OPERATOR RANGES OF SHIFTS AND C*-ALGEBRAS (STRANGE RANGE, QUASI-SIMILARITY, LATTICE)

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OPERATOR RANGES OF SHIFTS AND $C^*$-ALGEBRAS (STRANGE RANGE, QUASI-SIMILARITY, LATTICE)

Abstract
It is shown that $\text{Lat}(\mathcal{L}(\mathcal{H}^\infty(\mathbb{N}))(\mathcal{S}))$, the invariant operator ranges of the commutant of the unilateral shift, is a proper sub-lattice of the lattice of invariant operator ranges of the unilateral shift, $\mathcal{S}$. The notion of a strange operator range for $\mathcal{S}$ of order $n$ where $n \in \mathbb{N}$ is introduced and it is demonstrated that there exist strange ranges for $\mathcal{S}$ of every order. This is done by deriving an operator range condition which is sufficient to insure that a pair of quasi-similar compressions of shifts really be similar. A set of operator ranges which forms a sub-lattice of $\text{Lat}(\mathcal{L}(\mathcal{H}^\infty(\mathbb{N}))(\mathcal{S}))$ is introduced, which is conjectured to be $\text{Lat}(\mathcal{L}(\mathcal{H}^\infty(\mathbb{N}))(\mathcal{S}))$. The conjecture is shown to be equivalent to the assertion that the image of $\mathcal{S}$ under certain homomorphisms of $\mathcal{H}^\infty(\mathbb{N})$ into $\mathcal{B}(\mathcal{H})$ is similar to a contraction.

It is proven that the ranges of operators from a commutative $C^*$-algebra form a lattice under intersection and vector sum. If $P$ and $Q$ are projections in $\mathcal{B}(\mathcal{H})$ with non-zero intersection and so that the angle between their ranges is $0$, then it is shown that the ranges of the operators in the $C^*$-algebra generated by $P$ and $Q$ does not contain the intersection of the ranges of $P$ and $Q$. Thus, non-commutative $C^*$-algebras need not have ranges which form a lattice. The question of whether the ranges of operators from different kinds of algebras form lattices is taken up and examples are provided.

It is proven that any pair of subspaces of a Hilbert space can be the ranges of a pair of commuting operators. A family of one dimensional subspaces of a Hilbert space, $\mathcal{H}$, is shown to representable as the set of ranges of a family of commuting operators if and only if for each subspace the linear span of the union of the remaining sub-spaces is not dense in $\mathcal{H}$. The sets of three subspaces of $\mathcal{C}(3)$ which can be the ranges of commuting operators are characterized.

Keywords
Mathematics

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OPERATOR RANGES OF SHIFTS AND C*-ALGEBRAS

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DISSERTATION

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LIST OF SYMBOLS

\[ \mathbb{N} \] \{1, 2, 3, \ldots\}  \\
\[ \mathbb{N}^* \] \{1, 2, 3, \ldots\}  \\
\[ \mathbb{C} \] the set of complex numbers  \\
\[ \mathbb{D} \] \{z \in \mathbb{C} : |z| < 1\}  \\
\[ \partial \mathbb{D} \] \{z \in \mathbb{C} : |z| = 1\}  \\
\[ H, K \] complex Hilbert spaces  \\
\[ \langle f, g \rangle \] the inner product of \( f \) and \( g \)  \\
\[ \| \cdot \| \] the norm of a vector, the supremum norm of a linear transformation  \\
\[ \mathcal{B}(H, K) \] the algebra of all bounded linear transformations from \( H \) to \( K \)  \\
\[ \mathcal{B}(H) \] the algebra of all bounded linear transformations on \( H \)  \\
\[ \overline{M} \] the norm closure of \( M \subseteq H \)  \\
\[ M \subseteq H \] \( M \) is a subspace of \( H \)  \\
\[ M^\perp \] the orthogonal complement of \( M \) in \( H \)  \\
\[ P_M \] the orthogonal projection from \( H \) onto \( M \subseteq H \)  \\
\[ (M, N) \] the angle between \( M \) and \( N \) \((M, N \subseteq H)\)  \\
\[ \text{ran}(A) \] \( AH \), the range of the operator \( A \)  \\
\[ \text{ker}(A) \] the null space of \( A \)  \\
\[ A|_M \] the restriction of \( A \in \mathcal{B}(H) \) to \( M \subseteq H \)  \\
\[ \sigma(A) \] the spectrum of \( A \)  \\
\[ A_n \to A \] the sequence of \( A_n \)'s converges to \( A \) in the supremum of norm of \( \mathcal{B}(H) \)  \\
\[ A_\lambda \to A(s) \] the net of \( A_\lambda \)'s converges to \( A \) in the strong operator topology on \( \mathcal{B}(H) \)


\[ A' \]

the commutant of \( A \)

\[ \text{Lat}(A) \]

the lattice of invariant subspaces of \( A \subseteq B(H) \)

\[ \text{Lat}(A_1, \ldots, A_n) \]

\( \text{Lat}(\{A_1, \ldots, A_n\}) \), where \( A_i \in B(H) \)

for \( i = 1, \ldots, n \)

\[ \text{Lat}_{1/2}(A) \]

the lattice of invariant operator ranges of \( A \subseteq B(H) \)

\[ \text{Lat}_{1/2}(A_1, \ldots, A_n) \]

\( \text{Lat}_{1/2}(\{A_1, \ldots, A_n\}) \), where \( A_i \in B(H) \)

for \( i = 1, \ldots, n \)

\[ C^*(A) \]

the \( C^* \)-algebra generated by \( A \subseteq B(H) \)

\[ C^*(A_1, \ldots, A_n) \]

\( C^*(\{A_1, \ldots, A_n\}) \)

\[ C^*_1(A) \]

the unital \( C^* \)-algebra generated by \( A \subseteq B(H) \)

\[ C^*_1(A_1, \ldots, A_n) \]

\( C^*_1(\{A_1, \ldots, A_n\}) \)

\[ A \sim B \]

A is similar to B

\[ A \sim_q B \]

A is quasi-similar to B

\[ Z(f) \]

the zero set of the \( C \)-valued function, \( f \)

\[ M_f \]

the operator, acting on a Hilbert space of \( C \)-valued functions, given by pointwise multiplication with the \( C \)-valued function, \( f \)

\[ dm \]

normalized arclength measure on \( \mathbb{3D} \)

\[ L^p, 1 \leq p \leq \infty \]

the Lebesque spaces \( L^p(\mathbb{3D}, dm) \), \( 1 \leq p \leq \infty \)

\[ H^p, p = 2, \infty \]

the classical Hardy spaces on \( \mathbb{3D} \)

\[ C^{(\infty)} \]

\( H^2 \)

\[ P \]

the projection of \( L^2 \) onto \( H^2 \)

\[ W \]

the bilateral shift on \( L^2 \)

\[ S \]

the unilateral shift on \( H^2 \)

vi
the analytic Toeplitz operator induced by \( f \in L^\infty \)

\[
T_f
\]

\[
\bigwedge_{k=1}^n \phi_k, n \in \mathbb{N} \cup \{\infty\}
\]

the greatest common inner factor of the sequence \( \{\phi_k\}_{k=1}^n \)

\[
L^2(\mathbb{C}(n)), n \in \mathbb{N} \cup \{\infty\}
\]

the direct sum of \( n \) copies of \( L^2 \)

\[
H^2(\mathbb{C}(n)), n \in \mathbb{N} \cup \{\infty\}
\]

the direct sum of \( n \) copies of \( H^2 \)

\[
W(n), n \in \mathbb{N} \cup \{\infty\}
\]

the bilateral shift of multiplicity \( n \) acting on the direct sum of \( n \) copies of \( L^2 \)

\[
S(n), n \in \mathbb{N} \cup \{\infty\}
\]

the unilateral shift of multiplicity \( n \) acting on the direct sum of \( n \) copies of \( H^2 \)

\[
K
\]

the direct sum of countably many copies of \( L^2 \)

\[
H
\]

the direct sum of countably many copies of \( H^2 \)

\[
p^{(\infty)}
\]

the projection of \( K \) onto \( H \)

\[
L^\infty(M_{mn}), m, n \in \mathbb{N} \cup \{\infty\}
\]

the set of \( m \times n \) matrices with entries from \( L^\infty \) yielding \( m \)-a.e. bounded linear transformations from \( \mathbb{C}(n) \) to \( \mathbb{C}(m) \)

\[
\hat{\Omega}
\]

the multiplication operator on \( B(L^2(\mathbb{C}(n)), L^2(\mathbb{C}(m))) \) induced by \( \Omega \) in \( L^\infty(M_{mn}) \)

\[
\tilde{\Omega}
\]

the multiplication operator on \( B(H^2(\mathbb{C}(n)), H^2(\mathbb{C}(m))) \) induced by \( \Omega \) in \( H^\infty(M_{mn}) \)
\( H(\Theta) \) \( \sim H^2(\mathbf{c}^{(n)})' \) \( \text{H(\Theta)} \), where \( \sim H^2(\mathbf{c}^{(n)})' \) is in \( \text{Lat}(S^{(n)}) \).

\( S(\Theta) \)

the compression of \( S^{(n)} \) to \( H(\Theta) \)

\( L_{\{\phi_n\}} \)

the linear transformation from \( L^2 \) to \( K \) given by \( L_{\{\phi_n\}}(g) = \sum_{n=1}^{\infty} \phi_n g \)

\( T_{\{\phi_n\}} \)

the linear transformation from \( H \) to \( H^2 \) given by \( T_{\{\phi\}} \left( \sum_{n=1}^{\infty} \phi_n g_n \right) = \sum_{n=1}^{\infty} \phi_n g_n \)
ABSTRACT

OPERATOR RANGES OF SHIFTS AND $C^*$-ALGEBRAS

by

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University of New Hampshire, May, 1986

It is shown that $\text{Lat}_{1/2}(\mathcal{H}^\infty(S))$, the invariant operator ranges of the commutant of the unilateral shift, is a proper sub-lattice of the lattice of invariant operator ranges of the unilateral shift, $S$. The notion of a strange operator range for $S$ of order $n$ where $n \in \mathbb{N}$ is introduced and it is demonstrated that there exist strange ranges for $S$ of every order. This is done by deriving an operator range condition which is sufficient to insure that a pair of quasi-similar compressions of shifts really be similar. A set of operator ranges which forms a sub-lattice of $\text{Lat}_{1/2}(\mathcal{H}^\infty(S))$ is introduced, which is conjectured to be $\text{Lat}_{1/2}(\mathcal{H}^\infty(S))$. The conjecture is shown to be equivalent to the assertion that the image of $S$ under certain homomorphisms of $\mathcal{H}^\infty(S)$ into $B(\mathcal{H})$ is similar to a contraction.

It is proven that the ranges of operators from a commutative $C^*$-algebra form a lattice under intersection and vector sum. If $P$ and $Q$ are projections in $B(\mathcal{H})$ with non zero intersection and so that the angle between their ranges is 0, then it is shown that the ranges of the operators in the $C^*$-algebra generated by $P$ and $Q$ does not contain the intersection of the ranges of $P$ and $Q$. Thus, non-commutative $C^*$-algebras need not have ranges which form a lattice. The
question of whether the ranges of operators from different kinds of algebras form lattices is taken up and examples are provided.

It is proven that any pair of subspaces of a Hilbert space can be the ranges of a pair of commuting operators. A family of one dimensional subspaces of a Hilbert space, \( H \), is shown to representable as the set of ranges of a family of commuting operators if and only if for each subspace the linear span of the union of the remaining subspaces is not dense in \( H \). The sets of three subspaces of \( \mathbb{C}^3 \) which can be the ranges of commuting operators are characterized.
INTRODUCTION

In this chapter, we introduce notation, concepts and facts which will be used often in the sequel; as general references, we offer [9], [11], [20] and [23]. Throughout what follows \( H \) and \( K \) will denote arbitrary complex Hilbert spaces. The symbol \( \langle \cdot , \cdot \rangle \) will be used for the inner product in the Hilbert space containing the arguments and \( \| \cdot \| \) will represent the associated norm.

All orthonormal bases of a Hilbert space, \( H \), have the same cardinality, which is called the dimension of the space and is written \( \dim(H) \). In case \( \dim(H) = \aleph_0 \), \( H \) is said to be separable. Note that a Hilbert space is determined, up to isomorphism, by its dimension.

A Hilbert space, \( H \), has a natural topology arising from its norm. The norm or strong topology is the topology given by the norm-induced metric; that is, a sequence \( \{h_n\}_{n=1}^\infty \) in \( H \) converges in norm (or strongly) to \( h \) if \( \|h_n - h\| \to 0 \) as \( n \) goes to \( \infty \). If \( M \subseteq H \), then \( M^\perp \) will denote the closure of \( M \) in the norm topology. The term subspace will always mean a norm closed linear submanifold of a Hilbert space. If \( M \) is a subspace of \( H \), denoted \( M \subseteq H \), then \( M^\perp \), the orthogonal complement of \( M \), is \( \{h \in H : \langle h, m \rangle = 0 \text{ for each } m \in M\} \subseteq H \). If \( M \subseteq H \), then for each \( h \in H \) there exist unique \( f \in M \) and \( g \in M^\perp \), such that \( h = f + g \). This is called a direct sum decomposition and written \( h = f \oplus g \); we also write \( H = M \oplus M^\perp \).

Suppose that \( P, Q, E \subseteq H \) with \( P \cap Q = E \); furthermore, let \( M \) and \( N \) be the subspaces of \( H \) given by \( P \cap E^\perp \) and \( Q \cap E^\perp \), respectively. By the angle between \( P \) and \( Q \), written \( \angle(P, Q) \),
we shall mean $\inf \{ \| m - n \| : m \in M, n \in N \text{ with } \| m \| = \| n \| = 1 \}$. A virtually equivalent definition can be found preceding Lemma 15-27 of [7].

By an operator, we shall always mean a (norm to norm) continuous linear transformation between Hilbert spaces. If $A$ is an operator from $H$ to $K$, the norm of $A$, denoted $\| A \|$, is the supremum $\sup \{ \| Ah \| : h \in H, \| h \| = 1 \}$. It follows immediately that for every $h \in H$, $\| Ah \| \leq \| A \| \cdot \| h \|$. (Note that that the symbol $\| \cdot \|$ will be used to represent operator norms together with those for different Hilbert spaces; the context invariably makes the meaning clear.) A simple consequence of linearity is the fact that a linear transformation is continuous if and only if $\| A \| < \infty$; therefore, continuous and bounded will be used interchangeably. By $B(H, K)$ we shall mean the collection of bounded linear transformations from $H$ to $K$; when $H = K$ we shall denote the set of operators on $H$ by $B(H)$.

The operator norm introduced above is not a misnomer; that is, it makes $B(H, K)$ into a normed vector space. That $B(H, K)$ is complete in this norm is a standard result (see [21], Theorem 4.1). The topology induced by that norm is called the norm topology of $B(H, K)$; a sequence $\{ A_n \}_{n=1}^{\infty}$ is said to converge to $A$ in norm, if $\| A_n - A \| \to 0$ as $n \to \infty$; it is written $A_n \to A$. Two other topologies on $B(H, K)$ will appear in the sequel; since the topologies need not be first countable, we must describe their convergent nets. A net of operators, $\{ A_\lambda \}_{\lambda \in \Lambda}$, converges to $A$ strongly, which is written $A_n \to A(s)$, if for every $h \in H$, $\| (A_\lambda - A)h \| \to 0$. A net of operators $\{ A_\lambda \}_{\lambda \in \Lambda}$ is said to converge weakly to $A$, written $A_\lambda \to A(w)$, provided that for every $h \in H$ and $k \in K$ we have
\[ \langle (A^\lambda - A)h, k \rangle \to 0. \]

Certain sorts of operators will occur frequently in what follows; we proceed to briefly describe these classes of operators. Associated with each \( A \in B(H, K) \) is a unique operator \( A^* \in B(H, K) \), called the adjoint of \( A \), so that for each \( h \in H \) and \( k \in K \) we have \( \langle Ah, k \rangle = \langle h, A^* k \rangle \). Note that \( (A^*)^* = A \). Unitary operators are the Hilbert space isomorphisms. An operator \( A \) is said to be unitary in case its adjoint is its inverse; thus, \( A^* A = 1_H \) and \( AA^* = 1_K \). If only the former holds \( A \) is called an isometry; equivalently, \( A \) is an isometry if \( \|Ah\| = \|h\| \) for all \( h \in H \). If there is a subspace \( M \subseteq H \) so that \( A|_M \), the restriction of \( A \) to \( M \), is an isometry and \( A|_M^\perp = 0 \), then \( A \) is called a partial isometry. In this case, \( M \) is called its initial space and its final space is \( \text{ran}(A) = \{Ah : h \in H\} \subseteq K \).

Now, suppose that \( A \in B(H) \). When \( A \) commutes with its adjoint, \( A \) is said to be normal; in particular, if we have \( A = A^* \), then \( A \) is called self-adjoint or hermitian. \( A \) is a positive operator, written \( A \geq 0 \), if (it is hermitian and) for every \( h \in H \) we have \( \langle Ah, h \rangle \geq 0 \). If \( A = A^2 = A^* \), then \( A \) is called a projection or an orthogonal projection. It is readily ascertained that a projection is a positive operator and that its range is a subspace of \( H \). If fact, there is a 1-1 correspondence between the subspaces of \( H \) and the projections in \( B(H) \) having them as ranges. If \( M \subseteq H \), then by \( P_M \) we shall mean the orthogonal projection onto \( M \); observe that the projection onto \( M^\perp \) is given by \( 1 - P_M \). Finally, we point out that if \( A \) is a partial isometry, then \( A^*A \) and \( AA^* \) are the projections onto the initial and final spaces of \( A \), respectively.
A helpful tool in representing an operator is the idea of operator matrix. Before introducing this concept we need notion of the direct sum of a sequence of Hilbert spaces. Suppose that \( \{H_n\}_{n=1}^{\infty} \) is a sequence of Hilbert spaces. Then the direct sum of the \( H_n \)'s, denoted \( \bigoplus_{n=1}^{\infty} H_n \), is the set of sequences, \( \bigoplus_{n=1}^{\infty} h_n \), so that for each \( n \in \mathbb{N} \), \( h_n \in H_n \) and with \( \sum_{n=1}^{\infty} \|h_n\|^2 < \infty \). It follows readily that \( \bigoplus_{n=1}^{\infty} H_n \) is a Hilbert space with inner product
\[
\langle \sum_{n=1}^{\infty} h_n, \sum_{n=1}^{\infty} g_n \rangle = \sum_{n=1}^{\infty} \langle h_n, g_n \rangle.
\]
Now, a subspace of a Hilbert space is itself a Hilbert space, so the previously given notion of direct sum is gotten by setting \( H_n = \{0\} \) for \( n > 2 \).

Suppose that \( A \in B(H, K) \) with \( H \) and \( K \) being the direct sum of the sequences \( \{H_n\}_{n=1}^{\infty} \) and \( \{K_m\}_{m=1}^{\infty} \) of Hilbert spaces, respectively. For each \( n, m \in \mathbb{N} \), let \( P_n \) be the projection onto \( H_n \) and set
\[
A_{nm} = P_n A |_{K_m}; \quad \text{relative to the direct sum decompositions given above}
\]
\( A \) is represented by the matrix of operators whose entry in row \( n \) and column \( m \) is \( A_{nm} \) for each \( n, m \in \mathbb{N} \). Composition of operators written as operator matrices is accomplished by "matrix multiplication".

