notin U_n[B]$ and $b \notin U_n[A]$ so that $k' = \min\{k: a \notin U_k(b) \text{ and } b \notin U_k(a)\} \leq n$. Thus $1/2^n \leq 1/2^k = d(a,b)$. This property holds for any $a \in A$, $b \in B$, so $d[A,B] \geq 1/2^n$. Since d separates disjoint compact sets, d is a K-semimetric for $(X,\mathcal{T})$.

<u>IV.2.8 Remark</u>. Note that since IV.2.7(i) implies that $(X,\mathcal{T})$ is Hausdorff, we still have $\cap\{\overline{U_n(p)}: n \in \mathbb{N}\} = \{p\}$. This theorem provides an alternative proof to Theorem I.2.13(i), that is, any K-semimetrizable space is Hausdorff.

We conclude this section with a characterization of open semimetrizable spaces.

<u>IV.2.9 Definition</u>. A topological space $(X,\mathcal{T})$ is *open semimetrizable* iff there is a semimetric d for $(X,\mathcal{T})$ such that $\mathcal{S}_d \subsetneq \mathcal{T}$.

IV.2.10 Theorem. A topological space $(X,\mathcal{T})$ is open semimetrizable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{U_n(p): n \in \mathbb{N}\} = \{p\}$;

(ii) if $x_n \in U_n(p)$, then $x_n \to p$ (in $\mathcal{T}$); and

(iii) $p \in U_n(q)$ iff $q \in U_n(p)$.

Proof. Suppose that d is an open semimetric for the topological space $(X,\mathcal{T})$. For each $n \in \mathbb{N}$, $p \in X$, let $U_n(p) = S_d(p,1/2^n)$. Then $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ with properties (i) - (iii). Note that property (iii) make properties (ii) and (iii) of the previous theorems in this section equivalent.

To prove the converse, define a distance function d as follows:

$$d(p,q) = 1/2^n, \text{ where } n = \min\{k: p \notin U_k(q)\}.$$

Since $U_n(p) = S_d(p,1/2^n)$, we conclude that d is a semimetric for $(X,\mathcal{T})$ and, hence, $(X,\mathcal{T})$ is open semimetrizable.

3. Developable Semimetrizable Spaces

Here we seek to characterize the developable semimetrizable spaces we have studied in this work. We begin with a neighborhood characterization of developable semimetrizable spaces.

IV.3.1 Theorem. A topological space $(X,\mathcal{T})$ is developable semimetrizable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{U_n(p): n \in \mathbb{N}\} = \{p\}$; and

(ii) if $x_n, p \in U_n(y_n)$ for some $y_n \in X$, then $x_n \to p$ (in $\mathcal{T}$).

Proof. Suppose d is a developable semimetric for $(X,\mathcal{T})$. For each $p \in X$, and $n \in \mathbb{N}$, choose an open set $G_n(p)$ such that $\delta_d[G_n(p)] < 1/2^n$. Let $U_n(p) = \cap\{G_i(p): i = 1,2, . . . n\}$. Then $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ with conditions (i) and (ii).

Conversely, suppose that $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ with conditions (i) and (ii). Let $\mathcal{U}_n = \{U_n(p): p \in X\}$. One has immediately:

(ii') if $x_n \in st(p,\mathcal{U}_n)$ for each $n \in \mathbb{N}$, then $x_n \to p$.

Thus, from IV.1.2, $\{st(p, \mathcal{U}_n): n \in \mathbb{N}\}$ is a local base for p in $(X,\mathcal{T})$; consequently,

$\{p\} = \cap\{st(p, \mathcal{U}_n): n \in \mathbb{N}\}$, since $\cap\{U_n(p): n \in \mathbb{N}\} = \{p\}$.

Now define a distance function d for X as follows:

$$d(p,q) = 1/2^n, \text{ where } n = \min\{k: p \notin st(q,\mathcal{U}_k)\}.$$

Since $S_d(p,1/2^n) = st(p,\mathcal{U}_n)$, d is a semimetric for $(X,\mathcal{T})$. Each $U \in \mathcal{U}_n$ has diameter less than $1/2^n$; therefore, d is developable.

<u>IV.3.2 Remark</u>. *Concerning developable spaces*

Recall (see Remark II.3.2) that a topological space $(X,\mathcal{T})$ is developable iff there is a sequence $\{\mathcal{G}_n\}$ of open covers such that:

for any $p \in X$, $\{st(p,\mathcal{G}_n): n \in \mathbb{N}\}$ is a local base for p.

Thus, from our proof, we note that:

(a) a topological space $(X,\mathcal{T})$ is developable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ with IV.3.1(ii), that is if $x_n, p \in U_n(y_n)$ for some $y_n \in X$, then $x_n \to p$ (in $\mathcal{T}$).

Hence,

(b) a topological space $(X,\mathcal{T})$ is developable semimetrizable iff it is a developable T_1-space.

As in Section 1 of this chapter, we continue our study by strengthening the separation property.

