Winter 1979

ANALYSIS AND SIMULATION OF STEPPING MOTOR SYSTEMS USING A PHASE PLANE APPROACH

RAYMOND GERARD GAUTHIER

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Nov. 24, 1979

Date
DEDICATION

I would like to dedicate this work to my wife, Linda, for without her patience, understanding and financial support through the years this dissertation work would have been far more difficult to complete.
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ABSTRACT

ANALYSIS AND SIMULATION
OF STEPPING MOTOR SYSTEMS
USING A PHASE PLANE APPROACH

BY

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University of New Hampshire, December, 1979

Stepping motors have become an increasingly popular electromechanical interface device, due to their increased reliability and lower costs over the D.C. Servo motor. The inherent open loop mode of stepping motor operation results, under certain circumstances, in a failure mode in which the output of the stepper no longer follows the input commands. In the past, it has been difficult for the designer to define the failure modes and it has not been obvious how to avoid them.

It is the intent of this dissertation to present a mathematical model for a class of stepping motors, the permanent magnet stepping motors. Analysis reveals that a second order nonlinear model adequately describes the major failure modes for stepping motors. The phase plane,
a plot of motor velocity versus position, is used to graphically display the computer solutions of the mathematical model. This method offers the advantage of organizing the solutions in a very compact format and brings order to an otherwise complex problem. By using the phase plane approach, one cannot only predict when the motor will fail, but optimum step sequences can be readily obtained by graphical means.

The second order model is based on the assumption that the current rise time in the motor windings is insignificant compared to the on-time of the windings. A more complex model which includes these effects is developed. This results in a third order nonlinear model, for which the phase plane is no longer adequate to display the solutions. It is then necessary to use a phase space or a three-dimensional plot. It is shown, however, that for step sequences, one can project the three space onto the phase plane to show the effects of the driver electronics. Most of the same organizational advantages apply to the projected three space as was found for the phase plane.
List of Symbols

A  =  number of cycles of torque-angle curve per revolution of motor.

a  =  motor acceleration (rad/sec^2)

B  =  Motor and load damping (oz in sec)

BEMF  =  Back electromotive force voltage

\( \frac{d\theta}{dt} \)  =  motor velocity (rad/sec)

\( \frac{d^2\theta}{dt^2} \)  =  motor acceleration (rad/sec^2)

\( \frac{d\theta}{d\tau} \)  =  normalized motor velocity

\( \frac{d^2\theta}{d\tau^2} \)  =  normalized motor acceleration

DZ  =  angular dead zone due to friction (radians)

E  =  supply voltage (volts)

FINTIM  =  normalized execution time for step sequence

q  =  gravitational constant (32.2 ft/sec^2)

i  =  current (amps)

I_{EQ}  =  reflected inertia (oz in sec^2)

\( \bar{i} \)  =  i/I_R, normalized current

i_n  =  current in the n motor phase (amps)

I_R  =  rated current per phase (amps)

i_s  =  stator current (amps)

J  =  motor and load inertia (oz in sec^2)

J_L  =  load inertia (oz in sec^2)

J_m  =  motor inertia (oz in sec^2)

J_o  =  L \omega_n/RI, normalized current rise time constant
K_b ≡ BEMF constant (volts/rad/sec)
K_T ≡ torque constant (oz-in/amp)
L ≡ winding inductance (henrys)
L_s ≡ stator winding inductance (henrys)
L_o ≡ stator self inductance (henrys)
L_m ≡ stator mutual inductance (henrys)
m ≡ integer
M ≡ mass (1b sec^2)
MWC ≡ maximum work curve
n ≡ integer
N ≡ \frac{V_S}{V_R}, overdrive voltage ratio, motor supply voltage divided by motor rated voltage
N_R ≡ number of rotor teeth
NSTPS ≡ number of motor steps
P ≡ number of rotor teeth
p ≡ number of stator phases
PM ≡ permanent magnet motor
r ≡ radius (inches)
R ≡ resistance (ohms)
R_d ≡ driver resistance (ohms)
R_s ≡ stator phase resistance (ohms)
S ≡ \ddot{\theta}/\dot{\theta}, slope of a trajectory in the phase plane
T ≡ period between two events (seconds)
T_A ≡ acceleration torque (oz in)
\ddot{T}_A ≡ T_A/T_m, normalized acceleration torque
T_f ≡ friction (oz in)
\ddot{T}_f ≡ T_f/T_m, normalized friction
\( T_L \equiv \) unidirectional load torque (oz in)
\( T_m \equiv \) peak torque of the motor torque angle curve (oz in)
\( T_n(\theta) \equiv \) torque-angle curve due to the \( n^{th} \) phase
\( T_s \equiv \) stiction torque (oz in)
\( \overline{T_s} \equiv \frac{T_s}{T_m} \), normalized stiction torque
\( T_T(\theta) \equiv \) total torque angle curve for any number of phases
\( V \equiv \) voltage (volts)
\( V_n \equiv \) \( n^{th} \) phase voltage (volts)
\( V_R \equiv \) motor rated voltage (volts)
\( V_S \equiv \) supply voltage (volts)
\( VR \equiv \) variable reluctance stepping motor
\( W_n \equiv \) natural frequency (rad/sec)
\( \text{WNT} \equiv \) dimensionless stepping period
\( x \equiv \) position (inches)
\( ZSI \equiv \) zero slope isocline
\( ZWC \equiv \) zero work curve
\( \theta \equiv \) rotor angle (radians)
\( \theta_s \equiv \) motor step angle (radians)
\( \theta \equiv A \theta, \) normalized rotor angle (radians)
\( \dot{\theta} \equiv \) rotor velocity (radians/sec)
\( \dot{\theta} \equiv \) normalized rotor velocity
\( \ddot{\theta} \equiv \) rotor acceleration (radians/sec\(^2\))
\( \ddot{\theta} \equiv \) normalized rotor acceleration
\( \tau \equiv \frac{w_n}{t}, \) dimensionless time
\( \tau_s \equiv \) dimensionless stepping period
\( \omega \equiv \) normalized motor velocity
\( \beta \equiv \frac{R}{w_n} \), normalized current rise time constant
\[ \rho \equiv \text{damping ratio} \]
\[ \gamma \equiv \text{normalized BEMF constant} \]
\[ \Omega \equiv \text{dimensionless rotor velocity} \]
\[ \psi \equiv \text{electrical flux density step angle (radians)} \]
CHAPTER I

INTRODUCTION

Since the conception of digital computation, the field of electronics has grown astronomically from vacuum tubes to single chip computers. This sudden surge of electronic hardware advancement has left the mechanical interface devices technically behind times. The lower costs of manufacturing electronic components have also contributed to the emphasis of improving the interface devices, since the mechanical parts of any system must now assume a larger portion of the total system cost. These two factors have influenced the designer's choice of interface devices, in order to upgrade performance and lower costs.

When digital information is available, and it is required to produce a mechanical displacement from that digital information, it is quite convenient to use stepping motors. The only information needed by the stepping motor drive circuit is the direction and number of steps to be executed and the rate of stepping. Often this rate information is obtained directly from the computer clock by dividing computer clock pulses to produce the desired stepping signal. All that is needed then is to feed the appropriate number of these pulses to input of the shift register to cause the proper stepping direction. In this way the stepping motor often forms a very convenient link from a digital computer or other digital logic systems to mechanical devices.

There is a problem in the application of stepping motors. It is possible to devise step sequences that the motor and drive will
fail to execute correctly. By observing rotor motion, it is difficult
to see why the motor fails to execute these step sequences and often it appears to run backwards or erratically. It may suddenly go out of step when previously the system was able to execute that sequence. Further, for a stepping sequence of a given number of steps, the motor will gain steps or lose steps while for another sequence of different length and equal rate it appears to operate satisfactorily. Thus, it is easy to see why the comment has been made that there is a certain amount of witchcraft in understanding the operation of stepping motors.

Unlike most control devices, the stepping motor will appear to operate satisfactorily most of the time, failing only on occasion. This makes the reasons for failure difficult to detect. Most other types of electromechanical positioning systems use position feedback sensors to insure accurate positioning. With this control strategy, most failures are easier to diagnose because when failures do occur they are more repeatable.

However, the stepping motor uses no external position feedback. Work is being done to try to detect from the current signals applied to the stator windings whether or not the motor has executed the desired step command.\(^{(1)}\) Most of these methods require that the motor be operated with a voltage driver circuit. This type of driver has somewhat less optimum characteristics than a current driver, but this compromise in performance is necessary in order to obtain the feedback feature without the use of an encoder. In some motor application the motor is connected to an encoder to indicate when a step is executed or when stator switching should occur. With an encoder the number of steps executed is measured and compared to the number of steps commanded.
and the difference is stored in an error register. The stepping sequence is continued until the desired number of steps is executed as indicated by a zero count in the error register. In these situations one begins to wonder if the stepping motor is really operating in the most effective way. If digital feedback is required, one might well ask why the task could not be performed by a conventional DC motor with encoder feedback which usually has lower cost and higher efficiency.

The purpose of this work is to determine how a stepping motor behaves. Mathematical descriptions for motor behavior will be developed and verified experimentally. It will be possible to show that the mathematical descriptions or equations do describe motor behavior precisely enough to show why the motor fails to step for certain input commands. It will be shown why these failures occur and the various failure situations will be identified analytically and experimentally. It will be seen that the motor fails to step because of a single cause. Once it has been established why the motor fails to step, this information will be used to predict when it will fail to execute a step sequence. That is, the model will be used to extrapolate and predict what step sequences the motor will fail to execute, knowing these sequences will allow the designer to avoid them.

The stepping motor is functionally a device which directly converts electrical digital information into mechanical rotation. There are two basic types of stepping motors, the variable reluctance (VR) stepping motor and the permanent magnet (PM) stepping motor. The electro-mechanical principles by which these two types of motors operate are quite different. However, the essential part of their mechanical behavior is very similar. While these notes are based
primarily on the physics of the permanent magnet stepping motor, the methods presented herein can be used to characterize the variable reluctance motor.

The permanent magnet motor consists of a permanent magnet rotor with two or more phases. Passing current through one or more of the stator phases will cause the rotor to take on a series of discrete positions. The stator phase currents establish a zero torque equilibrium point for the angular rotation of the rotor. That is, there is a rotor angular position called a stable equilibrium point where the electromechanical torque balances the external torques. As the external torque is increased or decreased the rotor will deflect from this position. A plot of this external torque versus deflection angle is called the windup or torque-angle curve. As the excitation to the stator phases is changed to step the motor (with a fixed external torque) this stable equilibrium point can be made to shift in equal increments. In this way, the position of the rotor shifts from one location to another. This process is known as stepping. This shift will be clockwise or counterclockwise, determined by the sequence in which phases are excited. The stator excitation sequence is determined by a digital circuit which usually consists of a shift register that has as many stages as there are stator phases. An input stepping signal causes the register to shift to the right or left depending on the desired direction of rotation. Each register stage provides a signal to a stator phase drive. These signals are amplified by drivers or power amplifiers which connect the appropriate stator phase windings to the power supply. Usually the current required by the stator windings is considerably in excess of the current that can be switched
by the logic elements in the shift register, thus power amplifiers are needed.

The variable reluctance (VR) motor does not have the permanent magnet rotor; however, when a given set of stator phase windings are excited, there is a preferred location of the rotor, such that a minimum-reluctance path is provided in the air gap, so the VR motor behaves in a very similar way to the PM motor. That is, there are a series of rotor locations where there will be no electro-mechanical driving torque on the rotor. In the absence of external torques, the rotor will assume an equilibrium position at one of these points. An external torque applied to the rotor will cause a deflection from this equilibrium position. Thus the variable reluctance motor and the permanent magnet motor behave in a similar way. Although the dynamic relationships between rotor angle and phase currents are somewhat different, the basic analytical approach developed here for the PM motor can also be applied to the VR motor.

Mathematical models will be used to describe the electro-mechanical behavior of the stepping motor. These models will be verified experimentally and then used to organize experimental results in a way which will allow the designer to predict motor behavior. That is, the designer can predict whether a stepping motor will be useful in a given application without actually building the system and trying it. The objective of this is to produce design information which will allow one to apply stepping motors to meet system design objectives. In order to do this the motors will be described in ways that are different from those now used by most motor manufacturers. As will be shown these descriptions can be easily determined
experimentally for any motor. Further, the computer can be used to organize this information in a way which will allow the designer to try a given motor in a given application and predict whether that motor will deliver the desired system performance. In this way a design approach for the application of stepping motors will be presented.
A permanent magnet stepping motor can have several basic configurations. The rotor can be formed in a cylinder with permanent magnet north and south poles on its surface. These can be salient poles or the poles may simply be established on a cylinder of magnetic material by locally magnetizing the rotor material. The stator structure is a series of electromagnets which are called phases. The number of phases indicates the number of stator windings which can be energized. An n phase stator has n windings that can be energized. The phase windings can be independent or they can have a common connection. The independent phases allow the voltage across, or current through, the windings to be reversed by reversing the way they are connected to the supply. With a common connection, current can only be reversed by exciting the phases with a supply of reversed polarity. The current reversing method of driving the windings is called a bipolar drive while the other is a unipolar drive. Figure II-1 shows several types of PM stepping motor configurations.

A second type of PM stepping motor is shown in Figure II-2. This type is usually used for small stepping angles. The rotor is an axial permanent magnet, with a cap of m teeth on each end. The teeth on the two caps are staggered by a 1/2 tooth space. One cap forms a set of m north poles on one end of the rotor and the other cap forms m
Figure II-1
PM Steppers, 2 Pole Rotors
FIGURE II-2

AXIAL PERMANENT MAGNET ROTOR
south poles on the other end. The stator has \( m \) poles. As with the first type, it may have \( n \) different stator windings or phases.

For both configurations, PM steppers will have a torque-angle curve which has as many cycles per revolution as the number of north poles on the rotor. If the same number of stator phases is excited each time, the rotor will make as many steps as stator phases per cycle of the torque-angle curve. By alternately exciting one or two phases it is possible to **half step** to cause the rotor to make twice as many steps per cycle of the torque-angle curve as the number of phases. Other combinations of phases can be executed to get even smaller steps as will be discussed later.

2-B Variable Reluctance Motors

The Variable Reluctance (VR) motor has no permanent magnetic material in the rotor. The rotor may be laminated or solid. A single stack rotor has a different number of teeth than the stator. As with the PM motor, the stator may have \( n \) phases. In principle, it operates like a solenoid. When a stator phase is excited the rotor moves to minimize the reluctance or reduce the air gap of the energized winding.

Several rotors may be stacked end on end. For \( \ell \) rotor stacks each rotor is staggered by \( \frac{1}{\ell} \) rotor tooth pitch. In this stacked configuration the number of rotor and stator teeth are equal. Usually each stack is controlled by a single stator winding or phase. Thus a three stack motor usually has three phases. The multiple stack design decreases the mutual coupling between stator windings.

Figure II-3 shows the configuration for a single stack VR motor; notice the different number of teeth on the stator and rotor.
Figure II-3

Single Stack VR Motor
For the VR motor, the number of rotor teeth equals the number of cycles of the torque-angle curve per revolution. As before, the number of stator phases equals the number of steps per cycle of the torque-angle curve when the same number of phases is excited for each step. Reversing the phase current has no effect on motor position so that unipolar drives are used exclusively for VR motors.

2-C Driver Configurations

The stator phases can be excited by a voltage driver or a current driver. The voltage drive connects a voltage source across the winding to be energized. Since the motion of the rotor causes a BEMF voltage to be induced in the stator phases the resulting phase current is speed dependent. Further, the phase inductance limits the speed with which the current can be switched on and off. For these reasons the phase currents in a voltage drive diminish as the motor is operated at higher speeds. A current drive uses a power amplifier with current feedback or a high voltage drive with a large series resistor to keep stator phase current constant and allow it to be turned off and on suddenly. However, at very high speeds the current drive ceases to operate effectively and behaves as a voltage drive, because of the BEMF voltage generated in the windings.

There are two general types of drivers. (3) The first of these is a unipolar drive in which each phase is either excited with a voltage or open circuited. In this type of drive each phase is connected to a supply voltage $E$ through a transistor, see Figure II-4a. When the transistor base current is high, the transistor acts as a short circuit between the lower end of the coil and ground.
Figure II-4
Drivers
This causes current to flow through the winding. When the transistor base current is low, then the transistor acts as a very high resistance to ground. This is equivalent to open circuiting the winding when it is not being energized and supplying it with a constant voltage when it is being energized. Another way of driving the winding is with a bipolar drive as shown in Figure II-4b. This drive operates by switching one end of a stator phase winding is connected to the ground and the other end is either connected to a positive or a negative supply through a transistor. When both transistors are open the winding is essentially unenergized. When one or the other of the transistor base currents is high then that particular winding is connected to the appropriate supply. In this way it is possible to reverse the current flow through the winding by connecting it either to the Positive or Negative voltage source. This method of driving is used for PM motors when all phases are energized and allows easy reversal of the current in each of the phases.

The bipolar drive with all phases energized produces the maximum motor torque from a given supply voltage, although, motor thermal limitations usually will determine maximum current and hence maximum torque. However, the bipolar arrangement requires a positive and negative supply and usually employs more transistors than the unipolar approach.

