MULTIPLE SCATTERING FROM A RANDOM AND INHOMOGENEOUS SLAB

HACI-MURAT HUBEY

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MULTIPLE SCATTERING FROM A RANDOM
AND INHOMOGENEOUS SLAB

by

HACI-MURAT HUBEY
B.S. in Mechanical Engineering, NJIT, 1971
Master of Science, NJIT, 1979

A DISSERTATION

Submitted to the University of New Hampshire
In Partial Fulfillment of
The Requirements of the Degree of

Doctor of Philosophy
In Engineering
Graduate School

September, 1982
This dissertation has been examined and approved.

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July 29, 1982
Date
"If you cannot join them, beat them!"

Julian Schwinger

This modest contribution is dedicated to my family and friends.
ACKNOWLEDGEMENTS

I would like to thank my advisor and friend Dr. Kondagunta Sivaprasad for his unflinching support encouragement and patience which made this dissertation possible.

Special thanks are rightfully offered in humility to Visiting Professor, Dr. R. Vasudevan for his contributions from his vast store of knowledge and, incisive mathematical analysis of physical problems.

I would like to offer my sincere gratitude to Dr. B. Celikkol for his financial support. In addition I would like to thank Professors W. Mosberg, F. Glanz, R. Swift, T. Wang, R. Valentine and J. Wilson for their valuable teachings during my stay at UNH. I would also like to express my appreciation to fellow students and friends Dr. P. Harvey, Dr. K. Sundqvist and Dr. R. Carrier for their efforts at furthering my understanding of physical and mathematical concepts and to Tom Fackelman for his beautiful graphics. Last but not least I would like to thank Mrs. Nan Collins for her patience and understanding in the typing of the manuscript.

This project was partly supported by NSF grant DDP.77-18000.
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NOMENCLATURE

$\mathbf{E}(\mathbf{r},t)$  Electric Field

$G(\mathbf{r},\mathbf{r}_1)$  3-D Green's Function

$G_0(\mathbf{r},\mathbf{r}_1)$  3-D Free Space Green's Function

$\mathbf{H}(\mathbf{r},t)$  Magnetic Field

$I$  Identity Operator

$\mathbf{J}(\mathbf{r},t)$  Conduction Current

$L(\cdot)$  Linear Operator

$L_0(\cdot)$  Deterministic Linear Operator

$L_1(\cdot)$  Stochastic Linear Operator

$Q(z)$  Reflection Function of Ghandour

$R(z)$  Reflection Function

$R_n(z)$  nth Order-of-Scattering Reflection Function

$R_\chi(\tau)$  Auto-correlation of $\chi$

$S_\chi(\omega)$  Spectral Density of $\chi$

$a(z)$  Real part of $Q(z)$

$b(z)$  Imaginary part of $Q(z)$

$c$  Speed of Light in Free Space

$k(z)$  Wave Number of Inhomogeneous Media

$p(a,b,z)$  Probability Density

$W(t)$  Gaussian White Noise

$u(z)$  Right-Traveling Wave

$v(z)$  Left-Traveling Wave

$u_n(z)$  Nth-Order Right-Traveling Wave

$v_n(z)$  Nth-Order Left-Traveling Wave
\[ \alpha \quad \text{Small Perturbation Parameter} \]
\[ \beta \quad \text{Wiener Process i.e. } W = \frac{d\beta}{dt} \]
\[ \delta(t-t_0) \quad \text{Dirac Delta-Function} \]
\[ \delta_{ij} \quad \text{Kronecker Delta} \]
\[ \varepsilon \quad \text{Permittivity} \]
\[ \varepsilon_r \quad \text{Relative Permittivity} \]
\[ \varepsilon_0 \quad \text{Permittivity of Free Space} \]
\[ \sigma \quad \text{Conductivity} \]
\[ \rho \quad \text{Charge Density} \]
\[ \rho(\vec{r},t) \quad \text{Source or Forcing term} \]
\[ \gamma \quad \text{Small Perturbation Parameter} \]
\[ \chi(\vec{r},t) \quad \text{Field} \]
\[ \chi_s \quad \text{Scattered Field} \]
\[ \chi_i \quad \text{Incident Field} \]
\[ \delta_x \quad \text{Fluctuating Component of Field} \]
\[ \mu \quad \text{Magnetic Permeability} \]
\[ \mu_0 \quad \text{Permeability of Free Space} \]
\[ \nabla \quad \text{Del Operator} \]
\[ \langle \cdot \rangle \quad \text{Average} \]
\[ \langle \langle \cdot \rangle \rangle \quad \text{Cumulant Average} \]
\[ (\cdot)^\top \quad \text{Transpose} \]
\[ (\cdot) \quad \text{Space Derivative} \]
\[ (\cdot) \quad \text{Time Derivative} \]
\[ (\cdot)^* \quad \text{Complex Conjugate} \]
\[ (\cdot)^{-1} \quad \text{Inverse} \]
ABSTRACT

MULTIPLE SCATTERING FROM A RANDOM AND INHOMOGENEOUS SLAB

by

Haci-Murat Hubey
University of New Hampshire, September 1982

The reflection and transmission functions of an inhomogeneous slab are calculated by labeling the wave by the number of reflections it has undergone in the medium. The Riccati equation, satisfied by the reflected amplitude, is decomposed into a finite set of linear equations by taking into account the number of scatterings taking place inside the medium and is solved by a novel iterative approach. The order-of-scattering reflection functions method presented here and the coupled integral equation approach of the Bremmer solutions for the internal fluxes are set in a unified frame. The solutions of the Riccati equation are computed numerically for the reflection function using the order-of-scattering technique, and it is demonstrated that the method leads to better convergence and stability as compared to usual linearization methods.

The Order-of-Scattering Solutions (BVU Series) of the Riccati Equation for the reflection function of an inhomogeneous slab are used to calculate the mean power intensity for a medium having random fluctuations of its dielectric constant. A solution for a Fokker-Planck-like differential equation for the joint probability density of the real and imaginary parts of the reflection function is obtained.
using a technique due to Van Kampen for nonlinear equations with multiplicative noise. The mean reflected power calculated from the probability density function is in agreement with that derived from the BVU solution. These results are in good agreement with the works of earlier investigators.
CHAPTER I. REVIEW

A. INTRODUCTION

The subject matter of this dissertation deals with a new interdisciplinary field termed "radio-glaciology" - that is; the probing of ice by means of radio waves. It is one of the two major developments of technique in glaciology in recent years. The other is the successful drilling through the complete depth of the Antarctic and Greenland ice sheets, which has brought up for inspection cores of ice showing the layers of snow which fell during the last 100,000 years, thereby revealing a highly detailed record of the past climate of the Earth for that period. The former development proceeded from the discovery that polar ice masses are sufficiently transparent to radio waves in the frequency range 1-500 MHz, so that their interiors could be explored in a much faster and clearer way than was possible with the older time-consuming seismic techniques.

In the beginning, airborne radio echo surveying had been used principally as a technique for depth sounding of the polar ice sheets, but in the deep inland ice of Greenland and the Antarctic, partial reflections from intermediate layers were observed. The presence of internal layering generated a great deal of interest among glaciologists and has been the subject of much debate since the nature of the layers themselves is geophysically significant.

The origin of the internal reflections have not been positively determined as of the present. There may, in fact, be several causes of
internal reflections. Variations in permittivity due to density variations have been suggested by Clough [1977]. Other possible causes could be depositions of impurities as suggested by Paren and Robin [1975]. Some theoretical work along the lines of "layering" of ice have been done by Harrison [1973], Sivaprasad [1976].

In any case, a better understanding of these reflections is needed to increase their value as a glaciological tool. The purpose of this dissertation is to develop suitable theoretical models for the observed partial reflections. Hence, the dissertation is comprised of various aspects of wave propagation and scattering from inhomogeneous and random media.

The various problems associated with wave propagation may be best classified as follows:

- Forward (Direct) solutions vs Inverse solutions
- Deterministic (Inhomogeneous) vs Stochastic Solutions
- Internal vs External Solutions

The forward (or direct) solution implies a solution for the field \( \chi(\vec{r},t) \) (internal or external, or deterministic or stochastic). The inverse solution involves uniquely solving for the material properties of the medium characterized by the wave number \( k^2(\vec{r}) \), having a knowledge of the field \( \chi \).

This thesis is concerned only with direct solutions of the wave equation. The direct solutions may further be split up into solutions for the field inside a material medium (internal solutions) or outside of the medium in which case it becomes a scattering problem, (i.e. we are then interested in the waves reflected from or transmitted through the medium). A major portion of this thesis is concerned with showing
the relationships of these internal and external solutions. It should be noted that the parameters characterizing the medium may be random functions. Both the deterministic (inhomogeneous) case and the stochastic case can be solved through similar methods. These methods will be developed and their relationships to each other and to other published works will be shown.

B. THE WAVE EQUATION

Maxwell's Equations of electromagnetism are given by

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  
(I.1)

\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \]  
(I.2)

\[ \nabla \cdot \mathbf{B} = 0 \]  
(I.3)

\[ \nabla \cdot \mathbf{D} = \rho \]  
(I.4)

For regions in which conduction currents exist (i.e. material media)

\[ \mathbf{J} = \sigma \mathbf{E} \]  
(I.5)

Substituting into (I.2), taking a time derivative of the result and taking the curl of (I.1) results in

\[ \nabla \times \frac{\partial \mathbf{H}}{\partial t} = \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma \frac{\partial \mathbf{E}}{\partial t} \]  
(I.6)

\[ \nabla \times \nabla \times \mathbf{E} = -\mu \nabla \times \frac{\partial \mathbf{H}}{\partial t} \]  
(I.7)

whence

\[ \nabla \times \nabla \times \mathbf{E} = -\mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu \sigma \frac{\partial \mathbf{E}}{\partial t} \]  
(I.8)

Using Eq. (I.4) and the vector identity

\[ \nabla \times \nabla \times \mathbf{E} = \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} \]  
(I.9)
one obtains the dissipative wave equation

$$\nabla^2 \vec{E} - \mu \sigma \frac{\partial \vec{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad (I.10)$$

Interchanging the operations on \( \vec{E} \) and \( \vec{H} \) one can show that the magnetic field \( \vec{H} \) also obeys the dissipative wave equation. Temporal Fourier Transform of Eq. (I.10) results in

$$\{ \nabla^2 + k_0^2 \left[ \frac{i \sigma}{\omega \varepsilon_0} + \varepsilon_r \right] \} \vec{E}(\vec{r}, \omega) = 0 \quad (I.11)$$

where \( k_0 = \frac{\omega}{c} \); \( c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu \varepsilon}} \)

and \( \varepsilon = \varepsilon_0 \varepsilon_r \)

If the conductivity \( \sigma \), and/or the relative permittivity \( \varepsilon_r \) of the medium are functions of space, we can write

$$\{ \nabla^2 + k^2(\vec{r}) \} \vec{E}(\vec{r}, \omega) = 0 \quad (I.12)$$

where \( k^2(\vec{r}) = k_0^2 \left[ \varepsilon_r(\vec{r}) + \frac{i \sigma(\vec{r})}{\omega \varepsilon_0} \right] \)

Thus the problem is reduced to solutions of the Helmholtz Equation

$$\{ \nabla^2 + k^2(\vec{r}) \} \chi(\vec{r}, \omega) = f(\vec{r}, \omega) \quad (I.13)$$

C. LITERATURE REVIEW

a. Deterministic Solutions

To obtain a formal and hence exact solution of Eq. (I.13) we can rewrite it as

$$\{ \nabla^2 + k_0^2 - V(\vec{r}) \} \chi(\vec{r}, \omega) = 0 \quad (I.14)$$

where \( V(\vec{r}) = -(k^2(\vec{r}) - k_0^2) = \) Scattering Potential

Splitting the total field \( \chi(\vec{r}, \omega) \) into the sum of an incident field \( \chi_i(\vec{r}, \omega) \) and a scattered field \( \chi_s(\vec{r}, \omega) \), Eq. (I.14) becomes
\[
\{v^2 + k_0^2 - V(\vec{r})\} \chi_s(\vec{r}, \omega) = V(\vec{r}) \chi_i(\vec{r}, \omega) \quad (I.15)
\]

If the Green's function for Eq. (I.15) can be calculated; that is if the solution of the equation
\[
\{v^2 - k_0^2 - V(\vec{r})\} G(\vec{r}, \vec{r}_1, \omega) = \delta(\vec{r}-\vec{r}_1) \quad (I.16)
\]
can be found, then the scattered field is given by
\[
\chi_s(\vec{r}, \omega) = \int G(\vec{r}, \vec{r}_1; \omega) V(\vec{r}_1) \chi_i(\vec{r}_1; \omega) d^3\vec{r}_1 \quad (I.17)
\]
To solve Eq. (I.16) we rewrite it as
\[
\{v^2 + k_0^2\} G(\vec{r}, \vec{r}_1; \omega) = \delta(\vec{r}-\vec{r}_1) + V(\vec{r}) G(\vec{r}, \vec{r}_1; \omega) \quad (I.18)
\]
Formally treating the right hand side of Eq. (I.18) as a source term, i.e.
\[
\rho(\vec{r}) = \delta(\vec{r}-\vec{r}_1) + V(\vec{r}) G(\vec{r}, \vec{r}_1; \omega) \quad (I.19)
\]
The solution of Eq. (I.16) is then given by
\[
G(\vec{r}, \vec{r}_1; \omega) = \int G_0(\vec{r}-\vec{r}_1; \omega) \rho(\vec{r}_1) d^3\vec{r}_1 \quad (I.20)
\]
where \(G_0(\vec{r}-\vec{r}_1)\) is the solution of
\[
\{v^2 + k_0^2\} G_0(\vec{r}-\vec{r}_1; \omega) = \delta(\vec{r}-\vec{r}_1) \quad (I.21)
\]
Substituting Eq. (I.19) into Eq. (I.20), we obtain
\[
G(\vec{r}, \vec{r}_1, \omega) = G_0(\vec{r}-\vec{r}_1, \omega) + \int G_0(\vec{r}-\vec{r}_2, \omega) V(\vec{r}_2) G(\vec{r}_2, \vec{r}_1, \omega) d^3\vec{r}_2 \quad (I.22)
\]
Since the scattering potential \(V(\vec{r})\) is in general restricted to a volume, the integral has finite limits and Eq. (I.22) is a Fredholm Equation of the second kind.

