Fall 1982

ON A CONJUGATE CLASS OF SUBGROUPS DETERMINED BY A FORMATION

MARK CHALLIS HOFMANN

Follow this and additional works at: https://scholars.unh.edu/dissertation

Recommended Citation
https://scholars.unh.edu/dissertation/1338

This Dissertation is brought to you for free and open access by the Student Scholarship at University of New Hampshire Scholars' Repository. It has been accepted for inclusion in Doctoral Dissertations by an authorized administrator of University of New Hampshire Scholars' Repository. For more information, please contact nicole.hentz@unh.edu.
INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document photographed is “Missing Page(s)”. If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.

2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.

3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of “sectioning” the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.

4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.

5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.

University Microfilms International
300 N. Zeeb Road
Ann Arbor, MI 48106
Hofmann, Mark Challis

ON A CONJUGATE CLASS OF SUBGROUPS DETERMINED BY A FORMATION

University of New Hampshire

Ph.D. 1982

University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106
ON A CONJUGATE CLASS OF SUBGROUPS DETERMINED BY A FORMATION

BY

Mark C. Hofmann
M.S., University of New Hampshire, 1980

DISSERTATION

Submitted to the University of New Hampshire in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy in Mathematics

September, 1982
This thesis has been examined and approved.

Homer Bechtell
Thesis director, Homer F. Bechtell, Jr.
Professor of Mathematics

Eric A. Nordgren, Professor of Mathematics

Donovan H. Van Osdol, Professor of Mathematics

Samuel D. Shore, Associate Professor of Mathematics

Marie A. Gaudard, Assistant Professor of Mathematics

June 22, 1982
ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor Homer F. Bechtell for his constant guidance and encouragement.

He also wishes to thank Linda Hofmann for understanding and most of all for being there.
# TABLE OF CONTENTS

| ACKNOWLEDGEMENTS                                | iii  |
| ABSTRACT                                       | v    |
| CHAPTER                                        | PAGE |
| INTRODUCTION                                   | 1    |
| I. FUNDAMENTAL CONCEPTS                        | 10   |
| II. THE DEFINITION AND BASIC STRUCTURE OF f-SUBGROUPS | 33   |
| III. THE RELATIONSHIP OF AN f-SUBGROUP TO ITS CORE | 62   |
| IV. OTHER CHARACTERIZATIONS OF f-SUBGROUPS      | 81   |
| V. f-SUBGROUPS AND FORMATIONS                  | 107  |
| VI. FURTHER QUESTIONS                          | 125  |
| NOTATION                                       | 129  |
| BIBLIOGRAPHY                                   | 132  |
ABSTRACT

ON A CONJUGATE CLASS OF SUBGROUPS
DETERMINED BY A FORMATION

by

MARK C. HOFMANN

University of New Hampshire, September, 1982

This thesis is an investigation of the interrelationships between a formation $f$, a finite solvable group $G$, and $G_f$ the residual of $f$ in $G$. This study is developed by introducing the $f$-subgroups. It is proven that the $f$-subgroups of $G$ form a characteristic conjugacy class of CAR-subgroups of $G$. Moreover these subgroups generate $G_f$. As a result, $G$ is an element of the formation $f$ if and only if an $f$-subgroup is equal to the identity subgroup.

It is established that an $f$-subgroup is a product of known subgroups of the $f$-residual. The covering and avoidance properties of $f$-subgroups are examined and the extent to which these properties characterize the $f$-subgroups is found.

Next, it is proven that an $f$-subgroup is a prefrattini subgroup when $f$ is the formation of solvable nC-groups. Consequently, for the results obtained for $f$-subgroups corresponding results are valid for the prefrattini subgroups.

It is determined that a group $G$ belongs to the formation $f$ if
and only if $G$ has a series $1 = N_0 \leq N_1 \leq \ldots \leq N_n = G$ such that $N_{i+1}/N_i$ is a maximal nilpotent normal subgroup of $G/N_i$ and the core of an $f$-subgroup of $G/N_i$ is the identity subgroup for $i = 0, 1, \ldots, n-1$. This reduces to a corresponding result by G. Zacher when $f$ is the solvable $nC$-groups.

Other CAR-subgroups that generate the $f$-residual are examined. An $f$-subgroup is proven to be the intersection of certain CAR-subgroups of the $f$-residual. A result by H. Bechtell is obtained as a corollary when $f$ is the solvable $nC$-groups and an $f$-subgroup is a prefrattini subgroup.

A formation $f$ of finite solvable groups is saturated if and only if for each group $G$ every chief factor of the form $G_{f}/K$ is complemented. In this work totally nonsaturated formations are defined as those formations in which $G_{f}/K$ is never complemented for any group $G$. It is proven that the structure of $f$-subgroups determine if a formation is saturated, totally nonsaturated, or neither of these two types of formations.
INTRODUCTION

In 1962 W. Gaschutz [11] developed for finite solvable groups a characteristic conjugacy class of subgroups called the prefrattini subgroups. This class is characteristic in the sense that the class of conjugate subgroups formed by the prefrattini subgroups of a group G is invariant under all automorphisms of G. The prefrattini subgroups represent an effort to construct a subgroup with properties similar to the frattini subgroup, which is the intersection of all maximal subgroups of the group, and also having the property that the subgroup is preserved under homomorphism. The largest normal subgroup contained in a prefrattini subgroup is the frattini subgroup. It is the intersection of all the prefrattini subgroups of a group. In a note at the end of W. Gaschutz's paper it is mentioned that a prefrattini subgroup of a group is the identity subgroup if and only if the group splits over every normal subgroup. The class of solvable groups which split over every normal subgroup is a formation, the formation of solvable nC-groups [7]. A formation is a class of groups $\mathcal{f}$ with the properties: 1) if $G \in \mathcal{f}$ then every epimorphic image of G belongs to $\mathcal{f}$; 2) if $N, M \triangleleft G$, $G/N \in \mathcal{f}$ and $G/M \in \mathcal{f}$ then $G/(N \cap M) \in \mathcal{f}$. As a consequence of properties 1) and 2), it may be shown that any group G has a normal subgroup N of least order such that $G/N \in \mathcal{f}$. Such a subgroup, called the $\mathcal{f}$-residual, will be denoted by $G_{\mathcal{f}}$.

In a subsequent paper H. Bechtell [2] proved that a prefrattini subgroup of a group generates the residual for the formation of solvable
nC-groups. That is, the smallest normal subgroup of a group which contains the prefrattini subgroup is the residual of the formation of solvable nC-groups.

In this work the relationships between an arbitrary formation $f$, a finite solvable group $G$, and the residual $G_f$ are studied. This study is developed by introducing a new characteristic conjugacy class of subgroups for finite solvable groups. For an arbitrary formation $f$, a class of subgroups called $f$-subgroups is defined. These subgroups generate the $f$-residual of a group $G$. Consequently, a group $G$ belongs to the formation $f$ if and only if the $f$-subgroup is trivial. The largest normal subgroup contained in an $f$-subgroup, the core of an $f$-subgroup, will play a significant role in the structure and behavior of an $f$-subgroup.

The primary motivation for examining the formation $f$, and the $f$-residual, in this manner stems from the relationship of the prefrattini and frattini subgroups to the formation of solvable nC-groups. In fact, if $\kappa$ is the formation of solvable nC-groups a $\kappa$-subgroup is precisely a prefrattini subgroup. In this thesis, many results for $f$-subgroups, the $f$-residual, and the core of an $f$-subgroup, when restricted to the formation of solvable nC-groups, reduce to known results about the prefrattini subgroups, the $\kappa$-residual, and the frattini subgroup. In addition other results are derived for $f$-subgroups which shed new light on the relationships among the solvable nC-groups, the prefrattini subgroups and the frattini subgroup. Thus $f$-subgroups, rather than being a generalization of the prefrattini subgroups, are a conjugate class of subgroups defined by a formation $f$ for which the prefrattini subgroups appear as a special
case for the formation of solvable nC-groups.

The study of group properties via a conjugate class of subgroups undertaken here descends from work done by Phillip Hall. Finite solvable groups may be characterized in two distinct ways. They are those groups which have a chief series all of whose factors are abelian. This characterization through the factor structure of the group is a commutatorial one, for such groups have derived series terminating with the identity element. Also, finite solvable groups may be characterized by their Sylow subgroups. A finite group is solvable if and only if it has a set of Sylow subgroups \( \{S_1, S_2, \ldots, S_n\} \), one for each prime integer dividing the order of the group, such that \( S_i S_j = S_j S_i \) for \( i, j = 1, 2, \ldots, n \). Such a set of Sylow subgroups is called a Sylow system of \( G \). This definition involves the manner in which the Sylow subgroups of \( G \) are contained in \( G \). That these characterizations are equivalent is due to P. Hall. In [17], P. Hall introduced a characteristic conjugacy class of subgroups called system normalizers. One of these subgroups, defined with respect to a given Sylow system of a group \( G \), is the set of all elements of \( G \) which map every element of the Sylow system onto itself through conjugation. System normalizers link the Sylow structure to the factor structure through cover and avoidance properties. A subgroup \( Q \) covers a factor \( H/K \) if \( H \leq QK \) and \( Q \) avoids \( H/K \) if \( Q \cap H \leq K \). A system normalizer covers all central chief factors, that is, factors \( H/K \) satisfying \( H/K \leq Z(G/K) \). System normalizers avoid eccentric, that is, noncentral chief factors. A system normalizer has many other important properties. Two of these are of particular importance. A system normalizer is preserved under homomorphism and a Sylow system of a group reduces into the system
normalizer it defines. That is, if $D(s)$ is a system normalizer defined with respect to the Sylow system $S = \{S_1, S_2, \ldots, S_n\}$, then $\{D(s) \cap S_1, D(s) \cap S_2, \ldots, D(s) \cap S_n\}$ is a Sylow system of $D(s)$.

The investigation of group structure through the use of conjugate classes of subgroups was continued by R. Carter [4]. R. Carter developed a class of subgroups which relate maximal nilpotent subgroups to system normalizers. This class of subgroups was extended by W. Gaschütz [12] to the case of locally defined formations. These are formations defined in the following manner. Let there correspond to each prime integer $p$ a formation $g(p)$. Let $g$ be a class of finite solvable groups such that: $G \in g$ if and only if, for each prime integer $p$ dividing the order of $G$ and each $p$-chief factor $H/K$ of $G$, $A_G(H/K) \cong G/C_G(H/K) \in g(p)$. The class $g$ is a formation and it is said to be locally defined by $\{g(p)\}$. A formation $g$ is said to be saturated if $G/\phi(G) \in g$ implies $G \in g$, for which $\phi(G)$ is the Frattini subgroup of $G$. It is known [12; 21] that a formation is saturated if and only if it is locally defined. Gaschütz extended the class of Carter subgroups to $g$-covering subgroups for $g$ a locally defined formation. When $g$ is the formation of nilpotent groups a $g$-covering subgroup is a Carter subgroup.

In an effort to produce a class of subgroups that have the same relationship to $g$-covering subgroups as system normalizers have to Carter subgroups, R. Carter and T. Hawkes defined $g$-normalizers [5]. For the formation $g$ of nilpotent groups, $g$-normalizers coincide with system normalizers. The $g$-normalizers link the Sylow structure and factor structure of a group $G$ to the formation $g$. A $g$-normalizer belongs to the formation $g$ and it has a covering and avoidance property. A $p$-chief
factor $H/K$ is covered by a $g$-normalizer if $G/C_G(H/K) \in g(p)$; otherwise $H/K$ is avoided by a $g$-normalizer. The $g$-normalizers have properties similar to system normalizers in that, they are preserved under homomorphism and the Sylow system used to define a $g$-normalizer reduces into the $g$-normalizer.

The primary motivation for this work, as stated earlier, stems from a class of subgroups introduced by W. Gaschutz. He defined through a relatively complex method involving $G$-modules, the prefrattini subgroups. In a related paper T. Hawkes [18] gave an equivalent although much simplified definition. The prefrattini subgroup of a group $G$, defined with respect to a given Sylow system, is the intersection of all maximal subgroups of $G$ into which the Sylow system reduces. The prefrattini subgroups form a characteristic conjugacy class. They are preserved under homomorphism and the Sylow system used to define a prefrattini subgroup reduces into that prefrattini subgroup. Prefrattini subgroups have a covering and avoidance property. A chief factor $H/K$ is said to be complemented if there exists a maximal subgroup $M$ of $G$ such that $G/K = (H/K)(M/K)$ and $H \cap M = K$. A noncomplemented chief factor is called a frattini chief factor, since $H/K$ is noncomplemented if and only if $H/K \leq \Phi(G/K)$. A prefrattini subgroup covers frattini chief factors and avoids complemented chief factors. For reemphasis, the largest normal subgroup of $G$ contained in a prefrattini subgroup is the frattini subgroup and the smallest normal subgroup of $G$ that contains a prefrattini subgroup is the residual in $G$ of the formation of solvable nC-groups.

The $g$-prefrattini subgroups presented by T. Hawkes in [18] result from the fusion of the cover and avoidance properties of $g$-normalizers
and prefrattini subgroups. The $g$-prefrattini subgroups, defined for a saturated formation $g$, cover those chief factors that are covered by either a prefrattini subgroup or a $g$-normalizer. They avoid all other chief factors; those factors avoided by both $g$-normalizers and prefrattini subgroups. In fact a $g$-prefrattini subgroup is the product of a $g$-normalizer and a prefrattini subgroup. The $g$-prefrattini subgroups form a characteristic conjugacy class, they are preserved under homomorphisms and the Sylow system used to define a $g$-prefrattini subgroup reduces into the $g$-prefrattini subgroup.

An element of any of the classes of subgroups discussed above is defined with respect to a Sylow system. Each subgroup has a covering and avoidance property. Each subgroup in some manner is connected to a formation, either by definition, or implicitly, as in the case of prefrattini subgroups. Consequently, each of the classes links the Sylow structure and the factor structure to some formation.

With regard to the above classes of groups and formation theory in general two points should be made. First, the classes of groups primarily deal with saturated formations. The exception is the prefrattini subgroups. However, the connection of prefrattini subgroups to the nonsaturated formation of solvable $nC$-groups appears as a consequence of the properties of the prefrattini subgroup rather than as an explicit connection through definition. Second, very few results have dealt with the relationship between the structure of the residual of an arbitrary formation and its embedding in the group. Exceptions are the specialized case of the residual being abelian and the results of H. Bechtell dealing with the relationship of a prefrattini subgroup to the solvable $nC$-residual.
The class of subgroups introduced in this thesis are defined for any formation \( f \), either saturated or nonsaturated. The class enables one to study the relationships among the Sylow structure, the factor structure and the formation \( f \). Emphasis will be placed on the residual of the formation, its structure, and its position in the group.

Each class of subgroups discussed so far, as well as having a covering and avoidance property, has a reduction property. That is, if a Sylow system is used to define a class element, then the Sylow system reduces into the element. Such classes of groups have been extensively investigated by M. Tomkinson [22;23]. Each one of which is a class of CAR-subgroups, CAR denoting covering, avoiding and reducing. The \( f \)-subgroups are CAR-subgroups, and the results of Tomkinson are relied upon heavily. They enable one to deduce important conclusions about covering and avoidance properties and the permutability of certain CAR-subgroups.

This dissertation is divided into six chapters. The first chapter, Fundamental Concepts, contains known or easily proven results that are necessary for the development of the \( f \)-subgroups and their properties.

In Chapter II, \( f \)-subgroups are defined in terms of a CAR-subgroup. The cover and avoidance properties of an \( f \)-subgroup are developed. Alternate characterizations of an \( f \)-subgroup are given which relate the structure of an \( f \)-subgroup with a certain \( g \)-prefrattini subgroup and to subgroups of the \( f \)-residual. Two results of fundamental importance given in this chapter are that an \( f \)-subgroup generates the \( f \)-residual and that a group \( G \in f \) if and only if the \( f \)-subgroups are equal to the identity subgroup. In this chapter the connection between \( f \)-subgroups and the prefrrattini subgroups is shown; namely if \( k \) is the formation of solvable \( nC \)-groups, a \( k \)-subgroup is a prefrrattini subgroup.
The extent to which cover and avoidance properties and a Sylow system may characterize an $f$-subgroup is fully determined. Finally it is proven that an $f$-subgroup is a characteristic conjugacy class of subgroups of $G$.

Chapter III focuses attention on the core of an $f$-subgroup. Many covering and avoidance properties and structure theorems for prefrattini subgroups can be explained by the properties of a frattini subgroup. For $f$-subgroups, the analogue of the frattini subgroup is the core of an $f$-subgroup. It is highly significant in explaining $f$-subgroup properties. The structure of the core is given in terms of known subgroups of the $f$-residual. Necessary and sufficient conditions are given for the core of an $f$-subgroup to equal an $f$-subgroup. A characterization of the core is given by identifying its elements with a well-defined property. This characterization provides for a new definition of the frattini subgroup through the use of nongenerators, but not in the traditional sense. In the second section of Chapter III a chain condition on a group $G$ is established that forces $G$ to be in the formation $f$. The result reduces to a well known result by G. Zacher [24] whenever $f$ is taken to be the solvable $nC$-groups. Zacher's original proof did not rely on the theory of prefrattini subgroups. In the third section, a chain of subgroups is used to find yet another characterization of the $f$-subgroup. An $f$-subgroup is shown to be the product of the cores of $f$-subgroups of elements of the chain. This result allows a characterization of the prefrattini subgroup as a product of frattini subgroups of chain elements.

In the fourth chapter other CAR-subgroups of a group and CAR-subgroups of the $f$-residual are used to determine the structure of an $f$-
subgroup. It is shown that if a group is a direct product, then an $f$-subgroup is the direct product of the $f$-subgroups of the factor groups. In the second section of this chapter, an $f$-subgroup is proven to be the intersection of a particular class of CAR-subgroups of the $f$-residual. This reduces to a known result by H. Bechtell [2] for the prefattini subgroup when $f$ is taken to be the formation of solvable nC-groups. In this third section it is proven that an $f$-subgroup is the intersection of a certain collection of maximal subgroups of the $f$-residual. In the final section a negative answer is given to whether or not a prefattini subgroup is the product of a prefattini subgroup of the solvable nC-residual and a system normalizer of the residual.

The next chapter, Chapter V, contains results on the interaction between $f$-subgroups and formations. First the relationship of $f$-subgroups to other CAR-subgroups which generate the residual is examined. Then $f$-subgroups are studied closely whenever $f$ is a normal subgroup inherited formation. Finally $f$-subgroups are used to characterize saturated formations. A dual to the concept of saturated formation is defined. It is shown that the formation of solvable nC-groups is the "smallest" formation dual to saturated formations.

The last chapter contains unanswered questions which are motivated by this work.
CHAPTER I

FUNDAMENTAL CONCEPTS

In this chapter essential background material is provided. If a result is known, no proof is offered. Proofs are included for results for which no reference is available.

Some knowledge of group theory is assumed such as the definitions and basic properties of nilpotent groups, solvable groups, chief series, the frattini subgroup $\phi(G)$, the center of a group $Z(G)$, the hypercenter of a group $Z_\infty(G)$, the commutator subgroup $G'$, and the hypercommutator subgroup $K_\infty(G)$.

1. Basic Results

This section contains general group theoretic results which appear frequently in proofs in succeeding chapters.

1.1 Theorem: [10] For a group $G$, $G' \cap Z_\infty(G) \leq \phi(G)$.

1.2 Theorem: [19] Suppose that $A$ and $B$ are permutable subgroups of a group $G$. If $C$ is a subgroup of $G$ for which $A \leq C$, then the modular identity, $AB \cap C = A(B \cap C)$ is satisfied.

1.3 Definition: The subgroup $A$ of the group $G$ is called abnormal in $G$ if for all $g \in G$, $g \in <A,A^g>$. 
1.4 Theorem: [19] The following conditions are equivalent:
(i) A is abnormal in the group G;
(ii) $A \leq H \leq G$ and $A \leq H \cap H^g$ implies that $g \in H$.

1.5 Definition: [19] If M is a subgroup of the group G, core M will denote the largest normal subgroup of G contained in M. Moreover, $\text{core } M = \bigcap_{g \in G} M^g$.

1.6 Theorem: [19] A normal subgroup A in a finite group G is nilpotent if and only if $A/(A \cap \Phi(G))$ is nilpotent.

1.7 Definition: Let G be a group with normal subgroup A. If $G = AC$ and $A \cap C = 1$ then G splits over A by C. The subgroup C is a complement of A, write $G = [A]C$.

1.8 Theorem: [19] Let G be a primitive permutation group of degree n and N a minimal normal subgroup of G. If N is solvable then:
(i) N is regular and elementary abelian. The degree n of G is a power of a prime integer p.
(ii) $G = [N]G_1$ for some $G_1 \leq G$.
(iii) $C_G(N) = N$.
(iv) N is the only minimal normal subgroup of G.
(v) $G_1$ has no normal subgroups of p-power order.
(vi) If G is solvable, all complements of N in G are conjugate.
1.9 Theorem: [10] If $A$ is an abelian normal subgroup of a finite group $G$ and $A \cap \phi(G) = 1$, then $G = [A]B$ for some subgroup $B$ of $G$. Moreover, $A$ is a direct product of minimal normal subgroups of $G$.

1.10 Theorem: [19] If for an abelian subgroup $N$ normal in a finite group $G$, $[G:N]$ and $|N|$ are relatively prime, then $G$ splits over $N$ and all complements of $N$ are conjugate.

1.11 Definition: A factor $M/N$ is called complemented if there exists a subgroup $H$ such that $G/N = [M/N]H/N$. Otherwise it is called a Frattini chief factor.

1.12 Definition: The sockel $S(G)$, of a finite group $G$, is defined to be the product of all minimal normal subgroups of $G$, if $G \neq 1$; and $S(G) = 1$ if $G = 1$.

1.13 Theorem: Let $N$ be a normal subgroup of the group $G$ and $N < S(G)$. Then there exists a normal subgroup $M$ of $G$ such that $S(G) = N \times M$.

Proof: Let $K_1, K_2, \ldots, K_n$ be all the minimal normal subgroups of $G$ such that $K_i \cap N = 1$, $i=1,2,\ldots,n$. Then $S(G)$ is generated by $N$ and these minimal normal subgroups. It will be shown that there is a subset \{i_1, i_2, \ldots, i_m\} of \{1,2,\ldots,n\} such that $S(G) = N \times K_{i_1} \times K_{i_2} \times \ldots \times K_{i_n}$. Let $U$ denote the set of all nonempty subsets \{i_1, i_2, \ldots, i_m\} of \{1,\ldots,n\} such that $i_1, i_2, \ldots, i_m$ are distinct and $N K_{i_1} K_{i_2} \ldots K_{i_n} = N \times K_{i_1} \times K_{i_2} \times \ldots \times K_{i_n}$. 


Since \( K_i \cap N = 1 \), \( \{i\} \in U \) for each \( i=1,2,\ldots,n \). Choose \( \{i_1,i_2,\ldots,i_m\} \in U \) with \( m \) as large as possible, and let \( L = N \times K_{i_1} \times K_{i_2} \times \cdots \times K_{i_m} \). Assume that \( S(G) \neq L \), then there exists \( l \in \{1,2,\ldots,n\} \) such that \( K_l \not\subseteq L \). Hence \( K_l \cap L = 1 \) and \( LK_l = L \times K_l \). Let \( i_{m+1} = 1 \). Since \( K_{i_j} \subseteq L \) for each \( j=1,\ldots,m, i_{m+1} \) is distinct from \( i_1,\ldots,i_m \) and \( N K_{i_1} K_{i_2} \cdots K_{i_m} K_{i_{m+1}} = N \times K_{i_1} \times K_{i_2} \times \cdots \times K_{i_m} \times K_{i_{m+1}} \). This contradicts the choice of \( m \). Therefore \( L = S(G) \), and so there exists \( \{i_1,\ldots,i_m\} \) such that \( S(G) = N \times K_{i_1} \times K_{i_2} \times \cdots \times K_{i_m} \). By letting \( M = K_{i_1} \times K_{i_2} \times \cdots \times K_{i_m} \), \( M \triangleright G \) and \( S(G) = N \times M \).

2. Covering and Avoidance

The concept of covering and avoidance was introduced by Phillip Hall [17]. It plays a major role in the following work on \( f \)-subgroups.

2.1 Definition: Let \( K \leq H \leq G \) and \( A \leq G \). Then

(i) \( A \) covers \( H/K \) if \( |H:K| = |H \cap A:K \cap A| \); equivalently, \( K(H \cap A) = H \lor H \leq KA \).

(ii) \( A \) avoids \( H/K \) if \( |H \cap A:K \cap A| = 1 \); equivalently, \( K(H \cap A) = K \lor H \cap A = K \).

2.2 Theorem: [19] Let \( G \geq A_1 \geq A_2 \geq \cdots \geq A_K \) and \( B \leq G \). If \( B \) covers (avoids) \( A_i/A_{i+1} \) for \( i=1,2,\ldots,K-1 \) then \( B \) covers (avoids) \( A_1/A_K \).

2.3 Theorem: Let \( L \) be a subgroup of \( G \) with the property that in a given series \( C \) of \( G \) if \( H/K \in C \), either \( L \) covers \( H/K \) or \( L \) avoids \( H/K \).
Then the order of $L$ is equal to the product of the orders of the factors of $C$ covered by $L$.

**Proof:** Let $G = G_0 > G_1 > \ldots > G_n = 1$ be the series $C$. Then, $|G| = [G:G_1][G_1:G_2] \ldots [G_{n-1}:G_n]$. So, $L = G_0 \cap L \geq G_1 \cap L \geq \ldots \geq G_n \cap L = 1$ is a series for $L$ and $|L| = [G_0 \cap L:G_1 \cap L][G_1 \cap L:G_2 \cap L] \ldots [G_{n-1} \cap L:G_n \cap L]$. If $L$ avoids $G_i/G_{i+1}$, then $[G_i \cap L:G_i+1 \cap L] = 1$ by Definition 2.1. If $L$ covers $G_i/G_{i+1}$, then Definition 2.1 implies $[G_i \cap L:G_{i+1} \cap L] = [G_i:G_{i+1}]$. Consequently, $|L|$ is the product of the orders of factors of $C$ covered by $L$.

**2.4 Theorem:** Let $L$ be a subgroup of $G$ and $C$ a chief series of $G$. Let $B$ be a subset of all the chief factors from $C$. If $L$ covers every factor from the set $B$ and the order of $L$ is equal to the product of the orders of the factors in $B$ then $L$ avoids all other factors of $C$.

**Proof:** Using the same series for $L$ as in the proof of Theorem 2.3, if $[G_i \cap L:G_{i+1} \cap L] \neq 1$ for some factor $G_i/G_{i+1} \not\in B$, then $|L|$ is greater than the product of the orders of factors from $B$. Hence if $G_i/G_{i+1} \not\in B$, then $[G_i \cap L:G_{i+1} \cap L] = 1$. By Definition 2.1, $L$ avoids $G_i/G_{i+1}$.

**3. Sylow Systems and System Normalizers**

In this section and for the remainder of Chapter I all groups under consideration are assumed to be **finite** and **solvable**.
3.1 Definition: Let \( |G| = \prod_{i=1}^{r} p_i^{a_i} \) be the prime factorization of the order of the group G.

(i) A collection \( S = \{S_{p_1}, S_{p_2}, \ldots, S_{p_r}\} \) of Sylow subgroups of G is called a Sylow system of G provided that \( S_{p_i} S_{p_j} = S_{p_j} S_{p_i} \) for all \( i, j \).

(ii) For a Sylow system \( S \) and prime integer \( p_i \), the subgroup \( S^{p_i} = S_{p_1} S_{p_2} \ldots S_{p_i-1} S_{p_i+1} \ldots S_{p_r} \) is called a Sylow \( p_i \)-complement of G.

The collection \( K = \{S^{p_1}, S^{p_2}, \ldots, S^{p_r}\} \) of Sylow complements of G is called a complement system of G.

3.2 Theorem: [16] The finite group G is solvable if and only if G has a Sylow system.

3.3 Theorem: [16] Any two Sylow systems of a solvable group are conjugate.

By definition every Sylow system gives rise to a complement system and through intersections each complement system may be used to recreate the original Sylow system. Consequently, for a Sylow system \( S = \{S_{p_1}, \ldots, S_{p_n}\} \) the convention of stating that \( S^{p_i} \in S \) will be adopted where it is understood that \( S^{p_i} = S_{p_1} S_{p_2} \ldots S_{p_i-1} S_{p_i+1} \ldots S_{p_n} \).

3.4 Definition: Let \( S = \{S_{p_i} | p_i \text{ dividing } |G|\} \) be a Sylow system of G and \( K = \{S^{p_i} | p_i \text{ dividing } |G|\} \) be the associated complement system. Then \( N(S) = \bigcap_1^r N_G(S_{p_i}) \) and \( N(K) = \bigcap_1^r N_G(S^{p_i}) \). The group \( N(S) \) is the system normalizer of the Sylow system \( S \). The group \( N(K) \) is the system normalizer of the complement system \( K \).
As a result of the definition of a Sylow $p_i$-complement and the fact that $S_{p_i} = \bigcap_{j \neq i} S_j$, $N(S) = N(K)$. So both may be referred to as a system normalizer.

3.5 Definition: Let $S = \{S_p \mid p \text{ dividing } |G|\}$ be a Sylow system of $G$, $A$ a subgroup of $G$. The Sylow system $S$ is said to reduce into $A$, or $S$ is reducible in $A$, if and only if $S \cap A = \{S_p \cap A \mid p \text{ dividing } |A|\}$ is a Sylow system of $A$. Equivalently, $S$ reduces into $A$ if and only if $\{S_p \cap A \mid p \text{ dividing } |A|\}$ is a complement system of $A$.

3.6 Theorem: [17] Let $S$ be a Sylow system of the group $G$.

(i) $S$ is reducible in every normal subgroup of $G$.

(ii) $S$ is reducible in the system normalizer $D = N(S)$.

3.7 Theorem: [17]

(i) Each system normalizer of the group $G$ is nilpotent.

(ii) All system normalizers of $G$ are conjugate in $G$.

(iii) Let $N(S)$ be the system normalizer of the Sylow system $S = \{S_p \mid p \text{ dividing } |G|\}$ of $G$ and let $K$ be a normal subgroup of $G$. Then $N(S)K/K$ is the system normalizer of the Sylow system $SK/K = \{S_pK/K \mid p \text{ dividing } |G/K|\}$ of $G/K$.

3.8 Definition:

(i) The chief factor $H/K$ of the group $G$ is a central chief factor if for all $h \in H$ and $g \in G$, $h^gK = hK$. Therefore $H/K$ is central whenever $H/K \leq Z(G/K)$ and whenever $[H,G] \leq K$.

(ii) Each noncentral chief factor of $G$ is called eccentric.
3.9 Theorem: [17] Each system normalizer of the group G covers every central chief factor of G and avoids every eccentric chief factor of G.

3.10 Theorem: [17]

(i) No proper normal subgroup of the group G contains a system normalizer of G.

(ii) The group G is generated by the system normalizers of G.

(iii) If $S$ is a Sylow system of G, then $\bigcap_{g \in G} N_G(N(g(S))) = Z_\infty(G)$.

3.11 Theorem: [17] Let M be a maximal but nonnormal subgroup of the group G, and $S$ a Sylow system of G.

(i) If $[G:M] = p^n$, and $S^p \in S$, then $S$ reduces into M if and only if $S^p \leq M$.

(ii) If $S$ is a Sylow system of G which reduces into M, then $N_G(S) \leq N_M(S \cap M)$, for which $S \cap M$ is the reduction of $S$ in M.

Implicit in Theorem 3.11 is that the index of a maximal subgroup is the power of a prime. While this is not true in general, it is valid for finite solvable groups. Part (ii) is of significance for it implies that $N_G(S) \leq M$.

3.12 Definition: Let $S$ be a Sylow system of the group G and $A$ a subgroup of G such that $S$ reduces into $A$. Then $N_G(A \cap S) = \bigcap_{p} N_G(S^p \cap A)$ is the relative system normalizer of $A \cap S$. 
3.13 Theorem: [17] Let $S$ be a Sylow system of the group $G$ and $A$ a subgroup of $G$. If $S$ reduces into $A$, then $N_G(A \cap S) \cap A = N_A(A \cap S)$.

3.14 Theorem: [17] Let $N$ and $K$ be normal subgroups of the group $G$ and let $N_G(S)$ be the relative system normalizer of the Sylow system $S = \{S_p | p \text{ dividing } |N|\}$ of $N$.

(i) $N_G(S)/K$ is the relative system normalizer of the Sylow system $SK/K = \{S_pK/K | p \text{ dividing } |NK/K|\}$ of $NK/K$ in the group $G/K$.

(ii) $G = N_G(S)N$.

3.15 Theorem: Let $S_p$ be a Sylow $p$-complement of $G$, then $N_G(S^p)$ is an abnormal subgroup of $G$.

Proof: Let $A = N_G(S^p)$, according to Theorem 1.4 it must be shown that for each $g \in G$, $g \in <A,A^g>$. Let $g \in G$ and $H = <A,A^g>$, then $S^p$ and $(S^p)^g$ are Sylow $p$-complements of $H$. Theorem 3.3 implies that there exists $h \in H$ such that $(S^p)^g = (S^p)^h$. Therefore $gh^{-1} \in N_G(S^p) = A \leq H$. Hence $g \in H = <A,A^g>$.

4. Formations and Saturated Formations

This section provides the basic results concerning formations, which were defined by W. Gaschutz [12].

4.1 Definition: A class $\mathcal{F}$ of groups is called a formation if
the following properties are satisfied:

(i) $G \in \mathcal{F}$ implies $G^\theta \in \mathcal{F}$ for every homomorphism $\theta$ of $G$.

(ii) If $N_1$ and $N_2$ are normal subgroups of $G$ and $G/N_1, G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$.

4.2 Theorem: [12] If $\mathcal{F}$ is a formation, then in each group $G$ there exists a uniquely determined normal subgroup $G^\mathcal{F}$ such that, $G/G^\mathcal{F} \in \mathcal{F}$ and if $G/N \in \mathcal{F}$, then $G^\mathcal{F} \leq N$. The subgroup $G^\mathcal{F}$ is called the $\mathcal{F}$-residual of $G$.

4.3 Definition: [13] A nonempty formation $\mathcal{F}$ is called saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$.

4.4 Definition: For each prime $p$, let there correspond a (possibly empty) formation $g(p)$. The class of groups $g$ that satisfy the following properties is called the formation locally defined by $\{g(p)\}$.

(i) If $g(p) = \emptyset$ then $p \nmid |G|$. 

(ii) If $g(p) \neq \emptyset$ and $H/K$ is a chief factor of $G$ whose order is divisible by $p$, then $\text{Aut}_G(H/K) \cong G/C_G(H/K)$ is a group belonging to the formation $g(p)$.

That a locally defined class of groups is indeed a formation is due to W. Gaschutz [12]. The following result by W. Gaschutz and U. Lubeseder connects locally defined formations and saturated formations.
4.5 Theorem: [12;21] A formation $g$ is locally defined if and only if $g$ is a saturated formation.