In the process of stepping, the excitation to the stator windings is changed to cause the torque-angle curve to shift to a new location. This in turn causes the rotor to move to a new location. This is basically the stepping process. If the stator windings are driven with a voltage source, then because of the inductance of the
winding, it takes time for the current to build up in one winding and
decay in the other winding to establish the new torque-angle curve.
This process increases the time it takes to jump the torque angle
curve to the new position. It would therefore seem reasonable this
would slow down the stepping process. Indeed most high performance
drives are designed to establish the stator phase currents as quickly
as possible. Thus most stator drives are what may be called current
drives. That is, they are designed to minimize the transient in
establishing the new current pattern in the stator windings. This
can be accomplished in two basic ways:

1) Current amplifier. A reference signal is used to control
an amplifier that excites one of the stator phases. The current
through the phase is sensed and fed back to force the current in that
phase to be proportional to the reference signal. This causes the
drive voltage to be very large at the start of the step to minimize
inductive effects.\(^4\) To be most effective, the driver should be
capable of applying positive and negative voltages to the stator
phases. It also minimizes BEMF effects at high motor speeds. Further,
phase windings are either energized by the logic voltage or open
circuited by the absence of it.

2) Voltage amplifier with series resistor. It is desirable
to excite the stator with a large voltage when the winding is first
energized to minimize the current build-up time. After current has
been established in the winding this starting voltage is too high and
will cause excessive heating. Thus a current limiting resistor is
connected to the driver in series with the stator winding. In effect,
this increases the stator circuit resistance and thereby reduces the
current build-up time. This is the simplest form of current drive.

Each of these methods can be mechanized by power transistors which either connect a winding to the power supply or open circuit it depending on the step sequence. In each of these cases, the stator windings are excited with a driver which minimizes the transient that occurs in the establishment of the torque-angle pattern at a new equilibrium location. To obtain very high stepping rates, it is generally agreed that the current drive approach is a far superior approach to establishing the torque-angle pattern in motor, than the voltage drive. In describing the behavior of the motor with a current drive, it will be assumed that the current build-up due to the overall time constant of the stator windings will be very small.

Figure II-5 shows a permanent magnet stepping motor structure with a four phase stator structure and a two-pole rotor.\(^{(5)}\) The rotor will align with the direction of maximum flux density established by the stator phases. One phase at a time, two phases at a time, or all four phases at a time may be excited to produce the stepping sequence. For example, referring to Figure II-5, when phase 1 is energized by current \(i_1\), it becomes an electromagnet. If the direction of current flow is as shown, then the upper end of the stator phase becomes the north pole and the lower end becomes the south pole and the rotor is shown in its equilibrium position. If the rotor is deflected clockwise from this position there will be a counterclockwise torque tending to bring it back to the equilibrium position. If it is rotated counterclockwise there will be a clockwise restoring torque, due to the interaction of stator and rotor fields, tending to bring it back to equilibrium. If the rotor is rotated 180° then it
Figure II-5

Basic PM Stepping Motor
is at an unstable equilibrium position and a slight CCW rotation will cause the rotor to turn CCW back to the original position. Also this unstable equilibrium point, a CW torque will cause the motor to continue to rotate CW until it has completed one revolution to return to the original position shown in Figure II-5.

If phase 2 is excited with a current in the direction shown and phase 1 is unexcited then the motor will step 90° to line up with stator winding number 2. In this way the stepper has been caused to step 90° in the clockwise direction.

If the excitation were shifted from phase 1 to phase 4, then the rotor would move in a counterclockwise direction to line up with the pole of phase 4. Notice again if the rotor is deflected by an external torque in a clockwise direction there is a counterclockwise electromechanical torque tending to bring it back in alignment with the pole of phase 4. If it is deflected in the counterclockwise direction there is a clockwise electromechanical torque tending to bring it back in alignment with the phase 4 pole. Thus the torque-angle curve for the rotor or "windup" curve when pole 4 is excited is the same as it was when the pole 1 was excited but it has been shifted 90° CCW.

This is the mode of stepping the motor. It can be shown that successively exciting the windings during the stepping process has the effect on the rotor of shifting the torque-angle curve an amount equal to the step size, in this case 90°. Notice too that it would be possible to obtain a larger torque from the motor by exciting phase 1 with a current in the direction shown and phase 3 in the direction opposite to that shown to reinforce the field produced by winding
one. In this way one might expect to double the torque available from the motor. In this motor, if two phases at a time are excited, double the torque is obtained from the rotor. Carrying this a bit further one might excite phases one and two with current in the direction shown and phases three and four with current in the opposite direction to produce an equilibrium position of the rotor which was half-way between the poles of phases one and two. In this case an even larger torque results from the motor, approximately the square root of two times the torque obtained by exciting two opposite phases at a time. The motor can then assume a series of equilibrium positions which are located in this drawing at 45°, 135°, 225°, and 315°. As long as all four phases are excited at the same time, the maximum torque will result from this electromagnetic configuration. On the other hand, it should be pointed out that the losses which result from this form of excitation are twice as large as those which result from exciting two phases at a time, which in turn are twice as large as those which result from one phase at a time. The major source of loss in a permanent magnet motor, in the steady state, would be the $i^2R$ losses in the stator phases. With any given motor structure there is a trade-off between the amount of current and the number of phases which can be excited at one time and the heat dissipation properties of the motor itself. It is possible to alternately drive one phase at a time and then two phases at a time to double the number of detent positions per revolution of the motor. This can be seen by comparing what happens when two phases at a time are excited and when one phase at a time is excited. For this motor the step angle will be 45° instead of 90°, but notice too, for a constant applied voltage,
that the magnitude of the torque-angle curve will shift in size and will be largest for two windings excited and smaller for one winding excited. This will cause the dynamic behavior of the motor to differ from step to step. This mode of driving the stepping motor is known as half-stepping. Half-step behavior can be analyzed and displayed by using the phase plane. It is a way of doubling the number of steps per revolution with any given electromagnetic structure, thus increasing the system resolution.

By varying the excitation current to the windings in small steps, the motor can be made to step at much more than the number of phases per cycle of the torque-angle curve. This is called mini-stepping. It is used to provide a very small step size from a standard motor and results in smoother response. The drive system is much more complex and requires a combination of digital and analog circuitry. In order to accomplish these small steps, the phase currents must be varied in a precise way. The approach causes the angle of the maximum flux density vector, established by the stator phase excitation, to shift in much smaller steps.

2-D Stepping Process

The permanent magnet stepping motor stator phase currents establish a flux density distribution in the air gap. For a given excitation of the stator phases. There is a direction of maximum flux density in the air gap, determined by number of phases excited. This direction will be designated by the flux density vector and its magnitude corresponds to the maximum flux density in the air gap. One of the sets of the permanent magnet poles of the rotor lines up
with this flux density vector when there are no external loads, as was shown in Figure II-5 for a 2-pole rotor. As an external load is applied to the motor, the rotor deflects through an angle which depends upon the external load. The torque-angle curve is a plot of the external torque required to deflect the rotor as a function of the angle of deflection for a given set of motor phase currents, see Figure II-6. The number of cycles \( A \) that the torque angle curve of a motor makes per revolution is fixed for any given motor structure. \( A \) is also the number of rotor teeth, \( N_R \), and from geometry, there are \( N_R \) possible locations in one revolution of rotor that are stable.

In the stepping process, this torque-angle curve or wind-up curve of the motor is shifted when the phase excitation is changed. For a full step of the motor, this curve is shifted through an angle which depends upon the number of cycles of the torque-angle curve in one revolution and the number of phases. The full step shift is equal to the period (in mechanical degrees) of the torque-angle curve divided by the number of phases. By certain stator excitation combinations, it is possible to step the motor at one-half or one-quarter or even one-eighth of this angle. Thus the stepping angle of the motor can be defined in terms of fractions of the period of the torque angle curve. Figure II-7 shows the torque angle curve and the corresponding flux density vector shifted during four full steps of a four phase stepping motor. Whenever the motor is stepped one cycle of the torque-angle curve, the flux density vector makes one revolution in the air gap. For each step the flux density vector is located at an angle which is \( (A) \) times the full step mechanical angle of the rotor and its amplitude is proportional to the magnitude of the torque angle curve. Thus the
FIGURE II-6
TORQUE-ANGLE CURVE
FIGURE II-7
STEPPING PROCESS
flux density vector can be used to indicate the shift angle of the torque-angle curve and at the same time show the amplitude of the torque-angle curve.

The relationships between stator current and the torque-angle curve can be more explicitly defined. Figure II-8 shows the stator equivalent circuit for a stepping motor. The stator circuit contains a resistance, an inductance and a back emf (BEMF) voltage source which represents the voltage induced in the stator winding due to the speed of the motor. There may also be a mutual inductance voltage induced in the stator due to the rate of change of current in one of the other stator windings. Mutual inductance affects tend to be transitory and do not materially influence the behavior of the motor except under very particular circumstances. Equation II-1 is the relationship for the current through, and the voltage across any given stator phase, \( n \).

\[
V_n = R_i n + L \frac{d i_n}{dt} - K_b \hat{\Theta} \sin (A \Theta - \frac{(n-1)2\pi}{p})
\]  

(II-1)

The current which flows in any given stator phase results in a torque-angle curve due to that current which is designated as \( T_n \). The torque-angle curve is nearly sinusoidal for a permanent magnet stepper and its location and magnitude depends upon the particular phase excited and the current through that phase, see Equation II-2. It is assumed here that angle is increasing in a clockwise direction and that the phases are numbered in a clockwise direction starting with number one.
FIGURE II-8

STATOR EQUIVALENT CIRCUIT

\[ K_b \dot{\theta} \sin(A\theta - (N-1)2\pi/P) \]
\[ T_n(\theta) = K_T i_n \sin \left( A\theta - \left( \frac{n-1}{p} \right) 2\pi \right) \]  

\[ (II-2) \]

At very slow speeds the voltage applied to any given stator phase is very nearly proportional to the current in that phase, if the current rise time in the windings is neglected. At high speeds the current in the phase is degraded by the BEMF, but this will be accounted for by a damping torque as a function of speed. From this, it is possible to write an expression for the torque produced by any given stator phase in terms of the applied voltage assuming that the phase is magnetically unsaturated. This is shown in Equation II-3.

\[ T_n(\theta) = \frac{V_n}{R} K_T \sin \left( A\theta - \left( \frac{n-1}{p} \right) 2\pi \right) \]  

\[ (II-3) \]

The total torque developed by the motor is the vector sum of all of the torques produced by each of the stator phases. This can be written as shown in Equation II-4 and results in a torque-angle curve which is sinusoidal. This total torque has a peak value of \( T_m \) and has a zero torque or equilibrium point located at some angle \( \psi \).

\[ T_T(\theta) = \sum_{n=1}^{p} \frac{V_n}{R} K_T \sin \left( A\theta - \left( \frac{n-1}{p} \right) 2\pi \right) \]

\[ T_T(\theta) = T_m \sin \left( A\theta - \psi \right) \]  

\[ (II-4) \]

This can be illustrated by several examples. For example, for a four phase motor it is possible to operate the motor with a current in only one of the stator phases at a time. This results in a torque angle curve shown in Figure II-9a which is marked No. 1. To full step the four phase motor one would switch the current off in phase one and switch the current on in phase two. This would result in a torque-angle curve, marked No. 3 on figure II-9A, that is shifted by
FIGURE II-9A
FOUR PHASE, HALF STEP
TORQUE-ANGLE CURVE

FIGURE II-9B
FIVE PHASE, HALF STEP
TORQUE-ANGLE CURVE
90 degrees and is equal in amplitude to curve No. 1. For two phases excited, phase one and two, the torque-angle curve becomes larger in amplitude, by the square root of 2, but has been shifted by $\frac{45}{A}$ degrees, marked No. 2 on figure II-9a. The period $A$ is dependent on motor structure and not on excitation current. Thus when half stepping a four phase motor the peak torque varies every other step by the square root of 2.

To full step the five phase motor, one phase on at a time, one can switch the current from phase one to phase two, which will result in a phase shift of $\frac{72}{A}$ degrees in the torque-angle curve from No. 1 to No. 3 on figure II-9B. If one were to energize phase four with a negative current the resulting torque angle curve, No. 2, would produce a phase shift of $\frac{36}{A}$ or a half-step. Thus for half-stepping a five phase motor, one can energize one phase at a time using positive and negative currents, since the geometry of five phase motor is such that ten primary flux density vector directions are possible. Thus half-stepping may be accomplished without any variation in torque. Table II-1 is a list of the phase sequence, peak torque and electrical step angle $A_\theta$ for half-stepping the four phase and five phase motors. Notice that torque-angle curve magnitude varies when half-stepping a four phase motor but can be made constant when half-stepping a five phase motor.

The torque angle curve for a stepping motor then has a magnitude and phase angle which depends upon the number and direction of the phases excited. This is an important consideration because it means that, not only does the amount of current and number of phases energized determine the peak $T_m$ of the torque angle curve, but the
### TABLE II-1

FOUR AND FIVE HALF STEP SEQUENCES

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direction of the phase currents also influences the amplitude of the
torque angle curve and the location $\psi$ of zero torque or the equilibrium
point of the rotor. It will be shown later that the amplitude and
shift in step angle of the torque angle curve plays an important part
in the dynamic behavior of the motor. In general, it will be shown
that the larger the peak of the torque-angle curve, the higher will be
the motor natural frequency and the lower will be its damping ratio
or, in other words, the more oscillatory its behavior will be. Further­
more, the higher the peak of the torque-angle curve, the less will be
the effect of a given amount of friction upon motor performance.

For the four phase motor, full stepping can be accomplished by
energizing one or two phases at a time with a unipolar drive. With a
bipolar drive, that is with the availability of bi-directional current
in the phase windings, up to four phases at a time may be used. Re­
ferring to Figure II-5, the torque produced by energizing phase 3 with
negative current is equal to the torque produced by energizing phase 1
with positive current. Half stepping the four phase motor is accomplish­
ed by energizing one then two phases at a time producing a non-uniform
peak torque between steps, see Table II-1.

For the five phase motor, there are five step per cycle of
the torque-angle curve. If the same number of phases are excited with
a unidirection current at every step; that is, one every time, two
every time, three every time, or four every time, the motor can be
said to be full stepping. If it is desired to half-step the five
phase motor, then it is possible to shift the torque angle curve an
amount equal to one half the full step angle by using positive and
negative currents in one of the phases every other step. Thus it is
possible to half step by energizing two, three, four, or five phases at a time, and in this way produce a half-stepping mode of operation. A similar approach can be used to produce quarter-stepping. This is shown for a five phase motor in Figure II-10 where two phases at a time are excited and in this way, the motor is caused to execute one quarter of a step with each phase switching. Notice that the amplitude of the flux density vector associated with each torque-angle curve is not constant, hence neither is the peak of the torque angle curve. So, in the quarter-stepping process, there has been a change in the amplitude of the torque angle curve as well as one quarter as much shift in the location of the torque angle curve, relative to full stepping, with each step. The percent increase in peak torque decreases as one energized more phases on for any particular sequence. This is illustrated for a different sequence shown in Figure II-11, for a four-five phase sequence. In this way, it can be shown that the larger the number of phases excited, the smaller will be the change in the peak of the torque-angle curve for quarter-stepping behavior. In general, this can be said for a stepping motor for any number of phases. The more phases that are available to be excited, the smaller will be the difference in the peak of the torque-angle curve when these phases are excited to produce a submultiple stepping behavior. Note that for a four phase stepper it is possible to quarter-step but if the phase currents are constant, then the step angles are not uniform and the peak torque is not constant.

In addition, it is possible to show that if the voltage applied to any stator phase or the resistance of any stator phase, or the torque constant of any stator phase is different from those of other
### Flux Density Vectors

![Diagram of flux density vectors]

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**Figure II-10**

Quarter step sequence - two phases on.
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<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>

FIGURE II-11
QUARTER STEP SEQUENCE - 4/5 PHASES ON
stator phases, there will be a shift in the amplitude and angle of the flux density vector from that desired when that particular phase is excited. This occurs because, referring back to Equation II-4, the magnitude of the torque component due to that particular stator phase is different for one phase relative to the others. In actual practice there is a slight shift in the location of the zero point or the equilibrium point of the torque angle curve as well as a change in amplitude when submultiple stepping. This has the effect of causing uneven stepping. In many power amplifiers it is possible to individually adjust the currents to each of the phases. When this is done, this affect can be reduced or eliminated providing that the voltage, resistance, or torque constant does not change over a period of time.

It can be further shown from Equation 4 that the location of the torque angle curve and its peak is determined entirely by the phase currents. It is possible to get a change in the location of the torque angle curve, which can be made as small as desired by continuously varying the current in the stator phases. By so doing, one changes the torque $T_n$ for each of the stator phases in almost a continuous or proportional manner in the stepping process. By using Equation II-4, it is possible to determine the necessary voltage on each of the stator phases in order to obtain a particular torque angle curve location and magnitude. By properly adjusting the currents in each of the phases, it is possible to make the torque angle curve location step through very small increments and the peak of the torque-angle curve $T_m$ be constant. This is the process of micro-stepping. In order to accomplish this with amplifiers of reasonable size and cost, it is necessary to pulse-width modulate the voltage on each of the
stator phases and use a current feedback loop to insure that the current in each of these phases has the desired magnitude. This operates the power transistors in their low loss regions which allows much smaller, lower cost transistors to be used to control a given motor.

As the motor structure changes from three to four to five phases, it is possible to obtain smaller and smaller changes in the location of the torque angle curve for the motor with a fixed current applied to each of the stator phases. This is illustrated in Equation II-4 but it can be shown in a more graphic way. If there are a given number of phases (p) available and each of these phases can have applied to it a current in one direction, a current in the other direction or zero current (3 states), it is possible to show that there are $p^3$ combinations of stator excitation.