IV.3.3 Theorem. A topological space $(X,\mathcal{T})$ is developable semimetrizable and Hausdorff iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{\overline{U_n(p)}: n \in \mathbb{N}\} = \{p\}$; and

(ii) if $x_n, p \in U_n(y_n)$ for some $y_n \in X$, then $x_n \to p$ (in $\mathcal{T}$).

Proof. We again use the local characterization for Hausdorff spaces, and the proof follows easily, as in Theorem IV.3.1.

IV.3.4 Corollary. A topological space $(X,\mathcal{T})$ is developable semimetrizable and Hausdorff iff it is a developable, Hausdorff space.

IV.3.5 Remark. *Another Proof of Hodel's Theorem (IV.3.3)*

First note that (i) and (ii) of Theorem IV.3.3 are equivalent to:

(i') $\{p\} = \cap\{\overline{st(p,\mathcal{U}_n)}: n \in \mathbb{N}\}$ and

(ii') if $x_n \in st(p,\mathcal{U}_n)$ for each $n \in \mathbb{N}$, then x_n clusters at p.

But, (i') holds iff $(X,\mathcal{T})$ has a G_δ^*-diagonal, and in the presence of (i'), (ii') is equivalent to $(X,\mathcal{T})$ being a $w\Delta$-space.

IV.3.6 Theorem. A topological space $(X,\mathcal{T})$ is regular and developable semimetrizable iff there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{U_n(p): n \in \mathbb{N}\} = \{p\}$;

(i') for every $n \in \mathbb{N}$, there is an $m \geq n$ such that $\overline{U_m(p)} \subsetneq U_n(p)$; and

(ii) if $x_n, p \in U_n(y_n)$ for some $y_n \in X$, then $x_n \to p$ (in $\mathcal{T}$).

Proof. We again use the local characterization for a regular space, and the proof follows easily as in Theorem IV.3.1.

IV.3.7 Corollary. A topological space $(X,\mathcal{T})$ is regular and developable semimetrizable iff it is a Moore space, i.e., it is regular, Hausdorff and developable.

IV.3.8 Open Question. Note that we have characterized developable, K-semimetrizable spaces. Namely,

(i) for disjoint compact $A,B \subsetneq X$, there is an $n \in \mathbb{N}$ such that $U_n[A] \cap B = \varnothing$, where $U_n[A] = \cup\{U_n(a): a \in A\}$ and

(ii) if $x_n, p \in U_n(y_n)$ for some $y_n \in X$, then $x_n \to p$ (in $\mathcal{T}$).

An open question is to find a neighborhood characterization for K-developable semimetrizable spaces.

4. <u>γ-spaces</u>

The class of γ-spaces has a long history as one generalization of metric spaces. Interest in these spaces was sparked by Ribeiro's false proof of what became the <u>γ-space conjecture</u>: is every Hausdorff γ-space quasimetrizable? Note that if this conjecture had been true, it would have extended Frink's metrization theorem to a quasimetrization theorem. However, Fox [18] has constructed a completely regular counterexample to this conjecture.

We begin with a neighborhood characterization of these spaces.

<u>IV.4.1 Definition</u>. $(X,\mathcal{T})$ is a <u>γ-space</u> [32] iff there is a neighborhood structure $\{U_n(p)\colon n \in \mathbb{N},\ p \in X\}$ for $(X,\mathcal{T})$ such that if $x_n \in U_n(y_n)$ and $y_n \in U_n(p)$ then $x_n \to p$ (in $\mathcal{T}$).

<u>IV.4.2 Theorem</u>. A topological space $(X,\mathcal{T})$ is a γ-space iff there is a neighborhood structure $\{U_n(p)\colon n \in \mathbb{N},\ p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) for disjoint $K,F \subseteq X$, K compact, F closed, there is an $n \in \mathbb{N}$ such that $U_n[K] \cap F = \emptyset$; and

(ii) if $x_n \in U_n(p)$, then $x_n \to p$ (in $\mathcal{T}$).

Proof. Suppose $\{U_n(p): n \in \mathbb{N}, p \in X\}$ is a neighborhood structure for $(X,\mathcal{T})$ as in IV.4.1. Then condition (ii) follows immediately. If condition (i) fails, then there are disjoint $K,F \subsetneq X$, K compact, F closed such that for every $n \in \mathbb{N}$, $U_n[K] \cap F \neq \varnothing$. Thus, there are sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \in U_n(y_n)$, $x_n \in F$, and $y_n \in K$. Since K is compact, th?re is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to $p \in K$. Since $x_{n_k} \in U_k(y_{n_k})$, $y_{n_k} \in U_k(p)$, and $(X,\mathcal{T})$ is a γ-space, $x_{n_k} \to p \in F$. Moreover, since $x_{n_k} \in F$ and F is closed, we have $p \in F$. Thus, $K \cap F \neq \varnothing$, which is a contradiction.