Iteratively substituting \(G(\vec{r}, \vec{r}_1, \omega)\) for itself in Eq. (I.22) one can obtain the solution given by the Neumann Series
In the zeroth approximation, which is known as the Born Approximation, we ignore all the integral terms of Eq. (1.23) resulting in the solution for $\chi_5(r,\omega)$ as

$$\chi_5(r,\omega) = \int G_0(\vec{r}-\vec{r}_1,\omega)V(\vec{r}_1)\chi_4(\vec{r}_1,\omega)d^3r_1$$

(1.24)

Physically this means that the rescattered terms are ignored. To take into account multiple scattering effects, in this thesis, we will work with a Bremmer-like series solutions. Furthermore, in this thesis only scattering from a slab will be considered. For such a case, the potential will vary in only one direction and will be restricted to a range, say $W<z<L$ (See Fig. I.1), so that

$$V(\vec{r}) = V(z)$$

(1.25)

For this case one can simplify the problem by taking a spatial Hankel transform (or equivalently a two-dimensional Fourier transform). Then Eq. (1.16) reduces to

$$\left\{ \frac{3}{az^2} + k_z^2 - V(z) \right\} g(z,z_1) = \delta(z-z_1)$$

(1.26)

where

$$k_z^2 = -k_0^2 - k_x^2 - k_y^2$$

and the k's refer to the respective wave numbers. Formally operating, one can then construct the three-dimensional Green's function as

$$G(\vec{r}-\vec{r}_1) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{-ik_x(x-x_1)} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} e^{-ik_y(y-y_1)} g(z,z_1,k_z)$$

(1.27)

for the Cartesian coordinate system or
Fig. 1.1

Slab Problem Configuration
\[ G(r-R_1) = \frac{1}{2\pi} \sum_{m} \int_{-\infty}^{\infty} \left[ j_{m}(\beta \rho) j_{m}(\beta \rho_1) g(z,z_1,kz) \right] d\beta \]  

for the cylindrical coordinate system \((j_{m}(\beta \rho) \text{ denotes Bessel function}).\)

(For other representations see for example ... Peter Harvey).

It should be noted that in both representations \(g(z,z_1;kz)\) implicitly satisfies the boundary conditions imposed by the introduction of the slab. To find \(g(z,z_1;kz)\) we can repeat the analysis analogous to that for the three-dimensional case. We'll get a Fredholm Equation given by

\[ g(z,z_1) = g_0(z-z_1) + \int_{L}^{W} g_0(z-z_2)V(z_2)g(z_2,z_1)dz_2 \]

where \(g_0(z-z_1)\) is the solution of

\[ \left\{ \frac{\partial^2}{\partial z^2} + k^2 \right\} g_0(z-z_1) = \delta(z-z_1) \]

and is given by.

\[ g_0(z-z_1) = e^{-ikz|z-z_1|} \]

b. Stochastic Solutions

Stochastic differential equations are generally defined as differential equations involving random elements. Since stochastic processes describing natural phenomenon are understood as families of ordinary functions, stochastic differential equations can be visualized as representing families of deterministic sample differential equations. The ultimate goal in the solution of random differential equations is the complete statistical description of the output \(x(t)\) from the statistical knowledge of the coefficients, the initial conditions and the input. Often, however, "solving" a stochastic differential equation means the determination of a limited amount of information about the solution.
process, such as expectation, correlation function or spectral density. From a mathematical as well as physical point of view, the characterization of stochastic differential equations is strongly dependent upon the manner in which the randomness enters the equations. It is thus convenient to distinguish three basic types of stochastic differential equations [Syski [1967].

1) Random initial conditions or boundary conditions
2) Random forcing functions
3) Random coefficients

The first two cases are fundamentally simpler than the case of stochastic coefficients because of the deterministic relationship of the statistical properties of the solution to the statistical properties of the elements of randomness.

i) Random Initial Conditions

The passage from deterministic differential equations to stochastic ones is easiest when random elements enter only through initial conditions. If \( \bar{x}(t) \) is a solution of the differential equation

\[
\frac{d\bar{x}}{dt} = f(\bar{x}(t),\bar{y}(t),t)
\]  

(1.32)
satisfying the stochastic initial conditions

\( \bar{x}(0) = \bar{x}_0 \) and \( \dot{\bar{x}}(0) = \dot{\bar{x}}_0 \), the solution is

\[
\bar{x}(t) = g(\bar{x}_0,\dot{\bar{x}}_0,t)
\]

(1.33)

Since, once started, the random variable \( \bar{x}(t) \) develops according to the deterministic law described by the function \( g \), the joint distribution function of \( \bar{x}(t) \) and its derivative \( \dot{\bar{x}}(t) \) can be found from the joint
distribution function of $x_0$ and $\dot{x}_0$ by standard methods of change of variables in double integrals. The distribution function of $\dot{x}(t)$ is then obtained as a marginal distribution.

ii) **Random Forcing Function**

To treat the case of a random forcing function, we consider again the sample functions and/or sample solution properties of the stochastic process described by

$$\frac{d\dot{x}}{dt} = f(\dot{x}(t), \dot{y}(t), t)$$  \hspace{1cm} (I.34)

where $\dot{y}(t)$ is a random process. A special case of Eq. (I.34) is

$$\frac{d\dot{x}}{dt} = f(\dot{x}(t), t) + G(\dot{x}(t), t) \dot{\bar{w}}(t)$$  \hspace{1cm} (I.35)

where $\dot{\bar{w}}(t)$ is an n-dimensional vector stochastic process whose components are Gaussian white noise. With the formal representation of the white noise as

$$\dot{\bar{w}}(t) = \frac{d\bar{\delta}}{dt}$$  \hspace{1cm} (I.36)

where $\bar{\delta}(t)$ is a Brownian motion or Wiener Process, we can write Eq. (I.35) as (Ito representation)

$$d\dot{x}(t) = f(\dot{x}(t), t) \, dt + G(\dot{x}(t), t) \, d\bar{\delta}(t)$$  \hspace{1cm} (I.37)

or in integral representation

$$\dot{x}(t) - \dot{x}(t_0) = \int_{t_0}^{t} f(\dot{x}(s), s) ds + \int_{t_0}^{t} G(\dot{x}(s), s) d\bar{\delta}(s)$$  \hspace{1cm} (I.38)

A further specialization of Eq. (I.35) is (i.e. linear equation)

$$\frac{d\dot{x}}{dt} = F(t)\dot{x}(t) + W(t)$$  \hspace{1cm} (I.39)

Following the Ito formulation, a formal interpretation of Eq. (I.39) is
\[ \dot{x}(t) = F(t)x(t)dt + \dot{\beta}(t) \]  \hspace{1cm} (I.40)

which in integral form is
\[ \dot{x}(t) = \int_{t_0}^{t} F(s)x(s)ds + \dot{\beta}(t) \]  \hspace{1cm} (I.41)

It has an explicit mean-square solution representation
\[ \dot{x}(t) = G(t,t_0)x_0 + \int_{t_0}^{t} G(t,s)W(s)ds \]  \hspace{1cm} (I.42)

If we consider only the particular solution
\[ \dot{x}(t) = \int_{t_0}^{t} G(t,s)W(s)ds = \int_{t_0}^{t} G(t,s)d\beta(s) \]  \hspace{1cm} (I.43)

we can then use Eq. (1.43) to determine the moments of the solution; the first two being the most important.

It is easy to show that
\[ \langle \dot{x}(t) \rangle = \int_{t_0}^{t} G(t,s)<W(s)>ds \]  \hspace{1cm} (I.44)

and
\[ R_X(t,s) = \langle \dot{x}(t)\dot{x}^T(s) \rangle = \int_{t_0}^{s} \int_{t_0}^{t} G(t,u)R_W(u,v)G^T(s,v)dudv \]  \hspace{1cm} (I.45)

A special case occurs when the input is stationary and the system is time-invariant. Then Eq. (I.43) can be written as
\[ \dot{x}(t) = G(t)W(t-x)dx \]  \hspace{1cm} (I.46)

Since \( W(t) \) is stationary, its mean is constant and it follows directly from above that \( \langle \dot{x}(t) \rangle \) is also constant. Then, the correlation function matrix of \( \dot{x}(t) \) is
\[ R_X(t,\tau) = \int_{0}^{\infty} \int_{0}^{\infty} G(u)R_W(t-\tau-u+v)G^T(v)dudv \]  \hspace{1cm} (I.47)

Taking a Fourier Transform, and making use of the definition of the Spectral Density \( S_X(\omega) \)
\[ S_X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega \tau} R_X(\tau) d\tau \quad (I.48) \]

we get the important relationship
\[ S_X(\omega) = G^*(\omega) S_Y(\omega) G^T(\omega) \quad (I.49) \]

Since the Wave Problem is one in which the coefficients are random, the random forcing model can be used to solve it only with heroic assumptions. In the next section, random coefficients case will be discussed.

iii) Random Coefficients

1. Operator Formalism

To facilitate the discussion of solution of random differential equation with random coefficients, it is best to start off with the development of operator formalism for differential equations. For the Sturm-Liouville equation
\[ L(z)\chi(z) = \left( \frac{d}{dz} p(z) \frac{d}{dz} -q(z) - \lambda \omega(z) \right) \chi(z) = f(z) \quad (I.50) \]
we would like to define an operator \( L^{-1}(z) \) such that
\[ L(z)L^{-1}(z) = I \quad (I.51) \]
so that we can write the solution as
\[ \chi(z) = L^{-1}(z)f(z) \quad (I.52) \]
Since \( L(z) \) is a differential operator, \( L^{-1}(z) \) will be an integral operator
\[ L^{-1}(z)f(z) = \int g(z,z_1)f(z_1)dz_1 \quad (I.53) \]
The kernel of this operator is defined as the Green's function of the Sturm-Liouville problem. We can find the Green's function as follows:
\[ \chi(z) = I_X(z) = L(z)L^{-1}(z)\chi(z) = L(z)\int g(z,z_1)\chi(z_1)dz_1 \quad (I.54) \]
Interchanging the order of differentiation and integration

\[ x(z) = \oint [L(z)g(z,z_1)]x(z_1)dz_1 \]  

(1.55)

From the above it is clear that

\[ L(z)g(z,z_1) = \delta(z-z_1) \]  

(1.56)

which is the defining equation for the Green's function.

2. Random Green's Function

The theory of differential equations with random coefficients is connected with the concepts of random operators or random Green's functions. We are concerned with the solutions of equations of form

\[ L(z)x(z) = f(z) \]  

(1.57)

where

\[ L(z) = \sum \alpha_n(z) \frac{d^n}{dz^n} \]  

(1.58)

and \( \alpha_n(z) \) and \( f(z) \) are stochastic processes. We would like to find \( L^{-1}(z) \), the random Green's function analogous to an ordinary Green's function. In some problems only some of the coefficients are random, so that the operator \( L(z) \) can be written as a sum of a deterministic operator and a stochastic operator.