4.6 Theorem: [8] Let $f$ be a formation and consider the group $G = G_1 \times G_2 \times \ldots \times G_n$. Then $G_f = G_1^f \times G_2^f \times \ldots \times G_n^f$, for which $G_i^f$ is the $f$-residual of the factor $G_i$ for $i = 1, 2, \ldots, n$.

4.7 Definition: For a formation $f$, let $c(f)$ be the class of all groups $G$ in which a chief factor $H/K$ of $G$ is complemented whenever $H \leq G_f$.

4.8 Theorem: [3] The class $c(f)$ is a formation.

5. $g$-normalizers

R. Carter and T. Hawkes extended the notion of a system normalizer to that of a $g$-normalizer for a saturated formation $g$. For the formation $N$ of nilpotent groups, an $N$-normalizer is a system normalizer. In the following let $g$ be a formation locally defined by $\{g(p)\}$.

5.1 Definition: Let $S$ be a Sylow system of the group $G$ and $G_{g(p)}$ the $g(p)$-residual of $G$. Define $T^p = G_{g(p)} \cap S^p$ for $S^p$. The Sylow $p$-complement from $S$. The set $T = \{T^p | p \text{ dividing } |G|\}$ is called a $g$-system of $G$. The normalizer of $T$ in $G D^G(S) = \bigcap_p N_G(T^p)$ is called the $g$-normalizer of $G$ associated with $S$. 
5.2 Definition: A p-chief factor $H/K$ of $G$ is said to be $g$-central if $\text{Aut}_G(H/K) \in g(p)$ and $g$-eccentric otherwise.

5.3 Definition: A maximal subgroup will be called $p$-maximal if its index is a power of the prime $p$.

5.4 Definition: A $p$-maximal subgroup of $G$ will be called $g$-normal if $M/\text{core } M \in g(p)$ and $g$-abnormal otherwise.

The following theorem provides the connection between $g$-normal subgroups and $g$-central chief factors.

5.5 Theorem: [5] A maximal subgroup of $G$ is $g$-normal if and only if it complements a $g$-central chief factor of $G$.

That a $g$-system normalizer behaves in a manner similar to a system normalizer is pointed out in the next two theorems.

5.6 Theorem: [5] A $g$-system normalizer of the group $G$ covers every $g$-central chief factor of $G$ and avoids every $g$-eccentric chief factor of $G$.

5.7 Theorem: [5]

(i) Let $N$ be a normal subgroup of the group $G$ and $S$ a Sylow system of $G$. If $D^g(S)$ is the $g$-normalizer of $G$ associated with $S$, then $D^g(S)N/N$ is the $g$-normalizer of $G/N$ associated with $SN/N$.

(ii) All $g$-normalizers of $G$ are conjugate in $G$. 

Theorem 5.7 allows the conclusion that $g$-normalizers are preserved under homomorphisms.

5.8 Theorem: [5] The $g$-normalizers of the group $G$ belong to the formation $g$.

The relationship between $g$-normalizers and $g$-abnormal subgroups is now strengthened.

5.9 Theorem: [5] A maximal subgroup $M$ of the group $G$ contains some $g$-normalizer of $G$ if and only if $M$ is $g$-abnormal in $G$.

5.10 Theorem: [5] A group $G$ belongs to the formation $g$ if and only if every minimal normal subgroup of $G/\phi(G)$ is $g$-central.

5.11 Definition: A $g$-abnormal maximal subgroup $M$ of the group $G$ such that $F(G)M = G$, for which $F(G)$ is the Fitting subgroup of $G$, is called a $g$-critical maximal subgroup.

5.12 Definition: A chief factor $H/K$ of the group $G$ is called $g$-critical if it is a complemented $g$-eccentric chief factor such that every chief factor of $G$ below $K$ is either $g$-central or frattini.

5.13 Theorem: [5] A group $G$ has $g$-critical maximal subgroups if and only if $G \nmid g$. 
5.14 Theorem: [5] A maximal subgroup of the group G is \( g \)-critical if and only if it complements a \( g \)-critical chief factor of G.

5.15 Theorem: [5] If M is a \( g \)-critical maximal subgroup of the group G, then each \( g \)-normalizer of M is a \( g \)-normalizer of G. In particular if \( S \) reduces into M, then the \( g \)-normalizer of M associated with \( S \cap M \) is the \( g \)-normalizer of G associated with \( S \).

6. Prefrattini Subgroups

Although the prefrattini subgroup was first defined by W. Gaschütz [11], the next definition is preferred.

6.1 Definition: Let \( S \) be a Sylow system of the group G and \( S^p \) the associated Sylow p-complement of G. For a fixed prime integer p define \( W_p(S) = \cap \{M | M \text{ is maximal in } G \text{ and } M \supseteq S^p \} \). The prefrattini subgroup of G associated with \( S \) is defined to be \( W(S) = \cap_{p} W_p(S) \).

Recall from Definition 1.11 that a chief factor \( H/K \) of G is called complemented if \( H/K \) has a complement in the group \( G/K \) and frattini otherwise.

Prefrattini subgroups have properties similar to system normalizers and $g$-normalizers.

**6.3 Theorem: [11]**

(i) If $N$ is a normal subgroup of the group $G$ and $W(s)$ is the prefrattini subgroup of $G$ associated with the Sylow system $S$, then $W(s)N/N$ is the prefrattini subgroup of $G/N$ associated with the Sylow system $sN/N$.

(ii) All prefrattini subgroups of $G$ are conjugate in $G$.

The prefrattini subgroups' definition was motivated by the fact that frattini subgroups are not necessarily preserved under homomorphism. However the frattini subgroup plays a very important role in describing properties of prefrattini subgroups.

**6.4 Theorem: [11]** The intersection of all prefrattini subgroups of the group $G$ is the frattini subgroup of $G$.

An implication of Theorem 6.4 is that the core of a prefrattini subgroup is the frattini subgroup since Theorem 6.3(ii) implies that the intersection of all the prefrattini subgroups of $G$ is equal to the intersection of all the conjugates of one prefrattini subgroup. This fact and Theorem 6.3(i) explain the covering and avoidance property of prefrattini subgroups. For if $H/K$ is frattini, then $H/K \leq \phi(G/K) \leq W(s)K/K$. Hence $H/K$ is covered by $W(s)$. If $H/K$ is complemented, then $H/K \cap W(s)K/K = 1$, else $H/K \cap \phi(G/K) \neq 1$. Thus $H/K$ is avoided by $W(s)$. 
6.5 Definition: [7] Let \( \kappa \) denote the collection of solvable groups that split over every normal subgroup. Then \( \kappa \) is a formation, the formation of solvable nC-groups.

6.6 Theorem: [11] A prefrattini subgroup of the group \( G \) is the identity subgroup if and only if each normal subgroup of \( G \) is complemented. That is, for any Sylow system \( S \) of \( G \), \( W(S) = 1 \) if and only if \( G \in \kappa \).

The interrelationship between prefrattini subgroups and the formation of solvable nC-groups was further developed by H. Bechtell. In the following let \( G_\kappa \) denote the residual in the group \( G \) of the formation \( \kappa \) of solvable nC-groups.

6.7 Theorem: [2] The \( \kappa \)-residual \( G_\kappa \) is the normal closure of the prefrattini subgroups in the group \( G \).

6.8 Theorem: [2] The \( \kappa \)-residual \( G_\kappa \) is nilpotent if and only if \( G_\kappa = \phi(G) \).

6.9 Theorem: [2]

(i) The prefrattini subgroups of \( G \) are a conjugate class in \( G_\kappa \).

(ii) The prefrattini subgroup \( W(S) \) of \( G \) associated with the Sylow system \( S \) contains a system normalizer of \( G_\kappa \). In particular, if \( D_{G_\kappa}(S) \) is the system normalizer of the reduction of \( S \) into \( G_\kappa \), then \( D_{G_\kappa}(S) \leq W(S) \).
H. Bechtell has also established that the solvable nC-groups are a normal formation [1], that is, if \( G \) is a solvable nC-group and \( N \triangleleft G \), then \( N \) is a solvable nC-group. This property is reflected by the prefattini subgroups.

6.10 Theorem: [2] If \( M \) is a normal subgroup of the group \( G \) and if \( M \) contains \( G_\kappa \), then each prefattini subgroup of \( M \) is contained in a prefattini subgroup of \( G \).

7. \( g \)-Prefattini Subgroups

The first generalization of the prefattini subgroups is due to T. Hawkes [18].

7.1 Definition: Let \( S \) be a Sylow system of the group \( G \), \( S^p \) the associated Sylow \( p \)-complement of \( G \), and \( g \) a locally defined formation. Denote by \( M^p \) the set of \( g \)-abnormal maximal subgroups of \( G \) containing \( S^p \). Set \( W^g_p(S) = \bigcap \{ M | M \in M^p \} \). Then \( W^g_p(S) = \bigcap W^g_p(S) \) is the \( g \)-prefattini subgroup of \( G \) associated with \( S \).

7.2 Theorem: [18] If \( (M^p)^* \) is a subset of \( M^p \) comprising exactly one complement of each complemented \( g \)-eccentric \( p \)-chief factor in a given chief series of \( G \), then \( W^g_p(S) = \bigcap \{ M | M \in (M^p)^* \} \).

By using the convention that if \( g(p) = \phi \), then every \( p \)-chief factor is \( g \)-eccentric and every \( p \)-maximal subgroup is \( g \)-abnormal the
definition of a prefrattini subgroup is recovered from Definition 7.1. Consequently for Theorem 7.2 and for the results that follow, corresponding results also hold for the prefrattini subgroups.

The next two results are corollaries of Theorem 7.2.

7.3 Corollary: [18] Let \( S \) be a Sylow system of the group \( G \). If \( N \) is a set of \( g \)-abnormal maximal subgroups of \( G \), each containing a Sylow \( p \)-complement of \( S \) for some \( p \), and if \( N \) contains at least one complement of each complemented \( g \)-eccentric chief factor of a given chief series of \( G \), then \( W^G(s) = \cap \{ M | M \in N \} \).

7.4 Corollary: [18] A \( g \)-abnormal maximal subgroup \( M \) of the group \( G \) contains \( W^G(s) \) whenever the Sylow system \( S \) reduces into \( M \).

7.5 Theorem: [18]
(i) If \( N \) is a normal subgroup of the group \( G \) and \( W^G(s) \) is the \( g \)-prefrattini subgroup of \( G \) associated with the Sylow system \( S \), then \( W^G(s)N/N \) is the \( g \)-prefrattini subgroup of \( G/N \) associated with the Sylow system \( SN/N \).

(ii) All \( g \)-prefrattini subgroups of \( G \) are conjugate in \( G \).

The structure of \( g \)-prefrattini subgroups, with respect to \( g \)-normalizers and prefrattini subgroups, is discovered through consideration of covering and avoidance properties.
7.6 Theorem: [18] A $g$-prefrattini subgroup avoids every complemented $g$-eccentric chief factor of $G$ and covers the rest.

Theorem 7.6 states that a $g$-prefrattini subgroup avoids chief factors that are simultaneously avoided by a $g$-normalizer and a prefrattini subgroup. It covers chief factors that are covered either by a $g$-normalizer or by a prefrattini subgroup. T. Hawkes uses this fact to prove that a $g$-prefrattini subgroup is the product of a $g$-normalizer and a prefrattini subgroup.

7.7 Theorem: [18] If $D^g(S)$ is a $g$-normalizer of $G$ and $W(S)$ is a prefrattini subgroup of $G$ both associated with the same Sylow system $S$, then $D^g(S)$ and $W(S)$ permute and $D^g(S)W(S) = W^g(S)$.

8. CAR-subgroups

System normalizers, prefrattini subgroups and $g$-system normalizers each have a covering and avoidance property. They each are defined as intersections of subgroups such that every subgroup contains a Sylow $p$-complement for some prime integer $p$. M. J. Tomkinson has studied the general properties of subgroups satisfying these criteria. In Tomkinson's study, a broader class of groups than finite solvable groups was used.

8.1 Definition: [22] Denote by $U$ the largest subgroup closed class of locally finite groups satisfying the following properties:

(i) If $G \in U$, then $G$ has a finite series $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$
with locally nilpotent factors.

(ii) If \( G \in \mathcal{U} \) and \( \pi \) is a set of primes, then the Sylow \( \pi \)-subgroups of \( G \) are conjugate in \( G \).

Finite groups are locally finite, and finite nilpotent factors are locally nilpotent, hence finite solvable groups satisfy property (i). Property (ii) also holds in finite solvable groups [15]. Consequently, finite solvable groups are \( U \)-groups. So all results in [22] are valid for finite solvable groups.

8.2 Definition: Let \( S \) be a Sylow system of \( G \). A set \( B = \{B_p\} \) of subgroups of \( G \), one for each prime integer \( p \) dividing the order of \( G \), is called a CAR-system associated with \( S \) if the following conditions are satisfied:

(i) \( B_p \supseteq S^p \) for each prime \( p \).

(ii) \( B_p \) either covers or avoids each chief factor of \( G \).

The intersection \( B = \bigcap B_p \) is called a CAR-subgroup associated with \( S \) (or an SCAR-subgroup).

The origin of the acronym CAR becomes apparent from the following theorem:

8.3 Theorem: [22] Let \( B = \{B_p\} \) be an SCAR-system of the group \( G \) and \( B = \bigcap B_p \) be the corresponding SCAR-subgroup. Then

(i) \( S \) reduces into \( B \) and \( B \cap S_p = B_p \cap S_p \), and

(ii) \( B \) covers those \( p \)-chief factors covered by \( B_p \) and avoids those which are avoided by \( B_p \).
A CAR-subgroup then has a covering and avoidance property and a reduction property. The next theorem implies that the image of a CAR-subgroup is also a CAR-subgroup.

8.4 Theorem: [22] If \( B = \{B_p\} \) is an SCAR-system of the group \( G \) and \( N \) is a normal subgroup of \( G \), then \( BN/N = \{B_pN/N\} \) is an \( SN/N \) CAR-system of \( G/N \). If \( B \) and \( \overline{B} \) are the corresponding CAR-subgroups, then \( \overline{B} = BN/N \).

The reason that CAR-subgroups are applied to this work is that general results about covering and avoidance and permutability of subgroups have been obtained. These results have useful applications in the development of these properties for \( f \)-subgroups.

8.5 Theorem: [22] Let \( B \) be an SCAR-system of the group \( G \) and, for each prime integer \( p \) dividing the order of \( G \), let \( V_p \) be a normal subgroup of \( G \).

(i) \( K = \{B_pV_p\} \) is an SCAR-system of \( G \); in particular \( V = \{S^pV_p\} \) is an SCAR-system of \( G \).

(ii) If \( K = \bigcap_p \{B_pV_p\} \) and \( V = \bigcap_p \{S^pV_p\} \), then \( K = BV \).

(iii) \( K \) avoids the \( p \)-chief factor \( H/K \) of \( G \) if and only if \( V \) avoids \( H/K \) and \( B \) avoids \( H_pV_p/K_pV_p \).

(iv) \( I = \{B_p \cap S^pV_p\} \) is an SCAR-system of \( G \).

(v) If \( I = \bigcap_p \{B_p \cap S^pV_p\} = B \cap V \), then \( I \) covers the \( p \)-chief factor \( H/K \) if and only if \( V \) covers \( H/K \) and \( B \) covers \( (H \cap V_p)/(K \cap V_p) \).
8.6 Definition: [22] A CAR-subgroup B is called perspective if whenever H/K and \( H_1/K_1 \) are chief factors of the group G such that \( H \cap K_1 = K, HK_1 = H \) and B covers \( H_1/K_1 \), then B covers H/K.

8.7 Theorem: [22] Let \( \{ B_p \} \) be a perspective SCAR-system of the group G and let \( V = \{ S^p V \} \) where \( V_p < G \). If \( V = \bigcap_p S^p V_p \) and \( B = \bigcap_p B_p \) then the following properties are satisfied:

(i) \( K = BV \) avoids those chief factors which are avoided by both V and B and covers the rest.

(ii) \( I = B \cap V \) covers those chief factors which are covered by both V and B and avoids the rest.

8.8 Theorem: [22] If \( g \) is a locally defined formation, then a \( g \)-normalizer of a group G associated with the Sylow system \( S \) is a perspective SCAR-subgroup.

In [6] G. Chambers defines a strongly pronormal subgroup. In [22] M. Tomkinson gives a necessary and sufficient condition for a subgroup of a group to be strongly pronormal. For the purposes here, this condition will be taken as the definition of strongly pronormal.

8.9 Definition: [22, Theorem 2.5] For each prime integer p dividing the order of the group G, let \( V_p \) be a normal subgroup of G. If \( S \) is a Sylow system of G then \( V = \bigcap_p S^p V_p \) is called a strongly pronormal SCAR-subgroup of G.
8.10 Theorem: [22] If $V$ is a strongly pronormal SCAR-subgroup of $G$, then $V$ permutes with every SCAR-subgroup of $G$.

Definition 8.9 and Theorem 8.10 are used to prove that SCAR-subgroups permute with the associated Sylow $p$-subgroups and Sylow $p$-complements.

8.11 Theorem: Let $S$ be a Sylow system of the group $G$, let $S_p$ be a Sylow $p$-complement from $S$ and $S_p$ a Sylow $p$-subgroup from $S$. Then $S_p$ and $S_p$ are strongly pronormal SCAR-subgroups.

Proof: For $q \neq p$ let $V_q = G$. Let $V_p = 1$, then $S_p = \cap S_r^r V_r$. By letting $V_q = 1$ for $q \neq p$ and $V_p = G$, $S_p = \cap S_r^r V_r$. According to Definition 8.9 $S_p$ and $S_p$ are strongly pronormal SCAR-subgroups of $G$.

8.12 Corollary: Let $S$ be a Sylow system of the group $G$, then $S_p$ and $S_p$ permute with every SCAR-subgroup of $G$.

Proof: Apply Theorem 8.10 to $S_p$ and $S_p$. 
CHAPTER II

THE DEFINITION AND BASIC STRUCTURE OF $\mathcal{f}$-SUBGROUPS

The definition of an $\mathcal{f}$-subgroup is given in this chapter's first section. The cover and avoidance properties are then investigated through M. Tomkinson's results [22], and by considering a certain $g$-prefrattini subgroup. The remaining three sections are devoted to showing that the $\mathcal{f}$-subgroups satisfy the properties mentioned in the Introduction. In particular, $\mathcal{f}$-subgroups are a characteristic conjugacy class of subgroups defined for any formation $\mathcal{f}$. The subgroups generate the $\mathcal{f}$-residual and a group $G \in \mathcal{f}$ if and only if the $\mathcal{f}$-subgroups of $G$ are trivial. It is established that an $\mathcal{f}$-subgroup can be identified with the product of known subgroups of the $\mathcal{f}$-residual. A consequence of this result is that a $\mathcal{k}$-subgroup is a prefrattini subgroup when $\mathcal{k}$ is taken to be the formation of solvable nC-groups.

All groups under consideration are finite and solvable. The formation $\mathcal{f}$ unless otherwise indicated, represents an arbitrary nonempty formation.

1. The Definition of an $\mathcal{f}$-subgroup

The definition of an $\mathcal{f}$-subgroup developed here is as a CAR-subgroup. The advantage in taking this approach is that covering and avoidance properties may be immediately deduced from M. Tomkinson's work [22].
It is desired that $f$-subgroups link Sylow structure, factor structure, and a formation $f$. The $f$-subgroups will be defined with respect to a Sylow system and an arbitrary formation $f$. The cover and avoidance properties provide the necessary connection to factor structure.

In order to introduce $f$-subgroups, the formation $f$ is used to define a saturated formation.

1.1 Definition: Let $f$ be a nonempty formation. Define the formation $h$ locally by $h(p) = f$ for every prime integer $p$.

Since $h$ is locally defined it is proper to speak of $h$-normal maximal subgroups and $h$-central chief factors. With $h$ defined as above, Definition I.5.4 states that a maximal subgroup $M$ is $h$-normal if $M/\text{core } M \in f$ and $h$-abnormal otherwise. Definition I.5.2 states that a chief factor $H/K$ is $h$-central if $\text{Aut}_G(H/K) \supseteq G/C_G(H/K) \in f$. Thus $H/K$ is $h$-central if and only if $G_f \leq C_G(H/K)$. Consequently, $h$-central chief factors are those factors centralized by the $f$-residual $G_f$. The $h$-eccentric chief factors are those factors not centralized by $G_f$.

For the remainder of this work, it is assumed that $h$ is the formation defined in Definition 1.1, for $f$ an arbitrary nonempty formation.

1.2 Definition: For a formation $f$ and a Sylow system $S$ of a group $G$;

(i) $M_p(S) = \cap \{M\mid M \text{ an } h\text{-abnormal maximal subgroup of } G, S^P \leq M, S^P \in S\}$
(ii) \( J_f(s) = \bigcap_p (M_p(s) \cap S^P G_p) \)

The subgroup \( J_f(s) \) is called the \( f \)-subgroup of \( G \) associated with \( S \).

When confusion may not possibly arise an \( f \)-subgroup \( J_f(s) \) will be denoted simply by \( J(s) \).

To prove that \( J_f(s) \) is an SCAR-subgroup of a group \( G \) two lemmas are required.

1.3 Lemma: If \( K \) is a normal subgroup of the group \( G \) and \( S K/K \) is the image of the Sylow system \( S \) of \( G \) in \( G/K \), then \( M_p(S) K/K = M_p(S K/K) \).

Proof: Let \( C \) be a chief series of \( G \) containing the normal subgroup \( K \). Let \( N \) be a set of maximal subgroups of \( G \) chosen so that \( N \) contains, for each \( \pi \)-eccentric \( \pi \)-chief factor of \( C \), exactly one complement \( M \) with \( S^P \leq M \). Theorem 1.7.2 implies that \( M_p(s) = \bigcap M \). It is sufficient to prove that \( (\bigcap M) K/K = \bigcap MK/K \). For if \( M \) is an \( \pi \)-abnormal maximal subgroup of \( G \), then either \( MK/K = G/K \) or \( M/K \) is an \( \pi \)-abnormal maximal subgroup of \( G/K \). Suppose that \( N \) is the minimal normal subgroup contained in the chief series \( C \). Then either \( N \) is complemented in \( G \) or \( N \leq \phi(G) \). If \( N \leq \phi(G) \), then \( N \leq M \) for all maximal subgroups \( M \). Hence, \((\bigcap M) N/N = (\bigcap M) N/N = (\bigcap M) N/N \). If \( M \) is complemented by \( M_1 \in N \), then \( N \leq M \) for each \( M \in N \) and \( M \neq M_1 \). Therefore \((\bigcap M) N/N = (\bigcap M) N/N \). Inductively assume that for \( H \in C \), \( H < K \), \((\bigcap M) H/H = (\bigcap M) H/H \). Let \( K/L \) be a chief factor from \( C \). The
group K/L is minimal normal in G/L. By applying the argument used above

\[ \left( \bigcap_{M \in N} \frac{(M \cap ML/L)K/L}{K/L} \right) \]

on N, to K/L in the group G/L, one concludes

\[ \frac{(\bigcap_{M \in N} M/L)K/L}{K/L} = \frac{(\bigcap_{M \in N} M/L)K/L}{K/L} \]

Moreover, \( (\bigcap_{M \in N} M/K)K = \bigcap_{M \in N} M/L )K/L \). Thus \( (\bigcap_{M \in N} M/K)K = \bigcap_{M \in N} M/L )K/L \) and the proof is complete.

1.4 Lemma: If \( S \) is a Sylow system of the group G, then \( \{M_p(S)\} \) is an SCAR-system of G.

Proof: It is necessary to satisfy the conditions in Definition I.8.2. By Definition 1.2, \( S^p \leq M_p(S) \) for each prime integer p. So, it will suffice to show that \( M_p(S) \) covers or avoids each chief factor of G.

Let H/K be a chief factor of G. If there exists a maximal \( \pi \)-abnormal subgroup, with \( S^p \leq M \), and M complementing H/K, then M avoids H/K. That is, \( M \cap H \leq K \). Definition 1.2 implies \( M_p(S) \leq M \), hence \( M_p(S) \cap H \leq K \).

Therefore \( M_p(S) \) avoids H/K.

Suppose that H/K is not complemented by such a maximal subgroup. Then each \( \pi \)-abnormal maximal subgroup M that contains \( S^p \) satisfies, \( H \leq MK \). Consequently, for each such M, \( H/K \leq MK/K \). Therefore \( \frac{H/K \cap \bigcap_{M \in N} \{MK/K| \text{M an } \pi \text{-abnormal maximal subgroup of } G, S^p \leq M\}}{MK/K} = \frac{\bigcap_{M \in N} \{MK/K| \text{M an } \pi \text{-abnormal maximal subgroup of } G/K, S^pK/K \leq M/K\}}{MK/K} \). This latter intersection is \( M_p(SK/K) \). Hence \( H/K \leq \frac{M_p(SK/K)}{MK/K} \).

By Lemma 1.3, \( M_p(S)K/K = M_p(SK/K) \). It follows that \( H/K \leq \frac{M_p(S)K/K}{MK/K} \) and \( H \leq M_p(S)K. \)
Thus $H/K$ is covered by $M_p(s)$.

It has been shown that every chief factor of $G$ is either covered or avoided by $M_p(s)$. The proof is now complete.

The SCAR-subgroup of $G$ generated by $\{M_p(s)\}$ is $\bigcap_{p} M_p(s)$. From Definitions 1.2 and I.7.1, $\bigcap_{p} M_p(s) = \hat{W^h}(s)$, the $h$-prefrattini subgroup of $G$ associated with $s$.

1.5 Theorem: Let $f$ be an arbitrary formation and $\mathcal{S}$ a Sylow system of the group $G$. The $f$-subgroup $J_f(s)$ associated with $s$ is an SCAR-subgroup of $G$ relative to the SCAR-system \(\{M_p(s) \cap S^D G_f\}\).

Proof: By Lemma 1.4, \(\{M_p(s)\}\) is an SCAR-system of $G$. From Theorem I.8.5(iv), \(\{M_p(s) \cap S^D G_f\}\) is an SCAR-system of $G$. The SCAR-subgroup relative to the SCAR-system \(\{M_p(s) \cap S^D G_f\}\) is $J_f(s)$, since

\[
J_f(s) = \bigcap_{p} (M_p(s) \cap S^D G_f).
\]

Note that since $G_f$ is a normal subgroup of $G$, \(\{S^D G_f/G_f|S^D \in \mathcal{S}\}\) contains a subset that is a complement system of $G/G_f$. The intersection over all elements of a complement system is the identity element, hence $\bigcap_{p} (S_p G_f/G_p) = G_f$. Therefore\( J_f(s) = \bigcap_{p} (M_p(s) \cap S^D G_f) = (\bigcap_{p} M_p(s)) \cap (\bigcap_{p} S^D G_f) = \bigcap_{p} M_p(s) \cap G_f\). As stated above, $\bigcap_{p} M_p(s) = \hat{W^h}(s)$. So it is valid that $J_f(s) = \hat{W^h}(s) \cap G_f$. Consequently, an $f$-subgroup associated with $\mathcal{S}$ is a subgroup of the $f$-residual and it is also contained in the $h$-prefrattini subgroup associated with $s$. It will be established in Chapter IV that $J_f(s)$ is in fact a CAR-subgroup of the $f$-residual.
1.6 Theorem: Let $f$ be a formation and $S$ a Sylow system of the group $G$. The $f$-subgroup $J_f(S)$ is equal to $W^h(S) \cap G_f$.

Proof: See the comments preceding the theorem.

The fact that $J_f(S)$ is an SCAR-subgroup allows one to draw conclusions concerning the permutability of $J_f(S)$ with the Sylow system $S$, and it enables one to determine covering and avoidance properties of $J_f(S)$.

1.7 Theorem: Let $S$ be a Sylow system of the group $G$. For each $S_p, S^p \in S, J_f(S)$ permutes with $S_p, S^p, S_p \cap G_f$ and $S^p \cap G_f$.

Proof: By Corollary I.8.12 $S^p$ and $S_p$ permute with every SCAR-subgroup. Thus $S^p J_f(S) = J_f(S) S^p$ and $S_p J_f(S) = J_f(S) S_p$. Moreover $(S^p \cap G_f) J_f(S) = S^p J_f(S) \cap G_f$ by the modular identity I.1.2. Thus $(S^p \cap G_f) J_f(S) = S^p J_f(S) \cap G_f = J_f(S) S^p \cap G_f = J_f(S) (S^p \cap G_f)$. Similarly $(S_p \cap G_f) J_f(S) = J_f(S) (S_p \cap G_f)$.

Covering and avoidance properties of $f$-subgroups depend upon the properties of the $f$-residual and the $h$-prefrattini subgroups. Theorem I.7.6 implies that $W^h(S)$ avoids every complemented $h$-eccentric chief factor of $G$ and it covers every frattini chief factor and every $h$-central chief factor of $G$. 
1.8 Theorem: An $f$-subgroup $J_f(S)$ of $G$ avoids every complemented $h$-eccentric chief factor of $G$ below $G_f$ and covers all other chief factors of $G$ below $G_f$.

Proof: If $H/K$ is a complemented $h$-eccentric chief factor of $G$ with $H \leq G_f$, then $H/K$ is avoided by $W^h(S)$. Thus $W^h(S) \cap H \leq K$ and $W^h(S) \cap G_f \cap H \leq K$. Consequently, $J_f(S)$ avoids $H/K$ since $J_f(S) = W^h(S) \cap G_f$ by Theorem 1.6. Otherwise if $H/K$ is either frattini or $h$-central with $H \leq G_f$, then $W^h(S)$ covers $H/K$. This implies that $H \leq W^h(S)K$. From the modular identity 1.1.2, it follows that $H \leq W^h(S)K \cap G_f = K(W^h(S) \cap G_f) = K J_f(S)$. Therefore $J_f(S) = W^h(S) \cap G_f$ covers $H/K$.

1.9 Theorem: The $f$-subgroup $J_f(S)$ of $G$ covers the chief factor $H/K$ of $G$ if and only if $G_f$ covers $H/K$ and $(H \cap G_f)/(K \cap G_f)$ is either a frattini or $h$-central chief factor of $G$. The $f$-subgroup $J_f(S)$ avoids all other chief factors of $G$.

Proof: According to Theorem I.8.5(v), $J_f(S)$ covers $H/K$ if and only if $G_f = \bigcap_p G_p$ covers $H/K$ and $W^h(S) = \bigcap_p M_p(S)$ covers $(H \cap G_f)/(K \cap G_f)$. If $G_f$ covers $H/K$ and $(H \cap G_f)/(K \cap G_f)$ is either a frattini or $h$-central chief factor of $G$, then $W^h(S)$ covers $(H \cap G_f)/(K \cap G_f)$. Therefore $J_f(S)$ covers $H/K$. Conversely, if $J_f(S)$ covers $H/K$, then by Theorem I.8.5(v), $G_f$ covers $H/K$ and $W^h(S)$ covers $(H \cap G_f)/(K \cap G_f)$. To complete the proof, if it is shown that $(H \cap G_f)/(K \cap G_f)$ is a chief factor of $G$, then it must either be $h$-central or frattini since it is covered by $W^h(S)$. Since $G_f$ covers $H/K$, $(H \cap G_f)/(K \cap G_f)$ is a nontrivial factor. Suppose that $K \cap G_f \leq L < H \cap G_f$ with $L < G$. Consider the factor $L/K \leq H/K$. Either
LK = H or L ≤ K since H/K is a chief factor of G. If L ≤ K, then L ≤ K \cap G_f. So L = K \cap G_f. Assume that LK = H, then H = H \cap LK = (H \cap L)K by the modular identity I.1.2. Applying the modular identity again, H \cap G_f = (H \cap L)K \cap G_f = (H \cap L)(K \cap G_f) = (K \cap G_f)L = L. A contradiction has arisen since L < H \cap G_f. Consequently, if K \cap G_f ≤ L < H \cap G_f, then L = K \cap G_f. Therefore (H \cap G_f)/(K \cap G_f) is a chief factor of G.

A simpler covering and avoidance property may be found for \( f \)-subgroups which does not require M. Tomkinson's results. However, further information is required about the \( h \)-prefrattini subgroup \( W^h(S) \).

1.10 Lemma: Let \( S \) be a Sylow system of the group \( G \). If \( W(S) \) is the prefrattini subgroup of \( G \) associated with \( S \) and \( D^h(S) \) is the \( h \)-normalizer of \( G \) associated with \( S \), then \( W^h(S) = W(S)D^h(S) \). Furthermore, \( D^h(S) \) is a relative system normalizer of the \( f \)-residual \( G_f \).

Proof: By Theorem I.7.7, \( W^h(S) = W(S)D^h(S) \). It need only be shown that \( D^h(S) \) is a relative system normalizer of \( G_f \). By Definition I.5.1, an \( h \)-system of \( G \) associated with \( S \) is \( T = \{ G_h(p) \cap S^p | S^p \in S \} \). The formation \( h \) is locally defined by \( f(p) = f \) for all primes \( p \). This implies that \( G_h(p) = G_f \) for all primes \( p \). Hence \( T = \{ G_f \cap S^p | S^p \in S \} \), which is precisely the reduction of \( S \) into \( G_f \). The normalizer of \( T \) is \( D^h(S) = \cap_{p} N_G(G_f \cap S^p) \). By Definition I.3.12, this is the relative system normalizer of the Sylow system \( T \) of \( G_f \).

1.11 Theorem: The \( f \)-subgroup \( J_f(S) \) of a group \( G \) covers those chief factors covered by both \( W^h(S) \) and \( G_f \) and avoids all other chief
factors.

Proof: Lemma 1.10 implies that $W^h(s)$ contains $D^h(s)$ a relative system normalizer of $G_f$. By Theorem 1.3.14(ii), $G = G_f D^h(s)$. It follows that $G = G_f W^h(s)$. Let $c$ be a chief series of $G$. Note that $|G| = |G| |W_f(s)|/|G_f \cap W^h(s)|$. By Theorem 1.2.3, $|G_f|$ is equal to the product of the orders of factors from $c$ that are covered by $G_f$. Also, $W^h(s)$ is equal to the product of the orders of factors from $c$ covered by $W^h(s)$. It follows that $|G_f \cap W^h(s)|$ is equal to the product of the orders of factors from $c$ covered simultaneously by $G_f$ and $W^h(s)$. Since by Theorem 1.6, $G_f \cap W^h(s) = J_f(s)$, $G_f \cap W^h(s)$ covers or avoids each chief factor of $G$. It is contained in both $G_f$ and $W^h(s)$ and so it avoids those factors from $c$ avoided by either $G_f$ or $W^h(s)$. If $G_f \cap W^h(s)$ avoided more than these factors, then Theorem 1.2.3 implies its order would be less than the product of the orders of factors of $c$ covered simultaneously by $G_f$ and $W^h(s)$. Therefore $G_f \cap W^h(s) = J_f(s)$ covers those factors from $c$ which are covered by both $W^h(s)$ and $G_f$ and avoids all other chief factors of $c$. Since $c$ was arbitrarily chosen the result holds.