If \( A \in B(H, K) \) with \( M \leq H \) and \( N \leq K \), the compression of \( A \) to \( M \) and \( N \) is given by \( P_N A |_M \). In terms of the operator matrix of \( A \) relative to \( H = M \oplus M^\perp \) and \( K = N \oplus N^\perp \), we see that this compression is \( A_{11} \). Suppose that \( A_n \in B(H_n, K_n) \) for each \( n = 1, \ldots, k \). By the direct sum of the \( A_n \), written \( \bigoplus_{n=1}^{\infty} A_n \), we mean the operator whose domain and codomain are the direct sums of the \( H_n \)'s and \( K_n \)'s, respectively, and whose matrix has the \( A_n \)'s along the main diagonal and 0's elsewhere.
A basic notion in operator theory is that of spectrum. If \( A \in B(H) \), then the set \( \{ \lambda \in \mathbb{C} : A - \lambda \text{ is not invertible} \} \) is called the spectrum of \( A \) and is denoted \( \sigma(A) \). That \( \sigma(A) \) is a non-empty, compact subset of \( \mathbb{C} \) is a standard result (see [22], Corollary 18.4.3 and Theorem 18.6). If \( A \) is a normal operator then there is a powerful functional calculus for bounded Borel functions of \( A \). By restricting our attention to continuous functions of \( A \), which are all that will be required in the sequel, a somewhat sharper statement of the properties of this functional calculus is available. When \( X \) is a compact Hausdorff space, \( C(X) \) represents the set of continuous complex valued functions on \( X \). The result below is excerpted from [21], Section 12.24 and Problem 123 of [11]:

0.1 Theorem. Let \( A \in B(H) \), with \( A \) normal and \( g \in C(\sigma(A)) \). There is a homomorphism of the algebra, \( C(\sigma(A)) \), into \( B(H) \) sending \( g \) to \( g(A) \) with the following properties:

\( a) \) If \( g \equiv 1 \), then \( g(A) = 1_H \) and if \( g(z) = z \) then \( g(A) = A \).

\( b) \) If \( \bar{g} \) is the complex conjugate of \( g \), \( \bar{g}(A) = g(A)^* \).

\( c) \) \( \sigma(g(A)) = g(\sigma(A)) = \{ g(\lambda) : \lambda \in \sigma(A) \} \).

\( d) \) \( \| g(A) \| = \sup \{ |g(\lambda)| : \lambda \in \sigma(A) \} \).

\( e) \) If \( \{ g_n \}_{n=1}^{\infty} \) is a sequence in \( C(\sigma(A)) \) which converges uniformly to \( g \), then \( \| g_n(A) - g(A) \| \to 0 \) as \( n \to \infty \).

\( f) \) If \( B \in B(H) \) and \( AB = BA \), then for every \( g \in C(\sigma(A)) \) we have \( g(A)B =Bg(A) \).
Our main applications of Theorem 0.1 will involve positive operators and projections. Two facts about the spectra of these sorts of operators are necessary. We note that a hermitian operator is positive if and only its spectrum is contained in the non-negative real axis and that a hermitian operator is a projection exactly when its spectrum is contained in \([0,1]\) (see [20], Corollary 1.7). If \(A \in \mathcal{B}(\mathcal{H})\), \(A \geq 0\) and \(f\) is the usual square root function restricted to \(\sigma(A)\), then \(f(A)\) or \(A^{1/2}\) is the positive square root of \(A\). Now, assume that \(A\) is normal and let \(g\) be a continuous characteristic function on \(\sigma(A)\). From part (c) of Theorem 0.1, we conclude that \(\sigma(g(A)) \subseteq \{0,1\}\), so by the above it follows that \(g(A)\) is a projection. In particular, if \(0\) is an isolated point of \(\sigma(A)\) and \(g\) is the characteristic function of \(\{0\}\) on \(\sigma(A)\), we conclude that \(g(A)\) is a projection. Indeed, it is the projection onto \(\ker(A)\), the eigenspace of \(0\) (see [20], Proposition 1.2). If we set \(h = 1 - g\), the characteristic function of \(\sigma(A) \setminus \{0\}\), we have \(h(A) = 1 - g(A)\), which means \(h(A)\) is the projection onto \((\ker(A))'\). A very simple argument shows that \((\ker(A))'\) is \(\text{ran}(A^*)'\), the norm closure of \(\text{ran}(A^*)\) (see [21], Theorem 12.10). Thus, if \(A\) is self adjoint and has closed range, the above shows that \(h(A)\) is the projection onto \(\text{ran}(A)\).

The concept of operator range is central to each of the remaining chapters. As the name suggests, \(R\), a linear submanifold of \(\mathcal{H}\), is called an operator range if there exists \(A \in \mathcal{B}(\mathcal{X}, \mathcal{H})\) such that \(R = AK\). Naturally, there is considerable latitude in the choice of the operator used to produce \(R\). It is often necessary to replace the operator given by the definition with another operator having the same
range, but with one or another desirable property. For example, it can sometimes be helpful to have a 1-1 operator with a given range. This is easily accomplished for if $A$ is the operator given by the definition we merely replace $A$ by $A|\ker(A)$. Producing a positive operator with the same range is another common requirement. To prove that this can always be done, let $A$ be as above. We note that the polar decomposition (see [7], Theorem 5-8) gives the existence of a partial isometry, $V$, so that $A = QV$ with $\ker(V^*) = \ker(Q)$ where $Q = (AA^*)^{1/2}$. By an argument like that required above, it follows that $(\ker(Q))^\perp = (\ker(V^*))^\perp = \text{ran}(V)$. Thus the final space of $V$ is $(\ker(Q))^\perp$, which means $\text{ran}(Q) = \text{ran}(QV)$.

The result below gives necessary and sufficient conditions for one operator range to be contained in another and for a pair of operators to have the same range. The proof can be found in [5], Theorem 2.1 and its Corollary 1. The containment result was first proven by Douglas (see [4], Theorem 1).

**0.2 Theorem.** Suppose that $A$ and $B$ are bounded operators.

(a) $A|\mathcal{H} \subseteq B|\mathcal{K}$ if and only there exists $C \in B(\mathcal{H}, \mathcal{K})$ with $\ker(A) = \ker(C)$ and such that $A = BC$.

(b) $A|\mathcal{H} = B|\mathcal{K}$ if and only there exists $C \in B(\mathcal{H}, \mathcal{K})$ such that $A = BC$ with $\ker(A) = \ker(C)$ and so that $C|\ker(A)^\perp : (\ker(A))^\perp \rightarrow (\ker(B))^\perp$ is invertible.

If $R = A|\mathcal{H}$ and $S = B|\mathcal{H}$ with $A, B \in B(\mathcal{H})$, then the vector sum of $R$ and $S$, denoted $R + S$, will mean $\{r + s : r \in R, s \in S\}$.
The next fact (see [5], Theorem 2.2) gives a useful representation of $R + S$ as an operator range. With notation as above, we have

0.3 Theorem. If $R = AH$ and $S = BH$ with $A, B \in B(H)$, then

$$R + S = (AA^* + BB^*)^{1/2}H.$$

Our next goal is the introduction of a construction which produces the intersection of two operator ranges. To this end, if $T \in B(H)$, then by $T^+$ we shall mean the linear transformation from $TH$ to $(\ker(T))^1$ given by $T^+(Tx) = x$, for each $x \in (\ker(T))^1$. If $A, B \in B(H)$ with $A$ and $B$ positive, then $A^{1/2}_H, B^{1/2}_H \subset A^{1/2}_H + B^{1/2}_H = (A + B)^{1/2}_H$ follows from the previous result. The Closed Graph Theorem (see [10], Problem 58) yields the boundedness of $C = ((A + B)^{1/2})^+ A^{1/2}$ and $D = ((A + B)^{1/2})^+ B^{1/2}$. The parallel sum of $A$ and $B$, written $A:B$, is given by $A:B = A^{1/2}(C^*D)B^{1/2}$. In effect, result can be found in Theorem 4.2 of [5]:

0.4 Theorem. If $R = AH$ and $S = BH$ with $A, B \in B(H)$, $A$ and $B$ positive, then $R \cap S = (A^2 : B^2)^{1/2}H$.

A more transparent approach to constructing the intersection of a pair of operator ranges is the following. If $R$ and $S$ are as in Theorem 0.3, then let $X:H \otimes H \to H$ have the operator matrix $(A-B)$. Letting $\pi^1$ represent the projection onto the first summand in $H \otimes H$, yields $R \cap S = A\pi^1(\ker(X))$.

If $A \subseteq B(H)$, then $M \subseteq H$ is said to be an invariant subspace for $A$ in case $AM \subseteq M$ for every $A \in A$. The set of all such is
denoted by $\text{Lat}(A)$; we will write $\text{Lat}(A)$ when $A$ is $\{A\}$. Similarly, if $R$ is an operator range and for each $A$ in $A$, $AR \subseteq R$ we will call $R$ an invariant operator range for $A$. The collection of all such will be called $\text{Lat}_{1/2}(A)$, with an analogous convention in effect in case $A$ contains a single operator. Because each closed subspace is the range of its associated projection, we have $\text{Lat}(A) \subseteq \text{Lat}_{1/2}(A)$.

There is another collection of operator ranges which is always contained in $\text{Lat}_{1/2}(A)$. Observe that when $B \in B(H)$ and $B$ commutes with each member of $A$, then for every $A \in A$ we have $ABH = BAH \subseteq BH$. The term commutant of $A$, written $A'$, refers to $\{B \in B(H) : AB = BA, \text{ for } A \in A\}$. Thus, the range of every member of $A'$ is invariant under $A$. An operator range, $R$, is called strange for $A$ in case $R \in \text{Lat}_{1/2}(A)$ that cannot be realized as the range of any operator in $A'$.

Not much is known about the invariant operator ranges for specific operators. Most of the results of this nature are due to Ong (see [16] and [17]).

Of particular importance for some of what follows is the notion of $C^*$-algebra of operators. A $C^*$-algebra of operators, $A$, is a subalgebra of $B(H)$ that is norm closed and self-adjoint; the final condition means that if $A \in A$ then $A^* \in A$. (A self-adjoint subalgebra of $B(H)$ that is closed in the weak operator topology is known as a von Neumann algebra of operators.) If $A$ contains the identity operator, it is said to be a unital $C^*$-algebra. If $S$ is a nonempty set of operators, the (unital) $C^*$-algebra generated by $S$ is the intersection of all (unital) $C^*$-algebras containing $S$; clearly, it is a
(unital) C*-algebra. C*(S) and C_1(S) will denote the C*-algebra and the unital C*-algebra generated by S, respectively. When S = \{S_1, \ldots, S_n\}, C*(S_1, \ldots, S_n) and C_1(S_1, \ldots, S_n) will be written.

We now give a more concrete characterization of the C*-algebra generated by a set of operators. If S \subseteq B(\mathcal{H}) and S = B(\mathcal{H}) with, then by Theorem 0.3 \[ R + S = (AA^* + BB^*)^{1/2}. \] Since \(\sigma(AA^* + BB^*)\) is a compact subset of \([0, \infty)\), the Weierstrass Approximation Theorem (see [12], Corollary 7.31) implies that there is a sequence of polynomials, \(\{f_n\}_{n=1}^\infty\), converging uniformly on \(\sigma(AA^* + BB^*)\) to the usual square root function. From Theorem 0.1 part (e), we conclude that \(f_n(AA^* + BB^*) \to (AA^* + BB^*)^{1/2}\); thus, \((AA^* + BB^*)^{1/2} \in C^*(A, B)\). It follows that the collection of ranges of operators from a C*-algebra is necessarily closed under vector sums; that is, if A, B \in A \subseteq B(\mathcal{H}) and A is a C*-algebra, then there exists C \in A such that \(AH + BH = CH\).

We now turn our attention to shift operators. Let \(\partial D\) be \(\{z \in \mathbb{C} : |z| = 1\}\) and let \(dm\) be normalized Lebesgue measure \((d\theta/2\pi)\) on \(\partial D\). For 1 \leq p \leq \infty, \(L^p\) shall mean the Lebesgue spaces, \(L^p(\partial D, dm)\). The space \(L^2\) is a Hilbert space with the inner product given by \(\langle f, g \rangle = \int f \overline{g} \, dm\), for every \(f, g \in L^2\). In a natural
way $L^\infty$ can be identified with a subalgebra of $B(L^2)$. If $g \in L^\infty$ then $M_g$, the operator of multiplication by $g$, is given by $[M_g(f)](z) = f(z)g(z)$ for every $f \in L^2$ and all $z \in \mathbb{D}$ . We observe that $\|M_g\| = \|g\|_\infty$ (see [11], Problem 64). The set of multiplication operators induced by members of $L^\infty$ forms a maximal abelian weakly closed subalgebra of $B(L^2)$ (see [20], Theorem 1.20 and Corollary 1.21); it is also self-adjoint since for each $g \in L^\infty$ the adjoint of multiplication by $g$ is multiplication by $\overline{g}$, the complex conjugate of $g$. (Frequently, we shall write $g$ when $M_g$ is our intent; this abuse of notation causes no confusion in practice.) The multiplication operator induced by the function $g(z) = z$ on $\mathbb{D}$ is called the bilateral shift and is denoted by $W$.

The collection of functions given by $e_n(z) = z^n$ for each $n \in \mathbb{Z}$ play a distinguished role in $L^2$; that is, they form an orthonormal basis. If $f \in L^2$, then for every $n \in \mathbb{Z}$, $f_n = \langle f, e_n \rangle$ is called the $n$th Fourier coefficient of $f$. Every $f$ in $L^2$ can be represented uniquely by a series of the form $\sum_{n=-\infty}^{\infty} \theta f_n e_n$, which is known as its Fourier series. One of the important features of the Fourier representation of $f$ is that $\|f\|^2 = \sum_{n=-\infty}^{\infty} |f_n|^2$.

We now introduce the classical Hardy spaces, $H^p$ and $H^\infty$. For $p = 2, \infty$, we set $H^p = \{g \in L^p : \langle g, e_{-n} \rangle = 0 \text{ for all } n \in \mathbb{N} \}$; $H^p$ is the set of functions in $L^p$ whose negatively indexed Fourier coefficients vanish. Since $H^2$ is the intersection of the kernels of the bounded linear functionals of $\langle \cdot, e_{-n} \rangle$, it is clear that $H^2 \subseteq L^2$; $P$ will denote the projection of $L^2$ onto $H^2$. The following characterization of $H^2$ as a collection of holomorphic
functions in $D$, the unit disk in $\mathbb{C}$. If $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ for $z \in D$, then $f \in H^2$ if and only if $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$. For details about the correspondence between the two versions of $H^2$, the reader is referred to Theorem 17.10 of [22].

If $g \in L^\infty$ the operator defined by $T_g = PM_g |H^2$ is called the Toeplitz operator induced by $g$; when $g \in H^\infty$ the corresponding Toeplitz operator will be called analytic. We note that if $g \in H^\infty$, then $M_g H^2 \subset H^2$, so that $T_g = M_g |H^2$. It is easily seen that $g(z) = z$ is in $H^\infty$ (see [22], Corollary 10.12), so $T_g = W|H^2$ is an analytic Toeplitz operator. This operator is called the unilateral shift operator and will be denoted by $S$.

There is a functional calculus for $H^\infty$ functions of $S$. Suppose that $f \in H^\infty$; since $H^\infty \subset H^2$ it may be assumed that $f$ has a power series expansion $\sum_{n=0}^{\infty} \alpha_n z^n$ valid for $z \in D$. The operator-valued power series, $\sum_{n=0}^{\infty} \alpha_n S^n$, is what is meant by $f(S)$; the Cesaro means of the partial sums of the series converge to $f(S)$ in the weak operator topology. Note that the identification of $S$ with multiplication by $z$ naturally extends to the correspondence of $f(S)$ and $T_f$. Thus, $H^\infty(S)$ will be used to denote the set of analytic Toeplitz operators.

The next result characterizes the commutants of $W$ and $S$ (see [11], Problems 146 and 147). We note that a multiplication operator associated with a function in $L^\infty$ is also known as the Laurent operator induced by that function.
0.5 Theorem. The commutant of $W$ is the set of Laurent operators and the commutant of $S$ is the set of analytic Toeplitz operators.

If $\phi \in H^\infty$ and for $z \in \partial D$, $|\phi(z)| = 1$ (m-a.e.), then $\phi$ is called an inner function. If $\phi$ and $\psi$ are inner, we say that $\phi$ divides $\psi$, written $\phi|\psi$ if there is an inner function $\omega$ such that $\psi = \omega \phi$. Note that for every $f \in H^P$ there is an inner function, unique up to multiplication by a constant function of modulus 1, called the inner factor of $f$. The inner factor of $f$, $f_\phi$, is the maximal inner function dividing $f$ in the sense that if $g \in H^P$ and $\phi$ is an inner function such that $f = \phi g$, then $\phi|f_\phi$. If \( \{f_k\}_{k=1}^n \) is a finite or infinite sequence of inner functions, the greatest common inner factor of the collection is defined in the obvious way and is denoted $f_1 \wedge f_2 \ldots \wedge f_n$; if $n = 2$, we write $f_1 \wedge f_2$.

The inner functions play the major role in the famous theorem of Beurling, identifying the invariant subspaces of $S$ (see [1], Theorem IV or [11], Problem 157).

0.6 Theorem. $\text{Lat}(S) = \{0\} \cup \{\phi H^2 : \phi, \text{ an inner function}\}$.

We shall now consider a special type of inner function, called a Blaschke product. Suppose $\{\alpha_n\}_{n=1}^\infty$ is a sequence of non-zero members of $D$ with $\sum_{n=0}^\infty (1 - |\alpha_n|) < \infty$. The Blaschke product associated with the sequence of the sequence of $\alpha_n$'s is given by

$$\prod_{n=1}^\infty \frac{\alpha_n - z}{1 - \frac{\alpha_n}{\alpha_n} z} \left|\frac{|\alpha_n|}{\alpha_n}\right|. \text{ This product is an inner function having as}$$
its zero sequence, including multiplicities, the sequence of \( \alpha_n \)'s (see [22], Theorem 15.21). A product of the form given above or one such multiplied by \( z^n \) for \( n \in \mathbb{N} \) will be called an infinite Blaschke product. Replacing the infinite products of the above with finite products of the same form or with the function identically 1 yields functions that will also be called Blaschke products.

We now turn to shifts of higher multiplicity. For \( n \in \mathbb{N} \), let \( L^2(\mathbb{C}^n) \) be the \( n \)-fold direct sum of copies of \( L^2 \) and let \( W^{(n)} \) be the \( n \)-fold direct sum of copies of \( W \); likewise, let \( H^2(\mathbb{C}^n) \) be the \( n \)-fold direct sum of copies of \( H^2 \) and let \( S^{(n)} \) be the \( n \)-fold direct sum of copies of \( S \). \( W^{(n)} \) and \( S^{(n)} \) are known, respectively, as the bilateral and unilateral shifts of multiplicity \( n \). Let

\[
\mathbf{K} = \sum_{n=1}^{\infty} \oplus L^2 \quad \text{and} \quad W^{(\infty)} = \sum_{n=1}^{\infty} \oplus W; \quad \text{also let} \quad \mathbf{H} = \sum_{n=1}^{\infty} \oplus H^2 \quad \text{and} \quad S^{(\infty)} = \sum_{n=1}^{\infty} \oplus S.
\]

We shall call \( W^{(\infty)} \) bilateral shift of infinite multiplicity and \( S^{(\infty)} \) the unilateral shift of infinite multiplicity.

