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On tensor autoequivalences of graded fusion categories

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ON TENSOR AUTOEQUIVALENCES OF GRADED FUSION CATEGORIES

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ABSTRACT

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Fusion categories generalize the representation theory of finite groups. The simplest examples of fusion categories come from finite groups, their representations, and their cohomology. In general, it is useful to examine group theoretical features of fusion categories such as groups of (isomorphism classes of) tensor invertible objects, and gradings by finite groups. Indeed, every fusion category has a maximal pointed subcategory (generated by tensor invertible objects) and a universal grading by a finite group. We use such features to study tensor autoequivalences.

Pointed fusion categories: categories for which all simple objects are tensor invertible, provide our prototype for graded fusion categories. Such categories are well understood; each pointed fusion category has tensor multiplication determined (up to equivalence) by a finite group, and tensor associativity determined (up to equivalence) by a third cohomology class. There is a well known description of the groups of tensor autoequivalences of pointed fusion categories in terms of this data. We use this description to write exact sequences for computing Brauer-Picard groups of pointed fusion categories. We also generalize this description to write exact sequences which describe groups of tensor autoequivalences of graded fusion categories.

CHAPTER 1

Introduction

Tensor categories are a relatively new and very beautiful area of mathematics. The study of tensor categories subsumes many interesting classical disciplines such as the representation theory of groups and Lie algebras, as well as the more recently developed quantum groups. They have useful application in the study of knots, operator algebras, non-commutative geometry, topological quantum field theory, and higher categories. It has become apparent that classification of tensor categories would be both interesting and useful for mathematicians working in these areas, however at the moment such a classification is out of reach. Algebraists working in this area have set upon the far more manageable problem of generating and classifying families of fusion categories. This represents a significant reduction in difficulty, although we are still in the early stages of this program, building classification techniques, and examples.

In the classification of tensor categories we make certain simplifying assumptions. The nicest categories are those constructed from group theoretical data: groups, group representations, and group cohomology. From the group theoretical perspective, first examples of tensor categories include the representation categories $\mathsf{Rep}(G)$, and graded vector spaces $\mathcal{C}(G, \omega)$. Next, one might look at the Morita equivalence classes of these. Such categories were classified [**O1**] and we call these group theoretical tensor categories.

An important piece of group theoretical data on a tensor category is a G-grading. This is a decomposition of a tensor category $\mathcal{C}$ into full Abelian subcategories $\mathcal{C}_g$ for each $g \in G$, so that the tensor product restricts to a family of functors

$$\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$$

and the trivial component $\mathcal{C}_e$ is a tensor subcategory. We call a G-graded tensor category $\mathcal{C}$ such that $\mathcal{C}_e = \mathcal{D}$, a *$G$-extension of $\mathcal{D}$*. Such extensions were studied and classified with group theoretical data in [**ENO1**]. Indeed a G-extension of a fusion category $\mathcal{D}$ is determined up to equivalence by a certain group homomorphism and a pair of elements from torsors over certain cohomology groups, given that certain cohomological obstructions vanish (see [**ENO1**] for details).

The group theoretic data attached to an extension of a fusion category $\mathcal{D}$ are not immediately accessible. There is a *grading homomorphism* with the grading group as its domain which describes the grading components. This takes values in the (abstractly defined) *Brauer-Picard group* $\mathsf{BrPic}(\mathcal{D})$ of (equivalence classes of) invertible $\mathcal{D}$-bimodule categories. Furthermore, there are cohomological considerations which describe the associative tensor product, the coherent associativity constraint, and when those exist. These depend on the grading homomorphism, and an action of the Brauer-Picard group on the center $\mathcal{Z}(\mathcal{D})$ of $\mathcal{D}$.

The key insight of [**ENO1**] is the canonical isomorphism

$$\mathsf{BrPic}(\mathcal{D}) \cong \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D})) \tag{1.1}$$

for each fusion category $\mathcal{D}$, which was generalized to finite tensor categories in [**DN**]. This provides a much more concrete and computable presentation of $\mathsf{BrPic}(\mathcal{D})$. In particular, the action of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D}))$ can be studied with techniques of representation theory. Dmitri Nikshych and Brianna Riepel studied the action of the Brauer-Picard group on the categorical Lagrangian Grassmanian of the center of a pointed category in [**NR**] and used these methods to compute the Brauer-Picard groups of the categories of graded vector spaces for various finite groups. In the non-semisimple setting, Costel-Gabriel Bontea and Dmitri Nikshich studied a projective representation in [**BN**] to compute the Brauer-Picard group of a certain family of finite symmetric tensor categories.

The purpose of this work is to examine tensor autoequivalence groups $\mathsf{Aut}(\mathcal{D})$ of fusion categories $\mathcal{D}$ when $\mathcal{D}$ is graded. Our methods make use of the group homomorphisms related to the center construction and restriction to invariant tensor subcategories.

This work is organized as follows:

Chapter 2 will cover the preliminary material. We explicitly define the structures, constructions and techniques which will be useful in our examination of the Brauer-Picard groups of group theoretical and graded fusion categories. Our techniques make heavy use of group cohomology, so we provide a brief review of the basic theory in Section 2.1, and introduce features which play an important role. Sections 2.2 through 2.6 introduce the theory of fusion categories, and their module categories. Sections 2.7, 2.8, and 2.9 introduce the foci of this work: the Brauer-Picard group of invertible bimodule categories, centers of group theoretical categories, and the data associated to a group extension of a fusion category.

Chapter 3 develops methods for computing the group $\mathsf{BrPic}(\mathcal{C})$ when $\mathcal{C}$ is a fusion category, using the induction homomorphism (Definition 3.1.1)

$$\mathsf{ind} : \mathsf{Aut}(\mathcal{C}) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C})) \cong \mathsf{BrPic}(\mathcal{C}) \tag{1.2}$$

from the group of tensor autoequivalences (Definition 2.3.16) of $\mathcal{C}$ to the group of braided autoequivalences (Definition 2.4.12) of its center (Definition 2.4.7). In Section 3.1 we describe the kernel of (1.2) (Proposition 3.3), we describe the image of (1.2) for pointed (Definition 2.3.12) fusion categories (Corollary 3.1.7), and when it is surjective (Corollary 3.1.9). In Sections 3.2, 3.3, and 3.4 we consider the group theoretical fusion categories constructed from elementary Abelian p-groups for even and odd primes p. We specialize our results (Corollary 3.3.5) on the image and kernel of (1.2), and write exact sequences for computing $\mathsf{BrPic}(\mathcal{C}(V_n, \omega))$ (Theorem 3.3.6, and Corollary 3.3.10) where V_n is an elementary Abelian p-group. In Section 3.5, we use our methods to determine short exact sequences for Brauer-Picard groups of extra special p-groups, extending the results of [**NR**]. Finally we describe how metric modular Lie algebras [**S**] are a rich source of interesting examples of pointed fusion categories for which our methods are effective.

Chapter 4 examines the restriction of tensor autoequivalences to the trivial component $\mathcal{D}$ of an invariant (Definition 2.9.12) graded fusion category $\mathcal{C}$. There is a group homomorphism

$$\mathsf{Res}_{\mathcal{D}}^{\mathcal{C}} : \mathsf{Aut}(\mathcal{C}) \to \mathsf{Aut}(\mathcal{D}) \tag{1.3}$$

for each invariant graded fusion category $\mathcal{C}$ and trivial component fusion subcategory $\mathcal{D}$. In Section 4.1 we describe the canonical isomorphism (1.1) and the data associated to the grading (Subsection 2.9.3) in a manner useful for our purposes. In Section 4.3 we describe (Theorem 4.3.1) the kernel of homomorphism (1.3). In Section 4.5 we describe (Theorem 4.5.10) the image of homomorphism (1.3).

CHAPTER 2

Preliminaries

Throughout this dissertation we will work over an algebraically closed field k of characteristic zero unless otherwise specified. We will write $k^\times$ to denote the group of units of k.

2.1. Relevant group cohomology

Let G be a finite group, A an Abelian group with an action of G. We will write both G and A multiplicatively, and the left action of G as $g.a = {}^g a$ for $g \in G$ and $a \in A$.

2.1.1. Ordinary group cohomology.

Definition 2.1.1. A function $f : G^n \to A$ is called an *n-cochain* on G in A. We will denote $C^n(G, A)$, the Abelian group of n-cochains on G in A with pointwise multiplication.

Definition 2.1.2. For every natural number n there is a homomorphism

$$d^n : C^n(G, A) \to C^{n+1}(G, A)$$

called the *coboundary operator* defined for $f \in C^n(G, A)$ and $g_1, \dots, g_{n+1} \in G$

$$\begin{aligned} d^n f(g_1, \dots, g_{n+1}) :=& g_1.f(g_2, \dots, g_{n+1}) \\ & \left(\Pi_{i=1}^n f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})^{(-1)^i}\right) f(g_1, \dots, g_n)^{(-1)^{n+1}}. \end{aligned}$$

Definition 2.1.3. We call:

- $Z^n(G, A) := \mathrm{Ker}\,(d^n)$ the group of *n-cocycles* on G in A.
- $B^n(G, A) := \mathrm{Im}(d^{n-1})$ the group of *n-coboundaries* on G in A.
- $H^n(G, A) := Z^n(G, A)/B^n(G, A)$ the n^{th} *cohomology group* of G in A.

We call n-cocycles which are equivalent in $H^n(G, A)$ *cohomologous*.

Remark 2.1.4. There is a natural action of $\mathsf{Aut}(G)$ on $H^n(G, A)$ by precomposition on cocycles. Given a cohomology class $\omega \in H^n(G, A)$ we denote its stabilizer $\mathsf{Stab}(\omega) \subset \mathsf{Aut}(G)$.

Remark 2.1.5. Low dimensional group cohomology has common applications in group theory [**B**, Chapter V]:

- $Z^1(G, A)$ parameterizes homomorphic sections $G \to A \rtimes G$. Given a 1-cocycle $\eta \in Z^1(G, A)$ we write:

$$G \to A \rtimes G : g \mapsto (\eta(g), g).$$

  Cohomologous 1-cocycles induce sections which differ by conjugation by an element in A.
- $Z^2(G, A)$ parameterizes extensions of G by A. We illustrate this in Subsection 2.8.3. Isomorphisms of short exact sequences which are the identity on A and G can be constructed between extensions induced by cohomologous 2-cocycles.
- Elements of $Z^3(G, A)$ corresponds to crossed modules. Cohomologous 3-cocycles correspond to equivalent crossed modules.

Remark 2.1.6. [**B**, Chapter III] Given a short exact sequence of G-modules:

$$0 \to A \to B \to C \to 0,$$

there is a long exact sequence in cohomology:

$$\ldots \to H^{n-1}(G, C) \to H^n(G, A) \to H^n(G, B) \to H^n(G, C) \to H^{n+1}(G, A) \to \ldots$$

Here $H^0(G, X) = X^G$, the group of invariants in X under the action of G. In the case when the action of G on X is trivial, $H^1(G, X) = \mathsf{Hom}(G, X)$, the group of homomorphisms $G \to X$.

2.1.2. Useful properties.

Definition 2.1.7. Let G be an Abelian group. We denote the group of alternating bicharacters on G with values in A as $Alt^2(G, A)$. There is *alternating homomorphism*

$$Alt : H^2(G, A) \to Alt^2(G, A)$$

defined on 2-cocycles $\mu \in Z^2(G, A)$

$$Alt(\mu)\,(g, h) := \frac{\mu(g, h)}{\mu(h, g)}.$$

Since 2-coboundaries are symmetric, this map is well defined over cohomology classes.

Proposition 2.1.8. [**K1**, Theorem 2.6.7] *For G a finite Abelian group, the alternating map*

$$Alt : H^2(G, k^\times) \to Alt^2(G, k^\times)$$

is an isomorphism.

Proposition 2.1.9. [**K2**, Corollary 9.9.5] *Let G be a finite group which acts trivially on both $k^\times$ and $\mathbb{Z}$. There is an isomorphism*

$$H^m(G, k^\times) \cong H^{m+1}(G, \mathbb{Z}). \tag{2.1}$$

Given the identification (2.1), and the fact that $\mathbb{Z}$ is a principle ideal domain, we may make use of the *Künneth formula* for direct products in group cohomology.

Proposition 2.1.10. [**B**, Corollary 6.5.8] *If G and G' are Abelian, there is a split-exact sequence*

$$0 \to \bigoplus_{p+q=n} H^p(G, \mathbb{Z}) \otimes_{\mathbb{Z}} H^q(G', \mathbb{Z}) \to H^n(G \times G', \mathbb{Z})$$

$$\to \bigoplus_{p+q=n-1} Tor_1^{\mathbb{Z}}(H^p(G, \mathbb{Z}), H^q(G', \mathbb{Z})) \to 0$$

Definition 2.1.11. Let R be a commutative ring, there is a product on the total cohomology group $H^*(G,R) := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} H^n(G,R)$

$$\cup : H^*(G,R) \otimes H^*(G,R) \to H^*(G,R)$$

called the *cup product*, making $H^*(G,R)$ into a graded commutative R-algebra. This product defined on cocycles $\rho \in Z^p(G,R)$ and $\sigma \in Z^q(G,R)$ is

$$\rho \cup \sigma(g_1,\ldots,g_{p+q}) := \rho(g_1,\ldots,g_p)\sigma(g_{p+1},\ldots,g_{p+q}).$$

Graded commutativity means that $\rho \cup \sigma = (-1)^{pq}\sigma \cup \rho$. See [**B**, Chapter V §3] for details.

2.1.3. The central product of groups. Let K_1 and K_2 be finite groups, let A be a finite Abelian group, and G_1 and G_2 central extensions of K_1 and K_2 by A with respective second cohomology class $\kappa_1 \in H^2(K_1,A)$ and $\kappa_2 \in H^2(K_2,A)$. There are obvious second cohomology classes $\hat{\kappa}_1$, $\hat{\kappa}_2 \in H^2(K_1 \times K_2, A)$ corresponding to central extensions $G_1 \times K_2$ and $K_1 \times G_2$ respectively.

Definition 2.1.12. The *central product* $G_1 * G_2$ is the quotient of $G_1 \times G_2$ by the subgroup of elements $\{(a,-a) \in G_1 \times G_2 | a \in A\}$.

Proposition 2.1.13. *The central product $G_1 * G_1$ is an extension of $K_1 \times K_2$ by A with corresponding second cohomology class $\hat{\kappa}_1 + \hat{\kappa}_2 \in H^2(K_1 \times K_2, A)$*

Proof. The quotient map described in Definition 2.1.12 restricts to $A \times A$ sending $(a,b) \mapsto a+b$. This maps

$$(\kappa_1,\kappa_2) \in H^2(K_1 \times K_2, A \times A) \longmapsto \hat{\kappa}_1 + \hat{\kappa}_2 \in H^2(K_1 \times K_2, A),$$

which determines the cohomology class corresponding to the central product $G_1 * G_2$. □

2.1.4. Abelian cohomology. Let G and A be Abelian groups where G acts trivially on A.

DEFINITION 2.1.14. An *Abelian 3-cocycle* on G with coefficients in A is a pair (ω, c), where $\omega : G^3 \to A$ is a normalized 3-cocycle and $c : G^2 \to A$ is a function satisfying

$$\omega(y,z,x)c(x,yz)\omega(x,y,z) = c(x,z)\omega(y,x,z)c(x,y),$$

$$\omega(z,x,y)^{-1}c(x,yz)\omega(x,y,z)^{-1} = c(x,z)\omega(x,z,y)^{-1}c(y,z).$$

for all $x,\ y,\ z \in G$. Let $Z^3_{ab}(G,A)$ be the group of Abelian 3-cocycles.

DEFINITION 2.1.15. For any normalized 2-cochain $\eta \in C^2(G,A)$, the *Abelian coboundary* of η is the Abelian 3-cocycle $d(\eta) = (\omega, c)$ defined by the equations

$$\omega(x,y,z) = \frac{\eta(y,z)\eta(x,yz)}{\eta(xy,z)\eta(x,y)},$$

$$c(x,y) = \frac{\eta(x,y)}{\eta(y,x)}.$$

Let $B^3_{ab}(G,A)$ be the group of Abelian 3-coboundaries.

DEFINITION 2.1.16. The *Abelian third cohomology group* $H^3_{ab}(G,\, A)$ is the group of Abelian 3-cocycles modulo Abelian 3-coboundaries.

REMARK 2.1.17. $\mathsf{Aut}(G)$ acts on $H^3_{ab}(G,A)$ in the obvious way.

DEFINITION 2.1.18. We say that a function $q : G \to A$ is a *quadratic form* on G in A if it satisfies $q(g) = q(g^{-1})$ for all $g \in G$, and the symmetric function

$$b(g,h) := \frac{q(gh)}{q(g)q(h)}, \qquad g,h \in G,$$

is a bicharacter.

We denote the group of quadratic forms on G in A as $Quad(G,A)$.

PROPOSITION 2.1.19. *(Eilenberg and MacLane,* [**JS**]*) There is an isomorphism*

$$H^3_{ab}(G,A) \to Quad(G,A) : (\omega, c) \mapsto q_c, \tag{2.2}$$

where $q_c(g) := c(g,g)$ *for* $g \in G$.

REMARK 2.1.20. Let $(\omega, c) \in H^3_{ab}(G, A)$ be an Abelian third cohomology class, and $q_c \in Quad(G, A)$ its image under the isomorphism (2.2). We identify the stabilizer subgroup $\mathsf{Stab}(\omega, c) \subset \mathsf{Aut}(G)$ with $O(G, q_c)$ the orthogonal group with respect to q_c.

2.2. Abelian categories

DEFINITION 2.2.1. We say that a category $\mathcal{C}$ is *enriched over Abelian groups* if for every X, Y, $Z \in \mathcal{C}$, the sets $Hom(X, Y)$ have the structure of Abelian groups, and composition

$$\circ : Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z)$$

is biadditive.

DEFINITION 2.2.2. We say an object in $\mathcal{C}$ is *initial* if it has a unique morphism to each object in $\mathcal{C}$ and an object is *terminal* if it has a unique morphism from each object in $\mathcal{C}$. We call an object in $\mathcal{C}$ that is both initial and terminal a *zero object*.

DEFINITION 2.2.3. A category $\mathcal{C}$ enriched over Abelian groups, with a zero object 0 is called *additive* if for all objects $X_1, X_2 \in \mathcal{C}$ there exists an object $Y \in \mathcal{C}$ and morphisms $p_1 : Y \to X_1$, $p_2 : Y \to X_2$, $i_1 : X_1 \to Y$, $i_2 : X_2 \to Y$, such that $p_1 i_1 = \mathrm{id}_{X_1}$, $p_2 i_2 = id_{X_2}$, and $i_1 p_1 + i_2 p_2 = id_Y$.

The object Y is unique up to unique isomorphism, is denoted $X_1 \oplus X_2$ and is called the *direct sum* of X_1 and X_2.

DEFINITION 2.2.4. Let k be a field. An additive category $\mathcal{C}$ enriched over k-vector spaces, that is for each X, $Y \in \mathcal{C}$, $Hom(X, Y)$ is a k-vector space and composition is k-bilinear, is called a *k-linear* category.

DEFINITION 2.2.5. A functor $F : \mathcal{C} \to \mathcal{D}$ between k-linear categories is called *k-linear* if each map

$$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y)), \quad X, Y \in \mathcal{C}$$

is a k-linear transformation.

DEFINITION 2.2.6. We say that an additive category $\mathcal{C}$ is *Abelian* if for every morphism $f \in Hom(X, Y)$ in $\mathcal{C}$ there is a sequence

$$\ker(f) \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} \text{Coker}\,(f)$$

where $f = ji$ and $I = \text{Coker}\,(k) = \ker(c)$. The object $I \in \mathcal{C}$ is called the image of f and is denoted $Im(f)$.

REMARK 2.2.7. [**Mi**] Every Abelian category is equivalent a full subcategory of modules in $R - Mod$ for some ring R.

DEFINITION 2.2.8. A *subobject* of $Y \in \mathcal{C}$ is a pair (X, i) where $X \in \mathcal{C}$ and $i : X \to Y$ is a monomorphism.

An object $X \in \mathcal{C}$ is *simple* if every subobject of X is either isomorphic to X, or a zero object, and *semisimple* if it is isomorphic to a direct sum of simple objects.

We say that $\mathcal{C}$ is *semisimple* if all of its objects are semisimple.

DEFINITION 2.2.9. We say that a semisimple k-linear Abelian category $\mathcal{C}$ is *finite* if the following two conditions are satisfied:

i. For every pair of objects X, $Y \in \mathcal{C}$, the k-vector space $Hom(X, Y)$ is finite dimensional.
ii. There are finitely many isomorphism classes of simple objects.

DEFINITION 2.2.10. [**De**] Let $\mathcal{C}$ and $\mathcal{D}$ be finite semisimple k-linear Abelian categories. *Deligne's tensor product* $\mathcal{C} \boxtimes \mathcal{D}$ is a finite semisimple k-linear Abelian category which is universal for the functor assigning to every k-linear Abelian category $\mathcal{A}$, the category of bilinear bifunctors $\mathcal{C} \times \mathcal{D} \to \mathcal{A}$. That is, there is a bifunctor

$$\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D} : (X, Y) \mapsto X \boxtimes Y$$

so that for any bilinear bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$, there exists a unique functor $\bar{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ which satisfies $F = \bar{F} \circ \boxtimes$.

2.3. Tensor categories

The main references for this section are [**BK, EGNO**].

DEFINITION 2.3.1. A *tensor category* $(\mathcal{C}, \otimes, a, 1, l, r)$ is a k-linear Abelian category with additional structure:

i. A *tensor product* bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
ii. A natural family of isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ called the *associativity constraint*.
iii. A *unit object* 1 with natural isomorphisms $l_X : 1 \otimes X \to X$, $r_X : X \otimes 1 \to X$ called the left and right *unit constraints*.

These must satisfy the *pentagon axiom*:

(2.3)
$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{a_{W \otimes X, Y, Z}} & & \searrow^{a_{W,X,Y} \otimes id_Z} & \\ (W \otimes X) \otimes (Y \otimes Z) & & & & (W \otimes (X \otimes Y)) \otimes Z \\ \downarrow a_{W,X,Y \otimes Z} & & & & \downarrow a_{W, X \otimes Y, Z} \\ W \otimes (X \otimes (Y \otimes Z)) & & \xleftarrow[id_W \otimes a_{X,Y,Z}]{} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

and *triangle axiom*:

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{a_{X,1,Y}} & X \otimes (1 \otimes Y) \\ & {}_{r_X \otimes id_Y} \searrow \quad \swarrow {}_{id_X \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

for all objects W, X, Y, $Z \in \mathcal{C}$.

DEFINITION 2.3.2. We say that a tensor category $\mathcal{C}$ is *strict* if for all objects X, Y, $Z \in \mathcal{C}$, there are equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $X \otimes 1 = X = 1 \otimes X$, and the associativity and unit constraints are identity morphisms.

EXAMPLE 2.3.3. Let $\mathcal{A}$ be a k-linear Abelian category. The category $\mathrm{End}(\mathcal{A})$ of k-linear endofunctors is a strict tensor category.

REMARK 2.3.4. Frequently in our computations, we will use MacLane's strictness theorem (see [**Ma, EGNO, JS**]). It states that any tensor category is tensor equivalent (see Definition 2.3.16) to a strict tensor category. This allows us to assume that the associativity and unit constraints are trivial. See Example 2.6.13 for an illustration of this equivalence.

2.3.1. Fusion categories.

DEFINITION 2.3.5. An object $X \in \mathcal{C}$ has a *left dual* X^* if there exist morphisms $ev_X : X^* \otimes X \to 1$ and $coev_X : 1 \to X \otimes X^*$ called the *evaluation* and *co-evaluation*, so that the compositions

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X$$

$$X^* \xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^*$$

are identity morphisms.

DEFINITION 2.3.6. An object $X \in \mathcal{C}$ has a right dual ${}^*X$ if there exists morphisms $ev'_X : X \otimes {}^*X \to 1$ and $coev'_X : 1 \to {}^*X \otimes X$ called the *evaluation* and *co-evaluation*, so that the compositions

$$X \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} X$$

$${}^*X \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a^{-1}_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} {}^*X$$

are identity morphisms.

Remark 2.3.7. [**EGNO**, Proposition 2.10.5] If $X \in \mathcal{C}$ has a left (respectively right) dual object, then it is unique up to a unique isomorphism.

Remark 2.3.8. Both the left and right duals are contravariant on morphisms. Let W, $X \in \mathcal{C}$ be objects with left duals: W^* and X^*. Let $\phi : W \to X$ be a morphism in $\mathcal{C}$. We define the left dual $\phi^* : X^* \to W^*$ of ϕ through the composition:

$$
\begin{array}{ccccc}
X^* & \xrightarrow{id_{X^*} \otimes coev_W} & X^* \otimes (W \otimes W^*) & \xrightarrow{a^{-1}_{X^*,W,W^*}} & (X^* \otimes W) \otimes W^* \\
\downarrow{\scriptstyle \phi^*} & & & & \downarrow{\scriptstyle (id_{X^*} \otimes \phi) \otimes id_{W^*}} \\
W^* & & \xleftarrow{\qquad ev_X \otimes id_{W^*} \qquad} & & (X^* \otimes X) \otimes W^*.
\end{array}
$$

Similarly, let Y, $Z \in \mathcal{C}$ be objects with right duals: ${}^*Y$ and ${}^*Z$. Let $\psi : Y \to Z$ be a morphism in $\mathcal{C}$. We define the right dual ${}^*\psi :{}^* Z \to{}^* Y$ is defined by the composition:

$$
\begin{array}{ccccc}
{}^*Z & \xrightarrow{coev'_Y \otimes id_{{}^*Z}} & ({}^*Y \otimes Y) \otimes {}^*Z & \xrightarrow{a_{{}^*Y,Y,{}^*Z}} & {}^*Y \otimes (Y \otimes {}^*Z) \\
\downarrow{\scriptstyle {}^*\psi} & & & & \downarrow{\scriptstyle id_{{}^*Y} \otimes (\psi \otimes id_{{}^*Z})} \\
{}^*Y & & \xleftarrow{\qquad id_{{}^*Y} \otimes ev'_Z \qquad} & & {}^*Y \otimes (Z \otimes {}^*Z).
\end{array}
$$

Definition 2.3.9. A tensor category is called *rigid* if each object $X \in \mathcal{C}$ has both left and right dual.

Definition 2.3.10. A *fusion category* is a finite, semisimple, rigid tensor category with simple unit object.

Example 2.3.11. Let G be a finite group, let $\mathsf{Rep}(G)$ be its category of finite dimensional k-linear representations. This is a fusion category with the usual tensor product and duals of group representation, here the unit object is the trivial representation.

2.3.2. Pointed fusion categories.

Definition 2.3.12. We say that a fusion category is *pointed* if all its simple objects are invertible with respect to the tensor product. That is $X \otimes X^* \cong 1$ for all simple objects $X \in \mathcal{C}$.

DEFINITION 2.3.13. For a fusion category $\mathcal{C}$, let $\mathcal{C}_{pt}$ denote its maximal pointed fusion subcategory. That is the full tensor subcategory generated by invertible simple objects of $\mathcal{C}$. We denote the group of (isomorphism classes of) invertible objects by $\mathsf{Inv}(\mathcal{C})$.