If all of these possibilities are explored by a computer simulation, then for each combination, the total peak torque $T_m$ and the location $\psi$ of the zero point of the torque-angle curve can be determined. The results of such a simulation can be represented graphically by a polar plot where the direction of the flux density vector and its magnitude are displayed allowing 360° for one cycle of the torque angle curve. With this arrangement it is possible then to demonstrate all of the equilibrium positions and indicate the peak magnitude of the torque angle curve for any given stator excitation combination. Figures II-12A-12C show several of these, for a three phase, a four phase and a five phase stepping motor. In figures II-12A and II-12B are shown the flux density vector directions and amplitudes for full-stepping, half-stepping and quarter-stepping the three and four phase
FIGURE II-12A

THREE PHASE MOTOR
FLUX DENSITY VECTORS
FIGURE II-12B

FOUR PHASE MOTOR
FLUX DENSITY VECTORS
FIGURE II-12C

FIVE PHASE MOTOR
FLUX DENSITY VECTORS
motors, respectfully. It is clear from figure II-12C that it is possible to establish a larger number of discreet positions, in one cycle of the torque angle curve as the number of phases of the motor is increased to five. In fact eighth-stepping with small variations in step size and peak torque is possible. It is further possible to show that more directions are available for the flux density vector or smaller steps can be made with a single phase excited if an odd number of stator phases are employed in the motor design. This is true because a three phase or a five phase stator winding has the possibility of establishing twice as many directions as there are phases, since it is possible to reverse the current in each of the phase windings to produce a new flux density direction. Whereas, for a four phase stepper, there are really only four possible directions for a single phase excited, regardless of current direction. There are six possible flux density directions for a single phase excited for a three phase stepper and ten for a five phase stepper. One might even consider that relative to the three and five phase steppers a four phase stepping motor is equivalent to a two phase motor with the possibility of current reversal to obtain four different flux density vector directions. This provides a considerable advantage in being able to step the motor through very small angles. This advantage carries through to half and quarter stepping as well. Any motor can be microstepped to obtain very small step angles, however, if small step angles can be obtained with constant phase currents, the control scheme can be much simpler than a microstepping control.
CHAPTER III

MATHEMATICAL MODEL DEVELOPMENT

3-A Equations of Motion

Regardless of the mechanical structure or the driver configuration, the stepping motor is a device which has a set of equilibrium positions equal to the number of steps per revolution. These equilibrium positions are determined by the stator winding excitation. The number of positions per revolution is known as the number of steps per revolution. For a fixed stator excitation pattern there is a torque-angle curve or windup curve about each equilibrium position for the rotor. If a gradually increasing external torque is applied to the rotor, the rotor deflection angle will increase. If the angle from the zero torque position is measured and both positive and negative torques are applied, a torque versus angle relationship results which is a periodic function of angle. There will be a number of locations around the rotation of the motor at which there is zero torque. See Figure III-1. Some of these locations are stable equilibrium points and the others are unstable equilibrium points. The unstable equilibrium points and the stable equilibrium points will alternate. This can be seen by energizing a set of stator phases and turning the rotor by hand. One will feel an increase in torque as one turns the rotor until suddenly the motor appears to run with the torque and jump to a new location. This pattern will repeat itself over and over as the rotor is turned with a fixed stator excitation pattern. Further, for a PM motor the
Figure III-1
Experimental Torque-Angle Curve
maximum torque can be assumed proportional to stator current until the stator phase saturates, see figure III-2.

Starting then with the torque-angle curve of the motor which represents the torque available to restore the motor to its equilibrium position, one may write the mechanical equations which describe the dynamic behavior of the motor. The torques on the motor fall into several categories.

1. Load and motor inertia, \( J = J_L + J_m \), causes an inertial torque.

2. The load and motor mechanical damping, \( B = B_L + B_m \), contribute a damping torque on the motor.

3. There may be a velocity independent load torque \( T_L \). For example, with a feed screw and nut driving a gravity load, there is a torque which is in a given direction regardless of the velocity. The positive direction for \( T_L \) is assumed to be opposite to the positive angle direction.

4. The motor is also required to overcome the friction torque. Friction may be characterized in many ways, but probably the most acceptable description of friction is to break it up into two parts. A high initial stiction torque \( T_s + T_F \) which decreases as soon as the two surfaces are sliding relative to one another. This representation of friction depends on the direction of velocity and is shown in Figure III-3. The starting friction \( T_s + T_F \) will be higher than cumb friction \( T_F \) and both will oppose velocity. This may be represented
Figure III-2
Experimental Torque-Current Curve
Figure III-3
Friction Model
mathematically as in equation III-1.

\[ T_{\text{friction}} = \frac{\ddot{\theta}}{|\dot{\theta}|} \left[ T_f + T_s(u(|\dot{\theta}|) - u(|\dot{\theta}| - \delta)) \right] \]

where \( u(x) = 0, \ x < 0 \)
\[ = 1, \ x > 0 \] (III-1)

These torques can be summed and equated to the inertial torque. The resulting mechanical second order differential equation is non-linear and represents all of the torques applied to the motor.

\[ J\ddot{\theta} = -B\dot{\theta} - T(\theta) - T_L - \frac{\dot{\theta}}{|\dot{\theta}|} \left[ T_f + T_s(u(|\dot{\theta}|) - u(|\dot{\theta}| - \delta)) \right] \]

where
\( u(|\dot{\theta}| - \delta) = \) a unit step occurring when \(|\dot{\theta}| = \delta\)
\( J = \) motor and load inertia
\( B = \) mechanical damping
\( T(\theta) = \) restoring torque due to the stator magnetic field
\( T_L = \) velocity independent load torque
\( T_F = \) coloumb friction
\( T_s = \) stiction torque
\( \delta = \) small velocity \( \rightarrow 0 \)

The torque-angle curve or windup curve of the motor due to the effect of the stator excitation upon the rotor depends on the stator currents. This kind of dependence may manifest itself in several ways. For a PM motor, neglecting eddy currents the magnitude of the torque-angle curve is proportional to current and the torque angle curve is nearly sinusoidal.

\[ T(\theta) = K_T I \sin \theta \] (III-3)

For the purposes of this development it will be assumed that the
stator windings are unsaturated so that the relationship between stator current and the magnitude of the torque-angle curve is a linear one. This is usually true if the stator current is equal to or less than the rated current, see figure III-2. Further it will be assumed that the torque-angle curve is sinusoidal in shape. On the other hand, without any loss in generality, it is perfectly possible to represent the torque angle curve by a Fourier Series. This may be appropriate in the model of variable reluctance (VR) stepping motors since the torque-angle curve tends to be much more non-sinusoidal than that for a permanent magnet stepping motor. Likewise it is also perfectly reasonable to make the coefficients of the Fourier Series dependent upon current. This can be determined by measuring the torque-angle curve for any given motor for a number of stator currents and then representing each of these torque-angle curves by Fourier Series. Using least square error methods, an equation can be developed for the coefficient of the fundamental, and each of harmonics of the resulting Fourier Series. In this way a Fourier Series consisting of a fundamental and several harmonics, in which the amplitude of each would depend upon current, could be used to represent the torque-angle curve of a stepping motor with a non-sinusoidal torque angle curve whose shape was not independent of current magnitude.

Thus, for a single valued nonlinear torque-current relation and non-sinusoidal torque angle curve:

\[ T(\theta) = K_T [f_1(I)\sin A\theta + f_{2s}(I)\sin 2A\theta + f_{2c}(I)\cos 2A\theta] \]

\[ = K_T \sum_{n=1}^{\infty} [f_{ns}(I)\sin nA\theta + f_{nc}(I)\cos nA\theta] \]

(III-4)

In this way it is possible to account for saturation effects and also
the changes in the shape of the torque angle curve which will occur in variable reluctance stepping motors as the current changes. This adds considerable complexity to the model; however, it will not alter the method of analysis that will be developed. The phase plane methods that will be used to describe stepping motor behavior will not be changed by using a more complex torque-angle relationship. A further complexity can also be introduced over that shown in equation III-3. In equation III-3 it is assumed that only one stator phase is excited at a time. If more than one phase is excited, the torque-angle curve that results is the vector sum of the torque-angle curves produced by each of the windings itself. For example if all four phases of the motor are excited at one time, the torque-angle curve which results can be expressed by equation III-5.

\[
T(\theta) = K_T \sum_{n=1}^{4} i_n \sin(A\theta - \frac{(n-1)\pi}{2}) \]

\[
= K_T [i_1 \sin A\theta - i_2 \cos A\theta - i_3 \sin A\theta + i_4 \cos A\theta] \quad (III-5)
\]

\[T_1(\theta) = K_T I \sin A\theta, \quad 1 \text{ phase at a time } i_1 = I, i_2 = i_3 = i_4 = 0\]

\[T_2(\theta) = 2K_T I \sin A\theta, \quad 2 \text{ phases at a time } i_1 = -i_3 = I, i_2 = i_4 = 0\]

\[T_4(\theta) = 2K_T I[\sin A - \cos A\theta], \quad 4 \text{ phases at a time } i_1 = i_2 = I = -i_3 = -i_4\]

\[= 2\sqrt{2} K_T I \sin(A\theta - \frac{\pi}{4})\]

Here each stator current contributes to the torque angle curve. These expressions describe the mechanical part of the stepping motor behavior and the torque produced by stator excitation.

It is even possible to account for eddy current effects in the rotor poles by making the torque-angle curve velocity dependent.\(^{(8)}\)

Thus, several more complex representations for the torque-
angle curve may be used, however, in this analysis it will be assumed:

\[ T(\theta) = K_T i \sin A\theta \]  \hspace{1cm} (III-6)

where

- \( K_T \) = current-torque constant
- \( i \) = stator current
- \( A \) = number of cycles of the \( T-\theta \) curve per rotor revolution
- \( \theta \) = actual shaft rotation in radians

Substituting this expression for \( T(\theta) \) into equation III-2 gives the governing equation for the mechanical model of the motor and is shown as equation III-7.

\[ J\ddot{\theta} + B\dot{\theta} + K_T i \sin A\theta = -T_L - \frac{\dot{\theta}}{|\theta|} [T_f + T_s(u(|\dot{\theta}|) - u(|\dot{\theta}| - \delta))] \]  \hspace{1cm} (III-7)

Stator windings have resistance and inductance by virtue of the fact that the stator windings are wound on an iron core. Further, there are voltages induced in the stator windings due to the relative motion of the permanent magnet rotor and the stator. These voltages, called back electromotive force (BEMF) voltage, depend upon the location of the rotor relative to the stator and its speed and in the case of the VR motor also stator current. In this way it is possible to write for each of the stator phases an equation which relates the voltage applied to the stator current.

The BEMF voltages will in general, display an angle dependence which is nearly the same as the torque-angle curves in shape. Since equation III-5 is the assumed torque-angle curve, the following BEMF voltage equations can be written.
PM motor stator for each phase:

\[ V_{\text{BEMF1}} = K_b \dot{\theta} \sin A \theta \]
\[ V_{\text{BEMF2}} = K_b \dot{\theta} \sin(A \theta - \frac{\pi}{2}) \]
\[ V_{\text{BEMF3}} = K_b \dot{\theta} \sin(A \theta - \pi) \]
\[ V_{\text{BEMF4}} = K_b \dot{\theta} \sin(A \theta - \frac{3\pi}{2}) \]

BEMF voltages

\[ = K_b \dot{\theta} \sin A \theta \]
\[ = -K_b \dot{\theta} \cos A \theta \]
\[ = -K_b \dot{\theta} \sin A \theta \]
\[ = K_b \dot{\theta} \cos A \theta \]

It will be assumed that the driver which is connected to each of the stator windings can be characterized by voltage source with a series resistance. This is done by using a Thevenin equivalent to the stator drive circuit. Many current drivers can be assumed to have this model. A true current source drive will change the stator current nearly instantaneously and maintain it nearly constant regardless of the speed. For this type of drive, equation III-7 describes system behavior completely. However even when the drive is not a true current source, because of the large effective circuit resistance, it will be possible to neglect the inductive effects in the windings. On the other hand it will not be possible to neglect the BEMF generated in the stator windings as this voltage will have a significant effect upon the behavior of the motor at high speeds.

In Figure III-4 an equivalent circuit is developed for each stator phase of the motor shown in Figure II-5. It should be noted that if the torque-angle curve of the motor is non-sinusoidal, the BEMF is also going to be a non-sinusoidal function of speed and angle. Thus it may be necessary to represent sinusoidal terms in the BEMF relations by a Fourier Series. Perhaps in the case of the VR motor, if the rotor is laminated, the BEMF voltages are much less than those in permanent magnet motor and may be neglected except at all but the very highest operating speeds. It should also be noted in the de-
Figure III-4
Stator Equivalent Circuits
velopment of this model the mutual inductance effects between stator windings have been neglected. It has been shown that these effects are small except in single stack VR motors. The chief effect of mutual inductance between stator phases is a short voltage transient which occurs in one of the phases due to another phase being turned on or off. Because they are of short duration their effect on mechanical motion is negligible. The self-inductive effects will be ignored in the initial development of this model, because of the very large driver amplifier resistance. It can also be assumed that the mutual inductance effects in the PM motor can be neglected for the same reason. Thus as long as the driver consists of a voltage source and a very large series resistance, the mutual as well as the self-inductive effects will be ignored in the development of this model. In fact, as long as the motor behavior is being described at rates equal to or less than the stop-start rates, it is probably safe to state that even the BEMF effects can be neglected. However, when describing the motor at slewing speeds, this is no longer a valid assumption as the BEMF can become substantial, even with a current drive. This can be seen by observing the currents in the stator windings during stepping. At low rates these currents appear to be nearly square waves. At higher rates approaching slewing, the effects of BEMF become more pronounced. Even though the inductive time constant is still small compared to the stator winding current on-time, the stator currents are no longer square waves. For this reason the BEMF cannot be neglected even though the inductive time constant is still small compared to current on-time.

In case of the variable reluctance motor, the peak of the
torque-angle curve will tend to be proportional to the square of the current,\(^{(9)}\) See appendix A. Consequently the linear torque-current relationship assumed for the PM stepping motor will not apply to the variable reluctance motor. On the other hand this increased complexity can be easily handled within the model as it does not change the order of the model, it simply means that the current-torque relationship is not a linear one.

Referring to the ideal element model of Figure III-4 for the stepping motor windings, it should be noted that the stator windings may be excited in four basic ways by a drive circuit. In general, stator winding drivers consist of fast switching power transistors. That is, the voltage applied to the windings is switched by means of these power transistors from zero to some magnitude. However, it is more complex than this. There are several possibilities for the switching procedure:

1) The voltage source connected to a given winding may be connected or disconnected. That is, the voltage source may be connected to the winding during the excitation of that winding and will be disconnected during the non-excitation period. When it is disconnected, no current flows on that particular winding. This situation is representative of most transistor driver circuits and can be used when running with one or two phases on.

2) The voltage across the winding will be shifted from a positive voltage to a negative voltage. This causes a current to flow in the winding at all times. That is, the voltage source in the model may be plus E or minus E.
This is commonly used when all four phases of a PM motor are excited.

3) The transistor drive for the winding may have a inductive voltage protection or power return diode which is connected to allow currents to flow around the winding when the winding polarity is opposite to the normal polarity. In this case when the driver transistor is on, the winding is connected to the voltage source. When the transistor is off, the winding is short circuited by the diode for voltages of reversed polarity caused by the stator winding inductance or BEMF. Mathematically, depending on diode location, this situation can be identical to case (4).

4) The source voltage for a winding circuit may be switched from zero to E. The implication here is that the winding has a voltage E applied to it or it is short circuited.

First looking at the cases where the unenergized windings are open circuited, there is no current flow in unenergized windings due to BEMF. If all four stator phases are energized simultaneously then the effect of the BEMF is to increase the motor damping. If only a single phase or two phases are energized simultaneously, the effect of the BEMF is to change the shape of the torque angle curve with speed. Thus these two cases are considered further.