For notational convenience, we set \( \mathbb{C}^\infty \equiv H^2, \quad L^2(\mathbb{C}^\infty) \equiv \mathbf{K} \) and \( H^2(\mathbb{C}^\infty) \equiv \mathbf{H} \). For each \( m, n \in \mathbb{N} \cup \{\infty\} \) set \( M_{mn} = \mathcal{B}(\mathbb{C}^m, \mathbb{C}^n) \); in case, \( m = n \), we shall write \( M_n \). By \( L^\infty(M_{mn}) \) we shall mean the set of matrices with \( m \) rows and \( n \) columns—intending countably many in case \( m \) or \( n \) is \( \infty \)—consisting of entries from \( L^\infty \) and so that if \( \Omega \) is one such, then \( \{ \| \Omega(z) \| : z \in \mathbb{D} \} \) is essentially bounded. We define \( H^\infty(M_{mn}) \) similarly and, analogous to the scalar case, its members are called matrix-valued analytic functions. With every \( \Omega \in L^\infty(M_{mn}) \), we associate an operator \( \hat{\Omega} \in \mathcal{B}(L^2(\mathbb{C}^n), L^2(\mathbb{C}^m)) \).
such that for every \( f \in L^2(\mathbb{C}^n) \) and every \( z \in \partial D \) we define 
\[
[\hat{\Omega} f](z) = \Omega(z) f(z).
\]
The mapping that sends \( \hat{\Omega} \) to \( \hat{\Omega} \) is an adjoint preserving algebra monomorphism (see [20], Theorem 3.17). Note that if \( \Omega \in L^\infty(M_{mn}) \) and \( \hat{\Omega}(H^2(\mathbb{C}^n)) \subseteq H^2(\mathbb{C}^m) \) then \( \Omega \in H^\infty(M_{mn}) \). When 
\[\tilde{\Omega} = \hat{\Omega}|H^2(\mathbb{C}^n), \text{ with } \Omega \in H^\infty(M_{mn}), \]
it is known as an operator valued analytic function. Also observe that the operators \( \hat{\Omega} \) and \( \tilde{\Omega} \) are the vector analogues of the Laurent and analytic Toeplitz operators of the scalar case. Just as in the scalar case, \( \Omega \) will be written in place of \( \hat{\Omega} \) or \( \tilde{\Omega} \) if context makes the correct interpretation clear.

If \( \Omega \in H^\infty(M_{mn}) \) and for almost every \( z \in \partial D \), \( \Omega(z) \) is a partial isometry with a common initial space, then \( \Omega \) is known as rigid; in case \( \Omega(z) \) is m-a.e. unitary, the term matrix inner function will be applied. If \( \Omega \) is rigid and \( N \leq \mathbb{C}^n \) is the common initial space of the \( \Omega(z) \), then \( \tilde{\Omega} \) is a partial isometry having 
\[H^2(N) = \{ f \in H^2(\mathbb{C}^n) : f(z) \in N \text{ m-a.e.} \}, \]
as its initial space. The operators induced by rigid and matrix inner functions will also be called rigid and inner.

The following results extend those of Theorems 0.5 and 0.6 to the vector-valued context introduced above (see [20], Corollaries 3.19 and 3.20; also [7], Theorem 12-22):

0.7 Theorem. For each \( n \in \mathbb{N} \cup \{\infty\} \), \( \{W^{(n)}\}' = L^\infty(M_n) \) \( \{W^{(n)}\}' = H^\infty(M_n) \).

0.8 Theorem. For each \( n \in \mathbb{N} \cup \{\infty\} \), \( \text{Lat}(S^{(n)}) \) is given by \( \{0\} \cup \{ \Theta H^2(\mathbb{C}^n) : \Theta \in H^\infty(M_n), \Theta \text{ rigid} \}. \)
Using the notation above, let $E = \{H^2(\mathbb{C}^n) : n \in \mathbb{N} \cup \{\infty\}\}$, with $H^2(\mathbb{C}^1) \equiv H^2$. If $E \in E$, let $S_E$ be the unilateral shift of the appropriate multiplicity. If $\Theta \in B(E)$, where $\Theta$ is a rigid function then by Theorem 0.8, we have $\Theta E \in \text{Lat}(S_E)$. We set $H(\Theta) = (\Theta E)^{\perp}$ and let $S(\Theta)$ be the compression of $S_E$ to $H(\Theta)$. The result below identifies those operators that are unitarily equivalent to a compression of a unilateral shift (see [7], Theorem 13-1). With notation as above, we have

0.9 Theorem. If $T \in B(H)$ with $\|T\| \leq 1$, then there exist $E \in E$, $\Theta \in B(E)$ with $\Theta$ rigid and $U \in B(H,E)$, a unitary operator, such that $T = U^* S(\Theta) U$ if and only if for each $h \in H$, $\| (T^*)^n h \| \to 0$.

An operator whose norm does not exceed 1 is called a contraction. A contraction that has no unitary direct summand is called a completely non-unitary contraction. In particular, if $T$ is a contraction and $(T^*)^n \to 0(s)$, then $T$ is completely non-unitary.

Let $E_1, E_2 \in E$ and let $S_1$ and $S_2$ be the corresponding shifts. If $\Omega \in B(E_1, E_2)$ and $\Omega$ is analytic, then $\Omega S_1 = S_2 \Omega$. In this case, $\Omega$ is said to intertwine $S_1$ and $S_2$. The result below identifies those operators that intertwine a pair of compressed shifts. A different version of the result below appears as Theorem 14-8 of [7], but virtually the same proof works for the one given below.
0.10 **Theorem.** Suppose that \( \Theta_1 \) and \( \Theta_2 \) are rigid operators acting on \( E_1 \) and \( E_2 \), respectively. If \( X \in B(H(\Theta_1), H(\Theta_2)) \) with \( XS(\Theta_1) = S(\Theta_2)X \), there is an operator valued analytic function, \( \Omega \in B(E_1, E_2) \) so that \( X = P_{H(\Theta_2)} \Omega|_{H(\Theta_1)}, \Omega S_1 = S_2 \Omega \) and \( \Omega(\Theta_1 E_1) \subseteq \Theta_2 E_2 \). Conversely, if \( \Omega \) intertwines \( S_1 \) and \( S_2 \), \( \Omega(\Theta_1 E_1) \subseteq \Theta_2 E_2 \) and \( X \) is the compression of \( \Omega \) to \( H(\Theta_1) \) and \( H(\Theta_2) \), then \( X \) intertwines \( S(\Theta_1) \) and \( S(\Theta_2) \).

We note that the conclusion above also holds in case either \( \Theta_1 \) or \( \Theta_2 \) is 0. In particular, suppose \( XS(\Theta_1) = S_2X \); that is, \( \Theta_2 = 0 \). It follows that there exists \( \Omega \in B(E_1, E_2) \) so that \( X = \Omega|_{H(\Theta_1)} \) and \( \Omega \Theta_1 H_1 = \{0\} \). Thus, \( \text{ran}(\Omega) = \text{ran}(X) \).

As operator that is 1-1 and has dense range is called a quasi-affinity. In case \( A \) and \( B \) are operators and there are quasi-affinities \( X \) and \( Y \) with \( AX = XB \) and \( YA = BY \), then \( A \) and \( B \) are called quasi-similar; this is written \( A \sim B \). Of course, if there is an invertible operator, \( Z \), so that \( AZ = AB \) (and \( Z^{-1}A = BZ^{-1} \)), then \( A \) and \( B \) are said to be similar, which is denoted \( A \sim B \).
CHAPTER I

ON THE INVARIANT OPERATOR RANGES OF THE
COMMUTANT OF THE UNILATERAL SHIFT

Let $H^2$ and $H^\infty$ be the classical Hardy spaces and let $S$ be the unilateral shift operator on $H^2$. We will study the collection of operator ranges which are left invariant by all the operators in $H^\infty(S)$. We will begin showing that $\text{Lat}_{1/2}(S)$ properly contains $\text{Lat}_{1/2}(H^\infty(S))$. Next, we prove that there is an extensive hierarchy of strange ranges for $S$. Extending an idea of Davidson (see [2], p. 106), we produce a sub-lattice of $\text{Lat}_{1/2}(H^\infty(S))$ which we conjecture is the set of invariant operator ranges for $H^\infty(S)$. Finally, we connect this conjecture with an unsolved problem of Halmos (see [10], Problem 6).

Before taking up the relationship between $\text{Lat}_{1/2}(H^\infty(S))$ and $\text{Lat}_{1/2}(S)$, we note a surprising fact about $\text{Lat}_{1/2}(S)$. Theorem 4 of [15] shows that $S$ has an uncountable set of dense invariant operator ranges so that the intersection of any pair is $\{0\}$. Note that this behavior is radically different from that of the invariant subspaces of $S$, since an immediate consequence of Theorem 0.8 is the fact that any two non-zero members of $\text{Lat}(S)$ have a non-zero intersection.

To show that, $\text{Lat}_{1/2}(H^\infty(S)) \neq \text{Lat}_{1/2}(S)$ for each $r$ in $(0, 1)$ consider the operator $D_r \in B(H^2)$ given by $D_r = \sum_{n=0}^{\infty} \theta r^n$. To see that each of the $D_r H^2 \in \text{Lat}_{1/2}(S)$, observe that for each $r$, we have $SD_r = D_r(r^{-1}S)$. Since $r^n \to 0$ each of the $D_r$ is a compact operator (see [11], Problem 171). We assert that no range in
Lat_{1/2}(H^\infty(S)) is that of a compact diagonal operator. Suppose that
D \in B(H^2) is given by D = \sum_{n=0}^{\infty} \Theta d_n is compact and that
DH^2 \in Lat_{1/2}(H^\infty(S)). Since D can be chosen to be positive and in-
jective, we may assume \( d_n > 0 \), for all \( n \in \mathbb{N}^* \); compactness means
that \( d_n \to 0 \). Thus, there is a subsequence \( \{d_{nk}\}_{k=0}^{\infty} \) of the diagonal
sequence of D such that for every \( k \), \( d_{nk} < (k + 1)^{-3/2} \). For each
\( n \in \mathbb{N}^* \), let \( \alpha_n = \begin{cases} (k+1)^{-2}, & \text{if } n = n_k \\ 0, & \text{otherwise} \end{cases} \). Clearly, \( \sum_{n=0}^{\infty} \alpha_n e_n \in H^\infty \),
so \( \sum_{n=0}^{\infty} \alpha_n s^n \in H^\infty(S) \). Now, \( e_0 = D \left( \frac{1}{d_0} e_0 \right) \), whence it follows
that \( \sum_{n=0}^{\infty} \alpha_n e_n = \left( \sum_{n=0}^{\infty} \alpha_n s^n \right) e_0 \in \text{ran} \left( \sum_{n=0}^{\infty} \alpha_n s^n \right) D \). However, if
\( \sum_{n=0}^{\infty} \alpha_n e_n \in \text{ran}(D) \), then \( \sum_{n=0}^{\infty} \frac{\alpha_n}{d_n} e_n \in H^2 \). This cannot be, for
\[
\left\| \sum_{n=0}^{\infty} \frac{\alpha_n}{d_n} e_n \right\|^2 = \sum_{n=0}^{\infty} \frac{\alpha_n}{d_n} \left( \sum_{k=0}^{\infty} \frac{1}{k} \right) \geq \sum_{k=0}^{\infty} \frac{1}{k},
\]
a contradiction. The question of whether there are any ranges of com-
 pact operators in Lat_{1/2}(H^\infty(S)) remains.

Recall that \( H^\infty(S) \) is the weakly closed algebra generated by \( S \).
Since a sequence of operators which converges in norm necessarily con-
verges weakly (to the same limit), it follows that \( A(S) \), the norm
closed algebra generated by \( S \), is a subalgebra \( H^\infty(S) \). Clearly,
this implies that \( Lat_{1/2}(H^\infty(S)) \subseteq Lat_{1/2}(A(S)) \). A result of Ong
(see [18], Corollary 2) states that these lattices are equal.

In [6], Foia \_ showed that both \( S \) and \( S^* \) have strange ranges.
We begin the preliminaries which will enable us to produce many more
such ranges for \( S \).
Suppose $\Theta_1$ and $\Theta_2$ are rigid and act on $H_1 = H^2(\mathbb{C}^n)$ and $H_2 = H^2(\mathbb{C}^m)$, respectively. Let $P_1$ and $P_2$ be the projections $H(\Theta_1)$ and $H(\Theta_2)$, respectively. If $S(\Theta_1) \sim q S(\Theta_2)$, there are $X_{12} \in B(H(\Theta_2), H(\Theta_1))$ and $X_{21} \in B(H(\Theta_1), H(\Theta_2))$ where $X_{12}$ and $X_{21}$ are 1-1, have dense ranges and with

$$S(\Theta_1)X_{12} = X_{12}S(\Theta_2) \quad \text{and} \quad X_{21}S(\Theta_1) = S(\Theta_2)X_{21}.$$  

An appeal to Theorem 0.10, yields the existence of $\Omega_{12} \in H^\infty(M_{nm})$ such that $X_{12} = P_1 \Omega_{12} | H(\Theta_1)$ with $\Omega_{12}(\Theta_2 H_2) \subseteq \Theta_1 H_1$; similarly, there exists $\Omega_{21} \in H^\infty(M_{mn})$ such that $X_{21} = P_2 \Omega_{21} | H(\Theta_2)$ with $\Omega_{21}(\Theta_1 H_1) \subseteq \Theta_2 H_2$.

We will denote the identity in $B(H_1)$ by $I_1$ and will let $I_n$ be the $n \times n$ identity matrix. This notation and that of the paragraph above will be used throughout our construction of strange ranges for $S$.

The proposition below contains necessary and sufficient conditions for a pair of quasi-similar shifts to actually be similar. In effect, these results can be found in Theorems 14.10 and 14.11 of [7].

1.1 Proposition. If $S(\Theta_1) \sim q S(\Theta_2)$ then the following are equivalent:

(a) $S(\Theta_1) \sim S(\Theta_2)$

(b) There exists $\Gamma \in H^\infty(M_n)$ such that
$$\tilde{\Omega}_{12} \tilde{\Omega}_{21} + \tilde{\Theta}_1 \Gamma = I_1$$

(c) There exists $\Gamma \in H^\infty(M_n)$ such that
$$\Omega_{12} \Omega_{21} + \Theta_1 \Gamma = I_n.$$
Proof: The equivalence of (b) and (c) is apparent from the fact that the mapping which sends \( \Omega \in H^\infty(M_n) \) into \( \tilde{\Omega} \in \mathcal{F}(H_1) \) is an algebra monomorphism which takes \( 1_n \) into \( I_1 \). To show (a) implies (b), assume \( S(\Theta_1) \sim S(\Theta_2) \). Thus, \( X_{12} = (X_{21})^{-1} \), which means that \( X_{12} X_{21} \) is the identity on \( H(\Theta_1) \). However, for every \( g \in H(\Theta_1) \),

\[
X_{12} X_{21} g = (P_1 \tilde{\Omega}_{12} | H(\Theta_1)) P_2 \tilde{\Omega}_{21} g
= P_1 \tilde{\Omega}_{12} \tilde{\Omega}_{21} g - P_1 \tilde{\Omega}_{12} (1 - P_2) \tilde{\Omega}_{21} g
= (P_1 \tilde{\Omega}_{12} \tilde{\Omega}_{21} | H(\Theta_1)) g,
\]

since \( \tilde{\Omega}_{12} (\tilde{\Theta}_2 H_2) \leq \tilde{\Theta}_1 H_1 \) means that \( P_1 \tilde{\Omega}_{12} (1 - P_2) = 0 \). Hence,

\[
I_1 | H(\Theta_1) = P_1 \tilde{\Omega}_{12} \tilde{\Omega}_{21} | H(\Theta_1) \quad \text{or equivalently,}
(*) \quad P_1 (I_1 - \tilde{\Omega}_{12} \tilde{\Omega}_{21}) | H(\Theta_1) = 0.
\]

Now consider \( \Gamma = \tilde{\Theta}_1^*(1_n - \tilde{\Omega}_{12} \tilde{\Omega}_{21}) \in L^\infty(M_n) \); we show that \( \Gamma \in H^\infty(M_n) \) by proving that \( \Gamma H_1 \leq H_1 \). Let \( f \in H_1 \) and write \( f = g \Theta \tilde{\Theta}_1 h \) relative \( H(\Theta_1) \Theta \tilde{\Theta}_1 H_1 \). By (*), it follows that

\[
(\tilde{\Theta}_1 - \tilde{\Omega}_{12} \tilde{\Omega}_{21}) g = \tilde{\Theta}_1 k \text{ for some } k \in H_1;
\]

also, because

\[
\tilde{\Omega}_{12} \tilde{\Omega}_{21} \Theta_1 H_1 \triangleleft \tilde{\Theta}_1 H_2 \triangleleft \tilde{\Theta}_1 H_1
\]

we conclude that for some \( 1 \in H_1 \)

we have \( \tilde{\Omega}_{12} \tilde{\Omega}_{21} \tilde{\Theta}_1 = \tilde{\Theta}_1 \). Thus, we have

\[
\hat{\Gamma} h = (\tilde{\Theta}_1)^* (I_1 - \tilde{\Omega}_{12} \tilde{\Omega}_{21}) (g + \tilde{\Theta}_1 f) = (\tilde{\Theta}_1)^* \tilde{\Theta}_1 k + (\tilde{\Theta}_1)^* \tilde{\Omega}_{12} \tilde{\Omega}_{21} \tilde{\Theta}_1 f.
\]

Moreover, \( \tilde{\Theta}_1 \) being a partial isometry means that \( (\tilde{\Theta}_1)^* \tilde{\Theta}_1 \) and \( \tilde{\Theta}_1 (\tilde{\Theta}_1)^* \) are projections onto subspaces of \( H_1 \), so there exist
$u, v \in H_1$ with $(\vec{\partial}_1)^* \vec{\partial}_1 k = u$ and $\vec{\partial}_1 (\vec{\partial}_1)^* l = v$. Consequently, 

$\hat{\Gamma} h = u + v \in H_1$ which means $\Gamma \in H^\infty(M_n)$. Also, note that

$\vec{\partial}_1 (\vec{\partial}_1)^* = I - P_2$ implies that $\vec{\partial}_1 \hat{\Gamma} = (I - P_2)(I_1 - \vec{\Omega}_{12} \vec{\Omega}_{21})$.

Coupled with (*), the above yields $\vec{\partial}_1 \hat{\Gamma} = I_1 - \vec{\Omega}_{12} \vec{\Omega}_{21}$ or as desired $\vec{\Omega}_{12} \vec{\Omega}_{21} + \vec{\partial}_1 \hat{\Gamma} = I_1$.

Now, assume that $S(\vec{\partial}_1) \sim q S(\vec{\partial}_2)$ and that there exists $\Gamma \in H^\infty(M_n)$ such that $\vec{\Omega}_{12} \vec{\Omega}_{21} + \vec{\partial}_1 \hat{\Gamma} = I_1$. By compressing to $H(\vec{\partial}_1)$ we get $P_1 \vec{\Omega}_{12} \vec{\Omega}_{21} | H(\vec{\partial}_1) = I_1 | H(\vec{\partial}_1)$. Exactly as above, we conclude that $X_{12} \ X_{21} = P_1 \vec{\Omega}_{12} \vec{\Omega}_{21} | H(\vec{\partial}_1)$, so we see that $X_{12}$ and $X_{21}$ are invertible and that $S(\vec{\partial}_1) \sim S(\vec{\partial}_2)$.