EXAMPLE 2.3.14. Let G be a finite group, and $\omega \in Z^3(G, k^\times)$ a 3-cocycle. We denote $\mathcal{C}(G, \omega)$, the category of G-graded vector spaces with associativity constraint determined by ω.

The objects of $\mathcal{C}(G, \omega)$ are G-graded vector spaces $V = \bigoplus_{g \in G} V_g$. Morphisms are grading preserving linear transformations. The tensor product is defined component-wise, so that

$$(V \otimes W)_g = \bigoplus_{xy=g} V_x \otimes W_y.$$

Simple objects are 1 dimensional spaces k_g concentrated in each component $g \in G$. The associativity constraint, defined over simple objects corresponds to a scalar $\omega(x, y, z) \in k^\times$ for each triple $x, y, z \in G$:

$$\omega(x, y, z) id_{k_{xyz}} : (k_x \otimes k_y) \otimes k_z \to k_x \otimes (k_y \otimes k_z)$$

This satisfies the pentagon and triangle axioms if and only if ω is a normalized 3-cocycle. We will see that equivalent 3-cocycles yield tensor equivalent fusion categories.

REMARK 2.3.15. Example 2.3.14 illustrates the non-trivial nature of Mac Lane's strictness theorem. The category $\mathcal{C}(G, \omega)$ for non-trivial ω is certainly not isomorphic to a strict fusion category, but is equivalent to one.

2.3.3. Tensor functors.

DEFINITION 2.3.16. A *tensor functor* is a triple (F, J, ϕ) where $F : \mathcal{C} \to \mathcal{D}$ is a k-linear functor between tensor categories, $J(X, Y) : F(X) \otimes F(Y) \to F(X \otimes Y)$ is a natural family of isomorphisms called the *tensor functor structure* satisfying a hexagon axiom:

$$\begin{array}{ccc} (F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{a_{F(X),F(Y),F(Z)}} & F(X)\otimes(F(Y)\otimes F(Z)) \\ \Big\downarrow {\scriptstyle J(X,Y)\otimes id_{F(Z)}} & & \Big\downarrow {\scriptstyle id_{F(X)}\otimes J(Y,Z)} \\ F(X\otimes Y)\otimes F(Z) & & F(X)\otimes F(Y\otimes Z) \\ \Big\downarrow {\scriptstyle J(X\otimes Y,Z)} & & \Big\downarrow {\scriptstyle J(X,Y\otimes Z)} \\ F((X\otimes Y)\otimes Z) & \xrightarrow[F(a_{X,Y,Z})]{} & F(X\otimes(Y\otimes Z)) \end{array} \tag{2.4}$$

for all X, Y, $Z\in\mathcal{C}$, and $\phi: F(1_{\mathcal{C}})\to 1_{\mathcal{D}}$ is an isomorphism satisfying:

$$\begin{array}{ccc} F(1_{\mathcal{C}})\otimes F(X) & \xrightarrow{J_{1_{\mathcal{C}},X}} & F(1_{\mathcal{C}}\otimes X) \\ \Big\downarrow {\scriptstyle \phi\otimes \mathrm{id}_{F(X)}} & & \Big\downarrow {\scriptstyle F(l_X)} \\ 1_{\mathcal{D}}\otimes F(X) & \xrightarrow[l_{F(X)}]{} & F(X) \end{array}$$

and

$$\begin{array}{ccc} F(X)\otimes F(1_{\mathcal{C}}) & \xrightarrow{J_{X,1_{\mathcal{C}}}} & F(X\otimes 1_{\mathcal{C}}) \\ \Big\downarrow {\scriptstyle \mathrm{id}_{F(X)}\otimes\phi} & & \Big\downarrow {\scriptstyle F(r_X)} \\ F(X)\otimes 1_{\mathcal{D}} & \xrightarrow[r_{F(X)}]{} & F(X). \end{array}$$

EXAMPLE 2.3.17. Let $\phi: H\to G$ be a homomorphism of groups. There is a tensor functor $F:\mathsf{Rep}(G)\to\mathsf{Rep}(H)$ along ϕ. For a G-representation (V,ρ), where V is a k-vector space and ρ is a homomorphism $G\to GL(V)$, we define an H-representation $F(V,\rho):=(V,\rho\circ\phi)$. Here tensor functor structure is trivial.

EXAMPLE 2.3.18. Let $\phi: H\to G$ be a homomorphism of finite groups, let $\omega\in H^3(G,k^\times)$ and $\varpi\in H^3(H,k^\times)$ be third cohomology classes so that $\omega\circ\phi^{\times 3}=\varpi$ in $H^3(H,k^\times)$. This means that given cocycle representatives for ϖ and ω, there exists a 2-cochain $\eta\in C^2(H,k^\times)$ satisfying $\omega\circ\phi^{\times 3}=\varpi d^2\eta$.

We define a tensor functor $F_{\phi,\eta}:\mathcal{C}(H,\varpi)\to\mathcal{C}(G,\omega)$ over simple objects and extend to the whole category via k-linearity. This functor acts on simple objects by the group homomorphism $F_{\phi,\eta}(k_g):=k_{\phi(g)}$. The action on morphisms follows from k-linearity. Tensor functor structure is defined

$$\eta_{x,y} id_{k_{\phi(xy)}} : F_{\phi,\eta}(k_x) \otimes F_{\phi,\eta}(k_y) \to F_{\phi,\eta}(k_{xy})$$

In this case the unit object isomorphism is trivial.

DEFINITION 2.3.19. A *morphism of tensor functors* $\sigma : (F_1, J_1, \phi_1) \to (F_2, J_2, \phi_2)$ consists of a natural transformation $\sigma : F_1 \to F_2$ satisfying

$$\begin{array}{ccc} F_1(X) \otimes F_1(Y) & \xrightarrow{J_1(X,Y)} & F_1(X \otimes Y) \\ \downarrow{\scriptstyle \sigma(X)\otimes\sigma(Y)} & & \downarrow{\scriptstyle \sigma(X\otimes Y)} \\ F_2(X) \otimes F_2(Y) & \xrightarrow[J_2(X,Y)]{} & F_2(X \otimes Y) \end{array}$$

for all X, $Y \in \mathcal{C}$, and $\phi_1 = \phi_2\sigma(1)$.

REMARK 2.3.20. Let $\mathcal{C}$ be a tensor category. We denote the group of (isomorphism classes of) tensor autoequivalences of $\mathcal{C}$ as $\mathsf{Aut}(\mathcal{C})$.

DEFINITION 2.3.21. A category $\mathcal{C}$ is called *skeletal* if all isomorphic pairs of objects are identical. The axiom of choice implies that any category is equivalent to a skeletal category (see [**N**, Section 4.6]).

2.3.4. Tensor autoequivalences of pointed fusion categories. Let G be a finite group and let $\omega \in Z^3(G, k^\times)$ be a 3-cocycle.

DEFINITION 2.3.22. Let $\mathsf{Stab}(\omega) \subset \mathsf{Aut}(G)$ be the subgroup of automorphisms $a \in \mathsf{Aut}(G)$ such that $\omega \circ (a \times a \times a)$ and ω are cohomologous.

The following result is well known (see [**EGNO**, Section 2.6], [**NR**, Proposition 4.1]).

PROPOSITION 2.3.23. *There is a short exact sequence*

$$0 \to H^2(G, k^\times) \to \mathsf{Aut}(\mathcal{C}(G, \omega)) \to \mathsf{Stab}(\omega) \to 0. \tag{2.5}$$

Here $H^2(G, k^\times)$ parameterizes the tensor functor structures on the identity endofunctor of $\mathcal{C}(G, \omega)$. Given $a \in \mathsf{Stab}(\omega)$ and a 2-cochain μ such that

$$d^2(\mu) = \frac{\omega \circ (a \times a \times a)}{\omega}$$

let $F_{a,\mu}$ denote the corresponding autoequivalence of $\mathcal{C}(G, \omega)$.

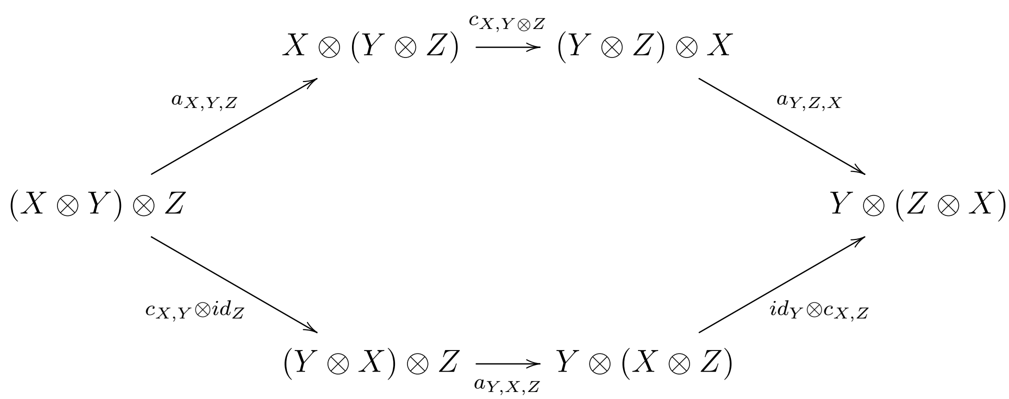
2.4. Braided categories

Definition 2.4.1. A *braided category* is a tensor category equipped with a natural family of isomorphisms

$$c_{X,Y} : X \otimes Y \to Y \otimes X, \qquad X, Y \in \mathcal{C},$$

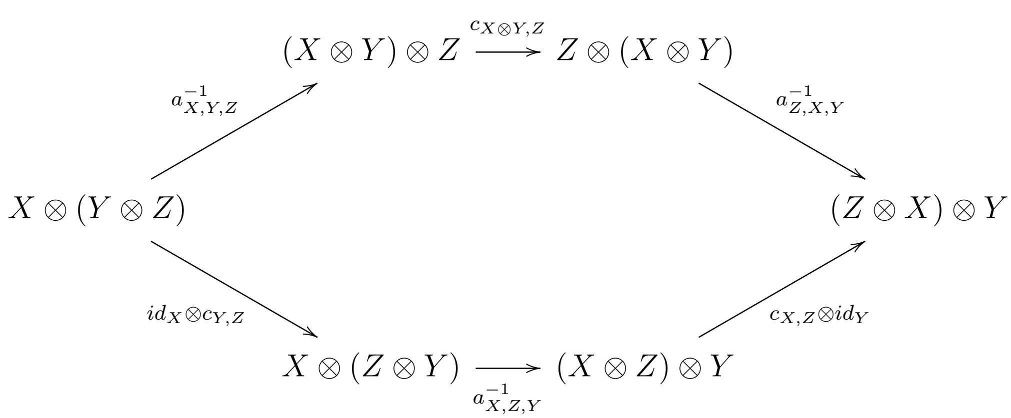
called the *braiding*, satisfying:

(2.6)



and

(2.7)



for all $X, Y, Z \in \mathcal{C}$.

Definition 2.4.2. Objects X and Y in a braided category $\mathcal{C}$ are said to *centralize* each other if

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}.$$

Definition 2.4.3. Let $\mathcal{C}$ be a braided fusion category. The *centralizer* $\mathcal{D}'$ of a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is the full fusion subcategory of objects in $\mathcal{C}$ which centralize each object in $\mathcal{D}$. When $\mathcal{D} = \mathcal{C}$ we call this the *symmetric center*.

We say that a braiding on $\mathcal{C}$ is *symmetric* if $\mathcal{C}' = \mathcal{C}$ and that the braiding on $\mathcal{C}$ is *non-degenerate* if $\mathcal{C}' = \mathsf{Vec}$, the subcategory generated by the unit object.

Example 2.4.4. Let G be a finite group, the representation category $\mathsf{Rep}(G)$ is a symmetric braided fusion category with the transposition map as its braiding.

Example 2.4.5. Let A be an Abelian group and $(\omega, c) \in Z^3_{ab}(A, k^\times)$ an Abelian 3-cocycle on A in $k^\times$. We give the fusion category $\mathcal{C}(A, \omega)$ braided category structure with braiding isomorphisms determined by the 2-cochain $c \in C^2(A, k^\times)$:

$$c_{x,y} id_{k_{xy}} : k_x \otimes k_y \to k_y \otimes k_x$$

and write $\mathcal{C}(A, \omega, c)$ to denote this braided fusion category. We will see that cohomologous Abelian 3-cocycles determine braided equivalent categories.

Remark 2.4.6. [**JS**, Theorem 3.3] Pointed braided fusion categories are parameterized up to braided equivalence by third Abelian cohomology. We make use of the canonical isomorphism (2.2) and denote a pointed braided category $\mathcal{C}$ with $\mathsf{Inv}(\mathcal{C}) = A$ and canonical quadratic form q as $\mathcal{C} = \mathcal{C}(A, q)$.

2.4.1. The center construction.

Definition 2.4.7. For every fusion category $\mathcal{C}$ we may construct a braided fusion category $\mathcal{Z}(\mathcal{C})$ called the *center* of $\mathcal{C}$. Objects consist of pairs $(Z, \gamma) \in \mathcal{Z}(\mathcal{C})$ where $Z \in \mathcal{C}$ and

$$\gamma_X : X \otimes Z \to Z \otimes X \qquad \text{for } X \in \mathcal{C}$$

is a natural family of isomorphisms called the *half-braiding* over $\mathcal{C}$ which satisfies:

$$
\begin{array}{ccccc}
 & (X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X \otimes Y}} & Z \otimes (X \otimes Y) & \\
\nearrow^{a^{-1}_{X,Y,Z}} & & & & \searrow^{a^{-1}_{Z,X,Y}} \\
X \otimes (Y \otimes Z) & & & & (Z \otimes X) \otimes Y \\
\searrow_{id_X \otimes \gamma_Y} & & & & \nearrow_{\gamma_X \otimes id_Y} \\
 & X \otimes (Z \otimes Y) & \xrightarrow[a^{-1}_{X,Z,Y}]{} & (X \otimes Z) \otimes Y &
\end{array}
$$

for each $X, Y \in \mathcal{C}$. Morphisms are induced from morphisms in $\mathcal{C}$. A morphism between (Z_1, γ) and (Z_2, ϑ) in $\mathcal{Z}(\mathcal{C})$ is a morphism $f : Z_1 \to Z_2$ in $\mathcal{C}$ such that:

$$
\begin{array}{ccc}
X \otimes Z_1 & \xrightarrow{\gamma_X} & Z_1 \otimes X \\
{\scriptstyle id_X \otimes f}\downarrow & & \downarrow{\scriptstyle f \otimes id_X} \\
X \otimes Z_2 & \xrightarrow[\vartheta_X]{} & Z_2 \otimes X
\end{array}
$$

commutes for all $X \in \mathcal{C}$. The tensor product in $\mathcal{Z}(\mathcal{C})$ is defined:

$$(Z_1, \gamma) \otimes (Z_2, \vartheta) := (Z_1 \otimes Z_2, \gamma * \vartheta),$$

where $Z_1 \otimes Z_2$ is the usual tensor product in $\mathcal{C}$ and $\gamma * \vartheta$ is the composition:

$$
\begin{array}{ccccc}
X \otimes (Z_1 \otimes Z_2) & \xrightarrow{a^{-1}_{X,Z_1,Z_2}} & (X \otimes Z_1) \otimes Z_2 & \xrightarrow{\gamma_X \otimes id_{Z_2}} & (Z_1 \otimes X) \otimes Z_2 \\
{\scriptstyle \gamma * \vartheta_X}\downarrow & & & & \downarrow{\scriptstyle a_{Z_1,X,Z_2}} \\
(Z_1 \otimes Z_2) \otimes X & \xleftarrow[a^{-1}_{Z_1,Z_2,X}]{} & Z_1 \otimes (Z_2 \otimes X) & \xleftarrow[id_{Z_1} \otimes \vartheta_X]{} & Z_1 \otimes (X \otimes Z_2)
\end{array}
$$

Duals are inherited from $\mathcal{C}$ so that left duals are $(Z, \gamma)^* := (Z^*, \bar{\gamma})$ where $\bar{\gamma}_X = (\gamma^{-1}_{{}^*X})^*$, and right duals are similar.

Definition 2.4.8. There is an obvious *forgetful tensor functor*

$$\mathcal{Z}(\mathcal{C}) \to \mathcal{C} : (Z, \gamma) \mapsto Z.$$

Remark 2.4.9. [**Mu2**, Corollary 5.13] The center $\mathcal{Z}(\mathcal{C})$ of a fusion category $\mathcal{C}$ is a non-degenerate braided fusion category.

Example 2.4.10. [**EGNO**, Example 8.5.4] The center $\mathcal{Z}(\mathcal{C}(G,1))$ of the category of G-graded vector spaces with trivial associativity constraint can be described as the category of G equivariant vector bundles supported on the elements of G.

Objects in $\mathcal{Z}(\mathcal{C}(G,1))$ come equipped with half-braiding isomorphisms

$$\gamma_g : k_g \otimes V \to V \otimes k_g \quad \text{for } g \in G$$

This is equivalent to the presence of a family of isomorphisms

$$\rho : k_g \otimes V \otimes k_{g^{-1}} \to V$$

satisfying certain conditions. Since conjugation permutes gradings, simple objects $V \in \mathcal{Z}(\mathcal{C}(G,1))$ must then be supported on a conjugacy class, represented by $r \in G$. The family of half-braiding isomorphisms equips V with the structure of a G-representation, induced by an irreducible representation of $C_G(r)$ the centralizer of r in G.

Example 2.4.11. When A is an Abelian group $\mathcal{Z}(\mathcal{C}(A,1)) \cong \mathcal{C}(A \oplus \widehat{A}, 1)$ as a tensor category since conjugacy classes in A are trivial, and irreducible representations of A correspond to the character group $\widehat{A}$. The half braiding comes from evaluation $\widehat{A} \times A \to k^{\times}$ as follows

$$\chi_2(a_1)\mathrm{id}_{(a_1a_2,\chi_1\chi_2)} : (a_1, \chi_1) \otimes (a_2, \chi_2) \to (a_2, \chi_2) \otimes (a_1, \chi_1)$$

for (a_1, χ_1), (a_2, χ_2) simple objects in $\mathcal{C}(A \oplus \widehat{A}, 1)$.

2.4.2. Braided functors.

DEFINITION 2.4.12. A tensor functor $(F, J) : \mathcal{C} \to \mathcal{D}$ between braided categories $\mathcal{C}$ and $\mathcal{D}$ with braidings c and $\bar{c}$ respectively, is braided if it satisfies:

$$\begin{CD} F(X) \otimes F(Y) @>{\bar{c}_{F(X),F(Y)}}>> F(Y) \otimes F(X) \\ @V{J(X,Y)}VV @VV{J(Y,X)}V \\ F(X \otimes Y) @>>{F(c_{X,Y})}> F(Y \otimes X). \end{CD} \tag{2.8}$$

REMARK 2.4.13. Let $\mathcal{C}$ be a braided tensor category. We denote the group of (isomorphism classes of) braided autoequivalences of $\mathcal{C}$ as $\mathsf{Aut}^{br}(\mathcal{C})$.

EXAMPLE 2.4.14. Braided tensor functors between pointed fusion categories must respect Abelian third cohomology classes (see Subsection 2.1.4). With respect to the isomorphism (2.2) these correspond to orthogonal homomorphisms. Indeed there is an equivalence [**EGNO**, Theorem 8.4.11], [**JS**, Theorem 3.3] between the category of pointed braided categories with isomorphism classes of braided functors, and the category of pre-metric groups with orthogonal homomorphisms.

2.5. Algebras in a fusion category

Let $\mathcal{C}$ be a fusion category.

DEFINITION 2.5.1. An *algebra* in $\mathcal{C}$ is a triple (A, m, u) where $A \in \mathcal{C}$ and $m : A \otimes A \to A$ and $u : 1 \to A$ are morphisms called the *multiplication* and *unit* which satisfy:

$$\begin{array}{ccccc} (A \otimes A) \otimes A & & \xrightarrow{a_{A,A,A}} & & A \otimes (A \otimes A) \\ \downarrow{\scriptstyle m \otimes \mathrm{id}_A} & & & & \downarrow{\scriptstyle \mathrm{id}_A \otimes m} \\ A \otimes A & & & & A \otimes A \\ & \searrow{\scriptstyle m} & & \swarrow{\scriptstyle m} & \\ & & A, & & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{1}\otimes A & \xrightarrow{\;l_A\;} & A \\ {\scriptstyle u\otimes\mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} \\ A\otimes A & \xrightarrow{\;m\;} & A, \end{array} \qquad \begin{array}{ccc} A\otimes \mathbf{1} & \xrightarrow{\;r_A\;} & A \\ {\scriptstyle \mathrm{id}_A\otimes u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} \\ A\otimes A & \xrightarrow{\;m\;} & A. \end{array}$$

Here, a, l, r denote the associativity and unit constraints of $\mathcal{C}$.

EXAMPLE 2.5.2. This definition gives rise to both familiar and unfamiliar algebra objects.

- When $\mathcal{C} = \mathsf{Vec}$, we recover the usual definition of a finite-dimensional unital associative algebra.
- When $\mathcal{C} = \mathsf{Rep}(G)$, algebras are equiped with an action of G by algebra automorphisms, we call these *G-algebras*.
- The regular representation in $\mathsf{Rep}(G)$ can be given the structure of the algebra of functions $k(G)$ on G.
- The regular algebra $k(G)$ for $\mathsf{Rep}(G) \subset \mathcal{Z}(\mathcal{C}(G,\omega))$ (see Definition 2.8.3) is called a *Lagrangian algebra* and has useful properties(see [**DMNO, NR**]).
- When $\mathcal{C} = \mathcal{C}(G,1)$, algebras are G-graded unital associative algebras in the usual sense. In particular, group algebras $kH = \bigoplus_{h\in H} k_h$ for subgroups $H \subset G$ are algebras in $\mathcal{C}$.
- When $\mathcal{C} = \mathcal{C}(G,\omega)$, let H be a subgroup of G such that $\omega|H = d^2\psi$ for a 2-cochain $\psi \in C^2(H,k^\times)$. We may define the *twisted group algebra*

  $$(kH)_\psi := \bigoplus_{h\in H} k_h$$

  with multiplication

  $$\bigoplus_{g,h\in H} \psi(g,h)\mathrm{id}_{k_{gh}} : \bigoplus_{g,h\in H} k_g \otimes k_h \to \bigoplus_{gh\in H} k_{gh}.$$

  When ψ is a 2-cocycle, this is a familiar associative algebra from group theory, otherwise this is only associative in $\mathcal{C}(G,\omega)$.

- Let $X \in \mathcal{C}$, we can define an algebra $A := X \otimes X^*$ where multiplication m comes from evaluation $ev_X : X^* \otimes X \to 1$, and the unit is coevaluation $u = coev_X$.

DEFINITION 2.5.3. A *right module* over an algebra (A, m, u) in $\mathcal{C}$ is a pair (M, p), where M is an object in $\mathcal{C}$ and $p : M \otimes A \to M$ satisfying:

$$\begin{array}{ccc}
(M \otimes A) \otimes A & \xrightarrow{a_{M,A,A}} & M \otimes (A \otimes A) \\
\downarrow{\scriptstyle p \otimes \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_M \otimes m} \\
M \otimes A & & M \otimes A \\
& \searrow^{p} \quad \swarrow^{p} & \\
& M &
\end{array}$$

and

$$\begin{array}{ccc}
M \otimes \mathbf{1} & \xrightarrow{r_M} & M \\
\downarrow{\scriptstyle \mathrm{id}_M \otimes u} & & \downarrow{\scriptstyle \mathrm{id}_M} \\
M \otimes A & \xrightarrow{p} & M.
\end{array}$$

REMARK 2.5.4. The definition of a left module in $\mathcal{C}$ is similar.

EXAMPLE 2.5.5. We consider module objects.

- When $\mathcal{C} = \mathsf{Vec}$ and $A \in \mathsf{Vec}$ is a finite dimensional unital associative k algebra, we recover the usual definition of a module over A.
- When $\mathcal{C} = \mathcal{C}(G, 1)$ and $A \in \mathcal{C}(G, 1)$ is a finite dimensional unital associative G-graded k algebra, modules over A are G-graded.
- Let X, $M \in \mathcal{C}$, we can define right modules over the algebra $X \otimes X^*$ by $M \otimes X^*$ where the right action p comes from evaluation $ev_X : X^* \otimes X \to 1$.
- Let X, $M \in \mathcal{C}$, we can define left modules over the algebra $X \otimes X^*$ by $X \otimes M$ where the left action p comes from evaluation $ev_X : X^* \otimes X \to 1$.

Definition 2.5.6. Let A and B be algebras in $\mathcal{C}$. An (A, B)-*bimodule* in $\mathcal{C}$ is a triple (M, p, q), where (M, p) is left A-module, and (M, q) is right B-module satisfying:

$$\begin{array}{ccccc} (A \otimes M) \otimes B & & \xrightarrow{a_{A,M,B}} & & A \otimes (M \otimes B) \\ \downarrow{\scriptstyle p\otimes \mathrm{id}_B} & & & & \downarrow{\scriptstyle \mathrm{id}_A \otimes q} \\ M \otimes B & & & & A \otimes M \\ & \searrow^{q} & & \swarrow^{p} & \\ & & M & & \end{array}$$

Example 2.5.7. Let X, Y, $M \in \mathcal{C}$, define algebras $A = X \otimes X^*$ and $B = Y \otimes Y^*$ as above. We can define an (A, B)-bimodule $X \otimes M \otimes Y^*$ where the left and right module isomorphisms p and q come from evaluation as described in previous examples.

Remark 2.5.8. We denote $\mathcal{C}_A$ the category of right A-modules, ${}_A\mathcal{C}$ the category of left A-modules, and ${}_A\mathcal{C}_A$ the category of A-bimodules in $\mathcal{C}$ with A-module or A-bimodule homomorphisms.

Definition 2.5.9. Let A be an algebra in $\mathcal{C}$ and let (M, p) and (N, q) be right and left modules respectively over A. The *tensor product* $M \otimes_A N$ over A is an object in $\mathcal{C}$ defined as the coequalizer

$$M \otimes A \otimes N \underset{\mathrm{id}_M \otimes q}{\overset{p\otimes \mathrm{id}_N}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_A N.$$

Definition 2.5.10. An algebra A in a fusion category $\mathcal{C}$ is called *separable* if the multiplication morphism $A \otimes A \to A$ splits as a morphism of A-bimodules.

Remark 2.5.11. [**EGNO**, Proposition 7.11.1][**O2**, §3] Let $A \in \mathcal{C}$ be a separable algebra, then ${}_A\mathcal{C}_A$ is a fusion category under $\otimes_A$ with unit object A.

2.6. Module categories

Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories.

Definition 2.6.1. A left $\mathcal{C}$*-module category* $(\mathcal{M}, \otimes, m, l)$, is a k-linear Abelian category with additional structure:

i. A bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$.

ii. A natural family of isomorphisms, called the *module associativity constraint*

$$m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$$

for X, $Y \in \mathcal{C}$ and $M \in \mathcal{M}$.

iii. A *module unit constraint* $l_M : 1 \otimes M \to M$.