The voltage applied to the nth phase $V_n$ can be divided by motor and driver resistance. So that the equations of Figure III-4 which have been developed for the motor can be written as:
\[
\begin{align*}
\frac{V_n}{R_d+R_s} &= i_n + \frac{L_s}{R_d+R_s} \frac{di_n}{dt} - \frac{K_b \dot{\theta}}{R_s+R_d} \sin(A \theta - \frac{(n-1)\pi}{2}) \quad n = 1, 2, 3, 4 \\
&\text{(III-9)}
\end{align*}
\]

Since \(L_s/(R_d+R_s)\) is small the inductive effect will be neglected so that:

\[
\begin{align*}
i_n &= \frac{V_n}{R_d+R_s} + \frac{K_b \dot{\theta}}{R_s+R_d} \sin(A \theta - \frac{(n-1)\pi}{2}) \quad n = 1, 2, 3, 4 \\
&\text{(III-10)}
\end{align*}
\]

This is combined with the torque equation 6 to yield

\[
\begin{align*}
T(\theta) &= K_T \sum_{n=1}^{4} i_n \sin(A \theta - \frac{(n-1)\pi}{2}) \\
&= K_T \sum_{n=1}^{4} \left[ \frac{V_n}{R_d+R_s} \sin(A \theta - \frac{(n-1)\pi}{2}) + \frac{K_b \dot{\theta}}{R_s+R_d} \sin^2(A \theta - \frac{(n-1)\pi}{2}) \right] \\
&\text{(III-11)}
\end{align*}
\]

There are three ways of driving the motor for full step operation, one, two, and four phases at a time. These three cases are now considered for the off phases open circuited, (a) For 1 phase excitation, the equivalent circuit is shown in figure III-5.

\[
i_2 = i_3 = i_4 = 0
\]

\[
i_1 = \frac{V}{R_s+R_d} + \frac{K_b \dot{\theta}}{R_s+R_d} \sin(A \theta)
\]

\[
T(\theta) = \frac{K_T V}{R_d-R_s} \sin(A \theta) + \frac{K_T K_b \dot{\theta}}{R_s+R_d} \sin^2(A \theta) \\
&\text{(III-12)}
\]

(b) For 2 phase excitation, the equivalent circuit is shown in figure III-6.

\[
i_2 = i_4 = 0
\]
Figure III-5
One Phase Excited
Equivalent Circuit

Figure III-6
Two Phases Excited
Equivalent Circuit

Figure III-7
Four Phases Excited
Equivalent Circuit
\[ i_1 = \frac{V}{R_s + R_d} + \frac{K_b \dot{\theta}}{R_s + R_d} \sin(A\theta) \]

\[ i_3 = \frac{-V}{R_s + R_d} + \frac{K_b \dot{\theta}}{R_s + R_d} \sin(A\theta - \pi) \]

\[ T(\theta) = K_T i_1 \sin(A\theta) + K_T i_3 \sin(A\theta - \pi) \]

\[ T(\theta) = K_T \left[ \frac{V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \right] \sin(A\theta) \]

\[ + K_T \left[ \frac{-V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \right] \sin(A\theta - \pi) \]

but \( \sin(A\theta - \pi) = -\sin(A\theta) \)

\[ T(\theta) = \frac{2K_T V}{R_d + R_s} \sin(A\theta) + \frac{2K_T K_b \dot{\theta}}{R_d + R_s} \sin^2(A\theta) \quad \text{(III-13)} \]

(c) For 4 phase excitation:

\[ i_1 = \frac{V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta) \]

\[ i_2 = \frac{+V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta - \frac{\pi}{2}) \]

\[ i_3 = \frac{-V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta - \pi) \]

\[ i_4 = \frac{-V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta - \frac{3\pi}{2}) \]

\[ T(\theta) = \frac{VK_T}{R_d + R_s} \left[ \sin(A\theta) + \sin(A\theta - \frac{\pi}{2}) - \sin(A\theta - \pi) - \sin(A\theta - \frac{3\pi}{2}) \right] \]

\[ + \frac{K_b K_T \dot{\theta}}{R_d + R_s} \left[ \sin^2(A\theta) + \sin^2(A\theta - \frac{\pi}{2}) + \sin^2(A\theta - \pi) + \sin^2(A\theta - \frac{3\pi}{2}) \right] \]

but \( \sin(A\theta - \pi) = -\sin(A\theta) \)

\[ \sin(A\theta - \frac{3\pi}{2}) = -\sin(A\theta - \frac{\pi}{2}) = +\cos(A\theta) \]

\[ T(\theta) = \frac{VK_T}{R_d + R_s} \left[ 2 \sin(A\theta) - 2 \cos(A\theta) \right] + \frac{2K_b K_T \dot{\theta}}{R_d + R_s} \]
\[
\sin(A\theta) - \cos(A\theta) = \sqrt{2} \sin(A\theta - \frac{\pi}{4})
\]

\[
T(\theta) = \frac{2\sqrt{2} V K_T}{R_d + R_s} \sin(A\theta - \frac{\pi}{4}) + \frac{2K_b K_T \dot{\theta}}{R_d + R_s}
\]  

(III-14)

This results in the following set of system equations for the three cases based on the torque equation.

\[
J\ddot{\theta} + B\dot{\theta} + T(\theta) = -T_L - [T_f + T_s(u(\dot{\theta}) - u(\dot{\theta} - \delta))] \frac{\dot{\theta}}{|\dot{\theta}|}
\]

(a) \[J\ddot{\theta} + B\dot{\theta} + \frac{K_b K_T \dot{\theta}}{R_d + R_s} \sin^2(A\theta) + \frac{K_T V}{R_d + R_s} \sin(A\theta) = ... 1 \text{ phase} \]  

(III-15)

(b) \[J\ddot{\theta} + B\dot{\theta} + \frac{2K_b K_T \dot{\theta}}{R_d + R_s} \sin^2(A\theta) + \frac{2K_T V}{R_d + R_s} \sin(A\theta) = ... 2 \text{ phase} \]  

(III-16)

(c) \[J\ddot{\theta} + B\dot{\theta} + \frac{2K_b K_T \dot{\theta}}{R_d + R_s} + \frac{2\sqrt{2} V K_T}{R_d + R_s} \sin(A\theta - \frac{\pi}{4}) = ... 4 \text{ phases} \]  

(III-17)

If the unenergized windings are all short circuited, the equivalent circuit, for one phase excited is shown in figure III-7.

one phase:

\[
V_1 = V \quad V_2 = V_3 = V_4 = 0
\]

\[
i_1 = \frac{V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta)
\]

\[
i_2 = \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta - \frac{\pi}{2})
\]

\[
i_3 = \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta - \pi)
\]

\[
i_4 = \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta - \frac{3\pi}{2})
\]

\[
T(\theta) = K_T \left[\frac{V}{R_d + R_s} + \frac{K_b \dot{\theta}}{R_d + R_s} \sin(A\theta)\right] \sin(A\theta) + \frac{K_T K_b \dot{\theta}}{R_d + R_s} \left[\sin^2(A\theta - \frac{\pi}{2})
\]

+ \sin^2(A\theta - \pi) + \sin^2(A\theta - \frac{3\pi}{2})\]
\[ T(\theta) = \frac{K_{r}V}{R_{d}+R_{s}} \sin(A\theta) + \frac{2K_{r}K_{b}\dot{\theta}}{R_{d}+R_{s}} \]  

(III-18)

for 2 phases excited, the others short circuited,

\[ T(\theta) = \frac{2K_{r}V}{R_{d}+R_{s}} \sin(A\theta) + \frac{2K_{r}K_{b}\dot{\theta}}{R_{d}+R_{s}} \]  

(III-19)

for 4 phases excited:

\[ T(\theta) = \frac{2\sqrt{2}K_{r}V}{R_{d}+R_{s}} \sin(A\theta - \frac{\pi}{4}) + \frac{2K_{r}K_{b}\dot{\theta}}{R_{d}+R_{s}} \]  

(III-20)

Thus, there are two different system equation forms:

\[ \ddot{J}_{\theta} + (B + \frac{2K_{d}K_{r}}{R_{d}+R_{s}})\dot{\theta} + T_{m} \sin(A\theta) = -T_{L} - T_{f} + ... \]  

(III-21)

\[ \ddot{J}_{\theta} + [B + \frac{KK_{d}K_{r}}{R_{d}+R_{s}} \sin^{2}(A\theta)]\dot{\theta} + T_{m} \sin(A\theta) = -T_{L} - T_{f} +... \]  

(III-22)

In the case of the diode protection circuit it is necessary to represent the unenergized winding situation by an equivalent circuit as shown below.

![Figure III-8](image)

In the case of the diode protection circuit it is necessary to represent the unenergized winding situation by an equivalent circuit as shown below.

Figure III-8

Here, when the winding is deenergized, the BEMF voltage will cause a current to flow when it is of one polarity and no current to flow if it is the other polarity. As before, the large series resistance inhibits the inductive effects, thus the system model will still be second order, however, the BEMF effects will be polarity
dependent. Thus, with the exception of the case with diode protection, two forms of torque equations apply. These are shown in equations III-21 and III-22.

If the unenergized phases are shorted, the effective damping of the motor is increased and the model is simpler, also it does not matter if 1, 2, or 4 phases are energized as far as the form of damping term is concerned.

For a PM motor with phase one energized, the electrical circuit equation is:

\[ V = L \frac{di}{dt} + R_T i - K_b \dot{\theta} \sin A \theta \]  

or,

\[ \frac{V}{R_T} = L \frac{di}{dt} + i - \frac{K_b}{R_T} \dot{\theta} \sin A \theta \]  

where

- \( i \) = current in the energized phase winding
- \( V \) = total driver voltage across motor phase and series resistance
- \( L \) = motor inductance
- \( K_b \) = BEMF constant in volts/rad/sec
- \( R_T \) = motor and driver resistance

If the \( L/R_T \) time constant of the windings is sufficiently small, the \( \frac{L}{R_T} \frac{di}{dt} \) term in equation III-24 can be neglected. Thus assuming a rapid current rise time, the current equation becomes

\[ i = \frac{V}{R_T} + \frac{K_b}{R_T} \dot{\theta} \sin A \theta \]  

Substituting this into equation III-7 and defining \( A \theta = \theta \) yields the total governing equation for the stepping motor with one phase energized.
\[
\frac{J}{A} \ddot{\theta} + \left[ \frac{B}{A} + \frac{K_T K_b}{A R_T} \sin^2 \theta \right] \dot{\theta} + \frac{K_T V}{R_T} \sin \theta =
- T_L - T_F \frac{\dot{\theta}}{|\dot{\theta}|} - T_s \frac{\dot{\theta}}{|\dot{\theta}|} \left[ u(\dot{\theta}) - u(\dot{\theta} - \bar{\delta}) \right]
\] (III-26)

It must be remembered that this equation is only valid for a single phase drive with a small \( L/R_T \) and with no saturation of the stator iron.

In order to put equation III-26 into the most useful form, it is desirable to make the entire equation dimensionless. This permits the model to be dependent on only a few dimensionless parameters instead of many dimensional ones and permits a global picture of stepping motor behavior to be more easily discerned.

To do this, the entire equation is divided through by \( K_T V/R_T \), which is also equivalent to the maximum static holding torque. Also defined are a number of dimensionless parameters:

\[
\frac{J R_T}{A K_T V} \ddot{\theta} + \left[ \frac{B R_T}{K_T AV} + \frac{K_b}{A V} \sin^2 \theta \right] \dot{\theta} + \sin \theta =
- \frac{T_L R_T}{K_T V} - \frac{T_F R_T}{K_T V} \frac{\dot{\theta}}{|\dot{\theta}|} - \frac{T_s R_T}{K_T V} \frac{\dot{\theta}}{|\dot{\theta}|} \left[ u(\dot{\theta}) - u(\dot{\theta} - \bar{\delta}) \right]
\] (III-27)

where \( \bar{\delta} = \frac{\delta}{\omega_n A} \)

let:
\[
\frac{T_L R_T}{K_T V} \equiv \bar{T}_L = \text{dimensionless velocity independent load torque}
\] (III-28)
\[
\frac{T_F R_T}{K_T V} \equiv \bar{T}_f = \text{dimensionless coloumb friction torque}
\] (III-29)
\[
\frac{T_s R_T}{K_T V} \equiv \bar{T}_s = \text{dimensionless stiction torque}
\] (III-30)
If $\bar{T}_f$, $\bar{T}_s$, and $\bar{T}_L$ are set equal to zero and equation III-27 is linearized about $\theta = 0$, for small changes $\Delta \theta$, the resulting equation is in the form of a second order linear differential equation.

$$\frac{JR_T}{AK_T} \frac{d^2 \Delta \theta}{dt^2} + \frac{BR_T}{AK_TV} \frac{d \Delta \theta}{dt} + \Delta \theta = 0 \quad (III-31)$$

Since for small changes from $\theta = 0$

$$\sin \theta \rightarrow \theta$$

$$\sin^2 \theta \rightarrow 0 \text{ (small compared to } \theta)$$

Equation 31 can now be written in standard second order form, i.e.,

$$\frac{1}{\omega_n^2} \ddot{x} + 2\zeta \frac{1}{\omega_n} \dot{x} + x = 0 \quad \text{where } x = \Delta \theta \quad (III-32)$$

Making equations III-31 and III-32 equivalent yields the natural frequency of the system

$$\omega_n = \sqrt[2]{\frac{K_{TVA}}{JR_T}} \quad (III-33)$$

The dimensionless damping ratio can be found

$$\frac{2\zeta}{\omega_n} = \frac{BR_T}{K_{TVA}}$$

or

$$\zeta = \frac{BR_T}{2K_{TVA}} \sqrt[2]{\frac{K_{TVA}}{JR_T}} = \frac{B}{2} \sqrt[2]{\frac{R_T}{JK_{TVA}}} \quad (III-34)$$

$\zeta$, the mechanical damping ratio term, is dimensionless. Since the equation was linearized to make this analogy, a dimensionless parameter must be defined to take care of the BEMF term. Let

$$\frac{\gamma}{\omega_n} = \frac{K_b}{AV}$$
Then
\[ \gamma = K_b \sqrt[3]{\frac{K_T}{AVJR_T}} \quad (III-35) \]

Upon checking the units, \( \gamma \) is found to be dimensionless and represents the BEMF term. Thus the entire equation has been written in terms of only six numbers. In order to make the entire equation dimensionless, it is necessary to eliminate the time dimension. This is done by defining a new dimensionless time, \( \tau \), which is dependent upon the natural frequency of the system, \( \omega_n \).

\[ \tau \equiv \omega_n t \quad (III-36) \]

By substituting equations III-33 through III-36 back into equation III-27, a new, dimensionless equation is created which depends on only five dimensionless motor parameters. The time scale depends on the system natural frequency.

\[ \frac{d^2 \phi}{d\tau^2} + \left[ 2 \zeta + \gamma \sin^2 \phi \right] \frac{d\phi}{d\tau} + \sin \phi = - \bar{T}_L - \bar{T}_f \frac{\ddot{\phi}}{\dot{\phi}} - \bar{T}_S \frac{\ddot{\phi}}{\dot{\phi}} \left[ u(\dot{\phi}) - u(\dot{\phi} - \ddot{\phi}) \right] \quad (III-37) \]

This equation makes the results of the analysis much easier to generalize since the solutions depend on only five dimensionless parameters, \( \zeta, \gamma, \bar{T}_L, \bar{T}_S \) and \( \bar{T}_f \), instead of ten dimensional parameters. Since it is second order, only two initial conditions must be specified, \( \phi \) and \( \dot{\phi} \), to determine a unique solution to the equation.

The foregoing development is predicated on two assumptions. The first is that the motor inductive time constant is small, secondly that the motor iron is not saturated. If the second assumption were not true, then in equation III-3, \( K_T I \) would be a nonlinear function of
I. The shape of the torque angle curve is sinusoidal, however, any periodic shape is acceptable (for example, a Fourier Series). That series would replace the \( \sin \theta \) and \( \sin^2 \theta \) terms in equation III-37. The model can be made considerably more complex, however, it will be shown that for most purposes this model is sufficiently accurate.

3-B Phase Plane Methods

The state space has been used since the early 1940's to give an overall picture of nonlinear system behavior. Several investigators have used it to study the stepping motor\(^{10-17}\). It is used to display any number of solutions to a differential equation, be it linear or nonlinear, and graphically shows global system behavior. It is also used as a very simple method of determining nonlinear transient behavior in a system. For a second order system, the state space is a plane with two coordinate axes. If the axes are velocity and position, the state plane is known as a phase plane. Any given point in the phase plane represents the two initial conditions for a second order system. There is a unique solution, consisting of one and only one trajectory in the phase plane, that passes through that point.

3-B.1 Simple Phase Plane Example

Consider a simple mechanical system consisting of a pendulum.

![Figure III-9](image-url)
FIGURE III-10
TRANSLATION FROM THE TIME RESPONSE TO THE PHASE PLANE
The system differential equation is:
\[ m r^2 \ddot{\theta} = -m g r \sin \theta - B \dot{\theta} \]  
(III-38)

The following observations can be made:

a. If the pendulum is deflected and released it will oscillate and finally come to rest. Its response is given by the solution to the differential equation for an initial pendulum displacement.

\[ m r^2 \ddot{\theta} + B \dot{\theta} + m g r \sin \theta = 0 \quad \dot{\theta}(0) = 0 \quad \theta(0) \neq 0 \]  
(III-39)

For small damping this will be a spiral in the phase plane.

b. If the pendulum is released at \( \theta = 0 \) with an initial velocity, the initial conditions are \( \theta(0) = 0 \) and \( \dot{\theta}(0) \neq 0 \). The differential equation solution can be found and again it will oscillate and come to rest.

c. A number of \( \theta(0), \dot{\theta}(0) \) combinations can be selected and the time response of the system can be plotted, however, it is hard to get much insight into overall behavior.

If one were to plot the displacement time response with the initial conditions \( \theta(0), \dot{\theta}(0) = 0 \), for the pendulum the results would look like those shown in figure III-10. Similarly the velocity-time response could be plotted, see figure III-10, note that one half cycle of the velocity has a 90° phase shift relative to the position response as would be expected. The phase plane plot of velocity versus position can now be derived from these two time responses. Starting with the initial condition, zero velocity for an initial position marked as point 1, these coordinates can be translated to the phase plane. Continuing with the given marked points one can translate the entire
response onto the phase plane. These points were taken as the zero crossings, but if they would have been taken at equal intervals of time, then timing marks would also been represented on the phase plane.

One can use the phase plane to plot the pendulum responses for any set of $\theta(0), \dot{\theta}(0)$ to get an overall picture of response.

a. Let $\dot{\theta}(t) = \omega(t)$ be one variable, $\theta(t)$ another, then $\frac{d\theta}{dt} = \omega(t)$.

The system differential equation becomes:

$$\frac{d\omega(t)}{dt} = -\frac{B}{mr^2} \omega(t) - \frac{g}{r} \sin \theta(t)$$

(III-40)

Thus for a given mass $m$ and radius $r$, $\frac{d\omega}{dt}, \frac{d\theta}{dt}$ depend only on $\omega, \theta$. Using these two equations, for any point in the $\omega$ versus $\theta$ plane one can find the values of $\frac{d\omega}{dt}$ and $\frac{d\theta}{dt}$.

b. Likewise a solution to the differential equation for a given $\theta(0), \omega(0)$ will have a path in this plane called a trajectory which proceeds from the $\theta(0), \omega(0)$ point in the plane to $\omega = \theta = 0$.

c. Further, the tangent to this path or trajectory has a slope where

$$s = \frac{d\omega}{d\theta} = \frac{\frac{d\omega}{dt}}{\frac{d\theta}{dt}} = \frac{-\frac{B}{mr^2} \omega(t) - \frac{g}{r} \sin \theta(t)}{\omega(t)}$$

(III-41)

Notice every point in the $\omega-\theta$ plane has only one slope value except where $\frac{d\omega}{d\theta} = \frac{0}{0}$

d. These points where $\frac{d\omega}{d\theta} = \frac{0}{0}$ are called singularities and they occur when, $\frac{d\omega}{dt} = \frac{d\theta}{dt} = 0$. In this case, where;

$$\omega = 0, \sin \theta = 0$$

that is $\theta = 0, \pm \pi, \pm 2\pi, \pm 3\pi ...$
Notice when $\omega=0$, $\theta=0$ the pendulum is at rest and if deflected slightly will return to $\omega=0$, $\theta=0$ thus one can say $\omega=0$, $\theta=0$ is a stable equilibrium point.