Thus (dropping \( z \)-dependence for convenience)

\[ \{L_0 + L_1\}x = f \]  

(1.59)

This equation can be changed into an integral equation

\[ x = L_0^{-1}f - L_0^{-1}L_1x \]  

(1.60)

where \( x \), the field, is an element of an infinite-dimensional vector space \( H \), \( L_0^{-1} \) and \( L_1 \) are linear operators in \( H \) corresponding respectively to a deterministic (free) propagation and a stochastic interaction, and \( f \) is a non-random source term. In the above problem, one is not interested in
the complete solution $x$, but only in the projection $P_X$ of $x$ on some linear subspace of $H$. For the random equation, the projection operator is simply the averaging; $P_X \rightarrow <x>$. 

Furthermore

$$P L^{-1}_0 = L^{-1}_0 P = L^{-1}_0 \quad P L^{-1}_1 = 0 \quad P f = f \quad (I.61)$$

The equations above simply express the fact that the free space Green's function and the source are non-random and that $L_1$ is a centered (zero-mean) random process. The projector $P$ is also called the smoothing operator. Now, any $x \in H$ can be written as the sum of a mean field $<x>$ and a fluctuating field $\delta x$.

$$x = <x> + \delta x = P_x + (I-P)x \quad (I.62)$$

We can now derive an equation for the mean field only. Applying $P$ to Eq. (I.60) and using Eqs. (I.61) and (I.62), we get

$$P_x = P L^{-1}_0 f - L^{-1}_0 P L_1 P x - L^{-1}_0 P L_1 \delta x \quad (I.63)$$

which becomes

$$<x> = L^{-1}_0 f - L^{-1}_0 P L_1 \delta x \quad (I.64)$$

Operating with $(I-P)$ on Eq. (I.60) gives

$$x - P_x = L^{-1}_0 f - L^{-1}_0 f - L^{-1}_0 (I-P) L_1 x$$

$$\delta x = - L^{-1}_0 (I-P) L_1 (<x> + \delta x) \quad (I.65)$$

By formal iteration, we get the solution for the fluctuating field to be

$$\delta x = \sum_{n=1}^{\infty} [-L^{-1}_0 (I-P) L_1]^n <x> \quad (I.66)$$

Substituting into Eq. (I.64) we obtain

$$<x> = L^{-1}_0 f + L^{-1}_0 M <x> \quad (I.67)$$

where
\[ M = - \sum_{n=1}^{\infty} PL_1 \left[ -L_0^{-1}(I-P)L_1 \right]^nP \]  

(I.68)

is the mass operator or the intensity operator. This equation is called the Dyson Equation. Perturbation solution of the Dyson Equation for the mean field [for wave propagation] has been obtained by Tsang and Kong [1976], [1979] and Tan and Fung [1979], using a two-variable expansion technique. It can be shown that finite-order iterations in the \((I-P)H\) space are equivalent to the summation of infinite subseries of the mean field [Bharucha-Reid [1968]]. For example, the "first-order" smoothing approximation

\[ M = PL_1L_0^{-1}L_1P \]  

(I.69)

\[ \langle x \rangle = L_0^{-1}f + L_0^{-1}PL_1L_0^{-1}L_1P\langle x \rangle \]  

(I.70)

corresponds to the formal summation

\[ \langle x \rangle = \sum_{n=0}^{\infty} \left[ L_0^{-1}PL_1L_0^{-1}L_1P \right]^nL_0^{-1}P = \]  

(I.71)

\[ = \sum_{n=0}^{\infty} \left[ L_0^{-1}\langle L_1L_0^{-1}L_1 \rangle \right]^nL_0^{-1}P \]

Eq. (I.71) can be obtained in a different manner. Consider the equation

\[ Lx = (L_0 + \alpha L_1)x = f \]  

(I.72)

where \( \alpha \) is a small parameter. We can write it as

\[ L_0(I + \alpha L_0^{-1}L_1)x = f \]  

(I.73)

so that

\[ x = (L_0 + \alpha L_1)^{-1}f = (I + \alpha L_0^{-1}L_1)^{-1}L_0^{-1}f \]  

(I.74)

Since the inverse of an Identity operator plus an infinitesimal operator is given by [Cushing [1975]].
\[(I + \alpha L_0^{-1} L_1)^{-1} = I + \sum_{n=1}^{\infty} (-\alpha L_0^{-1} L_1)^n \] (I.75)

we can then obtain

\[(L_0 + \alpha L_1)^{-1} = \sum_{n=0}^{\infty} (-\alpha L_0^{-1} L_1)^n L_0^{-1} \] (I.76)

This result can be seen in Van Kampen [1976]. If we average, we get the inverse to be

\[< (L_0 + \alpha L_1)^{-1} > = \sum_{m=0}^{\infty} (-\alpha)^2 m [L_0^{-1} < L_1 L_0^{-1} L_1 >] m L_0^{-1} \] (I.77)

This result has been obtained by LoDato [1972] using renormalized Projection Operator techniques. Similar results for stochastic Green's functions have been obtained by Adomian [1970]. If we keep only the terms up to \( \alpha^2 \) order, and average, then

\[< L^{-1} > = L_0^{-1} + \alpha^2 L_0^{-1} < L_1 L_0^{-1} L_1 > L_0^{-1} \] (I.78)

This is Bourret's integral equation [Bourret, 1962]. Since

\[< L^{-1} >^{-1} = L_0 - \alpha^2 < L_1 L_0^{-1} L_1 > \] (I.79)

We can then get an integro-differential equation for the mean field (correct to \( \alpha^2 \) order)

\[\{ L_0 - \alpha^2 < L_1 L_0^{-1} L_1 > \} < \chi > = f \] (I.80)

This equation is due originally to Keller [1964]. It can also be seen in Tatarski and Gershenstein [1963]. However, the justification offered by Keller at the time involved "illegal" decoupling of the field from the operator. Since then, "legal" derivations of this result have been presented by Adomian [1970], Bourret [1962], LoDato [1972] and Van Kampen 1976.
For the one dimensional wave equation
\[ \left( \frac{\partial^2}{\partial z^2} + k^2(1+\alpha\epsilon(z)) \right)\chi(z) = 0 \] (I.81)

Eq. (I.80) written explicitly is
\[ \left( \frac{\partial^2}{\partial z^2} + k^2 \right)\langle \chi \rangle - \alpha^2 k^4 \int g(z-z_1)\langle \epsilon(z)\epsilon(z_1)\rangle \langle \chi(z_1) \rangle dz_1 = 0 \] (I.82)

where \( g(z-z_1) \) is the Green's function of Eq. (I.81) with \( \alpha=0 \). Solutions for
\[ \langle \epsilon(z)\epsilon(z_1)\rangle = A\delta(z-z_1) \] (I.83)

and
\[ \langle \epsilon(z)\epsilon(z_1)\rangle = e^{-|z-z_1|/a} \] (I.84)

are given in Kupiec, et al [1969]. Other approximate results are also given in Acquista [1978].

D. THE REFLECTION FUNCTION

Rather than finding the scattered field \( \chi_s \), it is sometimes more meaningful physically to solve for the Reflection Function \( R(z) \) defined by
\[ \chi_s(z) = R(z)\chi_i(z) \] (I.85)

We know that in the region \( L<z<W \), the field obeys the equation (for one-dimensional case)
\[ \left( \frac{d^2}{dz^2} + k^2(z) \right)\chi(z) = 0 \] (I.86)

Again, splitting the field into an incident and a scattered wave, Eq. (I.86) can be written as
\[ \left( \frac{d^2}{dz^2} + k_1^2 \right)\chi_s(z) = -\left( k^2-k_1^2 \right)\chi(z) \] (I.87)

so that
\[ \chi_s(z) = \int (k^2(z_1)-k_1^2)\chi(z_1)g(z,z_1)dz_1 \] (I.88)
With the incident wave given by
\[ \chi_i(z) = e^{i k_1 (z-w)} \] (I.89)
and
\[ g(z, z_1) = \frac{i}{2k_1} e^{ik_1(z_1-z)} \] (I.90)
for \( z < L \) and \( w = 0 \).

The reflection function for the slab then becomes
\[ R(L) = \frac{i}{2k_1} \int_0^L (k^2(s) - k_1^2) \chi(s) e^{ik_1 s} ds \] (I.91)

This result can be seen in Wang [1965]. Thus the evaluation of the field inside the medium also allows one to calculate the reflected wave. The next chapter will deal with direct means of solutions for the Reflection Function.
CHAPTER II. REFLECTION FUNCTION SOLUTIONS

In Chapter I, a review was given of the direct (deterministic, inhomogeneous or stochastic) solutions of the wave equation. These fall under the category of internal solutions. However, in radio-echo sounding or seismic sounding, the waves outside the random or inhomogeneous medium are of primary interest. These are the external solutions or the reflected waves.

In this chapter the derivations of the equation for the reflected waves or the reflection function $R(z)$ and the associated internal solutions given by Bremmer Series [Bremmer, 1951] will be given.

Additionally, a new set of transport equations for the internal fluxes and a new Riccati Equation for the reflected waves, appropriate for the problem under consideration, will be derived. Solutions of these equations via a new linearization method will be given and their relationships to each other and to the JWKB Approximation will be shown. Comparisons will be made to other perturbation solutions. Numerical simulations showing the convergence properties of these series to the usual linearization algorithms will be demonstrated.

A. DERIVATION OF THE REFLECTION FUNCTION

In the treatment of the reduced wave equation

$$\frac{d^2\chi}{dz^2} + k^2(z)\chi = 0 \quad (II.1)$$

it is common to write it as [See Van Kampen [1976], Kubo [1963] or Fox [1978]],
\[
\frac{d\chi}{dt} = \chi = ik\chi \tag{II.2}
\]

for \( k = \text{const} \) or for \( k = \text{stochastic process} \)

whence

\[
\chi'' = ik\chi = -k^2\chi \tag{II.3}
\]

However if \( k = k(z) \), then Eq. (II.1) can no longer be written as in (II.2).

Then using the Bremmer-Splitting [Bremmer [1951]], field \( \chi \) can be represented as a sum of a right-traveling and a left-traveling wave

\[
\chi(z) = u(z) + v(z) \tag{II.4}
\]

It can then be shown [See for example Atkinson [1960]], that \( u(z) \) and \( v(z) \) obey a coupled set of linear first order transport equations

\[
\begin{align*}
\dot{u} &= (ik - \frac{k}{2k})u + \frac{k}{2k}v \\
\dot{v} &= -(ik + \frac{k}{2k})v + \frac{k}{2k}u
\end{align*} \tag{II.5a,b}
\]

To verify this, we note

\[
\dot{\chi} = \dot{u} + \dot{v} = ik(u-v) \tag{II.6}
\]

\[
\frac{d^2\chi}{dz^2} = ik(u-v) + ik(\dot{u}+\dot{v}) = -k^2(u+v)
\]

\[
= -k^2\chi \tag{II.7}
\]

For the Reflection Coefficient \( R(z) = \frac{v(z)}{u(z)} \) Eq. (II.7) becomes,

\[
\frac{dR}{dz} = \dot{R} = \frac{\dot{vu}-v\dot{u}}{u^2} = \frac{\dot{v}}{u} - R \frac{\dot{u}}{u} \tag{II.8}
\]

Substituting for \( \dot{v} \) and \( \dot{u} \) from Eqs. (II.5) and (II.6), Eq. (II.8) simplifies to

\[
\dot{R} = \frac{k}{2k} (1-R^2) - 2ikR \tag{II.9}
\]

Equation (II.9) is the usual Riccati Equation for the Reflection Coefficient. This equation was originally derived by Stokes, Bellman and Wing [1975] using wavelet counting techniques. It can also be obtained
directly form Eq. (II.1) by making the Liouville transformation

\[ x = \exp \left[ \int_0^z y(s) ds \right] \quad \text{(II.10)} \]

followed by the substitution

\[ y = ik \frac{1-R}{1+R} \quad \text{(II.11)} \]

Making the further substitution in Eq. (II.9).

\[ R = Se^{ikz} \quad \text{(II.12)} \]

we arrive at,

\[ \frac{dS}{dz} = \frac{k_1 \gamma \varepsilon(z)}{2i \sqrt{\varepsilon_0}} \left[ Se^{-ikz} + e^{ikz} \right]^{2} \quad \text{(II.13)} \]

for

\[ k^2 = k_1^2 [1+\gamma \varepsilon(z)] \quad \text{(II.14)} \]

where \( \gamma \) is a small parameter.