We now derive an operator range condition which is sufficient for $S(\vec{\partial}_1) \sim S(\vec{\partial}_2)$ given that $S(\vec{\partial}_1) \sim q S(\vec{\partial}_2)$. Let $\theta_0$ be a scaler inner function and let $S(\theta_0)$ be the compression of $S$ to $H(\theta_0)$. We begin by introducing some notation which will pervade the analysis which follows. Let $k_1 = n$, $k_2 = m$ and $P_0$ be the projection onto $H(\theta_0)$. As in the discussion preceding Proposition 1.1 if $S(\vec{\partial}_j) \sim q S(\theta_0)$ for $j = 1, 2$ then we have $X_{0j} \in B(H(\vec{\partial}_j), H(\theta_0))$ and $\Omega_{0j} \in H^\infty(M_{k_j})$ so that $X_{0j}$ is $1-1$ with dense range, $X_{0j} = P_0 \Omega_{0j} | H(\vec{\partial}_j)$, $\Omega_{0j}(\vec{\partial}_j H_j) \subseteq \theta_0 H^2$ and $S(\theta_0) X_{0j} = X_{0j} S(\vec{\partial}_j)$.

1.2 Theorem. If $S(\vec{\partial}_1) \sim q S(\theta_0)$ and $S(\vec{\partial}_2) \sim q S(\theta_0)$, then $\text{ran}(\Omega_{01}) = \text{ran}(\Omega_{02})$ implies $S(\vec{\partial}_1) \sim S(\vec{\partial}_2)$. 

Proof: Assume that \( S(\Theta_1) \sim S(\Theta_0) \), \( S(\Theta_2) \sim S(\Theta_0) \) and that \( \text{ran}(\Omega_{01}) = \text{ran}(\Omega_{02}) \). Then there exists \( A \in B(H_2, H_1) \) such that \( \text{ran}(A) = \ker(\Omega_{01})^\perp \), \( A|\ker(\Omega_{02})^\perp \) is invertible and with \( \Omega_{02} = \Omega_{01} A \).

We set \( X_{12} = P_1 A P_2 \) and claim that \( X_{02} = X_{01} X_{12} \). To show this we observe that \( \Omega_{01}(\Theta_1) \subset \Theta_0 H^2 \) implies that \( P_0 \Omega_{01} P_1 H_1 = 0 \).

Supposing that \( g \in H(\Theta_2) \) and arguing as in the beginning of the proof of Proposition 1.1 yields \( X_{01} X_{12} g = (P_0 \Omega_{01} | H(\Theta_1)) P_1 A g = P_0 \Omega_{01} A g \).

However, this means that \( X_{01} X_{12} g = P_0 \Omega_{02} g = X_{02} g \), verifying the claim. Whence \( X_{12} S(\Theta_2) = X_{02} S(\Theta_2) = S(\Theta_0) X_{02} = S(\Theta_0) X_{01} X_{12} = X_{01} S(\Theta_1) X_{12} \) follows. Because \( X_{01} \) is \( 1-1 \) it may be cancelled from the first and last expressions above giving \( X_{12} S(\Theta_2) = S(\Theta_1) X_{12} \).

It remains to show that \( X_{12} \) is invertible. That \( X_{12} \) is \( 1-1 \) is immediate from \( X_{02} = X_{01} X_{12} \) and the fact that \( X_{02} \) is \( 1-1 \).

We now show that \( X_{12} \) is onto. Let \( h \in H(\Theta_1) \) and consider \( X_{01} h \). Clearly, \( \text{ran}(X_{01}) = \text{ran}(X_{02}) \) follows from the corresponding conditions for \( \Omega_{01} \) and \( \Omega_{02} \) and the fact that \( \Omega_{0j}(\Theta_j H_j) \subset \Theta_0 H^2 \) for \( j = 1, 2 \).

Hence, there is a \( g \) in \( H(\Theta_2) \) for which \( X_{02} g = X_{01} h \); thus, \( X_{01} h = X_{02} g = X_{01} X_{12} g \) so \( X_{01} \) being \( 1-1 \) means that \( h = X_{12} g \).

The use of Theorem 1.2 for our ends requires an array of matrix inner functions acting on \( H = H^2(C^n) \) with \( n \geq 2 \) so that if \( \Theta \) is a scalar inner function, \( \Theta \), with \( S(\Theta) \sim S(\Theta) \). The subsequent proposition will provide such a collection, but first we introduce some notation. If \( \psi_1, \ldots, \psi_n \) are scalar inner functions then for
1.3 **Proposition.** Suppose that \( \psi_1, \ldots, \psi_n \) are scalar inner functions such that \( \psi_k \wedge \psi_k = 1 \) for each \( k = 1, \ldots, n \). Let

\[
\Theta_1 = \sum_{k=1}^{n} \Theta \psi_k, \quad \Theta = \prod_{k=1}^{n} \psi_k \quad \text{and} \quad \Omega \in H^\infty(M_n) \quad \text{be given by} \quad \Omega = (\psi_k)_{k=1}^{n}.
\]

Then \( S(\Theta) \sim S(\Theta) \) and \( X = P\Theta|H(\Theta) \) implements the quasi-similarity, where \( P \) is the projection onto \( \Theta H^2 \).

**Proof:** We commence by showing that \( X \) intertwines \( S(\Theta) \) and \( S(\Theta) \). That \( S\Omega = \Omega S(n) \) is immediate and \( \Omega \Theta H = \Theta H^2 \) so by the "converse part" of Theorem 0.10, we have \( S(\Theta)X = XS(\Theta) \). It remains to show that \( X \) is a quasi-affinity.

To prove that \( X \) is 1-1, let \( g = \sum_{k=1}^{n} \Theta g_k \in H(\Theta) \); then for each \( k \), \( g_k \perp \psi_k H^2 \). Letting \( P \) denote the projection onto \( (\Theta H^2) \perp \), we have, \( Xg = P\Theta g = P\left(\sum_{k=1}^{n} \psi_k g_k\right) \) so \( g \in \ker(X) \) means that

\[
\sum_{k=1}^{n} \psi_k g_k \in \Theta H^2.
\]

Since each \( g_k \perp \psi_k H^2 \) we observe that

\[
| \langle \sum_{k=1}^{n} \psi_k g_k, \Theta \rangle | \leq \sum_{k=1}^{n} | \langle \psi_k g_k, \prod_{j=1}^{n} \psi_j \rangle | = \sum_{k=1}^{n} | \langle g_k, \psi_k \rangle | = 0;
\]

hence, if \( g \in \ker(X) \) it follows that \( \sum_{k=1}^{n} \psi_k g_k = 0 \). For each \( j = 1, \ldots, n \) we find that \( \psi_j g_j = \sum_{k=1}^{n} \psi_k g_k \). However, for each \( j, \psi_j \) divides all the terms on the right hand side of the above, which along with \( \psi_j \wedge \psi_j = 1 \) means that \( \psi_j |g_j| \) and we conclude that \( g_j = 0 \). Thus, as desired, \( g = 0 \).
To finish the proof that \(X\) is a quasi-affinity we begin by showing that \(Ω\) has dense range. It is clear that if \(K = \text{ran}(Ω)\)
then \(K \in \text{Lat}(S)\); therefore, by Theorem 0.6, there is a scalar inner function \(θ'\) with \(K = θ'H^2\). Now, for each \(k = 1, \ldots, n\) it follows that \(ψ_k H^2 \subseteq \text{ran}(Ω) \subseteq K\); thus, for each \(k\), we have \(θ'|ψ_k\). Because \(\sum_{k=1}^n ψ_k = 1\) implies that \(θ = 1\), it follows that \(K = H^2\).

Taking up the corresponding question for \(X\), suppose that \(g \in H(θ)\) with \(g \perp \text{ran}(X)\). Then, for every \(f \in H(θ)\) we have \(\langle g, PΩf \rangle = 0\) and thus \(\langle g, Ωf \rangle = 0\). That \(g \perp ΩGH\) follows from the fact that \(Ω \cap H = θ H^2\); thus, we have \(g \perp \text{ran}(Ω)\) which yields \(g = 0\) because \(Ω\) has dense range. Thus, the range of \(X\) is dense, which completes the proof that \(X\) is a quasi-affinity.

Suppose that \(ψ_k\) for \(k = 1, \ldots, n\) is a collection of scalar inner functions such that \(ψ_k \wedge ψ_k = 1\) for \(k = 1, \ldots, n\). Now, set \(Θ_0 = \sum_{k=1}^n θ_k \wedge ψ_k, Ω_0 = \prod_{k=1}^n ψ_k\) and \(Ω_01 = (ψ_k)_{k=1}^n \in H^∞(M_{ln})\). Proposition 1.3 insures that \(S(Θ_1) \sim S(Θ_0)\) and that the compression of \(Ω_01\) to \(H(Θ_1)\) and \(H(Θ_0)\) implements the quasi-similarity. Our goal is to find a condition on the \(ψ_1, \ldots, ψ_n\) which precludes the existence of \(Ω_02 \in H^∞(M_{lm})\) so that \(Ω_02 H^2 = Ω_01 H^1\), where \(m < n\).

Suppose that \(Ω_02\) is as above and let \(N\) be the subspace of \(H^2\) given by \(\{h \in H^2 : Ω_01 h \in θ_0 H^2\}\). Note that \(N \subseteq \text{Lat}(S^{(m)})\) for if \(f \in N\), then \(Ω_02 S^{(m)} f = S Ω_02 f = S θ_0 g = θ_0 g\), for some \(g \in H^2\).

The fact that \(Ω_02\) has dense range means \(N \neq \{0\}\) so from Theorem 0.8 we see that there exists \(Ω_2 \in H^∞(M_m)\) with \(Ω_2\) rigid and so that \(N = Ω_2 H^2\). Let \(X_02\) be as above. Since
\( \Omega_{02} S^{(m)} = S \Omega_{02} \) and \( \Omega_{02}(\Theta_2 H_2) \in \theta_0 H_2 \), from Theorem 0.10 the conclusion that \( X_{02} \) intertwines \( S(\Theta_2) \) and \( S(\Theta_0) \) follows. To see that \( X_{02} \) is 1-1 note that if \( f \in H(\Theta_2) \) and \( X_{02}f = 0 \) we have \( \Omega_{02}f \in \theta_0 H_2^{(2)} \) so \( f \in \Theta_2 H_2 \) which forces \( f \) to be 0. That \( X_{02} \) has dense range follows, as in the proof of Proposition 1.3, from the fact that \( \Omega_{02} \) does; hence, \( S(\Theta_2) \sim S(\Theta_0) \). Applying Theorem 1.2, we conclude that \( S(\Theta_1) \sim S(\Theta_2) \). Thus, the argument just given shows that a sufficient condition on \( \psi_1, \ldots, \psi_n \) is that they be selected so that \( S(\Theta_1) \) cannot be similar to \( S(\Theta_2) \) for any rigid \( \Theta_2 \in H^\infty(M_m) \) with \( m < n \).

It is not immediately obvious how to select the \( \psi_k \)'s so as to satisfy this condition. Thus, a requirement more directly related to \( \psi_1, \ldots, \psi_n \) is sought. Towards that end, we press on with the line of reasoning used above. If \( S(\Theta_1) \) is similar to \( S(\Theta_2) \), then by Proposition 1.1 there exists \( \Gamma \in H^\infty(M_n) \) such that \( \Omega_{12} \Omega_{21} + \Theta_1 \Gamma = 1_n \); equivalently, for every \( z \in \mathbb{D} \), \( (1_n - \Theta_1 \Gamma)(z) = \Omega_{12}(z) \Omega_{21}(z) \). Because the ranks of \( \Omega_{12}(z) \) and \( \Omega_{21}(z) \) are both at most \( m < n \), it follows that \( \det(1_n - \Theta_1 \Gamma) = 0 \).

The following technical result gives a useful representation for the determinant of an operator which is of the form above:

Claim: Suppose that \( \phi, \Delta \in H^\infty(M_m) \) with \( \phi = \sum_{i=1}^{m} \Theta_i \phi_i \) and \( \Delta = (\delta_{ij})_{i,j=1}^{m} \) and let \( 1_m \) be the \( m \times m \) identity matrix. There exist \( \omega_1, \ldots, \omega_m \in H^\infty \) such that \( \det(1_m - \phi \Delta) = 1 - \sum_{i=1}^{m} \omega_i \phi_i \).
Proof of claim: We proceed by induction on \( m \); the case with \( m = 2 \) is dispatched by a simple computation. Now, assume that for \( 2 \leq k < m \) that the result holds and consider \( \det(l_m - \Delta) \).

Let \( \phi' \) and \( \Delta' \) be the matrices derived from \( \phi \) and \( \Delta \), respectively, by deleting their first row and first column. Since the determinant of any member of \( H^\infty(M_m) \) is in \( H^\infty \), expanding by its first column yields \( \alpha_2, \ldots, \alpha_m \in H^\infty \) such that

\[
\det(l_m - \phi\Delta) = (1 - \phi_1 \delta_{11}) \det(l_{m-1} - \phi' \Delta') - \sum_{i=2}^{m} \alpha_i \phi_i.
\]

Applying the inductive hypothesis yields \( \beta_2, \ldots, \beta_m \) in \( H^\infty \) such that

\[
\det(l_m - \phi\Delta) = (1 - \phi_1 \delta_{11})(1 - \sum_{i=2}^{m} \beta_i \phi_i) - \sum_{i=2}^{m} \alpha_i \phi_i.
\]

Setting \( \omega_1 = \delta_{11} \) and \( \omega_i = \alpha_i - \phi_1 \delta_{11} \beta_i \) for \( i = 2, \ldots, m \) yields the desired result.

Applying the claim to the evaluation of \( \det(l_n - \Theta_1 \Gamma) \) means that there exist \( \omega_1, \ldots, \omega_n \in H^\infty \) such that \( \sum_{k=1}^{n} \omega_k \psi_k = 1 \).

The results above are summarized in the following way:

1.4 Theorem: Let \( \psi_1, \ldots, \psi_n \) be scalar inner functions such that \( \psi_k \wedge \psi_k = 1 \) for \( k = 1, \ldots, n \); set \( \Theta_1 = \sum_{k=1}^{n} \Theta \psi_k \in H^\infty(M_n) \).

A necessary condition for the existence of \( \Theta_2 \) in \( \in H^\infty(M_m) \) with \( m < n \), where \( \Theta_2 \) is rigid and so that \( S(\Theta_1) \sim S(\Theta_2) \) is the existence of \( \omega_1, \ldots, \omega_n \in H^\infty \) such that \( \sum_{k=1}^{n} \omega_k \psi_k = 1 \).
1.5 Remark: Proposition 5 of [24] gives essentially the same result in the case when \( m = 1 \) but without the requirement that \( \Theta_1 \) be diagonal.

The next corollary is the major tool in the construction of strange operator ranges for \( S \).

1.6 Corollary: Let \( \psi_1, \ldots, \psi_n \) be as above and suppose that for every \( \omega_1, \ldots, \omega_n \in \mathbb{H}^\infty \) there exists \( z \in D \) such that

\[
\sum_{k=1}^{n} \omega_k(z) \psi_k(z) \neq 1.
\]

If \( \Omega_{01} = (\psi_k)_{k=1}^{n} \in \mathbb{H}^\infty(M_{1n}) \), there does not exist \( \Omega_{02} \in \mathbb{H}^\infty(M_{1m}) \) with \( m < n \) such that \( \Omega_{02} H_2 = \Omega_{01} H_1 \).

Proof: Proceeding by contradiction, assume the hypothesis and that there does exist \( \Omega_{02} \in \mathbb{H}^\infty(M_{1m}) \) with \( m < n \) such that \( \Omega_{02} H_2 = \Omega_{01} H_1 \).

As in the discussion above, it follows that \( S(\Omega_1) \sim S(\Omega_2) \), which is a contradiction since the necessary condition of Theorem 1.4 does not hold.

If \( \Omega \in \mathbb{H}^\infty(M_{1,n+1}) \) and there does not exist \( \Omega' \in \mathbb{H}^\infty(M_{1m}) \) with \( m \leq n \) and \( \Omega' H^2(C^m) = \Omega H^2(C^{n+1}) \), then \( \Omega H^2(C^{n+1}) \) will be called a strange operator range for \( S \) of order \( n \). With this terminology, the ranges produced by Folaş are strange for \( S \) of order 1. In the sequel, Corollary 1.6 will be employed in showing that there are strange ranges for \( S \) of every finite order.

The construction of the ranges indicated above requires the production of infinite Blaschke products having certain properties. A sequence, \( \{B_k\}_{k=1}^{\infty} \), of infinite Blaschke products satisfying the con-
ditions below for each \( k \in \mathbb{N} \) will be displayed:

(a) if \( B_k(z) = 0 \), then \( B_j(z) \neq 0 \) for \( j \neq k \)

(b) if \( \{z_i\}_{i=1}^{\infty} \) is the zero sequence of \( B_k \), then for each \( j \in \mathbb{N} \), \( B_j(z_i) \to 0 \) as \( i \to \infty \)

(c) \( \prod_{k=1}^{\infty} B_k \) is a convergent infinite product.

Before constructing the products described above, it will be shown that such a collection is enough to produce strange ranges for \( S \) of every order.

Suppose that \( \{B_k\}_{k=1}^{\infty} \) is a sequence of infinite Blaschke products satisfying (a), (b) and (c). Because of property (c), we may set

\[
\theta_0 = \prod_{k=1}^{\infty} B_k.
\]

For each \( n \in \mathbb{N} \), let

\[
\Theta_n = \left( \sum_{k=1}^{\infty} \phi_k B_k \right) \prod_{k=n+1}^{\infty} B_k
\]

and

\[
\Omega_n = (\omega_k)_{k=1}^{n+1} \in H(M_{1,n+1}) \text{ where } \omega_k = \frac{\theta_0}{B_k}, \text{ for } k = 1, \ldots, n \text{ and } \omega_{n+1} = \prod_{k=1}^{n} B_k.
\]

Property (a) insures that the hypothesis of Proposition 1.3 is satisfied so we conclude that \( S(\Theta_n) \sim S(\theta_0) \). Suppose that there exist \( \phi_1, \ldots, \phi_{n+1} \in H^\infty \) such that

\[
\left( \sum_{k=1}^{n} \phi_k B_k \right) + \phi_{n+1} \prod_{k=n+1}^{\infty} B_k = 1
\]

and let \( \{z_i\}_{i=1}^{\infty} \) be the zero sequence of \( B_1 \). Because the \( \phi_k \)'s are in \( H^\infty \) and since the \( B_k \)'s satisfy property (b), we see that the values at the \( z_i \)'s of every term in the sum above must go to 0 as \( i \to \infty \), which is contradiction. Thus, no such \( \phi_1, \ldots, \phi_{n+1} \) can exist, which by Corollary 1.6 implies that each \( \Omega_n \) is strange of order \( n \).

We now turn to the matter of constructing the infinite Blaschke products meeting conditions (a), (b) and (c). The first step is to
introduce the pseudo-hyperbolic metric on \( D \). For each \( z, w \in D \),
their pseudo-hyperbolic separation is \( \rho(z; w) = \frac{|z - w|}{|1 - \overline{z}w|} \); a proof
that \( \rho \) is a metric can be found in [8], p. 5. The construction will
proceed by induction.