When $\omega=0$, $\theta=\pi$ the pendulum is at rest but it is upside down. If it is deflected slightly it will swing down and away from this point thus this is an unstable equilibrium point.

In the same way $\omega=0$, $\theta=0$, $\pm 2\pi$, $\pm 4\pi$ ... are stable equilibrium points and $\theta = \pm \pi$, $\pm 3\pi$ ... are unstable equilibrium points. This can be shown in the phase plane.

As long as pendulum initial position for $\omega=0$ satisfies $|\theta| < \pi$ it will return to $\omega=0$, $\theta=0$. If the initial velocity and displacement is just right one can make the pendulum stand upside down. Thus the trajectory goes to $\omega=0$, $\theta=\pm \pi$, $\pm 3\pi$ see Figure III-12. With a particular initial positive velocity, $\omega$, one trajectory goes to $\pi$, $0$ in Figure III-12. With an initial negative velocity, $\omega$, one trajectory goes to $-\pi$, $0$ in Figure III-12. These trajectories are called separatrices because they separate those trajectories that go to zero and those that go to $+2\pi$ or $-2\pi$ respectively.
f. If the pendulum is disturbed from an unstable point it will go to a stable equilibrium point.

g. Given initial conditions $\omega(0), \theta(0)$ between two separatrices, the trajectory will go to the stable singularity bounded by those separatrix. Thus the pendulum will go to $\omega=\theta=0$ when started between separatrix 1 and 2.

h. If in setting an initial condition a separatrix is crossed then the pendulum will go to another stable singularity. If the right separatrix 1 is crossed then the trajectory will go to $\omega=0, \theta=2\pi$. The pendulum will have gained one revolution. Once the pendulum is in motion it will not cross a separatrix unless it is distributed by an external force.

i. Separatrices are like ridges, stable singularities like holes. The trajectory follows a path that a ball will follow as it goes into the hole in such a topography. If the ball crosses a ridge the trajectory will go to another hole.

The simple pendulum system is analogous to the stepping motor. It also illustrates some of the basic notions of the phase plane.
a. Starting with any initial conditions $\omega(0), \theta(0)$ one gets a unique path in the $\omega$ vs $\theta$ plane called a "trajectory"

b. Singularities occur where the slope of a trajectory is not defined. For the pendulum they all occur when pendulum is at rest, and vertical either the mass is down or up (stable and unstable).

c. Separatrices separate the plane into regions where all trajectories within that region go to one singularity.

d. Putting time markers on plot will relate the trajectories to time responses. Thus time responses could be plotted from the phase plane trajectories and vice versa.

3-C Phase Plane for the Stepping Motor

The solutions to the dimensionless stepping motor differential equation III-37 can be plotted in the same way as the pendulum response. If dimensionless motor velocity $\frac{d\theta}{dt}$ is plotted versus dimensionless motor position $\theta$, a global picture results from a phase portrait showing many solutions.

For the stepping motor, like the pendulum, there are an infinite number of singularities in the phase plane, both stable and unstable. The stability of each singularity is defined by the torque-angle curve of the motor. Think of the curve as being like a spring near the singularity. For a stable singularity, the torque-angle curve crosses the zero torque point with a positive slope (local positive spring rate). For an unstable singularity the slope is negative (local negative spring rate).

The singularities are places in the phase plane where the
motor can be at rest. This occurs when $\ddot{\theta} = \dot{\theta} = 0$. For no friction when $\ddot{\theta} = \dot{\theta} = 0$, Equation III-37 becomes

$$\sin \theta = -T_L$$

(III-42)

or

$$\theta = \sin^{-1}(-T_L) \pm 2\pi m$$

(III-43)

and

$$\theta = \pi - \sin^{-1}(-T_L) \pm 2\pi m$$

(III-44)

where $m = 0, 1, 2, 3, \ldots$

If the equations are linearized about each of these singularities, it can be shown that the singularities of equation III-43 are stable focal points to which trajectories spiral and all other singularities equation 44, are unstable saddle points. That is the linearized system differential equation near stable focal points

$$\ddot{\theta} + 2\zeta \dot{\theta} + \theta = 0$$

for small motions

(III-45)

near saddle points

$$\ddot{\theta} + 2\zeta \dot{\theta} - \theta = 0$$

for small motions

(III-46)

If the motor is placed on the unstable singularity it will remain there until disturbed. This can be visualized more easily by thinking of the pendulum. The stable singularity corresponds to the situation when the pendulum remains at the bottom of its swing and the unstable singularity corresponds to the behavior of the pendulum at the top of its swing when it is inverted. When the pendulum is inverted it will stay, if balanced very carefully because of friction, but upon the least disturbance it will fall to the stable singularity. The same situation occurs for the stepping motor at a saddle point.

By placing the trajectories for several initial conditions on the same phase plane, a "phase portrait" is obtained (Figure III-13).
Since the trajectories will not cross, only a few of the infinite number of possible trajectories are needed in this portrait. For the stepping motor, the shape of this portrait depends on five parameters, \( z, y, \bar{T}_f, \bar{T}_s, \) and \( \bar{T}_L \). In this way the response of the motor is much easier to visualize. Figure III-13 has the following characteristics.

1. It is a phase plane portrait for \( z = .05, y = .01, \bar{T}_f = \bar{T}_s = \bar{T}_L = 0 \)
2. It is a picture of \( \frac{d\theta}{dt} \) vs \( \theta \) for a finite number of \( \frac{d\theta(0)}{dt}, \theta(0) \) pairs.
3. Timing markers occur every \( \Delta t = .2 \), boxes for separatrices and crosses for other trajectories.
4. Stable singularities are at \( 0, \pm 2\pi, \pm 4\pi \ldots \)
   Unstable singularities are at \( \pm \pi, \pm 3\pi, \pm 5\pi \ldots \)
5. Separatrices divide the plane up into regions with one stable singularly inside each region to which trajectories in that region spiral. The time markers on separatrices are small boxes .2 dimensionless seconds apart.
6. When \( \frac{d\theta}{dt} > 0 \), \( \theta \) is increasing thus the trajectories move to right as \( t \) increases in the top half plane and to the left in the bottom half plane, where \( \frac{d\theta}{dt} < 0 \).
7. In the top half plane, the motor will accelerate whenever the slope \( \frac{d\dot{\theta}}{d\theta} \) is positive and slow down where it is negative. Since \( \frac{d\theta}{dt} = \dot{\theta} \) is positive in the top half plane, then if \( \frac{d\dot{\theta}}{d\theta} = \frac{d\dot{\theta}}{d\theta} \) is positive \( \frac{d\dot{\theta}}{dt} = \ddot{\theta} \) must be positive. In the bottom half plane the opposite is true.
8. Any trajectory initially on one side of a separatrix must always remain on that side. It is important to note that the only reason for stepping motor failure (i.e., not executing the re-
Figure III-13
Stepping Motor Phase Portrait
\( \zeta = 0.05 \gamma = 0.01 T_L = 0. T_F = 0. \)
quired number of steps) is a crossing of a separatrix during stepping.

10. The vertical scale for all phase planes is rad/dimensionless second and the horizontal scale is radians (it will be shown that stepping causes the trajectories to cross a separatrix).

Figures III-14 through III-25 show phase planes for various values of \( \zeta, \gamma, \bar{T}_f, \bar{T}_S \) and \( \bar{T}_L \). Each parameter will be varied separately to show the effect of that parameter on the shape of the trajectories in the phase plane.

3-C.1 Effect of Damping

Figures III-14 through III-16 show the effect of increasing mechanical damping for a typical PM motor. From these phase plane portraits the following observations about the effect of motor damping ratio \( \zeta \) can be made:

a. As \( \zeta \) increases separatrices are farther apart. This means that to run at any given speed as \( \zeta \) increases must cross fewer separatrices.

b. Trajectories spiral into stable singularities more rapidly as \( \zeta \) increases. Thus single step oscillations decrease faster as \( \zeta \) increases.

c. The region of the top half of the phase plane where the motor can accelerate, (positive trajectory slope, where \( \frac{d\dot{\theta}}{d\theta} > 0 \)), is smaller as \( \zeta \) increases.

d. The maximum speed \( \dot{\theta}_{\text{max}} \) at which acceleration can occur (max velocity for which \( \frac{d\dot{\theta}}{d\theta} > 0 \)) becomes smaller as \( \zeta \) increases. Thus slewing speed decreases as \( \zeta \) increases. Thus increasing damping ratio will improve single step response but will reduce maximum slewing speed.
Figure III-14
Phase Portrait, $\zeta = .01$
Figure III-15
Phase Portrait, $\zeta = 0.1$
Figure III-16
Phase Portrait, $\zeta = .2$
3-C.2 Effect of Load

The velocity independent load torque $T_L$ can be investigated using the phase portraits. Figures III-17 through III-20 show the effects of increases in load torque $T_L$.

a. The separatrices in the bottom half plane crowd together while those in the top half plane spread apart. Thus more separatrices are crossed when moving with the load torque for a given absolute velocity.

b. For $\zeta = .05$ and $\gamma = .01$ at $T_L = .14$ the separatrices pinch off and a region on the phase plane exists where the trajectories go toward $-\infty$. If the system enters one of these regions, it will run backwards under load with an oscillating speed. It should be noted that as $T_L$ (the velocity independent load torque) is increased, the separatrices move down closer to the equilibrium points, and at about $T_L = .14$ for $\zeta = .05 \gamma = .01$, the separatrices actually circle around the equilibrium position. The shaded area in Figure III-20 is an area between two separatrices which does not contain a stable singularity except at $-\infty$. Any time the motor is placed in this area, the motor will follow that trajectory. However, any trajectory in this area leads to a singularity at $-\infty$ in the phase plane. This means that the motor can actually be driven backwards under a load which is much less than the maximum torque it can deliver. This is very apt to happen when driving a gravity load.

c. The trajectories spiral into the stable singularities faster as $T_L$ increases.

d. For a torque opposing motion in the positive $\omega$ direction, the
Figure III-17
Phase Portrait, $T_L = .1$
Figure III-18
Phase Portrait, $\bar{T}_L = .2$
Figure III-19
Phase Portrait, $T_L = 0.3$
Figure III-20
Phase Portrait, $T_L = .4$
stable singularities move to the left toward the unstable singularities.
e. The path in the phase plane which leads to the stable singularity becomes smaller.
f. The maximum speed at which the motor can accelerate against a load torque becomes smaller so that slewing speed decreases.

3-C.3 Effect of Friction

To see the effect of a constant load torque and friction upon the singularities in the phase plane it is useful to plot the torque-angle curve and the load torque plus friction versus angle, see Figure III-21. The angles at which the load torque plus friction equals the torque-angle curve values are the equilibrium points. As load torque increases these move together and to the left as shown in equations III-43 and III-44. Since the load torque plus friction has one magnitude in one direction and another in the opposite direction ($\bar{T}_L + \bar{T}_f + \bar{T}_S$ when $\hat{\theta} > 0$ and $\bar{T}_L - \bar{T}_f - \bar{T}_S$ when $\hat{\theta} < 0$) this establishes a band of angles about each equilibrium point. With friction there will be a band of singularities or "dead-zone", when $\hat{\theta} = 0$, for which there will be no motion. Because of the stiction, the band will be wider for velocity increasing in magnitude from zero and when the velocity is decreasing towards zero. Since the system is at rest when $\dot{\theta}=0$, the effect of stiction is to increase the dead zone for trajectories at the $\dot{\theta}=0$ axis near an equilibrium point. The smaller dead zone due to $\bar{T}_F$ alone will not be seen in the phase plane. Thus $\bar{T}_f + \bar{T}_S$ will determine the dead zone. For nonzero velocity $\bar{T}_S$ has no effect. For this reason $\bar{T}_S$ will not effect phase plane shape, it will only increase the dead zone.
DEAD ZONE DUE TO FRICTION

INCREASED DEAD ZONE DUE TO STICION

Figure III-21
Torque-Angle Curve and Dead Zone
Figures III-22 through III-24 show the effect of increasing motor and load column friction $\bar{T}_f$. These effects can be summarized:

a. Increasing friction $\bar{T}_f$ causes the separatrices to be further apart. In this respect increased friction has an effect similar to increased mechanical damping.

b. Increasing $\bar{T}_f$ decreases the maximum speed at which motor can accelerate and so reduces maximum slewing speed.

c. Friction also causes a dead zone or region about all the equilibrium points in which the motor can sit without motion. The width of the zone is

$$DZ = \pm \sin^{-1}(\bar{T}_f + \bar{T}_s) \quad (III-48)$$

Figure III-24 illustrates this deadzone effect dramatically where it is seen that the deadzone appears about all the equilibrium points. This deadzone decreases motor stepping accuracy because the final motor position about the equilibrium points will be anywhere within the band defined by equation III-48. Thus load friction can strongly influence stepping accuracy. Other factors determined by motor manufacturing tolerances effect accuracy, however $\bar{T}_f + \bar{T}_s$ is usually the dominating effect. (18)

3-D Summary

The effect of any motor parameter on the dimensionless parameter which describe motor behavior is summarized in Figure III-25. This chart is also useful to predict the effect of a postulated change in the system.

The phase portraits shown earlier demonstrated that the dimensionless parameters could be related to motor behavior. Specifically
Figure III-22
Phase Portrait, $T_r = .2$
Figure III-23
Phase Portrait, $T_p = .3$
Figure III-24
Phase Portrait, $T_p = .4$
### Dimensional Parameters Increasing

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**Effect of Physical Parameters on Dimensionless Parameters**

*Figure III-25*
1. Decreasing damping ratio ($\zeta$), allows faster slewing, causes longer transients, the motor may gain steps for short fixed period sequences and can lose steps when stepping at its natural frequency.

2. Decreasing BEMF constant ($\gamma$), has an effect similar to decreasing damping ratio $\zeta$.

3. Decreasing natural frequency ($\omega_n$), causes the time marks on the phase portrait to represent longer real times. A smaller $\omega_n$ makes the dimensionless velocity represent smaller real velocities.

4. Decreasing friction ($\bar{T}_f, \bar{T}_s$) causes a smaller dead zone about each singularity. Makes the motor appear to have a lower damping ratio $\zeta$.

5. Decreasing load torque ($\bar{T}_L$), reduces the chance of the motor running backwards under load, reduces the shift of the equilibrium points, and allows the motor to run at faster fixed period rates against the load.

Upon close inspection of many different phase portraits for various values of the parameters, it is possible to draw some general conclusions about stepping motor behavior as motor parameters are changed.

1) As $\bar{T}_f$ is increased, the separatrices move farther away from each other with fewer separatrices below any given velocity, as shown in Figure III-26. This effectively reduces the chance that the motor will gain steps, but at the same time it decreases the maximum slew speed which can be attained. Increasing $\bar{T}_f$ also makes the dead zones about the equilibrium positions wider, hence decreasing position accuracy when the motor is at rest. It also causes the trajectories to spiral in towards equilibrium position more rapidly so that it
Effect of $T_f$ on Phase Portrait Shape

Figure III-26
looks as if the motor had additional mechanical damping. As $\tilde{T}_s$, the stiction, is increased, the dead zone increases with no other effect on trajectories.

2) If the damping ratio, $\zeta$, is increased (Figure III-27), the separatrices move away from the position axis and are spaced farther apart, which makes it harder to gain steps at a stepping rate near the start-stop rate, the maximum fixed period stepping rate without losing steps. This also causes the response to be less oscillatory and the settling time is reduced. However, increasing the damping ratio also reduces the maximum slewing rate, and for high speed applications, this is unacceptable. In addition, there is no dead zone associated with an increase in $\zeta$ so that static accuracy is not effected. Lastly, increased $\zeta$ reduces the maximum start-stop rate.

3) If $\tilde{T}_L$, the velocity independent load, is increased (Figure III-28) the separatrices all begin to move down toward the position axis and at some value of $\tilde{T}_L$ determined by the other parameters, the separatrices can actually encircle the stable singularities and produce an area which leads to a singularity at $-\infty$, as has previously been described. It will also be noted that the unstable equilibrium point to the left of any stable singularity moves closer to that stable singularity as $\tilde{T}_L$ is increased. This makes it easier for the motor to fail by crossing a separatrix to the left to lose steps. At a value of torque equal to $\tilde{T}_L = .707$ for a sinusoidal torque-angle curve and four phase motor, and the motor will no longer be able to step, because this is the maximum average torque that is available. Increasing $\tilde{T}_L$ will also decrease the maximum slewing speed.
Effect of $\zeta$ on Phase Portrait Shape

Figure III-27
Effect of $T_1$ on Phase Portrait Shape

Figure III-28
Effect of $\gamma$ on Phase Portrait Shape

Figure III-29
4) By increasing the BEMF constant, $\gamma$, the trajectories tend to take on a steeper slope at high speeds (Figure III-29). This moves the separatrices away from the position axis, making it harder for the motor to gain steps in a short stepping sequence. It also reduces the maximum speed at which the motor may operate, and for low damping becomes the dominating factor in determining the maximum speed.

5) Since the natural frequency of the system changes the time scale on all plots, it has no effect on phase portrait shape, but the higher $\omega_n$ becomes, the faster the system will respond and the faster will be the actual start-stop and slewing rates.