Equation (II.13) was originally derived and solved for stochastic \( \varepsilon(z) \) by Papanicolaou [1971]. It can also be seen in Ryzhov [1976].

For the case of many particles, using wavelet counting techniques, Bellman, Vasudevan and Ueno [1973] have derived a vector Riccati Equation which can be decomposed into an iterative set of first-order linear differential equations. For one particle (i.e. Wave Equation) the equations reduce to

\[ \dot{R}_{2n+1} = \frac{k}{2k} \delta_{n,0} - 2i k R_{2n+1} - \frac{k}{2k} \sum_{m=0}^{n-1} R_{2m+1} R_2(n-m)-1 \quad \text{(II.15)} \]

where \( n \) is the number of internal reflections a wave goes through before emerging from the incident surface, and \( \delta_{n,0} \) is the Kronecker delta.

Thus \( R_{2n+1} \) is the wave amplitude that is reflected back after going through \( 2n+1 \) internal reversals of direction. We also note that the B.C. which is
appear only with the equation for $R_1$. This method of linearization of the Riccati Equation is mathematically attractive in that only the lower orders, which have already been solved, occur in the solutions of the higher order equations. We will now derive an approximate and different equation for $R(z)$. From Eqs. (II.4) and (II.6) we obtain

$$u = \frac{x}{2} + \frac{x}{2i k(z)}$$  \hspace{1cm} (II.17)$$

$$v = \frac{x}{2} - \frac{x}{2i k(z)}$$  \hspace{1cm} (II.18)$$

The wave is incident at the interface of Region I and II (at $z = o$) with unit amplitude ($u(o) = 1$) and there is no wave entering from the other end ($v(L) = 0$).

As an approximation when $k_1 \neq k_2$ [See Bellman and Wing (1975)]

$$u = (\dot{x} + ik_1x)/2ik_1$$  \hspace{1cm} (II.19a)$$

$$v = (ik_2 \dot{x} - x)/2ik_2$$  \hspace{1cm} (II.19b)$$

From Eq. (II.14) we get

$$\dot{x} = 2ik_1u - ik_1x = -2ik_2v + ik_2x$$  \hspace{1cm} (II.20)$$

whence

$$x = \frac{2(k_2v + k_1u)}{k_1 + k_2}$$  \hspace{1cm} (II.21)$$

From (II.14a) and (II.15) and Eq. (II.1)

$$\ddot{u} = \frac{x'' + ik_1 \dot{x}}{2i k_1} = \frac{1}{2ik_1} [-k_2^2 \dot{x} + ik_1(-2ik_2v + ik_2x)]$$  \hspace{1cm} (II.22)$$

Substituting for $x$ from (II.16) and rearranging

$$\ddot{u} = \frac{1}{ik_1(k_1 + k_2)} [-k_1(k^2 + k_1k_2)u - k_2(k^2 - k_1^2)v]$$  \hspace{1cm} (II.23a)$$
Similar analysis for $v$ gives

$$
\dot{v} = \frac{1}{ik_2(k_1+k_2)} [k_2(k^2+k_1k_2)v + k_1(k^2-k_2^2)u] \tag{II.23b}
$$

When the medium in Region I is identical with Region III, i.e. $k_1=k_2$ and assuming Eq. (II.14), Eq. (II.23) reduce to

$$
du(z) = \frac{ik_1}{2} [(2+\gamma\varepsilon(z))u(z) + \gamma\varepsilon(z)v(z)] \tag{II.24a}
$$

$$
dv(z) = \frac{-ik_1}{2} [\gamma\varepsilon(z)u(z) + (2+\gamma\varepsilon(z))v(z)] \tag{II.24b}
$$

with the B.C.

$$
u(o) = 1 \quad \text{and} \quad v(L) = 0 \quad (0<z<L)
$$

Using the definition of the Reflection function

$$
R(z) = \frac{v(z)}{u(z)}
$$

and substituting for $u(z)$ and $v(z)$ from Eq. (II.24) we obtain the Riccati Equation

$$
\frac{dR(z)}{dz} = \frac{-ik_1\gamma\varepsilon(z)}{2} - ik_1(2+\gamma\varepsilon(z))R(z) - \frac{ik_1}{2} \gamma\varepsilon(z)R^2(z) \tag{II.25}
$$

A similar equation for $R$ has been obtained by Ryzhov [1973].

\textit{a. Internal Solutions}

Integrating Equations (II.24) we have

$$
u(z) = \frac{ik_1\gamma}{2} \int_0^z \varepsilon(z_1)v(z_1)\exp\left[\frac{ik_1}{2} \int_{z_1}^z (2+\gamma\varepsilon(z_2))dz_2\right]dz_1
$$

$$
+ \exp\left[\frac{ik_1}{2} \int_0^z (2+\gamma\varepsilon(z_2))dz_2\right]\tag{II.26a}
$$

$$
v(z) = \frac{ik_1\gamma}{2} \int_0^L \varepsilon(z_1)u(z_1)\exp\left[\frac{ik_1}{2} \int_{z_1}^z (2+\gamma\varepsilon(z_2))dz_2\right]dz_1
$$

$$
+ \exp\left[\frac{ik_1}{2} \int_0^L (2+\gamma\varepsilon(z_2))dz_2\right]\tag{II.26b}$$
We now label the right and left amplitudes by the number of back scatterings they have undergone inside the medium \((0,L)\).

Let \(u_{2n}\) represent the right-traveling amplitude with \(2n\) back scatterings and \(v_{2n+1}\) represent the left-traveling amplitude with \((2n+1)\) backscatterings at any point in the medium. Hence

\[
\chi(z) = \sum_{n=0}^{\infty} (u_{2n} + v_{2n+1}) \tag{II.27}
\]

The corresponding Bremmer like differential equation for \(u_{2n}\) and \(v_{2n+1}\) are

\[
\frac{d}{dz} u_{2n}(z) = (ik_1/2)[(2+\gamma \varepsilon(z)) u_{2n}(z) + \gamma \varepsilon(z) v_{2n-1}(z)] \tag{II.28a}
\]

\[
\frac{d}{dz} v_{2n+1}(z) = (-ik_1/2)[\gamma \varepsilon(z) u_{2n}(z) + (2+\gamma \varepsilon(z)) v_{2n+1}(z)] \tag{II.28b}
\]

with \(u_{2n}(0) = \delta_{n,0}\) and \(v_{2n+1}(L) = 0\) for all \(n\). Note that for \(u_0(z)\) there is no back scattering from the \(v\) flux. Hence \(u_{2n}(z)\) and \(v_{2n+1}(z)\) can be now successively solved. The integral equations for these decomposed amplitudes are easily seen to be

\[
u_{2n}(z) = \delta_{n,0} \exp \left[ \frac{ik_1}{2} \int_0^z (2+\gamma \varepsilon(z_1))dz_1 \right] \tag{II.29a}
\]

\[
+ (1-\delta_{n,0}) \frac{ik_1 \gamma}{2} \int_0^z \varepsilon(z_1) v_{2n-1}(z_1) \exp \left[ \frac{ik_1}{2} \int_{z_1}^z (2+\gamma \varepsilon(z_2))dz_2 \right]dz_1
\]

\[
v_{2n+1}(z) = \frac{ik_1 \gamma}{2} \int_z^L \varepsilon(z_1) u_{2n}(z_1) \exp \left[ \frac{ik_1}{2} \int_z^{z_1} (2+\gamma \varepsilon(z_2))dz_2 \right]dz_1 \tag{II.29b}
\]

where \(n = 0,1,2...\) and \(\delta_{n,0}\) is the Kronecker delta function. From the above, we see that the W.K.B. approximations for the solution inside region II is given by
\[ u_0(z) = \exp[(ik_1 z) + \frac{ik_1 \gamma}{2} \int_0^z 0(z_1)dz_1] \]

\[ \sim \exp(i \int_0^Z k(\sigma)d\sigma) \text{ since } k_1(1 + \frac{\gamma 0(z)}{2}) \sim k(z) \]

When \[ [k_1/k(z)]^{1/2} \sim 1 \], this agrees with the first term, \( u_0^B \) of the Bremner series, (the usual W.K.B. solution).

\[ u_0^B(z) = [k_1/k(z)]^{1/2} \exp[i \int_0^Z k(\sigma)d\sigma] \]

The factor \((ik_1 \gamma 0/2)\) as seen from the differential equations (II.24) is the amount of back scattering amplitude per unit input and is equivalent to \((k/2k(z))\) appearing in the analysis of Bremner [1951].

If the slab occupies the region \([W,L]\) then the W.K.B. solution for \( u_0(z) \) and \( v_1(z) \) are

\[ u_0(z) = u_0(z;W) = \exp[ik_1/2 (\int_W^Z (z+0(z_1)) dz_1)] \]

\[ v_1(z;L) = \frac{ik_1 \gamma}{2} \int_z^L 0(z_1) u_0(z_1;W) u_0^*(z;z_1)dz_1 \]

where \( u_0^*(z;z_1) = u_0(z_1;z) \)

Going up the hierarchy, we obtain the following set of equations for \( u_{2n} \) and \( v_{2n+1} \) for deterministic profiles \( 0(z) \) with \( z\in(W,L) \)

\[ u_{2n}(z) = u_0(z;W) \delta_{n,0} + (ik_1 \gamma/2) \int_W^Z 0(z_1) v_{2n-1}(z_1) u_0(z;z_1)dz_1 \]

\[ v_{2n+1}(z) = (ik_1 \gamma/2) \int_z^L 0(z_1) u_{2n}(z_1) u_0^*(z;z_1)dz_1 \]

for \( n = 0,1,2 \ldots \). The interpretations of the above equations are clear.

In Eq. (II.34) \( u_0(z) \) is the nonscattered flux incident at the left edge and going up to \( z \). The second integral describes the \((2n-1)\) scattered left moving flux \( v_{2n-1}(z_1) \) that is scattered at \( z_1 \) once more and is con-
verted into a right moving flux and reaches without further scattering to z by the multiplying factor $u_0(z; z_1)$. Similarly Eq. (II.35) can be interpreted for $v_{2n+1}$. The reflection function for the medium, $[W,L]$, corresponding to single reflection is given by

$$R_1(W;L) = v_1(W) = (ik_1 \gamma/2) \int_W^L \epsilon(z_1) u_0(z_1; W) u_0^*(W; z_1) \, dz_1 \quad (II.36)$$

for unit input at the interface at $W$. Similarly for the reflection function for three reflections inside the medium is

$$R_3(W;L) = v_3(W) = (ik_1 \gamma/2) \int_W^L \epsilon(z_1) u_2(z_1) u_0^*(W; z_1) \, dz_1 \quad (II.37)$$

and

$$R_{2n+1}(W;L) = v_{2n+1}(W) = (ik_1 \gamma/2) \int_W^L \epsilon(z_1) u_{2n}(z_1) u_0^*(W; z_1) \, dz_1 \quad (II.38)$$

The total reflection function $R = \sum_{n=0}^{\infty} R_{2n+1}(W;L)$, is given by $v(W)$. Similarly one can easily see that determination of different orders of $u_{2n}$ leads to the computation of $T_{2n}$, the transmission function. Thus in this section we have concerned ourselves in the determination of the internal fields, $u_{2n}$ and $v_{2n+1}$, in a successive fashion made possible by the ordering or scattering method for the flux inside the medium.

**b. External Solutions**

In this section we deal with the external solutions for the Reflection Function given by the Riccati Equation (II.25) in a direct formalism and show the relationship to the Bremmer Series. The method of arriving at the solutions of the Riccati Equation (II.9) in an iterative manner by decomposing it into a set of linear equations - nth equation yielding the solution of the wave that has undergone n reflections inside the medium - has been analyzed extensively by Bellman.
et al. [1973]. The set of linear equations for the Riccati Equation (II.25) is thus given by

\[ \dot{R}_{2n+1}(z;L) = -(ik_1\gamma e/2)\delta_{n,0} - (ik_1\gamma e+2ik_1) R_{2n+1} - (ik_1\gamma e/2) \sum_{m=1}^{n} R_{2m-1} R_{2(n-m)+1} \]  

(II.39)

with the B.C. \( R_{2n+1}(L;L) = 0 \) and

\[ R(z;L) = \sum_{n=0}^{\infty} R_{2n+1}(z;L) \]

The general solution is given by

\[ R_{2n+1}(z;L) = \int_{z}^{L} \left[ (ik_1\gamma e(z_1)/2)\delta_{n,0} + (ik_1\gamma e(z_1)/2) \sum_{m=1}^{n} (R_{2m-1} R_{2(n-m)+1}) \right] \]

\[ \cdot \exp[2ik_1(z_1-z) + ik_1 \int_{z}^{z_1} \epsilon(z_2)dz_2]dz_1 \]

(II.40)

We find that \( R_1 \), the reflected contribution due to waves that have undergone one reflection inside the medium is given by

\[ R_1(z;L) = \left( \frac{ik_1\gamma}{2} \right) \int_{z}^{L} \epsilon(z_1)u_0(z_1;z)u_0^*(z;z_1)dz_1 \]

(II.41)

We note that this is different from Equation (II.33) as expected since \( R_1(z;L) \) gives the reflected wave at any point \( z \). However it should be noted that \( R_1(W;L) = v_1(W) \), at \( W \), the left end of the medium. The next order-of-scattering \( R_3(z;L) \) is given in terms of \( R_1(z;L) \) as

\[ R_3(z;L) = (ik_1\gamma/2) \int_{z}^{L} \epsilon(z_1)R_1^2(z_1;L)u_0(z_1;z)u_0^*(z;z_1)dz_1 \]

(II.42)

Hence \( R_{2n+1}(W;L) \), the \((2n+1)\) order reflection function can be computed from (II.40).