These must satisfy:

$$\begin{tikzcd}[column sep=small]
 & ((X \otimes Y) \otimes Z) \otimes M \arrow[dl, "m_{X\otimes Y,Z,M}"'] \arrow[dr, "a_{X,Y,Z}\otimes id_M"] & \\
(X \otimes Y) \otimes (Z \otimes M) \arrow[d, "m_{X,Y,Z\otimes M}"'] & & (X \otimes (Y \otimes Z)) \otimes M \arrow[d, "m_{X,Y\otimes Z,M}"] \\
X \otimes (Y \otimes (Z \otimes M)) & & X \otimes ((Y \otimes Z) \otimes M) \arrow[ll, "\mathrm{id}_X \otimes m_{Y,Z,M}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
(X \otimes 1) \otimes M \arrow[rr, "m_{X,1,M}"] \arrow[dr, "r_X \otimes \mathrm{id}_M"'] & & X \otimes (1 \otimes M) \arrow[dl, "\mathrm{id}_X \otimes l_M"] \\
 & X \otimes M &
\end{tikzcd}$$

for all objects X, Y, $Z \in \mathcal{C}$ and $M \in \mathcal{M}$.

Remark 2.6.2. The definition for *right module categories* is similar. Alternatively, let $\mathcal{C}^{op}$ be the category $\mathcal{C}$ with opposite tensor multiplication

$$X \otimes^{op} Y := Y \otimes X \quad \text{for all } X, Y \in \mathcal{C}$$

with $\otimes$ the usual tensor product in $\mathcal{C}$, and $\otimes^{op}$ the opposite tensor product; any left $\mathcal{C}$-module category $\mathcal{M}$ is right $\mathcal{C}^{op}$-module.

Definition 2.6.3. A $(\mathcal{C},\mathcal{D})$-*bimodule category* is a k-linear Abelian category $\mathcal{M}$ which is both left $\mathcal{C}$-module, and right $\mathcal{D}$-module, with left module associativity constraint

$$m_{W,X,M} : (W \otimes X) \otimes M \to W \otimes (X \otimes M)$$

and right module associativity constraint

$$n_{M,Y,Z} : (M \otimes Y) \otimes Z \to M \otimes (Y \otimes Z)$$

compatible with a natural family of isomorphisms

$$b_{X,M,Y} : (X \otimes M) \otimes Y \to X \otimes (M \otimes Y)$$

called the *middle associativity constraint* satisfying

$$\begin{array}{ccc} & ((W \otimes X) \otimes M) \otimes Y & \\ {\scriptstyle m_{W,X,M} \otimes \mathrm{id}_Z} \swarrow & & \searrow {\scriptstyle b_{W \otimes X,M,Y}} \\ (W \otimes (X \otimes M)) \otimes Y & & (W \otimes X) \otimes (M \otimes Y) \\ \downarrow {\scriptstyle b_{W,X \otimes M,Y}} & & \downarrow {\scriptstyle m_{W,X,M \otimes Y}} \\ W \otimes ((X \otimes M) \otimes Y) & \xrightarrow[\mathrm{id}_W \otimes b_{X,M,Y}]{} & W \otimes (X \otimes (M \otimes Y)) \end{array}$$

and

$$\begin{array}{ccc} & ((X \otimes M) \otimes Y) \otimes Z & \\ {\scriptstyle b_{X,M,Y} \otimes \mathrm{id}_Z} \swarrow & & \searrow {\scriptstyle n_{X \otimes M,Y,Z}} \\ (X \otimes (M \otimes Y)) \otimes Z & & (X \otimes M) \otimes (Y \otimes Z) \\ \downarrow {\scriptstyle b_{X,M \otimes Y,Z}} & & \downarrow {\scriptstyle b_{X,M,Y \otimes Z}} \\ X \otimes ((M \otimes Y) \otimes Z) & \xrightarrow[\mathrm{id}_X \otimes n_{M,Y,Z}]{} & X \otimes (M \otimes (Y \otimes Z)) \end{array}$$

for all $W,\ X \in \mathcal{C},\ Y,\ Z \in \mathcal{D}$ and $M \in \mathcal{M}$.

REMARK 2.6.4. A $(\mathcal{C}, \mathcal{D})$-bimodule category is the same thing as a left $\mathcal{C} \boxtimes \mathcal{D}^{op}$-module category.

EXAMPLE 2.6.5. Any tensor category $\mathcal{C}$ is bimodule over itself. This is called the *regular module category* over $\mathcal{C}$.

EXAMPLE 2.6.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor, and let $\mathcal{M}$ be a module category over $\mathcal{D}$. Then $\mathcal{M}$ is a module category over $\mathcal{C}$ through F and the action of $\mathcal{D}$.

2.6.1. Module functors.

DEFINITION 2.6.7. Let $(\mathcal{M}, m)$ and $(\mathcal{N}, n)$ be left module categories over $\mathcal{C}$. A $\mathcal{C}$-module functor consisits of a functor $F : \mathcal{M} \to \mathcal{N}$ and a natural family of isomorphisms

$$s_{X,M} : X \otimes F(M) \to F(X \otimes M)$$

called the *module functor structure* of F which satisfies:

$$\begin{array}{ccccc} & & (X \otimes Y) \otimes F(M) & & \\ & \swarrow^{n_{X,Y,F(M)}} & & \searrow^{s_{X \otimes Y, M}} & \\ X \otimes (Y \otimes F(M)) & & & & F((X \otimes Y) \otimes M) \\ \downarrow{\scriptstyle \mathrm{id}_X \otimes s_{Y,M}} & & & & \downarrow{\scriptstyle F(m_{X,Y,M})} \\ X \otimes F(Y \otimes M) & & \xrightarrow[s_{X, Y \otimes M}]{} & & F(X \otimes (Y \otimes M)) \end{array}$$

and

$$\begin{array}{ccc} 1 \otimes F(M) & \xrightarrow{s_{1,M}} & F(1 \otimes M) \\ & {\scriptstyle l_{F(M)}} \searrow \quad \swarrow {\scriptstyle F(l_M)} & \\ & F(M) & \end{array}$$

for each $X,\ Y \in \mathcal{C}$ and $M \in \mathcal{M}$.

Definition 2.6.8. Let $(F,s),(G,t) : \mathcal{M} \to \mathcal{N}$ be module functors. A morphism of module functors $\nu : (F,s) \to (G,t)$ is a natural transformation $F \to G$ satisfying:

$$\begin{array}{ccc} X \otimes F(M) & \xrightarrow{s_{X,M}} & F(X \otimes M) \\ \Big\downarrow {\scriptstyle \mathrm{id}_X \otimes \nu_M} & & \Big\downarrow {\scriptstyle \nu_{X \otimes M}} \\ X \otimes G(M) & \xrightarrow[t_{X,M}]{} & G(X \otimes M) \end{array}$$

for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

Definition 2.6.9. Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{C},\mathcal{D})$-bimodule categories. A $(\mathcal{C},\mathcal{D})$-bimodule functor $F : \mathcal{M} \to \mathcal{N}$ is a left $\mathcal{C} \boxtimes \mathcal{D}^{op}$-module functor.

Example 2.6.10. Let $\mathcal{M}$ be a $(\mathcal{C},\mathcal{D})$-bimodule category. The right action of $X \in \mathcal{D}$ is a $\mathcal{C}$-module endofunctor, and similarly the left action of $Y \in \mathcal{C}$ is a $\mathcal{D}$-module endofunctor. In both of these cases, middle associativity serves as the module functor structure.

Definition 2.6.11. There is an obvious construction of a direct sum of module categories. We say that a module category $\mathcal{M}$ over $\mathcal{C}$ is *indecomposable* if it is not ($\mathcal{C}$-module) equivalent to a non trivial direct sum of module categories.

Definition 2.6.12. We denote the category $Fun_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ of left $\mathcal{C}$-module endofunctors as $\mathcal{C}^*_{\mathcal{M}}$ and call it the *dual category* to $\mathcal{C}$ with respect to $\mathcal{M}$. When $\mathcal{M}$ is indecomposable over $\mathcal{C}$ this is a tensor category with composition of functors acting as the tensor product, and adjoints as duals.

Example 2.6.13. There is a canonical tensor equivalence $\mathcal{C}^*_{\mathcal{C}} = Fun_{\mathcal{C}}(\mathcal{C},\mathcal{C}) \cong \mathcal{C}^{op}$ following the identifications described in Example 2.6.10. Note that each module functor F may be identified with a right action $- \otimes F(1)$. We note that the category $Fun_{\mathcal{C}}(\mathcal{C},\mathcal{C})$ is strict (see Remark 2.3.4).

Example 2.6.14. There is a canonical tensor equivalence $Fun_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C},\mathcal{C}) \cong \mathcal{Z}(\mathcal{C})$. We recall that left $(\mathcal{C},\mathcal{C}^{op})$-module categories are equivalent to $\mathcal{C}$-bimodule categories. Thus we may identify $(\mathcal{C}, \mathcal{C}^{op})$-module functors with the left action by objects in $\mathcal{C}$ equipped with left module functor structure. Here left module functor structure is equivalent to the half braiding.

DEFINITION 2.6.15. Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories. We say that $\mathcal{C}$ is *categorically Morita equivalent* to $\mathcal{D}$ if there is a $\mathcal{C}$-module category $\mathcal{M}$ such that $\mathcal{C}^*_{\mathcal{M}} \cong \mathcal{D}^{op}$. This determines an equivalence relation [**Mu2**, Proposition 4.6].

REMARK 2.6.16. [**ENO2**, Theorem 3.1] Two fusion categories are categorically Morita equivalent if and only if they have braided equivalent centers.

EXAMPLE 2.6.17. We observe that $\mathcal{C}(G, 1)$ is categorically Morita equivalent to $\mathsf{Rep}(G)$.

Consider the dual category over the category of finite dimensional vector spaces $\mathcal{C}(G,1)^*_{\mathsf{Vec}}$. Module functors correspond to a vector space V paired with module functor structure isomorphisms $k_g \otimes V \cong V \otimes k_g$ for each $g \in G$. These provide V with the structure of a G-representation.

2.7. The tensor product of module categories

Let $\mathcal{C}$ be a tensor category, $(\mathcal{M}, m)$ a right module category over $\mathcal{C}$, $(\mathcal{N}, n)$ a left module category over $\mathcal{C}$, and $\mathcal{A}$ a k-linear Abelian category.

DEFINITION 2.7.1. We say that a functor $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ is $\mathcal{C}$-*balanced* if there is a natural family of isomorphisms

$$b_{M,X,N} : F(M \otimes X \boxtimes N) \to F(M \boxtimes X \otimes N)$$

satisfying the following commutative diagram:

$$
\begin{array}{ccc}
F(M \otimes (X \otimes Y) \boxtimes N) & \xrightarrow{\quad m_{M,X,Y} \quad} & F((M \otimes X) \otimes Y \boxtimes N) \\
\Big\downarrow {\scriptstyle b_{M,X\otimes Y,N}} & & \Big\downarrow {\scriptstyle b_{M\otimes X,Y,N}} \\
F(M \boxtimes (X \otimes Y) \otimes N) & & F(M \otimes X \boxtimes Y \otimes N) \\
& {\scriptstyle n^{X,Y,N}} \searrow \quad \swarrow {\scriptstyle b_{M,X,Y\otimes N}} & \\
& F(M \boxtimes X \otimes (Y \otimes N)) &
\end{array}
$$

for all $M \in \mathcal{M}$, $N \in \mathcal{N}$, $X, Y \in \mathcal{C}$.

Definition 2.7.2. *Deligne's tensor product* is a k-linear Abelian category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ together with a $\mathcal{C}$-balanced functor

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$$

inducing, for every k-linear Abelian category $\mathcal{A}$, an equivalence between the category of $\mathcal{C}$-balanced functors from $\mathcal{M} \boxtimes \mathcal{N}$ to $\mathcal{A}$ and the category of functors from $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ to $\mathcal{A}$:

$$\mathrm{Fun}_{bal}(\mathcal{M} \boxtimes \mathcal{N},\, \mathcal{A}) \cong \mathrm{Fun}(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N},\, \mathcal{A}).$$

Remark 2.7.3. Equivalently, the balancing functor $B_{\mathcal{M},\mathcal{N}}$ is universal for all $\mathcal{C}$-balanced functors from $\mathcal{M} \boxtimes \mathcal{N}$ to Abelian categories. In other words, for any $\mathcal{C}$-balanced functor $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ there exists a unique functor $F' : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathcal{A}$ making the following diagram commutative

$$
\begin{array}{ccc}
\mathcal{M} \boxtimes \mathcal{N} & & \\
{\scriptstyle B_{\mathcal{M},\mathcal{N}}} \Big\downarrow & \searrow {\scriptstyle F} & \\
\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} & \overset{F'}{\dashrightarrow} & \mathcal{A}.
\end{array}
$$

Proposition 2.7.4. [**ENO1**, Proposition 3.5] *There is an equivalence of Abelian categories*

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathit{Fun}_{\mathcal{C}}(\mathcal{M}^{op}, \mathcal{N}). \tag{2.9}$$

Remark 2.7.5. If $\mathcal{M}$ is a $(\mathcal{D}, \mathcal{C})$-bimodule category, and $\mathcal{N}$ is a $(\mathcal{C}, \mathcal{E})$-bimodule category then (2.9) is an equivalence of $(\mathcal{D}, \mathcal{E})$-bimodule categories.

Remark 2.7.6. Let $\mathcal{M}$ be a right $\mathcal{C}$-module category, $\mathcal{N}$ a $(\mathcal{C}, \mathcal{D})$-bimodule category, and $\mathcal{K}$ a left $\mathcal{D}$-module category. Then there is a canonical equivalence

$$(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}) \boxtimes_{\mathcal{D}} \mathcal{K} \cong \mathcal{M} \boxtimes_{\mathcal{C}} (\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{K})$$

of categories. Hence the notation $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{K}$ will yield no ambiguity.

2.7.1. Invertible module categories.

Definition 2.7.7. We say that a $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{M}$ is *invertible* if and only if there is a $(\mathcal{D}, \mathcal{C})$-bimodule category $\mathcal{N}$ so that there are bimodule equivalences

$$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{C} \qquad \text{and} \qquad \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \mathcal{D}$$

to the regular $\mathcal{C}$- and $\mathcal{D}$-bimodule categories respectively.

Remark 2.7.8. Definition 2.6.15 and Proposition 2.7.4 establish that invertible $(\mathcal{C}, \mathcal{D})$-bimodule categories determine categorical Morita equivalences between $\mathcal{C}$ and $\mathcal{D}$.

Example 2.7.9. Let $T \in \mathsf{Aut}(\mathcal{C})$, we define a quasi-trivial module category $\mathcal{C}_T$ as in Example 2.6.6. Here $\mathcal{C}_T$ is the regular right module category (see Example 2.6.5), and the left action is defined as follows

$$\mathcal{C} \times \mathcal{C}_T \to \mathcal{C}_T : (X, M) \mapsto T(X) \otimes M$$

The associativity and unit constraints are inherited from tensor category $\mathcal{C}$ and tensor functor T. The tensor inverse is $\mathcal{C}_T^{op} \cong \mathcal{C}_{T^{-1}}$.

DEFINITION 2.7.10. We call the group of (equivalence classes of) invertible $\mathcal{C}$-bimodule categories the *Brauer-Picard group* of $\mathcal{C}$. It is denoted $\mathsf{BrPic}(\mathcal{C})$.

REMARK 2.7.11. [**ENO1**] When $\mathcal{C}$ is a fusion category, the Brauer-Picard group is finite.

REMARK 2.7.12. [**DN, ENO1, ENO2**] There is a canonical isomorphism of groups:

$$\mathsf{BrPic}(\mathcal{C}) \cong \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C})). \tag{2.10}$$

Indeed there is a canonical monoidal equivalence of monoidal categories:

$$\underline{\mathsf{BrPic}}(\mathcal{C}) \cong \underline{\mathsf{Aut}}^{br}(\mathcal{Z}(\mathcal{C})).$$

The objects of $\underline{\mathsf{Aut}}^{br}(\mathcal{Z}(\mathcal{C}))$ are braided tensor autoequivalences of $\mathcal{Z}(\mathcal{C})$ with tensor automorphisms as morphisms; the objects of $\underline{\mathsf{BrPic}}(\mathcal{C})$ are invertible $\mathcal{C}$-bimodule categories, and morphisms are (isomorphism classes of) $\mathcal{C}$-bimodule equivalences.

DEFINITION 2.7.13. When $\mathcal{C}$ is a braided tensor category, one-sided module categories can be given a canonical $\mathcal{C}$-bimodule structure. In this case the Brauer-Picard group has a subgroup of invertible one-sided module categories called the *Picard group* of $\mathcal{C}$. It is denoted $\mathsf{Pic}(\mathcal{C})$.

REMARK 2.7.14. [**DN**] There is a canonical isomorphism

$$\mathsf{Pic}(\mathcal{C}) \cong \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}), \mathcal{C}), \tag{2.11}$$

where $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}), \mathcal{C})$ is the group of (isomorphism classes of) braided tensor autoequivalences of $\mathcal{Z}(\mathcal{C})$ trivializable on $\mathcal{C} \subset \mathcal{Z}(\mathcal{C})$. The inclusion here is given by identifying $Z \in \mathcal{C}$ with $(Z, c_{-,Z}) \in \mathcal{Z}(\mathcal{C})$ for c the braiding on $\mathcal{C}$.

2.8. Group theoretical categories

The most well understood tensor categories come from the representation theory and cohomology of finite groups.

Definition 2.8.1. A fusion category is *group theoretical* if it is categorically Morita equivalent to a pointed fusion category.

2.8.1. Lagrangian subcategories. Let $\mathcal{C}$ be a non-degenerate braided fusion category.

Definition 2.8.2. A fusion subcategory $\mathcal{L} \subset \mathcal{C}$ is called *Tannakian* if $\mathcal{L} = \mathsf{Rep}(G)$ as a braided fusion category for some finite group G.

Definition 2.8.3. A fusion subcategory $\mathcal{L} \subset \mathcal{C}$ is called *Lagrangian* if $\mathcal{L}$ is Tannakian, and $\mathcal{L}$ coincides with its centralizer, i.e., $\mathcal{L} = \mathcal{L}'$. In this case the regular algebra $A = \mathrm{Fun}(G)$ of $\mathcal{L} = \mathsf{Rep}(G)$ is a *Lagrangian algebra* in $\mathcal{C}$ and the category $\mathcal{C}_A$ of left A-modules in $\mathcal{C}$ is equivalent to $\mathcal{C}(G, \omega)$ for some $\omega \in H^3(G, k^\times)$, so that there is a braided tensor equivalence

$$\mathcal{C} \cong \mathcal{Z}(\mathcal{C}(G, \omega)). \tag{2.12}$$

See [**DGNO2**, Section 4.4.10] for details.

Proposition 2.8.4. [**DGNO1**, Corollary 4.14] *A category $\mathcal{C}$ is group theoretical if and only if $\mathcal{Z}(\mathcal{C})$ contains a Lagrangian subcategory.*

2.8.2. Invertible objects of the center of a pointed fusion category. Let G be a finite group and let $\omega \in H^3(G, k^\times)$.

Let $Z(G)$ denote the center of G. For any $a \in Z(G)$ let

$$\beta_a(x, y) = \frac{\omega(a, x, y)\omega(x, y, a)}{\omega(x, a, y)}, \qquad x, y \in G. \tag{2.13}$$

It is known that β_a is a 2-cocycle and that the map

$$\beta : Z(G) \to H^2(G, k^\times) : a \mapsto \beta_a \tag{2.14}$$

is a group homomorphism.

Let G_{ab} denote the maximal Abelian quotient of G, i.e., $G_{ab} = G/[G, G]$.

The invertible objects of $\mathcal{Z}(\mathcal{C}(G, \omega))$ are well known, see, e.g., [**DPR**]. The exact sequence (2.15) in the next proposition can be found in [**GP**, Example 6.2]. We include its proof for the sake of completeness.

Proposition 2.8.5. *The following sequence*

$$0 \to \widehat{G_{ab}} \to \mathsf{Inv}(\mathcal{Z}(\mathcal{C}(G, \omega)) \xrightarrow{F} Z(G) \xrightarrow{\beta} H^2(G, k^\times) \tag{2.15}$$

is exact. Here F is given by the forgetful functor.

Proof. The central structures on the identity object of $\mathcal{C}(G, \omega)$ are parameterized by linear characters of G, i.e., by $\widehat{G_{ab}}$. It is clear that in order to have a central structure the invertible object of $\mathcal{C}(G, \omega)$ must correspond to an element of $Z(G)$. Finally, it follows from (2.13) that $a \in Z(G)$ admits a central structure if and only if β_a is a coboundary. □

Remark 2.8.6. Proposition 2.8.5 can also be derived from the exact sequence (3.3).

Corollary 2.8.7. *The category $\mathcal{Z}(\mathcal{C}(G, \omega))$ is pointed if and only if G is Abelian and β is zero.*

Corollary 2.8.8. *Suppose that G is Abelian. Then $\mathcal{Z}(\mathcal{C}(G, \omega))_{pt}$ is Lagrangian if and only if β is injective.*

Proof. Note that $\mathcal{Z}(\mathcal{C}(G, \omega))_{pt}$ contains the Lagrangian subcategory $\mathcal{L} = \mathsf{Rep}(G)$ consisting of central objects supported on $\mathbf{1}$. Clearly, $\mathcal{Z}(\mathcal{C}(G, \omega))_{pt} = \mathcal{L}$ if and only if the forgetful homomorphism $F : \mathsf{Inv}(\mathcal{Z}(\mathcal{C})) \to \mathsf{Inv}(\mathcal{C})$ is trivial, i.e., if and only if β is injective. □

2.8.3. Braided equivalences between centers of pointed categories. Let G be a finite group and let $A \subset G$ be a normal Abelian subgroup. Let $K := G/A$ be the quotient group, so that there is an extension

$$0 \to A \to G \to K \to 0. \tag{2.16}$$

Such an extension is determined up to an isomorphism by the action of K on G (denoted by $(x,\, a) \mapsto x \cdot a$ for $x \in K$, $a \in A$) and the cohomology class of a 2-cocycle $\kappa \in Z^2(K,\, A)$, so that elements of G are identified with pairs $(a,\, x) \in A \times K$ and the multiplication is given by

$$(a,\, x)\,(b,\, y) = (a(x \cdot b)\kappa(x,\, y),\, xy), \qquad a, b \in A,\; x, y \in K.$$

It was shown in [**MN, N**] that the fusion category $\mathcal{C}(G, 1)$ is categorically Morita equivalent to $\mathcal{C}(\widehat{A} \rtimes K, \omega)$, where $\widehat{A}$ is the dual of the K-module A and the 3-cocycle $\omega \in Z^3(\widehat{A} \rtimes K,\, k^\times)$ is defined by

$$\omega((\rho_1, x_1),\, (\rho_2, x_2),\, (\rho_3, x_3)) = \rho_1(\kappa(x_2,\, x_3)), \tag{2.17}$$

for all $\rho_1, \rho_2, \rho_3 \in \widehat{A}$ and $x_1, x_2, x_3 \in K$.

Remark 2.8.9. In fact, in [**MN, N, U**] *all* pairs of Morita equivalent pointed fusion categories (i.e., all pairs of twisted group doubles with braided tensor equivalent representation categories) were classified. We will only use the special case described above.

Thus, there exists a braided equivalence

$$\mathcal{Z}(\mathcal{C}(G, 1)) \cong \mathcal{Z}(\mathcal{C}(\widehat{A} \rtimes K, \omega)). \tag{2.18}$$

So for computational purposes (see Section 3.4) the group $\mathsf{BrPic}(\mathcal{C}(G, 1))$ can be replaced by $\mathsf{BrPic}(\mathcal{C}(\widehat{A} \rtimes K, \omega))$ substituting G for an easier group.

2.9. Graded fusion categories

Let G be a finite group.

Definition 2.9.1. We say that a fusion category $\mathcal{C}$ is G-*graded* if there is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

into a direct sum of full Abelian subcategories such that the tensor product maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$ for all g, $h \in G$. It follows that the trivial component $\mathcal{C}_e$ is a full tensor subcategory of $\mathcal{C}$, and each component $\mathcal{C}_g$ is a $\mathcal{C}_e$-bimodule category.

When $\mathcal{D} = \mathcal{C}_e$ we say that $\mathcal{C}$ is a G-*extension* of $\mathcal{D}$.

DEFINITION 2.9.2. A G-grading of $\mathcal{C}$ is *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$. In this dissertation we will always assume faithful gradings.

EXAMPLE 2.9.3. The category of G-graded vector spaces $\mathcal{C}(G, \omega)$ with associativity constraint determined by ω is a graded tensor category with faithful grading group G. Component subcategories correspond to graded vector spaces homogeneous in an element $g \in G$.

EXAMPLE 2.9.4. Let $\phi : H \to G$ be a surjective homomorphism of finite groups. The category of H-graded vector spaces $\mathcal{C}(H, \omega)$ is a graded tensor category with faithful grading group G.

EXAMPLE 2.9.5. The category of representations $\mathsf{Rep}(G)$ is graded by the character group $\widehat{Z(G)}$. We note that each irreducible representation of G restricts to a direct sum of (isomorphic) 1-dimensional representations of $Z(G)$.

EXAMPLE 2.9.6. Let G be a finite group. Denote $G_{ab} := G/[G, G]$, the maximal Abelian quotient of G with canonical projection $p : G \to G_{ab}$. Denote the group of linear characters of the center of G as $\widehat{Z(G)} := \mathsf{Hom}(Z(G), k^\times)$. The center $\mathcal{Z}(\mathcal{C}(G, 1))$ of the category of G-graded vector spaces is graded by $G_{ab} \times \widehat{Z(G)}$. We recall the description (Example 2.4.10) of objects in $\mathcal{Z}(\mathcal{C}(G, 1))$ as G-graded G-representations satisfying certain conditions. Let $Z \in \mathcal{Z}(\mathcal{C}(G, 1))$ be a simple object corresponding to (r, ρ) where $r \in G$ is a representative of a conjugacy class, and $\rho : C_G(r) \to GL(V)$ is an irreducible representation of $C_G(r)$.

- There is a G_{ab}-grading on $\mathcal{Z}(\mathcal{C}(G, 1))$ induced by the G-gradings of simple objects (each of which corresponds to a grading by elements of a conjugacy class of G). Z belongs to the grading component corresponding to $p(r) \in G_{ab}$.

- There is a $\widehat{Z(G)}$-grading on $\mathcal{Z}(\mathcal{C}(G,1))$ induced by the G-representation associated to each simple object (each of which is induced from a certain irreducible representation). By Schur's Lemma, an irreducible $C_G(r)$-representation (V,ρ) decomposes as the direct sum of isomorphic 1-dimensional $Z(G) \subset C_G(r)$ representations, and there is $\chi \in \widehat{Z(G)}$ such that $\rho|_{Z(G)} = \chi \mathrm{id}_V$. Z is an object in the grading component corresponding to $\chi \in \widehat{Z(G)}$.

These gradings are each quotients of a finer $G_{ab} \times \widehat{Z(G)}$ grading, such that Z described above is an object in the grading component corresponding to $(p(r), \chi)$.