The above observations are strictly trends which are seen in the phase portraits. The phase portrait for a given motor drive and load will give an overall view of the system and will show a designer what steps to take to improve the performance of his system. In order to get a picture of what will occur for various stepping sequences, it is easier to augment the phase portrait with a computer program which will graphically show the motor stepping response for any given number of steps at any time spacing $\tau_s$. This also has been accomplished and is the means by which this model will be verified in the experimental section of these notes.
A stepping motor steps as a result of the current being switched from one stator phase to another. The location of the next stator phase to be excited determining the direction of stepping, see Figure IV-1. In a four phase stepper there are four steps as $\theta$ goes from 0 to $2\pi$ radians. Every step, therefore in $\phi$, is a step of size $\phi_s = \frac{\pi}{2}$ radians. If the current buildup in the windings is very rapid and stepping in the positive direction is called for, then the motor's torque-angle curve immediately shifts its position to the right by $\frac{\pi}{2}$ radians.

Since the shape of the torque-angle curve has not changed, this is equivalent to shifting the phase portrait to the right an amount equal to the step size, in this case $\frac{\pi}{2}$ radians. However, this is also equivalent to shifting the phase plane trajectory of the motor $\frac{\pi}{2}$ radians to the left at the instant that the stator windings excitation is changed. **This is a critical concept!**

In the process of stepping the phase plane shifts from one location to a new location representing the equilibrium or zero torque point of the new torque angle curve. If the peak of the torque angle curve is the same for every step, then the original phase plane simply shifts to the right to step in a positive direction or to the left to step in a negative direction and the size of the step is equal to the
shift in the flux density vector $\Delta \psi$. By this process one might represent any stepping sequence as a shift from one phase plane to another, or one might represent a horizontal shift in the trajectory to the left in a given phase plane as the same as a shift of the whole phase plane. By doing this, it is possible to show that the origin in the phase plane is always the location to which the rotor should eventually go if it is stepping correctly. If the torque-angle curve of the stepping motor changes in amplitude from one step to another, then the shape of the phase plane will also change because of this change in torque amplitude and so it will be necessary to describe the motion of the motor with two or more phase planes in this situation.

When the stator excitation is shifted by the driver amplifiers to step the motor, the new torque equation is identical to the old one but has been shifted to the right or left by the step size. Hence, then the resulting motor response is shifted to the left or right in the phase plane an amount equal to the step size.

As long as the inductive effects can be neglected, and torque-angle curve shape and magnitude doesn't change from step to step, this is a valid way of representing motor stepping behavior. This representation has the following advantages.

1. A single phase plane portrait can be used to represent stepping behavior for any step sequence for a given motor and load.
2. The origin of the phase portrait represents the final desired position. Thus the horizontal distance from the origin to the motor trajectory is indicative of the instantaneous error.
3. Phase portrait shape can be related directly to the phenomena which occur during a given step sequence. Conversely if a given
Figure IV-1
Motor Windings and Torque-Angle Curves
motor parameter such as damping can be shown to alter the portrait shape, then the effect of that alternation on response can be associated with the parameter change.

4. Crossing a separatrix will cause the final position of the rotor to differ from the origin or desired final position unless subsequent steps bring the trajectory back across the separatrix. This is the mechanism of failure.

Thus the phase plane can be used to study the response of the motor to any step sequence. This approach is illustrated in Figure IV-2 for a 2 step sequence. Since the time markers are .2 dimensionless seconds apart, this plot shows the motor response to two clockwise step commands 1 dimensionless second apart. Clearly the motor can execute this sequence as it returns to the origin after the sequence is over.

As was shown earlier, the number of phases will equal the number of steps that occur for one cycle of the torque-angle curve. Thus step angle is defined by

$$\theta_s = \frac{2\pi}{PA} \quad \text{or in normalized angle} \quad \theta_s = \frac{2\pi}{P} \quad \text{(IV-1)}$$

where $P =$ number of stator phases. In Figure IV-3 a stepping sequence of ten steps is shown. With each of the marks on the trajectory representing $\tau = \omega_n t = .2$, for a stepping sequence with ten commands of period $T_s = 1$ dimensionless second, the motor steps and follows that trajectory for 5 marks until the next step occurs one dimensionless second later. This continues until the stepping sequence is terminated, at which time the motor follows the trajectory it is currently on to a stable singularity. If the motor does not fail, it will return to the same singularity in the phase plane from which it
Figure IV-2
Stepping by Shifting Trajectory
A Ten Step Sequence in the Phase Plane
started as long as stepping behavior is represented by a shift in angle of $\theta_s$ at each step. It must be remembered that the actual value of the rotor displacement should be

$$\theta = \theta(0) + \theta_s \times \text{number of steps commanded} \quad (IV-2)$$

This occurs because the trajectories were shifted rather than shifting the phase planes. By shifting the trajectories, the origin always remains as the desired final location. Looking at Figure IV-3 again, it will be noted that if stepping sequence is terminated at the end of 1, 2, 3, 4, 8, or 9 steps, the motor will return to the correct equilibrium point. However, if it is stopped after the 5th, 6th, or 7th step, the motor will have crossed the sepaatrix to the right of the singularity and the motor will gain four steps. This is due to the speed transient which occurs at the start of a stepping sequence. As the damping ratio, $\zeta$, becomes smaller, the transient is longer and there will be more sequence lengths for which the motor will gain steps. This effects the start-stop rate but more will be said about this later.

When stepping occurs, the governing equation changes due to the change in the position of the torque-angle curve which has been accounted for by subtracting $\theta_s$ from the motor angle in the phase plane at stepping instants $T_n$. The new angle definition is:

$$\theta \to \theta - \sum_{n=1}^{N} \theta_s \ u(\tau - \tau_n) \quad (IV-3)$$

Where

$$\theta_s = A \theta_s = \text{normalized step size} = \frac{360^\circ}{\# \text{phases}}$$

$$u(\tau - \tau_n) = \text{unit step occurring at time } \tau_n$$

$N = \text{number of steps}$
\( \tau_n = \) time of the \( n \)th step command, for a fixed stepping period \( \tau_s, \tau_n = n\tau_s. \)

Then the actual equation becomes

\[
\ddot{\phi} + [2\zeta + \gamma \sin^2(\phi - \sum_{n=1}^{N} \phi_s u(\tau - T_n))]\dot{\phi} \\
+ \sin(\phi - \sum_{n=1}^{N} \phi_s u(\tau - T_n)) = -\ddot{\tau}_L - (\ddot{\tau}_L + \dddot{\phi}_s (u(|\dot{\phi}|) - u(|\dot{\phi}| - \delta))) \frac{\dot{\phi}}{|\dot{\phi}|}
\]  

(IV-4)

This equation is valid for any stepping sequence for any given stepping motor with single phase excitation as long as the current in the motor windings changes almost instantaneously at step times \( T_n. \)

Actual motor response does not move discontinuously to the left, but progresses to the right for a positive stepping sequence. Its response could also be described by shifting the phase plane to the right at each step instant.  

(19, 20) All theoretical plots will be rearranged in this manner by means of a computer for easy comparison to measured data. By shifting the phase planes, it is easier to relate experimental responses to computer simulations, by shifting the trajectories, it is easier to see the causes of failure so that when comparing data to simulations, the phase planes for stepping sequences will be shown both ways.

4-B Experimental Step Sequence Measurements

Shifting the trajectory to the left in the phase plane an amount equal to the step size \( \phi_s \) at stepping instants is a convenient way of showing how motor stepping failures can occur. It allows high speed stepping to be shown very conveniently. It also can demonstrate methods of position feedback and shows optimum sequences. However,
when comparing measured phase plane responses to computer simulations other presentations are more convenient.

Velocity sensors are continuous, however, most position sensors produce an output voltage which is a periodic function of angle. A potentiometer can be connected to the rotor and a bridge circuit used to produce a voltage which is proportional to angle over about 360° see figure IV-4. This signal appears as shown in Figure IV-5. The bridge output voltage shifts every $2\pi$ radians of rotation so that the position signal repeats itself.

Several forms of computer simulated response presentations are convenient depending on the step size relative to sensor period for a 360° position sensor period.

1. If motor step size is small compared to $2\pi$ radians. In this case, except for long sequences, the measured motor phase plane response will appear to be the same as the calculated responses when the phase plane is shifted in the stepping direction an amount equal to the step size. It is important, of course, to position the sensor shift point away from the stepping motor phase plane region of interest.

2. If the motor makes only a few steps per revolution, then the measured phase planes will repeat every $2\pi$ radians as the potentiometer signal cycles. This can be accomplished in the simulations by subtracting $2\pi$ radians from the motor angle for every $2\pi$ radians of motion from some arbitrary angle.

Acceptable phase planes can be obtained using an integrator and a tachometer. The integrator must be reset after some maximum angle indicating signal has been reached. This has the same effect as the periodic position sensor.
Figure IV-5
Position Sensor Output

Figure IV-4
Position Sensor Bridge
Since the simulation results are produced by a computer, it is useful to reproduce the expected measured responses as closely as possible. However, except when comparing measured results to the simulations, motor stepping behavior will be described in terms of an instantaneous shift of position $\theta_s$ at the instant of a step command because it is this representation which most effectively shows the mechanism of failure.

An alternate approach to the experimental data taking would be with the use of an encoder and a mini-computer. Some preliminary work in this area has indicated that a minimum of 32K twelve or sixteen bit words are necessary to represent a reasonable amount of data. The choice of encoder resolution is a function of step angle and accuracy of desired results. Digitally one can compute the desired velocity from the position data, thus DMA is used to record the original encoder output at fixed intervals of time, to increase sampling rates. After a sequence has been inserted in memory then the position data is processed to plot the phase plane.

4-C Zero Slope Isocline

In addition to the trajectories in the phase portrait there is another set of curves which can be drawn to provide information about the trajectories, which are known as isoclines. All trajectories crossing a given isocline curve cross it with a fixed slope, thus isoclines are curves of constant trajectory slope. These isoclines can be used to plot a series of trajectories in the phase plane by hand so they are of little interest since the computer will be used to plot the trajectories. However, there is a particular isocline which
will provide insight into motor behavior. This is called the zero slope isocline (ZSI) and all trajectories cross it with a slope of zero. Thus, in the phase plane all points on the zero slope isocline means that the motor has zero acceleration at that point. This isocline can be determined easily by noting that the slope of any trajectory at any point in the plane is defined as
\[ S = \frac{d\dot{\theta}}{d\theta} \]  
(IV-5)
This can be expanded as follows:
\[ \frac{d\dot{\theta}}{d\theta} = \frac{d\dot{\theta}}{d\tau} = \frac{d\dot{\theta}}{d\theta} \frac{d\theta}{d\tau} = S \]  
(IV-6)
In the top half plane where \( \dot{\theta} > 0 \), when \( S > 0 \) the motor is accelerating when \( S < 0 \) its velocity is decreasing. By substituting the equivalent expressions from equation III-37 this becomes, for \( \dot{\theta} > 0 \):
\[ S = \frac{\dot{\theta}}{\ddot{\theta}} = \frac{-\ddot{T}_L - \ddot{T}_f - \sin\theta - \dot{\theta}[2\zeta + \gamma\sin^2\theta]}{\ddot{\theta}} \]  
(IV-7)
For the zero slope isocline, \( S = 0 \) and equation 10-7 can be solved for \( \ddot{\theta} \) in terms of \( \theta \) along the isocline to provide an equation for this isocline.
\[ \ddot{\theta} = \frac{-\ddot{T}_L - \ddot{T}_f - \sin\theta}{2\zeta + \gamma\sin^2\theta} \]  
(IV-8)
Since at \( \dot{\theta} = 0 \) the stiction term \( \ddot{T}_s \) would be present, this relation is not valid where \( \dot{\theta} = 0 \). Figure IV-6 is a phase plane portrait with the zero slope isocline plotted for \( \gamma = 0 \). It can be shown that inside the area bounded by the angle \( \theta \) axis and the zero slope isocline all trajectories must have a positive slope \( S > 0 \). Thus the zero slope isocline outlines the only region in the top half plane in which the motor may accelerate. In order to reach any speed, the motor must
FIGURE IV-6

ZERO SLOPE ISOCLINE FOR --- ZERO GAMMA
accelerate. To maintain any steady state speed, the motor must either have zero acceleration between steps which is impossible for steady state stepping motor operation, or it must accelerate during part of the time and decelerate for the remainder of the time between steps. For these reasons, when the motor is stepping at a constant step rate and after the start up transient is over, the motor trajectory must pass through the zero slope isocline between each step.

For the non-zero gamma case, that is to include the effects of the BEMF voltage, the zero slope isocline is altered. Since the BEMF voltage increases with speed the major effects occur at high speeds and will tend to limit the high speed current in the stator. Figure IV-7 shows the zero slope isocline for non-zero gamma, note the effect of lowering the ZSI, especially at minus \( \pi/2 \) where BEMF is a maximum. When using a voltage drive for high speed performance one must try to minimize the BEMF effects for the best performance.

It is also possible to find the maximum speed that the motor may attain by finding the top of the zero slope isocline. Above this point all trajectories must have a negative slope and hence the motor velocity can only diminish. Thus to maintain any stepping sequence, part of the cycle must be within the zero slope isocline. Hence, the maximum slew speed may be approximated by the peak of the zero slope isocline. For the non-zero GAMMA cases this value is taken at minus \( \pi/2 \). If the friction torque, \( \bar{T}_f \), is varied and the maximum value of the zero slope isocline is plotted versus \( \bar{T}_f \), for a given damping ratio \( \zeta \) and BEMF constant \( \gamma \) a resulting torque-slew speed curve is developed for a given motor and drive circuit. This is shown in Figure IV-8. The shape of this curve is very similar to manufacturers
FIGURE IV-7
ZERO SLOPE ISOCLINE—NONZERO GAMMA
\text{ZETA} = 0.03150 \quad \text{GAMMA} = 0.12620

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{friction_vs_max_speed.png}
\caption{Friction versus maximum speed, $\frac{de}{dt}$}
\label{fig:friction_vs_max_speed}
\end{figure}
pull-out curves. Figure IV-9 shows the peak of the zero slope isocline or the maximum theoretical slewing speed versus damping ratio for a typical 90° PM stepping. This curve illustrates why low damping is desirable since maximum slewing speed increases as \( \zeta \) decreases.

Figure IV-10 shows the maximum theoretical slewing speed versus \( \gamma \), the motor BEMF constant, for a typical PM stepping motor. As the drive approaches a perfect current source the effect of \( \gamma \) is diminished so that this also shows why the slewing speed will increase when the motor is driven from a current source.

It should be noted that the speed scale \( \frac{d\phi}{dt} \) max is a dimensionless one. The actual motor speed can be computed from the following definitions of the dimensionless time scale and angle scale. From equations III-33 and III-36

\[
\frac{d\phi}{dT} = \frac{d\theta}{KT\gamma} = \sqrt{\frac{JR_\gamma A}{KTV}} \frac{d\phi}{dt}
\]

where

Motor Speed = \( \frac{d\theta}{dt} = \sqrt{\frac{KTV}{JR_\gamma A}} \frac{d\phi}{dt} = \frac{\omega n}{A} \frac{d\phi}{dt} \) radians/sec

(IV-9)

for motor speed in steps per second:

\[
\text{motor speed} = \frac{d\theta}{\theta_s} = \frac{\omega n}{\theta_s A} \frac{d\phi}{dt} = \frac{2\omega n}{\pi} \frac{d\phi}{dt} \text{ steps/sec}
\]

(IV-10)

where \( \theta_s = \frac{\pi}{\theta_s A} \) for a four phase motor.

These equations allow the velocity scale of the dimensionless phase planes to be converted back to actual motor speed. Using these, it can be shown that for no friction or load and ignoring BEMF, the peak
TFP = 0.83753  \gamma = 0.12633

ZETA VERSUS MAXIMUM SPEED, $\frac{\theta}{dt}$

FIGURE IV-9

ZETA VERSUS MAXIMUM SPEED, $\frac{\theta}{dt}$
ZETA = 0.80153  TFP = 0.03758

FIGURE IV-10

GAMMA VERSUS MAXIMUM SPEED, $\frac{d\theta}{dt}$
torque \( T_m \) divided by the damping constant \( B \) equals the maximum slew speed.

The zero slope isocline gives a global picture of motor acceleration and maximum speed response and is useful in determining system limits for a given motor, drive, and load. Since motor damping, \( B \), is very small, very high slew speeds are possible. However, when the motor is connected to a mechanical drive and load which has some damping, the slew speed will drop dramatically. This explains why a motor and drive may perform well in prototype form with a simulated load inertia, but fail when the actual drive train is attached which exerts some additional friction and damping loads on the motor.

The zero slope isocline is a plot of instantaneous zero acceleration points and in order to evaluate the true performance of a stepping motor one must find the average torques available in a step interval, which is equivalent to the amount of work done in one step period. When the stator does positive work on the rotor, the motor must accelerate and conversely for negative work it must slow down. Thus for zero work over one step period, the motor must be in the steady state with a given average velocity and a particular lead angle.