For each \( n \in \mathbb{N} \), we set \( \alpha_{1n} = 1 - n^{-2} \); consequently,
\[
\sum_{n=1}^{\infty} |1 - \alpha_{1n}| = \sum_{n=1}^{\infty} n^{-2} < \infty \text{ which means that } \alpha_{1n}\text{ is the zero se-
quence of a Blaschke product. We let } B_1 \text{ be the Blaschke product with}
\{\alpha_{1n}\}_{n=1}^{\infty} \text{ as its zero sequence. Now, suppose that for } k \geq 1, \ B_k
\text{ having zero sequence } \{\alpha_{kn}\}_{n=1}^{\infty} \text{ has been formed. Then for each } n \in \mathbb{N}
\text{ we seek } \alpha_{k+1,n} \text{ such that } \alpha_{k+1,n} > \alpha_{k,n} \text{ and } \rho(\alpha_{k+1,n} ; \alpha_{k,n}) = 2^{-(n+k)}.
\rho(\alpha_{k+1,n} ; \alpha_{k,n}) = 2^{-(n+k)} \text{ It is easily seen that for each } n \in \mathbb{N} \text{ we}
\text{ must set } \alpha_{k+1,n} = \frac{1 + 2n + \alpha_{nk}}{2n + k + \alpha_{nk}} . \text{ The first criterion for choosing the}
\alpha_{k+1,n}^{'}s \text{ forces them to meet the condition for being the zero sequence}
of a Blaschke product because the } \alpha_{k,n} \text{ do so. We let } B_{k+1} \text{ be the}
Blaschke product with zero sequence } \{\alpha_{k+1,n}\}_{n=1}^{\infty} .

We now show that \( \{B_k\}_{k=1}^{\infty} \) is a sequence of infinite Blaschke pro-
products satisfying conditions (a) and (b). Since for each
\( n \in \mathbb{N} \) \( \{\alpha_{k,n}\}_{k=1}^{\infty} \) is a strictly increasing sequence and the same is true
of the zero sequence of \( B_1 \), to verify (a) it suffices to show that
\( \alpha_{k,n} < \alpha_{1,n+1} \) for each \( k, n \in \mathbb{N} \). Since for \( k = 1 \) this is clear,
we assume that \( k \geq 2 \). Thus, for each \( k \geq 2 \) we have
\[
\rho(\alpha_{1,n} ; \alpha_{k,n}) \leq \sum_{i=1}^{k-1} \rho(\alpha_{1,i+1,n} ; \alpha_{1,n}) = 2^{-n} \left( \sum_{i=1}^{k-1} 2^{-1} \right) < 2^{-n} ; \text{ also,}
\text{note that } \rho(\alpha_{1,n} ; \alpha_{1,n+1}) = \frac{2n+1}{2n(n+1)} > 2^{-n} . \text{ Now, we claim that if}
r, s, t \in (0, 1) \text{ with } r < s, r < t \text{ and } \rho(r, s) < \rho(r, t), \text{ then}
s < t. Since substantiating the claim amounts to no more than a brief exercise in elementary algebra we omit its proof. With the above, however, the claim yields the desired conclusion. Taking up condition (b), we first observe that if $k = j$, the assertion is trivial. For $k \neq j$, set $M$ and $M'$ equal to $\min(j, k)$ and $\max(j, k)$, respectively. For each $i$ in

$$\mathbb{N}, \quad |B_k(\alpha_j, i)| = \left| \prod_{n=1}^{\infty} \frac{\alpha_k, n - \alpha_j, i}{1 - \alpha_k, n \alpha_j, i} \right| \leq \rho(\alpha_j, i; \alpha_k, i),$$

because

$$\rho(\alpha_j, i; \alpha_k, i)$$
is the $n = i$ factor in the product and the rest of the factors have modulus at most 1. However,

$$\rho(\alpha_j, i; \alpha_k, i) \leq \sum_{m=M}^{M'-1} \rho(\alpha_{m+1}, i; \alpha_m, i) \leq |j - k| 2^{-(i+M)},$$

which means that $|B_k(\alpha_j, i)| \to 0$ as $i \to \infty$.

The sequence given above need not satisfy (c) because

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |1 - \alpha_k, n|$$
might not converge. But, if each $B_k$ is replaced by $B'_k$, where $B'_k$ is the Blaschke product constructed from a sub-sequence, $\{\beta_k, n\}_{n=1}^{\infty}$, of $\{\alpha_k, n\}_{n=1}^{\infty}$ with

$$\sum_{n=1}^{\infty} |1 - \beta_k, n| < 2^{-k},$$

then $\{B'_k\}_{k=1}^{\infty}$ obviously satisfies (c).

Moreover, it is evident that (a) and (b) hold equally well for these modified sequences so long as each sequence of $\beta_k, n$'s is constructed from the corresponding sequence of $\alpha_k, n$'s by removing finitely many entries from the beginning of the sequence.

Having constructed the Blaschke products needed to make the hierarchy of strange ranges for $S$, we turn our attention to the sub-lattice of $\text{Lat}_{1/2}(H(S))$ mentioned at the outset. If $\{\phi_n\}_{n=1}^{\infty}$
is a sequence of $H^\infty$ functions, let $T_{\{\phi_n\}}: H \rightarrow H^2$ be the linear
transformation with $T_{\{\phi_n\}}\left(\sum_{n=1}^{\infty} \phi_n h_n\right) = \sum_{n=1}^{\infty} \phi_n h_n$. The transfor-
mations given above need not be bounded so their domains of definition
are in general only submanifolds of $H$. If $A$ is the set of such
transformations as $\{\phi_n\}_{n=1}^{\infty}$ ranges over all sequences of $H^\infty$
functions, we set $T = A \cap B(H, H^2)$ and let $T_H = \{TH: T \in T\}$.

It is natural to inquire whether the sequences of $H^\infty$ functions
giving rise to operators in $T$ can be identified. The following re-
sult gives a sufficient condition for a sequence to be of that sort:

1.7 Proposition. If $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of $H^\infty$ functions with
$\sum_{n=1}^{\infty} |\phi_n|^2 \in L^\infty$ then $T_{\{\phi_n\}} \in T$; furthermore, we have
\[ \|T_{\{\phi_n\}}\|^2 \leq \|\sum_{n=1}^{\infty} |\phi_n|^2\|_\infty. \]

Proof: Assume the hypothesis and that $h = \sum_{n=1}^{\infty} \phi_n h_n \in H$. Now,
\[ \|T_{\{\phi_n\}} h\|^2 = \|\sum_{n=1}^{\infty} \phi_n h_n\|^2 \leq \sum_{n=1}^{\infty} \|\phi_n h_n\|^2 \leq \sum_{n=1}^{\infty} \|\phi_n\|^2 \|h_n\|^2; \]
applying the Cauchy-Schwarz inequality to the last of the above yields
\[ \|T_{\{\phi_n\}} h\|^2 \leq \left(\sum_{n=1}^{\infty} \|\phi_n\|^2\right) \left(\sum_{n=1}^{\infty} \|h_n\|^2\right). \]
Thus, we have
\[ \|T_{\{\phi_n\}} h\|^2 \leq \left(\sum_{n=1}^{\infty} \int |\phi_n|^2 \, dm\right) \|h\|^2 = \left(\int \sum_{n=1}^{\infty} |\phi_n|^2 \, dm\right) \|h\|^2, \]
where the equation is justified by appealing to a well-known corollary
of the Lebesgue Monotone Convergence Theorem (see [22], Theorem 1.27).
Thus, $\| T_{\{\phi_n\}} \mathbf{h} \|^2 \leq \sum_{n=1}^{\infty} |\phi_n|^2 \| \mathbf{h} \|^2 < \infty$.

The condition given above is also necessary but before we begin to establish this, we introduce more notation. If $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of $L^\infty$ functions, then we define the linear transformation $L_{\{\psi_n\}} : L^2 \rightarrow \mathbf{K}$ by $L_{\{\psi_n\}}(g) = \sum_{n=1}^{\infty} \psi_n g$; as above, this mapping need not be bounded.

Our interest in those linear transformations described above is their relationship to the adjoints of operators in $T$. The result below characterizes the sequences of $L^\infty$ functions which produce bounded linear transformations.

1.8 Lemma. If $\{\psi_n\}_{n=1}^{\infty}$ is a sequence of $L^\infty$ functions, then $L_{\{\psi_n\}} \in B(L^2, \mathbf{K})$ if and only if $\sum_{n=1}^{\infty} |\psi_n|^2 \in L^\infty$; moreover,

$$\| L_{\{\psi_n\}} \|^2 \leq \sum_{n=1}^{\infty} |\psi_n|^2 \| \mathbf{h} \|_\infty.$$  

**Proof:** Suppose that $\sum_{n=1}^{\infty} |\psi_n|^2 \in L^\infty$ and that $g \in L^2$. Arguing as above, $\| L_{\{\psi_n\}} g \|^2 = \sum_{n=1}^{\infty} \| \psi_n g \|^2 = \sum_{n=1}^{\infty} \int |\psi_n|^2 |g|^2 \, dm$

$$= \int \left( \sum_{n=1}^{\infty} |\psi_n|^2 \right) |g|^2 \, dm$$

$$\leq \sum_{n=1}^{\infty} |\psi_n|^2 \| \mathbf{h} \|_\infty \| g \|^2 < \infty.$$  

Therefore, $L_{\{\psi_n\}} \in B(L^2, \mathbf{K})$ and $\| L_{\{\psi_n\}} \| \leq \left( \sum_{n=1}^{\infty} |\psi_n|^2 \right)^{1/2} \| \mathbf{h} \|_\infty$. 

Alternatively, if $L_{\{\psi_n\}} \in B(L^2, K)$ then for every $g \in L^2$, as above, \[ \|L_{\{\psi_n\}} g\|^2 = \int \left( \sum_{n=1}^{\infty} |\psi_n|^2 \right) |g|^2 \, dm \leq \infty. \] Hence, for any $g \in L^2$, the above shows that \( \left( \sum_{n=1}^{\infty} |\psi_n|^2 \right)^{1/2} g \in L^2 \). But, the only measurable functions which induce bounded multiplication operators on $L^2$ are those in $L^\infty$ (Cf. [11], Problem 65). Thus, \( \sum_{n=1}^{\infty} |\psi_n|^2 \in L^\infty \) follows from \( \left( \sum_{n=1}^{\infty} |\psi_n|^2 \right)^{1/2} \in L^\infty \) and the fact that $L^\infty$ is an algebra.

The next two results will complete our preparation for proving the necessity of the condition in Proposition 1.10.

1.9 Lemma. $(1 - P^{(\infty)}) (W^{(\infty)})^k$ goes to 0 strongly as $k \to \infty$.

Proof: First, observe that $(1 - P)W^k \to 0(s)$. To see this, note that if $g = \sum_{n=-\infty}^{\infty} \Theta g_n e_n \in L^2$ then $\sum_{n=-\infty}^{\infty} |g_n|^2 < \infty$. Hence, for every $\delta > 0$ there exists $N \in \mathbb{Z}$ such that $\sum_{n=-\infty}^{N} |g_n|^2 < \delta$. If $g > 0$ let $N \in \mathbb{Z}$ be as above and set $N' = \min\{-1, -|N|\}$; then

\[
\| (1 - P)W^{N'} \|_1 f \|^2 \leq \sum_{n=-\infty}^{N'} |f_n|^2 < \delta, \quad \text{as desired.}
\]

Suppose $g = \sum_{n=-\infty}^{\infty} \Theta g_n \in K$ and let $\varepsilon > 0$ be given. There is $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \| f_n \|^2 < \frac{\varepsilon}{2}$. In addition, there exists a $k \in \mathbb{N}$ such that for $n = 1, 2, \ldots, N$ we have $\| (1 - P)W^k g_n \|^2 < \frac{\varepsilon}{2N}$. Hence,
\[ \| (1 - P^{(\infty)}) (W^{(\infty)})^k g \|^2 = \sum_{n=0}^{\infty} \| (1 - P) W^k g_n \|^2 \]

\[ \leq \sum_{n=0}^{N} \| (1 - P) W^k g_n \|^2 + \sum_{n=N+1}^{\infty} \| g_n \|^2 < N\left(\frac{\varepsilon}{2N}\right) + \frac{\varepsilon}{2} = \varepsilon \]

follows from the above and the fact that \( \| (1 - P) W^k \| = 1 \).

1.10 Proposition. Suppose \( \{\psi_n\}_{n=1}^{\infty} \) is a sequence of \( L^\infty \) functions. Then \( P^{(\infty)} L_{\{\psi_n\}} | H^2 \in B(H^2, H) \) if and only if

\[ \sum_{n=1}^{\infty} |\psi_n|^2 \in L^\infty \; \text{ and, } \| P^{(\infty)} L_{\{\psi_n\}} \| H^2 \leq \| \sum_{n=1}^{\infty} |\psi_n|^2 \|_\infty \].

**Proof:** Since \( \| P^{(\infty)} L_{\{\psi_n\}} \| H^2 \leq \| L_{\{\psi_n\}} \| \leq \left( \sum_{n=1}^{\infty} |\psi_n|^2 \right)^{1/2} \|_\infty \)

(by Lemma 1.8), if \( \sum_{n=1}^{\infty} |\psi_n|^2 \in L^\infty \) we have \( P^{(\infty)} L_{\{\psi_n\}} | H^2 \in B(H^2, H) \). On the other hand, if \( P^{(\infty)} L_{\{\psi_n\}} | H^2 \in B(H^2, H) \), then for each \( f \) in \( H^2 \) and each \( k \in \mathbb{N} \), it follows that

\[ \| P^{(\infty)} L_{\{\psi_n\}} f \| = \| P^{(\infty)} (W^{(\infty)})^k L_{\{\psi_n\}} f \| \]. By Lemma 1.9,

\[ \| (1 - P^{(\infty)}) (W^{(\infty)})^k L_{\{\psi_n\}} f \| \to 0 \text{ as } k \to \infty \] which implies that

\[ \| P^{(\infty)} (W^{(\infty)})^k L_{\{\psi_n\}} f \| \to \| L_{\{\psi_n\}} f \| \text{ as } k \to \infty \] since \( \| L_{\{\psi_n\}} f \|^2 \) is given by

\[ \| P^{(\infty)} (W^{(\infty)})^k L_{\{\psi_n\}} f \|^2 + \| (1 - P^{(\infty)}) (W^{(\infty)})^k L_{\{\psi_n\}} f \|^2 \].

Therefore, \( \| P^{(\infty)} L_{\{\psi_n\}} | H^2 \| = \| L_{\{\psi_n\}} | H^2 \| \). If \( g \in L^2 \), then set

\[ g = (1 - P)g \Theta P g \]. Note that \( P g \in H^2 \) and that there exists \( h \in H^2 \) so that \( (1 - P)g(z) = \overline{h(z)} \), for each \( z \in \partial D \). Also,
\[ |h(z)| = |h(z)| \] implies that \[ \|L_{\{\psi_n\}}(1 - p) g\| = \|L_{\{\psi_n\}} h\| \]. Hence,
\[ \|L_{\{\psi_n\}} g\| \leq \|L_{\{\psi_n\}} p g\| + \|L_{\{\psi_n\}} (1 - p) g\| \]
\[ \leq \|L_{\{\psi_n\}} \| h^2 \| p g\| + \|L_{\{\psi_n\}} \| h^2 \| h\| < \infty . \] Thus,
\[ L_{\{\psi_n\}} \in B(L^2, \mathcal{H}) \] and by Lemma 1.9 we have \[ \sum_{n=1}^{\infty} |\psi_n|^2 \in L^\infty . \]

We now prove the converse of Proposition 1.7.

1.11 Proposition. If \[ T_{\{\psi_n\}} \in B(\mathcal{H}, H^2) \], then \[ \sum_{n=1}^{\infty} |\phi_n|^2 \in L^\infty . \]

Proof: We claim \[ T_{\{\phi_n\}}^* = P^{(\infty)}_{\{\psi_n\}} \| H^2 \], where for each \( n \in \mathbb{N} \)
\[ \psi_n = \overline{\psi_n} \]. This is most easily seen by representing \( T_{\{\phi_n\}} \) as an
operator matrix with one row and countably many columns, \( (T_{\phi_n})_{n=1}^{\infty} \),
where for each \( n \in \mathbb{N} \), \( T_{\phi_n} \) is the analytic Toeplitz operator
associated with multiplication by \( \phi_n \). The adjoint of such an op-
erator has a matrix with countably many rows and one column with \( T_{\phi_n}^* \)
as the entry in row \( n \). Recall that \( T_{\phi_n}^* = T_{\psi_n} = PM_{\psi_n} \| H^2 \) ; hence,
\[ T_{\{\phi_n\}}^* \] is as asserted. This means that \( P^{(\infty)}_{\{\psi_n\}} \| H^2 \in B(H^2, \mathcal{H}) \).

Using Proposition 1.10, we conclude that \[ \sum_{n=1}^{\infty} |\psi_n|^2 = \sum_{n=1}^{\infty} |\phi_n|^2 \in L^\infty . \]

In view of the above, it follows that \( \mathcal{H} \) is the set of operator
ranges of the form \( T_{\{\phi_n\}} \mathcal{H} \) with \[ \sum_{n=1}^{\infty} |\phi_n|^2 \in L^\infty \] where for each
\( n \in \mathbb{N} \), \( \phi_n \in H^\infty \). The following is our major result concerning \( \mathcal{H} \)
1.12 **Theorem.** \( TH \) is a sub-lattice of \( \text{Lat}_{1/2}(H^∞(S)) \).

**Proof:** Let \( T\{\phi_n\} \in T \) and \( h = \sum_{n=1}^{∞} \Theta \cdot h_n \in H \). Then, for every \( \psi \in H^∞ \) and every \( z \in \mathbb{C} \) with \( |z| = 1 \), we have

\[
[\psi(S)T\{\phi_n\}h](z) = [\psi(S)\left(\sum_{n=1}^{∞} \phi_n h_n\right)](z) = \psi(z)\left(\sum_{n=1}^{∞} \phi_n(z)h_n(z)\right)
\]

\[
= \sum_{n=1}^{∞} \phi_n(z)\psi(z)h_n(z) = [T\{\phi_n\}\left(\sum_{n=1}^{∞} \Theta \cdot \psi(S)\right)h](z).
\]

Thus, \( TH \subseteq \text{Lat}_{1/2}(H^∞(S)) \). Now, assume that \( T\{\phi_n\}, T\{\psi_n\} \in T \).