It will be established later in this chapter than an $f$-subgroup can be uniquely identified by its permutability and cover and avoidance properties.
2. The Structure of an $f$-subgroup

It will now be proven that the formation $f$ is precisely the class of groups for which the $f$-subgroups are trivial. It will also be shown that an $f$-subgroup generates the $f$-residual. To prove the latter result it is established that an $f$-subgroup is the product of known subgroups of the $f$-residual.

2.1 Theorem: The group $G$ is an element of the formation $f$ if and only if $J_f(S) = 1$ for any Sylow system $S$ of $G$.

Proof: If $G \in f$ then $G_f = 1$. The subgroup $J_f(S)$ is contained in $G_f$. Therefore $J_f(S) = 1$. Conversely, suppose that $J_f(S) = 1$ and assume that $G_f \neq 1$, then there exists a chief factor of the form $G_f/K$. Since this factor is abelian, $G_f \leq C_G(G_f/K)$. Thus $G/C_G(H/K) \in f$. By Definition 1.1, $G_f/K$ is $h$-central. Theorem 1.8 implies that $J_f(S)$ covers $G_f/K$. A contradiction arises since $J_f(S) = 1$. Hence it must be the case that $G_f = 1$. Therefore if $J_f(S) = 1$, then $G_f = 1$ and $G \in f$.

2.2 Theorem: Let $S$ be a Sylow system of the group $G$. If $W(S)$ is the prefattini subgroup of $G$ associated with $S$, $D_{G_f}(S)$ is the system normalizer of the reduction of $S$ into $G_f$, and $J_f(S)$ is the $f$-subgroup associated with $S$, then $J_f(S) = D_{G_f}(S)(W(S) \cap G_f)$.

Proof: From Theorem 1.6 and Lemma 1.10, $J_f(S) = W(S)D^h(S) \cap G_f$. Notice that $D_{G_f}(S)(W(S) \cap G_f) \in <D_{G_f}(S), W(S) \cap G_f> \leq W(S)D^h(S) \cap G_f$, for
which \( \langle D_g(s), W(s) \cap G_f \rangle \) is the group generated by \( D_g(s) \) and \( W(s) \cap G_f \). If it is shown that \( D_g(s)(W(s) \cap G_f) \) and \( J_f(s) \) have the same order, then 
\[
D_g(s)(W(s) \cap G_f) = J_f(s).
\]
Consider \(|D_g(s)(W(s) \cap G_f)| = |D_g(s)||W(s) \cap G_f|/|W(s) \cap G_f \cap D_g(s)|\). Let \( c \) be the chief series \( G = G_0 \supset G_1 \supset \ldots \supset G_k \supset G_{k+1} \supset \ldots \supset G_n = 1 \).

By Theorem 1.3.13, \( D_g(s) = D^{h}(s) \cap G_f \). The group \( D^{h}(s) \cap G_f \) covers every \( h \)-central chief factor of \( c \) below \( G_f \) and it avoids all other factors of \( c \). By Theorem 1.2.3, \(|D_g(s)| = |D^{h}(s) \cap G_f|\) is equal to the product of the orders of \( h \)-central chief factors of \( c \) that lie below \( G_f \).

The group \( W(s) \cap G_f \) covers every frattini chief factor of \( c \) that lies below \( G_f \). It avoids all other factors of \( c \). According to Theorem 1.2.3 \(|W(s) \cap G_f|\) is equal to the product of the orders of the frattini factors from \( c \) that lie below \( G_f \).

Since \( D^{h}(s) \cap G_f = D_g(s) \), \( W(s) \cap G_f \cap D_g(s) = W(s) \cap G_f \cap D^{h}(s) \). The subgroup \( A = W(s) \cap G_f \cap D^{h}(s) \) avoids every factor avoided by \( D^{h}(s) \cap G_f \) and \( W(s) \cap G_f \). Hence \( A \) avoids every factor above \( G_f \) and it avoids every \( h \)-eccentric and every complemented chief factor below \( G_f \). Because \(|G| = [G:G_1][G_1:G_2] \ldots [G_{n-1}:G_n], |A| = [G_0 \cap A:G_i \cap A][G_i \cap A:G_{i+1} \cap A] \ldots [G_{n-1} \cap A:G_n \cap A] \).

If \( A \) avoids a chief factor \( G_i/G_{i+1} \) of \( c \), then \([G_i \cap A:G_{i+1} \cap A] = 1 \). It follows that \(|A| = [G_{i+1} \cap A:G_{i+2} \cap A][G_{i+2} \cap A:G_{i+3} \cap A] \ldots [G_{i+m-1} \cap A:G_{i+m} \cap A] \), for which \( G_{i,j}/G_{i,j+1} \) is \( h \)-central, frattini, and \( G_{i,j} \leq G_f \) for each \( j = 1, \ldots, m-1 \).

If \( A \) covers \( G_{i,j}/G_{i,j+1} \), then \([G_{i,j}:G_{i,j+1}] = [G_{i,j} \cap A:G_{i,j+1} \cap A] \). Otherwise, \([G_{i,j}:G_{i,j+1}] > [G_{i,j} \cap A:G_{i,j+1} \cap A] \). Consequently, \(|A|\) is less than or equal to the product of the orders of factors from \( c \) that are frattini, \( h \)-central, and below \( G_f \).

By combining the above, \(|D_g(s)(W(s) \cap G_f)|\) is greater than or equal to the product of the orders of chief factors of \( c \) which are below
$G_f$ and that are either $\lambda$-central or frattini. However, $|J_f(S)|$ is equal to this product by Theorems 1.8 and 1.2.3. Therefore $|D_{G_f}(S)(W(S) \cap G_f)| \geq |J_f(S)|$. Thus $|J_f(S)| = |D_{G_f}(S)(W(S) \cap G_f)|$ and $J_f(S) = D_{G_f}(S)(W(S) \cap G_f)$.

In Theorem 2.1, it was proven that a group $G \in J_f$ if and only if $J_f(S) = 1$. With Theorem 2.2 a stronger result is obtained.

2.3 Theorem: The $J_f(S)$ of the group $G$ generates the $f$-residual, that is $<<J_f(S)>> = G_f$, for which $<<J_f(S)>>$ is the normal closure of $J_f(S)$ in $G$.

Proof: By Theorem 2.3, $D_{G_f}(S) \leq J_f(S)$. So $<<D_{G_f}(S)>> \leq <<J_f(S)>> \leq G_f$. By Theorem 1.3.10(i), $<<D_{G_f}(S)>> = G_f$. Therefore $G_f = <<J_f(S)>>$.

To further examine the subgroup $J_f(S)$, consider again the formation $f$. Let $G_h$ denote the $f$-residual of a group $G$. The group $G \in f$ if and only if every chief factor of $G$ is $\lambda$-central. A chief factor is $\lambda$-central if and only if it is centralized by $G_f$. Note that $G_h \leq G_f$ since every factor above $G_f$ is centralized by $G_f$. In fact every factor above $K_\infty(G_f)$ the hypercommutator of $G_f$ is centralized by $G_f$. Moreover, $K_\infty(G_f)$ is characteristic in $G_f$ and hence normal in $G$. Therefore $G_h = K_\infty(G_f)$. Each chief factor of the form $K_\infty(G_f)/K$ is noncentral in $G_f$ and hence $\lambda$-eccentric. Thus $G_h = K_\infty(G_f)$.

The $f$-subgroups may be obtained by considering specific chief series of $G$. In the following let $C_1$ be a chief series of $G$ which contains $G_f$ and let $C_2$ be a chief series which contains $G_f$ and $K_\infty(G_f)$. 
Let $N_1$ be a set of $\mathcal{H}$-abnormal maximal subgroups of $G$, each containing a Sylow $p$-complement of $S$ for some $p$, and let $N_1$ contain at least one complement of each complemented $\mathcal{H}$-eccentric chief factor below $G_f$. Let $N_2$ be a set of maximal subgroups of $G$, each containing a Sylow $p$-complement of $S$ for some $p$ and each complementing a chief factor of $C_2$ below $K_{\infty}(G_f)$. Furthermore require that $N_2$ contains at least one complement of each complemented chief factor that lies below $K_{\infty}(G_f)$.

2.4 Theorem: For a group $G$ with Sylow system $s$, $J_f(S) = \cap\{M|M \in N_i\} \cap G_f$ for $i = 1, 2$.

Proof: By Theorem 1.6, $J_f(S) = W^\mathcal{H}(S) \cap G_f$. Consider the series $C_1$. Each chief factor above $G_f$ is centralized by $G_f$ and hence $\mathcal{H}$-central. This implies that $N_1$ is a set of maximal subgroups that satisfy the conditions of Corollary I.7.3. Thus $W^\mathcal{H}(S) = \cap\{M|M \in N_1\}$. Therefore $J_f(S) = \cap\{M|M \in N_1\} \cap G_f$. Now consider the series $C_2$. Each chief factor above $K_{\infty}(G_f) = G_h$ is $\mathcal{H}$-central. Let $H/K$ be a complemented chief factor of $C_2$ such that $H \leq K_{\infty}(G_f)$. Assume that $H/K$ is $\mathcal{H}$-central, then $H/K \leq Z(G_f/K)$. Note that $K_{\infty}(G_f/K) \leq G_f/K$. By Theorem I.1.1, $H/K \leq Z(G_f/K) \cap G_f/K \leq \phi(G_f/K)$. Hence $H/K \leq \phi(G_f/K) \leq \phi(G/K)$. A contradiction arises since $H/K$ is complemented in $G/K$. Consequently, the complemented factors below $K_{\infty}(G_f)$ are $\mathcal{H}$-eccentric and they are the only complemented $\mathcal{H}$-eccentric chief factors from the series $C_2$. Therefore the set $N_2$ also satisfies the conditions of Corollary I.7.3. By this result $W^\mathcal{H}(S) = \cap\{M|M \in N_2\}$ and so $J_f(S) = \cap\{M|M \in N_2\} \cap G_f$. 

As a result of Theorem 2.4 one only need consider either the complements of \( h \)-eccentric chief factors below \( G_f \) or the complements of chief factors below \( K_w(G_f) \) in order to determine \( J_f(s) \).

### 3. The Prefrattini Subgroup

In this section, \( \kappa \) will denote the formation of solvable nC-groups. It will be established that a \( \kappa \)-subgroup is a prefrattini subgroup. In consequence, several theorems for prefrattini subgroups appear as corollaries to results already obtained in this chapter.

**3.1 Theorem:** If \( W(s) \) is the prefrattini subgroup of a group \( G \) associated with the Sylow system \( s \) of \( G \), then \( W(s) = J_\kappa(s) \).

**Proof:** By Theorem 2.2, \( J_\kappa(s) = D_{G_\kappa}(s)(W(s) \cap G_\kappa) \), for which \( D_{G_\kappa}(s) \) is the system normalizer of the reduction of \( s \) into \( G_\kappa \). Theorem 1.6.9(ii) implies that \( D_{G_\kappa}(s) \leq W(s) \cap G_\kappa \) while Theorem 1.6.7 implies \( W(s) \leq G_\kappa \). Therefore \( J_\kappa(s) = W(s) \).

**3.2 Corollary:** [9;24] The prefrattini subgroup \( W(s) \) of \( G \) associated with the Sylow system \( s \) permutes with \( S_p, S^P, S^P \cap G_\kappa \) and \( S_p \cap G_\kappa \) for each \( S_p, S^P \in s \).

**Proof:** Apply Theorem 1.7 with \( f = \kappa \).

**3.3 Corollary:** A chief factor \( H/K \) of \( G \) is frattini if and only
if \((H \cap G_k) / (K \cap G_k)\) is a frattini chief factor of \(G\).

**Proof:** Let \(H / K\) be a frattini chief factor of \(G\). By Theorem I.6.2, \(W(s) = J_k(s)\) covers \(H / K\). By Theorem 1.9, \((H \cap G_k) / (K \cap G_k)\) is a frattini or \(\hat{h}\)-central chief factor of \(G\). If \((H \cap G_k) / (K \cap G_k)\) is \(\hat{h}\)-central, then \((H \cap G_k) / (K \cap G_k)\) is centralized by \(G_k\) since \(\hat{h}\) is locally defined by \(\hat{h}(p) = k\) for every prime \(p\). Consequently, \((H \cap G_k) / (K \cap G_k)\) is covered by \(D_{G_k}(s)\) a system normalizer of \(G_k\). Theorem I.6.9(ii) implies that \(D_{G_k}(s) \leq W(s)\) and so \(W(s)\) covers \((H \cap G_k) / (K \cap G_k)\). Now Theorem I.6.2 implies that \((H \cap G_k) / (K \cap G_k)\) is a frattini chief factor of \(G\).

Let \((H \cap G_k) / (K \cap G_k)\) be a frattini chief factor of \(G\). Assume that \(H / K\) is not covered by \(G_k\), then \(H \nsubseteq G_k\). So \(H / K \cap G_k K / K = K\) and \(H \cap G_k K = K\). By the modular identity I.1.2, \(K = K(H \cap G_k)\). It follows that \(H \cap G_k \leq K \cap G_k\). A contradiction has arisen, for by hypothesis \((H \cap G_k) / (K \cap G_k)\) is a chief factor. Hence \(H \subseteq G_k\) and \(G_k\) covers \(H / K\). Theorem 1.9 implies \(J_k(s) = W(s)\) covers \(H / K\). By Theorem I.6.2, \(H / K\) is a frattini chief factor. The proof is now complete.

4. The Uniqueness of an \(f\)-subgroup

A natural question for any subgroup with a covering and avoidance property is whether or not this property alone is sufficient to characterize the subgroup. It is not for \(f\)-subgroups. A counter example for prefattini subgroups will suffice. Such an example has been found by J. Gillam [14].
By introducing one additional property, \( f \)-subgroups are unique. The additional property is that a subgroup must permute with the Sylow \( p \)-complements of a Sylow system \( S \). What is achieved in this section is a proof of the following theorem:

4.1 Theorem: Let \( G \) be a group with Sylow system \( S \) and let \( X \) be a subgroup of \( G_f \). Then \( X = J_f(S) \) if and only if \( X \) satisfies the following conditions:

(i) \( X \) avoids complemented \( h \)-eccentric chief factors of \( G \) below \( G_f \) and covers all other chief factors below \( G_f \).

(ii) \( (S^p \cap G_f)X = X(S^p \cap G_f) \) for all \( S^p \in S \).

In order to prove Theorem 4.1, three lemmas are required. This first lemma along with Theorem 4.1 is used later in this section to show that \( f \)-subgroups form a characteristic conjugacy class of subgroups of \( G \). In the following, \( S \) denotes a Sylow system of \( G \) and \( SN/N \) denotes the Sylow system of \( GN/N \) determined by \( SN/N = \{S^pN/N| p \text{ divides } |G/N|, S^p \in S\} \).

4.2 Lemma: If \( N \) is a normal subgroup of the group \( G \) and \( J_f(SN/N) \) is the \( f \)-subgroup of \( G/N \) associated with \( SN/N \), then \( J_f(SN/N) = J_f(S)N/N \).

Proof: By Theorem 1.6, \( J_f(S) = G_f \cap W^h(S) \). Thus it will suffice to prove that \( (G_f \cap W^h(S))N/N = (G/N)_f \cap W^h(SN/N) \). Note that \( (G/N)_f = G_fN/N \) and by Theorem 1.7.5(i), \( W^h(SN/N) = W^h(S)N/N \). In view of the
fact that \((G_f \cap W^s)N/N \leq (G_f N/N) \cap (W^s N/N)\), if it is established that
the orders of these two groups are equal, then the result holds. If
\((G_f N/N) \cap (W^s N/N)\) avoids a chief factor then clearly \((G_f \cap W^s)N/N\)
avoids that factor. Suppose \((H/N)/(K/N)\) is a chief factor of \(G/N\) covered
by \((G_f N/N) \cap (W^s N/N)\). Theorem 1.11 implies that \((H/N)/(K/N)\) is covered
by \(G_f N/N\) and by \(W^s N/N\). It follows that \(H/N \leq G_f K/N\) and \(H \leq G_f K\). Thus
\(H/K\) is covered by \(G_f\). Similarly \(H/K\) is covered by \(W^s\). By Theorem 1.11,
\(H/K\) is covered by \(G_f \cap W^s\). Therefore \(H \leq (G_f \cap W^s)K\) and \(H/N \leq \(G_f \cap W^s)K/N\). Hence \((G_f \cap W^s)N/N\) covers \((H/N)/(K/N)\). The groups
\((G_f \cap W^s)N/N\) and \((G_f N/N) \cap (W^s N/N)\) then cover and avoid the same chief
factors of \(G/N\). By Theorem 1.2.3, they have the same order and so the
proof is complete.

4.3 Lemma: Let \(Y\) be a subgroup of the group \(G\) and \(Y \leq G_f\). Then \(Y\)
and \(J_f(s)\) cover and avoid the same \(p\)-chief factors of \(G\) if and only if
they cover and avoid the same \(p\)-chief factors of \(G\) below \(G_f\).

Proof: Suppose \(Y\) and \(J_f(s)\) cover and avoid the same \(p\)-chief factors of \(G\) below \(G_f\). Let \(H/K\) be a \(p\)-chief factor covered by \(J_f(s)\). By
Theorem 1.9, \(G_f\) covers \(H/K\) and \(J_f(s)\) covers the \(p\)-chief factor \((H \cap G_f)/\(K \cap G_f)\). By hypothesis \(Y\) covers \((H \cap G_f)/(K \cap G_f)\). Hence \((H \cap G_f) \leq Y(K \cap G_f)\)
and \((H \cap G_f)K \leq Y(K \cap G_f)K = YK\). By the modular identity 1.1.2, \((H \cap G_f)K = \(H \cap G_f)K = H\). Therefore \(H \leq YK\),
that is, \(Y\) covers \(H/K\).

If the \(p\)-chief factor \(H/K\) is avoided by \(J_f(s)\), then either \(G_f\)
avoids \(H/K\) or \(J_f(s)\) avoids \((H \cap G_f)/(K \cap G_f)\). If \(G_f\) avoids \(H/K\), then since
\(Y \leq G_f\), \(Y\) avoids \(H/K\). If \(G_f\) covers \(H/K\), then \(J_f(s)\) avoids the
p-chief factor \((H \cap G^\varphi)/(K \cap G^\varphi)\). By hypothesis \(Y\) then avoids \((H \cap G^\varphi)/(K \cap G^\varphi)\). Equivalently \(H \cap G^\varphi \cap Y \leq K \cap G^\varphi\). So \(H \cap Y \leq K \cap G^\varphi\) and thus \(H \cap Y \leq K\). Therefore \(Y\) avoids \(H/K\).

The converse follows immediately.

4.4 Lemma: Let \(Y\) be a subgroup of \(G^\varphi\) which avoids complemented \(\pi\)-eccentric p-chief factors of \(G\) below \(G^\varphi\). Suppose that \(Y\) covers all other chief factors of \(G\) below \(G^\varphi\). If \(S^D \in \mathcal{S}\) and \(S^D \cap G^\varphi \leq Y\), then \(Y \leq M_p(S)\). Moreover, \(Y = M_p(S) \cap G^\varphi\).

Proof: From Definition 1.2, \(M_p(S) = \cap\{M | M\) is maximal, \(M\) complements an \(\pi\)-eccentric p-chief factor, \(S^D \leq M\}\). Theorem 1.7.2 implies that one need only consider p-chief factors of series which contain \(G^\varphi\). Every chief factor above \(G^\varphi\) is centralized by \(G^\varphi\), and hence \(\pi\)-central. Thus \(M_p(S) = \cap\{M | M\) is maximal, \(M\) complements \(H/K\) an eccentric p-chief factor, \(H \leq G^\varphi, S^D \leq M\}\). It will suffice to show that if \(M\) is one of the maximal subgroups comprising this intersection then \(Y \leq M\). This is accomplished by using induction on the order of \(G\). So, let \(M\) be a complement of \(H/K\), for which \(H \leq G^\varphi, H/K\) is an \(\pi\)-eccentric p-chief factor and \(S^D \leq M\).

Case 1: Core \(M \neq 1\).

Let \(N \leq \text{core } M\) with \(N\) minimal normal in \(G\). First, the group \(YN/N\) has the same covering and avoidance properties in \(G/N\) as \(Y\) has in \(G\). For suppose \((H/N)/(K/N)\) is a p-chief factor of \(G/N\) with \(H/N \leq (G/N)^\varphi = G^\varphi N/N\). Then \(H/K\) is a p-chief factor of \(G\). If \((H/N)/(K/N)\) is \(\pi\)-eccentric
and complemented, then \( J_f(\mathcal{S}N/N) \) avoids \((H/N)/(K/N)\). Lemma 4.2 implies \( J_f(\mathcal{S})N/N \) avoids \((H/N)/(K/N)\) and so \((H/N) \cap (J_f(\mathcal{S})N/N) \leq K/N\). It follows that \( H \cap J_f(\mathcal{S})N = N(H \cap J_f(\mathcal{S})) \leq K \) by the modular identity 1.1.2. Consequently, \( H \cap J_f(\mathcal{S}) \leq K \). So \( J_f(\mathcal{S}) \) avoids \( H/K \). By Lemma 4.3, \( Y \) avoids \( H/K \). Therefore \( Y \cap H \leq K \). By the modular identity, \((Y \cap H)N = YN \cap H \leq K \). Hence \( (YN/N) \cap (H/N) \leq K/N \), that is, \( YN/N \) avoids \((H/N)/(K/N)\).

If \((H/N)/(K/N)\) is either \(h\)-central or frattini, then \( J_f(\mathcal{S}N/N) \) covers \((H/N)/(K/N)\). Lemma 4.2 implies \( J_f(\mathcal{S})N/N \) covers \((H/N)/(K/N)\). Thus \( H/N \leq (J_f(\mathcal{S})NKN)/N = J_f(\mathcal{S})K/N \). Hence \( H \leq J_f(\mathcal{S})K \), equivalently \( J_f(\mathcal{S}) \) covers \( H/K \). By Lemma 4.3, \( Y \) covers \( H/K \). Now, \( H \leq YK \) so, \( HN/N \leq (YNK)/N \). Therefore \( YN/N \) covers \((H/N)/(K/N)\). If \((H/N)/(K/N)\) is a \(q\)-chief factor for \(q \neq p\), then \((S^p \cap G_f)N/N \) covers \((H/N)/(K/N)\). Since \((S^p \cap G_f) \leq Y\), \((S^p \cap G_f)N/N \leq YN/N\). Therefore \( YN/N \) covers \((H/N)/(K/N)\). To employ induction note that \((S^pN/N) \cap (G_fN/N) = (S^p \cap G_f)N/N\) and so the Sylow \(p\)-complement of \(SN/N\) is contained in \(YN/N\). Inductively \( YN/N \leq M/N \). This implies that \( Y \leq M \).

Case 2: Core \(M = 1\).

Let \( N \) be a minimal normal subgroup of \( G \). It may be assumed that \( N \leq G_f \). For if \( G_f = 1 \), then \( Y = 1 \). Hence \( Y \leq M_p(\mathcal{S}) \). Since Core \( M = 1 \), the representation of \( G \) as a permutation group on the cosets of \( M \) is faithful and primitive. Consider \( G \) as a permutation group and apply Theorem I.1.8. Then \( N \) is the only minimal normal subgroup of \( G \), \( N \) is self-centralizing, all complements of \( N \) in \( G \) are conjugate, and \( M \) has no nontrivial normal subgroups of \(q\)-power order, for which \(|N| = q^n\).

Since \( M \) complements \( N \) and \( M \) is \(h\)-abnormal, Theorem I.5.5 implies that \( N \) is \(h\)-eccentric. Also, \( N \) is a \(p\)-group of \( G \), since \( S^p \leq M \) implies
\[ [G:M] = |N| = \rho^r \]. By hypothesis \( Y \) avoids \( N \). If \( N = G_f \), since \( Y \) avoids \( N, Y = 1 \) and so \( Y \leq M_p(S) \). Assume that \( N < G_f \). Let \( H/N \) be a chief factor of \( G \) with \( H \leq G_f \). Since \( N \) is self-centralizing, \( H/N \) is a \( q \)-chief factor for \( q \neq p \). By hypothesis, \( Y \) covers \( H/N \). Hence \( H = N(Y \cap H) \). In view of the fact that \( Y \cap N = 1 \), \( Y \cap N \cap H = 1 \). Thus \( Y \cap H \) is a complement of \( N \) in \( H \). Moreover, \( Y \leq N_G(Y \cap H) \) since \( Y \cap H \leq Y \).

Now it will be established that \( N_G(Y \cap H) \) is a complement of \( N \) in \( G \). By Theorem 1.1.10, all the complements of \( N \) in \( H \) are conjugate since \( (|N|, [H:N]) = 1 \) and \( N \) is abelian. For each \( g \in G \) \( (Y \cap H)^g \) is a complement of \( N \) in \( H \). It follows that there exists \( n \in N \), such that \( (Y \cap H)^g = (Y \cap H)^n \). Consequently \( G = N \cdot N_G(Y \cap H) \). Now it is necessary to show \( N \cap N_G(Y \cap H) = 1 \). Notice that \( N \cap N_G(Y \cap H) \leq N \cdot N_G(Y \cap H) = G \).

Since \( N \) is minimal normal in \( G \), either \( N \leq N_G(Y \cap H) \) or \( N \cap N_G(Y \cap H) = 1 \). If \( N \leq N_G(Y \cap H) \), then \( N \) normalizes \( Y \cap H \). In this case \( H = N \cdot (Y \cap H) \). This contradicts the fact that \( N \) is self-centralizing. It then must be the case that \( N \cap N_G(Y \cap H) = 1 \). Therefore \( N_G(Y \cap H) \) is a complement of \( N \) in \( G \).

It will now be proven that \( N_G(Y \cap H) \cap G_f = M \cap G_f \). The groups \( M \) and \( N_G(Y \cap H) \) are conjugate since all complements of \( N \) in \( G \) are conjugate. So there exists \( n \in N \) such that \( N_G(Y \cap H)^n = M \). Moreover \( (N_G(Y \cap H) \cap G_f)^n \leq N_G(Y \cap H)^n \cap G_f = M \cap G_f \). The subgroups \( M \cap G_f \) and \( N_G(Y \cap H) \cap G_f \) have the same order since they both complement \( N \) in \( G_f \). Therefore \( (N_G(Y \cap H) \cap G_f)^n \) and \( M \cap G_f \) have the same order. Hence \( (N_G(Y \cap H) \cap G_f)^n = M \cap G_f \). The proof proceeds by proving \( N_G(S^p \cap G_f) \leq M \cap G_f \) and \( N_G(S^p \cap G_f) \leq N_G(Y \cap H) \cap G_f = N_G(Y \cap H) \). To accomplish this, let \( B \) be a maximal subgroup of \( G \) complementing \( N \) with \( S^p \cap G_f \leq B \cap G_f \). Let \( L \) be a minimal normal subgroup of \( B \).
The core of $B$ is the identity since $B$ is conjugate to $M$ and core $M = 1$.

By Theorem I.1.8 $p / |L|$. Since $N \not\subseteq G_f$, $B \cap G_f \neq 1$ and $B \cap G_f \triangleleft B$. It follows that $L$ exists such that $L \leq B \cap G_f$, then $L \leq S^P \cap G_f$ since $p \nmid |L|$. Suppose $g \in N_{G_f}(S^P \cap G_f)$. Note that $G_f = N(B \cap G_f)$ since $B$ complements $N$ in $G$. Thus $g = bn$ for $b \in B \cap G_f$ and $n \in N$. Hence, $L^g = L^{bn} = L^n \leq NL$.

By the modular identity I.1.2, $L^g \leq NL \cap (S^P \cap G_f) = L(N \cap S^P \cap G_f) \leq L(N \cap B \cap G_f) = L$ since $N \cap B = 1$. Consequently, $g \in N_{G_f}(L)$. So $N_{G_f}(S^P \cap G_f) \leq N_{G_f}(L)$. However $N_{G_f}(L) = N_G(L) \cap G_f$ and by the modular identity, $N_G(L) = BN \cap N_L(B) = B(N \cap N_L(B))$. Also $N \cap N_G(L) < N N_G(L) \geq NB = G$. Either $N \cap N_G(L) = 1$ or $N \leq N_G(L)$ since $N$ is minimal normal in $G$. If the latter is true, then $L < NB = G$. This is a contradiction to the fact that core $B = 1$. Therefore $N \cap N_G(L) = 1$. Now since $B \leq N_G(L)$ and $G = [NB, B$ and $N_G(L)$ have the same order. Thus $B = N_G(L)$. This implies that $N_{G_f}(L) = G_f \cap N_G(L) = G_f \cap B$. Since $N_{G_f}(S^P \cap G_f) \leq N_{G_f}(L) \leq N_G(L) = B$, $N_{G_f}(S^P \cap G_f) \leq B \cap G_f$. Both $M$ and $N_G(Y \cap H)$ complement $N$ in $G$. Since $Y \leq N_G(Y \cap H)$, $S^P \cap G_f \leq Y \leq N_G(Y \cap H) \cap G_f$. Also $S^P \leq M$, so $S^P \cap G_f \leq M \cap G_f$. By letting $B = M$ and then letting $B = N_G(Y \cap H)$ it follows that, $N_{G_f}(S^P \cap G_f) \leq M \cap G_f$ and $N_{G_f}(S^P \cap G_f) \leq N_G(Y \cap H) \cap G_f = N_{G_f}(Y \cap H)$. By Theorem I.3.15, $N_{G_f}(S^P \cap G_f)$ is abnormal in $G_f$. It has been shown that $(N_G(Y \cap H) \cap G_f)^n = M \cap G_f$. Consequently $N_{G_f}(S^P \cap G_f) \leq (N_G(Y \cap H) \cap G_f)^n \cap (N_G(Y \cap H) \cap G_f)$. Theorem I.1.4 implies that $M \cap G_f = (N_G(Y \cap H) \cap G_f) = N_{G_f}(Y \cap H)$.

Since $Y \leq N_{G_f}(Y \cap H)$, then $Y \leq G_f \cap M$. Therefore $Y \leq M$. As stated in the introduction of the proof, this is sufficient to conclude $Y \leq M_p(\mathcal{S})$. By hypothesis, $Y$ and $M_p(\mathcal{S})$ cover and avoid the same chief factors below $G_f$. By Theorem I.2.3, $Y$ and $M_p(\mathcal{S}) \cap G_f$ have the same order.
Therefore \( Y = M_p(S) \cap G_f \).

The proof of Theorem 4.1 may now be achieved.

**Proof of 4.1:** By Theorem 1.8, \( J_f(S) \) has the required cover and avoidance property. Theorem 1.7 implies that \( J_f(S) \) has the required permutability property. Therefore \( J_f(S) \) satisfies conditions (i) and (ii).

Conversely let \( X \) be a subgroup of \( G_f \) satisfying conditions (i) and (ii). For a fixed prime \( p \), consider \( (S^p \cap G_f)X \). Since \( (S^p \cap G_f) \) covers each \( q \)-chief factor of \( G \) below \( G_f \), for \( q \neq p \), then \( (S^p \cap G_f)X \) covers these factors. Since \( X \) covers each \( h \)-central and each frattini chief factor of \( G \) below \( G_f \), then \( (S^p \cap G_f)X \) covers these factors. The proof proceeds by establishing that \( (S^p \cap G_f)X \) avoids those chief factors avoided by both \( S^p \cap G_f \) and \( X \). Let \( C \) be a chief series of \( G \) containing \( G_f \). Note that \( |(S^p \cap G_f)X| = |S^p \cap G_f| \cdot |X| / |S^p \cap G_f \cap X| \). By Theorem I.2.3, \( |S^p \cap G_f| \) is equal to the product of the orders of the \( q \)-chief factors from \( C \) that lie below \( G_f \). Also, \( |X| \) is equal to the product of the orders of the factors of \( C \) below \( G_f \) that are either \( h \)-central or frattini.

It is now necessary to show that \( |S^p \cap G_f \cap X| \) is the product of the orders of the \( q \)-chief factors of \( C \) below \( G_f \) that are either frattini or \( h \)-central. First, it is shown that \( S^p \) reduces into \( X \). Let \( X^p \) be a Sylow \( p \)-complement of \( X \). Since \( S^p \cap G_f \) is a Sylow \( p \)-complement of \( G_f \), it is also a Sylow \( p \)-complement of \( (S^p \cap G_f)X \). It follows that there exists \( x \in X \) such that \( X^p \leq (S^p \cap G_f)^X \). By a judicious choice of \( X^p \) it may be assumed that \( X^p \leq S^p \cap G_f \). Therefore \( S^p \cap G_f \cap X = S^p \cap X \) is a subgroup of...
X whose order is divisible only by primes q ≠ p and which contains \( X^p \).

Hence \( S^P \cap X = X^P \), that is, \( S^P \) reduces into \( X \). Let \( H/K \) be a chief factor from \( C \), for \( H \leq G_f \), such that \( H/K \) is covered by \( X \) and by \( S^P \). Since \( X \) covers \( H/K \), \(|H/K| = |(H \cap X)/(K \cap X)|\). Moreover \((H \cap X)/(K \cap X)\) is a q-chief factor, for \( q \neq p \), since \( S^P \) covers \( H/K \). Since \( S^P \cap X \) is a p-complement of \( X \), \( S^P \cap X \) covers \((H \cap X)/(K \cap X)\). Hence \( H \cap X \leq (S^P \cap X)(K \cap X) \) and so \((H \cap X)K \leq K(S^P \cap X)(K \cap X)\). Because \( X \) covers \( H/K \), \( H = K(X \cap H) \). Therefore \( H \leq K(S^P \cap X)(K \cap X) = K(S^P \cap X) \). Since \( X \leq G_f \), \( S^P \cap X = S^P \cap G_f \cap X \). So \( H \leq K(S^P \cap G_f \cap X) \).

Equivalently, \( S^P \cap G_f \cap X \) covers \( H/K \). Consequently \( S^P \cap G_f \cap X \) covers those factors that are covered by both \( S^P \cap G_f \) and \( X \). Since \((S^P \cap G_f \cap X) \leq X \) and \((S^P \cap G_f \cap X) \leq S^P \cap G_f \), \((S^P \cap G_f \cap X) \) avoids those factors avoided by either \( S^P \cap G_f \) or \( X \). By Theorem I.2.3, \(|S^P \cap G_f \cap X|\) is the product of the orders of the q-chief factors of \( C \) that are below \( G_f \) and either frattini or \( h \)-central.

Combining the above \(|(S^P \cap G_f)X|\) is equal to the product of the orders of the factors from \( C \) that are either \( h \)-central, or frattini, or q-chief factors, for \( q \neq p \). It follows from Theorem I.2.4 that \((S^P \cap G_f)X\) avoids all other chief factors of \( C \). Consequently, \((S^P \cap G_f)X\) satisfies the conditions of the subgroup \( Y \) in Lemma 4.4. This lemma implies that \((S^P \cap G_f)X = M_p(S) \cap G_f \) and so \( X \leq M_p(S) \cap G_f \). Since \( p \) was arbitrary,

\( X \leq \bigcap_p M_p(S) \cap G_f = J_f(S) \). By hypothesis, \( X \) and \( J_f(S) \) cover and avoid the same factors in any series that contains \( G_f \). By Theorem I.2.3, \( X \) and \( J_f(S) \) have the same order. Therefore \( X = J_f(S) \).