The imbedding analysis for the transmission function yields for this case [See for example Bellman and Wing (1975)].

\[ \frac{dT}{dz}(z;L) = (-ik_1T(z,L)/2) [(2+\gamma e(z)) + \gamma e(z) R(z,L)] \]

(II.49)
with $T(L,L) = 1$.

The order of scattering functions $T_{2n}(z,L)$ satisfy the differential equation

$$
\frac{dT_{2n}}{dz}(z,L) = -(ik_1/2) \left[ T_{2n}(2+\gamma \varepsilon(z)) + \gamma \varepsilon(z) \sum_{m=1}^{n} (T_{2(n-m)}(z,L) R_{2m-1}(z,L)) \right]
$$

(II.50)

with $T_0(L,L) = 1, T_{2n}(L,L) = 0$ for $n \neq 0$

These can be easily computed once $R_{2n+1}$ function have been determined for the medium.

The Bellman-Vasudevan-Ueno decomposition, Eq. (II.39) for the Riccati Equation (II.25), explicitly is

$$
\dot{R}_1 + ik_1(2+\gamma \varepsilon)R_1 = \frac{-ik_1}{2} \gamma \varepsilon \varepsilon
\dot{R}_3 + ik_1(2+\gamma \varepsilon)R_3 = \frac{-ik_1}{2} \epsilon R_1^2
\dot{R}_5 + ik_1(2+\gamma \varepsilon)R_5 = \frac{-ik_1}{2} \epsilon (2R_1R_3)
\dot{R}_7 + ik_1(2+\gamma \varepsilon)R_7 = \frac{-ik_1}{2} \epsilon (2R_1R_5+R_3^2)
\vdots
\vdots
$$

(II.51)

The formal solution of a first-order linear differential equation

$$
(d/dz + P(z)) \ R(z) = L(z)R(z) = Q(z)
$$

(II.52)

is given by

$$
R(z) = L^{-1}(z)Q(z) = \int_0^z dz_1 Q(z_1) \exp \left[ \int_z^{z_1} P(s) ds \right]
$$

(II.53)

Identifying

$$
P(z) = ik_1(2+\gamma \varepsilon)$$

$$Q(z) = \frac{-ik_1}{2} \gamma \varepsilon$$

(II.54)
The BVU Series solutions can be written in convenient operator formalism as

\[ R_1 = + L^{-1}Q \]
\[ R_3 = L^{-1}Q R_1^2 \]
\[ R_5 = 2L^{-1}QR_1R_3 \]
\[ R_7 = + 2L^{-1}QR_1R_5 + L^{-1}QR_3^2 \] (II.55)

B. COMPARISON WITH OTHER SOLUTIONS

a. Perturbation Solutions

i) Adomian Series

For the Non-Linear Stochastic Equation

\[ LR + N(R) = Q \] (II.56)

where \( L \) is a linear operator of form

\[ L = \sum_{n=0}^{\infty} \alpha_n(z) \frac{d^n}{dz^n} \] (II.57)

where one or all \( \alpha_n(z) \) may be random, \( N(R) \) is a non-linear term of the form \( \sum \beta_n(z)R^m \) and \( \beta_n(z) \) and \( Q(z) \) are stochastic coefficients, we can formally write the solution as [Adomian [1976]].

\[ R(z) = L_0^{-1}Q - L_0^{-1}L_1R - L_0^{-1}N(R) \] (II.58)

if we can write the linear operator \( L \) as

\[ L = L_0 + L_1 \] (II.59)

where \( L_0 \) is a deterministic operator and \( L_1 \) is a stochastic operator.

It is clear that identifying

\[ L_0 = \frac{d}{dz} + 2ik_1 \] (II.60)
\[ L_1 = i k_1 y e \]  
\[ N(R) = \frac{ik_1 y e R^2}{2} \]  
\[ Q(z) = \frac{-ik_1 y e}{2} \]  

makes Eq. (II.56) equivalent to the Riccati Eq. (II.25) with the formal solution given by (II.58). If we now write the solution \( R(z) \) to be given by

\[ R(z) = \sum_{n=0}^{\infty} (-1)^n R_n \]  

as done in Adomian [1976], we then have

\[ R = L_0^{-1} Q - L_0^{-1} L_1 R - L_0^{-1} QR^2 = \]
\[ = L_0^{-1} Q - L_0^{-1} L_1 (R_0 - R_1 + R_2 + \ldots) - L_0^{-1} Q (R_0 - R_1 + R_2 + \ldots)^2 \]
\[ = L_0^{-1} Q - L_0^{-1} L_1 R_0 + L_0^{-1} L_1 R_1 - L_0^{-1} L_1 R_2 + \ldots \]
\[ - L_0^{-1} Q (R_0^2 + R_1^2 + R_2^2 + \ldots) \]
\[ - 2R_0 R_1 + 2R_0 R_2 - 2R_0 R_3 + \ldots \]
\[ - 2R_1 R_2 + 2R_1 R_3 - 2R_1 R_4 + \ldots \]  

We note that the non-linearity brings in the cross product terms as in the BVU decomposition. As in the linear case, we identify

\[ R_0 = L_0^{-1} Q \]  

which is identical with \( R_1 \) of the BVU series. We can pick \( R_1 \) in terms of \( R_0 \) as

\[ R_1 = L_0^{-1} L_1 R_0 + L_0^{-1} QR_0^2 \]  

Continuing in the same manner

\[ R_2 = L_0^{-1} L_1 R_1 - L_0^{-1} QR_0^2 + 2L_0^{-1} QR_0 R_1 \]
Here, as in the BVU series, each of the $R_n$ can be calculated in terms of the preceding terms. If the random operator is zero (i.e. $L_1 = 0$), or if the average is zero ($<L_1>=0$), then

$$R_0 = L_0^{-1} Q$$

$$R_1 = L_0^{-1} Q R_0$$

$$R_2 = -L_0^{-1} Q R_0 + 2L_0^{-1} Q R_0 R_1$$

$$R_3 = L_0^{-1} Q R_2 + 2L_0^{-1} Q R_0 R_2 - 2L_0^{-1} Q R_1 R_2$$

ii) An Alternate Series

To identify the magnitudes of the different terms of the series, we can try an expansion of form

$$R = \sum (-\alpha)^n R_n$$

where $\alpha<<1$. Then we can write, as before

$$R = L_0^{-1} Q - L_0^{-1} L_1 R - L_0^{-1} Q R_2 =$$

$$= L_0^{-1} Q - L_0^{-1} L_1 (R_0 - \alpha R_1 + \alpha^2 R_2 \ldots) - L_0^{-1} Q (R_0 - \alpha R_1 + \alpha^2 R_2 \ldots)^2 =$$

Expanding the terms, and starting off with

$$R_0 = L_0^{-1} Q$$

$$R_1 = L_0^{-1} L_1 R_0 + L_0^{-1} Q R_0$$

we can collect terms of the approximate magnitudes by keeping track of the orders of $\alpha$, so that
\[ R_2 = L_0^{-1} L_1 R_1 + 2L_0^{-1} Q R_0 R_1 \]
\[ R_3 = L_0^{-1} L_1 R_2 + L_0^{-1} Q R_1^2 + 2L_0^{-1} Q R_0 R_2 \]  \hspace{1cm} (II.72)
\[ R_4 = L_0^{-1} L_1 R_3 + 2L_0^{-1} Q R_0 R_3 + 2L_0^{-1} Q R_1 R_2 \]
\[ R_5 = L_0^{-1} L_1 R_4 + L_0^{-1} Q R_2^2 + 2L_0^{-1} Q R_0 R_4 + 2L_0^{-1} Q R_1 R_3 \]

If \( L_1 = 0 \) or \( <L_1> = 0 \), then
\[ R_0 = L_0^{-1} Q \]
\[ R_1 = L_0^{-1} Q R_0^2 \]
\[ R_2 = 2L_0^{-1} Q R_0 R_1 \]  \hspace{1cm} (II.73)
\[ R_3 = L_0^{-1} Q R_1^2 + 2L_0^{-1} Q R_0 R_1 \]
\[ R_4 = 2L_0^{-1} Q R_0 R_3 + 2L_0^{-1} Q R_1 R_2 \]

It can be noted that the first few terms of the series Eq (II.55), Eq (II.70) and Eq (II.73) are similar, especially in a weakly inhomogeneous medium, where the higher order terms will be small.

**b. Quasilinearization**

This method, first employed by Kantorovich [See for example Bellman [1970], or Lee [1968]], is a variation of the Newton-Raphson iteration scheme. For a first-order differential equation
\[ \frac{dR}{dz} = f(R) = a + bR + cR^2 \]  \hspace{1cm} (II.75)
we make a Taylor expansion about \( R(z_0) \), and substitute into the rhs of Eq. (II.75). Thus
\[
\frac{dR}{dz} = f(R(z_0)) + (R - R(z_0)) \frac{df}{dR}_{R=R_0} = a + bR(z) + 2cR(z)R(z_0) - cR^2(z_0) \quad (II.76)
\]
so that the Riccati Equation is effectively linearized. For numerical solutions, Eq. (II.76) is discretized to give the general recursion relation
\[
\frac{dR}{dz}_{n+1} - [b + 2cR_n]R_{n+1} = a - cR_n^2 \quad (II.77)
\]
With the appropriate values, the Quasilinearization Equations for the Riccati Equation (II.25) are
\[
\frac{dR}{dz}_{n+1} - ik_1 [2 + \varepsilon(1 + R_n)]R_{n+1} = \frac{ik_1 \varepsilon}{2} [1 - R_n^2] \quad (II.78)
\]
The linear equations (II.78) are quite easily solved numerically.

C. NUMERICAL RESULTS

As an illustration of the effectiveness of the order-of-scattering techniques, we have computed numerically \(|R|\) for a slab. For the propagation vector profile defined by \(k^2 = k_1^2(1 + \gamma \varepsilon)\), we have taken \(\varepsilon(z) = \alpha z\) and \(\alpha z^2\) with \(k_1 = .0002\) for two different coefficients \(\alpha\). The profiles for \(\varepsilon(z)\) were chosen specifically with glacial ice in mind since the variations in \(\varepsilon(z)\) vary slowly and linearly as shown by experimental evidence Fig. (II.1) [Keliher and Ackley [1978]]. In both cases it is found that within three iterations the order of scattering techniques converges in the sense that successive scatterings decrease at least by an order of magnitude. As the depth increases for more than 200 meters, due to inherent computational difficulties one must use doubling techniques and other such devices. We have used two different profiles to analyze how the increase in inhomogeneity affects the efficiency of the method.
### TABLE II.1 - ORDER OF SCATTERING

**Table I.A** \( \varepsilon(z) = 0.25 \, z \)

<table>
<thead>
<tr>
<th>Depth (meter)</th>
<th>( R_1 )</th>
<th>( R_3 )</th>
<th>( R_5 )</th>
<th>( R_7 )</th>
<th>( R_9 )</th>
<th>( R_{11} )</th>
<th>( R )</th>
</tr>
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<tbody>
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<td>40</td>
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<td>.3441E-10</td>
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<tr>
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<tr>
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<td>.2554E+00</td>
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</table>

**Table I.B** \( \varepsilon(z) = 0.5 \, z \)

<table>
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<tr>
<th>Depth (meter)</th>
<th>( R_1 )</th>
<th>( R_3 )</th>
<th>( R_5 )</th>
<th>( R_7 )</th>
<th>( R_9 )</th>
<th>( R_{11} )</th>
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<tr>
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<td>.7900E+00</td>
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</tbody>
</table>

**Table I.C** \( \varepsilon(z) = 0.005 \, z^2 \)

<table>
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<th>Depth (meter)</th>
<th>( R_1 )</th>
<th>( R_3 )</th>
<th>( R_5 )</th>
<th>( R_7 )</th>
<th>( R_9 )</th>
<th>( R_{11} )</th>
<th>( R )</th>
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### Table II.2 - Linearization

#### Table II.A $\epsilon(z) = 0.25z$

<table>
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<tr>
<th>Depth (meter)</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>$R_4$</th>
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</tbody>
</table>

#### Table II.B $\epsilon(z) = 0.50z$

<table>
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<tr>
<th>Depth (meter)</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>$R_4$</th>
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</table>

#### Table II.C $\epsilon(z) = 0.005 z^2$

<table>
<thead>
<tr>
<th>Depth (meter)</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>$R_4$</th>
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</thead>
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</table>
It is found that this technique works satisfactorily as demonstrated in Table (II.1), Figs (II.2) and Figs (II.3).