For each \( n \in \mathbb{N}^+ \), let \( \theta_n = \begin{cases} \phi_k, & \text{if } n = 2k \\ \psi_k, & \text{if } n = 2k - 1 \end{cases} \); appealing to Proposition 1.11, gives

\[
\sum_{n=1}^{∞} |\theta_n|^2 = \sum_{n=1}^{∞} |\phi_n|^2 + \sum_{n=1}^{∞} |\psi_n|^2 \in L^∞,
\]

by Proposition 1.7 we conclude that \( T\{\theta_n\} \in T \). To show that \( TH \) is closed under sums, let \( r \in \text{ran}(T\{\phi_n\}) + \text{ran}(T\{\psi_n\}) \). Thus, there exist \( f, g \in H \) such that \( r = T\{\phi_n\}f + T\{\psi_n\}g \). If \( f = \sum_{n=1}^{∞} \Theta \cdot f_n \) and \( g = \sum_{n=1}^{∞} \Theta \cdot g_n \), let \( h = \sum_{n=1}^{∞} \Theta \cdot h_n \), where for each \( n \in \mathbb{N} \),

we set \( h_n = \begin{cases} \tilde{f}_k, & \text{if } n = 2k \\ \tilde{g}_k, & \text{if } n = 2k - 1 \end{cases} \). Clearly, \( h \in H \) and

\( r = T\{\theta_n\}h \). Alternatively, if \( s \in \text{ran}(T\{\theta_n\}) \), there exists \( h \in H \) so that \( s = T\{\theta_n\}h = \sum_{n=1}^{∞} \theta_n h_n = \sum_{n=1}^{∞} \phi_n h_{2n} + \sum_{n=1}^{∞} \psi_n h_{2n-1} \), which implies that \( s \in \text{ran}(T\{\phi_n\}) + \text{ran}(T\{\psi_n\}) \).
To show closure under intersections, with notation as above, let 
\( (T_{\phi_n}, -T_{\psi_n}) : H \oplus H \to H^2 \). Now, set \( N = \ker[(T_{\phi_n}, -T_{\psi_n})] \) and note that \( N \in \text{Lat}(S^{(\infty)} \oplus S^{(\infty)}) \). From Theorem 0.8, we conclude that for \( i, j = 1, 2 \) there exist \( \theta_{ij} \) in \( H^{(\infty)}(M_n) \) such that
\[
N = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} (H \oplus H). \quad \text{That}
\]
\( \text{ran}(T_{\phi_n}) \cap \text{ran}(T_{\psi_n}) = T_{\phi_n} \{ \theta_{11} H + \theta_{12} H \} \) is easily seen.
\( \theta_{11} \in \{ S^{(\infty)} \} \), we have \( \theta_{11} = (\theta_{ij})^{\infty}_{i=0, j=0} \) with \( \theta_{ij} \in H^{\infty} \) for all \( i, j \in \mathbb{N} \). Thus, \( T_{\phi_n} \theta_{11} = T_{\omega_n} \), where for every \( n \in \mathbb{N} \) we set \( \omega_n = \sum_{j=1}^{\infty} \phi_j \theta_{jn} \). Let \( \alpha_n \) be the canonical injection of \( H^2 \) into \( H \) as the nth summand, for each \( n \in \mathbb{N} \). Observe that
\( \theta_{11} \alpha_n : H^2 \to H \) and its matrix is given by the nth column of the matrix for \( \theta_{11} \). Clearly, \( \theta_{11} \alpha_n = \phi_n^{(\infty)} \theta_{11} \alpha_n \in B(H^2, H) \) so by Proposition 1.10, it follows that \( \sum_{j=1}^{\infty} |\theta_{jn}|^2 < L^\infty \). Therefore, for almost every \( z \in \partial D \) and for each \( n \in \mathbb{N} \),
\[ |\omega_n(z)|^2 \leq \sum_{j=1}^{\infty} |\phi_j(z) \theta_{jn}(z)|^2 \leq \| \sum_{j=1}^{\infty} |\phi_j|^2 \| \infty \| \sum_{j=1}^{\infty} |\theta_{jn}|^2 \| \infty \].

Since Proposition 1.11 insures that the first norm is finite and the above does likewise for the second we conclude that \( \omega_n \in H^{\infty} \) for all \( n \in \mathbb{N} \). Moreover, \( T_{\phi_n} \theta_{11} \in B(H, H^2) \) with the above means that \( T_{\phi_n} \theta_{11} \in T \). A similar argument shows that \( T_{\phi_n} \theta_{12} \in T \), so by the closure of \( TH \) under sums we conclude that \( TH \) is closed.
under intersections.

1.13 **Remark.** The collection of ranges given above was suggested by a conjecture of Davidson in [2], where it is asked whether

\[ \text{Lat}_{1/2}(H^\infty(S)) = \{ T[\phi_n] : \sum_{n=1}^\infty \| \phi_n \|_\infty < \infty \} \]

Theorem 1.12 indicates that \( T' \) may be a more natural conjecture. We nominate \( T' \) as a candidate for \( \text{Lat}_{1/2}(H^\infty(S)) \).

Our next goal is the display of a condition which is equivalent to \( T' = \text{Lat}_{1/2}(H^\infty(S)) \). We now introduce more notation. If \( R \in \text{Lat}_{1/2}(H^\infty(S)) \), then there exist a Hilbert space \( K \) and \( A \in B(K, H^2) \) such that \( R = AK \) with \( A \) 1-1. By a theorem of Foiaş ([6], p. 890), there exists an algebra homomorphism, \( \phi_A : H^\infty(S) \to B(H) \) such that for every \( \psi(S) \in H^\infty(S) \),

\[ \psi(S)A = A\phi_A(\psi(S)) \]

and so that \( \text{ran}(A) \perp \subset \ker(\phi_A(\psi(S))) \).

Before proceeding with the result below, we point out that if \( A \) is as above then if \( K \neq \{0\} \) it must be a separable infinite dimensional Hilbert space. Separability is clear because \( A \) is 1-1 with a separable codomain. If \( K \) is finite dimensional then \( AK \) is closed so \( AK \in \text{Lat}(S) \), but the only finite dimensional subspace in \( \text{Lat}(S) \) is \( \{0\} \), so \( K = \{0\} \).
1.14 Proposition. The following are equivalent:

(1) \( T^H = \text{Lat}_{1/2}(H^\infty(S)) \)

(2) If \( A \in \text{Lat}_{1/2}(H^\infty(S)) \) and \( A \) is 1-1 (with \( K \neq \{0\} \)), then \( \Phi_A(S) \) is similar to a contraction; that is, there exist \( B \in B(K, E) \) and \( C \in B(E) \) such that \( B \) is invertible, \( ||C|| < 1 \) and \( \Phi_A(S) = B^{-1}CB \).

Proof: We begin by showing that (1) implies (2). Suppose that \( A \in \text{Lat}_{1/2}(H^\infty(S)) \) and \( A \) is 1-1. By hypothesis, there exists \( T \in T \) with \( \text{ran}(A) = \text{ran}(T) \); let \( E = \ker(T)^\perp \) and set \( T_0 = T|E \).

Now, by Theorem 0.2 there exists \( B \in (K, E) \) so that \( B \) is invertible and \( A = T_0B \). Since \( SA = A\Phi_A(S) \), it follows that

\[
ST_0 = T_0B\Phi_A(S)B^{-1}. \quad \text{Thus, if } n \in \ker(T), \text{ then }
\]

\[
TS^{(\infty)}n = STn = 0 \quad \text{so } \ker(T) \in \text{ Lat}(S^{(\infty)}). \quad \text{Applying Theorem 0.8, there is } \Theta \in B(H) \text{ with } \Theta \text{ rigid and so that } \ker(T) \text{ is } \Theta H. \quad \text{With the usual notation, let } S(\Theta) = P_ES^{(\infty)}|E. \quad \text{Now, for every } e \in E \text{ we have } ST_0e = STe = TS^{(\infty)}e = T_0S(\Theta)e \text{ because } T(1 - P_E)S^{(\infty)}e = 0.
\]

Therefore, \( ST_0 = T_0S(\Theta) \), which yields \( T_0B\Phi_A(S)B^{-1} = T_0S(\Theta) \); however, \( T_0 \) is 1-1, so we conclude that \( \Phi_A(S) = B^{-1}S(\Theta)B \). Letting \( C = S(\Theta) \) finishes the argument since \( ||S(\Theta)|| \leq ||S^{(\infty)}|| = 1 \).

Turning to (2) implies (1), let \( R \in \text{Lat}_{1/2}(H^\infty(S)) \). As above, we may assume that \( R = AK \) with \( A \) being 1-1. By hypothesis, we have \( B' \), invertible, and \( C' \), a contraction, such that \( \Phi_A(S) = (B')^{-1}C'B' \). Now, \( K \) must be separable and infinite dimen-
sional, so there is a unitary operator \( U \in B(H, K) \); set \( B = U^{*}B' \)
and \( C = U^{*}C'U \). Therefore, \( C \in B(H) \), \( \| C \| = \| C' \| \)
and \( \phi_{A}(S) = B^{-1}CB \). Moreover, \( \phi_{A} \) being a homomorphism means that
for every \( n \in \mathbb{N} \) that \( S^{n}A = A\phi_{A}(S^{n}) = A(\phi_{A}(S))^{n} \). Along with the
above this yields \( S^{n}AB^{-1} = AB^{-1}C_{n} \), for every \( n \in \mathbb{N} \); by taking
adjoints for each \( n \in \mathbb{N} \), we have
\[
(AB^{-1})^{n}(S^{*})^{n} = (C^{*})^{n}(AB^{-1}) = (C^{*})^{n}(B^{-1})^{*}A^{*}.
\]
Fix \( h \in H^{2} \). An argument like that used at the outset of the proof of Lemma 1.12 shows that
\( \| (S^{*})^{n}h \| \to 0 \) as \( n \to \infty \) so it follows that \( \| (C^{*})^{n}(B^{-1})^{*}A^{*}h \| \to 0 \)
as well; hence, \( (C^{*})^{n}(B^{-1})^{*}A^{*} \to 0(s) \). Since \( A \) is 1-1, \( A^{*} \)
has dense range and \( B \) invertible means that \( (B^{-1})^{*}A^{*} \) also has
dense range; thus, we conclude that \( (C^{*})^{n} \to 0(s) \). Hence, \( C \) is a
completely non-unitary contraction and by Theorem 0.9, it follows that
there is a rigid function \( \Theta \in B(E) \) and a unitary operator
\( V \in B(E, H) \) so that \( C = VS(\Theta)V^{*} \). Thus, \( SAB^{-1}V = AB^{-1}VS(\Theta) \) and
applying Theorem 0.10, it follows that there exists \( T \in B(H, H^{2}) \) such
that \( T \) is analytic and \( AB^{-1}V = T \). Now, \( B^{-1}V \) is invertible so
\( \text{ran}(A) = \text{ran}(T) \). But \( T \) is precisely the set of analytic operators
in \( B(H, H^{2}) \), so \( R \in T \) which means that \( T \in \text{Lat}_{1/2}(H(\infty(S))) \).

1.15 Remark: In [11], Halmos' Problem 6 asks whether every poly-
nomially bounded operator is similar to a contraction. An operator,
\( A \) is said to be polynomially bounded if there is a constant \( C \) so
that \( \| p(A) \| \leq \| p \|_{\infty}C \), for any polynomial, \( p \). Let
\( AH \in \text{Lat}_{1/2}(H(\infty(S))) \) and let \( \phi_{A} \) be as above. The result of Foias
cited in the discussion preceding the last proposition also states
that $\Phi_A$ is bounded on every norm closed subspace of $H^\infty(S)$. If $A(S)$ is the norm closure of the polynomials in $H^\infty(S)$, then let the norm of the restriction of $\Phi_A$ to $A$ be $C$. Because $\Phi_A$ is a homomorphism, for all polynomials $p$, $p(\Phi_A(S)) = \Phi_A(p(S))$. In addition, for each $h \in H^2$, $\|p(S)h\|^2 = \int \|p(z)h(z)\|^2 dm \leq \|p\|_\infty^2 \|h\|^2$, $\|p(\Phi_A(S))\| \leq \|p(S)\| \leq \|p\|_\infty C$. Thus, a positive solution to Halmos' Problem 6 would imply that $\Phi_A(S)$ is similar to a contraction which by Proposition 1.16 would justify our conjecture that $\mathcal{TH} = \text{Lat}_{1/2}(H^\infty(S))$.

In [19], Paulsen has shown that every completely polynomially bounded operator is similar to a contraction. If for each $R \in \text{Lat}_{1/2}(H^\infty(S))$ there is an operator $A$ such that $R = AK$ and such that $\Phi_A$ is completely polynomially bounded, then the conjecture would again follow.
A theorem proven independently by Mackey (see [13]) and Dixmier (see [3]) asserts that the collection of ranges of all bounded operators acting on a Hilbert space \( H \) forms a lattice with the operations of vector sum and intersection and that this lattice is the lattice generated by the closed subspaces of \( H \). The question which will be addressed herein is that of determining if subalgebras of \( B(H) \) also have this property of having their ranges form a lattice under + and \( \cap \). The major significant positive result along these lines is that if \( A \) is a von Neumann algebra then \( \{AH : A \in A\} \) forms such a lattice (see [5], p. 261). In [5], Fillmore and Williams ask whether the same must be true of a \( C^* \)-algebra. We show that for a commutative \( C^* \)-algebra the result does hold. We also provide an example which shows that self-adjointness is necessary and another example showing that norm (or weak) closure is not necessary. Our major result is an example which shows that ranges of operators from a non-commutative \( C^* \)-algebra need not form a lattice. The key ingredient is a condition on projections \( P \) and \( Q \) in \( B(H) \) which is equivalent to \( C^*(P, Q) \) being an algebra whose ranges form a lattice; we mention other conditions which are equivalent to the latter. Finally, we produce an example which shows that for the results referred to above it is necessary for \( P \) and \( Q \) to be projections and another example which shows the extent of pathology possible in a non-commutative \( C^* \)-algebra.
We begin with the following theorem:

2.1 Theorem. If \( A \) is a commutative \( C^* \)-algebra of operators acting on a Hilbert space, \( H \), then \( \{AH : A \in A\} \) forms a lattice under + and \( \cap \).

Proof: If \( A \) is a commutative \( C^* \)-algebra, we may assume that \( A \) is the algebra of multiplications by functions in \( C_0(X) \) acting on \( L^2(X, \mu) \) where \( X \) is a locally compact Hausdorff space and \( \mu \) is a Borel measure (see [13], Theorems 3.2.2 and 10.2.1 and [22], Definition 3.16). If \( h \in C_0(X) \), since \( h L^2(X, \mu) = |h| L^2(X, \mu) \), we may assume that \( h \geq 0 \). Suppose \( f, g \in C_0(X) \) with \( f, g \geq 0 \); let \( fu + gv \in \text{ran}(M_f) + \text{ran}(M_g) \).

Claim: \( k = \frac{fu + gv}{(f^2 + g^2)^{1/2}} \in L^2(X, \mu) \).

Proof of claim: since \( \left\| \frac{fu}{(f^2 + g^2)^{1/2}} \right\|^2 = \int \frac{f^2}{f^2 + g^2} |u|^2 \, d\mu \)

\( f^2 \leq f^2 + g^2 \), implies that

\( \left\| \frac{fu}{(f^2 + g^2)^{1/2}} \right\|^2 \leq \|u\|^2 \). A similar argument shows that

\( \left\| \frac{gv}{(f^2 + g^2)^{1/2}} \right\|^2 \leq \|v\|^2 \). We conclude that

\( \left\| \frac{fu + gv}{(f^2 + g^2)^{1/2}} \right\| \leq \|u\| + \|v\| \leq \infty \).
Therefore, \( fu + gv \) is in the range of multiplication by 
\((f^2 + g^2)^{1/2} \in C_0(X)\). On the other hand, if \( m \in L^2(X, \mu) \) and 
\( h = (f^2 + g^2)^{1/2} m \), then we have \( h = fr + gs \) where,

\[
\begin{align*}
    r &= \frac{fm}{(f^2 + g^2)^{1/2}} \quad \text{and} \quad s = \frac{gm}{(f^2 + g^2)^{1/2}}. 
\end{align*}
\]

As above, we have 
\( \|r\|^2, \|s\|^2 \leq \|m\|^2 \), so \( h \in \text{ran}(M_f) + \text{ran}(M_g) \). Now, the range of multiplication by \((f^2 + g^2)^{1/2}\) is \( \text{ran}(M_f) + \text{ran}(M_g) \).

Letting \( Z(f) \) and \( Z(g) \) be the zero sets of \( f \) and \( g \), respectively, set \( h = \begin{cases} 0 & \text{if } x \in Z(f) \cap Z(g) \\ \frac{fg}{f+g} & \text{otherwise} \end{cases} \); an easy argument shows \( h \in C_0(X) \). We have \( hk = f \frac{gk}{f+g} = g \frac{fk}{f+g} \), for every \( k \in L^2(X, \mu) \). Since \( f, g \leq f + g \), as above we see that 
\[
\|\frac{gk}{f+g}\|^2, \|\frac{fk}{f+g}\|^2 \leq \|k\|^2; \text{ thus, } hk \in \text{ran}(M_f) \cap \text{ran}(M_g),
\]

Now, suppose \( m \in \text{ran}(M_f) \cap \text{ran}(M_g) \); i.e., there exist \( u, v \in L^2(X, \mu) \) such that \( m = fu = gv \). Setting \( k = u + v \), if \( x \notin Z(f) \cap Z(g) \) we have

\[
hk(x) = \left( \frac{fg}{f+g} \right) [u+v](x) = \left( \frac{mg + mf}{f+g} \right)(x) = m(x).
\]

Clearly, \( hk \) and \( m \) also agree on \( Z(f) \cap Z(g) \), so we conclude that \( m \in \text{ran}(M_h) \). Consequently, it follows that 
\( \text{ran}(M_f) \cap \text{ran}(M_g) = \text{ran}(M_h) \).
2.2 Remarks. (1) The constructions in the theorem are special cases of those given in Theorems 0.3 and 0.4 for the sum and intersection of a pair of operator ranges.

(2) If \( A \) is a unital \( C^* \)-algebra, then \( X \) in the above will be compact (see [13], Theorem 3.2.2).

2.3 Examples. (1) The following matrix example shows that for a non-self adjoint algebra of operators, commutativity is not enough to insure that the ranges of those operators will form a lattice. Let \( H = \mathbb{C}^5 \),

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

since \( AB = BA = 0 \), the algebra generated by \( A \) and \( B \) is commutative. In addition, \( A^2 = B^2 = 0 \) implies that the algebra generated by \( A \) and \( B \) is \( \{\alpha A + \beta B + \gamma 1 : \alpha, \beta, \gamma \in \mathbb{C} \} \). Now, \( \text{ran}(A) \cap \text{ran}(B) = \{(z, 0, 0, 0, 0) : z \in \mathbb{C} \} \), which cannot be the range of an operator from the algebra.

(2) The next example shows that a norm dense proper sub-algebra of a \( C^* \)-algebra can have ranges forming a lattice. Suppose \( H = L^2(0, 1) \) and that \( A \) is the algebra of multiplications by the polynomials in \( H \). If \( p \) and \( q \) are not identically 0, then \( \text{ran}(M_p) \cap \text{ran}(M_q) = \text{ran}(M_{\text{lcm}(p, q)}) \) and \( \text{ran}(M_p) + \text{ran}(M_q) = \text{ran}(M_{\text{gcd}(p, q)}) \). It is readily seen that
\[ \text{ran}(M_p) + \text{ran}(M_q) \subseteq \text{ran}(M_{\gcd(p, q)}) \] 
and the reverse inclusion follows the Euclidean algorithm. To verify the first assertion, we note that \( \text{ran}(M_q) \supseteq \text{ran}(M_{\lcm(p, q)}) \) is apparent. For the opposite inclusion, first consider the case where \( p \) divides \( q \). Then \( \text{ran}(M_p) \cap \text{ran}(M_q) = \text{ran}(M_q) \) and \( \lcm(p, q) = q \); the case with \( q \) dividing \( p \) gives an analogous result. Now, assume that \( p \) and \( q \) are relatively prime and that \( f \in \text{ran}(M_p) \cap \text{ran}(M_q) \); thus, there are \( g, h \in \mathbb{H} \) so that \( f = pg = qh \). Let \( p_1 = p/\gcd(p, q) \), \( q_1 = q/\gcd(p, q) \) and set 
\[ \varepsilon = \min\{|x-y| : x \in Z(p_1), y \in Z(q_1)\} \], where \( Z(p_1) \) and \( Z(q_1) \) are the zero sets of \( p_1 \) and \( q_1 \), respectively. Also, set 
\[ E = (0, 1) \cap \bigcup_{x \in Z(p_1)} [x - \varepsilon / 2, x + \varepsilon / 2] \] and \( F = (0, 1) \setminus E \). Then 
\[ \left\| \frac{h}{p_1} \right\|^2 = \int \left\| \frac{h}{p_1} \right\|^2 dx = \int \left\| \frac{h}{p_1} \right\|^2 dx + \int \left\| \frac{h}{p_1} \right\|^2 dx \]. Now, \((p_1)^{-1}\) is continuous on \( F \), which is compact, so there is an \( M > 0 \) so that for every \( x \in F \), \( \left\| p_1(x)^{-1} \right\| \leq M \). Thus, the second term in the last expression above is less than or equal to \( M^2 \|h\|^2 \). On the other hand, \( q_1 \) is bounded from below on \( E \), so there is an \( m > 0 \) such that for all \( x \in E \), \( \left| q_1(x) \right| \geq m \). But \( g = \frac{f}{p} = \frac{gh}{p} = \frac{q_1h}{p_1} \), so \( \|g\|^2 \geq m^2 \int \left\| \frac{h}{p_1} \right\|^2 dx \). Consequently, 
\[ \left\| \frac{h}{p_1} \right\|^2 \leq m^{-2} \|g\|^2 + M^2 \|h\|^2 < \infty \], which means \( h/p_1 \in \mathbb{H} \). Thus, 
\[ f = qh = qp_1(h/p_1) = [qp/\gcd(p, q)](h/p_1) = \lcm(p, q)(h/p_1) \]. Hence, 
\[ \text{ran}(M_p) \cap \text{ran}(M_q) \subseteq \text{ran}(M_{\lcm(p, q)}) \]. Clearly, \( A \) is not norm
closed, so it is dense in the \( C^* \)-algebra which it generates.