2.9.1. The universal grading. We can observe that any two gradings on a fusion category $\mathcal{C}$ admit a common refinement.

PROPOSITION 2.9.7. [**GN**, Corollary 3.7] *Every fusion category $\mathcal{C}$ has a canonical faithful grading group denoted $U(\mathcal{C})$ called its* universal grading group. *Any other faithful grading of $\mathcal{C}$ by a group G is determined by a surjective homomorphism $\pi : U(\mathcal{C}) \to G$.*

REMARK 2.9.8. Examples 2.9.3, 2.9.5, and 2.9.6 are all examples of the universal grading.

PROPOSITION 2.9.9. [**GN**, Proposition 3.9] *There is an isomorphism $\widehat{U(\mathcal{C})_{ab}} \cong \mathsf{Aut}(Id_{\mathcal{C}})$, where $\widehat{U(\mathcal{C})_{ab}}$ is the group of linear characters on the maximal Abelian quotient of $U(\mathcal{C})$, and $\mathsf{Aut}(Id_{\mathcal{C}})$ is the group of tensor automorphisms of the identity functor of $\mathcal{C}$.*

PROPOSITION 2.9.10. [**GN**, Theorem 6.3] *Let $\mathcal{C}$ be a non-degenerate braided fusion category. There is a canonical isomorphism $U(\mathcal{C}) \cong \widehat{\mathsf{Inv}(\mathcal{C})}$, where $\widehat{\mathsf{Inv}(\mathcal{C})}$ is the group of linear characters on the invertible objects of $\mathcal{C}$.*

REMARK 2.9.11. We form the identification $\widehat{U(\mathcal{C})} \cong \mathsf{Inv}(\mathcal{C})$ dual to that of Proposition 2.9.10 by way of the square braiding

$$c_{W_g,x} \circ c_{x,W_g} = \chi(g) id_{x \otimes W_g} \qquad \text{for } x \in \mathsf{Inv}(\mathcal{C}),\ g \in U(\mathcal{C}), \text{ and } W_g \in \mathcal{C}_g$$

so that

$$\mathsf{Inv}(\mathcal{C}) \to \widehat{U(\mathcal{C})} : x \mapsto \chi$$

is an isomorphism.

Definition 2.9.12. We say that a grading is *invariant* if any autoequivalence of $\mathcal{C}$ maps $\mathcal{C}_e$ to itself (and, hence, permutes the homogeneous components of the grading). Clearly, the universal grading is invariant. More generally, if $N \subset U(\mathcal{C})$ is a characteristic subgroup of $U(\mathcal{C})$ then the corresponding grading of $\mathcal{C}$ by $U(\mathcal{C})/N$ is invariant.

2.9.2. Crossed product categories. Let G be a finite group, we denote $Cat(G)$ the monoidal category whose objects are elements of G, the only morphisms are identities, and the tensor product is given by multiplication in G.

Definition 2.9.13. An *action* of a group G on a fusion category $\mathcal{C}$ is a monoidal functor

$$T : Cat(G) \to \mathsf{Aut}(\mathcal{C})$$

Here $\mathsf{Aut}(\mathcal{C})$ is a tensor category, where objects are tensor autoequivalences, and morphisms are tensor isomorphisms. We will use the notation $T_g X = {}^g X$ to describe the action of $g \in G$ on objects $X \in \mathcal{C}$.

Definition 2.9.14. Let G be a finite group acting on a fusion category $\mathcal{C}$. A *crossed product category* $\mathcal{C} \rtimes G$ is defined as follows. We set $\mathcal{C} \rtimes G = \mathcal{C} \boxtimes \mathcal{C}(G, 1)$ as a k-linear Abelian category. The tensor product is defined

$$(X \boxtimes g) \otimes (Y \boxtimes h) := (X \otimes {}^g Y) \boxtimes gh$$

for X, $Y \in \mathcal{C}$ and g, $h \in G$. The unit object is $1 \boxtimes e$ and the associativity and unit constraint come from those of $\mathcal{C}$.

Remark 2.9.15. The crossed product category $\mathcal{C} \rtimes G$ is a G-graded fusion category

$$\mathcal{C} \rtimes G = \bigoplus_{g \in G} (\mathcal{C} \rtimes G)_g,$$

where objects of $(\mathcal{C} \rtimes G)_g$ are $X \boxtimes g$ for objects $X \in \mathcal{C}$. In particular, this contains $\mathcal{C}$ as its trivial component $(\mathcal{C} \rtimes G)_e$.

EXAMPLE 2.9.16. For the trivial action of G on Vec we have $\mathsf{Vec} \rtimes G = \mathcal{C}(G, 1)$.

2.9.3. Classification of graded extensions of fusion categories. A description of G-extensions in terms of Brauer-Picard groups was obtained in [**ENO1**]. Below we reformulate this description in the way suitable for our purposes.

PROPOSITION 2.9.17. [**ENO1**, Theorem 6.1] *Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a G-extension. Then:*

(i) *each $\mathcal{C}_g$, $g \in G$ is an invertible $\mathcal{C}_e$-bimodule category;*

(ii) *the tensor product yields $\mathcal{C}_e$-bimodule equivalences:*

$$M_{g,h} : \mathcal{C}_g \boxtimes_{\mathcal{C}_e} \mathcal{C}_h \to \mathcal{C}_{gh}, \quad g, h \in G. \tag{2.19}$$

This gives rise to a homomorphism

$$G \to \mathsf{BrPic}(\mathcal{C}_e) : g \mapsto \mathcal{C}_g. \tag{2.20}$$

DEFINITION 2.9.18. We say that a grading is *injective* if homomorphism (2.20) is injective.

DEFINITION 2.9.19. We denote the monoidal category $\underline{\mathsf{BrPic}}(\mathcal{C}_e)$, which has invertible $\mathcal{C}_e$-bimodule categories as objects, the tensor product $\boxtimes_{\mathcal{C}_e}$ and morphisms are (isomorphism classes of) $\mathcal{C}_e$-bimodule equivalences.

The associativity constraint of $\mathcal{C}$ gives rise to an isomorphism of $\mathcal{C}_e$-bimodule equiavelences:

$$a_{f,g,h} : M_{f,gh}(\mathrm{id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h}) \cong M_{fg,h}(M_{f,g} \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{C}_h}). \tag{2.21}$$

This gives rise to a monoidal functor

$$Cat(G) \to \underline{\mathsf{BrPic}}(\mathcal{C}_e) : g \mapsto \mathcal{C}_g. \tag{2.22}$$

In view of isomorphism (2.10) the above data is the same thing as an action of G on $\mathcal{Z}(\mathcal{D})$, i.e., a monoidal functor

$$T : Cat(G) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D})) : g \mapsto T_g := \Phi(\mathcal{C}_g). \tag{2.23}$$

The above equivalences (2.19) give rise to the monoidal functor structure on T. According to (2.10) for every $g \in G$ we have natural isomorphisms of $\mathcal{D}$-bimodule endofunctors of $\mathcal{C}_g$:

$$T_g(Y) \otimes X \cong X \otimes Y, \qquad X \in \mathcal{C}_g,\, Y \in \mathcal{Z}(\mathcal{D}) \tag{2.24}$$

which we will refer to as a *module half braiding*.

REMARK 2.9.20. The action (2.23) alone does not determine an extension: for the existence of associativity constraint on $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ a certain obstruction $O_4(T) \in H^4(G,\, k^\times)$ must vanish. In this case G-extensions of $\mathcal{D}$ corresponding to the action (2.23) are parameterized by a torsor over $H^3(G,\, k^\times)$.

CHAPTER 3

Brauer Picard groups of fusion p-categories

3.1. Induction of central autoequivalences

Definition 3.1.1. Let $\mathcal{C}$ be a fusion category, let $\mathcal{Z}(\mathcal{C})$ be its center, with objects (Z, γ). There is a *central induction homomorphism*

$$\mathsf{ind} : \mathsf{Aut}(\mathcal{C}) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C})) : \alpha \mapsto \mathsf{ind}(\alpha),$$

where $\mathsf{ind}(\alpha)(Z, \gamma) = (\alpha(Z), \gamma^\alpha)$ and γ^α is defined by the following commutative diagram

$$\begin{array}{ccc}
X \otimes \alpha(Z) & \xrightarrow{\quad\gamma^\alpha_X\quad} & \alpha(Z) \otimes X \\
\downarrow & & \downarrow \\
\alpha(\alpha^{-1}(X)) \otimes \alpha(Z) & & \alpha(Z) \otimes \alpha(\alpha^{-1}(X)) \\
{\scriptstyle J_{\alpha^{-1}(X),Z}}\downarrow & & \downarrow{\scriptstyle J_{Z,\alpha^{-1}(X)}} \\
\alpha(\alpha^{-1}(X) \otimes Z) & \xrightarrow{\alpha(\gamma_{\alpha^{-1}(X)})} & \alpha(Z \otimes \alpha^{-1}(X)).
\end{array} \tag{3.1}$$

Here α^{-1} is a quasi-inverse of α and $J_{X,Y} : \alpha(X) \otimes \alpha(Z) \xrightarrow{\sim} \alpha(X \otimes Z)$ is the tensor functor structure of α.

3.1.1. The kernel of induction.

Definition 3.1.2. The forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ restricts to a *forgetful homomorphism* between groups of invertible objects:

$$F : \mathsf{Inv}(\mathcal{Z}(\mathcal{C})) \to \mathsf{Inv}(\mathcal{C}).$$

We note that there is a cannonical identification $\ker(F) \cong \widehat{U(\mathcal{C})_{ab}}$.

Definition 3.1.3. For any invertible object $X \in \mathcal{A}$ the conjugation by X is a tensor autoequivalence of $\mathcal{C}$, thus there is a group homomorphism

$$\mathsf{conj} : \mathsf{Inv}(\mathcal{C}) \to \mathsf{Aut}(\mathcal{C}). \tag{3.2}$$

Proposition 3.1.4. *The sequence of group homomorphisms*

$$0 \longrightarrow \widehat{U(\mathcal{C})_{ab}} \longrightarrow \mathsf{Inv}(\mathcal{Z}(\mathcal{C})) \xrightarrow{F} \mathsf{Inv}(\mathcal{C}) \xrightarrow{\mathsf{conj}} \mathsf{Aut}(\mathcal{C}) \xrightarrow{\mathsf{ind}} \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C})) \tag{3.3}$$

is exact.

Proof. The exactness at $\mathsf{Inv}(\mathcal{Z}(\mathcal{C}))$ is obvious. To see that the sequence is exact at $\mathsf{Inv}(\mathcal{C})$ observe that a natural tensor isomorphism between the conjugation functor

$$V \mapsto X \otimes V \otimes X^*$$

and $\mathrm{id}_{\mathcal{C}}$ is the same thing as a central structure on X. It remains to establish exactness at $\mathsf{Aut}(\mathcal{C})$. For $\alpha \in \mathsf{Aut}(\mathcal{C})$ let $\mathcal{C}_\alpha$ denote the invertible $\mathcal{C}$-bimodule category corresponding to the induced autoequivalence $\mathsf{ind}(\alpha) \in \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ under isomorphism (2.10). This category is equivalent to the regular category $\mathcal{C}$ as a right $\mathcal{C}$-module category and left action of $\mathcal{C}$ on $\mathcal{C}_\alpha$ is given by

$$(X, V) \mapsto \alpha(X) \otimes V, \tag{3.4}$$

for all $X \in \mathcal{C}$ and $V \in \mathcal{C}_\alpha$, see [**NR**, Example 6.4]. Braided autoequivalence $\mathsf{ind}(\alpha)$ is trivial if and only if there is an $\mathcal{C}$-bimodule equivalence between $\mathcal{C}_\alpha$ and $\mathcal{C}$. Such an equivalence is given by $V \mapsto X \otimes V$ for an invertible object X in $\mathcal{C}$ such that α is the conjugation by X. Thus, the result follows from isomorphism (2.10). □

3.1.2. The image of induction. Let $\mathcal{C}$ be a non-degenerate braided fusion category and let A be a *Lagrangian algebra* in $\mathcal{C}$. Let $\mathcal{C}_A$ be the category of left A-modules in $\mathcal{C}$. It is a fusion category with tensor product $\otimes_A$.

There is a canonical braided tensor equivalence

$$\iota_A : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}_A) : Z \mapsto A \otimes Z. \tag{3.5}$$

Let

$$\mathsf{ind}_A : \mathsf{Aut}(\mathcal{C}_A) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}_A)) \tag{3.6}$$

denote the induction homomorphism.

THEOREM 3.1.5. *Let α be a braided tensor autoequivalence of $\mathcal{C}$. The following conditions are equivalent:*

(i) *there is an algebra isomorphism $\alpha(A) \cong A$,*

(ii) *there is $\gamma \in \mathsf{Aut}(\mathcal{C}_A)$ such that $\alpha = \iota_A^{-1} \circ \mathsf{ind}_A(\gamma) \circ \iota_A$.*

PROOF. Suppose that there is an algebra isomorphism $\phi : A \xrightarrow{\sim} \alpha(A)$. Define an autoequivalence $\gamma \in \mathsf{Aut}(\mathcal{C}_A)$ as follows. Given an A-module X in $\mathcal{C}$ with the action $p : A \otimes X \to X$ we set $\gamma(X) = \alpha(X)$ as an object in $\mathcal{C}$, with the action

$$A \otimes \gamma(X) = A \otimes \alpha(X) \xrightarrow{\phi \otimes \mathrm{id}_{\alpha(X)}} \alpha(A) \otimes \alpha(X) \cong \alpha(A \otimes X) \xrightarrow{\alpha(p)} \alpha(X).$$

Then $\iota_A \alpha \iota_A^{-1}(A \otimes Z) \cong A \otimes \alpha(Z) = \gamma(A \otimes Z)$ as A-modules and its central structure is determined by that of Z. This means that $\iota_A \circ \alpha \circ \iota_A^{-1} = \mathsf{ind}_A(\gamma)$.

Conversely, suppose that $\alpha = \iota_A^{-1} \circ \mathsf{ind}_A(\gamma) \circ \iota_A$ for some $\gamma \in \mathsf{Aut}(\mathcal{C}_A)$. Let $F_A : \mathcal{Z}(\mathcal{C}_A) \to \mathcal{C}_A$ be the forgetful functor and let $I_A : \mathcal{C}_A \to \mathcal{Z}(\mathcal{C}_A)$ be its right adjoint. Note that $I_A(\mathbf{1})$ is a Lagrangian algebra in $\mathcal{Z}(\mathcal{C}_A)$ and $A \cong \iota_A^{-1}(I_A(\mathbf{1}))$. For any $\gamma \in \mathsf{Aut}(\mathcal{C}_A)$ we have a natural tensor isomorphism

$$F_A \circ \mathsf{ind}(\gamma) \cong \gamma \circ F_A. \tag{3.7}$$

Taking adjoints of both sides and replacing γ by its inverse we obtain natural isomorphism $\mathsf{ind}(\gamma) \circ I_A \cong I_A \circ \gamma$ satisfying a multiplicative property corresponding to (3.7) being an isomorphism of tensor functors. Applying both sides of the last isomorphism to $\mathbf{1}$ we obtain an algebra isomorphism

$$\mathsf{ind}(\gamma)\,(I_A(\mathbf{1})) \cong I_A(\mathbf{1}),$$

which is equivalent to $\alpha(A) \cong A$. □

REMARK 3.1.6. It follows from Theorem 3.1.5 that the image of the induction homomorphism (3.6) is the stabilizer of (the isomorphism class of) a Lagrangian algebra $A \in \mathcal{C}$ in $\mathsf{Aut}^{br}(\mathcal{C})$.

3.1.3. Image of induction in the pointed case. Let G be a finite group and let $\omega \in H^3(G,\, k^\times)$.

Let $\mathsf{OutStab}(\omega) = \mathsf{Stab}(\omega)/\,\mathsf{Inn}(G)$ denote the subgroup of $\mathsf{Out}(G)$ consisting of classes of automorphisms a such that $\omega \circ (a \times a \times a)$ is cohomologous to ω.

Let $I(G,\, \omega)$ be the image of induction $\mathsf{Aut}(\mathcal{C}(G,\, \omega)) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}(G,\, \omega)))$.

PROPOSITION 3.1.7. *There is an exact sequence*

$$Z(G) \xrightarrow{\beta} H^2(G,\, k^\times) \xrightarrow{\mathsf{ind}} I(G,\, \omega) \to \mathsf{OutStab}(\omega) \to 0 \tag{3.8}$$

PROOF. Using the exact sequence (3.3) we see that

$$I(G,\, \omega) \cong \mathsf{Aut}(\mathcal{C}(G,\, \omega))/\mathrm{Ker}(\mathsf{ind}) \cong \mathsf{Aut}(\mathcal{C}(G,\, \omega))/\mathrm{Im}(\mathsf{conj}),$$

where conj is given by (3.2). The image of conj in $\mathsf{Aut}(\mathcal{C}(G,\, \omega))$ is generated by $\mathsf{Inn}(G) = G/Z(G)$ and the image of $\beta : Z(G) \to H^2(G, k^\times) \subset \mathsf{Aut}(\mathcal{C}(G,\, \omega))$ defined in (2.14) (note that the conjugation by $a \in Z(G)$ gives rise to a non-trivial tensor autoequivalence of $\mathcal{C}(G,\, \omega)$ precisely when β_a is non-trivial in $H^2(G, k^\times)$). By (2.5), this implies the statement. □

COROLLARY 3.1.8. *We have an exact sequence*

$$0 \to \widehat{G_{ab}} \to \mathsf{Inv}(\mathcal{Z}(\mathcal{C}(G,\, \omega)) \xrightarrow{F} Z(G) \xrightarrow{\beta} H^2(G,\, k^\times) \xrightarrow{\mathsf{ind}} I(G,\, \omega) \to \mathsf{OutStab}(\omega) \to 0. \tag{3.9}$$

COROLLARY 3.1.9. *Let $\mathcal{C}$ be a non-degenerate braided fusion category and let $\mathcal{L} = \mathsf{Rep}(G)$ be a Lagrangian subcategory of $\mathcal{C}$ such that $\alpha(\mathcal{L}) \cong \mathcal{L}$ for every $\alpha \in \mathsf{Aut}^{br}(\mathcal{C})$. Then $\mathcal{C} \cong$*

$\mathcal{Z}(\mathcal{C}(G,\,\omega))$ for some $\omega \in H^3(G,\, k^\times)$ and the induction homomorphism

$$\mathsf{Aut}(\mathcal{C}(G,\,\omega)) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}(G,\,\omega)) \tag{3.10}$$

is surjective, i.e., $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}(G,\,\omega)) = I(G,\,\omega)$.

Proof. This follows from Theorem 3.1.5 and Proposition 3.1.7. □

3.2. Cohomology of elementary Abelian p-groups

Let p be a prime, let n be a positive integer, and let V_n denote the elementary Abelian p-group of order p^n. We have $\mathsf{Aut}(V_n) = GL_n(\mathbb{F}_p)$. Below we will also view V_n as an n-dimensional vector space over the field $\mathbb{F}_p$ with p elements. We will denote $V_n^* = \mathsf{Hom}(V_n,\, \mathbb{F}_p)$ the dual vector space.

As before, for $\omega \in H^3(V_n,\, k^\times)$ we denote $\mathcal{C}(V_n,\, \omega)$ the category of V_n-graded vector spaces with the associativity constraint twisted by ω.

For a vector space W we denote

$$\bigwedge(W) = \bigoplus_{i=0}^{\infty} \bigwedge^i(W) \quad \text{and} \quad \mathsf{S}(W) = \bigoplus_{i=0}^{\infty} \mathsf{S}^i(W)$$

the alternating and symmetric algebras of W. When W has a basis $\{w_1, \dots, w_n\}$ we also write $\bigwedge(W) = \bigwedge(w_1, \dots, w_n)$ and $\mathsf{S}(W) = \mathsf{S}(w_1, \dots, w_n)$.

3.2.1. The case when p is odd.

Let p be an odd prime.

The cohomology ring $H^\bullet(V_n,\, \mathbb{F}_p)$ is well known (see, e.g., [**A**]), namely

$$H^\bullet(V_n,\, \mathbb{F}_p) = \bigwedge(x_1, \dots, x_n) \otimes_{\mathbb{F}_p} \mathbb{F}_p[y_1, \dots, y_n],$$

where $\deg(x_i) = 1$ and $\deg(y_i) = 2$ for all $i = 1, \dots, n$.

The cocycles representing generators x_i and y_i can be explicitly described as follows (below we identify cocycles with the cohomology classes they represent). Define $x_i : V_n \to \mathbb{F}_p$ and $y_i : V_n \times V_n \to \mathbb{F}_p$ by

$$x_i(v) = v_i \tag{3.11}$$

and

$$y_i(u,\, v) = \begin{cases} 0 & \text{if } u_i + v_i < p, \\ 1 & \text{if } u_i + v_i \geq p, \end{cases} \tag{3.12}$$

for all $v = (v_1, \dots v_n)$, $u = (u_1, \dots u_n)$ in V_n and $i = 1, \dots n$. Here we view u_i, v_i as elements of $\{0, 1, \dots, p-1\}$ and add them as usual integers.

In particular,

$$H^3(V_n,\, \mathbb{F}_p) = \bigwedge^3 (x_1, \dots, x_n) \bigoplus \mathbb{F}_p \langle x_i \cup y_j \mid i, j = 1, \dots, n \rangle, \tag{3.13}$$

where $\cup$ denotes the cup product. Note that the second summand in (3.13) is isomorphic to $V_n^* \otimes V_n^*$ as a $GL_n(\mathbb{F}_p)$-module.

Proposition 3.2.1. *There is an isomorphism of $GL_n(\mathbb{F}_p)$-modules:*

$$H^3(V_n,\, k^\times) \cong \bigwedge^3 (V_n^*) \bigoplus \mathsf{S}^2(V_n^*). \tag{3.14}$$

Proof. The automorphism of $k^\times$ given by $\xi \mapsto \xi^p$ yields an exact sequence of Abelian groups:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow k^\times \longrightarrow k^\times \longrightarrow 0,$$

where $\mathbb{F}_p$ is identified with the group of pth roots of 1 in k. This, by functoriality, yields a long exact sequence of $GL_n(\mathbb{F}_p)$-modules:

$$\begin{aligned} \cdots \longrightarrow H^{m-1}(V_n,\, \mathbb{F}_p) \longrightarrow H^{m-1}(V_n,\, k^\times) \xrightarrow{0} H^{m-1}(V_n,\, k^\times) \\ \longrightarrow H^m(V_n,\, \mathbb{F}_p) \longrightarrow H^m(V_n,\, k^\times) \xrightarrow{0} H^m(V_n,\, k^\times) \longrightarrow \cdots \end{aligned}$$

Note that the map $H^{m-1}(V_n,\, k^\times) \to H^{m-1}(V_n,\, k^\times)$ induced by taking the pth power is zero since the exponent of $H^m(V_n,\, k^\times)$ is p. The latter fact follows from isomorphism $H^m(V_n,\, k^\times) = H^{m+1}(V_n,\, \mathbb{Z})$ and the Künneth formula for the direct product in cohomology.

In particular, there is a short exact sequence of $GL_n(\mathbb{F}_p)$-modules

$$0 \to H^2(V_n,\, k^\times) \xrightarrow{\delta} H^3(V_n,\, \mathbb{F}_p) \to H^3(V_n,\, k^\times) \to 0. \tag{3.15}$$

We claim that the image of inclusion

$$\delta : H^2(V_n,\, k^\times) = \bigwedge^2(V_n^*) \to H^3(V_n,\, \mathbb{F}_p) \tag{3.16}$$

is the subspace of $\mathbb{F}_p\langle x_i \cup y_j \mid i, j = 1, \ldots, n\rangle \subset H^3(V_n,\, \mathbb{F}_p)$ consisting of cohomology classes of the form $\sum_{i,j=1}^n a_{ij} x_i \cup y_j$, where $A = \{a_{ij}\}$ is an anti-symmetric matrix over $\mathbb{F}_p$.

Indeed, (3.16) is given by the connecting homomorphism which is explicitly computed as follows. Fix a primitive p-th root of unity ξ in k and consider homomorphism

$$\mathbb{F}_p \to k^\times : a \mapsto \xi^a.$$

A class in $H^2(V_n,\, k^\times)$ is represented by a 2-cocycle $\mu : V_n \times V_n \to k^\times$ given by

$$\mu(u,\, v) = \xi^{(Au,\, v)}, \qquad u,\, v \in V_n,$$

where $A = \{a_{ij}\}$ is an n-by-n matrix over $\mathbb{F}_p$ and $(Au,\, v) = \sum_{i,j}^n a_{ij} u_i v_j$ is viewed as a non-negative integer.

Let λ be a fixed pth root of ξ in k. Take a 2-cochain $\nu : V_n \times V_n \to k^\times$ defined by

$$\nu(u,\, v) = \lambda^{(Au,\, v)}, \qquad u,\, v \in V_n.$$

For non-negative integers a, b let us denote $\{a, b\}$ the integral part of $\frac{a+b}{p}$. We have

$$\begin{aligned} \nu(u,v)\nu(u',v) &= \nu(u+u',v)\, \xi^{\{(Au,v),(Au',v)\}}, \\ \nu(u,v)\nu(u,v') &= \nu(u,v+v')\, \xi^{\{(Au,v),(Au,v')\}}, \end{aligned}$$

for all $u,\, u',\, v,\, v' \in V_n$. Using these identities we compute the differential of ν:

$$d\nu(u,v,w) = \frac{\nu(u+v,w)\nu(u,v)}{\nu(u,v+w)\nu(v,w)} = \frac{\xi^{\{(Au,v),(Au,w)\}}}{\xi^{\{(Au,w),(Av,w)\}}}, \qquad u,\, v,\, w \in V_n.$$

Fix $k,\, l \in \{1, 2, \ldots, n\}$ and take A such that $a_{ij} = 1$ if $i = k$ and $j = l$ and $a_{ij} = 0$ otherwise. Then the previous calculation yields

$$d\nu(u, v, w) = \frac{\xi^{u_k\{v_l, w_l\}}}{\xi^{w_l\{u_k, v_k\}}}.$$

Since $d\nu(u,\, v,\, w) = \xi^{\delta(\mu)(u,v,w)}$, we conclude that in this case

$$\delta(\mu)(u,\, v,\, w) = u_k\{v_l, w_l\} - w_l\{u_k, v_k\}.$$

Comparing this with (3.11) and (3.12) (note that $y_i(u,\, v) = \{u_i,\, v_i\}$) we conclude that the image of δ is spanned by

$$x_k \cup y_l - x_l \cup y_k,\; k, l = 1, \ldots, n, \tag{3.17}$$

as claimed. The quotient of $V_n^* \otimes V_n^*$ by this space is isomorphic to $\mathsf{S}^2(V_n^*)$ via the symmetrization map. Since

$$H^3(V_n,\, k^\times) = H^3(V_n,\, \mathbb{F}_p)/\mathrm{Image}(\delta : H^2(V_n,\, k^\times) \to H^3(V_n,\, \mathbb{F}_p)),$$

the statement follows from the last claim and (3.13). □

3.2.2. The case when $p = 2$. It is known (see, e.g., [**A**]) that

$$H^\bullet(V_n,\, \mathbb{F}_2) = \mathbb{F}_2[x_1, \ldots, x_n],$$

where $x_i,\; i = 1, \ldots, n$ are one-dimensional generators represented by 1-cocycles

$$x_i(v) = v_i, \qquad \text{where } v = (v_1, \ldots v_n) \in V_n. \tag{3.18}$$

Proposition 3.2.2. *There is an isomorphism of $GL_n(\mathbb{F}_2)$-modules:*

$$H^3(V_n,\, k^\times) \cong \mathsf{S}^3(x_1, \ldots, x_n)/\mathbb{F}_2\langle x_i^2 x_j + x_i x_j^2 \mid\, i, j = 1, \ldots, n\rangle. \tag{3.19}$$

PROOF. The argument is similar to that of Proposition 3.2.1. There is a short exact sequence of $GL_n(\mathbb{F}_2)$-modules

$$0 \to H^2(V_n,\, k^\times) \to H^3(V_n,\, \mathbb{F}_2) \to H^3(V_n,\, k^\times) \to 0.$$

The same computation as in the proof of Proposition 3.2.1 shows image of inclusion of $H^2(V_n,\, k^\times) = \bigwedge^2(V_n^*)$ into $H^3(V_n,\, \mathbb{F}_p) = \mathsf{S}^3(x_1, \dots, x_n)$ is spanned by elements $x_k \cup y_l + x_l \cup y_k$ with $k, l = 1, \dots, n$ (see (3.17)). Since for $p = 2$ we have $y_k = x_k^2$, the result follows. □

3.3. $\mathsf{BrPic}(\mathcal{C}(V_n, \omega))$ when V_n is an elementary Abelian p-group

3.3.1. The case when p is odd. Given a cohomology class $\omega \in H^3(V_n,\, k^\times)$ denote

$$\omega = \omega_{alt} + \omega_{sym}, \qquad \omega_{alt} \in \bigwedge^3(V_n^*), \quad \omega_{sym} \in \mathsf{S}^2(V_n^*) \tag{3.20}$$

the decomposition of ω from (3.14).