4.5 Corollary: Let \( S \) be a Sylow system of the group \( G \), and let \( X \) be a subgroup of \( G_f \). Then \( X = J_f(S) \) if and only if \( X \) satisfies the following conditions:
(i) $X$ avoids complemented $\kappa$-eccentric chief factors of $G$ below $G_f$ and $X$ covers all other chief factors below $G_f$.

(ii) $S^P X = XS^P$ for all $S^P \in S$.

**Proof:** By Theorem 1.8, $J_f(S)$ satisfies (i). By Theorem 1.7 $J_f(S)$ satisfies (ii).

Conversely, let $X \leq G_f$ and suppose $X$ satisfies (i) and (ii). From the modular identity 1.1.2, $S^P X \cap G_f = (S^P \cap G_f)X$ and $XS^P \cap G_f = X(S^P \cap G_f)$ for all $S^P \in S$. Thus $(S^P \cap G_f)X = X(S^P \cap G_f)$ since $S^P X = XS^P$. Consequently $X$ satisfies the conditions in Theorem 4.1. Therefore $X = J_f(S)$.

Theorem 4.1 is extremely useful in investigating the structure and behavior of $f$-subgroups. A corollary of Corollary 4.5 is the following theorem originally established by J. Gillam.

4.6 Corollary: [14] Let $X$ be a subgroup of $G$. Then $X$ is theprefrattini subgroup of $G$ associated with the Sylow system $S$ if and only if $X$ satisfies the conditions:

(i) $X$ avoids complemented chief factors of $G$ and $X$ covers all other chief factors of $G$.

(ii) $S^P X = XS^P$ for all $S^P \in S$.

**Proof:** Let $\kappa$ be the formation of solvable nC-groups. Every factor above $G_\kappa$ is complemented. By hypothesis $X$ avoids each such factor. By Theorem I.2.2, then $X$ avoids $G/G_\kappa$. Hence $X = X \cap G \leq G_\kappa$. 
For the formation $k$, the complemented $\kappa$-eccentric chief factors below $G_k$ coincide with the complemented chief factors below $G_k$. Thus $X$ avoids all complemented $\kappa$-eccentric chief factors below $G_k$ and covers all other chief factors of $G$ below $G_k$. By Corollary 4.5, $X = J_k(s)$. Theorem 3.1 implies $X = W(s)$.

Conversely, $W(s)$ satisfies (i) by Theorem 1.6.2 and (ii) by Corollary 3.2.

The method of proof for Theorem 4.1 and for Corollary 4.5 could be handled in a different manner. In [23] M. Tomkinson introduces the concept of a constructable SCAR-subgroup. He then proves that, if $X$ is any subgroup of a group $G$ such that $X$ permutes with the Sylow system $S$ and $X$ covers and avoids the same chief factors as the constructable SCAR-subgroup $A(s)$, then $X = A(s)$. To introduce the notation for constructable subgroups and to show that $J_f(s)$ is a constructable SCAR-subgroup would require at least the equivalent amount of work as the approach taken. The method applied here allows this paper to be more self-contained.

Theorem 4.1 is now used to show that $f$-subgroups form a conjugate class.

4.7 Theorem: The $f$-subgroups are a characteristic conjugacy class of subgroups of $G$. Furthermore the $f$-subgroups are conjugate in $G_f$.

Proof: Let $J(s_1)$ and $J(s_2)$ be two $f$-subgroups of $G$. The Sylow systems $S_1$ and $S_2$ reduce into $G_f$. Hence there exists a $g \in G$ such that
\((S_1 \cap G_f) = (S_2 \cap G_f)^g\), for which \((S_2 \cap G_f)^g\) represents the Sylow system \((S_2 \cap G_f)^g = \{(S^p \cap G_f)^g | S^p \cap G_f \in S_2 \cap G_f\}\). Note that \(J(S_2)^g(S^p \cap G_f)^g = [J(S_2)(S^p \cap G_f)]^g = [(S^p \cap G_f)J(S_2)]^g = (S^p \cap G_f)^gJ(S_2)^g\) for all \(S^p \in S_2\).

It follows that \(J(S_2)^g\) permutes with the Sylow system \(S_1 \cap G_f\). It is clear that \(J(S_1)\) and \(J(S_2)^g\) cover and avoid the same factors of \(G\). By Theorem 4.1, \(J(S_1) = J(S_2)^g\). Therefore any two \(f\)-subgroups are conjugate in \(G_f\). On the other hand if \(J(S)\) is an \(f\)-subgroup of \(G\) and \(g \in G\), then \(J(S)^g\) permutes with the Sylow system \(S^g\) and \(J(S)^g\) covers and avoids the same factors as \(J(S^g)\). By Corollary 4.5, \(J(S)^g = J(S^g)\). Consequently the conjugate of any \(f\)-subgroup is also an \(f\)-subgroup. Hence the \(f\)-subgroups are a conjugate class in \(G\) and in \(G_f\).

To show that the class is characteristic let \(\theta\) be an automorphism of \(G\), then \(S^\theta = S^g\) for some \(g \in G\). Under the automorphism \(\theta\), frattini chief factors are mapped onto frattini chief factors and \(h\)-central chief factors are mapped onto \(h\)-central chief factors. Thus \(J(S)^\theta\) covers and avoids the same factors as \(J(S^g)\). Moreover, \(J(S)^\theta(S^p)^g = J(S)^\theta(S^p)^\theta = (J(S)S^p)^\theta = (S^pJ(S))^\theta = (S^p)^\theta J(S)^\theta = (S^p)^gJ(S)^\theta\). By Corollary 4.5, \(J(S)^\theta = J(S^g)\). Therefore the class of \(f\)-subgroups is invariant under automorphisms of \(G\). As a result, the \(f\)-subgroups are a characteristic conjugacy class in \(G\).

By Theorem 4.6, together with Theorem 2.2, the following result is valid:

4.8 Corollary: If \(G\) is a group with Sylow system \(S\), and \(f\)-subgroup \(J_f(S)\), and if \(\theta\) is an epimorphism of \(G\), then \(J_f(S)^\theta\) is the \(f\)-subgroup
of $G^\theta$ associated with the Sylow system $S^\theta$.

The next corollaries are applications of Theorem 4.7 and Corollary 4.8 to the formation of solvable nC-groups.

4.9 Corollary: [2;11] The prefrattini subgroups of a group $G$ are a characteristic conjugacy class in $G$ and in $G^\kappa$.

4.10 Corollary: [11] If $G$ is a group with Sylow system $S$, and prefrattini subgroup $W(S)$, and if $\phi$ is an epimorphism of $G$, then $W(S)^\phi$ is the prefrattini subgroup of $G^\phi$ associated with $S^\phi$.

The final theorem of this chapter will prove useful in induction arguments in the work to follow. Its proof also depends on Theorem 4.1.

4.11 Theorem: Let $N$ be a direct product of minimal normal subgroups of $G$ such that $N \leq G$ and $J_f(S) \cap N = 1$. If $M$ is a complement of $N$ then an $f$-subgroup of $M$ is an $f$-subgroup of $G$. In particular, if $J^M(S)$ is the $f$-subgroup of $M$ associated with the reduction of $S$ into $M$, then $J^M(S) = J(S)$.

Proof: Let $N = N_1 \times N_2 \times \ldots \times N_n$, with $N_i$ minimal normal in $G$ for $i = 1, \ldots, n$. Induction is used on the number of factors in $N$. First, let $N = N_1$, that is, $N$ is minimal normal in $G$. By hypothesis $G = [N]M$ for some maximal subgroup $M$ of $G$. If $N$ is a $p$-group, then there exists a
Sylow $p$-complement $S^p$ such that $S^p \leq M$. Let $S^p$ be an element of the Sylow system $S$ of $G$. By Theorem I.3.11(i), $S$ reduces into $M$. Since $N$ is avoided by $J(s)$ and $N \leq G_f$, Theorem 1.8 implies that $N$ is $\kappa$-eccentric.

By Theorem I.5.5, $M$ is an $\kappa$-abnormal maximal subgroup. According to Definition 1.2, $J(s) \leq M$. Consider the isomorphism $\theta$ given by $G/N = \frac{\cap N}{\cap N \cap N} = M$. Since $J(s) \leq M$, $(J(s)N/N)\theta = J(s)$. By Corollary 4.8, $J(s)$ is an $\mathfrak{f}$-subgroup of $M$. If $S^q \in S$, then $S^qJ(s) = J(s)S^q$. By the modular identity 1.2.1, $J(s)(S^q \cap M) = J(s)S^q \cap M = S^qJ(s) \cap M = (S^q \cap M)J(s)$ for all $S^q \in S$. Corollary 4.5 implies that $J(s)$ is equal to $J^M(s)$ the $\mathfrak{f}$-subgroup of $M$. Now, if $J^M(s_1)$ is another $\mathfrak{f}$-subgroup of $M$ it is conjugate to $J^M(s) = J(s)$ by Theorem 4.7. Hence $J^M(s_1)$ is also an $\mathfrak{f}$-subgroup of $G$.

Inductively assume that the result holds whenever $N$ consists of $k$ factors. Suppose $N = N_1 \times N_2 \times \ldots \times N_k \times N_{k+1}$ and $G = \prod N$. Then $G = N_{k+1}$. By induction, there exists a Sylow system $S$ that reduces into $N_{k+1}$ and $J^M(s) = J(s)$. The subgroup $N_{k+1}$ is minimal normal in $N_{k+1}$. For if $Q < N_{k+1}$ and $Q < N_{k+1}M$ then $Q < N_{k+1}M = G$ since $N$ is abelian. It follows that $(S \cap N_{k+1}M)^g$ a conjugate of $(S \cap N_{k+1}M)$ reduces into $M$. Therefore $S^g$ reduces into $M$. Now note that $N_{k+1}M/N_{k+1}M_f \cong (N_{k+1}M/N_{k+1})/N_{k+1}M_f/N_{k+1}M \cong (M/(N_{k+1}M))/(M_f/(N_{k+1}M)) \cong (M/M_f) \in \mathfrak{f}$. Hence $N_{k+1}M_f \geq (N_{k+1}M_f)$. Since $G/N = \prod N \cong M$, $G_f/N = M_f/N$. Hence $G_f = M_fN$. If $N_1 \times \ldots \times N_k = B$, then $G = MN_{k+1}$. Thus $G/B = (MN_{k+1})/B$ and $G_f/B = ((MN_{k+1})_fB)/B$. Therefore $M_fN = (MN_{k+1})_fB$. In view of the fact that $N_{k+1}B = N$, it follows that $M_fN_{k+1}B = (MN_{k+1})_fB$. Consequently $|M_fN_{k+1}| = |(MN_{k+1})_f|$. Hence $M_fN_{k+1} = (MN_{k+1})_f$. It is now clear that $N_{k+1} = (MN_{k+1})_f$.

Since $J(s) = J_{k+1}(s)$ and $J(s)$ avoids $N_{k+1}$, $J_{k+1}(s)$ avoids $N_{k+1}$.
Applying the first portion of the proof $J_{N_k+1}^M(S^g) = J^M(S^g)$. But Theorem 4.7 implies $J_{N_k+1}^M(S^g) = J_{N_k+1}^M(S^g)$. Hence $J^M(S^g) = J_{N_k+1}^M(S^g) = J(S^g) = J(S^g)$. The induction argument is now complete. If $J^M(S_1)$ is another $f$-subgroup of $M$, it is conjugate to $J^M(S^g) = J(S^g)$ and hence it is an $f$-subgroup of $G$. 
CHAPTER III
THE RELATIONSHIP OF AN $\mathcal{F}$-SUBGROUP TO ITS CORE

If $A$ is a subgroup of the group $G$ then the core of $A$ is defined to be the largest normal subgroup of $G$ contained in $A$. Hence $\operatorname{core} A = \bigcap A^g$. As a consequence of the fact that the $\mathcal{F}$-subgroups form a conjugate class, the core of an $\mathcal{F}$-subgroup of the group $G$ is the intersection of all the $\mathcal{F}$-subgroups of $G$.

This chapter is divided into three sections. In the first the structure of the core of an $\mathcal{F}$-subgroup is examined. The core is identified with known subgroups of the $\mathcal{F}$-residual. This result allows the exploration of the relationship between the $\mathcal{F}$-residual and the core of an $\mathcal{F}$-subgroup. By using nongenerating elements an alternate characterization of the core is given. This will prove to be needed in Chapter V. In the second section, the largest normal nilpotent subgroup avoided by an $\mathcal{F}$-subgroup is found whenever the core of an $\mathcal{F}$-subgroup is the identity subgroup. This result is used to determine a chain condition on a group $G$ which, when $G$ satisfies the condition, implies that $G$ belongs to the formation $\mathcal{F}$. A well known result by G. Zacher is obtained as a corollary. In the final section, $\mathcal{F}$-subgroups are related to $\chi$-critical subgroups. Another characterization of an $\mathcal{F}$-subgroup is given. As a consequence a new structure theorem for the prefrattini subgroup is obtained.

1. The Structure of the Core of an $\mathcal{F}$-subgroup

In the previous chapter the structure of an $\mathcal{F}$-subgroup was given
entirely in terms of the $\mathfrak{r}$-residual. Specifically, for a group $G$ with Sylow system $S$, $J_{\mathfrak{r}}(S) = D_{\mathfrak{r}}(S)(W(S) \cap G_{\mathfrak{r}})$ for which $D_{\mathfrak{r}}(S)$ is the system normalizer of the reduction of $S$ into $G_{\mathfrak{r}}$ and $W(S)$ is the prefattini subgroup of $G$ associated with $S$. This result will be used to show that the core of an $\mathfrak{r}$-subgroup may be described as the product of certain subgroups of $G_{\mathfrak{r}}$. Not surprisingly, it is the product of the cores of the subgroups $D_{\mathfrak{r}}(S)$ and $(W(S) \cap G_{\mathfrak{r}})$.

1.1 Theorem: For an $\mathfrak{r}$-subgroup $J_{\mathfrak{r}}(S)$ of the group $G$ core $J_{\mathfrak{r}}(S) = Z_{\infty}(G_{\mathfrak{r}})(\phi(G) \cap G_{\mathfrak{r}})$.

Proof: Theorem I.3.10(iii) implies that the core of $D_{\mathfrak{r}}(S)$ is $Z_{\infty}(G_{\mathfrak{r}})$, the hypercommutator of $G_{\mathfrak{r}}$. From Theorem I.6.4, the core of $W(S) \cap G_{\mathfrak{r}}$ is $\phi(G) \cap G_{\mathfrak{r}}$. Therefore $Z_{\infty}(G_{\mathfrak{r}})(\phi(G) \cap G_{\mathfrak{r}}) \leq D_{\mathfrak{r}}(S)(W(S) \cap G_{\mathfrak{r}}) = J_{\mathfrak{r}}(S)$. Hence $Z_{\infty}(G_{\mathfrak{r}})(\phi(G) \cap G_{\mathfrak{r}}) \leq \text{core } J_{\mathfrak{r}}(S)$.

The opposite inclusion will be obtained by using induction on the group order.

Case 1: $Z_{\infty}(G_{\mathfrak{r}}) \cap \phi(G) = 1$.

Let $x \in Z_{\infty}(G_{\mathfrak{r}})$ and $g \in G_{\mathfrak{r}}$. Then $[x,g] \in G_{\mathfrak{r}} \cap Z_{\infty}(G_{\mathfrak{r}}) \leq \phi(G_{\mathfrak{r}})$ by Theorem I.1.1. However, $\phi(G_{\mathfrak{r}}) \leq \phi(G)$, so $[x,g] \in \phi(G) \cap Z_{\infty}(G_{\mathfrak{r}}) = 1$. Thus $x \in Z(G_{\mathfrak{r}})$. Consequently, $Z_{\infty}(G_{\mathfrak{r}}) \leq Z(G_{\mathfrak{r}})$ and this implies that $Z_{\infty}(G_{\mathfrak{r}}) = Z(G_{\mathfrak{r}})$. As a result, $Z_{\infty}(G_{\mathfrak{r}})$ is an abelian normal subgroup of $G$ with $Z_{\infty}(G_{\mathfrak{r}}) \cap \phi(G) = 1$. By Theorem I.1.9, $G = [Z_{\infty}(G_{\mathfrak{r}})]A$ for an appropriate subgroup $A$. It follows that $G_{\mathfrak{r}} = Z_{\infty}(G_{\mathfrak{r}}) \times (G_{\mathfrak{r}} \cap A)$ and that $G_{\mathfrak{r}} = Z_{\infty}(G_{\mathfrak{r}})A_{\mathfrak{r}}$. Therefore $A_{\mathfrak{r}} = G_{\mathfrak{r}} \cap A$. Hence $G_{\mathfrak{r}} = Z_{\infty}(G_{\mathfrak{r}}) \times A_{\mathfrak{r}}$. Let $J^A(T)$ be an $\mathfrak{r}$-subgroup of $A$. Since $A \cong Z_{\infty}(G_{\mathfrak{r}})A/Z_{\infty}(G_{\mathfrak{r}}) = G/Z_{\infty}(G_{\mathfrak{r}})$. Corollary II.4.8 implies that $J^A(T)Z_{\infty}(G_{\mathfrak{r}})/Z_{\infty}(G_{\mathfrak{r}})$ is an $\mathfrak{r}$-subgroup of $G/Z_{\infty}(G_{\mathfrak{r}})$. By Theorem II.4.7,
$J^A(T)Z_\infty(G_f)/Z_\infty(G_f) = J(S^g)/Z_\infty(G_f)$ for which $S^g$ is a conjugate of the Sylow system $S$. Hence $J^A(T)Z_\infty(G_f) = J_f(S^g)$. Inductively core $J^A(T) = Z_\infty(A_{(f)})(\phi(A) \cap A_{(f)})$. Clearly $Z_\infty(A_{(f)}) = 1$. So core $J^A(T) = \phi(A) \cap A_{(f)}$. Thus core $J_f(S) = Z_\infty(G_f)(\phi(A) \cap A_{(f)})$. Since $\phi(A) \cap A_f \leq G$, then it follows that $\phi(A) \cap A_f \leq \phi(G) \cap G_f$. Therefore core $J_f(S) \leq Z_\infty(G_f)(\phi(G) \cap G_f)$.

Case 2: $N = Z_\infty(G_f) \cap \phi(G) \neq 1$.

By Lemma II.4.2, $J_f(SN/N) = J_f(S)/N$. Inductively core $J_f(S)/N = Z_\infty((G/N)_f)(\phi(G/N) \cap (G/N)_f)$. Note that, $(G/N)_f = G_f/N$, $\phi(G/N) = \phi(G)/N$, and $Z_\infty((G/N)_f) = Z_\infty(G_f)/N = Z_\infty(G_f)/N$. Therefore core $J_f(S)/N = (Z_\infty(G_f)/N)((\phi(G) \cap G_f)/N)$. Consequently, core $J_f(S) = Z_\infty(G_f)(\phi(G) \cap G_f)$.

The structure of the core of an $f$-subgroup and the fact that $f$-subgroups are preserved under homomorphism can be used to explain the covering and avoidance properties of $f$-subgroups. That is, let $H/K$ be a chief factor of $G$, then $H/K$ is minimal normal in $G/K$. The $f$-subgroup $J_f(S)$ avoids $H/K$ if and only if $(H/K) \cap (J_f(S)K/K) = K$. So $H/K$ is avoided by $J_f(S)$ if and only if $(H/K) \cap (\text{core}(J_f(S)K/K)) = K$. In fact $H/K$ is avoided by $J_f(S)$ if and only if $(H/K) \cap (Z_\infty(G_f)K/K) = K$ and $(H/K) \cap (\phi(G/K) \cap G_fK/K) = K$. Hence if either $G_f$ avoids $H/K$ or if $H/K$ is $\kappa$-eccentric and complemented, then $J_f(S)$ avoids $H/K$. Otherwise $J_f(S)$ covers $H/K$. Consequently, $J_f(S)$ avoids those chief factors that are avoided by either $G_f$ or by $W^h(S)$. It covers all other chief factors. This is just a restatement of Theorem II.1.11.

An important corollary is obtained from Theorem 1.1.

1.2 Corollary: The core of an $f$-subgroup is nilpotent.
Proof: By Theorem 1.1, core \( J_f(s) = Z_{\infty}(G_f)(\phi(G) \cap G_f) \). It follows that core \( J_f(s) \leq F(G) \), the Fitting subgroup of the group \( G \). Therefore core \( J_f(s) \) is nilpotent.

The nilpotence of the core of an \( f \)-subgroup is useful in examining the structure of the \( f \)-residual.

1.3 Theorem: For a group \( G \) with Sylow system \( S \) the following conditions are equivalent:

(i) \( G_f = J_f(S) \).

(ii) \( J_f(S) = \text{core } J_f(S) \).

(iii) \( G_f \) is nilpotent.

(iv) \( G \in \mathcal{h} \).

Proof: If \( G_f = J_f(S) \), then \( J_f(S) \) is a normal subgroup of \( G \). Consequently, \( J_f(S) = \text{core } J_f(S) \), that is, (i) implies (ii). If \( J_f(S) = \text{core } J_f(S) \), then \( J_f(S) \) is nilpotent by Corollary 1.2. By Theorem II.2.2, \( J_f(S) \) contains a system normalizer of \( G_f \). Theorem I.3.10 implies that \( J_f(S) = G_f \) since \( J_f(S) \triangleleft G_f \). Hence \( G_f \) is nilpotent. Therefore (ii) implies (iii). If \( G_f \) is nilpotent then \( K_{\infty}(G_f) = 1 \). By the comments following Theorem II.2.3, \( K_{\infty}(G_f) = G_H \). So \( G \in \mathcal{h} \). Therefore (iii) implies (iv). If \( G \in \mathcal{h} \), then every factor of \( G_f \) is \( \mathcal{h} \)-central. By Theorem II.1.8, \( J_f(S) \) covers every factor of \( G_f \). Theorem I.2.2 implies that \( J_f(S) \) covers \( G_f \). Consequently, \( J_f(S) = G_f \). Hence (iv) implies (i).

The main consequence of Theorem 1.3 is that the \( f \)-residual is nilpotent if and only if it coincides with an \( f \)-subgroup.
1.4 Corollary: [2] Let $\kappa$ denote the formation of solvable nC-groups. For a group $G$ with Sylow system $S$ the following conditions are equivalent:

(i) $G_{\kappa} = W(S)$.
(ii) $W(S) = \phi(G)$.
(iii) $G_{\kappa}$ is nilpotent.

Proof: Apply Theorem 1.3 with $f = \kappa$. By Theorem II.3.1, $J_{\kappa}(S) = W(S)$. By Theorem I.6.4, core $W(S) = \phi(G)$.

When $\kappa$ is the formation of solvable nC-groups the core of a $\kappa$-subgroup is the frattini subgroup. A frattini subgroup may be defined as the set of all nongenerators of a group $G$, for which an element $x$ is a nongenerator if for each nonempty set $X$ such that $\langle x, X \rangle = G$, $\langle X \rangle = G$. A natural question is whether or not a nongenerator-like definition exists for the core of an $f$-subgroup for an arbitrary formation. The answer is yes.

1.5 Definition: A maximal subgroup $M$ of a group $G$ is $\kappa^*$-normal if $M$ is $\kappa$-normal and $M$ does not contain $G_f$. Otherwise $M$ is called an $\kappa^*$-abnormal maximal subgroup.

Note that if $M$ is maximal, then $M$ is $\kappa^*$-abnormal if either $M$ is $\kappa$-abnormal or if $G_f \leq M$. The core of an $f$-subgroup can be described by $\kappa^*$-abnormal subgroups.
1.6 Lemma: If $J_f(S)$ is an $f$-subgroup of the group $G$, then
\[ \text{core } J_f(S) = \cap \{ M \mid M \text{ is an } h^*-abnormal maximal subgroup of } G \} \cap G_f. \]

Proof: By Theorem II.1.6, $J_f(S) = W^h(S) \cap G_f$. Therefore
\[ \text{core } J_f(S) = \cap \{ \text{core } J_f(S) \}^G = \cap \{ W^h(S) \}^G \cap G_f. \]

Theorem I.3.3 and Definition I.7.1 imply that $\cap \{ W^h(S) \}^G = \cap \{ M \mid M \text{ is an } h\text{-abnormal maximal subgroup of } G \}$. Hence, $\text{core } J_f(S) = \cap \{ M \mid M \text{ is an } h\text{-abnormal maximal subgroup of } G \} \cap G_f = (\cap \{ M \mid M \text{ is an } h\text{-abnormal maximal subgroup of } G \}) \cap \{ M \mid M \text{ is a maximal subgroup of } G \text{ such that } G_f \leq M \} \cap G_f$. Therefore $\text{core } J_f(S) = \cap \{ M \mid M \text{ is an } h^*-abnormal maximal subgroup of } G \} \cap G_f. \]

Nongenerating elements which are linked to an arbitrary formation $f$ are defined next.

1.7 Definition: Let $y$ be an element of $G_f$, such that if there exists a nonempty subset $X$ of the group $G$ with $G = \langle y, X \rangle$, then either $G = \langle X \rangle$ or $X \leq M$ for $M$ an $h^*$-normal maximal subgroup of $G$. The element $y$ is called an $f$-nongenerator of $G$. The set of all $f$-nongenerators of $G$ is denoted by $\phi_f(G)$.

1.8 Theorem: The set of all $f$-nongenerators $\phi_f(G)$ of a group $G$ is a subgroup of $G$.

Proof: Suppose $x, y \in \phi_f(G)$ and that $X$ is a nonempty subset of $G$ with $\langle xy^{-1}, X \rangle = G$. Then $\langle x, y^{-1}, X \rangle = G$. Since $x$ is an $f$-nongenerator,
either \( <y^{-1},X> = G \) or \( y^{-1} \) and \( X \) are contained in an \( \ast \)-normal maximal subgroup \( M \) of \( G \). If \( <y^{-1},X> = G \), then \( <y,X> = G \). Since \( y \) is an \( f \)-non-generator of \( G \), either \( <X> = G \) or \( X \subseteq M \), for an \( \ast \)-normal maximal subgroup \( M \) of \( G \). In any case \( <xy^{-1},X> = G \) implies that either \( <X> = G \) or that \( X \equiv M \), for an \( \ast \)-normal maximal subgroup \( M \) of \( G \). Therefore \( xy^{-1} \in \phi_f(G) \). Hence \( \phi_f(G) \) is a subgroup of \( G \).

As the next theorem indicates, the set of all \( f \)-nongenerators of the group \( G \), \( \phi_f(G) \), coincides with the core of an \( f \)-subgroup of \( G \). Consequently, a nongenerator definition of the core of an \( f \)-subgroup has been established.

1.9 Theorem: For a group \( G \) with Sylow system \( S \), core \( J_f(S) = \phi_f(G) \).

Proof: Let \( y \in \text{core } J_f(S) \) and \( X \) be a nonempty subset of \( G \) such that \( <y,X> = G \). If \( <X> \neq G \) there exists a subgroup \( M \) that is maximal with the property that \( y \notin M \) and \( X \subseteq M \). If there exists a subgroup \( H \) of \( G \) such that \( M < H < G \), then \( <y,X> \subseteq H \). But \( G = <y,X> \). Hence \( G = H \). Therefore \( M \) is a maximal subgroup of \( G \). If \( M \) is \( \ast \)-abnormal, then Theorem 1.6 implies that \( \text{core } J_f(S) \leq M \). Hence \( y \in M \). This contradicts the existence of \( M \). Therefore \( M \) is \( \ast \)-normal. Thus if \( <y,X> = G \), either \( <X> = G \) or \( X \equiv M \) for an \( \ast \)-normal maximal subgroup \( M \) of \( G \). Consequently, \( y \in \phi_f(G) \) and core \( J_f(S) \leq \phi_f(G) \).

Let \( y \) be an element of \( G \) such that \( y \notin \text{core } J_f(S) \). From Theorem 1.6 either \( y \notin G_f \) or \( y \notin M \) for some \( \ast \)-abnormal maximal subgroup \( M \) of \( G \).
If $y \notin G_f$, then $y \notin \phi_f(G)$ since $\phi_f(G) \leq G_f$. If $y \notin M$, then $G = \langle y, M \rangle$, $M = M \neq G$, and $M$ is not $\ast$-normal. Therefore $y \notin \phi_f(G)$. Consequently, if $y \notin \text{core } J_f(S)$, then $y \notin \phi_f(G)$. Thus $\phi_f(G) \leq \text{core } J_f(S)$. Hence $\phi_f(G) = \text{core } J_f(S)$.

The value of an $f$-nongenerator characterization of the core of an $f$-subgroup is not immediately apparent. It will prove to be significant in Chapter V when the connection between $f$-subgroups and normal formations is examined. In that process the following result will be required.

1.10 Theorem: Let $Y$ be a subset of the group $G$ contained in $\phi_f(G)$. If there exists a nonempty subset $X$ of $G$ such that $G = \langle Y, X \rangle$, then either $G = \langle X \rangle$ or $X \leq M$ for an $\ast$-normal maximal subgroup $M$ of $G$.

Proof: Since $Y$ is contained in $\phi_f(G)$, it must be contained in a subgroup of $\phi_f(G)$ generated by elements $x_1, x_2, \ldots, x_n$. If $G = \langle Y, X \rangle$, then $G = \langle Y, X \rangle \leq \langle x_1, x_2, \ldots, x_n, X \rangle = \langle x_1, x_2, \ldots, x_n, X \rangle$. Since $x_1 \in \phi_f(G)$, either $\langle x_2, x_3, \ldots, x_n, X \rangle = G$ or $\{x_2, \ldots, x_n\} \cup X \leq M$ for an $\ast$-normal maximal subgroup $M$ of $G$. Inductively assume that, either $\langle x_k, x_{k+1}, \ldots, x_n, X \rangle = G$ or $\{x_k, x_{k+1}, \ldots, x_n\} \cup X \leq M$ for an $\ast$-normal maximal subgroup $M$ of $G$. Since $x_k \in \phi_f(G)$ either $\langle x_{k+1}, x_{k+2}, \ldots, x_n, X \rangle = G$ or $\{x_{k+1}, \ldots, x_n\} \cup X \leq M$ for an $\ast$-normal maximal subgroup $M$ of $G$. Hence by induction, either $\langle X \rangle = G$ or $X \leq M$ for $M$ an $\ast$-normal maximal subgroup of $G$.

Since the core of a prefrattini subgroup is the frattini subgroup, it is interesting to note the connection of $k$-nongenerators, for $k$ the
formation of solvable nC-groups, to the usual nongenerators of a group. The following corollary provides a nongenerator description of the frattini subgroup which emphasizes its relationship with the solvable nC-groups.

1.11 Corollary: If $\kappa$ is the formation of solvable nC-groups, then for a group $G$ $\phi_{\kappa}(G) = \phi(G)$. An element of $G$ is a nongenerator if and only if it is a $\kappa$-nongenerator.

Proof: The result follows from Theorems 1.9 and II.3.1.

2. On a Result by G. Zacher

G. Zacher had obtained a chain condition which when applied to a solvable group $G$, implies that $G$ is an nC-group. In this section a similar result is obtained for an arbitrary formation $\mathcal{F}$. A chain condition is given which is necessary and sufficient to imply that $G$ is an element of the formation $\mathcal{F}$. A number of preliminary results are required.

2.1 Lemma: Let $G$ be a group and let $S(G_{\mathcal{F}})$ denote the sockel of $G_{\mathcal{F}}$. If $M$ is a normal subgroup of $G$ such that $M \leq S(G_{\mathcal{F}})$ and $M \cap \text{core } J_{\mathcal{F}}(S) = 1$, then $J_{\mathcal{F}}(S)$ avoids $M$.

Proof: Let $1 = M_0 \triangleleft M_1 \triangleleft \ldots \triangleleft M_n = M$ be a series for $M$ such that $M_{i+1}/M_i$ is a chief factor of $G$ for $i = 0, 1, \ldots, n-1$. It will be established that $J_{\mathcal{F}}(S)$ avoids $M$ by using induction on the number of terms in the series.
Since $M \cap \text{core } J_f(S) = 1$, $M_1 \cap \text{core } J_f(S) = 1$. If $J_f(S)$ covers $M_1$, then $M_1 \leq J_f(S)$. Hence $M_1 \leq \text{core } J_f(S)$. Therefore $J_f(S)$ avoids $M_1$. Assume inductively that $J_f(S)$ avoids $M_k$ for $k < n$. If it is proven that $M_{k+1}/M_k$ is complemented and $\bar{n}$-eccentric, then $J_f(S)$ avoids $M_{k+1}/M_k$ by Theorem II.1.8. Since $M_{k+1} \leq S(G_f)$, $M_{k+1}$ is abelian. By hypothesis, $\text{core } J_f(S) \cap M_{k+1} = 1$. Hence Theorem I.1 implies that $\phi(G) \cap G_f \cap M_{k+1} = \phi(G) \cap M_{k+1} = 1$. By Theorem I.1.9, $G$ splits over $M_{k+1}$. If $G = [M_{k+1}]A$, then $G/M_k = (M_{k+1}/M_k)(AM_k/M_k)$. By the modular identity I.1.2, $M_k \cap AM_k = M_k(M_k \cap A) = M_k$. Therefore $G/M_k = [M_{k+1}/M_k]AM_k/M_k$. So $M_{k+1}/M_k$ is a complemented chief factor. Assume that $M_{k+1}/M_k$ is an $\bar{n}$-central factor, then $M_{k+1}/M_k$ is centralized by $G_f$. Theorem I.1.13 implies that $M_{k+1} = M_k \times L$ for a minimal normal subgroup $L$. Since $(M_k \times L)/M_k$ is central in $G_f$, $[g, 1] \in M_k \cap L = 1$ for all $g \in G_f$, $1 \in L$. Hence $L \leq Z(G_f) \leq Z_\infty(G_f)$. So $L \leq \text{core } J_f(S) \cap M_{k+1} = 1$. This contradicts the fact that $M_{k+1}/M_k$ is a chief factor. Consequently, $M_{k+1}/M_k$ is an $\bar{n}$-eccentric chief factor. Therefore $J_f(S)$ avoids $M_{k+1}/M_k$. By the induction hypothesis, $J_f(S)$ avoids $M_k$. Theorem I.2.2 implies that $J_f(S)$ avoids $M_{k+1}$. By induction, $J_f(S)$ avoids $M_n = M$.

2.2 Theorem: For the sockel $S(G)$ of a group $G$ let $A(G) = S(G) \cap \text{core } J_f(S)$. Then there exists $L(G) \leq G$ having the following properties:

(i) $S(G) = L(G) \times A(G)$.

(ii) $J_f(S)$ avoids $L(G)$.

(iii) If $L(G) \neq 1$, no normal subgroup of $G$ which properly contains $L(G)$ and which is contained in $S(G)$ is avoided by $J_f(S)$. 


Proof: Note that \( A(G) \leq S(G) \) and \( A(G) \triangleleft G \). By Theorem I.1.13, there exists at least one normal subgroup of \( G \), denoted by \( L(G) \), such that \( S(G) = A(G) \times L(G) \). It will suffice to prove that \( L(G) \) satisfies (ii) and (iii).