We now derive a result which states when the \( C^* \)-algebra generated by a pair of projections has a set of ranges forming a lattice. Let \( P, Q \) and \( E \) be non-zero subspaces of \( H \) such that \( E = P \cap Q \) and let \( P, Q \) and \( E \) be the corresponding projections in \( B(H) \). In addition, set \( M = P \cap E^\perp \) and \( N = Q \cap E^\perp \). Setting \( P_1 = P|E^\perp \) and \( Q_1 = Q|E^\perp \) gives \( P = 1 \oplus P_1 \) and \( Q = 1 \oplus Q_1 \) relative to \( E \oplus E^\perp \); furthermore, relative to \( E^\perp = M \oplus E^\perp \), we have operator matrices

\[
P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_1 = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}.
\]

The notation above will be used throughout the discussions which follow.

It is easily seen that for any subspaces \( P \) and \( Q \) that \( \mu(P, Q) \in [0, \sqrt{2}] \cup \{-\infty\} \); the results below give consequences of the value of \( \mu(P, Q) \).

2.4 Lemma. If \( \mu(P, Q) \neq 0 \), then \( ||A|| < 1 \).

Proof: Proceeding contrapositively, we assume that \( ||A|| = 1 \). Since \( Q_1 \) is a projection, \( A > 0 \) so \( 1 \in \mathfrak{p}(\sigma(A)) \subseteq \pi(A) \), the approximate point spectrum of \( A \). Thus there exists a sequence \( \{x_k\}_{k=1}^\infty \) of unit vectors in \( M \) such that \( ||Ax_k - x_k|| \to 0 \) as \( k \) goes to \( \infty \). Now, for every \( k \in \mathbb{N} \), we have \( ||Q_1 x_k||^2 = ||Ax_k||^2 + ||Bx_k||^2 \leq 1 \);
since $\|Ax_k\| \to 1$ means that $\|Bx_k\| \to 0$, follows that $\|Q_1x_k\| \to 1$. For each $k \in \mathbb{N}$, set $m_k = x_k$ and $n_k = \frac{Q_1x_k}{\|Q_1x_k\|}$.

Therefore,

$$\|m_k - n_k\|^2 = \left\|x_k - \frac{Ax_k \oplus Bx_k}{\|Q_1x_k\|}\right\|^2$$

$$\leq \left\|x_k - \frac{Ax_k}{\|Q_1x_k\|}\right\|^2 + \frac{\|Bx_k\|^2}{\|Q_1x_k\|^2}.$$

Placing the right hand side over a common denominator, adding and subtracting $x_k$ in the first term of the numerator and using the triangle inequality, the Cauchy-Schwarz inequality and the fact that $x_k$ is a unit vector yields

$$\leq \left[1 - \|Q_1x_k\| + \|Ax_k - x_k\|\right]^2 + \frac{\|Bx_k\|^2}{\|Q_1x_k\|^2}.$$ 

Let $\varepsilon > 0$ be given and set $d = \max\left\{\frac{1}{\sqrt{2}}, 1 - \frac{\varepsilon}{\sqrt{12}}\right\}$; since $\|Q_1x_k\| + 1$, there exists $N_1$ such that $k > N_1$ implies that

$\|Q_1x_k\| > d$. For all $k > N_1$, we have

$$\|Q_1x_k\|^2 > \frac{1}{2} \text{ and } 1 - \|Q_1x_k\| < \frac{\varepsilon}{\sqrt{12}};$$

therefore, $k > N_1$ implies that

$$\|m_k - n_k\|^2 \leq 2\left(\frac{\varepsilon}{\sqrt{12}} + \|Ax_k - x_k\|\right)^2 + \|Bx_k\|^2.$$
Now, there exists $N_2 > N_1$ such that $k > N_2$ implies

$$\|Ax_k - x_k\| \leq \frac{e}{\sqrt{12}} \quad \text{and} \quad \|Bx_k\|^2 \leq \frac{e^2}{6}. \quad \text{Therefore, } k > N_2 \quad \text{means}$$

$$\|m_k - n_k\|^2 \leq \left[ \frac{e}{\sqrt{12}} + \frac{e}{\sqrt{12}} \right]^2 + \frac{e^2}{3} = e^2. \quad \text{Consequently,}$$

$$\|m_k - n_k\| \to 0 \quad \text{and we conclude that } \langle P, Q \rangle = 0.$$

2.5 Lemma. If $\langle P, Q \rangle = 0$, then there exists a sequence $\{p_k\}_{k=1}^{\infty}$ of unit vectors in $M$ such that $\|Q_1 - P_1p_k\| \to 0$.

Proof: Since $\langle P, Q \rangle = 0$, there exists sequences $\{p_k\}_{k=1}^{\infty}$ and $\{q_k\}_{k=1}^{\infty}$ consisting of unit vectors in $M$ and $N$, respectively, and $\|p_k - q_k\| \to 0$. Hence,

$$\|Q_1 - P_1p_k\| = \|Q_1(p_k - q_k + p_k) - p_k\|$$

$$= \|(Q_1 - 1)(p_k - q_k)\|$$

$$\leq \|Q_1 - 1\| \|p_k - q_k\| \to 0.$$

2.6 Lemma. If $\langle P, Q \rangle = 0$, then there exists a sequence $\{p_k\}_{k=1}^{\infty}$ of unit vectors in $M$ such that for every complex polynomial, $f$, in two non-commuting variables we have

$$\|\left(f(P_1, p_1) - f(P_1, Q_1)\right)p_k\| \to 0.$$
Proof: We proceed by induction on $d$, the degree of $f$. The cases $d = -\infty, 0$ are trivial and $d = 1$ follows readily from the previous lemma. Suppose, inductively, that the assertion holds for all polynomials of degree $\leq j$ and that $f$ has degree $j+1$ with $j \geq 1$.

For every $p \in M \subseteq P$ we have

$$\left\| (f(P_1, P_1) - f(P_1, Q_1))p \right\| = \left\| (g(P_1, P_1) - g(P_1, Q_1))p \right\|$$

where $\deg(g) \leq j$. Replacing the last $Q_1$ by $P_1 + Q_1 - P_1$ and applying the triangle inequality to the right hand side of the above for every $p \in M$ that right hand side becomes

$$\left\| [g(P_1, P_1) - g(P_1, Q_1)]p \right\| + \left\| g(P_1, Q_1) \right\| \| (Q_1 - P_1)p \|.$$ 

Applying the inductive hypothesis gives the existence of a sequence, $\{p_k\}_{k=1}^{\infty}$, of unit vectors in $M$ such that for every complex polynomial, $h$, in two non-commuting variables of degree $\leq j$ we have

$$\left\| [h(P_1, P_1) - h(P_1, Q_1)]p_k \right\| \to 0.$$ 

Because $\deg(g) \leq j$, it follows that $\left\| [g(P_1, P_1) - g(P_1, Q_1)]p_k \right\| \to 0$ as $k \to \infty$. Also, $j \geq 1$ implies $\| (Q_1 - P_1)p_k \| \to 0$ and we conclude that

$$\left\| [f(P_1, P_1) - f(P_1, Q_1)]p_k \right\| \to 0.$$ 

Before proceeding to the theorem below, we make a very simple observation which is crucial to its proof. If $f$ is a polynomial like in the previous lemma, we claim that one can compute

$$\left\| f(P_1, P_2) \right\|_M$$ 

by evaluating $\| f(P_1, P_1)p \|$, where $p$ is any vector unit in $M$. To see this, note that because $P_1$ is a projection
\( f(P_1, P_1) = \alpha P_1 + \beta \), where \( \alpha, \beta \in \mathbb{C} \) and the last term means the operator which sends \( p \) to \( \beta p \). Thus, for any \( p \in M \subseteq P \) we have
\[
f(P_1, P_1)p = \alpha P_1 p + \beta p = (\alpha + \beta)p.
\]
Hence, for any unit vector \( p \in M \), \( \|f(P_1, P_1)p\| = |\alpha + \beta| \). Because this value is the same for all unit vectors \( p \in M \), we have
\[
\|f(P_1, P_1)\|_{M} = \sup \left\{ \|f(P_1, P_1)p\| : p \in M, \|p\| = 1 \right\} = |\alpha + \beta|.
\]
Moreover, since the line of reasoning above applies to any projection, it follows that \( \|f(E, E)\|_{E} = |\alpha + \beta| \).

Recall that \( E \) is the orthogonal projection onto \( P \cap Q \), which is different from \( \{0\} \). The next result connects the value of \( \gamma(P, Q) \) with the question of whether or not \( E \) is in the \( C^* \)-algebra generated by \( P \) and \( Q \).

2.7 Theorem. The following are equivalent:

1. \( \gamma(P, Q) \neq 0 \)
2. \( E \in C^*(P, Q) \)
3. \( E \in C_1^*(P, Q) \)

Proof: Beginning with the (1) implies (2), we assume that \( \gamma(P, Q) > 0 \). Applying Lemma 2.4, we conclude that \( A \) must be strict contraction. A simple calculation reveals that for every \( n \in \mathbb{N} \) we have \( (P_1 Q_1)^n = \left( \begin{array}{cc} A^n & A^{n-1}B \\ 0 & 0 \end{array} \right) \), whence it follows that for every \( n \in \mathbb{N} \), \( \| (P_1 Q_1)^n \| \leq \| A \|^n + \| A \|^{n-1} \| B \| \). Because \( \| A \| < 1 \), we
conclude that \( \| (P_1Q_1)^n \| \to 0 \). However, for each

\[ n \in \mathbb{N}, \quad (PQ)^n = \sum (P^jQ_1)^n \] which means that \( \| (PQ)^n - E \| \to 0 \)

so we conclude that \( E \in C^*(P, Q) \). The case \( \wedge (P, Q) = -\infty \) is a triviality because in that case either \( P = E \) or \( Q = E \).

That (2) implies (3) follows immediately from the fact that

\[ C^*(P, Q) \subseteq C^*_1(P, Q) \].

For (3) implies (1), we proceed by contradiction; that is, we assume \( E \in C^*_1(P, Q) \) and that \( \wedge (P, Q) = 0 \). From the former, we get the existence of a sequence of polynomials, \( \{f_j\}_{j=1}^\infty \), such that \( \| f_j(P, Q) - E \| \to 0 \); it follows that \( \| f_j(P_1, Q_1) \| \to 0 \) and that \( \| f_j(E, E) - E \| \to 0 \). Now, for every \( j \in \mathbb{N} \) if \( p \in M \) with \( \| p \| = 1 \), we have

\[
\| f_j(P_1, P_1) - M \| = \| f_j(P_1, P_1)p \|
\leq \| f_j(P_1, Q_1)p \| + \| (f_j(P_1, P_1) - f_j(P_1, Q_1))p \|
\leq \| f_j(P_1, Q_1) \| + \| (f_j(P_1, P_1) - f_j(P_1, Q_1))p \|
\]

Let \( \varepsilon > 0 \) be given; then there exists a \( J \in \mathbb{N} \) so that \( j \geq J \)

implies that \( \| f_j(P_1, Q_1) \| < \frac{\varepsilon}{2} \). Fix \( j \in \mathbb{N} \) with \( j \geq J \). Using Lemma 2.6, yields a unit vector \( p \in M \) such that

\[
\| (f_j(P_1, P_1) - f_j(P_1, Q_1))p \| < \frac{\varepsilon}{2} \]

Hence, \( \| f_j(P_1, P_1) - M \| \to 0 \).

Note that for every \( j \in \mathbb{N} \) that \( \| f_j(E, E) - E \| = \| f_j(P_1, P_1) - E \| \to 0 \)

This means that \( \| f_j(E, E) - E \| \to 0 \); therefore, we conclude that
E = 0, which is a contradiction.

Theorem 2.7 shows that \( \mathcal{L}(P, Q) = 0 \) is sufficient to insure that \( E \notin \mathcal{C}_1^*(P, Q) \), but this need not necessarily imply that \( E \) is not the range of some other operator in \( \mathcal{C}_1^*(P, Q) \). The subsequent corollary rules out this possibility.

2.8 Corollary. If \( \mathcal{L}(P, Q) = 0 \), then \( E \) is not the range of any operator in \( \mathcal{C}_1^*(P, Q) \).

Proof: We use the contrapositive. Suppose \( E = AH \) with \( A \in \mathcal{C}_1^*(P, Q) \); it may be assumed that \( A > 0 \). By an argument like that referred to in the discussion following Theorem 0.1, we conclude that \( E = \ker(A)^\perp \). Now, let \( A_1 \in B(E) \) be given by \( A|E \); clearly, \( A_1 \) is invertible and \( A = A_1 + 0 \) (relative to \( H = E \oplus E^\perp \)). The spectrum of a direct sum is the union of the spectra of the summands (see [20], Proposition 0.3), so \( \sigma(A) = \sigma(A) \cup \sigma(0|E^\perp) \). Now, because \( 0 \notin \sigma(A_1) \) and \( \sigma(0|E^\perp) = \{0\} \), this means \( \sigma(\cdot) \) is a disjoint union of non-empty closed sets. Thus, we see that \( \chi_{\sigma}(A_1) \) is a continuous function on \( \sigma(A) \). Since \( A > 0 \), \( \sigma(A) \) is a compact subset of \([0, \infty)\), so the Weierstrass Approximation Theorem (see [12], Theorem 7.31) means there is a sequence \( \{f_j\}_{j=1}^{\infty} \) of polynomials which converges uniformly to \( \chi_{\sigma}(A_1) \) on \( \sigma(A) \). That \( \|f_j[A] - \chi_{\sigma}(A_1)[A]\| \to 0 \) is a result of Theorem 0.1, which means that \( \chi_{\sigma}(A_1)[A] \in \mathcal{C}_1^*(P, Q) \). However, \( \chi_{\sigma}(A_1)[A] = E \) so applying Theorem
2.7 yields $\langle P, Q \rangle > 0$.

2.9 Remarks. (1) The argument used to prove Theorem 2.7 also proves that $\langle PH, QH \rangle = 0$ is equivalent to the assertion that for every $T \in C^*(P, Q)$ if $TH \subseteq E''$ then $T = 0$.

(2) Suppose that $P$, $P_1$, $Q$ and $Q_1$ are as above and that $P + Q$ has closed range; clearly, $P_1 + Q_1$ also has closed range. By Corollary 3 of Theorem 2 in [5], it follows that $\text{ran}(P_1 + Q_1) = \text{ran}(P_1) + \text{ran}(Q_1)$. From the discussion following Theorem 2.7 in [5], we conclude that the angle between $\text{ran}(P_1)$ and $\text{ran}(Q_1)$ is positive. It follows immediately that $\langle P, Q \rangle > 0$.

Since closure under sums is automatic for the set of ranges from a $C^*$-algebra, $P + Q$ having closed range is sufficient to insure that the ranges from $C^*(P, Q)$ form a lattice.

2.10 Examples. (1) We now produce the example which answers the question of Fillmore and Williams in the negative.

Let $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and for every $j \in \mathbb{N}$ let $q_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{1}{j} & \frac{\sqrt{j-1}}{j} \\ 0 & \frac{\sqrt{j-1}}{j} & 1 \end{pmatrix}$; these matrices are those of projections relative to the standard basis for $\mathbb{C}^3$. Let $H = \mathbb{C}^2(\mathbb{C}^3)$ and consider projections $E = \sum_{j=1}^{\infty} \oplus e$, $P = \sum_{j=1}^{\infty} \oplus p$ and $\sum_{j=1}^{\infty} \oplus q_j$ in
$B(H)$. Direct computation indicates that $EH = PH \cap QH$. We let $\delta^k_j$ be the Kronecker delta and for each $k \in \mathbb{N}$ we choose

$$m^k_j = \sum_{j=1}^{\infty} \theta <0, \delta^k_j, 0> \quad \text{and} \quad n^k_j = \sum_{j=1}^{\infty} \theta \delta^k_j <0, \frac{\sqrt{j(j-1)}}{j}, \frac{\sqrt{j}}{j}>.$$ 

the sequences $\{m^k_j\}_{j=1}^{\infty}$ and $\{n^k_j\}_{j=1}^{\infty}$ are easily seen to consist of unit vectors from $M$ and $N$, respectively. Also, for every $k \in \mathbb{N}$, we have $\|m^k_j - n^k_j\| = \|<0, 1, 0> - <0, \frac{\sqrt{k(k-1)}}{k}, \frac{\sqrt{k}}{k}>\| \to 0$;

this means that $\langle PH, QH \rangle = 0$, so $E \notin C_1^*(P, Q)$. The corollary gives the conclusion that $EH$ cannot be the range of an operator in $C_1^*(P, Q)$. Therefore, $C_1^*(P, Q)$ is a $C^*$-algebra whose ranges fail to be closed under intersections and thus fails to be a lattice.

(2) Let $E, P$ and $Q$ be non-zero projections in $B(H)$ such that $EH = PH \cap QH$, $\langle PH, QH \rangle = 0$ and with $\text{rank}(E) = \text{rank}(Q)$; in addition, let $V$ be a partial isometry with initial space $EH$ and final space $QH$. Then $VH = QH$ so $\langle PH, VH \rangle = 0$, but $E = VV^* \in C^*(P, V)$. Hence, the phenomenon of the first example may not occur if the operators involved are not projections. An interesting question remains, however; namely, if $A$ and $B$ are positive operators in $B(H)$ so that $AH = PH$ and $BH = QH$, could it be that there exists $C$ in $C_1^*(A, B)$ with $CH = EH$? Observe that if there were such a $C$, then the functional calculus argument which proved Corollary 2.8 would show that $E$ itself must be in $C_1^*(A, B)$. 
(3) In the first example, \( \{0\} \) is the biggest operator range contained within the intersection, \( EH \). We now produce a \( C^* \)-algebra having a pair of operators the intersection of whose ranges fails to be the range of an operator from the algebra and where there is no largest range from the algebra contained within that intersection. Let \( E, P, \) and \( Q \) be as in example 2, plus the requirement that \( E \) be an infinite rank projection. Set \( A = C^*_1(\{P, Q\} \cup P_E) \), where \( P_E = \{T \in B(H) : T = T^* = T^2, TH \in EH \text{ and } \text{rank}(T) < \text{rank}(E)\} \). To show that \( E \notin A \) we assume the contrary. Thus, there exists a sequence of polynomials in \( P, Q, \) and elements of \( P_E \) which converges in norm to \( E \); we shall denote this sequence of polynomials by \( \{f_k(P, Q, P_E)\}_{k=1}^{\infty} \). Since for every \( T \in P_E \) it follows that \( PT = QT = T = TQ = TP \), we conclude that each of the polynomials in the sequence can be decomposed into a sum of a polynomial in \( P \) and \( Q \) and a polynomial in elements of \( P_E \). Thus, for each \( k \in \mathbb{N} \) there exist \( g_k(P, Q) \) and \( h_k(P_E) \) with \( f_k(P, Q, P_E) = g_k(P, Q) + h_k(P_E) \). Since the constant term from each of \( h_k \) can be added to that of the corresponding \( g_k \), we may assume that each \( h_k \)'s has a zero constant term. For every \( T \in P_E \) we have \( (1 - E)T = 0 \), whence it follows that \( (1 - E)f_k(P, Q, P_E) = g_k(P_1, Q_1) \). Since \( f_k(P, Q, P_E) \to E \), we see that \( (1 - E)f_k(P, Q, P_E) = g_k(P_1, Q_1) \to 0 \). Since \( \bigwedge (PH, QH) = 0 \), the argument employed in the proof of Theorem 2.7 shows that \( g_k(E, E) \to 0 \); therefore, \( h_k(P_E) \to E \). To show that this is impossible, we consider two cases. First, suppose that \( \text{rank}(E) = \mathcal{X}_o \). This means that \( E \) is compact, being a norm limit of finite rank operators, but the only compact projections are
those of finite rank (see [11], Problem 131, [20], Theorem 7.8 and 
[5], Theorem 2.5). Now, suppose that \( \text{rank}(E) > \kappa_0 \). Because for 
each \( T \in P_E \), \( \text{rank}(T) < \text{rank}(E) \), it follows that for each \( k \in \mathbb{N} \) 
that \( \text{rank}(h_k(P_E)) < \text{rank}(E) \). But, this means that the rank of the 
limit of the sequence of \( h_k(P_E) \)'s must also be strictly less than 
the rank of \( E \). We conclude that \( E \notin A \); moreover, the functional 
calculus argument used to prove Corollary 2.8 shows that \( EH \) cannot 
be the range of any operator in \( A \). Now, \( P, Q \in A \) so \( A \) is a 
\( \mathbb{C}^\ast \)-algebra whose ranges are not closed under intersections.