DEFINITION 3.3.1. Let V be a vector space. Consider the *interior derivation*

$$\iota : V \otimes \bigwedge^3(V^*) \to \bigwedge^2(V^*) : v \otimes \phi \mapsto \iota_v(\phi),$$

given by

$$\iota_v(x \wedge y \wedge z) = \langle v,\, z\rangle x \wedge y - \langle v,\, y\rangle x \wedge z + \langle v,\, x\rangle y \wedge z,$$

for all $v \in V$, $x, y, z \in V^*$, and extended to $\bigwedge^3(V^*)$ by linearity.

DEFINITION 3.3.2. The *radical* of $\phi \in \bigwedge^3(V^*)$ is defined as

$$\mathsf{Rad}(\phi) := \{u \in V \mid \iota_u(\phi) = 0\}. \tag{3.21}$$

We say that $\phi \in \bigwedge^3(V^*)$ is *non-degenerate* if $\mathsf{Rad}(\phi) = 0$.

PROPOSITION 3.3.3. *Let $v \in V_n$ and let $\omega \in H^3(V_n,\, k^\times)$. Then v, regarded as a simple object in $\mathcal{C}(V_n,\, \omega)$, is in the image of $\mathsf{Inv}(\mathcal{Z}(\mathcal{C}(V_n,\, \omega))) \to \mathsf{Inv}(\mathcal{C}(V_n,\, \omega))$ if and only if $v \in \mathsf{Rad}(\omega_{alt})$.*

PROOF. We claim that in this case the homomorphism

$$\beta : V_n \to H^2(V_n,\, k^\times) = \bigwedge^2(V_n^*)$$

defined by (2.13) and (2.14) is given by

$$\beta(v) = \iota_v(\omega_{alt}), \quad v \in V_n. \tag{3.22}$$

Let us first prove this claim when $\omega = \omega_{alt} \in \bigwedge^3(V_n^*)$ (i.e., when the symmetric part of ω in (3.20) is trivial). Such an ω is a linear combination of 3-cocycles ω_{ijk} (i, j, k are distinct elements of $\{1, \dots, n\}$), given by $\omega(u,\, v,\, w) = \xi^{u_i v_j w_k}$, where ξ is a fixed primitive pth root of 1 in k. So we may assume that $\omega = \omega_{ijk}$. We have

$$\beta(v) = v_i x_j \wedge x_k - v_j x_i \wedge x_k + v_k x_i \wedge x_j, \qquad v = (v_1, \dots, v_n) \in V_n,$$

where x_i are defined by (3.11). Thus, $\beta(v) = \iota_v(\omega_{ijk})$ and (3.22) is true in this case.

Next, let us prove the claim when $\omega = \omega_{sym} \in \mathsf{S}^2(V_n^*)$. We need to check that $\beta = 0 \in H^2(V_n,\, k^\times)$ in this case. We may assume that ω is the image of $x_i \cup y_j \in H^3(V_n, \mathbb{F}_p)$ under (3.15) i.e.,

$$\omega(u,\, v,\, w) = \xi^{x_i(u) y_j(v,w)}, \qquad u, v, w \in V_n.$$

Since y_j is symmetric (see (3.12)) we conclude that $\beta(v)$ is the image of $v_i y_j$ under the homomorphism $H^2(V_n,\, \mathbb{F}_p) \to H^2(V_n,\, k^\times)$ and, hence, is trivial.

So, (3.22) is true for all $\omega \in H^3(V_n,\, k^\times)$ and the result follows from Proposition 2.8.5. □

COROLLARY 3.3.4. *The category $\mathcal{Z}(\mathcal{C}(V_n,\, \omega))$ is pointed if and only if $\omega_{alt} = 0$.*

COROLLARY 3.3.5. *$\mathcal{Z}(\mathcal{C}(V_n,\, \omega))_{pt} = \mathsf{Rep}(V_n)$ if and only if ω_{alt} is non-degenerate. In this case $\mathcal{Z}(\mathcal{C}(V_n,\, \omega))_{pt}$ is the trivial component of the universal grading of $\mathcal{Z}(\mathcal{C}(V_n,\, \omega))$ and the universal grading group of $\mathcal{Z}(\mathcal{C}(V_n,\, \omega))$ is $\mathsf{Hom}(V_n,\, k^\times)$, the dual group of V_n.*

PROOF. This follows from Corollary 2.8.8. □

THEOREM 3.3.6. *Let $\omega \in H^3(V_n, k^\times)$ be such that ω_{alt} is non-degenerate. There is an exact sequence of groups:*

$$0 \to V_n \xrightarrow{\iota(\omega_{alt})} \bigwedge^2(V_n^*) \to \mathsf{BrPic}(\mathcal{C}(V_n, \omega)) \to \mathsf{Stab}_{V_n}(\omega) \to 0. \tag{3.23}$$

PROOF. This follows from Proposition 3.1.7 and Corollary 3.1.9 applied to $G = V_n$. □

Theorem 3.3.6 implies that the Brauer-Picard group of $\mathcal{C}(V_n, \omega)$ is an extension of

$$\mathsf{Stab}_{V_n}(\omega) = \mathsf{Stab}(\omega_{alt}) \cap \mathsf{Stab}(\omega_{sym})$$

by an elementary Abelian p-group.

3.3.2. The case when $p = 2$.

PROPOSITION 3.3.7. *There is a short exact sequence of $GL_n(\mathbb{F}_2)$-modules:*

$$0 \to \mathsf{S}^2(V_n^*) \to H^3(V_n, k^\times) \xrightarrow{\pi} \bigwedge^3(V_n^*) \to 0. \tag{3.24}$$

PROOF. Using Proposition 3.2.2 we see that $H^3(V_n, k^\times)$ contains a $GL_n(\mathbb{F}_2)$-submodule spanned by $x_i^2 x_j$, $i, j = 1, \ldots, n$ (modulo $\mathbb{F}_2\langle x_i^2 x_j + x_i x_j^2 \mid i, j = 1, \ldots, n\rangle$). Clearly, this submodule is isomorphic to $\mathsf{S}^2(V_n^*)$ and the corresponding quotient is isomorphic to $\bigwedge^3(V_n^*)$ (the cosets are represented by polynomials $x_i x_j x_k$, where $i, j, k = 1, \ldots, n$ are distinct). □

For $\omega \in H^3(V_n, k^\times)$ let

$$\omega_{alt} = \pi(\omega) \in \bigwedge^3(V_n^*). \tag{3.25}$$

PROPOSITION 3.3.8. *Let $v \in V_n$ and let $\omega \in H^3(V_n, k^\times)$. Then v, regarded as a simple object in $\mathcal{C}(V_n, \omega)$, is in the image of $\mathsf{Inv}(\mathcal{Z}(\mathcal{C}(V_n, \omega))) \to \mathsf{Inv}(\mathcal{C}(V_n, \omega))$ if and only if $v \in \mathsf{Rad}(\omega_{alt})$.*

PROOF. This is similar to proof of Proposition 3.3.3. □

COROLLARY 3.3.9. *The category $\mathcal{Z}(\mathcal{C}(V_n, \omega))$ is pointed if and only if $\omega_{alt} = 0$.*

Corollary 3.3.10. *Corollary 3.3.5 and Theorem 3.3.6 hold for* $p = 2$ *(but note that the meaning of* ω_{alt} *is different in this case, cf.* (3.25)*)*

Remark 3.3.11. The representation categories of twisted group doubles of elementary Abelian 2-groups and braided tensor equivalences between them were studied by Goff, Mason, and Ng in [**GMN**]. The above results about the third cohomology of V_n and invertible objects of $\mathcal{Z}(\mathcal{C}(V_n,\, \omega))$ can be derived from that paper.

3.4. Cocycles associated to extra special p-groups

Definition 3.4.1. Let p be a prime. A p-group G is called *extra special* if its center Z is cyclic of order p and G/Z is elementary abelian. Such groups are well known: for each positive integer n there exist precisely two non-isomorphic extra special p-groups of order p^{2n+1}.

Since G is a central extension of the form

$$0 \to \mathbb{F}_p \to G \to V_{2n} \to 0, \tag{3.26}$$

corresponding to some cohomology class $\kappa_G \in H^2(V_n,\, \mathbb{F}_p)$ we can apply the results of Section 2.8.3. Namely, there is a braided equivalence

$$\mathcal{Z}(\mathcal{C}(G,\, 1)) \cong \mathcal{Z}(\mathcal{C}(V_{2n+1},\, \omega_G)), \tag{3.27}$$

where $\omega_G \in H^3(G,\, k^\times)$ is obtained from κ_G as follows. Choose a generator x_0 of $H^1(\mathbb{F}_p,\, \mathbb{F}_p) = \mathbb{F}_p^*$ and consider the cup product

$$H^1(\mathbb{F}_p,\, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^2(V_n,\, \mathbb{F}_p) \to H^3(V_{2n+1},\, \mathbb{F}_p) : x_0 \otimes \kappa \mapsto x_0 \cup \kappa.$$

Proposition 3.4.2. ω_G *is the image of* $x_0 \cup \kappa_G$ *under the projection*

$$H^3(V_{2n+1},\, \mathbb{F}_p) \to H^3(V_{2n+1},\, k^\times).$$

Proof. This follows from (2.17). $\square$

3.4.1. Extra special p-groups when p is odd. Let p be an odd prime and let n be a positive integer. Let D and Q denote extra special groups of order p^{2n+1} and exponents p and p^2 respectively. They can be constructed as follows. Let $M = (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes \mathbb{Z}/p\mathbb{Z}$ and $N = \mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$ be non-Abelian groups of order p^3. Then D is the central product of n copies of M and Q is the central product of N and $n-1$ copies of M.

It is straightforward to compute cohomology classes κ_D, $\kappa_Q \in H^2(V_{2n}, \mathbb{F}_p)$ corresponding to extension (3.26) with $G = D, Q$. We have

$$\kappa_D = \sum_{i=1}^{n} x_{2i-1}\, x_{2i}, \tag{3.28}$$

$$\kappa_Q = \sum_{i=1}^{n} x_{2i-1}\, x_{2i} + y_1, \tag{3.29}$$

where generators x_i, y_i are defined by (3.11) and (3.12).

Recall from Proposition 3.2.1 that there is a $GL_{2n+1}(\mathbb{F}_p)$-module isomorphism

$$H^3(V_{2n+1},\, k^\times) \cong \bigwedge^3 (x_0,\, x_1, \ldots, x_{2n}) \oplus \mathsf{S}^2(z_0,\, z_1, \ldots, z_{2n}), \tag{3.30}$$

hence elements $\omega \in H^3(V_{2n+1}, k^\times)$ correspond to pairs $(\omega_{alt}, \omega_{sym})$, where ω_{alt} is a degree 3 element of the exterior algebra and ω_{sym} is a degree 2 element of the symmetric algebra.

Below we identify ω with its image under isomorphism (3.30).

PROPOSITION 3.4.3. *We have*

$$\omega_D = \left(x_0 \wedge \left(\sum_{i=1}^{n} x_{2i-1} \wedge x_{2i}, \right),\, 0 \right), \tag{3.31}$$

$$\omega_Q = \left(x_0 \wedge \left(\sum_{i=1}^{n} x_{2i-1} \wedge x_{2i}, \right),\, z_0 z_1 \right). \tag{3.32}$$

PROOF. This follows from equations (3.28), (3.29), and Proposition 3.4.2. □

3.4.2. Extra special 2-groups. Let D and Q denote extra special groups of order 2^{2n+1}. They can be constructed as follows. The group D is the central product of n copies of the dihedral group of order 8 and Q is the central product $n-1$ copies of the dihedral group of order 8 and one copy of the quaternion group.

It is straightforward to compute cohomology classes $\kappa_D,\, \kappa_Q \in H^2(V_{2n},\, \mathbb{F}_2)$ corresponding to extension (3.26) with $G = D,\, Q$. We have

$$\kappa_D = \sum_{i=1}^{n} x_{2i-1}\, x_{2i}, \tag{3.33}$$

$$\kappa_Q = \sum_{i=1}^{n} x_{2i-1}\, x_{2i} \,+\, x_1^2 \,+\, x_2^2, \tag{3.34}$$

where generators x_i are defined by (3.18).

Recall from Proposition 3.2.2 that there is an isomorphism of $GL_{2n+1}(\mathbb{F}_2)$-modules:

$$H^3(V_{2n+1},\, k^\times) \cong \mathsf{S}^3(x_0, x_1, \dots, x_{2n})/\mathbb{F}_2\langle x_i^2 x_j + x_i x_j^2 \mid i, j = 0, \dots, 2n\rangle. \tag{3.35}$$

Below we identify ω with its image under (3.35).

Proposition 3.4.4. *We have*

$$\omega_D = \sum_{i=1}^{n} x_0\, x_{2i-1}\, x_{2i}, \tag{3.36}$$

$$\omega_Q = \sum_{i=1}^{n} x_0\, x_{2i-1}\, x_{2i} \,+\, x_0\, x_1^2 \,+\, x_0\, x_2^2. \tag{3.37}$$

Proof. This follows from equations (3.33), (3.34), and Proposition 3.4.2. □

Recall from Proposition 3.3.7 that there is a short exact sequence of $GL_{2n+1}(\mathbb{F}_2)$-modules

$$0 \to \mathsf{S}^2(x_0,\, x_1, \dots, x_{2n}) \to H^3(V_{2n+1},\, k^\times) \xrightarrow{\pi} \bigwedge^3(x_0,\, x_1, \dots, x_{2n}) \to 0. \tag{3.38}$$

Proposition 3.4.5. *We have* $\pi(\omega_D) = \pi(\omega_Q) = \omega_{alt}$, *where*

$$\omega_{alt} = x_0 \wedge \left(\sum_{i=1}^{n} x_{2i-1} \wedge x_{2i}, \right). \tag{3.39}$$

Proof. The homomorphism $\pi : H^3(V_{2n+1},\, k^\times) \to \bigwedge^3(V_{2n+1}^*)$ is described explicitly in the proof of Proposition 3.3.7. The result is immediate from there. □

3.5. Brauer-Picard groups

3.5.1. Representation categories of extra special p-groups for odd p. It follows from Theorem 3.3.6 that the Brauer-Picard groups $\mathsf{BrPic}(\mathcal{C}(D,\,1))$ and $\mathsf{BrPic}(\mathcal{C}(Q,\,1))$ are extensions of $\mathsf{Stab}(\omega_D)$ and $\mathsf{Stab}(\omega_Q)$, respectively, by elementary Abelian p-groups. One can find these stabilizers using the explicit formulas (3.31) and (3.32). Below we do it for ω_D.

Let V be a finite dimensional vector space. For any subgroup $G \subset GL(V)$ define the corresponding affine group $AffG = V \rtimes G$.

Proposition 3.5.1. *Suppose $n > 1$. Then*

$$\mathsf{Stab}(\omega_D) \cong AffSp_{2n}(\mathbb{F}_p) \rtimes \mathbb{F}_p^\times.$$

Here $Sp_{2n}(\mathbb{F}_p)$ denotes the symplectic group.

Proof. Let $\{e_0,\, e_1, \ldots, e_{2n}\}$ be the standard basis of V_{2n+1} so that $\{x_0,\, x_1, \ldots, x_{2n}\}$ is the dual basis of V_{2n+1}^*. By the rank of $\sum_i a_i \wedge b_i \in \bigwedge^2 V^*$ we mean the rank of the associated linear endomorphism of v given by

$$V \to V : v \mapsto \sum_i \langle v,\, a_i \rangle b_i - \langle v,\, b_i \rangle a_i.$$

For every $g \in \mathsf{Stab}(\omega_D)$ and every $v \in V_{2n+1}$ the ranks of $\iota_{g(v)}(\omega_D)$ and $\iota_v(\omega_D)$ are equal. It follows that the span of $\{e_1, \ldots, e_{2n}\}$ is stable under $\mathsf{Stab}(\omega_D)$. Indeed, non-zero vectors v lying in this span are characterized by the property that $\iota_v(\omega_D)$ has rank 2. Let

$$s = \sum_{i=1}^{n} x_{2i-1} \wedge x_{2i}$$

(it corresponds to the symplectic form). An element $g \in \mathsf{Stab}(\omega_D)$ must map e_0 to $\lambda e_0 + \sum_{i=1}^{2n} v_i e_i$ and the dual of the restriction of g on the span of $\{e_1, \ldots, e_{2n}\}$ must map s to $\lambda^{-1} s$, where $v = (v_1, \ldots v_{2n})$ is an arbitrary vector in $\mathbb{F}_p^{2n}$ and $\lambda \in \mathbb{F}_p^\times$.

Thus, $\mathsf{Stab}(\omega_D)$ is generated by matrices of the form

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline v & M \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|ccc} \lambda & & 0 & \\ \hline & \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} & & 0 \\ 0 & & \ddots & \\ & 0 & & \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right],$$

where $v \in \mathbb{F}_p^{2n}$, $M \in Sp_{2n}(\mathbb{F}_p)$, and $\lambda \in \mathbb{F}_p^{\times}$. This clearly implies the statement. □

When $n = 1$, i.e., when D, Q are extra special groups of order p^3, there is a neat precise description of their Brauer-Picard groups.

PROPOSITION 3.5.2. *Let $n = 1$. Then*

$$\mathsf{Stab}(\omega_D) = SL_3(\mathbb{F}_p) \quad \textit{and} \quad \mathsf{Stab}(\omega_Q) = AffO_2(\mathbb{F}_p).$$

PROOF. For any $A \in GL_3(\mathbb{F}_p)$ we have

$$A(x_0 \wedge x_1 \wedge x_2) = \det(A)(x_0 \wedge x_1 \wedge x_2)$$

in $\bigwedge^3(x_0,\, x_1,\, x_2)$, which proves the first equality. The group of matrices in $SL_3(\mathbb{F}_p)$ stabilizing $z_0 z_1 \in \mathsf{S}^2(z_0,\, z_1,\, z_2)$ is precisely the 2-dimensional affine orthogonal group, which proves the second equality. □

COROLLARY 3.5.3. *Let D, Q be the extra special groups of order p^3 of exponents p and p^2, respectively. Then*

$$\mathsf{BrPic}(\mathcal{C}(D,\, 1)) \cong SL_3(\mathbb{F}_p) \quad \textit{and} \quad \mathsf{BrPic}(\mathcal{C}(Q,\, 1)) \cong AffO_2(\mathbb{F}_p). \tag{3.40}$$

PROOF. Note that the homomorphisms $\iota(\omega_{alt})$ for $\omega = \omega_D, \omega_Q$ in (3.23) are isomorphisms, so $\mathsf{BrPic}(\mathcal{C}(V_3,\, \omega)) = \mathsf{Stab}(\omega)$ and the isomorphisms follow from Proposition 3.5.2. □

REMARK 3.5.4. The first of isomorphisms in (3.40) was established by Riepel in [**R**] by different methods.

3.5.2. Representation categories of extra special 2-groups. In view of Corollary 3.3.10 the Brauer-Picard groups of $\mathcal{C}(D,\,1)$ and $\mathcal{C}(Q,\,1)$ are extensions of $\mathsf{Stab}(\omega_D)$ and $\mathsf{Stab}(\omega_Q)$, respectively, by elementary Abelian 2-groups. The above stabilizers can be found using the explicit formulas (3.36) and (3.37).

We consider $\mathsf{Stab}(\omega_D)$ and $\mathsf{Stab}(\omega_Q)$ as subgroups of $\mathsf{Stab}(\omega_{alt})$.

PROPOSITION 3.5.5. *Suppose $n > 1$. Then*

$$\mathsf{Stab}(\omega_{alt}) \cong AffSp_{2n}(\mathbb{F}_2).$$

PROOF. This is similar to the proof of Proposition 3.5.1. □

PROPOSITION 3.5.6. *Let $n > 1$. The projection from $AffSp_{2n}(\mathbb{F}_2)$ to $Sp_{2n}(\mathbb{F}_2)$ induces inclusions*

$$\mathsf{Stab}(\omega_D) \hookrightarrow Sp_{2n}(\mathbb{F}_2) \quad \textit{and} \quad \mathsf{Stab}(\omega_Q) \hookrightarrow Sp_{2n}(\mathbb{F}_2)$$

PROOF. A simple computation verifies that the kernel of this projection intersects both $\mathsf{Stab}(\omega_D)$ and $\mathsf{Stab}(\omega_Q)$ trivially. □

PROPOSITION 3.5.7. *Let $n > 1$. Then*

$$\mathsf{Stab}(\omega_D) \cong Sp_{2n}(\mathbb{F}_2) \quad \textit{and} \quad \mathsf{Stab}(\omega_Q) \cong Sp_{2n}(\mathbb{F}_2)$$

PROOF. Second cohomology classes κ_D and κ_Q defined in equations (3.33) and (3.34), represent the two equivalence classes of quadratic forms in even dimension [**W**, 3.4.7]. Their respective stabilizers, the orthogonal groups $O^+_{2n}(\mathbb{F}_2)$ and $O^-_{2n}(\mathbb{F}_2)$, are identified with subgroups of $\mathsf{Stab}(\omega_D)$ and $\mathsf{Stab}(\omega_Q)$.

It is known that $O^+_{2n}(\mathbb{F}_2)$ and $O^-_{2n}(\mathbb{F}_2)$ are maximal subgroups of $Sp_{2n}(\mathbb{F}_2)$ [**P**, Theorem 1.5].

Let $M \in GL(V_{2n+1})$ be defined by

$$M(v_0,\, v_1,\, v_2,\, v_3,\, v_4, \dots, v_{2n-1},\, v_{2n}) = (v_0,\, v_1,\, v_2,\, v_0 + v_3 + v_4,\, v_4, \dots, v_{2n-1},\, v_{2n}).$$

Note that $M \in \mathsf{Stab}(\omega_D)$ and $M \in \mathsf{Stab}(\omega_Q)$, however, it is not an element of either orthogonal subgroup. Thus the images of inclusions specified in Proposition 3.5.6 properly contain maximal subgroups. It follows that these are isomorphisms. □

PROPOSITION 3.5.8. *Let $n = 1$. Then*

$$\mathsf{Stab}(\omega_D) \cong S_4 \quad \textit{and} \quad \mathsf{Stab}(\omega_Q) = S_3.$$

PROOF. For any $\omega \in H^3(V_3, k^\times) = \mathsf{S}^3(x_0, x_1, x_2)/\mathbb{F}_2\langle x_i^2 x_j + x_i x_j^2 \mid i, j = 0, 1, 2\rangle$ the evaluation map

$$V_3 \to \mathbb{F}_2 : (v_0, v_1, v_2) \mapsto \omega(v_0, v_1, v_2)$$

is well defined and, furthermore, ω is completely determined by the set of vectors of V_3 mapped to 1 (cf. [**M**]).

For $\omega = \omega_D = x_0 x_1 x_2$ this set consists of the single vector (1, 1, 1). So $\mathsf{Stab}(\omega_D)$ is precisely the subgroup of automorphisms of V_3 fixing this vector, i.e.,

$$\mathsf{Stab}(\omega_D) \cong AffGL_2(\mathbb{F}_2) \cong \mathbb{F}_2^2 \rtimes GL_2(\mathbb{F}_2) \cong S_4.$$

For $\omega = \omega_Q = x_0 x_1 x_2 + x_0 x_1^2 + x_0 x_2^2$ this set consists of vectors (1, 1, 1), (1, 1, 0), and (1, 0, 1). The group $\mathsf{Stab}(\omega_Q)$ consists of automorphisms of V_3 permuting these vectors. Since they form a basis of V_3, we conclude $\mathsf{Stab}(\omega_Q) \cong S_3$. □

COROLLARY 3.5.9. *Let D, Q be the dihedral group and quaternion groups of order 8, respectively. Then*

$$\mathsf{BrPic}(\mathcal{C}(D, 1)) \cong S_4 \quad \textit{and} \quad \mathsf{BrPic}(\mathcal{C}(Q, 1)) \cong S_3. \tag{3.41}$$

PROOF. Note that the homomorphisms $\iota(\omega_{alt})$ for $\omega = \omega_D, \omega_Q$ in (3.23) are isomorphisms, so $\mathsf{BrPic}(\mathcal{C}(V_3, \omega)) = \mathsf{Stab}(\omega)$ and the isomorphisms follow from Proposition 3.5.8. □

REMARK 3.5.10. The isomorphisms in (3.41) were established in [**NR**] by different methods.

3.5.3. Pointed p-categories coming from metric modular Lie algebras. In view of isomorphism (3.14) one can produce interesting examples of 3-cocycles on elementary Abelian groups as follows.

Let F be a finite field of characteristic $p > 3$. Below we consider finite dimensional Lie algebras over F. We refer the reader to [**S**] for the theory of modular Lie algebras.

DEFINITION 3.5.11. A *pre-metric Lie algebra* is a Lie algebra $\mathfrak{g}$ equipped with an invariant symmetric bilinear form $(\,,\,)$, i.e., such that

$$([a,\, b],\, c) = (a,\, [b,\, c]),$$

for all $a, b, c \in \mathfrak{g}$. A *metric Lie algebra* is a pre-metric Lie algebra such that $(\,,\,)$ is non-degenerate.