For (ii), consider a chief factor \( H/K \) with \( K < H \leq L(G) \). Assume that \( J_f(S) \) covers \( H/K \). Theorem II.1.9 implies that \( J_f(S) \) covers the chief factor \( (H \cap G_f)/(K \cap G_f) \). By hypothesis, \( H \cap G_f \leq S(G) \). Hence \( H \cap G_f \) is an abelian normal subgroup of \( G \). Moreover \( \text{core } J_f(S) \cap L(G) = \text{core } J_f(S) \cap S(G) \cap L(G) = A(G) \cap L(G) = 1 \). Theorem 1.1 implies that \( \phi(G_f) \leq \text{core } J_f(S) \). Thus \( (H \cap G_f) \cap \phi(G_f) = 1 \). In view of the fact that \( H \cap G_f \) is abelian Theorem I.1.9 implies that \( H \cap G_f \) is the direct product of minimal normal subgroups of \( G_f \). Therefore \( H \cap G_f \leq S(G_f) \). Applying Lemma 2.1, \( J_f(S) \) avoids \( H \cap G_f \). Hence \( J_f(S) \) avoids \( (H \cap G_f)/(K \cap G_f) \). The factor \( (H \cap G_f)/(K \cap G_f) \) is covered and avoided by \( J_f(S) \). This implies that \( (H \cap G_f) = (K \cap G_f) \). A contradiction arises since \( (H \cap G_f)/(K \cap G_f) \) is a chief factor. Consequently, \( J_f(S) \) avoids \( H/K \). Since the factor \( H/K \) is arbitrary, \( J_f(S) \) avoids every factor of \( G \) below \( L(G) \). By Theorem I.2.2, \( J_f(S) \) avoids \( L(G) \).

For (iii), suppose that \( L(G) < H \leq S(G) \) and that \( H/L(G) \) is a chief factor of \( G \). Assume that \( J_f(S) \) avoids \( H/L(G) \). From (ii), \( J_f(S) \) avoids \( L(G) \). Theorem I.2.2 implies that \( J_f(S) \) avoids \( H \). Therefore \( H \cap J_f(S) = H \cap \text{core } J_f(S) = 1 \). Note that \( S(G) = A(G)L(G) = A(G)H \). Hence \( |S(G)| = |A(G)||L(G)| = |A(G)||H|/|H \cap A(G)| \). However \( |H \cap A(G)| = |H \cap \text{core } J_f(S) \cap S(G)| = 1 \). So \( |H| = |L(G)| \). Since \( L(G) \leq H \), then \( L(G) = H \). Since the factor \( H/L(G) \) is assumed to be nontrivial, a contradiction arises. The assumption that \( H/L(G) \) is avoided by \( J_f(S) \) is invalid. Therefore (iii) holds.
In Corollary 1.2 the core of an $f$-subgroup was proven to be nilpotent. Consequently, the core is the largest normal nilpotent subgroup of a group covered by an $f$-subgroup. In the following theorem whenever $\text{core} J_f(S) = 1$ the maximal normal nilpotent subgroup avoided by $J_f(S)$ is identified.

2.3 Theorem: For a group $G$ if $\text{core} J_f(S) = 1$, then the maximal normal nilpotent subgroup of $G$ avoided by $J_f(S)$ is the Fitting subgroup $F(G)$ of $G$.

Proof: Theorem 1.1 implies that $\phi(G_f) \leq \text{core} J_f(S)$. Hence $\phi(G_f) = 1$ since $\text{core} J_f(S) = 1$. Consequently, $F(G_f) = S(G_f)$. By Lemma 2.1, $J_f(S)$ avoids $F(G_f)$. Let $H/K$ be a chief factor of $G$ for which $F(G_f) \leq K < H \leq F(G)$. If $H/K$ does not exist, then $F(G_f) = F(G)$ and the result follows. If $H/K$ exists, assume that $J_f(S)$ covers $H/K$. Theorem II.1.9 implies that $J_f(S)$ covers the chief factor $(H \cap G_f)/(K \cap G_f)$. Note that $(H \cap G_f) \leq (F(G) \cap G_f) = F(G_f)$. Hence $J_f(S)$ avoids $(H \cap G_f)/(K \cap G_f)$ since $J_f(S)$ avoids $F(G_f)$. The factor $(H \cap G_f)/(K \cap G_f)$ is covered and avoided by $J_f(S)$. So $H \cap G_f = K \cap G_f$. This contradicts the fact that $(H \cap G_f)/(K \cap G_f)$ is a chief factor. Therefore $H/K$ must be avoided by $J_f(S)$. Since $H/K$ is arbitrary, Theorem I.2.2 implies that $J_f(S)$ avoids $F(G)/F(G_f)$. Since $F(G_f)$ is avoided by $J_f(S)$, $J_f(S)$ avoids $F(G)$. The Fitting subgroup is the maximal normal nilpotent subgroup of $G$. Hence the result follows.

The chain condition on a group $G$ which implies that $G$ is an element of the formation $f$ results from examining Fitting subgroups. In
a finite solvable group, $F(G) \neq 1$. Therefore there exists a chain of subgroups $1 = K_0 < K_1 < \ldots < K_m = G$ defined by $K_i = F(G/K_{i-1})$ for $i = 1, 2, \ldots, m$, it is called the Fitting chain of $G$.

2.4 Theorem: Let $f$ be a formation and let $1 = K_0 < K_1 < \ldots < K_m = G$ be the Fitting chain of the group $G$. Then $G \in f$ if and only if $\text{core}(J_f(S)K_i/K_i) = K_i$ for $i = 0, 1, \ldots, m-1$.

Proof: If $G \in f$, then by Theorem II.2.1, $J_f(S) = 1$. Therefore $J_f(S)K_i/K_i = K_i$ for each $i$. This implies that $\text{core}(J_f(S)K_i/K_i) = K_i$ for $i = 0, 1, \ldots, m-1$.

Conversely, suppose that $\text{core}(J_f(S)K_i/K_i) = K_i$ for $i = 0, 1, \ldots, m-1$. Consider $G/K_{m-1} = F(G/K_{m-1})$. Since $\text{core}(J_f(S)K_{m-1}/K_{m-1}) = K_{m-1}$, and $J_f(S)K_{m-1}/K_{m-1}$ is an $f$-subgroup of $G/K_{m-1}$ by Lemma II.4.2, from Theorem 2.3 it follows that $J_f(S)K_{m-1}/K_{m-1}$ avoids $G/K_{m-1}$. Thus $(J_f(S)K_{m-1}/K_{m-1}) \cap (G/K_{m-1}) = K_{m-1}$. Equivalently, $J_f(S)K_{m-1} \cap G \subseteq K_{m-1}$. Hence $J_f(S) \subseteq K_{m-1}$.

Inductively assume that $J_f(S) \subseteq K_{1}$ and consider $G/K_{1-1}$. Since $\text{core}(J_f(S)K_{1-1}/K_{1-1}) = K_{1-1}$ and $J_f(S)K_{1-1}/K_{1-1}$ is an $f$-subgroup of $G/K_{1-1}$ by Lemma II.4.2, Theorem 2.3 may be applied to $G/K_{1-1}$ to conclude $J_f(S)K_{1-1}/K_{1-1}$ avoids $F(G/K_{1-1}) = K_{1}/K_{1-1}$. Consequently, $(J_f(S)K_{1-1}/K_{1-1}) \cap (K_{1}/K_{1-1}) = K_{1-1}$. So $J_f(S)K_{1-1} \cap K_{1} \subseteq K_{1-1}$. Since $J_f(S) \subseteq K_{1}$, $J_f(S)K_{1-1} \subseteq K_{1-1}$. Therefore $J_f(S) \subseteq K_{1-1}$. By induction $J_f(S) \subseteq K_{0} = 1$. By Theorem II.2.1, $G \in G_f$.

The following result due to G. Zacher does not rely on the theory of prefarrattini subgroups.
2.5 Corollary: [24] Let $1 = K_0 < K_1 < \ldots < K_m = G$ be the Fitting chain of the group $G$ and let $\kappa$ denote the formation of solvable $nC$-groups. Then $G \in \kappa$ if and only if $\phi(G/K_i) = K_i$ for $i = 0, 1, \ldots, m-1$.

Proof: By Theorem II.3.1 $J_f(s) = W(s)$. Theorem I.6.4 implies that $\text{core } W(s) = \phi(G)$. Now apply Theorem 2.4 with $f = \kappa$.

3. $\kappa$-critical Maximal Subgroups

The system normalizers and $g$-normalizers are the minimal elements of particular chains of subgroups of a group. The $f$-subgroups can be obtained from chains quite similar to these although not as minimal chain elements. To arrive at this result, the $\kappa$-critical maximal subgroups must be examined in detail.

3.1 Lemma: For a group $G$, every minimal normal subgroup of $G/\text{core } J_f(s)$ is $\kappa$-central if and only if $G \in \kappa$.

Proof: Suppose that every minimal normal subgroup of $G/\text{core } J_f(s)$ is $\kappa$-central. If core $J_f(s) = G_f$, then Theorem 1.3 implies that $G \in \kappa$. Assume that core $J_f(s) < G_f$ and let $N/\text{core } J_f(s)$ be a minimal normal subgroup of $G/\text{core } J_f(s)$ such that $N \leq G_f$. By Theorem II.1.8, $J_f(s)$ covers $N/\text{core } J_f(s)$. Theorem I.2.2 implies $J_f(s)$ covers $N$. Therefore $N \leq J_f(s)$. So $N \leq \text{core } J_f(s)$. A contradiction arises since $N \neq \text{core } J_f(s)$. Consequently, $G_f = \text{core } J_f(s)$ and $G \in \kappa$.

Conversely, if $G \in \kappa$, then every chief factor of $G$ is $\kappa$-central.
Therefore every minimal normal subgroup of $G/\text{core } J_f(S)$ is $h$-central.

Recall Definition I.5.12 that an $h$-critical maximal subgroup is an $h$-abnormal maximal subgroup such that $F(G)M = G$, for which $F(G)$ is the Fitting subgroup of $G$.

3.2 Lemma: The group $G$ has an $h$-critical maximal subgroup if and only if $G$ has an $h$-critical maximal subgroup $M$ such that $G/\text{core } J_f(S) = [N/\text{core } J_f(S)] M/\text{core } J_f(S)$ for $N \leq G_f$.

Proof: If $G$ has an $h$-critical maximal subgroup, then $G \not\leq h$ by Theorem I.5.13. From Lemma 3.1, $G/\text{core } J_f(S)$ has at least one minimal core $J_f(S) = \text{core } J_f(S)$ or $N/\text{core } J_f(S) \leq G_f/\text{core } J_f(S)$. If the normal subgroup $\cap(G_f/\text{core } J_f(S)) = \text{core } J_f(S)$ or $N/\text{core } J_f(S) \leq G_f/\text{core } J_f(S)$. If the former is valid, then $G_f \leq C_G[N/\text{core } J_f(S)]$. Hence $N/\text{core } J_f(S)$ is $h$-central. A contradiction arises. Therefore $N/\text{core } J_f(S) \leq G_f/\text{core } J_f(S)$, that is, $N \leq G_f$. The factor $N/\text{core } J_f(S)$ is avoided by $J_f(S)$, for otherwise Theorem I.2.2 implies $N \leq \text{core } J_f(S)$. By Theorem II.1.8, $N/\text{core } J_f(S)$ is a complemented $h$-eccentric chief factor. Let the maximal subgroup $M$ be a complement of $N/\text{core } J_f(S)$. The chief factor $N/\text{core } J_f(S)$ is $h$-critical by Definition I.5.12. From Theorem I.5.14 it follows that $M$ is an $h$-critical maximal subgroup such that $G/\text{core } J_f(S) = [N/\text{core } J_f(S)] M/\text{core } J_f(S)$.

The converse follows immediately.

Next the $h$-critical maximal subgroups found in Lemma 3.2 are
related to $f$-subgroups.

3.3 Theorem: Let $M$ be an $\eta$-critical maximal subgroup of the group $G$ such that the Sylow system $S$ reduces into $M$ and $G/\text{core } J_f(s) = [N/\text{core } J(s)]M/\text{core } J(s)$ for $N \leq G_f$. Then $J(s) = J^M(s) \text{ core } J(s)$ for which $J^M(s)$ is the $f$-subgroup of $M$ associated with the reduction of $S$ into $M$.

Proof: Let $\overline{G} = G/\text{core } J(s)$, $\overline{J}(s) = J(s)/\text{core } J(s)$, $\overline{M} = M/\text{core } J(s)$, and $\overline{N} = N/\text{core } J(s)$. Then $\overline{N}$ is minimal normal and complemented in $\overline{G}$. Since $\overline{M}$ complements $\overline{N}$ and $\overline{M}$ is $\eta$-abnormal, $\overline{N}$ is $\eta$-eccentric by Theorem 1.5.5. Let $\overline{J}(s)$ be the $f$-subgroup of $\overline{M}$ associated with the reduction of $S$ core $J(s)/\text{core } J(s)$ into $\overline{M}$. Theorem II.4.11 implies that $\overline{J}(s) = J(s)$. However from Lemma II.4.2 $\overline{J}(s) = J^M(s) \text{ core } J(s)/\text{core } J(s)$. Hence $J(s)/\text{core } J(s) = J^M(s) \text{ core } J(s)/\text{core } J(s)$. Therefore $J(s) = J^M(s) \text{ core } J(s)$.

3.4 Theorem: Let $D^\eta(s)$ be an $\eta$-normalizer of the group $G$ associated with the Sylow system $S$ of $G$. Then $D^\eta(s)$ can be joined to $G$ by a chain of the form $D^\eta(s) = G_r < G_{r-1} < \ldots < G_1 < G_0 = G$, for which the subgroups $G_i$ satisfy the following conditions for $i = 1, 2, \ldots, r$:

(i) $G_i$ is an $\eta$-critical maximal subgroup of $G_{i-1}$.

(ii) $S$ reduces into $G_i$.

(iii) If $G \not\in \eta$ and if $J_{i-1}(s)$ is the $f$-subgroup of $G_{i-1}$ associated with the reduction of $S$ into $G_{i-1}$ then there exists $N_{i-1}/\text{core } J_{i-1}(s)$, such that $N_{i-1} \leq (G_{i-1})$ and $G_{i-1}/\text{core } J_{i-1}(s) = [N_{i-1}/\text{core } J_{i-1}(s)]$ $G_i/\text{core } J_{i-1}(s)$. 
Proof: If $G \in \mathfrak{h}$, then $D^h(s) = G$. Hence the chain is $D^h(s) = G$. If $G \notin \mathfrak{h}$ then Theorem 1.5.13 implies that $G$ has $\mathfrak{h}$-critical maximal subgroups. By Lemma 3.2, $G$ has an $\mathfrak{h}$-critical maximal subgroup $M$ such that $G/\text{core } J(s) = [N/\text{core } J(s)]M/\text{core } J(s)$ for $N \leq G$. By replacing $M$ with a suitable conjugate of $M$ if necessary, it may be assumed that $s$ reduces into $M$. Let $M = G_1$. By Theorem 1.5.15, $D^h(s)$ is an $\mathfrak{h}$-normalizer of $G_1$.

Since $|G_1| < |G|$ inductively it may be assumed that $D^h(s) = G_r < G_{r-1} < \ldots < G_2 < G_1$ for $G_i$ defined as in the hypothesis. Therefore $D^h(s) = G_r < G_{r-1} < \ldots < G_2 < G_1 < G_0 = G$ is the chain that is required.

A characterization of the $f$-subgroup of a group $G$ can be in terms of the cores of the $f$-subgroups of the chain elements $G_i$ defined in Theorem 3.4.

3.5 Theorem: Let $D^h(s) = G_r < G_{r-1} < \ldots < G_1 < G_0 = G$ be the chain of $\mathfrak{h}$-critical maximal subgroups of the group $G$ which exists by Theorem 3.4. Then the $f$-subgroup $J(s)$ of $G$ associated with $s$ satisfies $J(s) = (\text{core } J(s)\text{core } J_1(s)\text{core } J_2(s)\ldots)(\text{core } J_r(s)\ldots))$, for which $J_i(s)$ is the $f$-subgroup of $G_i$ associated with the reduction of $s$ into $G_i$ for $i = 1, \ldots, r$.

Proof: Use induction on the length of the chain. If $D^h(s) = G$ then $G \in \mathfrak{h}$. Hence by Theorem 1.3, $J(s) = \text{core } J(s)$. Therefore the theorem is valid. Suppose that $D^h(s) < G$. By Theorem 3.3, $J(s) = \text{core } J(s) J_1(s)$. Assume that $J(s) = (\text{core } J(s)\text{core } J_1(s)\ldots)(\text{core } J_{k-1}(s)\ldots)$ for $k < r$. Then applying Theorem 3.3 to $G_k$, $J_k(s) = J_{k+1}(s)$.
core \( J_k(s) \). Thus \( J(s) = (\text{core } J(s))(\text{core } J_1(s))(\ldots(\text{core } J_{k-1}(s))(\text{core } J_k(s) \\
J_{k+1}(s))\ldots))). \) By induction, \( J(s) = (\text{core } J(s))(\text{core } J_1(s))(\ldots(\text{core } J_{r-1}(s) \\
J_r(s))\ldots))). \) From Theorem 1.5.8, \( G_r = D^h(s) \in h. \) Hence by Theorem 1.3, \( J_r(s) = \text{core } J_r(s). \) Therefore \( J(s) = (\text{core } J(s))(\text{core } J_1(s))(\ldots(\text{core } J_{r-1}(s) \\
\text{core } J_r(s))\ldots))). \)

Theorem 3.5 is used to obtain a new characterization of the pre-frattini subgroup as a product of frattini subgroups. The chain of subgroups used to define this product is somewhat less restricted than that of Theorem 3.5.

3.6 Theorem: Let \( \kappa \) be the formation of solvable nC-groups and let \( h \) be locally defined by \( h(p) = \kappa \) for every prime integer \( p \). If \( D^h(s) = G_r < G_{r-1} \ldots < G_1 < G_0 = G \) is a chain of subgroups of the group \( G \) such that \( S \) reduces into \( G_i \) and \( G_i \) is an \( h \)-critical maximal subgroup of \( G_{i-1} \) for \( i = 1, \ldots, r \), then \( W(S) = (\phi(G)\phi(G_1)\phi(G_2)\ldots(\phi(G_r))\ldots))). \)

Proof: If \( G \in h \), then \( G = D^h(s) \). By Corollary 1.4, \( W(S) = \phi(G) \). Hence the result holds. If \( G \notin h \), let \( M \) be an \( h \)-critical maximal subgroup of \( G \). Note that \( F(G)/\phi(G) = (N_1/\phi(G))\times(N_2/\phi(G))\times\ldots\times(N_k/\phi(G)) \) for minimal normal subgroups \( N_1/\phi(G), N_2/\phi(G), \ldots, N_k/\phi(G) \) of \( G/\phi(G) \). Since \( M \) is \( h \)-critical, \( G = F(G)M \) by Definition 1.5.11. Thus there exists \( N_i \) such that \( G/\phi(G) = (N_i/\phi(G))(M/\phi(G)) \) and \( (N_i/\phi(G)) \cap (M/\phi(G)) = \phi(G). \) Therefore if \( M \) is any \( h \)-critical maximal subgroup, there exists \( N \) such that \( G/\phi(G) = [N/\phi(G)]M/\phi(G). \) Moreover, by Theorem 1.5.10, \( N/\phi(G) \) is \( h \)-eccentric. It follows that \( N/\phi(G) \leq G/\phi(G). \) Consequently, since
core $J^\kappa(S) = \text{core } W(S) = \phi(G)$, the chain in the hypothesis is a chain that satisfies the conditions in Theorem 3.4. Applying Theorem 3.5, $J^\kappa(S) = (\text{core } J(S)(\text{core } J_1(S)\ldots(\text{core } J_r(S))\ldots)))$. Therefore $W(S) = (\phi(G_1)(\phi(G_1)(\phi(G_2)(\ldots(\phi(G_r))\ldots))))$.

The result in Theorem 3.5 is also applied in a different manner. O. Kramer [20] defined a group $G$ to be $f$-critical if $G \notin f$, but every subgroup of $G$ belongs to $f$.

3.7 Corollary: [20] Let $f$ be a formation and $G$ an $f$-critical group. Then $G \in \mathcal{N}_f$ for the formation $\mathcal{N}_f$ of groups with nilpotent $f$-residual.

Proof: If $M$ is a maximal subgroup of $G$, then $M \in f$ since $G$ is $f$-critical. By Theorem II.2.1, $J^M(S) = 1$. By Theorem 3.5, $J(S) = \text{core } J(S)$. Hence by Theorem 1.3, $G_f$ is nilpotent. Therefore $G \in \mathcal{N}_f$. 
CHAPTER IV

OTHER CHARACTERIZATIONS OF \(f\)-SUBGROUPS

In this chapter alternate forms for the \(f\)-subgroups are given. The \(f\)-subgroups structure is explained through the use of other CAR-subgroups of the group and of the \(f\)-residual. It is proven first that an \(f\)-subgroup of a direct product is a direct product of \(f\)-subgroups of the direct factors. In the second section, an \(f\)-subgroup is proven to be a CAR-subgroup of the \(f\)-residual. This is achieved by considering an intersection of certain known CAR-subgroups of the \(f\)-residual and proving that the intersection is in fact an \(f\)-subgroup. In the following section, the question of whether or not \(f\)-subgroups are intersections of a collection of maximal subgroups of the \(f\)-residual is answered affirmatively. Then \(f\)-subgroups are shown not to be, in general, intersections of a collection of maximal subgroups of the group. Finally, the structure of an \(f\)-subgroup is related to a prefrattini subgroup and system normalizer of the \(f\)-residual.

1. Direct Products of \(f\)-subgroups

T. Hawkes and K. Doerk [8] have proven that for a formation \(f\) of finite solvable groups, if the group \(G\) is the direct product \(G = G_1 \times G_2 \times \ldots \times G_n\), then the \(f\)-residual is \(G_f = G_1^f \times G_2^f \times \ldots \times G_n^f\), for which \(G_i^f\) is the \(f\)-residual of the \(i\)th factor group \(G_i\). In order to prove that in such a case an \(f\)-subgroup of \(G\) is the direct product of \(f\)-subgroups of the factor groups, a preliminary result is necessary.
In the following, for a Sylow system $S$ of a group $G$ let $J^K(s)$ be the $f$-subgroup of $K$ associated with $S \cap K$, $J^{G/N}(s)$ be the $f$-subgroup of $G/N$ associated with $SN/N$ and $J^{K/N}(s)$ be the $f$-subgroup of $K/N$ associated with $(S \cap K)N/N$.

1.1 Lemma: Let $f$ be a formation and $K$ be a normal subgroup of the group $G$ for which $K_f \leq G_f$. Then $J^K(s) \leq J(s)$ if and only if $\text{core } J^{K/N}(s) \leq \text{core } J^{G/N}(s)$ for each normal subgroup $N$ of $G$ such that $N \leq K_f$.

Proof: Assume that $\text{core } J^{K/N}(s) \leq \text{core } J^{G/N}(s)$ for each normal subgroup $N$ of $G$ such that $N \leq K_f$. Use induction on the group order. The property that $\text{core } J^{K/N}(s) \leq \text{core } J^{G/N}(s)$ is preserved under the homomorphisms having kernels contained in $K_f$. For suppose that $\theta$ is such a homomorphism with kernel $L \leq K_f$. Then if $L \leq N \leq K_f$ for $N \triangleleft G$, then $\text{core } J^{K/N}(s) \leq \text{core } J^{G/N}(s)$. Since $K/N \cong (K/L)/(N/L)$ and $G/N \cong (G/L)/(N/L)$, Corollary 11.4.8 implies that $\text{core } J^{(K/L)/(N/L)}(s) \leq \text{core } J^{(G/L)/(N/L)}(s)$.

Case 1: $\text{core } J^K(s) \neq 1$.

Let $Q$ be a minimal normal subgroup of $G$ with $Q \leq \text{core } J^K(s)$. Then $Q \leq \text{core } J(s) \leq J(s)$. By induction $J^K(s)/Q \leq J(s)/Q$. Hence $J^K(s) \leq J(s)$.

Case 2: $\text{core } J^K(s) = 1$.

Let $Q$ be a minimal normal subgroup of $G$ with $Q \leq K_f$. If $Q$ does not exist, then $J^K(s) \leq K_f = 1$. Hence $J^K(s) \leq J(s)$. If $J(s)$ covers $Q$, then by induction $J^K(s)Q/Q \leq J(s)/Q$. So $J^K(s) \leq J(s)$. Suppose that $J(s)$ avoids $Q$. Since $\text{core } J^K(s) = 1$, Theorem III.1.1 implies that $\phi(K_f) \cap Q \leq \phi(K) \cap K_f \cap Q = \phi(K) \cap Q = 1$. By Theorem 1.1.9, $Q$ is a direct product of minimal normal subgroups of $K_f$. It follows that $Q$ is contained in the
sockel of $K_F$. By Lemma III.2.1, $J^K(S)$ avoids $Q$. Since $J(S)$ avoids $Q$, $Q$ is complemented in $G$ by Theorem II.1.8. Hence $G = [Q]L$ for a subgroup $L$ such that $S$ reduces into $L$. Therefore, $K = G \cap K = QL \cap K = Q(L \cap K)$ by the modular identity I.1.2. Since $Q \cap L \cap K \leq Q \cap L = 1$, $K = [Q](L \cap K)$. In view of the fact that $L \cap K < L$ and $S$ reduces into $L$, it follows that $S$ reduces into $L \cap K$. By Theorem II.4.11, $J^K \cap L(S) = J^K(S)$ and $J^L(S) = J(S)$.

If it is shown that $(L \cap K)_{\nu} \leq L_{\nu}$, then $J^K \cap L(S) \leq J^L(S)$ by induction.

Note that $(L \cap K)/(L_{\nu} \cap K) \cong ((L \cap K)Q/Q)/((L_{\nu} \cap K)Q/Q)$. By the modular identity I.1.2, $(L_{\nu} \cap K)Q = L_{\nu}Q \cap K$. Since $G/Q \cong LQ/Q$, $L_{\nu}Q/Q = G_{\nu}/Q$. Since $K_{\nu} \leq G_{\nu}$, $K_{\nu} \leq L_{\nu}Q \cap K$. Consequently, $(L \cap K)/(L_{\nu} \cap K) \cong K/(L_{\nu}Q \cap K) \in \nu$.

Therefore $(L \cap K)_{\nu} \leq L_{\nu} \cap K$. Thus $(L \cap K)_{\nu} \leq L_{\nu}$. So $J^K \cap L(S) \leq J^L(S)$.

Since $J^K \cap L(S) = J^K(S)$ and $J^L(S) = J(S)$, then $J^K(S) \leq J(S)$.

Conversely, suppose that $J^K(S) \leq J(S)$. Let $N$ be normal in $G$ and $N \leq K$. Corollary II.4.8 implies that $J^K/N(S) \leq J^G/N(S)$. Hence core $J^K/N(S) \leq J^G/N(S)$. It follows that core $J^K/N(S) \leq$ core $J^G/N(S)$.

1.2 Theorem: If the group $G$ is the direct product $G = G_1 \times G_2 \times \ldots \times G_K$ and $J_1(S)$ is the $f$-subgroup of $G_i$ associated with the reduction of $S$ into $G_i$ for $i = 1, 2, \ldots, K$, then $J(S) = J_1(S) \times J_2(S) \times \ldots \times J_n(S)$.

**Proof:** It is sufficient to prove the result for $n = 2$. An induction argument would then imply the result for $n = k$. So let $G = H \times K$. In an effort to ease notational problems, write $G = HK$ for which it is assumed that $H < G$, $K < G$, and $H \cap K = 1$. It is shown first that $J^H(S)J^K(S) \leq J(S)$. By Lemma I.4.6, $G_{\nu} = H_{\nu}K_{\nu}$. Hence $H_{\nu} \leq G_{\nu}$. Since $\phi(H) \leq \phi(G)$, then $\phi(H) \cap H_{\nu} \leq \phi(G) \cap G_{\nu}$. If $Q/R$ is a chief factor of $H$, then
it is also a chief factor of $G$. If $Q \leq Z_\infty(H_f)$, then $Q/R$ is centralized by $H_f$. So $Q/R$ is centralized by $G_f$. It follows that $Z_\infty(H_f) \leq Z_\infty(G_f)$. Consequently, $(\phi(H) \cap H_f)Z_\infty(H_f) \leq (\phi(G) \cap G_f)Z_\infty(G_f)$. Therefore core $J^H(s) \leq$ core $J(s)$ by Theorem III.1. For a normal subgroup $N$ of $G$ contained in $H_f$, $G/N = (H/N)(K/N)$. Moreover, $(H/N) \cap (K/N) = (H \cap K)/N = (H \cap K)/N = N$ by the modular identity I.1.2. Applying the above argument to $H/N$ in $G/N$, core $J^H/N(s) \leq$ core $J^G/N(s)$. By lemma 1.1, $J^H(s) \leq J(s)$. An analogous argument yields $J^K(s) \leq J(s)$. Therefore $J^H(s)J^K(s) \leq J(s)$.

To show that $J^H(s)J^K(s) = J(s)$, it is sufficient to prove that they have the same order. From Theorem I.2.3, this reduces to proving that $J(s)$ and $J^H(s)J^K(s)$ cover and avoid the same chief factors of a series $C$ of $G$. If $J(s)$ avoids a factor, then $J^H(s)J^K(s)$ also avoids that factor since $J^H(s)J^K(s) \leq J(s)$. Hence it suffices to prove that in a given series $C$, every factor covered by $J(s)$ is also covered by $J^H(s)J^K(s)$. Consider a chief series of $G$ of the form $G = G_k > G_{k-1} > \ldots > G_0 = G_f = H_fK_f > H_{k-1}K_f > \ldots > H_0K_f = K_f > K_{m-1} > \ldots > K_0 = 1$. Since $J(s)$ and $J^H(s)J^K(s)$ are contained in $G_f$, they avoid all factors above $G_f$. Suppose that $J(s)$ covers $H_iK_f/H_{i-1}K_f$. Then $H_iK_f \leq H_{i-1}K_fJ(s)$. Hence $H_iK \leq H_{i-1}KJ(s)$. Thus $J(s)K/K$ covers $H_iK/H_{i-1}K$. By Corollary II.4.8, $f$-subgroups are invariant under homomorphism. Since $HK/H \cong H$, this implies that $J(s)K/K = J^H(s)K/K$. It follows that $J^H(s)K/K$ covers $(H_iK/K)/(H_{i-1}K/K)$. If $\theta$ is the isomorphism mapping $HK/H$ onto $H$, then the image under $\theta$ of $J^H(s)K/K$ is $J^H(s)$. The image under $\theta$ of $(H_iK/K)/(H_{i-1}K/K)$ is $H_i/H_{i-1}$. Consequently, $J^H(s)$ covers $H_i/H_{i-1}$, that is, $H_i \leq H_{i-1}J^H(s)$. Hence $H_iK \leq H_{i-1}KJ^K(s)J^H(s)$. Therefore $J^H(s)J^K(s)$ covers $H_iK/H_{i-1}K$. Suppose that $J(s)$ covers $K_i/K_{i-1}$, then $K_i \leq J(s)K_{i-1}$ and $K_iH \leq J(s)K_{i-1}H$. So $J(s)H/H
covers $HK_i/HK_{i-1}$. By applying the above argument to $J^K(s)$ and $HK_i/HK_{i-1}$, one concludes that $K_i/K_{i-1}$ is covered by $J^K(s)$. Equivalently $K_i \leq K_{i-1}J^K(s)$. Hence $K_i \leq K_{i-1}J^K(s)J^H(s)$. Thus $J^K(s)J^H(s)$ covers $K_i/K_{i-1}$. In the given chief series, every factor covered by $J(s)$ is also covered by $J^H(s)J^K(s)$. Therefore $J(s) = J^H(s)J^K(s)$.

Because a $f$-subgroup is a prefattini subgroup for the formation $f$ of solvable $nC$-groups, the following corollary is a consequence of the theorem.

1.3 Corollary: If the group $G$ is the direct product $G = G_1 \times G_2 \times \ldots \times G_k$, $W(s)$ is the prefattini subgroup of $G$ associated with $s$, and $W_i(s)$ is the prefattini subgroup of the reduction of $s$ into $H_i$ for $i = 1, 2, \ldots, k$, then $W(s) = W_1(s) \times \ldots \times W_k(s)$.

2. On a Result by H. Bechtell

In this section an $f$-subgroup is characterized as the intersection of a particular collection of CAR-subgroups of the $f$-residual. As a consequence, an $f$-subgroup is not only a CAR-subgroup of a group $G$, but it is also a CAR-subgroup of the $f$-residual $G_f$. This characterization when applied to the prefattini subgroups yields a result by H. Bechtell.

2.1 Definition: Let $G_f = L_0 > L_1 > \ldots > L_n = 1$ denote a lower nilpotent series of $G_f$ of length $n$, that is, $L_j = K_{\infty}(L_{j-1})$. Define $\phi_j$ inductively
as follows: \( \phi_1 = \phi(G) \cap G_f \), \( \phi_j+1 \) is the subgroup of \( G \) such that \( \phi_{j+1}/L_j^* = \phi(G/L_j^*) \cap (G_f/L_j^*) \) for \( L_j^* = \phi_j L_{n-j} \). The series \( 1 = L_0^* \leq \phi_1 \leq L_1^* \leq \phi_2 < \cdots < L_{n-1}^* \leq \phi_n = G_f \) is a characteristic series of \( G_f \); it is called an \( f \)-series. For a Sylow system \( S \) of \( G_f \) and a lower nilpotent series \( \{L_j\} \) of \( G_f \), \( S \) reduces into a Sylow system \( S_j \) of \( L_j \) and into a relative system normalizer \( N_{G_f}(S_j) \). Using the terminology of H. Bechtell [2], call \( \{N_{G_f}(S_j)\}_{j=0, \ldots, n-2} \) an \( S \)-system of relative system normalizers of \( G_f \).

2.2 Definition: For a group \( G \) with associated \( S \)-system of relative system normalizers of \( G_f \) and \( f \)-series of \( G_f \), define \( A(S) = \bigcap_{j=1}^{n-1} \phi_j N_{G_f}(S_{n-j-1}) \). Call \( A(S) \) the \( A \)-subgroup associated with \( S \).

The goal of this section is to prove that in a group \( G \) \( A(S) = J_f(S) \).

Before this is accomplished it is proven that \( A(S) \) is an \( (S \cap G_f) \) CAR-subgroup of \( G_f \). The results of M. Tomkinson will play a major role.

2.3 Lemma: The class \( C_j \) of solvable groups for which \( L_j = 1 \) is a saturated formation.

Proof: Let \( G \in C_j \) and \( N < G \). If \( G_f = L_0 > L_1 > \cdots > L_j = 1 \) is a lower nilpotent series of \( G_f \), then \( G_f N/N = L_0 N/N \geq L_1 N/N \geq \cdots \geq L_j N/N = N \) is a lower nilpotent series of \( G_f N/N \). Consequently, if \( G \in C_j \), then \( G/N \in C_j \).