While there are many operators in \( A \) whose ranges are contained 
in \( EH = PH \cap QH \), it is not difficult to see that there can be no 
operator in \( A \) with that property whose range is maximal. To show 
this, suppose that we have \( S \in A \) such that \( SH \subset EH \) and for every 
\( A \in A \) if \( AH \subset EH \) then \( AH \subset SH \). Because \( \text{ran}(S) \) is properly con­
tained in \( \text{ran}(E) \) there is an \( e \in EH \setminus SH \). Naturally, \( SH \) must be 
properly contained in \( SH + (e \otimes e)H \), where \( e \otimes e \) is the pro-
jection onto the span of \( \{e\} \); but \( SH + (e \otimes e)H = (SS^\ast + (e \otimes e))^{1/2} \)
\( SH + (e \otimes e)H = (SS^\ast + (e \otimes e))^{1/2} \) and \( (SS^\ast + (e \otimes e))^{1/2} \in A \). 
This contradiction of the maximality of \( S \), indicates that no such 
\( S \) can exist.

(4) The results above have only dealt with operators 
having closed range. The question of what can be said along these 
lines for operators with non-closed ranges is an interesting one, but 
the techniques above make such strong use of the closedness of the 
ranges involved that it is unlikely they can be modified so as to 
bear on this more general problem.
CHAPTER 3

MISCELLANEA

This brief chapter contains a few results on a pair of unrelated subjects. We first consider countable sums of operator ranges. We then take up the question of which families of operator ranges can be represented as the ranges of commuting operators. We completely answer this question in the simplest of cases and provide some very limited results and examples in more general settings.

If \( \{A_n\}_{n=1}^{\infty} \) is a sequence of members of \( B(H) \), the sum of the ranges of the \( A_n \)'s will mean the smallest linear submanifold of \( H \) containing \( \bigcup_{n=1}^{\infty} A_n H \); such a sum will be written \( \sum_{n=1}^{\infty} A_n H \). If for each \( n \in \mathbb{N} \) we have \( A_n H \subseteq A_{n+1} H \) the \( A_n H \)'s are called an increasing sequence of operator ranges. The fact which follows is a useful technical device in dealing with countable sums of operator ranges.

3.1 Lemma. Every countable sum of operator ranges can be represented as the sum of an increasing sequence of operator ranges.

Proof: Suppose \( \sum_{n=1}^{\infty} A_n H \) is a sum of operator ranges; we may assume that for each \( n \in \mathbb{N} \) that \( A_n \geq 0 \). Now, \( \bigcup_{n=1}^{\infty} A_n H \) can be written as \( \bigcup_{n=1}^{\infty} \left( \bigcup_{k=1}^{n} A_k H \right) \) and the linear span of the latter is easily seen to be the same as that of \( \bigcup_{n=1}^{\infty} \left( \sum_{k=1}^{n} A_k H \right) \). For each \( n \in \mathbb{N} \), we set...
\[ C_n = \left( \sum_{k=1}^{n} A_k^n \right)^{1/2} \]; appealing to Theorem 0.3 yields the conclusion that \( C_n H = \sum_{k=1}^{n} A_k^n H \) for each \( n \in \mathbb{N} \). Of course, the \( C_n H \)'s form an increasing sequence of operator ranges so the observations above show that sum of the \( C_n \)'s produces the required representation.

The sum of a collection of countable sums of operator ranges is defined to be the linear span of the union of all the operator ranges in the summands. The result below shows that the set of countable sums of operator ranges in \( H \) has some structure.

**3.2 Proposition.** The collection of countable sums of operator ranges in \( H \) is closed under the formation of countable sums and finite intersections.

**Proof:** Since the union of a countable collection of countable sets is countable, closure under sums is obvious. Suppose \( R \) and \( S \) are countable sums of operator ranges in \( H \). By the previous result, it may be assumed that there exist sequences of positive operators \( \{A_n\}_{n=1}^{\infty} \) and \( \{B_n\}_{n=1}^{\infty} \) yielding realizations of \( R \) and \( S \), respectively, as sums of increasing sequences of operator ranges. Since these are increasing sums, it follows that for each \( x \in R \cap S \) there are \( m, n \in \mathbb{N} \) and \( g, h \in H \) such that \( x = A_m^g = B_n^h \). In addition, the sums being increasing means that we may choose \( m = n \), be setting the common value equal to the larger of the two given above; thus,
x ∈ A_n ⊗ B H_n. From Theorem 0.4, it follows that for each n ∈ N that there exists C_n ∈ B(H) with C_n H = A_n ⊗ B H_n, which with the above yields R ∧ S ⊆ \sum_{n=1}^{∞} E_n H. The opposite inclusion is obvious.

A more interesting question involves determining the objects generated by allowing the formation of countable intersections as well as countable sums of operator ranges; that is, the σ-lattice generated by the operator ranges in H. A useful tool in such an investigation would be a dualized version of the above; however, even under very special circumstances, it is not clear whether or not the sum of a pair of countable intersections of operator ranges can be a countable intersection of operator ranges.

Before considering operator ranges representable as the ranges of commuting operators, we introduce a bit of terminology. If M and N are subspaces of H with M + N = H and M ∩ N = \{0\}, then M and N are said to be complementary. For each h ∈ H there are unique m ∈ M and n ∈ N so that h is m + n. The projection, P, on M along N is the linear transformation given by P(m + n) = m; the Closed Graph Theorem (see [11], Problem 44) shows that P ∈ B(H).

The next result provides one of the answers alluded to above.

3.3 Proposition. Any pair of subspaces of a Hilbert space can be represented as the ranges of a pair of commuting operators.

Proof: Suppose that P, Q ⊆ H and set E = P ∩ N, M = P ∩ E^⊥, N = Q ∩ E^⊥ and K = (M + N)^-. In addition, let P, Q, E, M and
N be the orthogonal projections onto \( P, Q, E, M \) and \( N \), respectively. Consider \( X = (1 - N)|M \) and \( Y = M|(K \cap N^\perp) \). If 
\( Xm = 0 \), \( m \in M \cap N = \{0\} \), so \( X \) is 1-1. On the other hand, if 
\( Yk = 0 \), then \( k \) is orthogonal to \( M \); but, the only member of \( K \) 
which is orthogonal to both \( M \) and \( N \) is 0 so \( Y \) is also 1-1.

Since \( XM \subseteq K \cap N^\perp \) and \( Y(K \cap N^\perp) \subseteq M \), \( M \) and \( K \cap N^\perp \) have the 
same dimension. A similar argument gives the conclusion that \( N \) and 
\( K \cap N^\perp \) have the same dimension. Therefore, there exists \( U \in B(H) \) 
which is unitary and such that \( U \) restricted to \( E \oplus K^\perp \) is the 
identity, \( U(M) = K \cap M^\perp \) and \( U(N) = K \cap N^\perp \). Setting \( P' = PU^* \) and 
\( Q' = QU \), we have \( F = P'H \) and \( Q = Q'H \). Because 
\( P'E \oplus K^\perp = E \oplus K^\perp = Q' \) \( E \oplus K^\perp \), \( P' \) and \( Q' \) commute on \( E \oplus K^\perp \).

Moreover, \( P'Q'K = PU^*QK = PU^*N = P(K \cap M^\perp) = \{0\} \) and 
\( Q'P'K = QUPK = QUM = Q(K \cap N^\perp) = \{0\} \), which completes the proof that 
\( P' \) and \( Q' \) commute.

3.2 Remark. The situation is radically different if one of the 
operators is required to be normal. Assume \( A, B \in B(H) \) and that \( A \) 
is normal; in addition, let \( A = AH \) and \( B = BH \) with \( A, B \leq H \).

Note that \( P_A \in C_1^*(A) \) and \( P_B \in C_1^*(B) \). If \( A \) commutes with \( B \), 
then Fuglede's Theorem (see [20], Corollary 1.18) insures that \( A^* \) 
commutes with \( B \). Thus, any member of \( C_1^*(A) \) commutes with \( B \); 
in particular, we have \( P_A B = BP_A \). Taking adjoints in the last 
equation yields \( B^*P_A = P_A B^* \). As above, we conclude that \( P_A \) and 
\( P_B \) must commute.
There are two natural paths to pursue in the quest for a more general version of Proposition 3.3. What could be concluded if one or perhaps both of the closed ranges were replaced by non-closed operator ranges? The answer to this question is unclear. The other sort of extension involves using more than two subspaces. In the discussion below, we present some results along this line.

We begin by restricting our attention to one dimensional subspaces. Suppose \( M \leq H \) is one dimensional and let \( \mathbf{e} \) be a vector of norm 1 in \( M \). Each operator with range \( M \) is of the form \( \mathbf{e} \otimes f \), for a nonzero \( f \in H \), where \( \mathbf{e} \otimes f (h) = \langle \mathbf{f}, h \rangle \mathbf{e} \) for every \( h \in H \). Let \( \{ M_\lambda : \lambda \in \Lambda \} \) be a family of distinct one dimensional subspaces of \( H \) and for each \( \lambda \in \Lambda \) let \( \mathbf{e}_\lambda \) be a unit vector in \( M_\lambda \). Furthermore, assume that for each \( \lambda_0 \in \Lambda \) the span of \( \{ \mathbf{e}_\lambda : \lambda \in \Lambda, \lambda \neq \lambda_0 \} \) is not dense in \( H \). Thus, for each \( \lambda_0 \in \Lambda \), there exists \( f_\lambda \in H \) which is orthogonal to the span of \( \{ \mathbf{e}_\lambda : \lambda \in \Lambda, \lambda \neq \lambda_0 \} \) set \( A_{\lambda_0, \lambda} = \mathbf{e}_\lambda \otimes f_\lambda \). Clearly, for each \( \lambda \in \Lambda, M_\lambda = \text{ran}(A_\lambda) \); moreover, \( \lambda_0 \neq \lambda_1 \) implies that \( A_{\lambda_0, \lambda_1} A_{\lambda_1, \lambda_0} = 0 = A_{\lambda_1, \lambda_0} A_{\lambda_0, \lambda_1} \). On the other hand, suppose that \( M \) and \( N \) are distinct one dimensional subspaces of \( H \); let \( \mathbf{m} \) and \( \mathbf{n} \) be unit vectors in \( M \) and \( N \), respectively. If \( \mathbf{f} \) and \( \mathbf{g} \) are non-zero vectors in \( H \), then for every \( h \in H \) we have
\[
(m \otimes f)(n \otimes g)h = \langle \mathbf{g}, h \rangle \langle \mathbf{f}, n \rangle \mathbf{m} \quad \text{and} \quad (n \otimes g)(m \otimes f)h = \langle \mathbf{f}, h \rangle \langle \mathbf{g}, m \rangle \mathbf{n}.
\]
If \( m \otimes f \) and \( n \otimes g \) commute, then the fact that \( \mathbf{m} \) and \( \mathbf{n} \) are linearly independent means that both \( \langle \mathbf{g}, h \rangle \langle \mathbf{f}, n \rangle \) and \( \langle \mathbf{f}, h \rangle \langle \mathbf{g}, m \rangle \) must be 0 for every
In particular, the choice \( h = g \) forces \( \langle f, n \rangle = 0 \) and \( h = f \) means that \( \langle g, m \rangle = 0 \). Therefore, for any pair of commuting operators with ranges \( M \) and \( N \), it follows that

\[
\text{ran}(m \otimes f) \subset \ker(n \otimes g) \quad \text{and} \quad \text{ran}(n \otimes g) \subset \ker(m \otimes f)
\]

Now, assume that \( \{M_{\lambda}\}_{\lambda \in \Lambda} \) is a family of distinct one dimensional subspaces of \( H \) and for every \( \lambda \in \Lambda \), set \( M_{\lambda} = \text{ran}(A_{\lambda}) \). Suppose that the \( A_{\lambda} \)'s are a commuting family and let \( \lambda_0 \in \Lambda \). By the above, it follows that

\[
\bigcup \{M_{\lambda} : \lambda \neq \lambda_0\} \subseteq \ker(A_{\lambda_0}) \neq H
\]

the span of \( \{e_\lambda : \lambda \in \Lambda, \lambda \neq \lambda_0\} \) is not dense in \( H \), where the \( e_\lambda \) are as above.

Retaining the notation of the previous paragraph, the results of the discussion above are summarized as follows:

### 3.5 Proposition
Suppose that \( \{M_{\lambda}\}_{\lambda \in \Lambda} \) is a family of distinct one dimensional subspaces of \( H \). There exists a family of commuting operators \( \{A_{\lambda}\}_{\lambda \in \Lambda} \) with \( M_{\lambda} = \text{ran}(A_{\lambda}) \) for every \( \lambda \in \Lambda \) if and only if the span of \( \{e_\lambda : \lambda \in \Lambda, \lambda \neq \lambda_0\} \) is not dense in \( H \) for each \( \lambda_0 \in \Lambda \).

### 3.6 Remark
If \( H \) is \( n \)-dimensional, Proposition 3.5 says that any collection of \( n \) or fewer one dimensional subspaces can be represented as the ranges of a commuting family of operators.

If \( H \) is \( n \)-dimensional and the subspaces involved are not all one dimensional then not every collection of \( n \) or fewer subspaces is representable as the ranges of a set of commuting operators. We claim that if \( M_1, M_2 \) and \( M_3 \) are two dimensional subspaces of \( \mathbb{C}^3 \), a
necessary and sufficient condition for the existence of commuting operators, \( A_i \), with \( M_i = \text{ran}(A_i) \) for \( i = 1, 2, 3 \) is that

\[ M_1 \cap M_2 \cap M_3 = \{0\} \]

To verify this, we begin by assuming that

\[ M_1 \cap M_2 \cap M_3 = \{0\} \]

Now, let \( E_1 = M_2 \cap M_3 \), \( E_2 = M_1 \cap M_3 \), and \( E_3 = M_1 \cap M_2 \); note that \( \mathbb{C}^3 = E_1 + E_2 + E_3 \). Also,

\[ M_1 = E_2 + E_3, \quad M_2 = E_1 + E_3, \quad \text{and} \quad M_3 = E_1 + E_2 \]. For \( i = 1, 2, 3 \) let \( A_i \) be the projection on \( M_i \) along \( E_i \). The \( A_i \)'s obviously have the desired ranges and they commute since for \( i \neq j \), \( A_i A_j = 0 \). Alternatively, suppose that there are commuting \( A_i \)'s having the \( M_i \)'s as ranges but \( M_1 \cap M_2 \cap M_3 \neq \{0\} \). Let \( N \) represent the intersection of the \( M_i \)'s and consider

\[ E_i = M_i / N \] in \( \mathcal{H} = \mathbb{C}^3 / N \) for \( i = 1, 2, 3 \); clearly, the \( E_i \)'s are distinct one dimensional subspaces of \( \mathcal{H} \). If \( \pi \) is the canonical projection of \( \mathbb{C}^3 \) onto \( \mathcal{H} \), then the \( \pi A_i \)'s form a commuting family in \( \mathcal{H} \) and \( M_i / N = \text{ran}(\pi A_i) \) for \( i = 1, 2, 3 \). Since \( N \neq \{0\} \), it follows that \( \mathcal{H} \) is at most two dimensional. This contradiction of Proposition 3.5 finishes the justification of the assertion.

Very similar lines of reasoning establish analogous results for the other combinations of three subspaces in \( \mathbb{C}^3 \). If \( M_1 \) and \( M_2 \) are distinct two dimensional subspaces of \( \mathbb{C}^3 \) and \( M_3 \) is one dimensional, then the \( M_i \)'s are realizable as the ranges of a commuting family if and only if \( M_3 \subseteq M_1 \) or \( M_3 \subseteq M_2 \). On the other hand, if \( M_1 \) and \( M_2 \) are distinct one dimensional subspaces of \( \mathbb{C}^3 \) and \( M_3 \)
is two dimensional, the \( M_1 \)'s can be represented as the ranges of a set of commuting operators if and only if \( M_1 \subseteq M_3 \) or \( M_2 \subseteq M_3 \).

The conclusions above do not exhaust the possibilities in \( \mathbb{C}^3 \). For instance, let \( P_1, P_2 \) and \( P_3 \) be the orthogonal projections onto the first, second and third summands of \( \mathbb{C}^3 \), respectively; being mutually orthogonal they form a commuting family. The set \( \{ P_1, P_2, P_3, P_1 \oplus P_2, P_1 \oplus P_3, P_2 \oplus P_3 \} \) is a commuting family of operators having six subspaces of \( \mathbb{C}^3 \) as their ranges.

There is a common thread which runs through the discussion following Proposition 3.5. In each of the situations yielding a positive result, it is possible to find an algebraic basis for \( \mathbb{C}^3 \) having certain properties. If commuting operators are found having as their ranges a given set of subspaces, there is a basis \( \{ e_1, e_2, e_3 \} \) of \( \mathbb{C}^3 \) so that each of the subspaces can be represented as the algebraic direct sum of a combination of \( \mathbb{C}e_1, \mathbb{C}e_2 \) and \( \mathbb{C}e_3 \). Furthermore, the operators constructed are all diagonal with respect to that basis. If there is a basis of \( \mathbb{C}^n \), \( \{ e_1, \ldots, e_n \} \) so that each member of a collection of its subspaces can be written as an algebraic direct sum of a combination of \( \mathbb{C}e_1, \ldots, \mathbb{C}e_n \) the subspaces will be called simultaneously diagonalizable. That a simultaneously diagonalizable family of subspaces of \( \mathbb{C}^n \) is representable as the ranges of a set of commuting operators is more or less obvious. In fact, as the arguments above show, the operators can be chosen to be a commuting family of, not necessarily orthogonal, projections. Must every set of subspaces in \( \mathbb{C}^n \) which can be represented as the ranges of a commuting family of operators be of that form?
LIST OF REFERENCES