For a Lie algebra $\mathfrak{g}$ let $\mathsf{Aut}(\mathfrak{g})$ denote the group of Lie algebra automorphisms of $\mathfrak{g}$. For a pre-metric Lie algebra $\mathfrak{g}$ let $\mathsf{Aut}_m(\mathfrak{g}) \subset \mathsf{Aut}(\mathfrak{g})$ denote the group of Lie algebra automorphisms of $\mathfrak{g}$ preserving $(\,,\,)$.

Consider the following bilinear symmetric and trilinear alternating forms on $\mathfrak{g}$:

$$\tilde{\omega}_{sym}(a,\, b) = (a,\, b) \quad \text{and} \quad \tilde{\omega}_{alt}(a,\, b,\, c) = ([a,\, b],\, c), \qquad a, b, c \in \mathfrak{g},$$

and identify them with $\omega_{sym} \in \mathsf{S}^2(\mathfrak{g}^*)$ and $\omega_{alt} \in \bigwedge^3(\mathfrak{g}^*)$ by means of symmetrization and anti-symmetrization maps.

Let V denote the underlying additive group of $\mathfrak{g}$. It is an elementary Abelian p-group.

Set

$$\omega = \omega_{alt} + \omega_{sym} \in H^3(V,\, k^\times). \tag{3.42}$$

PROPOSITION 3.5.12. *Let $\mathfrak{g}$ be a metric Lie algebra such that $\mathfrak{g} = [\mathfrak{g},\, \mathfrak{g}]$ and let $\omega \in H^3(V,\, k^\times)$ be the 3-cocycle constructed above. Then $\mathsf{Stab}(\omega) \cong \mathsf{Aut}_m(\mathfrak{g})$.*

PROOF. It is clear that each $\phi \in \mathsf{Aut}_m(\mathfrak{g})$ stabilizes $\tilde{\omega}_{sym}$ and $\tilde{\omega}_{alt}$. Hence, it stabilizes ω. Conversely, if ψ is a group automorphism of V stabilizing ω then it must stabilize both ω_{sym} and ω_{alt}. The former condition means that ψ preserves $(\,,\,)$ and the latter one means that it is a Lie algebra homomorphism. □

PROPOSITION 3.5.13. *Let $\mathfrak{g}$ be a pre-metric Lie algebra. Then $\omega_{alt} \in \bigwedge^3(\mathfrak{g}^*)$ is non-degenerate if and only if $\mathfrak{g} = [\mathfrak{g},\, \mathfrak{g}]$ and $(\,,\,)$ is non-degenerate (i.e., $\mathfrak{g}$ is a metric Lie algebra).*

PROOF. Non-degeneracy of ω_{alt} is equivalent to non-degeneracy of the alternating trilinear form $\tilde{\omega}_{alt}$. This implies the statement. □

COROLLARY 3.5.14. *Suppose that $\mathfrak{g}$ is a metric Lie algebra such that $\mathfrak{g} = [\mathfrak{g},\, \mathfrak{g}]$. There is an exact sequence*

$$0 \to V \to \bigwedge^2(V^*) \to \mathsf{BrPic}(\mathcal{C}(V,\, \omega)) \to \mathsf{Aut}_m(\mathfrak{g}) \to 0. \tag{3.43}$$

PROOF. This follows from Theorem 3.3.6 and Propositions 3.5.12 and 3.5.13. □

One can take $(\,,\,)$ to be the Killing form of $\mathfrak{g}$:

$$(a,\, b) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(a)\,\mathrm{ad}(b)), \qquad a, b \in \mathfrak{g},$$

where ad denotes the adjoint representation of $\mathfrak{g}$. Let $\omega^{\mathfrak{g}} \in H^3(V,\, k^\times)$ denote the corresponding 3rd cohomology class defined by (3.42).

REMARK 3.5.15. If $\mathfrak{g}$ has a non-degenerate Killing form then $\mathfrak{g}$ is a direct sum of simple Lie algebras. Unlike for Lie algebras over $\mathbb{C}$, the converse to this statement is false in positive characteristic. All simple classical Lie algebras with a non-degenerate Killing over a field F with $\mathrm{char}(F) > 3$ are known. The necessary and sufficient condition is that $\mathrm{char}(F)$ should not divide the determinant of the Killing form of the corresponding simple complex Lie algebra, see [**S**, Chapter II §9].

It can even happen that every trace form of a simple $\mathfrak{g}$ is degenerate, this is the case, e.g., for $\mathfrak{g} = \mathfrak{sl}_n(F)$ when $\mathrm{char}(F)$ divides n.

PROPOSITION 3.5.16. *Let $\mathfrak{g}$ be a simple Lie algebra with a non-degenerate Killing form. We have an exact sequence*

$$0 \to V \to \bigwedge^2 (V^*) \to \mathsf{BrPic}(\mathcal{C}(V,\, \omega^{\mathfrak{g}})) \to \mathsf{Aut}(\mathfrak{g}) \to 0,$$

PROOF. It is clear that when (,) given by the Killing form of $\mathfrak{g}$ we have $\mathsf{Aut}_m(\mathfrak{g}) = \mathsf{Aut}(\mathfrak{g})$. The result follows from Corollary 3.5.14. □

REMARK 3.5.17. The automorphism groups of classical simple modular Lie algebras $\mathfrak{g}$ are known [**S**].

Thus, finite simple groups of Lie type naturally appear as composition factors of groups of autoequivalences and Brauer-Picard groups of pointed fusion categories.

EXAMPLE 3.5.18. Take $\mathfrak{g} = \mathfrak{sl}_2(F)$. Since p is odd, the Killing form is non-degenerate. Let

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

be a basis of $\mathfrak{g}$ so that

$$ad(e) = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; ad(f) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}, \; \text{and} \quad ad(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding 3-cocycle $\omega^{\mathfrak{g}}$ is given by

$$\omega^{\mathfrak{g}} = \left(8x_e \wedge x_f \wedge x_h,\, 4z_e z_f \,+\, 8z_h^2\right).$$

In particular, when $F = \mathbb{F}_p$ we have $\mathsf{BrPic}(\mathcal{C}(V,\, \omega^{\mathfrak{g}})) = SO_3(\mathbb{F}_p)$.

CHAPTER 4

Autoequivalences of graded extensions

Let G be a finite group. Let

$$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g, \qquad \mathcal{C}_e = \mathcal{D}, \tag{4.1}$$

be an invariant (Definition 2.9.12) extension of a fusion category $\mathcal{D}$. Let $\mathsf{Aut}(\mathcal{C})$ and $\mathsf{Aut}(\mathcal{D})$ denote the groups of tensor autoequivalences of $\mathcal{C}$ and $\mathcal{D}$ respectively. Recall that in this situation, the restriction homomorphism

$$\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}} : \mathsf{Aut}(\mathcal{C}) \to \mathsf{Aut}(\mathcal{D}) \tag{4.2}$$

is well defined.

The main goal of this chapter is to understand the image and kernel of (4.2).

4.1. Grading data

The grading (4.1) is a homomorphism $G \to \mathsf{BrPic}(\mathcal{D})$ (2.20), which lifts to a monoidal functor $Cat(G) \to \underline{\mathsf{BrPic}}(\mathcal{D})$ (Subsection 2.9.3). Indeed the grading is precisely a certain monoidal 2-functor [**ENO1**], however we make use of the description above and the canonical monoidal equivalence

$$\Phi : \underline{\mathsf{BrPic}}(\mathcal{D}) \to \underline{\mathsf{Aut}}^{br}(\mathcal{Z}(\mathcal{D})) \tag{4.3}$$

to encode grading data as an action of G on $\mathcal{Z}(\mathcal{D})$ by braided autoequivalences.

We define this action

$$T : Cat(G) \to \underline{\mathsf{Aut}}^{br}(\mathcal{Z}(\mathcal{D})) : g \mapsto T_g = \Phi(\mathcal{C}_g) \tag{4.4}$$

by making choices

$$\sigma_g(X,Z) : X \otimes Z \to T_g(Z) \otimes X, \qquad X \in \mathcal{C}_g,\ Z \in \mathcal{Z}(\mathcal{D}), \tag{4.5}$$

of *module half-braidings* (2.24) for $\mathcal{Z}(\mathcal{D})$ over $\mathcal{C}_g$ for each $g \in G$. Take $\mathcal{Z}(\mathcal{D})$ and the module action on each $\mathcal{C}_g$ to be strict, let J_g denote the tensor functor structure of T_g, and c the braiding on $\mathcal{Z}(\mathcal{D})$. The module half-braiding (4.5) satisfies

$$\begin{array}{ccc} X \otimes Z \otimes Z' & \xrightarrow{\sigma_g(X,Z)\otimes \mathrm{id}_{Z'}} & T_g(Z) \otimes X \otimes Z' \\ \Big\downarrow{\scriptstyle \sigma_g(X, Z\otimes Z')} & & \Big\downarrow{\scriptstyle \mathrm{id}_{T_g(Z)}\otimes \sigma_g(X,Z')} \\ T_g(Z \otimes Z') \otimes X & \xleftarrow[J_g(Z,Z')\otimes \mathrm{id}_X]{} & T_g(Z) \otimes T_g(Z') \otimes X, \end{array} \tag{4.6}$$

$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{\mathrm{id}_X \otimes c(Y,Z)} & X \otimes Z \otimes Y \\ & {\scriptstyle \sigma_g(X\otimes Y, Z)} \searrow \quad \swarrow {\scriptstyle \sigma_g(X,Z)\otimes \mathrm{id}_Y} & \\ & T_g(Z) \otimes X \otimes Y, & \end{array} \tag{4.7}$$

and

$$\begin{array}{ccc} Y \otimes X \otimes Z & \xrightarrow{\mathrm{id}_Y \otimes \sigma_g(X,Z)} & Y \otimes T_g(Z) \otimes X \\ & {\scriptstyle \sigma_g(Y\otimes X, Z)} \searrow \quad \swarrow {\scriptstyle c(Y,T_g(Z))\otimes \mathrm{id}_X} & \\ & T_g(Z) \otimes Y \otimes X, & \end{array} \tag{4.8}$$

for each $Y \in \mathcal{D}$, $Z, Z' \in \mathcal{Z}(\mathcal{D})$, $g \in G$, and each $X \in \mathcal{C}_g$.

REMARK 4.1.1. In general, an equivalence $(F,s) : \mathcal{M} \to \mathcal{N}$ of invertible $\mathcal{D}$-bimodule categories gives rise to an isomorphism $\Phi(F) : \Phi(\mathcal{M}) \to \Phi(N)$ of braided autoequivalences of $\mathcal{Z}(\mathcal{D})$. With Φ defined by making choices (4.5) of module half braidings denoted $\sigma_{\mathcal{M}}$ and

$\sigma_{\mathcal{N}}$ for $\mathcal{M}$ and $\mathcal{N}$, $\Phi(F) : \Phi(\mathcal{M}) \to \Phi(\mathcal{N})$ is the isomorphism which makes the diagram

$$\begin{array}{ccccc} F(M)\otimes Z & & \xrightarrow{\sigma_{\mathcal{N}}(F(M),Z)} & & \Phi(\mathcal{N})(Z)\otimes F(M) \\ \downarrow{\scriptstyle s(M,Z)} & & & & \downarrow{\scriptstyle s(\Phi(\mathcal{N})(Z),M)} \\ F(M\otimes Z) & \xrightarrow[F(\sigma_{\mathcal{M}}(M,Z))]{} & F(\Phi(\mathcal{M})(Z)\otimes M) & \xrightarrow[F(\Phi(F)(Z)\otimes \mathrm{id}_{\mathcal{M}})]{} & F(\Phi(\mathcal{N})(Z)\otimes M) \end{array}$$

commute for each $M \in \mathcal{M}$, and $Z \in \mathcal{Z}(\mathcal{D})$.

We extract the monoidal functor structure $\Phi(M_{g,h}) : \Phi(\mathcal{C}_g)\Phi(\mathcal{C}_h) \to \Phi(\mathcal{C}_{gh})$, $g, h \in G$, induced by the tensor product on $\mathcal{C}$, by taking $\mathcal{C}$ to be strict, and composing module half-braidings. This is the isomorphism $\gamma_{g,h} : T_gT_h \to T_{gh}$ so that the diagram

$$\begin{array}{ccc} X_g\otimes X_h\otimes Z & \xrightarrow{\mathrm{id}_{X_g}\otimes\sigma_h(X_h,Z)} & X_g\otimes T_g(Z)\otimes X_h \\ \downarrow{\scriptstyle \sigma_{gh}(X_g\otimes X_h,Z)} & & \downarrow{\scriptstyle \sigma_g(X_g,T_h(Z))\otimes\mathrm{id}_{X_h}} \\ T_{gh}(Z)\otimes X_g\otimes X_h & \xleftarrow[\gamma_{g,h}(Z)\otimes\mathrm{id}_{X_g\otimes X_h}]{} & T_gT_h(Z)\otimes X_g\otimes X_h \end{array} \tag{4.9}$$

commutes for each $Z \in \mathcal{Z}(\mathcal{D})$, $g, h \in G$, and each $X_g \in \mathcal{C}_g$, $X_h \in \mathcal{C}_h$.

Definition 4.1.2. Let $\mathcal{M}$ be a $\mathcal{D}$-bimodule category, let $F \in \mathsf{Aut}(\mathcal{D})$ and denote central induction (Definition 3.1.1) $\bar{F} := \mathsf{ind}(F) \in \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D}))$. We take $\mathcal{M}^F$ to be the $\mathcal{D}$-bimodule category obtained from $\mathcal{M}$ by "twisting" actions of $\mathcal{D}$ by F, i.e.,

$$X \odot M = F(X)\otimes M,\ M\odot X = M\otimes F(X), \qquad X \in \mathcal{D},\ M \in \mathcal{M},$$

where $\otimes$ denotes the original action of $\mathcal{D}$ and $\odot$ denotes the module action. If $\mathcal{M}$ is an invertible $\mathcal{D}$-bimodule category then so is $\mathcal{M}^F$. In this case $\mathcal{M}^F$ corresponds to the braided autoequivalence $\mathsf{ind}(F)^{-1} \circ \Phi(\mathcal{M}) \circ \mathsf{ind}(F)$ under isomorphism (2.10).

Remark 4.1.3. When the grading (4.1) is invariant, a tensor autoequivalence $\Gamma : \mathcal{C} \to \mathcal{C}$ determines:

- A tensor autoequivalence $F := \mathsf{Res}^{\mathcal{C}}_{\mathcal{D}}(\Gamma) : \mathcal{D} \to \mathcal{D}$.
- A group automorphism $a \in \mathsf{Aut}(G)$ which permutes the grading components.

- A family of $\mathcal{D}$-bimodule equivalences $F_g := \Gamma|_{\mathcal{C}_g} : \mathcal{C}_g \to \mathcal{C}^F_{a(g)}$, $g \in G$.
- Natural isomorphisms $M^F_{g,h} \circ F_g \boxtimes_{\mathcal{D}} F_h \to F_{gh} \circ M_{g,h}$, for each g, $h \in G$.

This data in terms of invertible $\mathcal{C}_e$-bimodule categories and their equivalences translates under (4.3) into an isomorphism of actions (i.e. a natural isomorphism of monoidal functors):

$$\bar{F}^{-1} \circ T_a \circ \bar{F} : Cat(G) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D})) : g \mapsto \bar{F}^{-1} T_{a(g)} \bar{F} = \Phi(\mathcal{C}^F_{a(g)})$$

$$T : Cat(G) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D})) : g \mapsto T_g = \Phi(\mathcal{C}_g).$$

We realize this as a family of isomorphisms $t_g : T_{a(g)} \,\mathsf{ind}(F) \to \mathsf{ind}(F) T_g$ satisfying

$$\begin{array}{ccccc} \Gamma(X_g) \otimes \bar{F}(Z) & \xrightarrow{\sigma_{a(g)}(\Gamma(X_g), \bar{F}(Z))} & T_{a(g)} \bar{F}(Z) \otimes \Gamma(X_g) & \xrightarrow{t_g(Z) \otimes \mathrm{id}_{\Gamma(X_g)}} & \bar{F} T_g(Z) \otimes \Gamma(X_g) \\ \downarrow{\scriptstyle J(X_g, Z)} & & & & \downarrow{\scriptstyle J(T_g(Z), X_g)} \\ \Gamma(X_g \otimes Z) & & \xrightarrow[\Gamma(\sigma_g(X_g, Z))]{} & & \Gamma(T_g(Z) \otimes X_g) \end{array}$$

for each $Z \in \mathcal{Z}(\mathcal{D})$, $g \in G$, and each $X_g \in \mathcal{C}_g$, where J is the tensor functor structure of Γ.

4.2. Canonical homomorphism associated to a graded fusion category

Let $\mathcal{C}$ be a graded fusion category (4.1) which extends $\mathcal{D} = \mathcal{C}_e$. In this section we consider the grading data (4.4) restricted to the maximal pointed subcategory (Definition 2.3.13) $\mathcal{Z}(\mathcal{D})_{pt} = \mathcal{C}(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})), \omega_{\mathcal{Z}(\mathcal{D})})$, where $\omega_{\mathcal{Z}(\mathcal{D})}$ is a representative of a canonical third cohomology class in $H^3(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})), k^\times)$.

The canonical action (4.4) restricts to an action

$$T : G \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D})_{pt}),\ g \mapsto T_g$$

of G on $\mathcal{C}(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})), \omega_{\mathcal{Z}(\mathcal{D})})$. The tensor functor structure of this restriction is given by a collection of 2-cochains $\mu_g \in C^2(G,\, k^\times)$, $g \in G$:

$$\mu_g(Y,\, W) :\, {}^gY \otimes {}^gW \xrightarrow{\sim} {}^g(Y \otimes W), \qquad Y, W \in \mathsf{Inv}(\mathcal{Z}(\mathcal{D})),$$

satisfying

$$d^2\mu_g(X,Y,W) = \frac{\omega_{\mathcal{Z}(\mathcal{D})}(X,\, Y,\, W)}{\omega_{\mathcal{Z}(\mathcal{D})}(\,{}^gX,\, {}^gY,\, {}^gW)}. \tag{4.10}$$

Next, natural isomorphisms of tensor functors:

$$\gamma_{g,h} : T_g T_h \to T_{gh}$$

restricted to $\mathcal{C}(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})), \omega_{\mathcal{Z}(\mathcal{D})})$ are given by a collection of 2-cocycles:

$$\gamma_Y = \{\gamma_{g,h}(Y)\}_{g,h\in G}, \quad \text{for} \quad Y \in \mathsf{Inv}(\mathcal{Z}(\mathcal{D})).$$

Tensor property of these isomorphisms translates to the relation

$$\frac{\mu_{gh}(Y,\, W)}{\gamma_{g,h}(Y\otimes W)} \;=\; \frac{\mu_h(Y,W)\,\mu_g({}^hY,\, {}^hW)}{\gamma_{g,h}(Y)\,\gamma_{g,h}(W)}, \tag{4.11}$$

for all $g, h \in G$, $Y, W \in \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))$.

Using the above action of G on $\mathcal{C}(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})), \omega_{\mathcal{Z}(\mathcal{D})})$ one can form the crossed product category (Definition 2.9.14)

$$\mathcal{C}(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})), \omega_{\mathcal{Z}(\mathcal{D})}) \rtimes G = \mathcal{C}(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})) \rtimes G, \tilde{\omega}), \tag{4.12}$$

where $\tilde{\omega}$ is a 3-cocycle on $\mathsf{Inv}(\mathcal{Z}(\mathcal{D})) \rtimes G$ given by

$$\tilde{\omega}((Z_1, g_1),\, (Z_2, g_2),\, (Z_3, g_3)) := \omega_{\mathcal{Z}(\mathcal{D})}(Z_1,\, {}^{g_1}Z_2,\, {}^{g_1g_2}Z_3)\, \gamma^{-1}_{g_1,g_2}(Z_3)\, \mu_{g_1}(Z_2,\, {}^{g_2}Z_3) \tag{4.13}$$

for all $Z_1, Z_2, Z_3 \in \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))$, $g_1, g_2, g_3 \in G$.

Let $z \in H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$ be a 1-cocycle with respect to the action T, i.e., a function $z : G \to \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))$ such that $z(gh) = {}^gz(h)z(g)$, $g, h \in G$. It determines a homomorphism $\iota_z : G \to \mathsf{Inv}(\mathcal{Z}(\mathcal{D})) \rtimes G$ right inverse to the canonical projection on G. Consequently, for any such z there is a fusion subcategory

$$\mathcal{C}(G, \omega_z) \subset \mathcal{C}(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})) \rtimes G, \tilde{\omega}), \qquad \text{where } \omega_z := \tilde{\omega} \circ (\iota_z \times \iota_z \times \iota_z). \tag{4.14}$$

Thus, we have a well-defined map

$$\tau : H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))) \to H^3(G,\, k^\times) : z \mapsto \omega_z. \tag{4.15}$$

LEMMA 4.2.1. *Let G be a finite group, let $\mathcal{C}_1$, $\mathcal{C}_2$ be fusion categories, and let*

$$T_i : G \to \mathsf{Aut}(\mathcal{C}_i),\ i = 1, 2$$

be actions of G. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a tensor functor such that there exists a natural isomorphism of monoidal functors $\nu : T_2 \cong F \circ T_1 \circ F^{-1}$. Then F can be canonically extended to a tensor functor $\tilde{F} : \mathcal{C}_1 \rtimes G \to \mathcal{C}_2 \rtimes G$ given by

$$\tilde{F}(X \boxtimes g) = F(X) \boxtimes g, \qquad X \in \mathcal{C}_1,\, g \in G.$$

PROOF. This is straightforward. The tensor structure of $\tilde{F}$ is defined combining that of F and isomorphism ν. □

REMARK 4.2.2. Lemma 4.2.1 applies to our situation. Namely, take $\mathcal{C}_1 = \mathcal{Z}(\mathcal{D})_{pt} \boxtimes \mathcal{Z}(\mathcal{D})_{pt}$ with diagonal action of G, $\mathcal{C}_2 = \mathcal{Z}(\mathcal{D})_{pt}$, and $F = \otimes : \mathcal{Z}(\mathcal{D})_{pt} \boxtimes \mathcal{Z}(\mathcal{D})_{pt} \to \mathcal{Z}(\mathcal{D})_{pt}$ (the tensor structure of F is given by the braiding of $\mathcal{Z}(\mathcal{D})_{pt}$). Thus, we have a canonical grading preserving tensor functor

$$(\mathcal{Z}(\mathcal{D})_{pt} \boxtimes \mathcal{Z}(\mathcal{D})_{pt}) \rtimes G \to \mathcal{Z}(\mathcal{D})_{pt} \rtimes G. \tag{4.16}$$

PROPOSITION 4.2.3. *The canonical map*

$$\tau : H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))) \to H^3(G,\, k^\times) \tag{4.17}$$

described in (4.15) *is a group homomorphism.*

PROOF. Let $z_1,\, z_2 \in H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$, there are fusion subcategories

$$\mathcal{C}(G, \omega_{z_i}) \subset \mathcal{Z}(\mathcal{D})_{pt} \rtimes G : g \mapsto z_i(g) \boxtimes g, \quad \text{for } i = 1,\, 2.$$

We need to show that $\omega_{z_1}\omega_{z_2}$ and $\omega_{z_1+z_2}$ defined by (4.14) are cohomologous 3-cocycles on G. We will use Remark 4.2.2.

We have an embedding of $(G \times G)$-graded categories:

$$\mathcal{C}(G,\, \omega_{z_1}) \boxtimes \mathcal{C}(G,\, \omega_{z_2}) \subset (\mathcal{Z}(\mathcal{D})_{pt} \rtimes G) \boxtimes (\mathcal{Z}(\mathcal{D})_{pt} \rtimes G).$$

Passing to diagonal subcategories we get an embedding of G-graded categories:

$$\mathcal{C}(G,\, \omega_{z_1}\omega_{z_2}) \subset (\mathcal{Z}(\mathcal{D})_{pt} \boxtimes \mathcal{Z}(\mathcal{D})_{pt}) \rtimes G.$$

Applying functor (4.16) we see that $\mathcal{C}(G,\, \omega_{z_1}\omega_{z_2})$ is equivalent to $\mathcal{C}(G,\, \omega_{z_1+z_2}) \subset \mathcal{Z}(\mathcal{D})_{pt} \rtimes G$ as a G-graded tensor category. This implies the result. □

REMARK 4.2.4. Suppose that $\omega_{\mathcal{Z}(\mathcal{D})} \in H^3(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})),\, k^\times)$ is trivial (this is automatically true, e.g., when $\mathcal{Z}(\mathcal{D})$ is the representation category of a finite dimensional Hopf algebra). Then $\tau = 0$.

4.3. The kernel of the restriction homomorphism

Assume that the grading (4.1) is invariant (Definition 2.9.12) and injective (Definition 2.9.18). In this case there can be no non-trivial permutation of grading components when $\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}}(\Gamma) = Id_{\mathcal{D}}$. Consequently, $\Gamma|_{\mathcal{C}_g}$ is a $\mathcal{D}$-bimodule autoequivalence for each $g \in G$, and may be identified with an element in $\mathsf{Inv}(\mathcal{Z}(\mathcal{D}))$. Let us describe the kernel of (4.2).

THEOREM 4.3.1. *There is an exact sequence*

$$H^2(G,\, k^\times) \to Ker(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}}) \to H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))) \xrightarrow{\tau} H^3(G,\, k^\times). \tag{4.18}$$

where τ is the homomorphism (4.17).

PROOF. The conditions on the grading imply that any $S \in \mathrm{Ker}\,(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}})$ satisfies $S(\mathcal{C}_g) = \mathcal{C}_g$ for all $g \in G$.

Any autoequivalence $S \in \mathrm{Ker}\,(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}})$ restricts to a $\mathcal{D}$-bimodule autoequivalence of each $\mathcal{C}_g$ and so there is $Z_g \in \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))$, $g \in G$ such that S on homogeneous objects is given by

$$S(X_g) = Z_g \otimes X_g, \qquad X_g \in \mathcal{C}_g. \tag{4.19}$$

Existence of $\mathcal{D}$-bimodule isomorphisms $S(X_g) \otimes S(X_h) \xrightarrow{\sim} S(X_g \otimes X_h)$ along with isomorphism (4.5) implies that

$$Z_{gh} \cong Z_g \otimes T_g(Z_h) \tag{4.20}$$

for all $g, h \in G$, i.e., that

$$z_S : G \to \mathsf{Inv}(\mathcal{Z}(\mathcal{D})) : g \mapsto z(g) = Z_g \tag{4.21}$$

is a 1-cocycle. In order for the autoequivalence S of $\mathcal{C}$ described in (4.19) to be tensor, one needs to choose isomorphisms (4.20) satisfying the tensor functor axiom (2.4). It is straightforward to check that this is possible precisely when 3-cocycle ω_{z_S} is cohomologically trivial, i.e., when 1-cocycle z_S from (4.21) belongs to the kernel of homomorphism (4.15).