Suppose \( G/N \in C_j \) and \( G/M \in C_j \) with \( M \cap N = 1 \). Then \( L_j M/M = M \) and \( L_j N/N = N \). Therefore \( L_j \leq M \) and \( L_j \leq N \). Thus \( L_j \leq M \cap N = 1 \). Hence \( G \in C_j \). By definition 1.4.1, \( C_j \) is a formation. To prove that \( C_j \) is saturated, suppose that \( G/\phi(G) \in C_j \). Then \( L_j-1 \phi(G)/\phi(G) \) is nilpotent. Since \( L_j-1/(L_j-1 \cap \phi(G)) \)
is isomorphic to $L_{j-1}\phi(G)/\phi(G)$, it is nilpotent. By Theorem I.1.6, $L_{j-1}$ is nilpotent, that is, $L_j = 1$. Therefore if $G/\phi(G) \in C_j$, then $G \in C_j$.

So by Definition I.4.3, $C_j$ is a saturated formation.

**2.4 Lemma:** For each $j = 0,1,\ldots,n-2$, $N_G^{f}(S_j)$ is an $s\text{CAR}$-subgroup of the group $G$ and a perspective $(S \cap G_f)$ $\text{CAR}$-subgroup of $G_f$, for which $S \cap G_f$ is the reduction of $S$ into $G_f$.

**Proof:** For $j = 0$, $N_G^{f}(S_0)$ is a system normalizer of $G_f$. Theorem I.8.8 implies that $N_G^{f}(S_0)$ is a perspective $S_0\text{CAR}$-subgroup of $G_f$. By definition, $S_0 = S \cap G_f$. Let $N_G(S_0)$ be the relative system normalizer of $S_0$ in $G$. By Theorem I.3.13, $N_G^{f}(S_0) = N_G(S_0) \cap G_f$. From Theorem I.8.5(iv) it follows that, $N_G^{f}(S_0) = N_G(S_0) \cap G_f = \bigcap_{p} (N_G(S_0) \cap S_p G_f)$ is an $s\text{CAR}$-subgroup of $G$. Therefore the result is valid for $j = 0$.

If $j > 0$, define a formation $K$ by $G \in K$ if and only if $L_{j-1} \leq C_G(H/K)$ for each chief factor $H/K$ of $G$. That is, $K$ is locally defined by $f(p) = C_{j-1}$ for every prime integer $p$. The formations $K$ and $C_j$ coincide. For if $G \in K$, then every factor below $L_{j-1}$ is centralized by $L_{j-1}$. This implies that $L_{j-1}$ is nilpotent. It follows that $L_j = 1$. Hence, $G \in C_j$. So $K \equiv C_j$. Conversely, if $G \in C_j$, then $L_j = 1$. Thus $L_{j-1}$ is nilpotent. If $H/K$ is a chief factor of $G$ with $H \leq L_{j-1}$, then $H/K \cap Z(L_{j-1}/K) \neq 1$ since $L_{j-1}/K$ is nilpotent. Because $H/K$ is minimal normal in $G/K$, then $H/K \leq Z(L_{j-1}/K)$. Consequently, $L_{j-1} \leq C_G(H/K)$. Clearly every chief factor above $L_{j-1}$ is centralized by $L_{j-1}$. Therefore $G \in K$. So $C_j \equiv K$.

Note that for every prime integer $p$, $L_{j-1}$ is the smallest normal subgroup $K$ such that $G/K \in C_{j-1} = f(p)$. By Definition I.5.1, a $C_j$-system
normalizer is $D^j(S) = \bigcap_p N_G(L_{j-1} \cap S^p)$. However, $\bigcap_p N_G(L_{j-1} \cap S^p)$ is a relative system normalizer of $L_{j-1}$ by Definition 1.3.12. Denote it by $N_G(S_j)$. It follows from Theorem I.8.8 that $N_G(S_j)$ is a perspective SCAR-subgroup of $G$. Applying Theorem I.3.13, $N_{G^p}(S_j) = N_G(S_j) \cap G^p$. Then Theorem I.8.5(iv) implies that $N_{G^p}(S_j) = \bigcap_p (N_G(S_j^p) \cap S^p G^p)$ is an SCAR-subgroup of $G$.

To complete the proof it is necessary to prove that $N_{G^p}(S_j)$ is a perspective $(S \cap G^p)$ CAR-subgroup of $G^p$. It has been shown that for any formation $f$ and group $G$, $N_G(S_j)$ is a perspective SCAR-subgroup of $G$. Apply this to the group $G^p$ and the formation $g$ defined by $G \in g$ if and only if $G = 1$. In this case $G^p = G$. Hence $N_{G^p}(S_j)$ is a perspective $(S \cap G^p)$ CAR-subgroup of $G^p$.

2.5 Lemma: The group $\phi_j N_{G^p}(S_{n-j-1})$ is an SCAR-subgroup of $G$ and an $(S \cap G^p)$ CAR-subgroup of $G^p$ for $i = 1, \ldots, n-1$.

Proof: Apply Lemma 2.4 and Theorem I.8.5(i) and (ii) with $V_p = \phi_j$ for all $p$.

2.6 Theorem: The $A$-subgroup $A(S) = \bigcap_{j=1}^{n-1} \phi_j N_{G^p}(S_{n-j-1})$ is an $(S \cap G^p)$ CAR-subgroup of $G^p$. Furthermore, if $H/K$ is a chief factor of $G^p$, $H/K$ is covered by $A(S)$ if and only if $H/K$ is covered by $\phi_j N_{G^p}(S_{n-j-1})$ for each $j = 1, \ldots, n-1$.

Proof: Let $A_1 = \bigcap_{j=1}^{n-1} \phi_j N_{G^p}(S_{n-j-1})$. For $i = 1$, $A_1 = \phi_1 N_{G^p}(S_{n-2})$. By Lemma 2.5 $A_1$ is an $(S \cap G^p)$ CAR-subgroup of $G^p$. 
Assume that the result holds for \( j = k \). Then \( A_k \) is an \((S \cap G_f)\) CAR-subgroup of \( G_f\) and a chief factor is covered by \( A_k\) if and only if it is covered by \( \phi_j N_{G_f}(S_{n-j-1}) \) for each \( j = 1, \ldots, k \). Consider the subgroup \( A_{k+1} \). It will be proven first that \( A_{k+1} \) has the required cover and avoidance property, then this fact will be utilized to show that \( A_{k+1} \) is an \((S \cap G_f)\) CAR-subgroup of \( G_f\). Let \( H/K\) be a chief factor of \( G_f\). If \( H/K\) is avoided by \( \phi_j N_{G_f}(S_{n-j-1}) \) for any \( j = 1, \ldots, k+1\), then \( A_{k+1}\) avoids \( H/K\) since \( A_{k+1} \) is contained in each \( \phi_j N_{G_f}(S_{n-j-1}) \). So suppose that \( H/K\) is covered by \( \phi_j N_{G_f}(S_{n-j-1}) \) for each \( j = 1,\ldots,k+1\). Inductively, \( A_k\) covers \( H/K\). Note that for \( j < k + 1, n - j - 1 < n - (k+1) - 1 \). Hence \( L_{n-(k+1)-1} \leq L_{n-j-1} \).

For each prime integer \( p \), \( S^p \cap L_{n-j-1} \leq S^p \cap L_{n-k-2} \). If \( x \in N_G(S^p \cap L_{n-k-2}) \) and \( y \in S^p \cap L_{n-j-1} \), then \( x^{-1}yx \in S^p \cap L_{n-k-2} \cap L_{n-j-1} = S^p \cap L_{n-j-1} \). Hence \( x \in N_{G_f}(S^p \cap L_{n-j-1}) \). Therefore for every \( p \), \( N_{G_f}(S^p) = N_{G_f}(S^p \cap L_{n-k-2}) \leq N_{G_f}(S^p \cap L_{n-j-1}) = N_{G_f}(S_{n-j-1}) \). Consequently, \( N_{G_f}(S_{n-(k+1)-1}) \leq N_{G_f}(S_{n-j-1}) \).

Also note that \( A_{k+1} = A_k \cap \phi_{k+1} N_{G_f}(S_{n-(k+1)-1}) \). Since \( N_{G_f}(S_{n-(k+1)-1}) \leq N_{G_f}(S_{n-(k+1)-1}) \) for \( j < k + 1 \), \( N_{G_f}(S_{n-(k+1)-1}) \leq A_k \). By the modular identity I.1.2, \( A_{k+1} = \frac{N_{G_f}(S_{n-(k+1)-1})}{(A_k \cap \phi_{k+1})} \). If \( H/K\) is covered by \( N_{G_f}(S_{n-(k+1)-1}) \), then \( H/K\) is covered by \( A_{k+1}\). Suppose that \( N_{G_f}(S_{n-(k+1)-1}) \) avoids \( H/K\). It is assumed that \( H/K\) is covered by \( \phi_{k+1} N_{G_f}(S_{n-(k+1)-1}) \). By Lemma 2.4 \( N_{G_f}(S_{n-(k+1)-1}) \) is a perspective \((S \cap G_f)\) CAR-subgroup of \( G_f\). From Theorem I.8.7(i) it follows that \( \phi_{k+1} \) covers \( H/K\). By hypothesis, \( A_k \) is an \((S \cap G_f)\) CAR-subgroup of \( G_f\). Theorem I.8.5(iv) and (v) imply that \( A_k \cap \phi_{k+1} \) is an \((S \cap G_f)\) CAR-subgroup of \( G_f\). Furthermore, \( A_k \cap \phi_{k+1} \) covers \( H/K\) if and only if \( \phi_{k+1} \) covers \( H/K\) and \( A_k \) covers \( (H \cap \phi_{k+1})/(K \cap \phi_{k+1}) \). So to prove that \( A_k \cap \phi_{k+1} \) covers \( H/K\) it need only be established that \( A_k \) covers \( (H \cap \phi_{k+1})/(K \cap \phi_{k+1}) \). By the induction hypothesis, \( A_k \) covers \( (H \cap \phi_{k+1})/(K \cap \phi_{k+1}) \) if
and only if $\phi_j N_{G_f}(s_{n-j-1})$ covers $(H \cap \phi_{k+1})/(K \cap \phi_{k+1})$ for each $j \leq k$. It has been assumed that $\phi_j N_{G_f}(s_{n-j-1})$ covers $H/K$ for each $j \leq k$. For a fixed $j \leq k$, Theorem I.8.7(i) implies that either $\phi_j$ covers $H/K$ or $N_{G_f}(s_{n-j-1})$ covers $H/K$, since $N_{G_f}(s_{n-j-1})$ is perspective by Lemma 2.4. If $\phi_j$ covers $H/K$, then $H \leq \phi_j K$. By definition, $\phi_j \leq \phi_{k+1}$. By the modular identity I.1.2, $H \cap \phi_{k+1} \leq \phi_j K \cap \phi_{k+1} = \phi_j (K \cap \phi_{k+1})$. Therefore $\phi_j$ covers $(H \cap \phi_{k+1})/(K \cap \phi_{k+1})$. If $N_{G_f}(s_{n-j-1})$ covers $H/K$, then since $N_{G_f}(s_{n-j-1})$ is perspective, from Definition I.8.6 it follows that $N_{G_f}(s_{n-j-1})$ covers $(H \cap \phi_{k+1})/(K \cap \phi_{k+1})$. In either case $\phi_j N_{G_f}(s_{n-j-1})$ covers $(H \cap \phi_{k+1})/(K \cap \phi_{k+1})$ for each $j \leq k$. Consequently $A_k$ covers $(H \cap \phi_{k+1})/(K \cap \phi_{k+1})$.

So $A_k \cap \phi_{k+1}$ covers $H/K$. It has been demonstrated that either $N_{G_f}(s_{n-(k+1)})$ covers $H/K$ or $A_k \cap \phi_{k+1}$ covers $H/K$. Hence $A_{k+1} = N_{G_f}(s_{n-(k+1)})(A_k \cap \phi_{k+1})$ covers $H/K$. Therefore if $\phi_j N_{G_f}(s_{n-j-1})$ covers $H/K$ for each $j \leq k + 1$, then $A_{k+1}$ covers $H/K$. Otherwise $H/K$ is avoided by $A_{k+1}$. By induction, $A(s) = A_{n-1}$ covers $H/K$ whenever $\phi_j N_{G_f}(s_{n-j-1})$ covers $H/K$ for each $j = 1, \ldots, n-2$; otherwise $A(s)$ avoids $H/K$.

To complete the proof it is necessary to prove that $A_{k+1}$ is an $(S \cap G_f)$ CAR-subgroup of $G_f$. In addition to the original induction hypothesis, assume that $\bigcap_{j=1}^{k+1} \phi_j N_{G_f}(S_{n-j-1})$ is an $(S \cap G_f)$ CAR-system for $A_k$. Note that for any $j$, $\bigcap_{\phi_j N_{G_f}(S_{n-j-1})} = \phi_j N_{G_f}(S_{n-j-1})$. Thus $\bigcap_{\phi_j N_{G_f}(S_{n-j-1})} = A_{k+1}$. Furthermore if $x \in S^P \cap G_f$ and $l \in L_{n-j-1} \cap S^P$, then $x^{-1}lx \in L_{n-j-1} \cap S^P$. Hence $x \in N_{G_f}(S_{n-j-1})$. Therefore $S^P \cap G_f \leq N_{G_f}(S_{n-j-1})$ for each $j$. Consequently, $S^P \cap G_f \leq \bigcap_{j=1}^{k+1} \phi_j N_{G_f}(S_{n-j-1})$. To prove that $A_{k+1}$ is an $(S \cap G_f)$ CAR-subgroup it remains to show that $\bigcap_{\phi_j N_{G_f}(S_{n-j-1})} = A_{k+1}$ covers or avoids each chief factor of $G$. Let $j=1$.
p be a fixed prime integer for p dividing the order of G. If a chief factor H/K of \( G_f \) has a q-power order \( q \neq p \), then \( A_{k+1}^p \) covers H/K since \( S^p \cap G_f \) covers H/K and \( S^p \cap G_f \leq A_{k+1}^p \). Let H/K be a p-chief factor of \( G_f \).

Inductively, \( A_k \) is an \( (S \cap G_f) \) CAR-subgroup of \( G_f \). Therefore \( A_k^p = \bigcap_{j=1}^{k} \phi_j N G_f (S_{n-j-1}^p) \) either covers or avoids H/K. Lemma 2.5 implies that \( \phi_{k+1} N G_f (S_{n-k-2}^p) \) is an \( (S \cap G_f) \) CAR-subgroup of \( G_f \) relative to the system \( \{ \phi_{k+1} N G_f (S_{n-k-2}^p) \} \). It follows that \( \phi_{k+1} N G_f (S_{n-k-2}^p) \) either covers or avoids H/K. Suppose that either \( \bigcap_{j=1}^{k} \phi_j N G_f (S_{n-j-1}^p) \) or \( \phi_{k+1} N G_f (S_{n-k-2}^p) \) avoids H/K. Then \( A_{k+1}^p \) avoids H/K since \( A_{k+1}^p \) is contained in both of these subgroups.

Suppose that both \( \bigcap_{j=1}^{k} \phi_j N G_f (S_{n-j-1}^p) \) and \( \phi_{k+1} N G_f (S_{n-k-2}^p) \) cover H/K. By Theorem 1.8.3, \( A_k \) covers H/K and \( \phi_{k+1} N G_f (S_{n-k-2}^p) \) covers H/K. The first section of the proof implies \( A_{k+1} \) covers H/K. Since \( A_{k+1} \leq A_{k+1}^p \), then \( A_{k+1}^p \) covers H/K. Consequently, \( A_{k+1}^p \) either covers or avoids each chief factor of \( G_f \). The group \( A_{k+1} \) is then an \( (S \cap G_f) \) CAR-subgroup of \( G_f \). By induction \( A(S) = A_{n-1} \) is an \( (S \cap G_f) \) CAR-subgroup of \( G_f \).

2.7 Theorem: The A-subgroup \( A(S) = \bigcap_{j=1}^{n-1} \phi_j N G_f (S_{n-j-1}^p) \) either covers or avoids each chief factor of the group G that lies below \( G_f \) and \( A(S) \) covers the chief factor H/K if and only if \( \phi_j N G_f (S_{n-j-1}^p) \) covers H/K for each \( j = 1, \ldots, n-1 \).

Proof: Let H/K be a chief factor of G with H \( \leq G_f \). If H/K is avoided by \( \phi_j N G_f (S_{n-j-1}^p) \) for any \( j = 1, \ldots, n-1 \), then \( A(S) \) avoids H/K since \( A(S) \) is contained in each of these subgroups. By Lemma 2.5, \( \phi_j N G_f (S_{n-j-1}^p) \) is an SCAR-subgroup of G for \( j = 1, 2, \ldots, n-1 \). It follows
that the only other possible case is that $\phi_j N_{G_f}(s_{n-j-1})$ covers $H/K$ for each $j = 1, 2, \ldots, n-1$. Lemma 2.6 implies that $A(s)$ covers $H/K$. Hence $A(s)$ covers $H/K$ if and only if $\phi_j N_{G_f}(s_{n-j-1})$ covers $H/K$ for each $j = 1, \ldots, n-1$.

In order to obtain $A(s) = J_f(s)$ one further lemma is required.

2.8 Lemma: If for a group $G$, $\phi(G_f) = 1$, then $G_f = [L_{n-1}] N_{G_f}(s_{n-2})$ and $G = [L_{n-1}] N_G(s_{n-2})$.

Proof: The group $G_f/L_{n-1}$ has a lower nilpotent series of length $n-1$, that is, $L_{n-2}/L_{n-1}$ is nilpotent. Therefore each Sylow subgroup of $L_{n-2}/L_{n-1}$ is characteristic in $L_{n-2}/L_{n-1}$. It follows that each Sylow subgroup of $L_{n-2}/L_{n-1}$ is normal in both $G_f/L_{n-1}$ and $G/L_{n-1}$. Consequently, $N_{G_f}(s_{n-2}/L_{n-1}) = G_f$ and $N_G(s_{n-2}/L_{n-1}) = G$. By Theorem 1.3.14, $N_{G_f}(s_{n-2}/L_{n-1}) = N_{G_f}(s_{n-2})/L_{n-1}$ and $N_G(s_{n-2}/L_{n-1}) = N_G(s_{n-2})/L_{n-1}$. So $G_f = N_{G_f}(s_{n-2})/L_{n-1}$ and $G = N_G(s_{n-2})/L_{n-1}$. Note that, $N_G(s_{n-2}) \cap L_{n-1} = N_G(s_{n-2}) \cap G_f \cap L_{n-1} = N_{G_f}(s_{n-2}) \cap L_{n-1}$. By way of contradiction assume that this intersection is nontrivial. Since $\phi(G_f) = 1$, $F(G_f)$ is abelian. Since $L_{n-1}$ is nilpotent and normal in $G_f$, $L_{n-1} \leq F(G_f)$. Therefore $L_{n-1}$ is abelian. Consider elements $g \in G_f$ and $x \in N_{G_f}(s_{n-2}) \cap L_{n-1}$, then $g = n!$ with $n \in N_{G_f}(s_{n-2})$ and $1 \in L_{n-1}$. So $g^{-1}x = 1^{-1}n^{-1}x n^{-1} = 1^{-1}y 1^{-1}$ for some $y \in N_{G_f}(s_{n-2}) \cap L_{n-1}$. Since $L_{n-1}$ is abelian, $1^{-1}y 1^{-1} = y$. Thus $g^{-1}x = y$ and hence $g^{-1}x \in N_{G_f}(s_{n-2}) \cap L_{n-1}$. Consequently, $N_{G_f}(s_{n-2}) \cap L_{n-1} \leq G_f$ and $N_{G_f}(s_{n-2}) \cap L_{n-1} \leq L_{n-2}$. Let $N$ be a proper minimal normal subgroup of $L_{n-2}$ such that $N \leq N_{G_f}(s_{n-2}) \cap L_{n-1}$. 
By Theorem 1.3.13, \( N_{G_f}(\mathcal{S}_{n-2}) \cap L_{n-2} = N_{L_{n-2}}(\mathcal{S}_{n-2}) \) is a system normalizer of \( L_{n-2} \). Hence \( N \leq Z(L_{n-2}) \). But then \( N \leq Z(L_{n-2}) \cap L_{n-1} \leq Z(L_{n-2}) \cap (L_{n-2})' \).

By Theorem 1.1.1, \( N \leq \phi(L_{n-2}) \leq \phi(G_f) = 1 \). A contradiction arises since \( N \neq 1 \). Therefore \( N_{G_f}(\mathcal{S}_{n-2}) \cap L_{n-1} = 1 \). It follows that \( G_f = [L_{n-1}] \)

\( N_{G_f}(\mathcal{S}_{n-2}) \) and \( G = [L_{n-1}] N_G(\mathcal{S}_{n-2}) \).

**2.9 Theorem:** For a group \( G \) and Sylow system \( \mathcal{S} \) of \( G \) let \( A(\mathcal{S}) \) be the subgroup of \( G \) defined by Definition 2.2, then \( A(\mathcal{S}) = J_f(\mathcal{S}) \).

**Proof:** Induction will be used on the group order.

Case 1: \( \phi(G) \cap G_f \neq 1 \).

Let \( N \) be a minimal normal subgroup of \( G \) such that \( N \leq \phi(G) \cap G_f \). The factor group \( G_f/N \) has the lower nilpotent series \( \{NL_j/N|j=0,\ldots,n\} = \{L_j|j=0,\ldots,n\} \). Note that \( \phi_1/N = (\phi(G) \cap G_f)/N = (\phi(G)/N) \cap (G_f/N) = \phi(G/N) \cap (G/N)_f = \phi_1(G/N) \). Denote \( \phi_1(G/N) \) by \( \phi_1 \) and the other components of the \( f \)-series of \( G_f/N \) by \( L^*_j \) and \( \phi_j \). Inductively assume that it has been proven that \( L^*_k = \phi_k L_{n-k}/N = L^*_k/N \) and \( \phi_k = \phi_k/N \). Then \( \phi_{k+1}/L_{k+1} = (\phi(G/N)/\phi_k L_{n-k}/N) \cap ((G_f/N)/(\phi_k L_{n-k}/N)) \cap (G_f/\phi_k L_{n-k}/N) = \phi_{k+1}/\phi_k L_{n-k} \). It follows that \( \phi_{k+1} = \phi_{k+1}/N \). Now \( L^*_{k+1} = \phi_{k+1} L_{n-k-1} = \phi_{k+1} L_{n-k-1}/N = L^*_{k+1}/N \). Hence \( L^*_{k+1}/N \). By induction, \( \phi_j = \phi_j/N \) and \( L^*_j = L^*_j/N \) for every \( j \). By Theorem 1.3.14(i)

\( N N_{G_f}(\mathcal{S}_{n-j-1})/N \) is a relative system normalizer of \( N L_{n-j-1}/N \). Hence for the

\( A(\mathcal{S}N/N) \) of \( G/N \), \( A(\mathcal{S}N/N) = \bigcap_{j=1}^{n-1} \phi_j(N N_{G_f}(\mathcal{S}_{n-j-1})/N) = \bigcap_{j=1}^{n-1} \phi_j N_{G_f}(\mathcal{S}_{n-j-1})/N = A(\mathcal{S})/N \). Since \( N \leq \phi(G) \cap G_f \),

\( N \leq J_f(\mathcal{S}) \) by Theorem III.1.1. By Lemma II.4.2 \( J_f(\mathcal{S}N/N) = J_f(\mathcal{S})/N \). By
induction \( A(SN/N) = J_f(SN/N) \). Therefore \( A(S)/N = J_f(S)/N \). Consequently, \( A(S) = J_f(S) \).

Case 2: \( \phi(G) \cap G_f = 1 \).

Since \( G_f \triangleleft G, \phi(G_f) \leq \phi(G) \). Hence \( \phi(G_f) = 1 \). Assume that \( N \) is a proper minimal normal subgroup such that \( N \leq \text{core } J_f(S) \cap L_{n-1} \). Since \( N \leq \text{core } J_f(S) \), Theorem II.1.8 implies that either \( N \leq \phi(G) \cap G_f \) or \( N \leq Z(G_f) \). Since \( \phi(G) \cap G_f = 1 \), then \( N \leq Z(G_f) \). Either \( G_f = 1 \), in which case \( A(S) = J_f(S) = 1 \), or \( N \leq L_{n-1} \leq L_1 = K_\infty(G_f) \). In the latter case \( N \leq Z(G_f) \cap K_\infty(G_f) \leq Z(G_f) \cap (G_f)' \). Then \( N \leq \phi(G_f) = 1 \) by Theorem I.1.1. A contradiction arises since it is assumed that \( N \neq 1 \). Therefore \( \phi(G_f) \cap J_f(S) \cap L_{n-1} = 1 \). Since \( \phi(G_f) = 1 \), \( F(G_f) = S(G_f) \), for which \( S(G_f) \) denotes the socle of \( G_f \). Since \( L_{n-1} \) is nilpotent, then \( L_{n-1} \leq S(G_f) \). By Lemma III.2.1 \( J_f(S) \) avoids \( L_{n-1} \). By Lemma 2.8, \( G = [L_{n-1} \cap N_G(S_{n-2}) \cap N_G(S_{n-2})] \). Since \( S \) reduces into \( N_G(S_{n-2}) \) and \( J_f(S) \) avoids \( L_{n-1} \), \( J_f(S) \) is the \( f \)-subgroup of \( N_G(S_{n-2}) \) associated with the reduction of \( S \) into \( N_G(S_{n-2}) \) by Theorem II.4.11. Since \( \phi_1 = \phi(G) \cap G_f = 1 \), then \( \phi_1 N_G(S_{n-2}) = N_G(S_{n-2}) \). Also \( N_G(S_{n-2}) \leq N_G(S_{n-2}) \). So \( A(S) \leq \phi_1 N_G(S_{n-2}) \leq N_G(S_{n-2}) \). If it is shown that \( A(S) \) is the \( A \)-subgroup of \( N_G(S_{n-2}) \) associated with the reduction of \( S \) into \( N_G(S_{n-2}) \), then \( A(S) = J_f(S) \) by induction.

Let \( \overline{A} \) be the \( A \)-subgroup of \( G/L_{n-1} \) associated with \( S_{n-1}/L_{n-1} \). It will be proven that \( \overline{A} = A(S) L_{n-1}/L_{n-1} \). If \( G_f = L_0 > L_1 > \cdots > L_{n-1} > L_n = 1 \) is the lower nilpotent series of \( G_f \), then \( G_f/L_{n-1} = (L_0/L_{n-1}) > \cdots > (L_{n-2}/L_{n-1}) > (L_{n-1}/L_{n-1}) = L_{n-1} \) is the lower nilpotent series of \( G_f/L_{n-1} \). Let \( \phi_f/L_{n-1} \) and \( \overline{f}/L_{n-1} \) denote the components of the \( f \)-series of \( G_f/L_{n-1} \). Then since \( \phi_1(G) = 1 \), \( \phi_1/L_{n-1} = \phi(G/L_{n-1}) \cap (G_f/L_{n-1}) = \phi(G/L_{n-1} \phi_1(G)) \cap \bigcap (G_f/L_{n-1} \phi_1(G)) \). But this last intersection equals \( \phi(G/L_{n-1}) \cap (G_f/L_{n-1}) = \phi_2/L_{n-1} \)
since \( \overline{L}^* = L_{n-1} \phi_1(G) \). Therefore \( \phi_1/L_{n-1} = \phi_2/L_{n-1} \), that is, \( \phi_1 = \phi_2 \).

Also \( \overline{L}^*/L_{n-1} = \overline{\phi_1} L_{n-2}/L_{n-1} = \phi_2 L_{n-2}/L_{n-1} = \overline{L}_2^*/L_{n-1} \). Hence \( \overline{L}_1^* = L_2^* \).

Assume inductively that \( \overline{\phi}_k = \phi_{k+1} \) and \( \overline{L}_k^* = L_{k+1}^* \). Then \( (\overline{\phi}_{k+1}/L_{n-1})/(\overline{L}_{k+1}/L_{n-1}) = (\overline{G}/L_{n-1})/(\overline{L}_{k+1}/L_{n-1}) \).

\( (\overline{G}/L_{n-1})/(L_{k+1}^*/L_{n-1}) \cap (\overline{G}_f/L_{n-1})/(L_{k+1}^*/L_{n-1}) \) \( \cong \phi(G/L_{k+1}^*) \cap (G_f/L_{k+1}^*) = \phi_k + 2/L_{k+1}^* \cong (\phi_k + 2/L_{n-1})/(L_{k+1}^*/L_{n-1}) = (\phi_k + 2/L_{n-1})/(\overline{L}_k^*/L_{n-1}) \). It follows that \( \overline{\phi}_{k+1} = \phi_{k+2} \). Also \( \overline{L}_k^*/L_{n-1} = \overline{\phi}_{k+1} L_{n-1}/(k+1)/L_{n-1} = \phi_{k+2}/L_{n-1}(k+2)/L_{n-1} \). Hence \( \overline{L}_k^* = L_{k+2}^* \). Therefore by induction \( \overline{\phi}_j = \phi_{j+1} \) for \( j = 0, 1, \ldots, n-2 \).

Let \( N_{G_f}(S_{n-j-1}) = N_{G_f}(S_{n-j-1} L_{n-1}/L_{n-1}) \). By Theorem I.3.14(i) \( N_{G_f}(S_{n-j-1}) = N_{G_f}(S_{n-j-1}) L_{n-1}/L_{n-1} \).

Since \( L_{n-1} \leq \phi_{j+1} \) for \( j = 1, \ldots, n-2 \), then \( \overline{A} = \cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1} = (\cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1}), \) Note that \( \overline{A} = AL_{n-1}/L_{n-1} = (\cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1}) L_{n-1} = (\cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1}) L_{n-1} = \overline{L}_{n-1} \).

Since \( \phi_1 = 1 \) and \( \phi_n \leq \phi_j \) for \( j = 2, \ldots, n-1 \), this last factor group equals \( (\cap_{j=1}^{n-1} \overline{\phi}_j N_{G_f}(S_{n-j-1})/L_{n-1}) \) by the modular identity I.1.2. By

\[ AL_{n-1}/L_{n-1} = (\cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1}) L_{n-1} = (\cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1}) L_{n-1} = \overline{L}_{n-1} \]

Lemma 2.8 \( G_f = N_{G_f}(S_{n-2}) L_{n-1} \). It follows that \( AL_{n-1}/L_{n-1} = (\cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1}) L_{n-1} = (\cap_{j=1}^{n-2} \phi_{j+1} N_{G_f}(S_{n-j-1})/L_{n-1}) L_{n-1} = \overline{L}_{n-1} \). Let \( k = j - 1 \). Then \( AL_{n-1}/L_{n-1} = (\cap_{k=1}^{n-2} \phi_{k+1} N_{G_f}(S_{n-(k+1)})/L_{n-1}) L_{n-1} = (\cap_{k=1}^{n-2} \phi_{k+1} N_{G_f}(S_{n-(k+1)})/L_{n-1}) L_{n-1} = \overline{L}_{n-1} \). From the above, this last intersection is also equal to \( \overline{A} \). Hence \( \overline{A} = AL_{n-1}/L_{n-1} \).
Let θ be the isomorphism given by $G/L_{n-1} = N_G(S_{n-2})L_{n-1}/L_{n-1}^\theta$. Since an $f$-series is a characteristic series and relative system normalizers are preserved under homomorphism, $\overline{A}^\theta$ is the A-subgroup of $(G/L_{n-1})^\theta$. Since $A \leq N_G(S_{n-2})$, then $\overline{A}^\theta = (A L_{n-1}/L_{n-1})^\theta = A(S)$. Moreover, since $(G/L_{n-1})^\theta = N_{G_f}(S_{n-2})$, then $A(S)$ is the A-subgroup of $N_{G_f}(S_{n-2})$ associated with the reduction of $S$ into $N_{G_f}(S_{n-2})$. By induction $A(S) = J_f(S)$.

2.10 Corollary: The $f$-subgroup $J_f(S)$ of a group $G$ is an $(S \cap G_f)$ CAR-subgroup of $G_f$.

Proof: By Theorem 2.9 $A(S) = J_f(S)$. By Theorem 2.6 $A(S)$ is an $(S \cap G_f)$ CAR-subgroup of $G_f$.

2.11 Corollary: The $f$-subgroup $J_f(S)$ of a group $G$ covers a chief factor $H/K$ of $G_f$ if and only if $\phi_j N_{G_f}(S_{n-j-1})$ covers $H/K$ for each $j = 1, ..., n-1$, for which $\phi_j$ and $N_{G_f}(S_{n-j-1})$ are defined as in Definition 2.1.

Proof: By Theorem 2.6 $A(S)$ has the required covering and avoidance property.

If $\kappa$ is the formation of solvable nC-groups, then $\phi(G) \leq G_\kappa$ and for $L_j^\kappa$, $\phi(G/L_j^\kappa) \leq G_\kappa/L_j^\kappa$. Thus in Definition 2.1, $\phi_1 = \phi(G)$ and $\phi_j+1/L_j^\kappa = \phi(G/L_j^\kappa)$ for which $L_j^\kappa = \phi_j L_{n-j}$. Therefore a $\kappa$-series $1 = L_0^\kappa \leq \phi_1 < L_1^\kappa \leq \phi_2 < ... < L_{n-1}^\kappa \leq \phi_n = G_\kappa$ coincides with the prefrattini series defined by H. Bechtell [2]. Furthermore Theorem 2.9 yields as a corollary this next result.
2.12 Corollary: [2] Let \( I(S) = \bigcap_{j=1}^{n-1} \phi_j N_{G_k}(S_{n-j-1}) \), for which \( \phi_j \) and \( N_{G_k}(S_{n-j-1}) \) are defined for the group \( G \) as in Definition 2.1. Then \( I(S) = W(S) \) the prefrattini subgroup of \( G \) associated with \( S \).

### 3. Maximal Subgroups of \( G_f \)

In Corollary 2.10, it is established that an \( f \)-subgroup of a group \( G \) is an \( (S \cap G_f) \) CAR-subgroup of \( G_f \). Certain CAR-subgroups of a group are intersections of maximal subgroups of the group. The primary examples are the prefrattini subgroups and \( g \)-prefrattini subgroups of a group. This raises the question of whether or not an \( f \)-subgroup can be expressed as the intersection of maximal subgroups of the \( f \)-residual.

#### 3.1 Definition: Let \( L(S) = \bigcap \{M|M \text{ is a maximal subgroup of } G_f, \ J_f(S) \leq M\} \).

It is proven in this section that \( L(S) = J_f(S) \).

#### 3.2 Definition: For a group \( G \), a chief factor \( H/K \) of \( G_f \) is \( L \)-complemented if \( H/K \) is complemented by a maximal subgroup \( M \) of \( G_f \) which contains an \( f \)-subgroup of \( G \).