4-D Zero Work Curve

The speed limits for stepping motors operating in the slewing mode depend to a large degree upon the power amplifier configuration. Many types of power amplifiers have been devised to control the high speed operation of stepping motors in the slewing mode. It seems generally understood that in order to obtain highest possible speeds
the motor must be driven from a current source and the load inertia must be minimized. Beyond this, analytical models for the high speed operation of the motor are rare. The phase plane can be used to study stepping motors and is especially useful for low speed operation of the motor at rates below the start-stop rate. At higher rates with an voltage drive the current buildup in the windings becomes a significant part of the on time of the winding. Further at very high speeds, current drives begin to be influenced by the BEMF voltage so that the effect of the current drive is reduced and it approaches the behavior of an voltage drive. The zero slope isocline showed regions of acceleration in the phase plane can be identified. In these regions of the rotor-stator lead angle versus motor speed plane a motor can speed up under the influence of a given stator phase. The boundary of these regions is called the zero slope isocline. It might be used to identify the limits of high speed operation of the motor. However, even if the current buildup time were not a factor, the stepping motor is influenced by a given stator phase for some portion of its angular rotation. During the acceleration mode the rotor lags behind the zero point of the torque angle curve established by the stator phase. During the deceleration mode it is ahead. In normal operation at high speeds the motor shifts back at each step instant relative to the torque angle curve established by the stator phases. This shifting is quite different from what one would expect from examining the behavior of a synchronous motor. The permanent magnet stepping motor is identical in structure to a synchronous motor. However, in the case of a synchronous motor or a stepping motor which is mini-stepping, the angle between the rotor and the zero point of the torque angle
curve in the steady state does not shift back and forth, even at high speeds. This shifting in the steady-state leads one to believe that an instantaneous picture of stepping motor behavior such as one might obtain from the phase plane is not going to provide insight into the limits of high speed steady-state behavior.

In order to address the high speed behavior of the motor, the following approach will be developed: The motor will be assumed to be traveling at a constant speed. Then the currents that would flow in the stator phases, when they are energized will be used to predict the torques which influence the rotor. Using this approach, it will be possible to show that operation at high speeds is stable only for a particular lag angle of the rotor relative to the stator phase being excited. A plot of this lag angle versus speed known as the "zero work curve" will outline the high speed limits of the motor in the phase plane. The model will include the inductive effects of the stator phases as well as the BEMF. It is a logical extension from the phase plane to a more complex representation for the motor that includes stator inductance.

Refering to figure IV-11, the phase plane for a given motor can be represented as a series of responses to a given initial disturbance. Further, the plane can be thought of as a plot of motor speed versus the lead angle of the rotor relative to the zero point of the torque angle curve established by the stator phase current and the rotor. In this picture (shown in Figure IV-11) the solid lines represent trajectories which go through the unstable equilibrium points of the motor. These are the points where the motor will turn either clockwise
ZETA = 0.0010  TL = 0.0000  TF = 0.0854  TS = 0.0000  GAMMA = 0.7840  DELT = 0.0100
MODEL #: 0

FIGURE IV-11
ACCELERATION REGION IN THE PHASE PLANE
or counterclockwise upon a small disturbance. The solid curves re-
represent boundary lines between the stable equilibrium points. If
during the stepping process the motor were to cross one of the solid
curves, the system response would proceed to a different final position.
The dotted lines represent trajectories for the motor and each dot
represents a time duration of a tenth of a dimensionless second. The
shaded region of this phase plane, is bounded by a curve known as the
zero slope isoline. The zero slope isocline is a curve on which the
response trajectories for the motor have zero slope. Inside this
curve the trajectories have a positive slope and the motor is capable
of acceleration, outside this curve, the trajectories have negative
slope and the motor will slow down. The particular motor represented
here has a very large amount of BEMF and a very small mechanical
damping. This causes the acceleration region to have horns on it.
From the phase plane analysis it is not clear whether the motor is
capable of operating in these horns, and so there is some abiguity
about the limits of high speed operation, even assuming that the
current buildup is very rapid. In constructing this phase plane, it has
been assumed that the current in the stator windings changes instaneous-
ly upon switching. If the L/R time constant of the motor is not small
compared to the on time of the windings, this assumption is not valid
and the phase plane representation will not be an exact representation.
Furthermore, at a high speed there is a fixed path in the phase plane,
which could be called a limit cycle, that has along its bottom boundary
a horizontal line equal to step size and along its top boundary a
trajectory. For a four phase motor, full stepping for example, the
horizontal distance in the phase plane would be 1.57 radians. To
operate the motor at a given velocity part of the step cycle must be in the acceleration region and in the deceleration region for the rest of the cycle, to yield an average zero acceleration. Hence motor operation is possible in the horns of the zero slope isocline, however it is still not clear how fast the motor is capable of operating.

In order to study the problem let us assume that the motor is operating at a constant speed governed by equation IV-12

$$A\theta = t_w$$  \(\text{(IV-12)}\)

That is, the angular acceleration of the rotor is zero. Under these circumstances let us then examine what happens when a given stator phase is turned on and the rotor moves with constant velocity. During this time the zero of torque angle curve established by the stator phase current will be constant relative to the stator, but the rotor will move relative to this torque angle curve. To examine this in more detail observe the circuit for one of the stator phases as shown in figure IV-12. In this circuit it is assumed that the phase is driven by a source of voltage \(V_s\) connected by a switch which represents the power transistor controlling the phase. The circuit of this phase may include the phase resistance as well as a series resistance in the case of a L/R drive or a sensing resistance in the case of a current drive, an inductance, and a BEMF generated by the relative motion of the rotor and the stator which depends upon their relative angles and the speed. If the switch were closed, a current would flow in this circuit which in turn would cause a torque on the rotor. One might think of the supply voltage as providing a current for rotation in a positive direction and the BEMF providing a voltage which would reduce this current or slow the motor down. Since the rotor speed is assumed
FIGURE IV-12
STATOR PHASE EQUIVALENT CIRCUIT
constant, one might plot the current in the stator phase versus the angle between the rotor and the stator-established torque-angle curve or phase current versus time. Either of these two scales can be used. For example, suppose that the current in the stator phase when it is energized is plotted versus angle for various speeds. If the inductance in the circuit is not assumed to be zero one can show that the current in this on phase will be dependent on BEMF, supply voltage and the total circuit impedance. Unless the drive has fly back diodes, and in the circuit shown in figure IV-12 this was not assumed, the currents in the stator phase cannot go negative because of the power transisters. For this reason even at very high speeds there will not be negative currents flowing in the stator phases as will be predicted by the model. Figure IV-13 shows this variation of current with angle at various speeds. It indicates that the current in the on stator phase can become much less than rated current at high speeds. The reduction in the magnitude of the current depends upon speed and angle. One may also include in this model a variation of inductance and the inductance may be made equal to

\[ L_0 + L_1 \cos A\theta \]  

(IV-13)

when this occurs there is a further reduction in the current in the stator phase with speed. This reduction extends over a wider rotor-stator angle making it more difficult for the motor to operate at these high speeds, however, it can be shown that a very large variation in stator inductance is necessary to cause a appreciable amount of stator reduction. For example in figure IV-14 it was assumed that there was a 100% variation in stator inductance. Comparing this to figure IV-13, the basic amplitudes of the currents at each speed are
ZETA = 0.015  GAMMA = 0.0980  J0 = 0.0300  J1 = 0.000

FIGURE IV-13

PHASE CURRENT VERSUS ANGLE
ZETA = 0.0015  GANMA = 0.0960  JO = 0.0300  J1 = 0.030

FIGURE IV-14

EFFECT OF INDUCTANCE VARIATION
similar, but the shape of the curves varies slightly and there is some phase lag in the case of the variable inductance.

In figures IV-13 and IV-14 a steady state speed was assumed and the current was plotted over 360 degrees around the zero torque position of the rotor. In practice, when full stepping the stator phase is only on through 90 electrical degrees for each step. At the instant of the step command one must assume or know the lead angle of the rotor and the speed. Thus the speed determines the curve in figure IV-13 or IV-14 which applies and the lead angle determines when the inductive buildup to that curve starts and which portion of that curve is valid during steady state conditions.

If a motor is operated at a fixed speed and the current in the on stator phases is measured and displayed as a function of time it is possible to show the effects of speed upon stator current for an L/R drive. Figure IV-15 shows this at a low speed and high speed for a given motor-drive combination. Also shown are the computed currents based on the model described above. At low speeds there is an initial stator current buildup in the windings due to inductance. It is assumed in this drive that the stator phase has no flyback diodes and turnoff of the current to the windings is almost instantaneous. At high speeds (shown in figure IV-15b) the current buildup occupies a longer portion of the on time of the winding and there is a serious reduction in the current through the winding by virtue of the voltage generated by the BEMF. Consequently figures IV-15 (a and b) illustrate very graphically why the BEMF and inductance limit the high speed current available to the motor. In order to study this effect in more detail a mathematical model will be built to examine motor behavior.
FIGURE IV-15A
CURRENT RISE TIME IN WINDINGS AT SLOW SPEEDS
FIGURE IV-15B
CURRENT RISE TIME IN WINDINGS AT HIGH SPEEDS
Refering to figure IV-12 the stator circuit, a loop equation can be written to result in equation IV-14.

\[ V_s - R_i i - L \frac{df}{dt} + K_b \dot{\theta} \sin \omega_0 = 0 \]  

(IV-14)

Equation IV-14 is valid for both an L/R drive and a current drive at high speeds where the difference between \( V_s \) the motor supply voltage and the BEMF limits current to be less than rated current. In solving the equations for the current drive case, one must recognize that the current will be limited to the rated current specified by the drive. First define the over-voltage ratio, \( N \), as the ratio of supply voltage to motor rated voltage, the voltage that would be placed on the motor for the equivalent current in a voltage drive.

In order to reduce the number of parameters in equation IV-14 several definitions are made:

\[ A \theta = \theta, \tau = \omega_n t \text{ and } \dot{\theta} = \omega_n \]

\[ N = \frac{V_S}{V_R}, \quad I_R = \frac{V_R}{R_T}, \quad I = \frac{I}{I_R} \]

\[ J_0 = \frac{L \omega_n}{R_T}, \quad \gamma = \frac{K_b \omega_n}{AV_R} \]

Substituting these dimensionless parameters into the equation for the stator circuit and solving for the rate of change of stator current one obtains equation IV-15

\[ \frac{dI}{dt} = \frac{1}{J_0} \left[ N + \gamma \dot{\theta} \sin \theta \tau - I \right] \]  

(IV-15)

Further it can be shown that because motor speed is constant, angle and time can be related.
Doing this allows one to write the differential equation of terms of angle rather than time as shown below.

\[
\frac{d\ddot{\theta}}{d\theta} = \frac{1}{J_0 \ddot{\theta}} \left[ N + \gamma \ddot{\theta} \sin \theta - \ddot{I} \right]
\]  

(IV-17)

It should also be noted that \( \dot{\theta} \times P \times \omega_n \times \frac{1}{2\pi} \) is the speed of the motor in full steps per second, where \( P \) is the number of stator phases. Furthermore, knowing the current it is possible to compute the instantaneous torque due to the stator current. This can be done from equation IV-18, where \( T_A \) is the torque available for acceleration, over and above the damping and friction torques needed to keep the motor in rotation.

\[
T_A = -K_T \sin \alpha - B\dot{\theta} - T_f
\]  

(IV-18)

by defining

\[
\bar{T}_f = \frac{T_f}{K_T T_R}, \quad \gamma = \frac{B\omega_n}{2AK_T T_R}, \quad \bar{T}_A = \frac{T_A}{K_T T_R}
\]

is possible to write this equation in dimensionless form also, where \( \bar{T}_A \) is the ratio of the peak of the instantaneous torque available for acceleration divided by the peak of the torque angle curve.

\[
\bar{T}_A = -\bar{I} \sin \theta - 2\zeta \dot{\theta} - \bar{T}_f
\]  

(IV-19)

If this torque is positive then the motor can accelerate. If it is negative then it must decelerate. Using these equations and a computer it is possible to solve for \( \bar{I} \) versus angle and \( \bar{T}_A \) versus angle. The results of this computation are shown in figure IV-16. In the current versus angle curve, when the motor is operating at low speeds there is very little reduction in the current over what one might expect to see if the motor had no inductance and no BEMF.
FIGURE IV-16
CURRENT AND TORQUE DURING
LOW SPEED STEPPING
Correspondingly the torque curve is positive for part of each cycle of the on time and negative for the rest. If one examines the area under this curve, that is torque times angle integrated over the on time of the stator phase, the result is the work done by the stator phase on the rotor during the period in which it is on. This number has great significance. If this work is positive then the rotor accelerated during this on period, if it is negative then the rotor decelerated. In Figure IV-17 the same sort of picture is developed for a much higher speed where the motor current is plotted versus angle at a particular turn on angle and the corresponding torque is plotted versus angle. In this case the area under the torque curve is negative and the motor would decelerate at this "turn on" angle for this operating speed. Consequently, at any given operating speed it is possible to vary the "turn on" angle of the stator phase and examine the area under the torque angle curve to determine if the work done by that stator phase, when turned on at that particular angle, is a positive or negative number. If the area under the torque curve is zero then the motor will undergo no net acceleration while the phase is on. In figure IV-18 shows the torque versus angle curve for several "turn on" angles at a given speed. As the turn on angle varies, the amount of torque available also varies and the area under the torque angle curve changes. As can be seen from figure IV-18 it is possible to vary the torque angle curve from -1.5 radians -.5 radians and go from a positive work exerted on the rotor to negative work. In this way it is possible to find the zero work "turn on" angle where the motor can operate in the steady state. This then allows one to determine the steady state "turn on" angle for any given speed, which is the
FIGURE IV-17

CURRENT AND TORQUE DURING
HIGH SPEED STEPPING
FIGURE IV-18
TORQUE CURVES FOR
VARIOUS LEAD ANGLES
angle of rotor relative to zero torque point of the torque angle curve after winding is on. If this process is continued for various speeds, a zero work curve can be found which is shown in figure IV-19. Also plotted in figure IV-19 is the zero slope isocline for the same motor assuming that the inductive time constant is zero. The zero work curve represents the curve of initial turn on angles versus speed for a given motor. The peak of this zero work curve defines the maximum speed at which the motor can operate. Also it shows that it is not possible for the motor to operate at speeds which are in the right horn of the zero slope isocline. This is an important conclusion. It can be shown in the phase plane that there are two sets of possible limit cycles for the motor around each horn and this zero work curve indicates that only one of these is stable. Furthermore, beyond the peak of the zero work curve it does not appear possible to operate the motor in a stable manner. In order to verify this concept the turn on angle was measured for a motor, driven by a L/R drive, operating at several speeds. This data was plotted on the calculated zero work curve, shown as X is on figure IV-20. 10 radians per dimensionless second, the motor lag angle suddenly shifts as motor step rate is increased gradually, this will cause a transient in motor operation lag angle. Also the motor operation shifts from a limit cycle on the right side of the zero slope isocline to one on the left side. It is quite possible that this sudden shift in lag angle with speed is related to the mid frequency resonance problem because there is a transient in motor operation at this point. It can also be shown that this is the point at which the BEMF starts to make the current zero in the stator phase during some portion of the stator phase on time.
\[ \text{FIGURE IV-19} \]
\text{ZERO WORK CURVE-ZERO SLOPE ISOCLINE}
FIGURE IV-20

EXPERIMENTAL DATA ON THE
ZERO WORK CURVE
Figure IV-21 is a summary of the zero work curve and ZSI in the speed versus angle plane for this motor. Inside the zero work curve, the work done by the stator phase on the rotor during the on time is positive, outside the zero work curve it is negative. Consequently, if the "turn on" angle for the motor at a given speed is inside the zero work curve, the rotor will speed up due to the positive work done on it during each on time. This will cause the rotor-stator angle to increase and move the motor rotor "turn on" angle to the right towards the zero work curve. If the motor is outside the zero work curve, it will slow down and the rotor-stator angle will decrease causing the motor to move to the left. This means then that the zero work curve is stable along its right hand boundary but unstable along its left hand boundary. Therefore there is only one set of stable limit cycles for the motor in the phase plane and that set lies along the right hand boundary of the zero work curve. As the motor operates at higher and higher speeds there is a smaller range of "turn on" angles for which there is positive work done on the motor. This must mean then, it is much more difficult to operate the motor at higher and higher speeds. Also it is much more sensitive to a disturbance which would cause the rotor-stator angle to cross the left side of the ZWC and become unstable or lose syncronism. When the supply voltage is no longer large enough to overcome the BEMF and control stator currents, the current drive will eventually look very much like an L/R drive. However, as long as the current drive maintains continuous control over stator current, it is possible to operate the motor at high speeds and maximum speed will be limited only by motor and load damping and friction.
FIGURE IV-21
ZERO WORK CURVE SUMMARY
In a current amplifier the most important parameter is the supply voltage or the overdrive ratio, \( N \). As \( N \) is increased two benefits are acquired, first the current rise time due to inductance is minimized because in an L/R drive \( R \) must also be increased to limit current and secondly the effect of BEMF is less. Figure IV-22 is a series of ZWC curves that were generated by increasing the overdrive ratio \( N \). The first curve, \( N = 1 \), representing the motor operating at rated voltage. From figure IV-22 the advantage of increasing the overdrive voltage can be seen as higher operating speeds.

Lastly, the damping ratio of the motor or the measure of mechanical damping has an effect on the zero work curve. In general higher damping will lower the zero work curve, see figure IV-23. In lowering the zero work curve the damping then causes the maximum speed of the motor to diminish. The same sort of effect might be expected from friction. As \( \gamma \) and L/R become smaller, the effect of damping and friction is more pronounced.