To compare the order of scattering method with other methods such as simple linearization we considered the algorithm described by

\[ R'_{n+1}(z) = \lambda(z) + m(z) R_{n+1} + CR_n^2 \]  \hspace{1cm} (II.79)

where \( n = 1,2,... \) etc. The true \( R = \lim_{n \to \infty} R_n \). The \( R_n \)'s are evaluated successively with some initial value \( R_1 \). It is found in Table II.2 and Figs (II.4) that the convergence is poor due to oscillations even with small orders.

It can be seen from Fig II.5 that the BVU series converges uniformly in contrast to the oscillatory behavior of the linearization methods.
Fig. II.1. Variation of Ice Density with Depth [From Keliher and Ackley [1978]]
Fig. II.2. Order-of-Scattering Solutions
Fig. II.3 Order of Magnitude Difference in the Reflection Coefficients for the BVU Method
Reflection Coefficient - $|R|$ (Logarithmic Scale)

Fig. II.4. Linearization Solutions for Various Iterations

\[ \varepsilon(z) = 0.5z \]
Reflection Coefficient ($|R|$)

$\varepsilon(z) = 0.005z^2$
$z = 190 \text{ m}$

$\varepsilon(z) = 0.5z$
$z = 200 \text{ m}$

$\varepsilon(z) = 0.25z$
$z = 200 \text{ m}$

Fig. II.5a. Oscillatory Convergence of the Linearization Method
Reflection Coefficient $|R| = |R_{2n+1}|$

$e(z) = 0.005z^2$

$z = 190$ m

$e(z) = 0.5z$

$z = 200$ m

$e(z) = 0.25z$

$z = 200$ m

The Order-of-Scattering

Fig. II.5b. Uniform Convergence of the BVU Method
CHAPTER III

STOCHASTIC SOLUTIONS

In Chapter I, stochastic solutions were reviewed for the internal fields. In Chapter II the reflection function solutions and their relationships to the Bremmer Series were shown. In addition, perturbation solutions of the Riccati equation for $R(z)$ (deterministic and stochastic) were discussed. In this chapter, the mean reflected power $\langle |R|^2 \rangle$, based on BVU series will be calculated to various orders of approximation. The probability density of the reflection process will be obtained through a Fokker-Planck equation which was derived using a method of Van Kampen [1976] for non-linear differential equations with multiplicative noise. The average reflected power $\langle |R|^2 \rangle$ calculated from this probability density will be shown to be identical to that derived from the BVU solutions. In addition $\langle |R|^2 \rangle$ will be shown to be qualitatively similar to the numerically integrated results of Papanicolau [1971], and asymptotic results of Ryzhov [1976] and Ghandour [1975], who have arrived at their solutions using alternate approaches.

A. SOLUTIONS VIA BELLMAN-VASUDEVAN-UENO SERIES

There are various approximate solutions available for the solution of the Riccati Equation for a stochastic case. Making the substitution

$$R = -[1+2i k_1 Q]$$

in the equation

(III.1)
\[
\frac{dR}{dz} = -2ik_1R - \left(\frac{ik_1\gamma e}{2}\right)(1+R)^2
\]  \hspace{1cm} (III.2)

we obtain

\[
\frac{dQ}{dz} = -[1+2ik_1Q + \gamma k_1^2eQ^2]
\]  \hspace{1cm} (III.3)

where \(\varepsilon = \varepsilon(z)\), is a stationary stochastic process such that

\[
\langle \varepsilon(z) \rangle = 0 \tag{III.4}
\]

\[
\langle \varepsilon(z)\varepsilon(z_1) \rangle = D\delta(z-z_1) \tag{III.5}
\]

Equation (III.3) must be solved with the initial condition \(Q(0) = -\frac{1}{2ik_1}\).

The solution of the equation with the first two terms (i.e. linear case) is given by

\[
Q(z) = e^{-Az} + \frac{1}{A}
\]  \hspace{1cm} (III.6)

where \(A = \frac{1}{2ik_1}\)

To find an approximate total solution we try a variation of constants, i.e.

\[
Q(z) = \phi(E,z)
\]  \hspace{1cm} (III.7)

then

\[
\frac{dQ}{dz} = \frac{d\phi}{dz} + \frac{\partial\phi}{\partial E} \frac{dE}{dz} = -[1+AQ+BQ^2]
\]  \hspace{1cm} (III.8)

where \(B = \gamma k_1^2e(z)\)

Since \(\frac{\partial\phi}{\partial z} = \frac{\partial Q}{\partial z} = -AEe^{-Az} = -[1+AQ]\)

\[
\frac{\partial\phi}{\partial E} = e^{-Az} \text{ and } \frac{\partial\phi}{\partial E} \frac{dE}{dz} = -BQ^2
\]  \hspace{1cm} (III.10)

We get an equation for the constant, \(E\), to be

\[
\frac{dE}{dz} = -B[E^2e^{-Az} + \frac{2E}{A} + \frac{e^{Az}}{A^2}]
\]  \hspace{1cm} (III.11)

Using a Taylor expansion for \(E(z)\)
\[ E(z) = E(0) + z \frac{dE}{dz} \bigg|_{z=0} \]  

(III.12)

and using Eq. (III.6), we find \( E(0) = 0 \). Hence

\[ E(z) = - \frac{Bz}{A^2} \]  

(III.13)

Substituting into Eq. (III.11) and keeping only linear terms in \( B \) (since \( B \ll 1 \)), we get

\[ E(z) = - \int_{0}^{z} \frac{B(x)e^{Ax}}{A^2} \, dx \]  

(III.14)

Thus

\[ Q(z) = \frac{\gamma}{4} \int_{0}^{z} \varepsilon(x)e^{2ik_1(x-z)} \, dx + \frac{1}{2ik_1} \]  

(III.15)

Using Eqs. (III.4) and (III.5), we can show that

\[ QQ^* = \frac{2}{16} \int_{0}^{L} \int_{0}^{L} dx \varepsilon(x)\varepsilon(y)e^{-2ik_1(x-y)} + \frac{1}{4k_1^2} = D_{\chi}^2 - \frac{1}{4} \]  

(III.16)

From the definition, Eq. (III.1)

\[ <RR^*> = [1 + <AQ> + <A^*Q^*> + <AA^*QQ^*>] \]  

(III.17)

where \(^*\) denotes complex conjugate. Substituting \( QQ^* \) from Eq. (III.16) in Eq. (III.17), we obtain

\[ <RR^*> = <|R|^2> = [AA^*<QQ^*>-1] = \frac{DK^2\gamma L}{4} \]  

(III.18)

This simple result is due to Ghandour [1975]. The appearance of the secular term shows that it is of limited validity. To show the relationship to BVU solutions, we obtain the average value of \( |R|^2 \) from

\[ <|R|^2> = <R_1R_1^*> \]  

(III.19)

From the solution of \( R_1 \) in Eq. (II.41)
\begin{equation}
<R_1 R_1^*> = \frac{k_1^2 \gamma^2}{4} \int_0^L dz_1 \int_0^L dz_2 \langle \epsilon(z_1) \epsilon(z_2) \rangle \exp \left[ \left( \frac{i k_1 \gamma}{2} \right) \int_{z_1}^{z_2} \epsilon(y) dy \right] \exp(2ik_1(z_2 - z_1))
\tag{III.20}
\end{equation}

The limits are $[0, L]$ since we are now considering reflection at left-end (i.e. $z = W = 0$). For a short length, we can expand the exponential inside the integrand and obtain $<R_1 R_1^*>$ correct to the $\gamma^2$ order.

\begin{equation}
<R_1 R_1^*> = \frac{(k_1^2 \gamma^2)}{4} \int_0^L dz_1 \int_0^L dz_2 \langle \epsilon(z_1) \epsilon(z_2) \rangle \exp[2ik_1(z_2 - z_1)]
\tag{III.21}
\end{equation}

With the assumption Eq. (III.5), Eq. (III.21) reduces to Eq. (III.18).

We can get better results by using the transport equations (II.24). Rewriting the equations for their complex conjugates and adding them, we obtain

\begin{equation}
\frac{d}{dz} (uu^*) = \frac{i k_1 \gamma \epsilon}{2} [v u^* - v^* u]
\tag{III.22a}
\end{equation}

\begin{equation}
\frac{d}{dz} (vv^*) = \frac{i k_1 \gamma \epsilon}{2} [u^* v - u v^*]
\tag{III.22b}
\end{equation}

Adding these equations results in the current conservation equation given by

\begin{equation}
\frac{d}{dz} (uu^* - vv^*) = 0
\tag{III.23}
\end{equation}

As $v = Ru$, we can write the above equation as

\begin{equation}
\frac{d}{dz} [(1 - RR^*) uu^*] = 0
\tag{III.24}
\end{equation}

Since $RR^* = |R|^2$, Eq. (III.24) becomes

\begin{equation}
uu^* \frac{d}{dz} (1 - |R|^2) + (1 - |R|^2) \frac{d}{dz} (uu^*) = 0
\tag{III.25}
\end{equation}

But from Eq. (III.22a)

\begin{equation}
\frac{d}{dz}(uu^*) = \frac{i k_1 \gamma \epsilon}{2} [R uu^* - R^* u^* u] = -k_1 \gamma \epsilon uu^* \text{Im}(R)
\tag{III.26}
\end{equation}
Thus Eq. (III.25) becomes

\[ \frac{1}{1-|R|^2} \frac{d}{dz} (1-|R|^2) = \frac{d}{dz} \ln(1-|R|^2) = k_1 \gamma \epsilon \text{Im}(R) \]  

(III.27)

This equation can also be obtained directly from the Riccati equation for R as will be shown below.

We have, for the general case [See Bellman and Wing [1975]].

\[ \frac{dR}{dz} = -\frac{1}{2i k_1} \left[ (k^2-k_1^2) + 2(k^2+k_1^2)R + (k^2-k_1^2)R^2 \right] \]  

(III.28)

Using Eq. (III.28) and its conjugate, we can obtain

\[ \frac{d}{dz} (RR^*) = \frac{1}{2i k_1} \left[ (k^2-k_1^2)(R^*-R) + (k^2-k_1^2)(RR^*-R^*R) \right] \]

\[ = \frac{1}{2i k_1} [k^2-k_1^2][1-RR^*](R-R^*) \]  

(III.29)

Substitution of Eq. (II.14) results in Eq. (III.27).

To make Eq. (III.27) somewhat tractable, we must make an approximation for \( \text{Im}(R) \). We can make this approximation from the Bellman-Vasudevan-Ueno solution for \( Q(z) \). If Eq. (III.3) is decomposed into a linearized set, we get

\[ \frac{dQ_1}{dz} + 2ik_1 Q_1 = -1 \]  

(III.30)

\[ \frac{dQ_3}{dz} + 2ik_1 Q_3 = -\gamma k_1^2 \epsilon Q_1^2 \]  

(III.31)

The solution for \( Q_1 \), satisfying the B.C. is

\[ Q_1 = \frac{-1}{2i k_1} \]  

(III.32)

We can rewrite Eq. (III.31) as

\[ \frac{d}{dz} [Q_3(z) \exp \left[ \int_0^z 2ik_1 ds \right]] = -\gamma k_1^2 \epsilon(z) e^{2ik_1 z} Q_1^2 \]  

(III.33)
Hence
\[ Q_3(z) = \frac{\gamma}{4} \int_0^z e(z_1) e^{2ik_1(z_1-z)} \, dz_1 \quad (\text{III.34}) \]

Thus
\[ R(z) = -[1+2ik_1(Q_1+Q_3)] = \frac{-ik_1\gamma}{2} \int_0^z e(z_1) e^{2ik_1(z_1-z)} \, dz_1 \quad (\text{III.35}) \]

Substituting the imaginary part of Eq. (III.35) into Eq. (III.27) results in
\[ |R|^2 = 1 - \exp\left[\frac{-k_1^2\gamma^2}{2} \int_0^L dz_1 \int_0^L dz_2 e(z_1) e(z_2) \cos 2k_1(z_2-z_1)\right] \quad (\text{III.36}) \]

An ensemble average of Eq. (III.36) yields
\[ \langle |R|^2 \rangle = 1 - \exp\left[\frac{-k_1^2\gamma^2}{2} \int_0^L dz_1 \int_0^L dz_2 \langle e(z_1) e(z_2) \rangle \cos 2k_1(z_2-z_1)\right] \quad (\text{III.37}) \]

Where the double bracket denotes cumulant average (See Van Kampen [1976] or Kubo [1963]). For a delta-correlated Gaussian process this reduces to
\[ \langle |R|^2 \rangle = 1 - \exp\left(-\frac{k_1^2\gamma DL}{2}\right) \quad (\text{III.38}) \]

This result yields zero for zero length and one for infinite length. It is qualitatively similar to that obtained by Papanicolaou [1971], Ghandour [1975] for a weakly correlated Ornstein-Uhlenbeck process, and Ryzhov [1976]. [See Fig III.1].