Finally, let us explain why the kernel of the homomorpshim $S \mapsto z_S$ is isomorphic to a quotient of $H^2(G, k^\times)$. Indeed, this kernel consists precisely of tensor autoeqivalences Γ of $\mathcal{C}$ such that F restricts to the identity tensor functor on $\mathcal{D}$ and to the identity $\mathcal{D}$-bimodule functor on every homogeneous component $\mathcal{C}_g$, $g \in G$. This implies that the tensor functor structure on Γ is an automorphism of the $\mathcal{D}$-bimodule and $\mathcal{D}$-balanced functor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$. Hence, it corresponds to a collection of automorphisms of $\mathcal{D}$-bimodule functors $M_{g,h} : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \to \mathcal{C}_{gh}$, i.e., the restriction of the tensor functor structure of Γ on $\mathcal{C}_g \times \mathcal{C}_h$ is given by a scalar $\nu(g, h)$, $g, h \in G$. The axioms of a tensor functor imply that ν is a 2-cocycle on G. Clearly, cohomologous cocycles give rise to isomorphic tensor functors. □

REMARK 4.3.2. The homomorphism $H^2(G, k^\times) \to \mathrm{Ker}\,(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}})$ is not typically injective. For instance, let Q be a finite group, and N a finite Abelian group. Consider an extension of Q by N

$$1 \to N \to G \to Q \to 1,$$

and consider corresponding the graded vector space $\mathcal{C}(G,1)$ as a Q-extension of $\mathcal{C}(N,1)$. In this case, the kernel of $H^2(G,k^\times) \to \mathrm{Ker}\,(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}})$ is described by the inflation-restriction exact sequence (See [**B**] for details).

Remark 4.3.3. Suppose that $\omega_{\mathcal{Z}(\mathcal{D})} \in H^3(\mathsf{Inv}(\mathcal{Z}(\mathcal{D})),\, k^\times)$ is trivial (this is automatic, e.g., when $\mathcal{D}$ is the representation category of a Hopf algebra). Then, in view of Remark 4.2.4 the exact sequence (4.18) becomes

$$H^2(G,\, k^\times) \to \mathrm{Ker}\,(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}}) \to H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))) \to 1, \tag{4.22}$$

and $\mathrm{Ker}\,(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}})$ is a quotient of the central extension corresponding to the cohomology class $\mu \in H^2(H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))),\, H^2(G,\, k^\times))$ given by

$$\mu(z,\, z')\,(g,\, h) = c(Z_g, Z'_h),$$

where $z : g \mapsto Z_g$, $z' : h \mapsto Z'_h$, $g, h \in G$, are elements of $H^1(G,\, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$, and c denotes the braiding of $\mathcal{Z}(\mathcal{D})$.

4.4. Quasi-tensor functors

In the following sections, we seek to reconstruct tensor autoequivalences component-wise from restriction data. In order to do that, we will need an intermediate step between k-linear functors (Definition 2.2.5), and tensor functors (Definition 2.3.16).

Definition 4.4.1. A *quasi-tensor functor* is a k-linear functor between tensor categories

$$F : \mathcal{C}_1 \to \mathcal{C}_2$$

with a natural *quasi-tensor isomorphism*

$$J(X,Y) : F(X) \otimes F(Y) \to F(X \otimes Y)$$

for each X, $Y \in \mathcal{C}_1$, which need not satisfy the associativity compatibility diagram (2.4).

REMARK 4.4.2. Let $(F_1, J_1) : \mathcal{C}_1 \to \mathcal{C}_2$ and $(F_2, J_2) : \mathcal{C}_2 \to \mathcal{C}_3$ be quasi-tensor functors between tensor categories. The composition

$$(F_2, J_2) \circ (F_1, J_1) : \mathcal{C}_1 \to \mathcal{C}_3$$

has quasi-tensor functor structure

$$F_2(J_1(X, Y))J_2(F_1(X), F_1(Y)) : F_2F_1(X) \otimes F_2F_1(Y) \to F_2F_1(X \otimes Y) \tag{4.23}$$

for each X, $Y \in \mathcal{C}_1$.

DEFINITION 4.4.3. Let $(F_1, J_1), (F_2, J_2) : \mathcal{C} \to \mathcal{C}'$ be quasi-tensor functors. A *morphism of quasi-tensor functors* $\sigma : (F_1, J_1) \to (F_2, J_2)$ consists of a natural transformation $\sigma : F_1 \to F_2$ satisfying

$$\begin{array}{ccc} F_1(X) \otimes F_1(Y) & \xrightarrow{J_1(X,Y)} & F_1(X \otimes Y) \\ \Big\downarrow{\scriptstyle \sigma(X)\otimes\sigma(Y)} & & \Big\downarrow{\scriptstyle \sigma(X\otimes Y)} \\ F_2(X) \otimes F_2(Y) & \xrightarrow[J_2(X,Y)]{} & F_2(X \otimes Y) \end{array}$$

for all X, $Y \in \mathcal{C}$.

DEFINITION 4.4.4. Quasi-tensor functors $(F, J) : \mathcal{C}_1 \to \mathcal{C}_2$ typically fail to satisfy the associativity compatibility diagram (2.4), however there is a natural automorphism

$$\Delta F : F(- \otimes (- \otimes -)) \to F(- \otimes (- \otimes -))$$

which satisfies

$$\Delta F(X, Y, Z)J(X, Y \otimes Z)J(Y, Z)\alpha_{F(Z),F(Y),F(Z)} = F(\alpha_{X,Y,Z})J(X \otimes Y, Z)J(X, Y)$$

for each X, Y, $Z \in \mathcal{C}_1$. We will call this the *defect* of (F, J).

REMARK 4.4.5. We will denote the defect of a quasi-tensor functor $(F, J) : \mathcal{C}_1 \to \mathcal{C}_2$ with a circle and an arrow, so that for each X, Y, $Z \in \mathcal{C}_1$, we write

$$\begin{array}{ccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & F(X) \otimes (F(Y) \otimes F(Z)) \\
\downarrow{\scriptstyle J(X,Y)} & & \downarrow{\scriptstyle J(Y,Z)} \\
F(X \otimes Y) \otimes F(Z) & \overset{\Delta F(X,Y,Z)}{\longrightarrow} & F(X) \otimes F(Y \otimes Z) \\
\downarrow{\scriptstyle J(X \otimes Y,Z)} & & \downarrow{\scriptstyle J(X,Y \otimes Z)} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z))
\end{array}$$

to indicate that the diagram

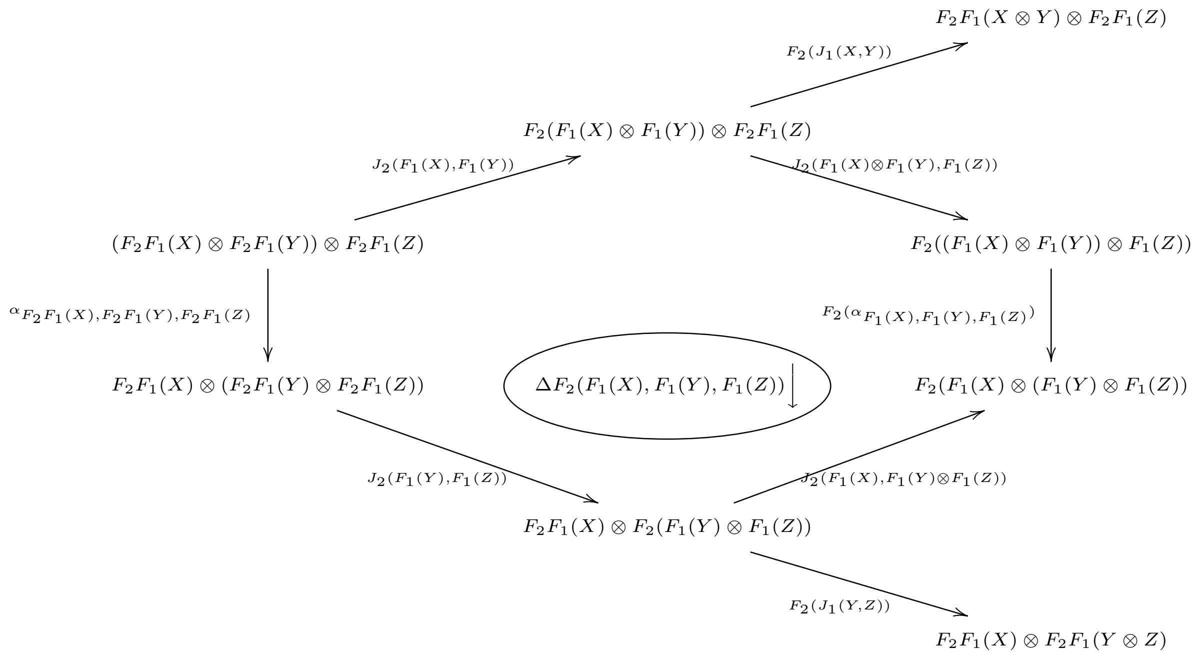
$$\begin{array}{ccccc}
(F(X) \otimes F(Y)) \otimes F(Z) & & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & & F(X) \otimes (F(Y) \otimes F(Z)) \\
\downarrow{\scriptstyle J(X,Y)} & & & & \downarrow{\scriptstyle J(Y,Z)} \\
F(X \otimes Y) \otimes F(Z) & & & & F(X) \otimes F(Y \otimes Z) \\
\downarrow{\scriptstyle J(X \otimes Y,Z)} & & & & \downarrow{\scriptstyle J(X,Y \otimes Z)} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z)) & \xleftarrow[\Delta F(X,Y,Z)]{} & F(X \otimes (Y \otimes Z))
\end{array}$$

commutes.

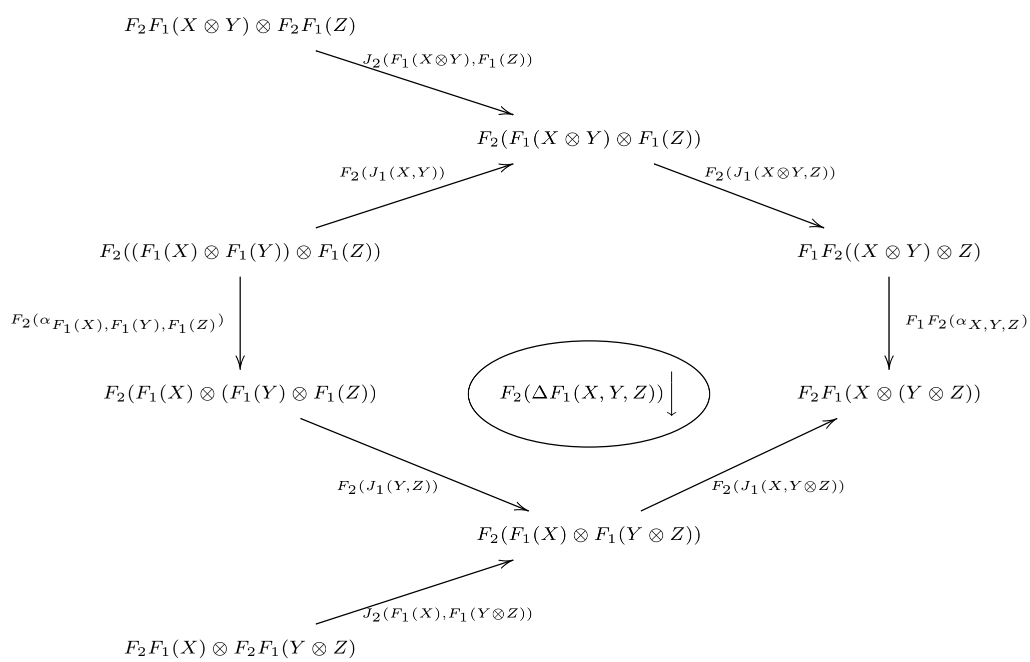
LEMMA 4.4.6. *Let $(F_1, J_1) : \mathcal{C}_1 \to \mathcal{C}_2$, and $(F_2, J_2) : \mathcal{C}_2 \to \mathcal{C}_3$ be quasi-tensor functors with defects ΔF_1 and ΔF_2. The defect of $(F_2, J_2) \circ (F_1, J_1)$ is*

$$\Delta F_2 F_1 = F_2(\Delta F_1) \circ F_2(J_1 \circ id_{F_1} \otimes J_1) \circ \Delta F_2 \left(F_1^{\times 3}\right) \circ F_2(id_{F_1} \otimes J_1^{-1} \circ J_1^{-1}). \tag{4.24}$$

Proof. Let X, Y, $Z \in \mathcal{C}_1$. We combine the (non-commuting) diagrams for the defect of (F_2, J_2):



and the defect of (F_1, J_1):



so that the upper and lower diagonal compositions are the quasi-tensor functor structure (4.23) for $(F_2, J_2) \circ (F_1, J_1)$. We see that the outer hexagon detects the defect $\Delta F_2 F_1$.

Trace the perimeters of the inner hexagons to determine the right hand side of the equation (4.24). Since each quadrilateral cell:

$$\begin{array}{ccc} F_2(F_1(X) \otimes F_1(Y)) \otimes F_2F_1(Z) & \xrightarrow{F_2(J_1(X,Y))} & F_2F_1(X \otimes Y) \otimes F_2F_1(Z) \\ \downarrow{\scriptstyle J_2(F_1(X)\otimes F_1(Y), F_1(Z))} & & \downarrow{\scriptstyle J_2(F_1(X\otimes Y), F_1(Z))} \\ F_2((F_1(X) \otimes F_1(Y)) \otimes F_1(Z)) & \xrightarrow[F_2(J_1(X,Y))]{} & F_2(F_1(X \otimes Y) \otimes F_1(Z)) \end{array}$$

and

$$\begin{array}{ccc} F_2F_1(X) \otimes F_2(F_1(Y) \otimes F_1(Z)) & \xrightarrow{J_2(F_1(X), F_1(Y)\otimes F_1(Z))} & F_2(F_1(X) \otimes (F_1(Y) \otimes F_1(Z))) \\ \downarrow{\scriptstyle F_2(J_1(Y,Z))} & & \downarrow{\scriptstyle F_2(J_1(Y,Z))} \\ F_2F_1(X) \otimes F_2F_1(Y \otimes Z) & \xrightarrow[J_2(F_1(X), F_1(Y\otimes Z))]{} & F_2(F_1(X) \otimes F_1(Y \otimes Z)) \end{array}$$

commutes by the naturality of the quasi-tensor isomorphisms J_1 and J_2, the outer perimeter determines the equation (4.24). □

4.5. The image of the restriction homomorphism

We continue to assume that the grading (4.1) is invariant (Definition 2.9.12). In this section we do not require that the grading is injective.

DEFINITION 4.5.1. Let $F \in \mathsf{Aut}(\mathcal{D})$ and $a \in \mathsf{Aut}(G)$, we say that a (quasi-) tensor autoequivalence $\Gamma : \mathcal{C} \to \mathcal{C}$ is a:

- *(quasi-) tensor extension* of F when there is an isomorphism $\Gamma|_{\mathcal{C}_e} \cong F$ of tensor functors.
- *(quasi-) tensor extension* of (F, a) when Γ extends F, and there are $\mathcal{D}$-bimodule equivalences $\Gamma|_{\mathcal{C}_g} : \mathcal{C}_g \to \mathcal{C}^F_{a(g)}$.

When Γ as above exists, we say that F (respectively (F, a)) *extends*.

DEFINITION 4.5.2. Let $\mathfrak{A}(\mathcal{C})$ denote the subgroup of $\mathsf{Aut}(\mathcal{D}) \times \mathsf{Aut}(G)$ consisting of pairs (F, a) for which there is an isomorphism of actions (i.e. a natural isomorphism between monoidal functors):

$$\begin{aligned}(4.25)\qquad &\bar{F}^{-1}\circ T_a\circ \bar{F}: Cat(G)\to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D})) : g\mapsto \bar{F}^{-1}T_{a(g)}\bar{F}=\Phi(\mathcal{C}^F_{a(g)})\\ \text{and}\qquad &T: Cat(G)\to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D})) : g\mapsto T_g=\Phi(\mathcal{C}_g).\end{aligned}$$

The action $T : Cat(G) \to \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{D}))$ is the canonical action (2.23) determined by the grading.

REMARK 4.5.3. When the grading (4.1) is injective (Definition 2.9.18), the automorphism $a \in \mathsf{Aut}(G)$ for each $F \in \mathsf{Aut}(\mathcal{D})$ when it exists, is necessarily unique, and satisfies $\mathcal{C}_g \cong \mathcal{C}^F_{a(g)}$ for each $g \in G$. In this case $\mathfrak{A}(\mathcal{C})$ may be identified with a subgroup of $\mathsf{Aut}(\mathcal{D})$.

LEMMA 4.5.4. *Each pair $(F, a) \in \mathfrak{A}(\mathcal{C})$ canonically induces a tensor autoequivalence of the category $\mathcal{Z}(\mathcal{D})_{pt} \rtimes G$ described at* (4.12)*. The action of $(F, a) \in \mathfrak{A}(\mathcal{C})$ is as follows:*

$$(F, a) : (X, g) \mapsto (\mathsf{ind}(F)(X), a(g))$$

with tensor functor structure determined by the isomorphism $\mathsf{ind}(F) \circ T_g \cong T_{a(g)} \circ \mathsf{ind}(F)$, and the tensor functor structure of $\mathsf{ind}(F)$.

PROOF. This follows immediately from Lemma 4.2.1. □

REMARK 4.5.5. The cohomology groups $H^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$ and $H^3(G, k^\times)$ are each right $\mathfrak{A}(\mathcal{C})$-modules as follows: $\mathsf{Aut}(G)$ acts on both $H^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$ and $H^3(G, k^\times)$ functorially (Remark 2.1.4), and each tensor autoequivalence $F \in \mathsf{Aut}(\mathcal{D})$ induces (Definition 3.1.1) a group automorphism of $\mathsf{Inv}(\mathcal{Z}(\mathcal{D}))$. The action of $\mathfrak{A}(\mathcal{C})$ then follows from the inclusion $\mathfrak{A}(\mathcal{C}) \subset \mathsf{Aut}(\mathcal{D}) \times \mathsf{Aut}(G)$.

COROLLARY 4.5.6. *The homomorphism* (4.15)

$$\tau : H^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))) \to H^3(G, k^\times)$$

is a morphism of right $\mathfrak{A}(\mathcal{C})$-modules.

Proof. $\mathfrak{A}(\mathcal{C})$ acts on the set of subcategories (4.14). We observe that this action coincides with the module action on $Z^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$, where each cocycle is identified with the subcategory it determines. Similarly, we observe that the induced action of $\mathfrak{A}(\mathcal{C})$ on the third cohomology classes associated to each subcategory is precisely that of $\mathfrak{A}(\mathcal{C})$ on $H^3(G, k^\times)$. Since the homomorphism (4.15) is specified by identifying first cohomology classes in $H^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$ with third cohomology classes in $H^3(G, k^\times)$ through the subcategories (4.14), it follows that τ is $\mathfrak{A}(\mathcal{C})$-module. □

Proposition 4.5.7. *A pair $F \in \mathsf{Aut}(\mathcal{D})$, $a \in \mathsf{Aut}(G)$ determines an element $(F, a) \in \mathfrak{A}(\mathcal{C})$ if and only if there exist choices of $\mathcal{D}$-bimodule equivalences*

$$F_g : \mathcal{C}_g \xrightarrow{\sim} \mathcal{C}^F_{a(g)}, \qquad g \in G, \tag{4.26}$$

and natural isomorphisms of $\mathcal{D}$-bimodule functors

$$\eta_{g,h} : M^F_{a(g),a(h)} \circ (F_g \boxtimes_{\mathcal{D}} F_h) \xrightarrow{\sim} F_{gh} \circ M_{g,h}, \qquad g, h \in G \tag{4.27}$$

where the functors $M^F_{a(g),a(h)}$ and $M_{g,h}$ are the $\mathcal{D}$-bimodule equivalences (2.19) *induced by the tensor product, the functor $F_e := F$, and the isomorphisms $\eta_{e,g}$ and $\eta_{g,e}$ are the left and right module functor structure of F_g for each $g \in G$.*

Proof. The monoidal equivalence $\underline{\mathsf{Aut}}^{br}(\mathcal{Z}(\mathcal{D})) \cong \underline{\mathsf{BrPic}}(\mathcal{D})$ identifies the isomorphism of monoidal functors (4.25) with the isomorphism classes of $\mathcal{D}$-bimodule equivalences (4.26) for which there exist isomorphisms (4.27).

A choice of $\mathcal{D}$-bimodule equivalences (4.26) determine a family of isomorphisms (see Remark 4.1.1)

$$\Phi(F_g) : \Phi(\mathcal{C}_g) \to \Phi(\mathcal{C}^F_{a(g)}) \quad g \in G$$

and the natural isomorphisms (4.27) ensure that these determine an isomorphism of monoidal functors. □

Remark 4.5.8. The choices (4.26) and (4.27) are equivalent to the construction of a quasi-tensor functor (Definition 4.5.1) extension of $(F, a) \in \mathfrak{A}(\mathcal{C})$ to

$$\left(\bigoplus_{g\in G} F_g, \bigoplus_{g,h\in G} \eta_{g,h}\right).$$

Let $X := \bigoplus_{g\in G} X_g$ and $Y := \bigoplus_{g\in G} Y_g$ be objects in $\mathcal{C}$ such that X_g, $Y_g \in \mathcal{C}_g$ for each $g \in G$, then

$$\bigoplus_{g\in G} F_g(X) := \bigoplus_{g\in G} F_g(X_g)$$

and

$$\left(\bigoplus_{g\in G} F_g(X_g)\right) \otimes \left(\bigoplus_{h\in G} F_h(Y_h)\right) = \bigoplus_{g,h\in G} (F_g(X_g) \otimes F_h(Y_h)) \xrightarrow{\bigoplus_{g,h\in G} \eta_{g,h}} \bigoplus_{g,h\in G} F_{gh}(X_g \otimes Y_h).$$

This satisfies the tensor functor structure diagram (2.4) precisely when its defect (Definition 4.4.4) is trivial.

Corollary 4.5.9. *The image* $Im\left(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}}\right)$ *of the restriction homomorphism* (4.2) *is a subgroup of the image of the canonical projection* $\mathfrak{A}(\mathcal{C}) \to \mathsf{Aut}(\mathcal{D})$*. When the grading* (4.1) *is injective,* $Im\left(\mathsf{Res}^{\mathcal{C}}_{\mathcal{D}}\right) \subset \mathfrak{A}(\mathcal{C})$ *(Remark 4.5.3).*

Theorem 4.5.10. *There is a canonical map* (4.31)

$$\kappa_{\mathcal{C}} : \mathfrak{A}(\mathcal{C}) \to Coker(\tau) \tag{4.28}$$

with the following properties:

(i) $\kappa_{\mathcal{C}}$ *is a* 1*-cocycle in the following sense: for* (F, a), $(f, b) \in \mathfrak{A}(\mathcal{C})$ *one has*

$$\kappa_{\mathcal{C}}(Ff, ab)_{g,h,i} = \kappa_{\mathcal{C}}(F, a)_{b(g),b(h),b(i)}\kappa_{\mathcal{C}}(f, b)_{g,h,i}.$$

(ii) *An element* $(F, a) \in \mathfrak{A}(\mathcal{C})$ *extends if and only if* $\kappa_{\mathcal{C}}(F, a)$ *is trivial.*

Proof. See Section 4.6. □

EXAMPLE 4.5.11. Let $\mathcal{C} = \mathcal{C}(G,\, \omega)$. Then $\mathcal{D} = \mathsf{Vec}$ is the trivial category,

$$\mathsf{Aut}(\mathcal{C};\, \mathcal{D}) = H^2(G,\, k^\times), \qquad \textit{and} \qquad \mathfrak{A}(\mathcal{C}(G,\, \omega)) = \mathsf{Aut}(G).$$

The map $\kappa_{\mathcal{C}} : \mathsf{Aut}(G) \to H^3(G,\, k^\times)$ is given by

$$\kappa_{\mathcal{C}}(a) = \frac{\omega \circ (a \times a \times a)}{\omega}$$

and there is a short exact sequence

$$1 \to H^2(G,\, k^\times) \to \mathsf{Aut}(\mathcal{C}(G,\omega)) \to \mathsf{Stab}(\omega) \to 1.$$

4.6. Proof of Theorem 4.5.10

In this section, we make choices $(F, a) \in \mathfrak{A}(\mathcal{C})$, $\mathcal{D}$-bimodule equivalences (4.26), and natural isomorphisms (4.27). From this data, we construct the quasi-tensor extension (Definition 4.5.1)

$$\Gamma := \left(\bigoplus_{g \in G} F_g,\, \bigoplus_{g,h \in G} \eta_{g,h} \right) \tag{4.29}$$

described in Remark 4.5.8. We examine its defect $\Delta\Gamma$ (Definition 4.4.4).

For our purposes, it is convenient to drop the multiplication functors, and Deligne's tensor product from our notation. We will keep track of their presence with parenthesis. For example, the $\mathcal{D}$-bimodule equivalence

$$M^F_{a(gh),a(i)} \circ M^F_{a(g),a(h)} \boxtimes_{\mathcal{D}} Id_{\mathcal{C}^F_{a(i)}} \circ F_g \boxtimes_{\mathcal{D}} F_h \boxtimes_{\mathcal{D}} F_i \;:\; \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{C}_i \to \mathcal{C}^F_{a(ghi)}$$

will be expressed as $(F_g F_h) F_i$, and similarly

$$F_{ghi} \circ M_{gh,i} \circ M_{g,h} \boxtimes_{\mathcal{D}} Id_{\mathcal{C}_i} \;:\; \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{C}_i \to \mathcal{C}^F_{a(ghi)}$$

will be expressed as $F_{(gh)i}$ for each $g,\, h,\, i \in G$.