#### 3.3 Lemma: Let \( C \) be a chief series of the group \( G \) containing \( G_f \). Below \( G_f \) refine \( C \) to a chief series \( C_0 \) of \( G_f \). If \( H/K \) is a chief factor from \( C_0 \) then \( H/K \) is \( L \)-complemented if and only if \( J_f(S) \) avoids \( H/K \).
Proof: If $H/K$ is $L$-complemented, then there exists a maximal subgroup $M$ of $G_f$ which complements $H/K$ and contains an $f$-subgroup. Since, by Theorem II.4.7, all $f$-subgroups of $G$ are conjugate in $G_f$, by replacing $M$ by a suitable conjugate of $M$ if necessary, it may be assumed that $J_f(S) \subseteq M$. Since $M$ complements $H/K$, then $M \cap H \subseteq K$. Therefore $J_f(S) \cap H \subseteq K$. So $J_f(S)$ avoids $H/K$.

To prove the converse, induction on the group order is used. First, let $N$ be a minimal normal subgroup of $G$, $N$ an element of the series $C$, and $H/K$ a chief factor of $C_0$ with $N \subseteq K$. Suppose that $J_f(S)$ avoids $H/K$. Then $J_f(S)N/N$ avoids $(H/N)/(K/N)$. By induction, $(H/N)/(K/N)$ is complemented by $M/N$ a maximal subgroup of $G/N$. Furthermore $M/N$ contains an $f$-subgroup of $G/N$. By Theorem II.4.7, all $f$-subgroups are conjugate in $G_f$. Hence there exists $g \in G_f$ such that $(J_f(S)N/N)^g = J_f(S)^gN/N \subseteq M/N$. Consequently, $J_f(S)^g \subseteq M$, that is, $M$ contains an $f$-subgroup of $G$. Thus the factor $H/K$ is an $L$-complemented chief factor. It remains to prove that the result holds whenever $H/K \in C_0$ and $H \subseteq N$. If $N$ is covered by $J_f(S)$, then the proof is complete. Assume that $J_f(S)$ avoids $N$. Then $N \cap \text{core } J_f(S) = 1$. By Theorem III.1.1, $\text{core } J_f(S) = Z_f(G_f)(\phi(G) \cap G_f)$. It follows that $N \cap \phi(G) = 1$ and $N \cap \phi(G_f) = 1$. Because $N$ is an abelian normal subgroup of $G_f$, Theorem I.1.9 implies that $N$ is the direct product of minimal normal subgroups of $G_f$ and $N$ is complemented in $G$. Thus $G = [N]M$ for which $M$ is maximal in $G$. By hypothesis, $J_f(S)$ avoids $N$. By Theorem II.1.8, $N$ is $\kappa$-eccentric. Theorem I.5.5 implies that $M$ is $\kappa$-abnormal. By Definition II.1.2, $M$ contains an $f$-subgroup of $G$. Without loss of generality, let $J_f(S) \subseteq M$. Since $N$ is the product of minimal normal subgroups of $G_f$, $N \subseteq S(G_f)$ the socle of $G_f$. By Theorem I.1.13 there exists $L \lhd G_f$ such
that $S(G_f) = H \times L$. Hence $N = N \cap S(G_f) = N \cap HL$. The modular identity I.1.2 implies that $N = H(N \cap L)$ and $N \cap L < G_f$. It also implies that $G_f = G \cap G_f = NM \cap G_f = N(M \cap G_f)$. Therefore $H(N \cap L)(M \cap G_f) = N(M \cap G_f) = G_f$.

But $H \cap (N \cap L)(M \cap G_f) = H \cap (N \cap (N \cap L)(M \cap G_f))$ which is equal to $H \cap ((N \cap L)(N \cap M \cap G_f))$ by another application of the modular identity.

However $N \cap M \cap G_f = 1$ and $H \cap L = 1$. So $H \cap (N \cap L)(M \cap G_f) = H \cap ((N \cap L)(N \cap M \cap G_f)) = H \cap N \cap L = 1$. Hence $G_f = [H][(N \cap L)(M \cap G_f)]$ and $G_f/K = [H/K][(N \cap L)(M \cap G_f)K]/K$. The group $(N \cap L)(M \cap G_f)K$ is maximal in $G_f$ and $J_f(S) \leq M \cap G_f \leq (N \cap L)(M \cap G_f)K$. Therefore $H/K$ is an $L$-complemented chief factor of $G_f$. Combining both arguments, if $H/K$ is any chief factor from $C_0$ that is avoided by $J_f(S)$, then $H/K$ is $L$-complemented.

3.4 Theorem: For each $f$-subgroup $J_f(S)$ of a group $G$, $J_f(S) = L(S)$. That is, $J_f(S)$ is the intersection of a well-defined set of maximal subgroups of $G_f$.

Proof: By Definition 3.1, $J_f(S) \leq L(S)$. It will suffice to prove that $|L(S)| = |J_f(S)|$. By Theorem I.2.3, this is equivalent to proving that in a given chief series, $L(S)$ and $J_f(S)$ cover and avoid the same chief factors. If $J_f(S)$ covers a factor, then $L(S)$ also covers that factor since $J_f(S) \leq L(S)$. Thus it is sufficient to prove that every factor avoided by $J_f(S)$ is also avoided by $L(S)$. Let $C$ be a chief series of $G$ containing $G_f$. Below $G_f$ refine $C$ to a chief series $C_0$ of $G_f$. Let $H/K \in C_0$. Suppose that $J_f(S)$ avoids $H/K$. Lemma 3.3 implies that $H/K$ is complemented by a maximal subgroup $M$ of $G_f$ which contains an $f$-subgroup. By replacing $M$ by a conjugate of $M$ if necessary, it may be assumed that
By Definition 3.1, \( L(S) \leq M \). Since \( M \) complements \( H/K \), \( H \cap M = K \). So \( H \cap L(S) \leq K \). Hence if \( J_f(S) \) avoids \( H/K \), then so does \( L(S) \). Therefore \( J_f(S) = L(S) \).

The prefattini subgroup of a group is the intersection of a well-defined set of maximal subgroups of the group. A prefattini subgroup is a \( \kappa \)-subgroup for the formation \( \kappa \) of solvable nC-groups. Is it possible that an \( f \)-subgroup is in general the intersection of a set of maximal subgroups of the group? The answer to this question is no. For if \( f \) is an arbitrary formation then \( G \in f \) if and only if \( J_f(S) = 1 \). If \( J_f(S) \) is an intersection of a well-defined set of maximal subgroups of the group \( G \), then \( J_f(S) = 1 \) implies that \( \phi(G) = 1 \). Consequently, any formation \( f \) would only have elements with trivial frattini subgroups. This is clearly false. A simple counter example is obtained by letting \( f = N \) the formation of nilpotent groups. Let \( G = \langle a | a^4 = 1 \rangle \), then \( \phi(G) = \{1, a^2\} \) and \( G \in N \).

The solvable nC-groups \( \kappa \) and the \( \kappa \)-subgroups, that is, the prefattini subgroups, have a special role. It is proven in the following theorem that if an \( f \)-subgroup is the intersection of a set of maximal subgroups for every group, then the formation \( f \) is a subformation of \( \kappa \).

3.5 Theorem: Let \( f \) be a formation such that for every group \( G \) an \( f \)-subgroup of \( G \) has the form \( J_f(S) = \cap \{ M | M \in m(G) \} \), for which \( m(G) \) is a subset of the set of maximal subgroups of \( G \). Then the formation \( f \) is a subformation of the solvable nC-groups \( \kappa \).

Proof: Let \( G \in f \) and \( 1 = K_0 < K_1 < \ldots < K_n = G \) be a Fitting chain of
the group $G$, that is, $K_i = F(G/K_{i-1})$ for $i = 1, 2, \ldots, n$. Then $G/K_i \in f$ for each $i = 1, 2, \ldots, n$. Thus the $f$-subgroups of $G/K_i$ are equal to the identity subgroup by Theorem II.2.1. Since an $f$-subgroup of $G/K_i$ is the intersection of maximal subgroups from $m(G/K_i)$, then $\phi(G/K_i) = 1$ for $i = 1, 2, \ldots, n$. By Corollary III.2.5 $G \in k$. Therefore $f$ is a subformation of $k$.

4. The Prefrattini Subgroup and System

Normalizer of the $f$-residual

Let $S$ be a Sylow system of the group $G$ and $k$ the formation of solvable $NC$-groups. Let $W(S)$, $W^*(S)$, and $D^*(S)$ denote respectively the prefrattini subgroup of $G$ associated with $S$, and the prefrattini subgroup of $G_k$ and system normalizer of $G_k$ associated with the reduction of $S$ into $G_k$. In many easily constructed examples $W(S) = W^*(S)D^*(S)$. That is, a prefrattini subgroup often appears as the product of a prefrattini subgroup and a system normalizer of the $k$-residual. In fact whenever $G_k$ is nilpotent $G_k = W(S)$ by Corollary III.1.4. In nilpotent groups each Sylow subgroup is normal in the group and so a system normalizer coincides with the group itself. Therefore if $G_k$ is nilpotent $G_k = D^*(S) = W(S)$.

A natural conjecture is that $W(S) = W^*(S)D^*(S)$. In this section a counter-example to this conjecture is given. Conditions which imply that $W(S) = W^*(S)D^*(S)$ and $J_f(S) = W^*(S)D^*(S)$ are then explored.

4.1 Example: Let $\alpha' = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and $\beta' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ for which $\alpha'$, $\beta' \in \text{SL}(2, 3)$. The quaternion group $Q_8$ of order 8 is isomorphic to
Set $\gamma' = \alpha' \beta'$. Then $\gamma' = (1 2 2)$. Consider $H = \text{SL}(2^2,3)$. Let

$$
\alpha = \begin{pmatrix}
0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 2 & 0
\end{pmatrix}
, \quad
\beta = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
, \quad
\gamma = \begin{pmatrix}
1 & 2 & 0 & 0 \\
2 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 2 & 2
\end{pmatrix}.$$

The elements $\alpha, \beta$, and $\gamma$ behave exactly as $\alpha', \beta'$, and $\gamma'$. Therefore $Q_\beta \cong <\alpha, \beta> = Q$.

Set $\xi = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix}$. Then $\xi^3 = 1$ and $\xi \in N_H(Q)$. Therefore $L = [Q]_{<\xi>}$ exists as a subgroup of $\text{SL}(2^2,3) = H$. Let $N = V_4(Z_3)$, that is, $N = Z_3 + Z_3 + Z_3 + Z_3$. Form the group $G = [N]L$. Consider the subspace $M$ of $N$ generated by the basis vectors $e_1 = (1,0,0,0)$ and $e_2 = (0,1,0,0)$. Then $M$ is a normal subgroup of $G$. Since $e_1^\alpha = e_2$ and $e_2^\alpha = 2e_1$, it is evident that $M$ is irreducible under $L$. Furthermore $M = \phi(G)$.

Let $S$ be the Sylow system of $G$ having Sylow 2-subgroup $Q$ and Sylow 3-subgroup $[N]_{<\xi>}$. With respect to this Sylow system, $W(S) = [M]_{<\alpha^2>}$. The $k$-residual is $G_k = [N]_{<\alpha^2>} \leq k$. Thus by Theorem 1.6.6, $W*(S) = 1$.

The system normalizer of $G_k$, associated with $S \cap G_k$ is $<\alpha^2>$. Therefore $W(S) = [M]_{<\alpha^2>} \neq <\alpha^2> = W*(S)D*(S)$.

Since a $k$-subgroup coincides with a prefarratti subgroup for $k$ the solvable nC-groups, the above example also illustrates that in general $J_f(S) \neq W*(S)D*(S)$, for which $W*(S)$ and $D*(S)$ are respectively the prefarratti subgroup and system normalizer of $G_f$, associated with the reduction of $S$ into $G_f$.

Under certain conditions $J_f(S) = W*(S)D*(S)$. A governing factor is the relationship of the cores of $W*(S)$ and $D*(S)$ to the core of $J_f(S)$. 
4.2 Theorem: Let $f$ be an arbitrary formation and $S$ a Sylow system of the group $G$. Denote the prefrattini subgroup and the system normalizer of $G_f$ associated with the reduction of $S$ into $G_f$ respectively by $W^*(S)$ and $D^*(S)$. The following conditions are equivalent:

(i) $J_f(S) = W^*(S)D^*(S)$.

(ii) $\text{core}(J_f(S)N/N) = \phi(G_fN/N)Z\langle G_fN/N \rangle$ for every normal subgroup $N$ of $G$.

(iii) $(\phi(G/N) \cap (G_fN/N))Z\langle G_fN/N \rangle = \phi(G_fN/N)Z\langle G_fN/N \rangle$ for every normal subgroup $N$ of $G$.

(iv) $\phi(G/N) \cap K\langle G_f/N \rangle = \phi(G_f/N) \cap K\langle G_f/N \rangle$ for every normal subgroup $N$ of $G$ for which $N \leq K\langle G_f/N \rangle$.

Proof: Suppose that $J_f(S) = W^*(S)D^*(S)$. By Corollary II.4.8 Theorem I.6.3(i) and Theorem I.3.7, $J_f(S)$, $W^*(S)$ and $D^*(S)$ are preserved under homomorphism. Hence if $N$ is normal in $G$, $J_f(S)N/N = (W^*(S)N/N) \cdot (D^*(S)N/N)$. To obtain (ii), it suffices to prove that $\text{core}((W^*(S)N/N)(D^*(S)N/N)) = \phi(G_fN/N)Z\langle G_fN/N \rangle$. Since $W^*(S)$ and $D^*(S)$ are preserved under homomorphism, it will be assumed that $N = 1$. The core of $W^*(S)$ is $\phi(G_f)$ by Theorem I.6.4. The core of $D^*(S)$ is $Z\langle G_f \rangle$ by Theorem I.3.10. So $\phi(G_f)Z\langle G_f \rangle \leq \text{core}(W^*(S)D^*(S))$. The opposite inclusion will be obtained by using induction on the group order.

Case 1: $Z\langle G_f \rangle \cap \phi(G_f) = 1$.

Let $x \in Z\langle G_f \rangle$ and $g \in G_f$. Then $[x,g] \in G_f \cap Z\langle G_f \rangle \leq \phi(G_f)$ by Theorem I.1.1. Hence $[x,g] \leq Z\langle G_f \rangle \cap \phi(G_f) = 1$. Therefore $x \in Z(G_f)$. It follows that $Z\langle G_f \rangle = Z(G_f)$. By Theorem I.1.9, $G_f = [Z\langle G_f \rangle]A$ for an appropriate subgroup $A$. Moreover $G_f = Z\langle G_f \rangle \times A$ since $Z\langle G_f \rangle = Z(G_f)$. By
Theorem I.6.3(i), $W^A(s)Z_\infty(G_f)/Z_\infty(G_f)$ is a prefrattini subgroup of $G_f/Z_\infty(G_f)$, for which $W^A(s)$ is the prefrattini subgroup associated with the reduction of $s$ into $A$. Similarly by Theorem I.3.7, $D^A(s)Z_\infty(G_f)/Z_\infty(G_f)$ is a system normalizer of $G_f/Z_\infty(G_f)$, for which $D^A(s)$ is the system normalizer of the reduction of $s$ into $A$. Therefore $(W^A(s)D^A(s)Z_\infty(G_f))/Z_\infty(G_f) = (W^*(s)D^*(s))/Z_\infty(G_f)$. Inductively core$(W^A(s)D^A(s)) = Z(A)\phi(A)$. Clearly $Z(A) = Z_\infty(A) = 1$. So core$(W^A(s)D^A(s)) = \phi(A)$. It follows that core$(W^*(s)D^*(s)) = \phi(A)Z_\infty(G_f)$. But $A < G_f$ so $\phi(A) \neq \phi(G_f)$. Consequently, core$(W^*(s)D^*(s)) < \phi(G_f)Z_\infty(G_f)$.

Case 2: $N = Z_\infty(G_f) \cap \phi(G_f) \neq 1$.

From above $W^*(s)D^*(s)/N = W^*(sN/N)D^*(sN/N)$. Inductively core$(W^*(s)D^*(s)/N) = Z_\infty(G_f/N)\phi(G_f/N)$. Note that $\phi(G_f/N) = \phi(G_f)/N$ and $Z_\infty(G_f/N) = Z_\infty(G_f)/N$. Therefore core$(W^*(s)D^*(s)/N) = (Z_\infty(G_f)/N)(\phi(G_f)/N)$.

So core$(W^*(s)D^*(s)) = Z_\infty(G_f)\phi(G_f)$.

Consequently, (i) implies (ii).

By Theorem III.1.1, core$(J_f(s)/N) = (\phi(G/N) \cap G_fN/N)(Z_\infty(G_fN/N))$. Hence (ii) implies (iii).

Assume that (iii) is valid. Then $(\phi(G) \cap G_f)Z_\infty(G_f) = \phi(G_f)Z_\infty(G_f)$.

Since $G_f < G$, $\phi(G_f) \leq \phi(G) \cap G_f$. Hence $\phi(G_f) \cap K_\infty(G_f) \leq \phi(G) \cap K_\infty(G_f)$. Let $H \leq \phi(G) \cap K_\infty(G_f)$ such that $H/(\phi(G_f) \cap K_\infty(G_f)) = H/K$ is a chief factor of $G_f$. If $H$ does not exist, then $\phi(G_f) \cap K_\infty(G_f) = \phi(G) \cap K_\infty(G_f)$. If $H$ exists, then $H \leq \phi(G_f)Z_\infty(G_f)$ since $H \leq \phi(G) \cap G_f$. Theorem I.8.8 implies that $D^*(s)$ is a perspective CAR-subgroup of $G_f$. By Theorem I.8.7, either $D^*(s)$ covers $H/K$ or $W^*(s)$ covers $H/K$. Therefore $H/K$ is either frattini or central in $G_f$. Since $H/K$ cannot be frattini, $H/K \leq Z(G_f/K)$. It follows that,

$H/K \leq Z(G_f/K) \cap K_\infty(G_f/K) \leq Z(G_f/K) \cap (G_f/K)' \leq \phi(G_f/K)$ by Theorem I.1.1.

This contradicts the choice of $H$ since, then $H/K \leq \phi(G_f/K) \cap K_\infty(G_f/K) = \phi(G_f/K)Z_\infty(G_f/K)$.
(ϕ(G,G∩G)/K) = K. Consequently, H does not exist. Therefore
(ϕ(G)∩K∞(G))/K = K. By applying the same argument to G/N
with N ∩ G and N ≤ K∞(G), ϕ(G/N)∩K∞(G)/N) = ϕ(G/N)∩K∞(G/N). So
(iii) implies (iv).

Assume that (iv) is valid. By Theorem I.6.10, W*(S) ≤ W(S)∩Gf.
Theorem II.2.2 implies that J*(S) = D*(S)(W(S)∩Gf). Therefore W*(S).
D*(S) ≤ J*(S). By Theorem I.2.3, it is sufficient to prove that J*(S)
and W*(S)D*(S) cover and avoid the same chief factors in a given series
C, for then |J*(S)| = |W*(S)D*(S)|. Every factor avoided by J*(S)
is also avoided by W*(S)D*(S) since W*(S)D*(S) ≤ J*(S). Hence it will
suffice to prove that in a given chief series every factor covered by
J*(S) is also covered by W*(S)D*(S). Let C be a chief series of G that
contains Gf and K∞(Gf). Every factor above Gf is avoided by J*(S) since
J*(S) ≤ Gf. Let H/K be a factor from C between Gf and K∞(Gf). Then H/K
is central in Gf. It follows that H/K ≤ Z(Gf/K) ≤ Z(Gf/K). Moreover,
since Z(Gf/K) ≤ D*(S)K/K, H/K ≤ D*(S)K/K. This implies that H ≤ D*(S)K,
that is, D*(S) covers H/K. Since D*(S) ≤ J*(S), then J*(S) covers H/K.
Let H/K be a factor from C such that H ≤ K∞(Gf). If H/K is covered by
J*(S), then H/K is either frattini or h-central by Theorem II.1.8. If
H/K is h-central, then H/K ≤ Z(Gf/K)∩K∞(Gf/K) ≤ Z(Gf/K)∩(Gf/K).
Theorem I.1.1 implies that H/K ≤ ϕ(G/K)∩K∞(Gf/K). If H/K is a frattini
chief factor then the same inclusion is valid. Since (iv) is valid, then
H/K ≤ ϕ(Gf/K)∩K∞(Gf/K) ≤ ϕ(Gf/K). Consequently, H/K ≤ W*(S)K/K. Equiv-
ally, W*(S) covers H/K. So every factor from C covered by J*(S) is
Therefore (iv) implies (i).
To return to the question of when does $W(s) = W^*(s) D^*(s)$, Theorem 4.2 yields the following corollary.

4.3 Corollary: Let $k$ be the formation of solvable nC-groups and $S$ a Sylow system of the group $G$. If $W^*(s)$ and $D^*(s)$ are respectively the prefattini subgroup, and system normalizer associated with the reduction of $S$ into $G_k$, then the following statements are equivalent:

(i) $W(s) = W^*(s) D^*(s)$.

(ii) $\text{core}(W(s) N / N) = \phi(G_k N / N) Z_{\infty}^k (G_k N / N)$ for every normal subgroup $N$ of $G$.

(iii) $\phi(G/N) = \phi(G_k N / N) Z_{\infty}^k (G_k N / N)$ for every normal subgroup $N$ of $G$.

(iv) $\phi(G/N) \cap K_{\infty}^k (G_k N / N) = \phi(G_k / N) \cap K_{\infty}^k (G_k N / N)$ for every normal subgroup $N$ of $G$ such that $N \leq K_{\infty}^k (G_k)$.

Proof: Apply Theorem 4.2 with $f = k$. By Theorem II.3.1 $J_k(s) = W(s)$. By Theorem I.6.4 core $W(s) = \phi(G)$. 
CHAPTER V

\(f\)-SUBGROUPS AND FORMATIONS

The \(f\)-subgroups are related to the formation \(f\) by definition. In this chapter, this relationship is examined in two directions for several specific types of formations. The first direction is to identify the effect of \(f\) on the structure of an \(f\)-subgroup. The second is to determine the extent to which the structure of the \(f\)-subgroups characterizes the formation \(f\).

In the first section, formations which are linked to CAR-subgroups by definition are examined. The \(f\)-subgroups are related to these CAR-subgroups. Normal formations are studied next, that is, formations in which each normal subgroup of a group in the formation is also in the formation. Necessary and sufficient conditions are obtained which imply that the \(f\)-subgroup of a normal subgroup is contained in the \(f\)-subgroup of the group. In the final section it is established that the structure of \(f\)-subgroups determine whether or not a formation is saturated. A dual to saturated formations is introduced. It is proven that the dual nature of these formations is reflected by the structure of the \(f\)-subgroups.

1. Formations Defined by CAR-subgroups

By Theorem II.2.3, the CAR-subgroup \(J_f(G)\) of a group \(G\) generates the residual of the formation \(f\). Are the \(f\)-subgroups the unique CAR-subgroups with this property or do other CAR-subgroups also generate
the $f$-residual? Other examples of CAR-subgroups of $G$ that generate $G_f$ are easily obtained. For example, the system normalizers of the $f$-residual are CAR-subgroups of the group $G$ and they generate $G_f$. However, according to Theorem II.4.1, an $f$-subgroup is the unique subgroup $X$ of $G_f$ satisfying the properties:

(i) $X$ avoids complemented $\kappa$-eccentric chief factors of $G$ and $X$ covers all other chief factors of $G$ below $G_f$.

(ii) $(S^P \cap G_f)X = X(S^P \cap G_f)$ for all $S^P \in S$.

If $X$ is any SCAR-subgroup of $G$ with $X \leq G_f$, then it satisfies property (ii). Therefore $J_f(S)$ is the unique SCAR-subgroup of $G$ contained in $G_f$ that satisfies property (i).

A question remains. If a formation $f$ is such that the residual is generated by a CAR-subgroup, what is the relationship between an $f$-subgroup and that CAR-subgroup. In examining this question, a function which sends a group onto a set of CAR-subgroups is required.

1.1 Definition: Let $f$ be a function on the class of all finite solvable groups such that $f$ maps a group $G$ onto a set of CAR-subgroups of $G$, denoted by $f(G) = \{ f(G, S) | f(G, S) \text{ is an } \text{S} \text{CAR-subgroup of } G \text{ for all Sylow systems } S \text{ of } G \}$. If for each epimorphism $\theta$ of $G$, $f(G, S)^\theta = f(G^\theta, S^\theta)$ for which $S^\theta = \{ (S^P)^\theta | S^P \in S \}$, then the function $f$ is called a CAR-function.

Inherent in the notation is that the set $f(G)$ contains precisely one SCAR-subgroup of the group $G$ for each Sylow system $S$ of $G$. However it is possible that $f(G, S_1) = f(G, S_2)$ for distinct Sylow systems $S_1$ and $S_2$. 
of G. The set $f(G)$ is a conjugate class of subgroups of G. For if $S_1$ and $S_2$ are Sylow systems of G, then there exists an inner automorphism $\theta$ of G such that $S_1^\theta = S_2$. Hence $f(G,S_1) = f(G^\theta,S_1^\theta) = f(G,S_2)$. Conversely, for an inner automorphism $\theta$ of G, $f(G,S) = f(G^\theta,S^\theta) = f(G,S^\theta) \in f(G)$. A CAR-function defines a formation.

1.2 Theorem: The class $F$ of groups for which $f(G,S) = 1$ for any Sylow system $S$ of G is a formation. The $F$-residual is $G_F = \langle f(G,S) \rangle$.

Proof: Note that since the set $f(G)$ is a conjugate class, $f(G,S) = 1$ for one Sylow system $S$ of G implies that $f(G,T) = 1$ for all Sylow systems $T$ of G. Consequently, $\langle f(G,S) \rangle = f(G)$. So $\langle f(G,S) \rangle$ depends only on $f$ and not on the Sylow system $S$.

First it is proven that $F$ is a formation. Suppose that $G \in F$. Then $f(G,S) = 1$. If $\theta$ is an epimorphism of G, then $f(G^\theta,S^\theta) = f(G,S)^\theta = 1$. Therefore $G^\theta \in F$. Suppose that N and M are normal subgroups of G with $N \cap M = 1$, and $G/N$, $G/M \in F$. By definition $f(G/N,S/N) = N$ and $f(G/M,S/M) = M$. If $\theta$ is the natural mapping which maps G onto $G/N$, then $f(G,S)^\theta = f(G,S)N/N$. It follows that $f(G,S)N/N = f(G^\theta,S^\theta) = f(G/N,S/N)$. Similarly, $f(G,S)M/M = f(G/M,S/M)$. Hence $f(G,S) \leq N$ and $f(G,S) \leq M$. Thus $f(G,S) \leq N \cap M = 1$. Therefore $G \in F$. The class $F$ satisfies the conditions of Definition 1.4.1. So $F$ is a formation.

It is now necessary to show that $G_F = \langle f(G,S) \rangle = f(G)$. Suppose that $G_F = 1$. Then $G \in F$. By definition $f(G,S) = 1$. Hence $\langle f(G,S) \rangle = 1$. If $G_F \neq 1$, then $G/G_F \in F$. Since $f(G,S)$ is preserved under homomorphism, $f(G,S)G_F/G_F = f(G/G_F,S_G/G_F) = G_F$. Therefore $f(G,S) \leq G_F$ and so $\langle f(G,S) \rangle \leq $
On the other hand in the group $G/\langle\langle f(G, s)\rangle\rangle$, $f(G/\langle\langle f(G, s)\rangle\rangle, s\langle\langle f(G, s)\rangle\rangle/\langle\langle f(G, s)\rangle\rangle) = f(G, s)\langle\langle f(G, s)\rangle\rangle/\langle\langle f(G, s)\rangle\rangle = \langle\langle f(G, s)\rangle\rangle$. It follows that $G/\langle\langle f(G, s)\rangle\rangle \in F$. So $G_F = \langle\langle f(G, s)\rangle\rangle$. Consequently, $G_F = \langle\langle f(G, s)\rangle\rangle = \langle f(G)\rangle$.

The CAR-function in Theorem 1.2 is said to induce the formation $F$. The formation $F$ defined in this manner is not necessarily saturated. For if $f$ is the function which maps a group $G$ onto its set of prefrattini subgroups, the formation $F$ is the formation of solvable nC-groups by Theorem I.6.6. This formation is not saturated. In fact every formation may be induced by a CAR-function. If $f$ is an arbitrary formation, then the function which maps a group $G$ onto its set of $f$-subgroups is a CAR-function. Applying Theorem II.2.1, it is seen that this function induces the formation $f$. Two additional examples of CAR-functions that induce an arbitrary formation $f$ are easily obtained. First the function $f$ which maps a group $G$ onto the set of system normalizers of the $f$-residual is a CAR-function. By Theorem I.3.10(i), these system normalizers generate the $f$-residual. Also the function defined by $f(G) = \{ G_f \}$ is a CAR-function which clearly induces the formation $f$.

In order to relate $F$-subgroups to CAR-functions that induce $F$, attention will be restricted to CAR-functions that satisfy an additional property.

1.3 Definition: Let $f$ be a CAR-function that induces the formation $F$ and $h$ the formation locally defined by $h(p) = F$ for all prime integers $p$. The function $f$ is called a consistent CAR-function if
f(G, S) = f(M, S \cap M) whenever M is an \( h \)-abnormal maximal subgroup into which S reduces.

1.4 Theorem: Let \( f \) be a consistent CAR-function that induces the formation \( F \). For each Sylow system \( S \) of the group \( G \), \( f(G, S) \leq J_F(S) \).

Proof: Let \( M \) be an \( h \)-abnormal maximal subgroup such that \( S \) reduces into \( M \), forming the Sylow system \( S \cap M \) of \( M \). By hypothesis, \( f(M, S \cap M) = f(G, S) \). Thus \( f(G, S) \leq M \). By Definition 1.7.1, \( W^h(S) \) is the intersection of all the \( h \)-abnormal maximal subgroups of \( G \) into which \( S \) reduces. It follows that \( f(G, S) \leq W^h(S) \). Since \( \langle \langle f(G, S) \rangle \rangle = G_F \) by Theorem 1.2, then \( f(G, S) \leq G_F \). Hence \( f(G, S) \leq W^h(S) \cap G_F \). By Theorem II.1.6 this intersection is \( J_F(S) \). Therefore \( f(G, S) \leq J_F(S) \).

The \( F \)-subgroups, for which \( F \) is induced by the CAR-function \( f \), can be characterized as resulting from consistent CAR-functions that satisfy one further property. This property reemphasizes the importance of the core of an \( f \)-subgroup.

1.5 Theorem: Let \( f \) be a consistent CAR-function and \( F \) the formation induced by \( f \). If \( \text{core } J_F(S) \leq \text{core } f(G, S) \) for every group \( G \), then \( J_F(S) = f(G, S) \) for every group \( G \).

Proof: By Theorem 1.4, \( f(G, S) \leq J_F(S) \). Therefore proving that these subgroups are equal can be accomplished by demonstrating that they have the same order. Theorem I.2.3 implies that it need only be
established that \( f(G,\mathcal{S}) \) and \( J_{\mathcal{F}}(\mathcal{S}) \) cover and avoid the same chief factors. If \( J_{\mathcal{F}}(\mathcal{S}) \) avoids a factor then \( f(G,\mathcal{S}) \) also avoids that factor since \( f(G,\mathcal{S}) \leq J_{\mathcal{F}}(\mathcal{S}) \). Suppose that the chief factor \( H/K \) of \( G \) is covered by \( J_{\mathcal{F}}(\mathcal{S}) \). Then \( H/K \leq \text{core}(J_{\mathcal{F}}(\mathcal{S})K/K) = \text{core}(J_{\mathcal{F}}G/K)(SK/K) \) since \( J_{\mathcal{F}}(\mathcal{S}) \) is invariant under homomorphism by Corollary 11.4.8. By hypothesis, \( \text{core}(J_{\mathcal{F}}G/K)(SK/K) \leq \text{core}(f(G/K,SK/K)). \) Since \( f(G,\mathcal{S}) \) is preserved under homomorphism by Definition 1.1, \( \text{core}(f(G/K,SK/K)) = \text{core}(f(G,\mathcal{S})K/K). \) It follows that \( H/K \leq \text{core}(f(G,\mathcal{S})K/K) \leq f(G,\mathcal{S})K/K. \) Equivalently, \( f(G,\mathcal{S}) \) covers \( H/K. \) Therefore \( J_{\mathcal{F}}(\mathcal{S}) \) and \( f(G,\mathcal{S}) \) cover and avoid the same chief factors.

2. Normal Formations

H. Bechtell [1] has proven that the formation of solvable nC-groups is a normal formation, that is, if \( G \) is a solvable nC-group and \( N < G \), then \( N \) is a solvable nC-group. This result can be obtained by proving that if \( N \) is a normal subgroup of a group \( G \), then a prefattini subgroup of \( N \) is contained in a prefattini subgroup of \( G \). So, for at least one formation \( \mathcal{F} \), if \( N < G \) then \( N_{\mathcal{F}} \leq G_{\mathcal{F}} \) if and only if \( J_{\mathcal{F}}^{N}(\mathcal{S}) \leq J_{\mathcal{F}}(\mathcal{S}) \), for which \( J_{\mathcal{F}}^{N}(\mathcal{S}) \) is the \( \mathcal{F} \)-subgroup of \( N \) associated with the reduction of \( \mathcal{S} \) into \( N \). This property does not hold in general. There exists a normal formation \( \mathcal{F} \), a group \( G \), and a normal subgroup \( N \) of \( G \) for which \( J_{\mathcal{F}}^{N}(\mathcal{S}) \nsubseteq J_{\mathcal{F}}(\mathcal{S}) \).

2.1 Example: Let \( G \) be the symmetric group of degree four and \( N \) the formation of nilpotent groups. Then the alternating group \( A_4 = K_\infty(G) = G_N \). If \( G_{\frac{N^2}{N}} \) denotes the \( \frac{N^2}{N} \)-residual of \( G_N \), then the Klein 4-group
$K_4 = K^2(G) = G_{2^2}$. For any Sylow system $S$, $J_N(S)$ is a cyclic group of order three whereas $J_N^A(S) = K_4$. Hence $N$ is a normal formation with $N = A_4 < G$ and $J_N^N(S) \neq J_N(S)$.

Since $N$ is a saturated formation and a subgroup inherited formation, this example also illustrates that $J_f^N(S) \leq J_f(S)$ for all $N < G$ is not a necessary condition for the normality of the formation $f$ even if $f$ is either a saturated or a subgroup inherited formation. For $f$ a normal formation necessary and sufficient conditions will be determined which imply that $J_f^N(S) \leq J_f(S)$ for each normal subgroup $N$ of $G$.

It is necessary to recall Definition 1.5 of Chapter III. A maximal subgroup of the group $G$ is $\ast$-normal if $M$ is $\ast$-normal and $G \not\triangleleft M$. Otherwise $M$ is $\ast$-abnormal.

2.2 Theorem: Let $K$ be a normal subgroup of a group $G$ and $f$ a normal formation, then core $J_f^K(S) \leq$ core $J_f(S)$ if the following condition is satisfied: If $M$ is an $\ast$-abnormal maximal subgroup of $G$ and $M \cap K \leq L$ for a maximal subgroup $L$ of $K$, then $L$ is $\ast$-abnormal.