B. SOLUTIONS USING VAN KAMPEN'S METHOD

An alternative procedure in getting stochastic solutions to random differential equation is to obtain the Fokker-Planck equation for the probability density of the random process. This has been done for the Helmholtz or the reduced wave equation by Papanicolaou and Keller [1971], Morrison, et al [1971] and Ryzhov [1976]. The general first order stochastic equation
Fig. III.1. Approximate Mean Reflected Power as a Function of Dimensionless Length
[From Ghandour [1975]]
\[ \frac{dx(z)}{dz} = f(x(z),z) + g(x(z),z) \frac{d\beta(z)}{dz} \quad (III.39) \]

or equivalently the Ito Stochastic Equation

\[ dx(z) = f(x(z),z)dz + g(x(z),z)d\beta(z) \quad (III.40) \]

is characterized by the density function \( p(x(z),z) \) and the transition probability density function \( p(x(z)|x(\xi)) \), since it is a Markov process (see for example Jazwinski [1970] or Srinivasan and Vasudevan [1972], or Soong [1973].

The forward Kolmogorov Equation or the Fokker-Planck Equation for the process is given by

\[ \frac{\partial p(x,z)}{\partial z} = - \frac{\partial}{\partial x} [f(x,z)p(x,z)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [p(x,z)g^2(x,z)] \quad (III.41) \]

If \( R(z) \) which has real and imaginary parts is represented as a vector process, the equation

\[ \frac{dR}{dz} = -2ik_1R - \frac{i}{2} (1+R)^2 \quad (III.42) \]

has the equivalent Ito form

\[ d\tilde{X}(z) = f(\tilde{X}(z),z) \, dz + G(\tilde{X}(z),z) \, d\beta(z) \quad (III.43) \]

For \( \tilde{\beta}(z) \) an \( m \)-dimensional vector Wiener Process (Brownian Motion Process) with

\[ \varepsilon(z) = \frac{d\beta(z)}{dz} \]

\[ <\varepsilon_i(z) > = 0 \quad (III.44) \]

\[ <\varepsilon_i(z)\varepsilon_j(z_1)> = Q_{ij} \]

The corresponding vector Fokker-Planck Equation becomes
Because of the non-linearity in Eq. (III.42), the corresponding Fokker-Planck Equation becomes quite impossible to solve. If the Fokker-Planck approach is to be used, a simpler equation needs to be obtained.

A simpler equation for the process can be derived using a technique for non-linear equations due to Van Kampen [1976]. For the general stochastic non-linear equation

\[ \frac{du}{dz} = F_\mu (u_1, u_2, \ldots, u_n, z) \quad (\mu = 1, 2 \ldots n) \]  

(III.46)

along with the initial conditions \( u_\mu (0) = a_\mu \), if we represent \( \bar{u} \) by a point in an \( n \)-dimensional "phase space", the density of such solutions \( \rho (\bar{u}, z) \) obeys the familiar continuity equation

\[ \frac{\partial \rho (\bar{u}, z)}{\partial z} = \frac{1}{\partial u_\mu} f_\mu (\bar{u}, z) \rho (\bar{u}, z) \]

(III.47)

The average of the density of solutions, \( \langle \rho (\bar{u}, z) \rangle \) through a lemma of Van Kampen can be shown to be identical with \( p (\bar{u}, z) \), the probability density of \( \bar{u} \). The probability density \( p (\bar{u}, z) \) then satisfies an equation of form

\[ \frac{\partial p (\bar{u}, z)}{\partial z} = K(\bar{u})p (\bar{u}, z) \]

(III.48)

where \( K(\bar{u}) \) is an operator acting only on the \( \bar{u} \) dependence of \( p (\bar{u}, z) \).

However, if we have

\[ \frac{d\bar{u}}{dz} = F_0 (\bar{u}) + \alpha F_1 (u, z) \]

(III.49)

where \( F_0 \) is the deterministic part of the operator and \( F_1 \) is the random part, then \( p(u, z) \) can be shown to satisfy
\[
\frac{\partial p(u,z)}{\partial z} = \nabla \cdot \left(-F_0(u) + \alpha^2 \int_0^\infty <F_1(u,z) \nabla_{-\tau} \cdot F_1(u^{\tau},z-\tau)> d\tau \right) \\
+ \alpha^2 \int_0^\infty <F_1(u,z) F_1(u^{\tau},z-\tau)> \Lambda(u^{\tau}) d\tau \right] p(u,z) 
\] (III.50)

In the above equation \( \nabla \) denotes differentiation with respect to \( u(z) \) and \( \nabla_{-\tau} \) denotes differentiation with respect to \( u^{\tau} \); that is \( u \) occurring at an earlier time or space. Additionally,

\[
\Lambda(u^{\tau}) = \nabla [\ln d(u^{\tau})/du] 
\] (III.51)
is the Jacobian determinant of the mapping from \( u^{\tau} \) to \( u \) obtained from the solutions for \( u \) keeping only the deterministic part of Eq. (III.49).

The solution for \( u \) (i.e. \( u^{\tau} \)) in Eq. (III.50) is the deterministic solutions of Eq. (III.49). The restrictions on the random operator are

\[
<F_1(u,z)> = 0 
\] (III.53)
\[
<F_1(u,z) F_1(u_{1},z_{1})> = 0 \text{ for } |z-z_{1}| > \tau_c 
\] (III.54)

where \( \tau_c \) = correlation length of the process.

To apply this method to the Riccati Equation (III.3), we write the vector equation for the real and imaginary parts of \( Q \).

\[
\frac{d}{dz} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1+2k_1b \\ -2k_1a \end{bmatrix} - k_1^2 \gamma \varepsilon(z) \begin{bmatrix} a^2-b^2 \\ 2ab \end{bmatrix} 
\] (III.55)

The initial conditions are \( a(0) = 0 \) and \( b(0) = \frac{1}{2k_1} \). To be able to use Van Kampen's method, we need a solution of the deterministic part alone. We have

\[
\frac{da}{dz} = -1+2k_1b 
\] (III.56)
\[
\frac{db}{dz} = -2k_1a 
\] (III.57)
The only solutions that satisfy the B.C. are

\[
\begin{align*}
  a(z) &= 0 \quad \text{(III.58)} \\
  b(z) &= \frac{1}{2k_1} \quad \text{(III.59)}
\end{align*}
\]

Thus

\[
\Lambda = \nabla \ln \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0 \quad \text{(III.60)}
\]

and Eq. (III.50) reduces to a special form

\[
\frac{\partial p}{\partial z} = \nabla \cdot \left( -F_0 + a^2 \int_0^\infty \langle F_1(u,z)\nabla \cdot F_1(u^-\tau,z-\tau) \rangle d\tau \right) p(u,z) \quad \text{(III.61)}
\]

The first term is given by

\[
\nabla \cdot \nabla F_0 p(u,z) = p \nabla \cdot \nabla F_0 + F_0 \cdot \nabla p = -(1-2k_1b) \frac{\partial p}{\partial a} + 2k_1 \frac{\partial p}{\partial b} \quad \text{(III.62)}
\]

The evaluation of the integrand results in

\[
\frac{\gamma^2}{16} \int_0^\infty \langle \epsilon(z)\epsilon(z-\tau) \rangle \frac{\partial^2}{\partial a^2} d\tau \quad \text{(III.63)}
\]

With the assumption Eq. (III.5), Eq. (III.50) finally reduces to

\[
\frac{\partial p}{\partial z} = \left(1-2k_1b\right) \frac{\partial p}{\partial a} + 2k_1 \frac{\partial p}{\partial b} + \frac{\gamma_D}{16} \frac{\partial^2 p}{\partial a^2} \quad \text{(III.64)}
\]

with the initial condition

\[
p(a,b,0) = \delta(a)\delta(b - \frac{1}{2k_1}) \quad \text{(III.65)}
\]

In the simplest approximation, using the solutions (III.58) and (III.59), Eq. (III.64) reduces to

\[
\frac{\partial p}{\partial z} = \frac{\gamma_D}{16} \frac{\partial^2 p}{\partial z^2} \quad \text{(III.66)}
\]

This is the usual diffusion equation with the solution
\[ p(a,z) = \frac{1}{(4\pi z)^{\frac{3}{2}}} \exp \left[ -\frac{a^2}{4\Omega z} \right] \]  

(III.67)

where \( \Omega = \frac{\gamma D}{16} \).

Then recalling the definition of \( R \) from Eq. (III.1), we see that

\[ <RR^*> = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 - 4k_1b + 4k_1^2(a^2 + b^2)]p(a,b,L) \]

(III.68)

Using Eq. (III.68), we find

\[ <RR^*> = \frac{2k_1^2}{1} \int_{-\infty}^{\infty} da \ a^2 \exp \left[ -\frac{a^2}{4\Omega z} \right] \]

(III.69)

The final result is then

\[ <RR^*> = <\lvert R \rvert^2> = \frac{\gamma^2 k_1^2 DL}{2} \]  

(III.70)

which qualitatively gives the Ghandour solution [1975] given by Eq. (III.21).

Better results can be obtained by solving Eq. (III.64) with the B.C. as given by Eq. (III.65). We take a spatial Laplace transform with the kernel \( e^{\xi z} \) yielding

\[ \xi p(a,b,\xi) - p(a,b,0) = \left( (1 - 2k_1b) \frac{\partial}{\partial a} + 2k_1a \frac{\partial}{\partial b} + \Omega^2 \right) p(a,b,\xi) \]  

(III.71)

Taking Fourier Transforms with the kernel \( e^{i(\alpha a + \beta b)} \) reduces the second-order equation (III.71) to a first-order equation in \( \alpha \) and \( \beta \).

\[ \left\{ \frac{\partial}{\partial \beta} - \beta \frac{\partial}{\partial \alpha} - \frac{1}{2k_1} (\Omega^2 + i\alpha + \xi) \right\} p(\alpha,\beta,\xi) = \frac{-1}{2k_1} \exp\{i\beta/2k_1\} \]  

(III.72)

This equation has the integrals

\[ \frac{d\alpha}{-\beta} = \frac{d\beta}{\alpha} = \frac{dp}{\frac{1}{2k_1} [\Omega^2 + i\alpha + \xi]p - \frac{e^{i\beta/2k_1}}{2k_1}} \]  

(III.73)

From the first set we get
\[ \alpha^2 + \beta^2 = r^2 = \text{constant} \]  

(III.74)

The second set gives

\[ \frac{dp}{d\alpha} = -\frac{[\Omega^2 + i\alpha + \xi]}{2k_1 \sqrt{r^2 - \alpha^2}} p + \frac{i \sqrt{r^2 - \alpha^2}}{2k_1 \sqrt{r^2 - \alpha^2}} \]  

(III.75)

which has the solution

\[ p(\alpha, \beta, \xi) = \frac{g(\alpha)}{2k_1} \int_0^\alpha f(\alpha') e^{\frac{1}{2k_1} \left[ \xi \left( \sin^{-1} \frac{\alpha}{|r|} - \sin^{-1} \frac{\alpha}{|r|} \right) \right]} d\alpha' \]  

(III.76)

where \[ g(\alpha) = \exp[-I_1(\alpha) - I_2(\alpha)] \]  

(III.77a)

\[ f(\alpha') = \frac{\exp[I_1(\alpha)/2k_1]}{(r^2 - \alpha^2)^{\frac{1}{2}}} \]  