Where the appropriate application of a natural isomorphism is clear from context, we will indicate its presence rather than write out the full expression. For instance, we write

$$(F_g F_h)F_i \xrightarrow{\eta_{g,h}} F_{(gh)}F_i$$

to indicate the presence of the natural isomorphism

$$\begin{array}{c} M^F_{a(gh),a(i)} \circ M^F_{a(g),a(h)} \boxtimes_{\mathcal{D}} Id_{\mathcal{C}^F_{a(i)}} \circ F_g \boxtimes_{\mathcal{D}} F_h \boxtimes_{\mathcal{D}} F_i \ . \\ \Big\downarrow {\scriptstyle M^F_{a(gh),a(i)}(\eta_{g,h}\boxtimes_{\mathcal{D}} F_i)} \\ M^F_{a(gh),a(i)} \circ F_{gh} \boxtimes_{\mathcal{D}} F_i \circ M_{g,h} \boxtimes_{\mathcal{D}} Id_{\mathcal{C}_i} \end{array}$$

Lemma 4.6.1. *Let* Γ *be as above* (4.29), *there is a* 3*-cochain* $\kappa \in C^3(G, k^\times)$ *satisfying*

$$\Delta\Gamma(X_g, X_h, X_i) = \kappa_{g,h,i} id_{\Gamma(X_g \otimes (X_h \otimes X_i))} \tag{4.30}$$

for each $g, h, i \in G$*, and each* $X_g \in \mathcal{C}_g$*,* $X_h \in \mathcal{C}_h$*,* $X_i \in \mathcal{C}_i$*.*

Proof. We consider the natural automorphism $\Delta\Gamma$ as a family of automorphisms of $\mathcal{D}$-bimodule equivalences:

$$\Delta\Gamma_{g,h,i} : F_{g(hi)} \to F_{g(hi)}, \qquad g,\ h,\ i \in G,$$

each of which may be identified with a scalar $\kappa_{g,h,i} \in k^\times$. □

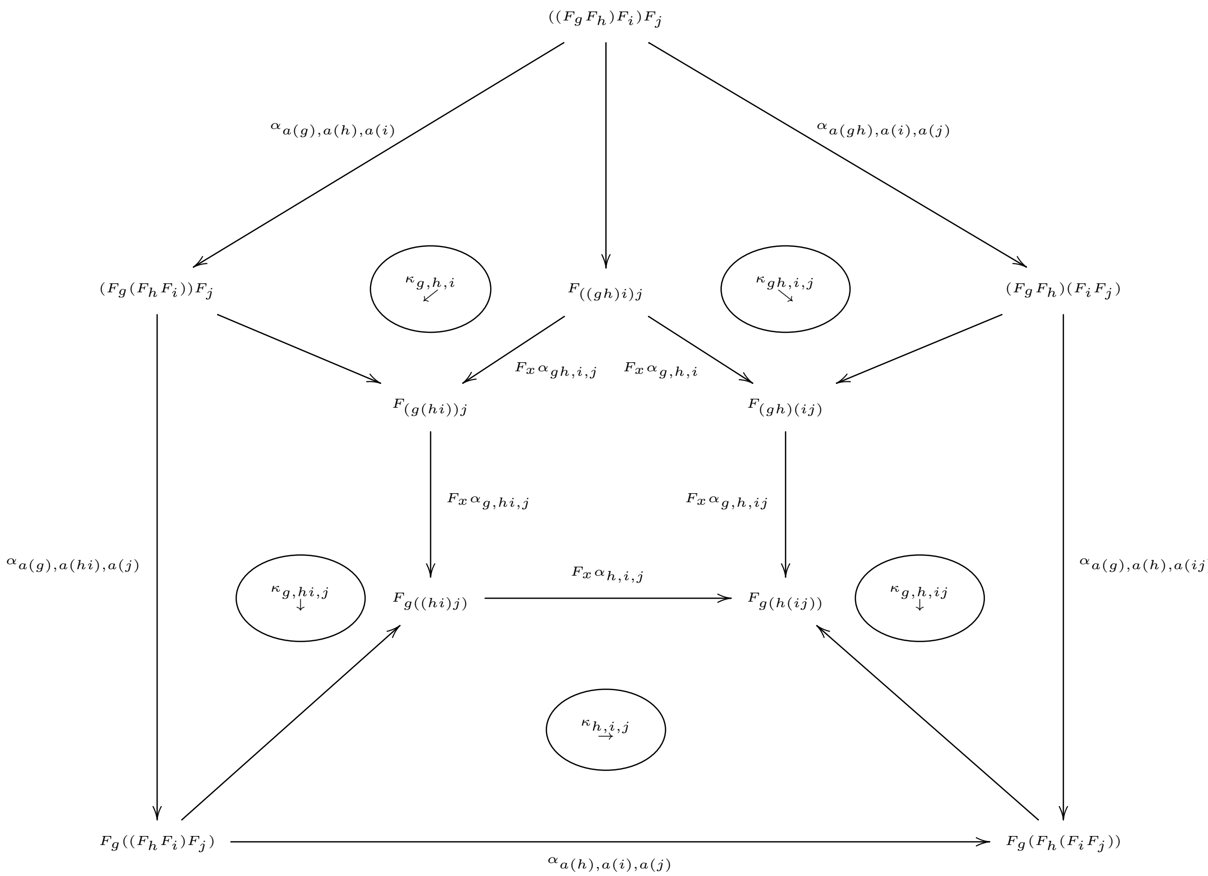
Corollary 4.6.2. *The* 3*-cochain* $\kappa \in C^3(G, k^\times)$ *determined by* (4.30) *is trivial if and only if the functor* (4.29) $\Gamma : \mathcal{C} \to \mathcal{C}$ *is a tensor autoequivalence which extends* $F \in \mathsf{Aut}(\mathcal{D})$ *(Definition 4.5.1).*

Lemma 4.6.3. *For fixed choices* $(F, a) \in \mathfrak{A}(\mathcal{C})$*,* (4.26) *and* (4.27)*, the* 3*-cochain*

$$\kappa : G^3 \to k^\times$$

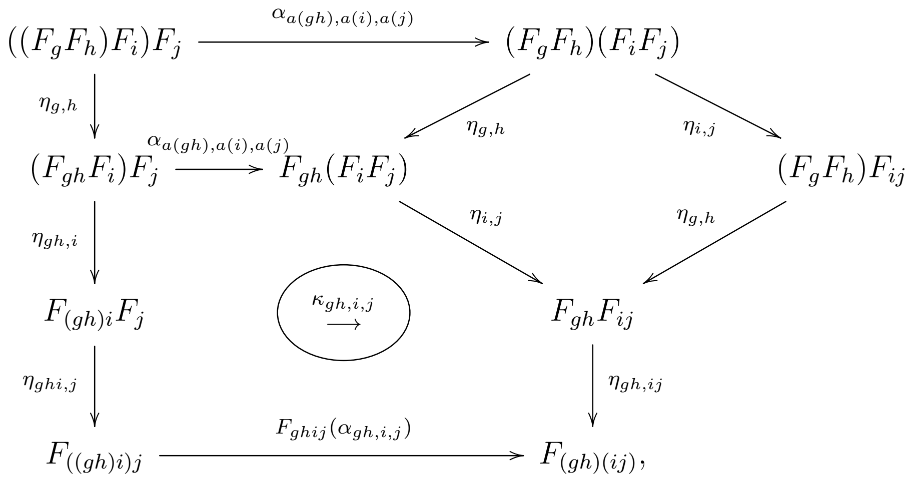
determined by (4.30) *is a* 3*-cocycle in* $Z^3(G, k^\times)$*.*

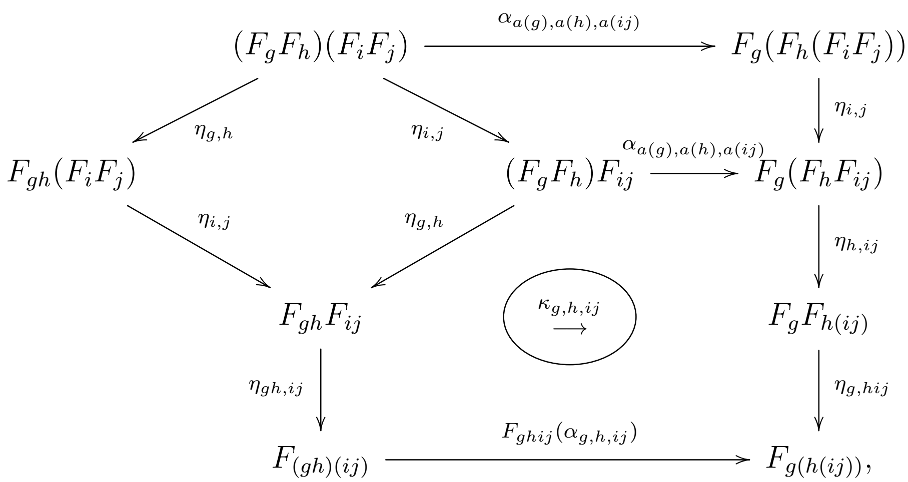
Proof. The 3-cocycle condition may be seen in the following pentagon diagram of natural isomorphisms between $\mathcal{D}$-bimodule functors. We caution that the quadrilateral faces only commute up to a scalar (Lemma 4.6.1), which we denote with our defect notation (Remark 4.4.5).

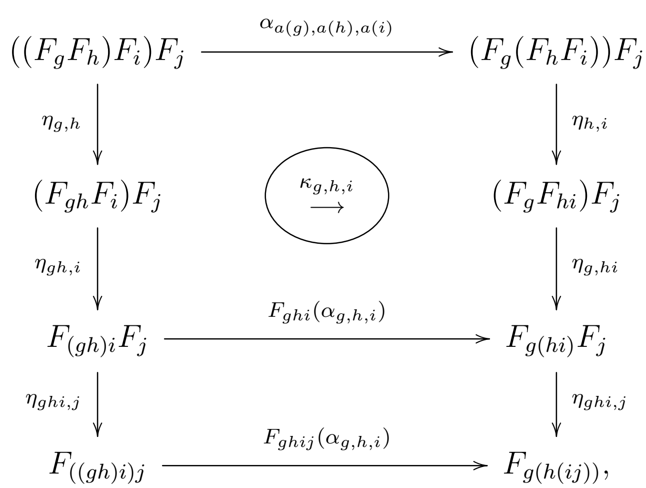


where $x = ghij$.

The outer and inner pentagons commute by the pentagon axiom (2.3). Each trapezoidal face commutes up to its κ factor, expressed in the diagrams below (one diagram for each of the five faces).


$((F_gF_h)F_i)F_j$
$\alpha_{a(gh),a(i),a(j)}$
$(F_gF_h)(F_iF_j)$
$\eta_{g,h}$
$\eta_{g,h}$
$\eta_{i,j}$
$(F_{gh}F_i)F_j$
$\alpha_{a(gh),a(i),a(j)}$
$F_{gh}(F_iF_j)$
$(F_gF_h)F_{ij}$
$\eta_{gh,i}$
$\eta_{i,j}$
$\eta_{g,h}$
$F_{(gh)i}F_j$
$\kappa_{gh,i,j}$
$F_{gh}F_{ij}$
$\eta_{ghi,j}$
$\eta_{gh,ij}$
$F_{((gh)i)j}$
$F_{ghij}(\alpha_{gh,i,j})$
$F_{(gh)(ij)}$,


$(F_gF_h)(F_iF_j)$
$\alpha_{a(g),a(h),a(ij)}$
$F_g(F_h(F_iF_j))$
$\eta_{g,h}$
$\eta_{i,j}$
$\eta_{i,j}$
$F_{gh}(F_iF_j)$
$(F_gF_h)F_{ij}$
$\alpha_{a(g),a(h),a(ij)}$
$F_g(F_hF_{ij})$
$\eta_{i,j}$
$\eta_{g,h}$
$\eta_{h,ij}$
$F_{gh}F_{ij}$
$\kappa_{g,h,ij}$
$F_gF_{h(ij)}$
$\eta_{gh,ij}$
$\eta_{g,hij}$
$F_{(gh)(ij)}$
$F_{ghij}(\alpha_{g,h,ij})$
$F_{g(h(ij))}$,


$((F_gF_h)F_i)F_j$
$\alpha_{a(g),a(h),a(i)}$
$(F_g(F_hF_i))F_j$
$\eta_{g,h}$
$\eta_{h,i}$
$(F_{gh}F_i)F_j$
$\kappa_{g,h,i}$
$(F_gF_{hi})F_j$
$\eta_{gh,i}$
$\eta_{g,hi}$
$F_{(gh)i}F_j$
$F_{ghi}(\alpha_{g,h,i})$
$F_{g(hi)}F_j$
$\eta_{ghi,j}$
$\eta_{ghi,j}$
$F_{((gh)i)j}$
$F_{ghij}(\alpha_{g,h,i})$
$F_{g(h(ij))}$,

$$
\begin{array}{ccc}
(F_g(F_hF_i))F_j & \xrightarrow{\alpha_{a(g),a(hi),a(j)}} & F_g((F_hF_i)F_j) \\
\downarrow{\scriptstyle \eta_{h,i}} & & \downarrow{\scriptstyle \eta_{h.i}} \\
(F_gF_{hi})F_j & \xrightarrow{\alpha_{a(g),a(hi),a(j)}} & F_g(F_{hi}F_j) \\
\downarrow{\scriptstyle \eta_{g,hi}} & & \downarrow{\scriptstyle \eta_{hi,j}} \\
F_{g(hi)}F_j & \overset{\kappa_{g,hi,j}}{\longrightarrow} & F_gF_{(hi)j} \\
\downarrow{\scriptstyle \eta_{ghi,j}} & & \downarrow{\scriptstyle \eta_{g,hij}} \\
F_{(g(hi))j} & \xrightarrow{F_{ghij}(\alpha_{g,hi,j})} & F_{g((hi)j)},
\end{array}
$$

$$
\begin{array}{ccc}
F_g((F_hF_i)F_j) & \xrightarrow{\alpha_{a(h),a(i),a(j)}} & F_g(F_h(F_iF_j)) \\
\downarrow{\scriptstyle \eta_{h,i}} & & \downarrow{\scriptstyle \eta_{i,j}} \\
F_g(F_{hi}F_j) & \overset{\kappa_{h,i,j}}{\longrightarrow} & F_g(F_hF_{ij}) \\
\downarrow{\scriptstyle \eta_{hi,j}} & & \downarrow{\scriptstyle \eta_{h,ij}} \\
F_gF_{(hi)j} & \xrightarrow{F_{hij}(\alpha_{h,i,j})} & F_gF_{h(ij)} \\
\downarrow{\scriptstyle \eta_{g,hij}} & & \downarrow{\scriptstyle \eta_{g,hij}} \\
F_{g((hi)j)} & \xrightarrow{F_{ghij}(\alpha_{h,i,j})} & F_{g(h(ij))}.
\end{array}
$$

□

Lemma 4.6.4. *Given a different choice*

$$\eta_{g,h} : M^F_{a(g),a(h)} \circ F_g \boxtimes_{\mathcal{D}} F_h \xrightarrow{\sim} F_{gh} \circ M_{g,h} \qquad g, h \in G$$

of isomorphism (4.27)*, the resulting κ differs from the original by a* 3*-coboundary.*

Proof. Any two choices of isomorphisms (4.27) differ by a family of automorphisms of $\mathcal{D}$-bimodule equivalences

$$F_{gh} \circ M_{g,h}$$

for each $g,\, h \in G - \{e\}$. Automorphisms of $\mathcal{D}$-bimodule equivalences may be identified with elements of $k^\times$. For any two choices η and η' of isomorphisms (4.27), there is a 2-cochain $\sigma \in C^2(G, k^\times)$ such that $\eta'_{g,h} = \sigma_{g,h}\eta_{g,h}$ for each $g,\, h \in G$.

We consider now κ' determined (4.30) by this new choice of isomorphisms (4.27)

$$\begin{array}{ccc}
(F_g F_h)F_i & \xrightarrow{\alpha_{a(g),a(h),a(i)}} & F_g(F_h F_i) \\
\downarrow{\scriptstyle \sigma_{g,h}\eta_{g,h}} & & \downarrow{\scriptstyle \sigma_{h,i}\eta_{h,i}} \\
F_{gh}F_i & \overset{\kappa'_{g,h,i}}{\longrightarrow} & F_g F_{hi} \\
\downarrow{\scriptstyle \sigma_{gh,i}\eta_{gh,i}} & & \downarrow{\scriptstyle \sigma_{g,hi}\eta_{g,hi}} \\
F_{(gh)i} & \xrightarrow[F_{ghi}(\alpha_{g,h,i})]{} & F_{g,(hi)}
\end{array}$$

since these maps are k-linear, we get

$$\begin{array}{ccc}
(F_g F_h)F_i & \xrightarrow{\alpha_{a(g),a(h),a(i)}} & F_g(F_h F_i) \\
\downarrow{\scriptstyle \eta_{g,h}} & & \downarrow{\scriptstyle \eta_{h,i}} \\
F_{gh}F_i & \overset{d^2(\sigma)\kappa'_{g,h,i}}{\longrightarrow} & F_g F_{hi} \\
\downarrow{\scriptstyle \eta_{gh,i}} & & \downarrow{\scriptstyle \eta_{g,hi}} \\
F_{(gh)i} & \xrightarrow[F_{ghi}(\alpha_{g,h,i})]{} & F_{g,(hi)}
\end{array}$$

and $\kappa'_{g,h,i} = d^2(\sigma)\kappa_{g,h,i}$ for each $g,\, h,\, i \in G$. □

Thus a pair $(F,\, a) \in \mathfrak{A}(\mathcal{C})$, and choice of equivalences (4.26), determine a cohomology class $[\kappa] \in H^3(G,\, k^\times)$ which does not depend on the choice of isomorphisms (4.27).

Corollary 4.6.5. *When the third cohomology class $[\kappa] \in H^3(G, k^\times)$ determined by $(F, a) \in \mathfrak{A}(\mathcal{C})$ and (4.26) is trivial, there exists a choice of isomorphisms (4.27) so that $\kappa \in Z^3(G, k^\times)$ is trivial.*

Remark 4.6.6. Fix $(F, a) \in \mathfrak{A}(\mathcal{C})$, the choice of (isomorphism classes of) $\mathcal{D}$-bimodule equivalences (4.26) is in a torsor over $H^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$. Explicitly, for any quasi-tensor functors Γ and Γ' extending (F, a), there is a quasi-tensor functor S which extends $(Id_{\mathcal{D}}, id_G) \in \mathfrak{A}(\mathcal{C})$ so that there is an isomorphism

$$\Gamma' \cong \Gamma S$$

as quasi-tensor functors (Definition 4.4.3). As in (4.19), S restricts to a $\mathcal{D}$-bimodule autoequivalence of each $\mathcal{C}_g$, so that there is $Z_g \in \mathsf{Inv}(\mathcal{Z}(\mathcal{D}))$ such that

$$S(X_g) = Z_g \otimes X_g, \qquad X_g \in \mathcal{C}_g,$$

for each $g \in G$. The quasi-tensor functor structure of S, and isomorphisms (4.5) ensure that

$$z : G \to \mathsf{Inv}(\mathcal{Z}(\mathcal{D})) : g \mapsto Z_g$$

is a 1-cocycle.

Lemma 4.6.7. *Changing the choice of $\mathcal{D}$-bimodule equivalences* (4.26) *by an element $z \in H^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$ changes the cohomology class $[\kappa] \in H^3(G, k^\times)$ to $[\kappa]\tau(z)$.*

Proof. Let $z \in H^1(G, \mathsf{Inv}(\mathcal{Z}(\mathcal{D})))$. Take a quasi-tensor equivalence S (as in Remark 4.6.6) which restricts to the left action

$$L_{Z_g} : \mathcal{C}_g \to \mathcal{C}_g : X_g \mapsto Z_g \otimes X_g$$

on each component so that

$$S := \left(\bigoplus_{g \in G} L_{Z_g}, \bigoplus_{g,h \in G} \zeta_{g,h} \right).$$

Let $X_g \in \mathcal{C}_g$, $X_h \in \mathcal{C}_h$ and $X_i \in \mathcal{C}_i$. We take $\mathcal{C}$ to be strict, and compute directly using isomorphisms (4.5), that the defect (Definition 4.4.4) in the associativity coherence diagram:

$$\begin{array}{ccc}
L_{Z_g}(X_g) \otimes L_{Z_h}(X_h) \otimes L_{Z_i}(X_i) & \xrightarrow{=} & L_{Z_g}(X_g) \otimes L_{Z_h}(X_h) \otimes L_{Z_i}(X_i) \\
\downarrow{\scriptstyle \zeta_{g,h}} & & \downarrow{\scriptstyle \zeta_{h,i}} \\
L_{Z_{gh}}(X_g \otimes X_h) \otimes L_{Z_i}(X_i) & \overset{\Delta S(X_g, X_h, X_i)}{\longrightarrow} & L_{Z_g}(X_g) \otimes L_{Z_{hi}}(X_h \otimes X_i) \\
\downarrow{\scriptstyle \zeta_{gh,i}} & & \downarrow{\scriptstyle \zeta_{g,hi}} \\
L_{Z_{ghi}}(X_g \otimes X_h \otimes X_i) & \xrightarrow[=]{} & L_{Z_{ghi}}(X_g \otimes X_h \otimes X_i)
\end{array}$$

is determined by the choice of isomorphisms $Z_g \otimes T_g(Z_h) \cong Z_{gh}$, $g, h \in G$, and an isomorphism:

$$Z_g \otimes T_g(Z_h) \otimes T_{gh}(Z_i) \cong Z_g \otimes T_g(Z_h \otimes T_h(Z_i)).$$

which are equivalent to the associativity constraint of (4.14), and determines (4.17) the cohomology class $\tau(z)$. The result follows from Lemma 4.24 and k-linearity of morphisms.

□

Thus, we have a well defined map

$$\kappa_{\mathcal{C}} : \mathfrak{A}(\mathcal{C}) \to \mathrm{Coker}\,(\tau) : (F,\, a) \mapsto [\kappa] + \mathrm{Im}(\tau). \tag{4.31}$$

Corollary 4.6.8. *When $\kappa_{\mathcal{C}}(F, a) \in$ Coker(τ) is trivial, there exists a choice of module equivalences* (4.26) *for which $[\kappa] \in H^3(G, k^\times)$ is trivial.*

Proof of Theorem 4.5.10(i).

Let $(F, a), (F', a') \in \mathfrak{A}(\mathcal{C})$, pick quasi-tensor extensions Γ and Γ' for each pair. Consider the defect $\Delta\Gamma'\Gamma$ as a family of automorphisms of $\mathcal{D}$-bimodule equivalences

$$\Delta\Gamma'\Gamma_{g,h,i} : F'_{a(ghi)} F_{g(hi)} \to F'_{a(ghi)} F_{g(hi)}$$

for each g, h, $i \in G$. By Lemma 4.6.1 and equation (4.24), this is determined by the scalar $\kappa'_{a(g),a(h),a(i)}\kappa_{g,h,i}$, which determines $\kappa_{\mathcal{C}}(F'F, a'a) = \kappa_{\mathcal{C}}(F', a')^{(F,a)}\ \kappa_{\mathcal{C}}(F, a) \in \operatorname{Coker}\tau$.

Proof of Theorem 4.5.10(ii). This follows from Corollary 4.6.8, Corollary 4.6.5, and Corollary 4.6.2.

Bibliography

[A] A. Adem, *Cohomology of finite groups*, 2nd edition, Springer, (2004).

[B] K. Brown, *Cohomology of groups*, Graduate texts in mathematics, **87** Springer-Verlag (1982).

[BK] B. Bakalov and A. Kirillov Jr. *Lectures on tensor categories and modular functors*, University Series Lectures, vol. 21, AMS (2001).

[BN] C. Bontea, D. Nikshych, *On the Brauer-Picard group of a finite symmetric tensor category*, J. Algebra, **440** (2015), 187–218.

[C] G. Carnovale *The Brauer group of modified supergroup algebras*, J. Algebra **305** (2006), no. 2, 993–1036.

[De] P. Deligne, *Catégories tannakiennes*, In: "The Grothendieck Festschrift, Vol. II" Progr. Math. **87** (1990), 111-195.

[DPR] R. Dijkgraaf, V. Pasquier, and P. Roche, *Quasi-Hopf algebras, group cohomology, and orbifold models*, Nuclear Phys. B, Proc. Suppl., **18**, (1991), no. 2, 60–72.

[DGNO1] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, Group-theoretical properties of nilpotent modular categories. e-print *arXiv:0704.0195* (2007).

[DGNO2] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, *On braided fusion categories I*, Selecta Mathematica, **16** (2010), no. 1, 1–119.

[DMNO] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik, *The Witt group of non-degenerate braided fusion categories*, Journal für die reine und angewandte Mathematik, **677** (2013), 135–177.

[DN] A. Davydov, D. Nikshych, *The Picard crossed module of a braided tensor category*, Algebra and Number Theory,**7** (2013), no. 6, 1365-1403.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, **205**, American Mathematical Society (2015).

[ENO1] P. Etingof, D. Nikshych, and V. Ostrik, *Fusion categories and homotopy theory*, Quantum Topology, **1** (2010), no. 3, 209–273.

[ENO2] P. Etingof, D. Nikshych, and V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, Adv. Math, **226** (2011), no. 1, 176–205

[GP] C. Galindo, J. Plavnik, *Tensor functors between Morita duals of fusion categories*, e-print *arXiv:1407.2783* (2014).

[GNN] S. Gelaki, D. Naidu, and D. Nikshych, *Centers of graded fusion categories*, Algebra and Number Theory, **3** (2009), no. 8, 959–990.

[GN] S. Gelaki, D. Nikshych, Nilpotent fusion categories. *Advances in Mathematics*, **217** (2008), no. 3, 1053–1071.

[GMN] C. Goff, G. Mason, and S.-H. Ng, *On the gauge equivalence of twisted quantum doubles of elementary abelian and extra-special* 2*-groups*. J. Algebra, **312** (2007), no. 2, 849-875.

[JS] A. Joyal, R. Street *Braided tensor categories*. Adv. Math., **102** 20-78,(1993).

[K1] G. Karpilovsky, *The Schur Multiplier*. Oxford University Press, New York (1987).

[K2] G. Karpilovsky, *Group representations Volume 2*. Mathematical Studies, **177**, North-Holland (1993).

[KP] G. Kac, V. Paljutkin. *Finite ring groups*, Trans. Moscow Math. Soc., (1966), 251–294.

[M] G. Mason, *Reed-Muller codes, the fourth cohomology group of a finite group, and the β-invariant*, J. Algebra, **312** (2007), no.$\tilde{1}$, 218–227.

[Ma] S. MacLane, *Categories for the working mathematician*, Graduate texts in mathematics, **5**, 2nd edition, Springer-Verlag (1998).

[Mi] B. Mitchell, *The full imbedding theorem*, Amer. J. Math. **86** (1964), 619-637.

[Mu1] M. Müger, *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Algebra **180** (2003), 81-157.

[Mu2] M. Müger, *From subfactors to categories and topology II. The quantum double of tensor categories and subfactors*, J. Pure Appl. Algebra **180** (2003), no. 1-2, 159-219.

[MM] E. Meir, E. Musicantov, *Module categories over graded fusion categories*, J. Pure and Applied Algebra, **216** (2012), no. 11, 2449–2466.

[MaN] I. Marshall, D. Nikshych, *On the Brauer-Picard groups of fusion categories*, e-print arXiv:1603.04318 (2016).

[MN] G. Mason, and S.-H. Ng, *Group Cohomology and Gauge Equivalence of some Twisted Quantum Doubles*, Trans. Amer. Math. Soc., **353** (2001), 3465–3509.

[N] D. Naidu, *Categorical Morita equivalence for group-theoretical categories*, Comm. Algebra **35**, (2007), no. 11, 3544–3565.

[NN] D. Naidu, D. Nikshych, *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*, Comm. Math. Phys., **279** (2008), 845-872.

[NNW] D. Naidu, D. Nikshych, and S. Witherspoon, *Fusion subcategories of representation categories of twisted quantum doubles of finite groups*, International Mathematics Research Notices (2009), no. 22, 4183–4219.

[NR] D. Nikshych, B. Riepel, *Categorical Lagrangian Grassmannians and Brauer-Picard groups of pointed fusion categories*, J. Algebra, **411** (2014), 191–214.

[O1] V. Ostrik, *Module categories over the Drinfeld double of a finite group.* Int. Math. Res. Not. (2003), 1507–1520.

[O2] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transformation Groups **8** (2003), 177-206.

[P] H. Pollatsek, *First cohomology groups of some linear groups over fields of characteristic two*, Illinois J. Math., **15** (1971), no. 3, 393–417

[R] B. Riepel, *Brauer-Picard groups of pointed fusion categories*, Ph.D. Thesis, University of New Hampshire (2014).

[S] G.B. Seligman, *Modular Lie Algebras*, Springer, (1967).

[U] B. Uribe, *On the Classification of Pointed Fusion Categories up to weak Morita Equivalence*, e-print *arXiv:1511.05522*(2015).

[W] R.A. Wilson, *The finite simple groups*, Springer, (2009).