To prove the converse, let $\{U_n(p): n \in \mathbb{N}, p \in X\}$ be a neighborhood structure for $(X,\mathcal{T})$ with conditions (i) and (ii). Our claim is that $(X,\mathcal{T})$ is a γ-space. Otherwise, for each $n \in \mathbb{N}$, there is $x_n \in U_n(y_n)$ and $y_n \in U_n(p)$, but $x_n \not\to p$ (in $\mathcal{T}$). Since $y_n \in U_n(p)$ for every $n \in \mathbb{N}$, $y_n \to p$ (in $\mathcal{T}$). However, $x_n \not\to p$. Thus, there is an open set G, containing p, such that $G \cap \{x_{n_k}: k \in \mathbb{N}\} = \varnothing$ and $y_{n_k} \in U_{n_k}(p)$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. If we let $F = X - G$ and $K = \{y_{n_k}: y_{n_k} \in G\} \cup \{p\}$, then $x_{n_k} \in U_{n_k}[K] \cap F$ for every $k \in \mathbb{N}$. Hence, $U_n[K] \cap F \neq \varnothing$ for every $n \in \mathbb{N}$, which is a contradiction.

Hodel has shown that a semistratifiable γ-space is developable. We show in the following theorem that, for Hausdorff spaces, we can say more.

IV.4.3 Theorem. If a topological space $(X,\mathcal{T})$ is Hausdorff, semistratifiable and a γ-space, then it is K-semimetrizable and developable semimetrizable.

Proof. Suppose $(X,\mathcal{T})$ is a Hausdorff, semistratifiable γ-space. Then there is a neighborhood structure $\{U_n(p): n \in \mathbb{N}, p \in X\}$ for $(X,\mathcal{T})$ such that:

(i) $\cap\{\overline{U_n(p)}: n \in \mathbb{N}\} = \{p\}$; and

(ii) if $p \in U_n(x_n)$, then $x_n \to p$ (in $\mathcal{T}$).

Again let $\mathcal{U}_n = \{U_n(p): p \in X\}$ and define two distance functions d_1 and d_2 for X as follows:

$d_1(p,q) = 1/2^n$, where $n = \min\{k: p \notin U_k(q)$ and $q \notin U_k(p)\}$, and

$d_2(p,q) = 1/2^n$, where $n = \min\{k: p \notin st(q,\mathcal{U}_k)\}$.

It follows that d_1 is a K-semimetric for $(X,\mathcal{T})$, and d_2 is a developable semimetric for $(X,\mathcal{T})$.

The proof of Theorem IV.4.3 follows easily from Theorem IV.4.2 and Remark IV.4.3. However, our interest stems from the construction of two distinct semimetrics for $(X,\mathcal{T})$, one a K-semimetric and the other a developable semimetric.

To conclude this section, we turn our attention back to several of the examples from Chapter I.

IV.4.4 Remark. *Concerning Theorem IV.4.3*

(a) Since Burke's Example (I.1.13) is Hausdorff and developable semimetrizable, but not K-semimetrizable, it is not a γ-space.

(b) Theorem IV.4.3 fails for T_1-spaces. Consider the Shore-Uhland Example (I.1.8). Letting $U_n(p) = S_d(p,p/2^n)$, one shows that it is a γ-space. It is developable semimetrizable, but not K-semimetrizable. Therefore, the Shore-Uhland Space is a counterexample to the conjecture that a T_1 developable semimetrizable γ-space is K-semimetrizable.

(c) Since the Isbell-Mrówka Spaces (I.1.14) are Hausdorff semistratifiable γ-spaces which are not K-developable semimetrizable, there need not be a single distance function which is both developable and K-semimetrizable.

CHAPTER V

CONCLUSION

In Chapter I we established the foundations for this work by presenting definitions and illustrative examples. The focus centered on developable semimetrizable, K-semimetrizable, and 1-continuously semimetrizable spaces.

Chapter II shows how our approach can establish the proofs for the classical metrization theorems by explicitly constructing metrics.

In our search for new results in semimetrizable spaces, we looked for characterizations of the spaces we studied.

In Chapter III we found characterizations which developed historically from the Alexandroff-Urysohn Metrization Theorem (II.3.1). These characterizations involve sequences of covers and diagonal conditions.

Alternatively, we found, in Chapter IV, characterizations that are given in terms of neighborhood structures. Such characterizations represent the spirit of the work of Fréchet and Hausdorff. Interest in these theorems stems from the "factorization" quality of the results.

In retrospect, we have found new characterizations or improved old characterizations of developable semimetrizable spaces and many other more restricted kinds of developable spaces. Our study of 1-continuously semimetrizable spaces remains quite incomplete.

Among others, we are left with the following open questions:

(1) Find a characterization, like those in Chapter III, for 1-continuously-developable (or developable-1-continuously) semimetrizable spaces.

(2) Find a neighborhood characterization for:

(a) 1-continuously semimetrizable spaces and

(b) K-developable semimetrizable spaces.

(3) We have found that separable, 1-continuously semimetrizable spaces are K-semimetrizable. Does the result hold if separable is omitted?

(4) Under what restrictions (if any) are the spaces we have considered γ-spaces?

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