(III.77b)

\[ I_1(\alpha) = \Omega \left[ \frac{-\alpha}{2} \sqrt{r^2 - \alpha^2} + \frac{r^2}{2} \sin^{-1} \frac{\alpha}{|r|} \right] \]  

(III.77c)

\[ I_2(\alpha) = -i \sqrt{r^2 - \alpha^2} \]  

(III.77d)

\[ I_3(\alpha) = \xi \sin^{-1} \frac{\alpha}{|r|} \]  

(III.77e)

We take an inverse spatial Laplace Transform of Eq. (III.76), so that

\[ p(\alpha, \beta, z) = \frac{g(\alpha)}{2k_1} \int_0^\alpha f(\alpha') \delta(z + \frac{1}{2k_1} \left[ \sin^{-1} \frac{\alpha'}{|r|} - \sin^{-1} \frac{\alpha}{|r|} \right]) d\alpha' \]  

(III.78)

where

\[ f(\alpha') = \exp[\Omega \left( \frac{-\alpha'}{2k_1} \sqrt{r^2 - \alpha^2} + \frac{r^2}{2} \sin^{-1} \frac{\alpha}{|r|} \right)] \]  

(III.79)

We can integrate Eq. (III.78) by the change of variable

\[ x = \frac{1}{2k_1} \sin^{-1} \frac{\alpha'}{|r|} \]  

(III.80)

We obtain

\[ \text{\ldots} \]
\[ p(\alpha, \beta, \xi) = g(\alpha) \exp\left(\frac{\Omega r^2}{4k_1^2} [-\sin 2k_1 y \cos 2k_1 y + 2k_1^2 \gamma]\right) \quad (\text{III.81}) \]

where
\[ y = \frac{1}{2k_1} \sin^{-1} \frac{\alpha}{|r|} - z \quad (\text{III.82}) \]

Further simplification results in the solution
\[ p(\alpha, \beta, z) = \exp[q(\alpha, \beta, z)] \quad (\text{III.83}) \]

where
\[ q(\alpha, \beta, z) = \left\{ \frac{\Omega}{4k_1} (1 - \frac{\cos 4k_1 z}{k_1}) \alpha \beta - \frac{\Lambda \cos 2k_1 z \sin 2k_1 z}{4k_1^2} (\alpha^2 - \beta^2) \right\} \]
\[ + \frac{i}{2k_1} \beta - \frac{\Omega r^2}{2} z = \ldots \]
\[ = \{A_0 \alpha \beta + A_1 (\alpha^2 - \beta^2) + A_2 \beta - A_3 z\} \quad (\text{III.84}) \]

The two-dimensional inverse Fourier Transform of Eq. (III.83) will yield the probability density \( p(\alpha, \beta, z) \), from which one can calculate \( \langle \text{RR}^* \rangle \) by using Eq. (III.66). The density \( p(\alpha, \beta, z) \) will be calculated later, since in practice it is easier to obtain \( \langle \text{RR}^* \rangle \) from
\[ \langle \text{RR}^* \rangle = 1 + \{4ik_1 \frac{\partial}{\partial \beta} - 4k_1^2 \left[ \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right] \} p(\alpha, \beta, z) \big|_{\alpha, \beta = 0} \quad (\text{III.85}) \]

Performing the indicated operations, we obtain
\[ \langle \text{RR}^* \rangle = 1 - e^{-A_3 z} = 1 - e^{-\frac{\Omega r^2}{2} z} \quad (\text{III.86}) \]

which is exactly the result Eq. (III.38) derived from the BVU solutions.
CHAPTER IV. CONCLUSION

A. SUMMARY OF RESULTS

Neumann Series solutions (for the internal fields) of the Helmholtz Equation, for the inhomogeneous or the random wave number \( k^2(z) \), have been amply reviewed in the earlier part of this dissertation. For both cases, Green's functions and the Fredholm Integral Equation formulations play a fundamental role.

However, in scattering (i.e. scattering) problems, the external fields are of primary interest. These Reflection Function solutions can be related to Green's functions and this has been shown in the review sections.

The scattering problem (from a slab) can also be formulated as a B.V. problem using Green's functions. If this approach is used, then one is usually forced to make use of the Leontevich B.C. [Usleghi [1978]].

\[
\frac{\partial x}{\partial z} - \frac{ik}{\zeta} x = 0
\]  

(IV.1)

There are several problems with this approach. The value of \( \zeta \) must already be known from the characteristics of the scattering medium. If the wave is scattered from an inhomogeneous or random medium then \( \zeta = \zeta(z) \) is a function of the length of the slab. Since \( \zeta(z) \) is related to the Reflection Function \( R(z) \), once \( R(z) \) is calculated, then the B.V. formulation becomes purely a mathematical exercise.

The approach in this dissertation, therefore, has been to calculate the Reflection Function in terms of the characteristics of the medium.
and the length of the scattering slab. The Helmholtz Equation, satisfied by the inhomogeneous slab (with the appropriate approximations for the wave number for glacial ice), has been solved by splitting the wave into two fluxes traveling in opposite directions. The resulting transport equations have been solved by the well known Bremmer Series, the first term of which is the JWKB approximation. A first order non-linear differential equation (Riccati Equation), taking into account the discontinuities at the boundaries, for the ratio of the fluxes - the ratio being the Local Reflection Function - has been derived. The resulting equation has been solved using a novel, iterative, linearization method due to Bellman, Vasudevan and Ueno (BVU) [1973]. The Bremmer Series solutions of the transport equations and the BVU Series solutions of the Riccati Equation have been set in a unified frame. Numerical solutions have been conducted demonstrating the uniform convergence properties of the BVU solutions vs. the oscillatory behavior of the usual linearization method.

For stochastic solutions, two different approaches; both based on the BVU Series and the Riccati Equation have been used. In the first method, a differential equation is derived for the reflected power $|R|^2$. A closed form solution is obtained by making appropriate approximations based on the BVU Series. The ensemble average of this solution yields the mean reflected power $<|R|^2>$ which is in very good agreement with the numerically integrated result of Papanicolaou [1971] and the approximate asymptotic Fokker-Planck results of Ghandour [1975] and Ryzhov [1976].

The second approach, also based on the Riccati Equation, was Van Kampen's [1976] technique for non-linear differential equations with multiplicative noise to derive an approximate Fokker-Planck equation for
the probability density of the reflection process. The exact solutions of the Fokker-Planck equation, obtained by using Fourier and Laplace Transforms and the method of characteristics, yields the joint probability density as a function of the real and imaginary parts of the Reflection Function and the length of the random slab. The mean reflected power $\langle |R|^2 \rangle$ computed from the characteristic function is shown to be identical to that derived earlier from the BVU Series using cumulant averaging techniques.

B. RECOMMENDATIONS

So far in this thesis, the problem of scattering has been treated from the viewpoint that the total a priori information on the material properties of the scattering medium (i.e. the slab) is available. A better approach would be to reconstruct the dielectric profile from the scattered (i.e. reflected and/or transmitted) waves; that is either the reflection coefficient or the impulse response of the medium. Some work has already been done in this field since it is of interest in a vast range of problems in the engineering sciences. However, no exact general inverse scattering theory that can yield practical solutions in a finite number of computation is available. The starting point for most of the work done in this area has been the Gelfand-Levitan [1955] and Marchenko [1963] theory for determining the potential of the Schrödinger Equation from the wave function form at large distances. The above authors show that for the one-dimensional time-independent Schrödinger equation

$$\frac{\partial^2}{\partial x^2} + (\omega^2 - V(x)) \chi = 0$$

(IV.3)
The potential can be determined from

\[ V(x) = 2 \frac{d}{dx} K(x, x) \]  

(IV.4)

where \( K(x, t) \) is the solution of the integral equation

\[ K(x, t) = -R(x+t) - \int_{-t}^{X} K(x, \tau) R(\tau+t) d\tau \]  

(IV.5)

where \( R(t) \) is the impulse response function of the medium. Ware and Aki [1969] by transforming the elastodynamic equations of motion (for either compressional or shear waves) into a Schrödinger equation have solved for the Lamé parameters by using the Gelfand-Levitan integral equation. They have also developed a discrete analog of the continuous solution. Berryman and Greene [1978] show the equivalence of the discrete and continuous inverse scattering problems and find a fast algorithm for the determination of the potential from the impulse response.

Recently, the inversion problem for the dissipative hyperbolic differential equation

\[ \left\{ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} + A(x) \frac{\partial}{\partial x} + B(x) \frac{\partial}{\partial t} + C(x) \right\} x = 0 \]  

(IV.6)

has been solved by Weston [1972]. It is shown that a dual set of generalized Gelfand-Levitan type integral equations are obtained. However, two incident waves (one from either side of the slab) are needed and the resultant reflected and transmitted waves are measured. It is also shown that the use of the transformation

\[ x = \int_{0}^{Z} \frac{1}{\mu_0 \varepsilon(s)} ds \]  

(IV.7)

allows one to use the same method to solve for the varying permittivity \( \varepsilon(z) \) and the conductivity \( \sigma(z) \) from the wave equation for the electric field, i.e.
A simpler equation (for scattering from cold plasma)

\[
\left\{ \frac{\partial^2}{\partial z^2} - \varepsilon(z)\mu_o \frac{\partial^2}{\partial t^2} - \sigma(z)\mu_o \frac{\partial}{\partial t} \right\} E = 0
\]  

(IV.8)

A simpler equation (for scattering from cold plasma)

\[
\left\{ \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - V(z) \right\} E = 0
\]  

(IV.9)

has also been treated by similar methods by Balanis [1972]. The inverse problem for the dissipative hyperbolic equation with discontinuous coefficients has been solved by Krueger [1976]. However, both the transmitted and reflected waves are needed. A simpler (and approximate) solution and algorithm which requires only the reflection coefficient or the impulse response of the medium is given by Coen [1981].

Further research into the use of radio reflections from glaciers should make use of the inversion solutions.
APPENDIX

For completeness, we now calculate the probability density p(a,b,z) of the process. We have from Eq. (III.84) and (III.83)

\[ p(a,\beta,z) = A_o \alpha \beta + A_1 (\alpha^2 - \beta^2) + A_2 \beta - A_3 z \]  \hspace{1cm} (A.1)

Substituting for \( r^2 \) in Eq. (A.1) results in

\[ p(a,\beta,z) = \exp\{-B_o \alpha^2 + B_1 \beta - B_2 \beta^2\} \]  \hspace{1cm} (A.2)

where

\[ B_o = \left[ \frac{\Omega z}{2} - A_1 \right] \]
\[ B_1 = [A_o \alpha + A_2] \]
\[ B_2 = \left[ \frac{\Omega z}{2} + A_1 \right] \]

so that a Fourier Transform with the Kernel \( \exp\{-i(\alpha a + \beta b)\} \) yields

\[ p(a,b,z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \exp\{-B_o \alpha^2 - i\alpha a + B_2 \beta^2 + B_1 \beta - ib\beta\} \]  \hspace{1cm} (A.4)

Completing the square of the argument in the second integral, and integrating we obtain

\[ p(a,b,z) = \frac{1}{4\pi^{3/2}B_2^{1/2}} \int_{-\infty}^{\infty} d\alpha \exp\{-B_o \alpha^2 - i\alpha a + B_2 \beta^2\} \]  \hspace{1cm} (A.4)

where

\[ B_3 = \frac{B_1 - ib}{2B_2} \]

Expansion of \( B_2 B_3^2 \) and substitution in Eq. (A.4) results in

\[ p(a,b,z) = H_0 \int_{-\infty}^{\infty} d\alpha \exp\{-C_0 \alpha^2 + C_1 \alpha\} \]  \hspace{1cm} (A.5)

where
Integration of Eq. (A.5) by completing the square gives the final result.

\[ p(a, b, z) = \Delta_0 \exp\left\{\frac{1}{4B_2} \left[ A_o^2 - 2iA_o a - b^2 \right] \right\} \]

where

\[ \Delta_0 = \frac{\exp\left\{ \frac{A_o^2}{4B_2} \right\} + \frac{A_o^2 A_2}{8B_2 C_0}}{2\pi \left( 4B_o B_2 - A_o^2 \right)^{1/2}} \]

\[ \Delta_1 = \frac{1}{2C_0} \]

\[ \Delta_2 = \frac{iA_o^2}{2C_0 B_2} \]

\[ \Delta_3 = \frac{A_o}{2C_0 B_2} \]

\[ \Delta_4 = \frac{iA_2}{2B_2} + \frac{iA_o^2 A_2}{4C_0 B_2^2} \]

\[ \Delta_5 = \frac{1}{4B_2} + \frac{A_o^2}{8C_0 B_2^2} \]
REFERENCES