Proof: Suppose that core $J^K(S) \ntriangleleft$ core $J(S)$. It will suffice to find a maximal subgroup $M$ of $G$ which does not satisfy the condition. If $G \in f$, then $K \in f$. By Theorem II.2.1 $J^K(S) = J(S) = 1$. Hence in this case core $J^K(S) \leq$ core $J(S)$. This contradicts the original assumption. Consequently, $G \ntriangleleft f$. Since core $J^K(S) \ntriangleleft$ core $J(S)$ and core $J^K(S) \leq G_f$, there exists an $\ast$-abnormal maximal subgroup $M$ of $G$ such that core $J^K(S) \ntriangleleft M$. Otherwise core $J^K(S)$ is contained in each $\ast$-abnormal maximal
subgroup and Definition II.1.2 would imply that core $J^K(S) \leq core J(S)$. It follows that $G = NM$ for $N = core J^K(S)$. By the modular identity I.1.2, $K = K \cap NM = N(K \cap M)$. By Theorem III.1.10, either $<K \cap M> = K \cap M = K$ or $K \cap M \leq L$ for $L$ an $h^*$-normal maximal subgroup of $K$. If $K \cap M = K$, then $K \leq M$. So core $J^K(S) \leq M$. This is a contradiction. Therefore $K \cap M \leq L$. Hence $M$ is a maximal subgroup that does not satisfy the condition.

One necessary and sufficient condition for $J^K(S) \leq J(S)$ is based on Lemma IV.1.1.

2.3 Corollary: Let $f$ be a normal formation and $K$ a normal subgroup of $G$. Then $J^K(S) \leq J(S)$ if and only if core $J^{K/N}(S) \leq core J^{G/N}(S)$ for each normal subgroup $N$ of $G$ such that $N \leq G_f$.

Proof: Since $f$ is a normal formation $K_f \leq G_f$. Therefore Lemma IV.1.1 applies.

Corollary 2.3 and Theorem 2.2 are used to find a second necessary and sufficient condition.

2.4 Theorem: For a normal formation $f$ and a normal subgroup $K$ of the group $G$, the following statements are equivalent:

(i) $J^K(S) \leq J_f(S)$.

(ii) If an $h^*$-abnormal maximal subgroup $M$ of a group $G$ is such that $M \cap K \leq L$ for a maximal subgroup $L$ of $G$, then $L$ is $h^*$-abnormal.
Proof: Suppose that (i) is valid. Then $J^K_f(s) \leq J_f(s)$. Let $M$ be an $h^*$-abnormal maximal subgroup of $G$. By Definition III.1.5, either $M$ is $h$-abnormal or $G_f \leq M$. Each case implies that $M$ has a conjugate $M^g$ such that $J_f(s) \leq M^g$. Consequently, $J_f(s)^{g^{-1}} \leq M$. Moreover, $(J^K_f(s))^{g^{-1}} \leq J_f(s)^{g^{-1}} \cap K \leq M \cap K$. Suppose that $M \cap K \leq L$ for $L$ maximal in $K$. Assume that $L$ is $h^*$-normal. Then $K_f \perp L$ and so $L$ complements a chief factor below $K_f$. Since $L$ is $h$-normal, the factor is $h$-central. Since $(J^K_f(s))^{g^{-1}} \leq L$, it avoids this factor. This is a contradiction since an $f$-subgroup of $K$ covers $h$-central factors below $K_f$ by Theorem II.1.8. Therefore $L$ is $h^*$-abnormal. Hence (ii) is valid.

Suppose that (ii) is valid. Let $N$ be a normal subgroup of $G$ and $N \leq K_f$. Let $M/N$ be an $h^*$-abnormal maximal subgroup of $G$ such that $(M/N) \cap (K/N) \leq L/N$ for $L/N$ a maximal subgroup of $K/N$. Since $M/N$ is $h^*$-abnormal either $M/N$ is $h$-abnormal or $G_fN/N \leq M/N$ by Definition III.1.5. If $M/N$ is $h$-abnormal, then $M$ is $h$-abnormal. If $G_fN/N \leq M/N$, then $G_f \leq M$. Therefore $M$ is an $h^*$-abnormal maximal subgroup of $G$. Since $M \cap K \leq L$ and $L$ is maximal in $K$, then $L$ is $h^*$-abnormal. It follows that $L/N$ is $h^*$-abnormal. By Theorem 2.2, $J^K_{f-N}(s) \leq \text{core}^{G-N}(s)$. Corollary 2.3 now yields $J^K_f(s) \leq J_f(s)$.

2.5 Corollary: [2] Let $k$ be the formation of solvable nC-groups and $K$ a normal subgroup of the group $G$. If $M$ is an $h^*$-abnormal maximal subgroup of $G$ and $M \cap G_K \leq L$ for $L$ a maximal subgroup of $K$, then $L$ is $h^*$-abnormal in $K$. Consequently, $W^K(s) \leq W(s)$ for which $W^K(s)$ is the prefrattini subgroup of $K$ associated with the reduction of $S$ into $K$.

Proof: If $M$ is $h^*$-abnormal in $K$, Definition III.1.5 states that either $G_K \leq M$ or $M$ is $h$-abnormal in $G$. If $G_K \leq M$, then $K_K \leq G_K \leq M$ since
\( K \) is a normal formation. Hence \( K \preceq M \cap G_K \preceq L \). If \( K \nmid L \), then \( L \) complements a chief factor of \( K \) below \( K \). By Theorem I.6.2, this factor is avoided by \( W^K(S) \). By Theorem I.6.9, \( W^K(S) \) contains a system normalizer of \( G_K \). Therefore the chief factor is avoided by a system normalizer of \( G_K \). Theorem I.3.9 implies that this chief factor is \( h \)-eccentric. It follows that \( L \) is \( h \)-abnormal by Theorem I.5.5. Both cases imply \( L \) is \( h^* \)-abnormal in \( K \). By Theorem 2.4, \( J^K_K(S) \preceq J^K_K(S) \). Since \( J^K_K(S) = W^K(S) \) and \( J^K_K(S) = W(S) \), then \( W^K(S) \preceq W(S) \).

3. \( f \)-subgroups and Saturated Formations

By Definition I.4.3, a formation \( f \) is saturated if for every group \( G \), \( G/\phi(G) \in f \) implies that \( G \in f \). Saturated formations may be characterized through the structure of the \( f \)-subgroups.

3.1 Theorem: A formation \( f \) of solvable groups is saturated if and only if for every group \( G \) for which \( G/\phi(G) \notin f \), \( J_f(S) \neq W(S) \cap G_f \).

Proof: Let \( f \) be a saturated formation and \( G \) a group with \( G_f \neq 1 \). Consider a chief factor \( G_f/K \) of \( G \). If \( G_f/K \leq \phi(G/K) \), then \( (G/K)/\phi(G/K) \in f \). Since \( f \) is saturated this implies that \( G/K \in f \). Hence \( G_f \leq K \). This is a contradiction. So \( G_f/K \cap \phi(G/K) = 1 \). It follows that \( G_f/K \) is a complemented chief factor. Theorem I.6.2 implies that \( W(S) \cap G_f \) avoids \( G_f/K \). The factor \( G_f/K \) is abelian and so centralized by \( G_f \), that is, \( G_f/K \) is \( h \)-central. By Theorem II.1.8, \( J_f(S) \) covers \( G_f/K \). Therefore \( J_f(S) \neq W(S) \cap G \) since they do not cover the same factors.
Conversely, if \( f \) is a nonsaturated formation, there exists a group \( G^* \) such that \( G^*/\phi(G^*) \in f \) and \( G^*/f \). In \( G^* \), \( G^* \leq \phi(G^*) \). So if \( G^*/K \) is a chief factor, then it is a frattini chief factor. Consider the group \( G = G^*/K \). In \( G \), \( G^* = G^*/K \) is a minimal normal subgroup. Therefore it is nilpotent. By Theorem III.1.3, \( G^* = J_f(S) \). Since \( G^* \leq \phi(G) \), \( G^* = W(S) \cap G^* \). Hence \( J_f(S) = W(S) \cap G^* \). Consequently, if \( f \) is a nonsaturated formation, then there exists a group \( G \) in which \( G^* \neq 1 \) and \( J_f(S) = W(S) \cap G^* \).

A formation \( f \) is saturated if and only if for any group \( G \) and chief factor \( G^*/K \), \( G^*/K \) is complemented. At the opposite extreme are formations in which for any group \( G \) and chief factor \( G^*/K \), \( G^*/K \) is a frattini chief factor.

3.2 Definition: A formation \( f \) in which for every group \( G \), each chief factor of the form \( G^*/K \) is frattini is called a totally nonsaturated formation.

An example of a totally nonsaturated formation is the formation of solvable \( nC \)-groups. It will be obtained that examples of totally nonsaturated formations are abundant. In a sense, the formation of solvable \( nC \)-groups \( k \) is the smallest such formation. If \( f \) is a totally nonsaturated formation, then \( G^*/K \leq G^*/K \) for each group \( G \). Totally nonsaturated formations also may be characterized through the structure of the correspondent \( f \)-subgroups. In order to do this it is useful to examine the formation \( c(f) \) introduced by D. Blessenohl and B. Brewster [3]. For a formation \( f \), \( c(f) \)
is the class of groups for which the $f$-residual has only complemented chief factors.

3.3 Theorem: A group $G$ is an element of the formation $c(f)$ if and only if $W(s) \cap G_f = 1$.

Proof: If $G \in c(f)$, then each chief factor below $G_f$ is complemented. By Theorem I.6.2 $W(s)$ avoids each such factor. Theorem I.2.2 implies that $W(s) \cap G_f = 1$.

Conversely, if $W(s) \cap G_f = 1$, then $W(s)$ avoids $G_f$. Hence $W(s)$ avoids each chief factor of $G$ below $G_f$. By Theorem I.6.2, each factor below $G_f$ is complemented. Therefore $G \in c(f)$.

3.4 Theorem: For each group $G$, $G_{c(f)} = \langle \langle W(s) \cap G \rangle \rangle$.

Proof: Let $\langle \langle W(s) \cap G_f \rangle \rangle = B$. Since a prefrattini subgroup and the $f$-residual are preserved under homomorphism, $W^G/B(s) \cap (G/B)_f = (W(s)B/B) \cap (G_f/B) = (W(s)B \cap G_f)/B$. By the modular identity I.1.2, $W(s)B \cap G_f = B(W(s) \cap G_f) = \langle \langle W(s) \cap G_f \rangle \rangle (W(s) \cap G_f) = \langle \langle W(s) \cap G_f \rangle \rangle = B$.

Hence by Theorem 3.3, $G/B = G/\langle \langle W(s) \cap G_f \rangle \rangle \in c(f)$. Therefore $G_{c(f)} \leq \langle \langle W(s) \cap G_f \rangle \rangle$.

Conversely, $G/G_{c(f)} \in c(f)$. By Theorem 3.3 $W^{G/G_{c(f)}}(s) \cap (G/G_{c(f)} f) = G/G_{c(f)}$. However, $W^{G/G_{c(f)}}(s) \cap (G/G_{c(f)} f) = (W(s)G_{c(f)} f)/(G_f/G_{c(f)}) = (G_{c(f)}(W(s) \cap G_f))/G_{c(f)}$ by the modular identity. It follows that $W(s) \cap G_f \leq G_{c(f)}$. Therefore $\langle \langle W(s) \cap G_f \rangle \rangle \leq G_{c(f)}$. Combining the two results, $G_{c(f)} = \langle \langle W(s) \cap G_f \rangle \rangle$. 
3.5 Theorem: The formation $f$ is totally nonsaturated if and only if $f = c(f)$.

Proof: If $f$ is totally nonsaturated, then for every group $G$ and chief factor $G_f/K$, $G_f/K$ is frattini. Therefore $G_{c(f)} = G_f$. Hence $f = c(f)$.

Conversely, if $f = c(f)$, then $G_f = G_{c(f)}$ for every group $G$. Thus each chief factor of the form $G_f/K$ is frattini. By Definition 3.2 $f$ is totally nonsaturated.

3.6 Corollary: For any formation $f$, $c(f)$ is a totally nonsaturated formation.

Proof: By Theorem 3.5, it suffices to prove that $c(f) = c(c(f))$. Since each factor of $G$ that lies between $G_f$ and $G_{c(f)}$ is complemented, Theorem 1.6.2 and Theorem I.2.2 imply that $W(S)$ avoids $G_f/G_{c(f)}$. It follows that $W(S) \cap G_f \leq G_{c(f)}$. Then $W(S) \cap G_f = W(S) \cap G_{c(f)}$. Consequently, $<W(S) \cap G_f> = <W(S) \cap G_{c(f)}>$. By Theorem 3.4 $G_{c(f)} = G_{c(c(f))}$. Therefore $c(f) = c(c(f))$.

It is clear that examples of totally nonsaturated formations are numerous. There corresponds to each formation $f$ the totally nonsaturated formation $c(f)$.

3.7 Theorem: The formation $f$ is totally nonsaturated if and only if $<W(S) \cap G_f> = G_f$ for every group $G$. 
Proof: If \( f \) is totally nonsaturated, then \( f = c(f) \) by Theorem 3.5. Then Theorem 3.4 yields \( <W(s) \cap G_f> = G_c(f) = G_f \). Suppose that \( G_f = <W(s) \cap G_f> \). Consider a chief factor \( G_f/K \) of \( G \). If \( G_f/K \) is complemented, then \( W(s) \) avoids \( G_f/K \) by Theorem 1.6.2. In this case, \( W(s) \cap G_f \leq K \). So \( <W(s) \cap G_f> \leq K \). This implies that \( G_f = K \). Hence there is a contradiction. It follows that \( G_f/K \) is a frattini chief factor. Therefore \( f \) is totally nonsaturated by Definition 3.2.

The duality of saturated and totally nonsaturated formations is reflected by the relationship of the prefrattini subgroups to the \( f \)-subgroups. This is evident in the comparison of the following theorem to Theorem 3.1.

3.8 Theorem: The formation \( f \) is totally nonsaturated if and only if \( J_f(s) = W(s) \cap G_f \) for every group \( G \).

Proof: Suppose that for every group \( G \), \( J_f(s) = W(s) \cap G_f \). By Theorem II.2.3, \( <J_f(s)> = G_f \). Hence \( G_f = <W(s) \cap G_f> \). By Theorem 3.7 \( f \) is totally nonsaturated.

Conversely, suppose that \( f \) is totally nonsaturated. By Theorem 3.7 \( G_f = <W(s) \cap G_f> \) for every group \( G \). For a group \( G \) assume that \( W(s) \cap G_f < J_f(s) \). Theorem I.2.3 implies that \( J_f(s) \) covers a chief factor of \( G \) below \( G_f \) that is avoided by \( W(s) \cap G_f \). Since the factor is avoided by \( W(s) \), Theorem I.6.2 implies that the factor is complemented. Since it is covered by \( J_f(s) \), the factor is \( h \)-central by Theorem II.1.8. It may be assumed that the chief factor in question is a minimal normal subgroup.
of G since $G_{\phi}$, $W(\phi)$, and $J_{\phi}(\phi)$ are preserved under homomorphism. Therefore $G = [N]M$ and $G/C_G(N) \in f$ for N a minimal normal subgroup of G. Since $G/C_G(N) \in f$, $G_{\phi} \leq C_G(N)$ and $N \leq Z(G_{\phi})$. By the modular identity 1.1.2, $G_{\phi} = NM \cap G_{\phi} = N(M \cap G_{\phi})$. Since $N \leq Z(G_{\phi})$, $G_{\phi} = N \cdot (M \cap G_{\phi})$. Replacing M by a suitable conjugate of M if necessary, it may be assumed that $W(\phi) \leq M$ by Definition 1.6.1. Thus $W(\phi) \cap G_{\phi} \leq M \cap G_{\phi}$. Theorem 3.7 indicates that the conjugates of $W(\phi) \cap G_{\phi}$ generate $G_{\phi}$. If $g \in G$, then $g = mn$ for $m \in M$ and $n \in N$. So $(W(\phi) \cap G_{\phi})^g = (W(\phi) \cap G_{\phi})^{hm} = (W(\phi) \cap G_{\phi})^m$ since $N \leq Z(G_{\phi})$. Moreover $(W(\phi) \cap G_{\phi})^m \leq (M \cap G_{\phi})^m = M \cap G_{\phi}$. It follows that every conjugate of $W(\phi) \cap G_{\phi}$ is contained in $M \cap G_{\phi}$. Consequently, $<W(\phi) \cap G_{\phi}> \leq M \cap G_{\phi} < G_{\phi}$. This is a contradiction since $G_{\phi} = <W(\phi) \cap G_{\phi}>$. The assumption that $W(\phi) \cap G_{\phi} < J_{\phi}(\phi)$ must be invalid. Therefore $W(\phi) \cap G_{\phi} = J_{\phi}(\phi)$.

3.9 Theorem: The formation f is totally nonsaturated if and only if $Z(G_{\phi}) \leq \phi(G)$ for every group G.

Proof: If f is totally nonsaturated, then $J_{\phi}(\phi) = W(\phi) \cap G_{\phi}$ by Theorem 3.8. Since $Z(G_{\phi}) \leq Z_{\phi}(G_{\phi})$ and $Z_{\phi}(G_{\phi}) \leq J_{\phi}(\phi)$ by Theorem III.1.1, then $Z(G_{\phi}) \leq W(\phi)$. Hence $Z(G_{\phi}) \leq \text{core } W(\phi) = \phi(G)$.

Conversely, suppose that $Z(G_{\phi}) \leq \phi(G)$ for every group G. Let $H/K$ be a chief factor of the group G such that $H \leq G_{\phi}$. Suppose that $H/K$ is covered by $J_{\phi}(\phi)$. By Theorem II.1.8, either $H/K$ is frattini or it is $h$-central. If $H/K$ is $h$-central, then $H/K \leq Z(G_{\phi}/K)$. But then $H/K \leq \phi(G/K)$ since by hypothesis $Z(G_{\phi}/K) \leq \phi(G/K)$. Consequently, $H/K$ is a frattini chief factor. It follows that every factor below $G_{\phi}$ that is covered by $J_{\phi}(\phi)$ is frattini. By Theorem I.6.2 each such factor is covered by
\( W(s) \cap G_f \). Since \( W(s) \cap G_f \leq J_f(s) \), every factor avoided by \( J_f(s) \) is also avoided by \( W(s) \cap G_f \). By Theorem I.2.3, \( |J_f(s)| = |W(s) \cap G_f| \). Hence \( J_f(s) = W(s) \cap G_f \). By Theorem 3.8 \( f \) is totally nonsaturated.

The following corollary summarizes the relationship between totally nonsaturated formations, \( f \)-subgroups, and the formation \( c(f) \).

3.10 Corollary: Let \( f \) be a formation. The following conditions are equivalent.

(i) \( f \) is totally nonsaturated.
(ii) \( f = c(f) \).
(iii) \( G_f = \langle \langle W(s) \cap G_f \rangle \rangle \) for every group \( G \).
(iv) \( J_f(s) = W(s) \cap G_f \) for every group \( G \).
(v) \( Z(G_f) \leq \phi(G) \) for every group \( G \).

3.11 Theorem: If \( f \) is a totally nonsaturated formation and \( k \) is the formation of solvable nC-groups, then \( G_f = G_k \cap G_f \) for each group \( G \). Thus \( G_f \leq G_k \).

Proof: If \( f \) is a totally nonsaturated formation, then \( G_f = \langle \langle W(s) \cap G_f \rangle \rangle \) by Corollary 3.10. Since \( W(s) \) is contained in \( G_k \) by Theorem I.6.7, then \( W(s) \cap G_f \leq G_k \cap G_f \). Therefore \( G_f = \langle \langle W(s) \cap G_f \rangle \rangle \leq G_k \cap G_f \). So \( G_f \leq G_k \).

The converse of Theorem 3.11 may be stated as follows: If \( G_f \leq G_k \) for every group \( G \), then \( f \) is totally nonsaturated. That this
statement is not valid in general, is illustrated by considering $f$ to be the formation of groups in which $G_k$ is nilpotent. The formation $f$ is saturated and $G_f = K_f(G_k) \leq G_k$.

The $f$-subgroups and totally nonsaturated formations can be used to characterize the solvable nC-groups.

3.12 Corollary: Let $f$ be a totally nonsaturated formation such that for every group $G$ an $f$-subgroup of $G$ has the form $J_f(S) = \cap\{M|M \in m(G)\}$, for which $m(G)$ is a subset of the set of maximal subgroups of $G$. Then $f$ is the formation $\kappa$ of solvable nC-groups.

Proof: Theorem IV.3.5 implies that $G_\kappa \leq G_f$ for every group $G$. By Theorem 3.11, $G_f \leq G_\kappa$. Hence $G_f = G_\kappa$ for every group $G$. Therefore $f = \kappa$.

Another consequence of the duality of saturated and nonsaturated formations is a result similar to Corollary 3.10, obtained for saturated formations.

3.13 Theorem: Let $f$ be a formation. The following conditions are equivalent:

(i) $f$ is saturated.
(ii) $G_f \ntriangleleft G_c(f)$ for each group $G \ntriangleleft f$.
(iii) $G_f \ntriangleleft \langle W(S) \cap G_f \rangle$ for each group $G \ntriangleleft f$.
(iv) $J_f(S) > W(S) \cap G_f$ for each group $G \ntriangleleft f$.
(v) Each group $G \ntriangleleft f$ has at least one chief factor below $G_f$ that is complemented in $G$ and centralized by $G_f$. 
Proof: Suppose that (i) is satisfied. Then if $G \nleq f$, then $G_f \neq 1$. Any chief factor of the form $G_f/K$ is complemented since $f$ is saturated. Furthermore $G_f/K$ is abelian and hence centralized by $G_f$. Therefore (i) implies (v). Suppose that (v) is valid. If $G_f \neq 1$, then $G$ has a chief factor $H/K$ such that $H \leq G_f$ and $H/K$ is $h$-central and complemented. By Theorem II.1.8, $H/K$ is covered by $J_f(s)$. By Theorem I.6.2, $H/K$ is avoided by $W(s) \cap G_f$. Consequently $W(s) \cap G_f \neq J_f(s)$. So $W(s) \cap G_f < J_f(s)$. Therefore (v) implies (iv). Suppose that (iv) is valid. In the proof of Theorem 3.8, it is proven that if $G_f = \langle W(s) \cap G_f \rangle$, then $J_f(s) = W(s) \cap G_f$. It follows that if $W(s) \cap G_f < J_f(s)$, then $G_f \neq \langle W(s) \cap G_f \rangle$, that is, $\langle W(s) \cap G_f \rangle < G_f$. Hence (iv) implies (iii). Since $W(s)$ avoids complemented chief factors and each factor of $G/G_c(f)$ is complemented, $W(s) \cap G_f \leq G_c(f)$. By Corollary 3.6, $c(f)$ is totally nonsaturated. Corollary 3.10 then indicates that $G_c(f) = \langle W(s) \cap G_c(f) \rangle = \langle W(s) \cap G_f \rangle$. Consequently if (iii) is satisfied, then $G_c(f) = \langle W(s) \cap G_f \rangle < G_f$. So (ii) is valid. Finally, assume that (ii) is valid. If $G_f \neq 1$, then $G_c(f) < G_f$. Hence every factor of the form $G_f/K$ is complemented. Equivalently, $f$ is saturated and (i) is valid.
CHAPTER VI

FURTHER QUESTIONS

Several directions exist for further study of the $f$-subgroups. One direction is to determine what further information about the structure of a group $G$ and residual $G_f$ can be obtained through the use of $f$-subgroups, and whether or not this structure can be related to well-defined characteristics of the formation $f$. Another direction is to determine if the $f$-subgroup definition can be extended to classes of groups other than formations of finite solvable groups.

The core of an $f$-subgroup and its analogue the frattini subgroup play a major role in this work on $f$-subgroups. A recurring property of the frattini subgroup is that $Z(G) \cap G' \leq \phi(G)$. Results on splitting such as Theorems 1.1.9 and 1.1.10 also occur frequently. This raises the question: Do analogous results exist for the core of an $f$-subgroup when $f$ is an arbitrary formation? If so, what additional information concerning the splitting properties of groups may be obtained?

The formation of solvable nC-groups has a special significance in formation theory. In Chapter V the solvable nC-groups $\kappa$ are determined to be the "smallest" totally nonsaturated formation. The formation $\kappa$ is related in a unique manner to another formation. Let $E$ be the class of groups for which $G \in E$ if and only if every Sylow subgroup of $G$ is elementary abelian. Then $E$ is a formation and the $E$-residual $G_E$ is the normal subgroup of $G$ generated by $\{ \phi(H) | H \leq G \}$. The residual $G_E$ is known as the
elementary commutator subgroup of $G$. The elementary commutator $G_E$ is generated by the cores of prefattinito subgroups of subgroups of $G$. In fact $G_E$ is generated by all the prefattinito subgroups of subgroups of $G$. It follows that $G_E$ is generated by the $\kappa$-subgroups of subgroups of $G$. Also, the formation $E$ is a subformation of $\kappa$. The formation $E$ is such that $c(E) = \kappa$ since every factor between $G_E$ and $G_\kappa$ is complemented. This poses several questions. Can either the prefattinito subgroups or $f$-subgroups further explain the relationship between $\kappa$ and $E$. For example when $G$ is nilpotent, $G_E = \phi(G) = W(S) = G_\kappa$ and hence $G_E = G_\kappa$. Let $f^*$ be the class of groups such that $G \in f^*$ if and only if core $J^H_f(S) = 1$ for every subgroup $H$ of $G$. For the formation $\kappa$, $\kappa^* = E$. In general is $f^*$ a formation? If so, is $G_{f^*}$ generated by the $f$-subgroups of all the subgroups of $G$, and is $c(f^*) = f$?

Totally nonsaturated formations are a dual of saturated formations. A natural question is whether or not there is a dual for the formation of solvable $nC$-groups. In particular is there a "smallest" saturated formation? Can the formations which are neither saturated nor totally nonsaturated be characterized by the structure of $f$-subgroups? That is, are some formations more closely related to saturated or totally nonsaturated formations than others? One approach to this last problem may be through $f$-projectors. For a class $L$ of groups a subgroup $L$ of the group $G$ is an $L$-projector if the following conditions are satisfied: (i) $L \in L$ and (ii) If $L \leq L_0 \leq G$ and $L_0/K \in L$ then $L_0 = LK$. W. Gaschutz [12] established that a formation $f$ of finite solvable groups is saturated if and only if every group $G$ has $f$-projectors. Combining this result with Theorem V.3.1, every group $G$ has $f$-projectors if and only if $W(S) \cap G_f \neq J_f(S)$ whenever $G$
is a group such that $G_{f} \neq 1$. One then wonders whether or not a group $G$ has $f$-projectors if and only if $W(S) \cap G_{f} \neq J_{f}(S)$, and whether or not any group for which $J_{f}(S) = W(S) \cap G_{f} \neq 1$ has an $f$-projector. Perhaps the degree to which a formation is saturated or totally nonsaturated can be based on $f$-projectors.

The formation of solvable nC-groups $k$ is a normal formation [2]. This property is reflected by the prefrattini subgroups. If $N$ is a normal subgroup of $G$ then a prefrattini subgroup of $N$ is contained in a prefrattini subgroup of $G$ [1]. If an $f$-subgroup of $N$ a normal subgroup of the group $G$ is contained in an $f$-subgroup of $G$, then $N_{f} \leq G_{f}$ since $f$-subgroups generate the $f$-residuals. Hence if this property holds for every normal subgroup in every group, then $f$ is a normal formation. This poses the question: When is $J_{f}^{N}(S) \leq J_{f}^{G}(S)$ for each group $G$ and normal subgroup $N$ of $G$? This question is partially answered in Chapter V. However, there it was necessary to assume that $f$ is a normal formation. Similarly, if for every group $G$ an $f$-subgroup of a subgroup of $G$ is always contained in an $f$-subgroup of $G$, then $f$ is a subgroup inherited formation. Can necessary and sufficient conditions be found for this condition on $f$-subgroups to hold?

If $h$ is the formation locally defined by $h(p) = f$ for all primes $p$, then a group $G \in h$ if and only if $G_{f} = J_{f}(S)$ by Theorem III.1.3. By Theorem IV.3.3, $G \in c(f)$ if and only if $W(S) \cap G_{f} = 1$. In both of these examples the group $G$ is an element of the formation $k$ if and only if the $f$-subgroups have a well-defined property. This suggests the possibility of investigating classes of groups in which the $f$-subgroups have a particular structure, and determining whether or not these classes of groups are formations.
As mentioned, a second direction of investigation is to extend the \( f \)-subgroup definition to classes of groups other than formations of finite solvable groups. M. Tomkinson \cite{22} extended the \( g \)-prefrattini subgroups to \( \chi \)-prefrattini subgroups. These subgroups are defined in the class of \( U \)-groups, a larger class than finite solvable groups. The question presents itself as to whether or not \( f \)-subgroups can be extended to \( U \)-groups.

For finite solvable groups a homomorph is a class of groups which is closed under epimorphic images. More clearly, \( H \) is a homomorph if whenever \( G \in H \) and \( \theta \) is a homomorphism of \( G \), then \( G^\theta \in H \). A saturated homomorph \( H \) (or Schunck class) is a homomorph in which each group \( G \) has at least one \( H \)-projector. Every formation is a homomorph by Definition I.4.1. W. Gaschutz's result on projectors for formations implies that every saturated formation is a saturated homomorph. It would be interesting to determine if an \( H \)-subgroup definition exists for a homomorph \( H \) which would coincide with the \( H \)-subgroups of this paper whenever \( H \) is a formation. If so, can the structure of these \( H \)-subgroups determine if the homomorph \( H \) is saturated.
<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \leq B$</td>
<td>$A$ is a subgroup of $B$</td>
</tr>
<tr>
<td>$A &lt; B$</td>
<td>$A$ is a proper subgroup of $B$</td>
</tr>
<tr>
<td>$A \triangleleft G$</td>
<td>$A$ is a normal subgroup of $G$</td>
</tr>
<tr>
<td>$X \subseteq G$</td>
<td>$X$ is a subset of $G$</td>
</tr>
<tr>
<td>$\langle a_1, a_2, \ldots, a_n \rangle$</td>
<td>subgroup generated by the elements $a_1, a_2, \ldots, a_n$</td>
</tr>
<tr>
<td>$\langle X \rangle$</td>
<td>subgroup generated by the set $X$</td>
</tr>
<tr>
<td>$\langle \langle A \rangle \rangle$</td>
<td>normal closure of $A$ in $G$</td>
</tr>
<tr>
<td>$\text{core } A$</td>
<td>largest normal subgroup of $G$ contained in $A$</td>
</tr>
<tr>
<td>$AB$ or $A \cdot B$</td>
<td>product of $A$ and $B$</td>
</tr>
<tr>
<td>$A \times B$</td>
<td>direct product of $A$ and $B$</td>
</tr>
<tr>
<td>$[A]B$</td>
<td>split product of $A$ by $B$</td>
</tr>
<tr>
<td>$G \cong H$</td>
<td>$G$ and $H$ are isomorphic groups</td>
</tr>
<tr>
<td>$G^\varnothing$</td>
<td>image of the group $G$ under the homomorphism $\varnothing$</td>
</tr>
<tr>
<td>$G/A$</td>
<td>quotient group</td>
</tr>
<tr>
<td>$X/Y$</td>
<td>elements of a set $X$ that are not in $Y$</td>
</tr>
<tr>
<td>$</td>
<td>G</td>
</tr>
<tr>
<td>$[G:N]$</td>
<td>index of $N$ in $G$</td>
</tr>
<tr>
<td>$m</td>
<td>n$</td>
</tr>
<tr>
<td>$h^g$</td>
<td>$g^{-1}hg$</td>
</tr>
<tr>
<td>$A^g$</td>
<td>${g^{-1}ag</td>
</tr>
<tr>
<td>$[a,b]$</td>
<td>$a^{-1}b^{-1}ab$</td>
</tr>
<tr>
<td>$[A,B]$</td>
<td>$\langle [a,b]</td>
</tr>
</tbody>
</table>
\[ N_G(A) \] G-normalizer of \( A \); \(<g|g \in G \text{ and } g^{-1}Ag = A>\)

\[ C_G(A) \] G-centralizer of \( A \); \(<g|g \in G \text{ and } [g,a] = 1 \text{ for all } a \in A>\)

\[ C_G(H/K) \] G-centralizer of \( H/K \);
\(<g|g \in G \text{ and } [g,h] \in K \text{ for all } h \in H>\)

\[ \text{Aut}_G(H/K) \] automorphism group of \( H/K \) generated by
\(<\theta|\theta \in G \text{ and } (hK)^{\theta g} = g^{-1}hgK \text{ for all } h \in H>\)

\( G' \) commutator subgroup of \( G \)

\( K_\infty(G) \) hypercommutator subgroup of \( G \)

\( Z(G) \) center of \( G \)

\( Z_\infty(G) \) hypercenter of \( G \)

\( \Phi(G) \) frattini subgroup of \( G \)

\( F(G) \) Fitting subgroup of \( G \)

\( S(G) \) sockel of \( G \)

\( Z_n \) ring of integers modulo \( n \)

\( SL(n,p) \) special linear group of degree \( n \) over a field of characteristic \( p \)

\( S \) Sylow system of a group

\( S_p \) Sylow \( p \)-subgroup

\( S^p \) Sylow \( p \)-complement

\( S \cap A \) reduction of \( S \) into \( A \)

\( S_{\text{in}N/N} \) image of \( S \) in \( G/N \)

\( S^g \) image of \( S \) under the inner automorphism induced by \( g \)

\( f \) an arbitrary formation

\( h \) formation locally defined by \( h(p) = f \) for all primes \( p \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>formation of solvable $nC$-groups</td>
</tr>
<tr>
<td>$G_f$</td>
<td>$f$-residual</td>
</tr>
<tr>
<td>$W(S)$</td>
<td>prefrettini subgroup of $G$ associated with $S$</td>
</tr>
<tr>
<td>$W^A(S)$</td>
<td>prefrettini subgroup of $A$ associated with $S \cap A$</td>
</tr>
<tr>
<td>$W(SN/N)$</td>
<td>prefrettini subgroup of $G/N$ associated with $SN/N$</td>
</tr>
<tr>
<td>$J_f(S)$</td>
<td>$f$-subgroup of $G$ associated with $S$</td>
</tr>
<tr>
<td>$J^A_f(S)$</td>
<td>$f$-subgroup of $A$ associated with $S \cap A$</td>
</tr>
<tr>
<td>$J_f(SN/N)$</td>
<td>$J^G_f(S)$ $f$-subgroup of $G/N$ associated with $SN/N$</td>
</tr>
<tr>
<td>$D(S)$ or $N_G(S)$</td>
<td>system normalizer of $S$</td>
</tr>
<tr>
<td>$D_A(S)$</td>
<td>system normalizer of $S \cap A$</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY


15. P. Hall, A note on soluble groups, J. London Math. Soc. 3(1928), 98-105.