In summary, the high speed behavior of the stepping motor is determined almost entirely by its BEMF and its L/R time constant for an L/R drive. If the motor is operated from a current drive then high speed behavior is determined by the damping and friction of the motor as long as the overdrive voltage ratio is sufficiently high. If the L/R time constant of the motor is small or a current drive is used, it is possible to use the phase plane to represent the motor. The zero slope isocline in this case defines a region where the motor can accelerate, but since each stator phase is on for a fixed period of time determined by the step size and speed is difficult to see from
FIGURE IV-22

EFFECT OF OVERDRIVE VOLTAGE ON THE ZWC
FIGURE IV-23
EFFECT OF DAMPING ON THE ZERO WORK CURVE
the phase plane what the actual maximum speed is. By assuming that
the motor is operated at a constant speed and examining motor behavior
for the stator phase on time it is possible to show the speed limitations
of the motor. A current flows in the winding when a given phase is
on and this causes a torque. If this torque is integrated over the on
time, the amount of work done by the stator phase on the rotor
during the period when that stator is on can be found. If this work
is positive the motor will accelerate and the angle between the rotor
and the stator established torque-angle curve will decrease. If the
work is negative the lag angle will increase. Hence, if the work is
zero the motor can be assumed to be in the steady state. This zero
work point can be determined at any given speed. It can be plotted
versus speed and angle. This curve also shows the speed limitation
of the motor as its peak represents the maximum speed at which the
motor can operate. It can be shown that with a current drive this
maximum speed is only limited by the damping and friction of the
motor, but with an L/R drive it is dependent upon the L/R time
constant of the motor and the BEMF. The larger that these two numbers
are, the smaller will be the peak of the zero work curve and the
lower will be maximum operating speed of the motor. This approach
can be used to study motors with one or two windings on. It can be
extended to deal with motors having flyback diodes. It can be used
to determine the steady-state high speed behavior of a motor-drive-
load combination. It would seem quite useful to use this approach to
evaluate the effect of a given drive upon the high speed behavior of a
given motor and load. Since this approach is based on the steady-
state behavior of the motor, it highlights the fact that maximum motor
speed is independent of inertia.
CHAPTER V

MODES OF FAILURE

5-A Introduction

There are five main modes of failure for any stepping motor. In each mode the only mechanism of failure is the crossing of a separatrix. Once a motor has crossed a separatrix to the left of its equilibrium position during stepping and it is being stepping in a positive direction, it will fail unless another step in the opposite direction brings it back across the separatrix. Thus failure occurs because all subsequent steps in the positive direction move the path of the motor in the phase plane further to the left. Only by applying a step command in the negative direction at the correct time can the motor be brought back so that it will spiral into the desired singularity. Since during a positive direction stepping sequence this is not practical, failure will be assured. It should be noted that if feedback were used to detect the occurrence of this type of failure, a negative, then positive step pair could be devised to prevent failure. Failures can also occur by crossing a separatrix to the right of the equilibrium position during a positive step sequence, but in this case the motor may not fail because another positive step in the sequence may bring it back across the separatrix.

Based on this observation, the motor can fail during a positive step sequence by:

1. Losing steps at a stepping frequency close to its
natural frequency or at some submultiple of that frequency because it crosses a separatrix to the left of the desired equilibrium position;

2. Losing steps at a stepping frequency too high for it to follow so that it falls back across a separatrix during startup.

3. Gaining steps if the motor step sequence is halted when it is to the right of the separatrix passing through \( \theta = \pi, 3\pi, 5\pi \ldots \).

4. Losing steps if, when the motor is slewing, it fails to remain in the zero work curve region (the only region where the motor may accelerate), in which case, the motor must declerate and falls back across one or more separatrices;

5. Running backwards due to a unidirectional load if the trajectory is in an area which leads to no finite singularity.

5-B Failure at the Natural Frequency

Figure V-1 shows a failure when stepping at the natural frequency of the motor. When the motor is stepped at or near its natural frequency, the trajectory makes almost one complete revolution or cycle between step commands. Since the damping ratio for most motors is very low, its amplitude does not decrease appreciably between steps. Thus when stepped again, the trajectory shifts even closer to the left separatrix. In the case shown in Figure V-1, when it is stepped the third time, it crosses the separatrix to the left and moves to the singularity to the
Failure at Natural Frequency
Figure V-1

Failure at Half the Natural Frequency
Figure V-2
left. The motor has been asked to execute three steps, however, it has lost one complete singularity. Since the step size for a four phase stepper is $\frac{\pi}{2}$, the final position is four steps behind the desired position. So instead of moving forward 3 steps it has moved backwards one step. At low damping ratios, for step periods, $T_s$, which are multiples of $2\pi$, the motor can lose steps. The basic method of failure is the same at lower frequencies. Figure V-2 shows a failure at 1/2 the motor natural frequency. It should be noted that if the damping ratio, $\zeta$, is high enough, then the motor amplitude decreases enough each cycle of its oscillation so that it cannot lose steps at its natural frequency. Thus for satisfactory operation at its natural frequency, motor damping should be increased.

When operating near the natural frequency of the motor, a sequence one might ask the motor to execute is one in which the motor is asked to switch on zero velocity everytime it has made one complete cycle in the phase plane. For a given damping ratio and friction, there will be a step size below which the motor cannot slide back across the separatrix and fail under these circumstances. Since motors can be half stepped or possess an odd number of phases or be micro-stepped to reduce step size, the effect of these strategies on low frequency resonance will be explored. This is a test sequence that one might ask the motor to execute at or near its resonant frequency. Figures V-3 and V-4 show two step sequences, one which was executed correctly and another that failed to execute this stepping sequence. If a fixed period sequence of pulses at or near the resonant frequency is applied to the motor, it will, in general, fail easier than it will for this particular sequence because of the velocity transient that
FIGURE V-3

STEP SEQUENCE IN THE PHASE PLANE
MODEL 0 ZETA = .0250  TF = .0000  GAMMA = .0000  
NPHASE = 5  WNT =1.0000  NSTPS = 4  XINER = 1.0000  FMTIM = 0.000

FIGURE V-4
STEP SEQUENCE THAT FAILED
occurs during the stepping process. However, this sequence can be used to explore the effect of step size on low frequency resonance.

Any given motor damping ratio is a function of the mechanical damping in the system, \( \zeta \), the peak of the torque angle curve, \( T_{\text{max}} \), the period of the torque angle curve, \( A \), and the rotor and load inertia, \( J \). It is possible to find the largest step size for which the motor can execute the test step sequence near the resonant frequency without failing for a particular damping ratio. Since the damping ratio is dependent on load inertia, it is further possible to represent the total system inertia along the damping scale. The simulation results can be plotted as shown in Figure V-5, which demonstrates that the smaller the damping ratio, which corresponds to larger inertia loads, the smaller must be the step size before a failure will occur at the resonant frequency. Also shown on this plot are the step sizes that would occur for a three, four, and five phase stepper operating at full, one-half, and one-quarter step modes. Clearly, the smaller the step size, the larger the inertia that the motor can drive without failure. This shows the advantage of stepping in small steps, such as one might accomplish with micro-stepping or by using a motor with a large number of stator phases. However, this particular plot was developed assuming that the torque angle curve was constant for every step. In the case of the four phase stepper, there are large variations in the torque angle curve between steps for half-stepping. When this occurs, the chances of the motor failing at its resonant frequency greatly increase, this is shown in Figure V-6. Figure V-6 shows a step sequence for a four phase motor which is half-stepped at a fixed period. Note that two different phase planes are used to develop this
FIGURE V-5

DAMPING RATIO VERSUS STEP SIZE
ZETA = 0.0400  TF = 0.0000  GAMMA = 0.0000
WNT = 6.4000  NSTPS = 8  FNTIM = 44.610

FIGURE V-6
HALF STEP SEQUENCE
FOUR PHASE MOTOR
step sequence. In the case of the five-phase stepper, half stepping

Thus the transient behavior of the five phase motor during half stepping is not as oscillatory as the four phase motor, see figure V-7. Figure V-8 shows more realistic values for stepping at the natural frequency at a fixed period for the four and five phase motors, full and half stepping and accounting for the variation in torque compared to the curve of figure V-5 for the assumed test sequence. Notice that with the five phase motor in the half stepping mode one can operate with a larger inertia without failure than even the zero velocity switching case of figure V-5. When half stepping a four phase motor, failure occurs with much less inertia. This demonstrates further the major advantages which occur when it is possible to step a motor through small steps and when it is possible to keep the torque angle curve nearly constant in executing these steps. Thus, a five phase stepper offers the following advantages when stepping near the natural frequency:

1. Smaller step sizes are possible without exercising continuous control over stator current, that is, micro-stepping.

2. The variation in the peak torque of the torque angle curve is smaller with a five phase stepper because a larger number of phases can be excited and still get half-stepping, quarter-stepping and eighth stepping sequences.

3. With these small step sizes, it is possible to drive much higher inertia loads with the stepping motor without failure at rates at or near the natural frequency.

The effect of friction is similar to that of damping, in that
ZETA = 0.0040  TF = 0.0000  GAMMA = 0.0020
WNT = 6.5000  NSTPS = 8  FNTIM = 45.510

FIGURE V-7
HALF STEP SEQUENCE
FIVE PHASE MOTOR
FIGURE V-6

DAMPING RATIO VERSUS STEP SIZE
the higher the system friction the less likely failure will occur at the natural frequency. The disadvantage of this, is that the static accuracy of the system will decrease with increasing friction.

5-C Start-Stop Rate Failures

If the stepping motor is operated at a fixed period step rate, then there will be a maximum rate which the motor will be able to execute before failure will occur. Figure V-9 illustrates a fixed period stepping rate that was too fast for this system to execute. Failure occurs because the left sepatrix is crossed and the motor deaccelerates. This sequenced may be continued but the motor will continually cross separatrix after separatrix. The motor velocity will be very erratic in this mode.

The start-stop rate of a stepping motor is usually defined as the maximum constant frequency step sequence that the motor can execute from a standstill to full speed and then a complete stop without failure. The start-stop rate depends on friction torque and inertia and is presented as a family of curves on a plot of friction torque versus stepping rate, sometimes several curves are presented, each representing a different value of external inertia Figure V-10. This is sometimes called a pull-in curve. In addition, this curve is specified for a long sequence of steps, because for low values of damping ratio $\zeta$, the motor will gain steps for some sequences consisting of a small number of steps. This is due to the long transient that occurs during start-up, before the motor reaches its steady state speed. Furthermore, these curves are often measured with voltage drives, the details of which are not always specified in the data sheets. Since most users
Failure Due to Rapid Stepping Rate

Figure V-9
DYNAMIC BIDIRECTIONAL CHARACTERISTICS
(START-STOP)

Figure V-10
Manufacturers Start-Stop Curves
develop their own driver amplifiers and may use current drives, it is difficult to use these curves. Thus these curves are useful if used only as a rough boundary for long step sequences and then only if the same driver is used. A problem encountered in using these curves is illustrated in Figure V-11. Consider a step sequence which for a large number of steps the motor executes successfully.

For a fixed period step sequence there will be a transient at the beginning of the sequence and then there will be a limit cycle which is being approached in Figure V-11a. As long as this limit cycle lies below a separatrix, then the step sequence will be executed correctly for a large number of steps. The transient of the sequence does cross a separatrix and thus if the sequence were terminated when above the separatrix a gaining of steps would occur, see figure V-11b.

If a sequence was defined such that its transient response was always below the separatrix, then any step sequence regardless of size could be executed at that rate or below with the exception of stepping at the natural frequency. An example of this is shown in figure V-12 where the response after the third step lies just below the separatrix. If this rate is called the no failure rate, then the relationship between this and the start-stop rate can be shown, see figure V-13. Figure V-13 is a plot of the normalized stepping rates versus damping ratio for zero friction. Note the large difference between the start-stop and no-failure rate at small damping ratios. Since friction is also an important factor in determining these rates, a family of curves can be made for various friction values, see figure V-14. One could go one step further with this and plot the friction versus stepping rate for a given damping ratio, this would be representative
Figure V-11a
Inaccuracy of Start-Stop Rate
Figure V-11b
STEP SEQUENCE AT THE NO-FAILURE RATE

Figure V-12
Figure V-13
Stepping Rate versus Damping Ratio

$T_F = 0$

Normalized Stepping Rate ($1/w_o t$)

Damping (Zeta)
Figure V-14
Stepping Rate Versus Damping for Various Frictions
of the manufacturers start-stop rate curves.

Most motors have very little mechanical damping, the damping ratio, \( \zeta \) is small, so that there is a long transient before reaching the steady-state in response to a step sequence of fixed frequency. Because of this transient, the motor is found to gain steps for sequences of particular lengths if it is stepped near its maximum start-stop speed. If the stepping sequence is terminated, the motor will follow the trajectory to the stable equilibrium point to the right of the desired equilibrium point. As the damping ratio increases, it becomes increasingly difficult to gain steps for fixed duration stepping sequences because the transient in coming up to speed is reduced.

Referring to Figure V-15 it is seen that the motor will reach a steady state path if enough steps are called for. This path can be called a "limit cycle." It is defined as any closed path that the motor repeatedly follows in the phase plane during stepping. The limit cycle has for part of its path a straight line of one step in length \( \theta_s \). The rest of its path is a single trajectory. The motor will only execute this limit cycle if the step period is constant, the load is constant and the stepping sequence duration is long enough for all transients to die out. For rates below the start-stop rates these limit cycles will lie below the separatrix for the equilibrium position thus when the sequence is terminated the motor will always return to the desired position. The vertical size of the limit cycle indicates the speed ripple at any input step rate. Thus these limit cycles provide insight into the steady state constant step period, \( T_s \), operation of stepping motors.
Limit Cycles
Figure V-15
The manufacturer's start-stop curves are an approximate tool for design, if the system being designed is similar to the test system that the manufacturer used to record the data. The phase plane would provide the same information in a more flexible way and would indicate situations where steps would be gained.

5-D Slew Mode Behavior

A stepping motor can be made to operate at speeds above the start-stop speed by slowly decreasing the time between step commands, thus increasing the stepping rate, making it possible to obtain very high stepping rates. This is known as ramping or slewing. To obtain the curve of maximum slewing speed versus frictional load torque, which is the most common curve given by the manufacturer (Figure V-16), the typical experimental method is to slowly increase the stepping rate to slew the motor up to a fixed speed above the stop-start speed and then gradually increase the load until the motor fails. Care should be exercised to measure the running load and not just static friction or stiction. This is commonly called a pull-out curve. However, these operating conditions are hardly typical of realistic load torque situations. In most applications the motor is subjected to a constant friction or velocity independent load.

Furthermore, operating the motor above its start-stop rate is risky at best. In this mode the motor must be brought up to speed slowly to avoid sliding back across a separatrix. It must also be decelerated carefully to avoid gaining steps. To run at high speeds it is necessary for the motor and load damping to be small as was shown previously. On the other hand, it can be shown that many
TORQUE vs SPEED

Figure V-16
Manufacturers Slewing Curve
separatrices must be crossed to attain these high speeds. To avoid gaining steps on stopping the motor must be nursed back across these separatrices.

Total system response is always a function of motor, load, and drive characteristics. If a true current drive is used, the system behavior becomes independent of the drive (as long as the same number of phases are energized with the same drive current as was used to obtain the data). In this case, the system response will depend on only the motor and load. However, the validity of any drive's constant current capability is doubtful at high speeds because of BEMF. As shown by the zero slope isocline and the zero work curve, the drive supply voltage is the limiting factor in high speed behavior. Once BEMF voltage becomes of the order of drive supply voltage the constant current capabilities of chopper amplifiers is lost and the available torque for acceleration decreases.

Two problems are encountered when running stepping motors at high speeds, the first is to supply the necessary current and the second is to provide the correct acceleration scheme to achieve high speed rates. The maximum speed of a given system can be determined by the zero work curve, see Figure V-17. Since at these high speeds there is very little mechanical damping, it is possible for the motor to undergo a disturbance, which will cause the system to go unstable. Many investigators (19-21) have classified this failure mode as the mid-frequency resonance. The zero work curve may provide insight into this failure mode. In figure V-17 the zero work curve for stepping motor system with a low value of gamma is shown. If this system were operating at a speed of 10 and it were asked to operate at a speed
of 11 then a larger lead angle would be called for. This would appear as a displacement disturbance to the system and with sufficiently low damping, the system may lose synchronism. In general the motor velocity will oscillate with growing amplitude, until the motor completely stalls. At speeds that are much greater then the start-stop rate the motor cannot reaccelerate to speed and the rotor will be stationary with the stator fields switching at some frequency given by the control system. The permanent magnet can become demagnetized in this process.

Another cause of instability can arise from dynamic imbalance in either the load or motor. Variations between phases can also contribute to instability. Since this can cause a periodic variation of torque, instability can result in low damping situations.

5-E Unidirectional Loads

In Figure V-18, a failure is shown for a three step sequence under a unidirectional load. If stepping is terminated after two steps, the motor executes both steps, but after three steps, the motor is on a trajectory which leads to a singularity at $-\infty$. The motor thus travels backwards under influence of the load. It should be noted for $\zeta = .05$ that this occurs for a dimensionless load torque as small as .14, or a torque equal to 14% of the maximum torque which the motor is capable of opposing. The load is assumed to be a velocity independent load. Typical of this would be a stepper called upon to lift a mass (a gravity load) with a ball screw.

5-F Relation To Manufacturers Specifications

It is difficult for the manufacturer to supply the user with
Failure due to Unidirectional Load

Figure V-18
all the information needed. A given motor may have some or all of the following application conditions. (Note the subscript \( m \) refers to motor parameters the subscript \( L \) to load parameters.)

1. Added load inertia
   \[ J = J_m + J_L \]

2. Added load friction
   \[ T_f = T_{fm} + T_{fL} \]

3. Added load damping
   \[ B = B_m + B_L \]

4. Added load stiction
   \[ T_s = T_{sm} + T_{sL} \]

5. Time varying load torques
   \[ T_L = T_L(t) \]

6. Velocity independent load torques
   \[ T_L = \text{constant} \]

7. Short sequences at the start-stop rate

8. Non constant step sequence periods, \( t_s \) not constant

9. Reversal sequences, forward and backward stepping

10. Sequences at motor and load resonant frequency \( t_s = \frac{2\pi}{\omega_n} \) or submultiples.

11. Driver amplifiers operating at other than rated current or voltage amplifiers with large series resistors \( \frac{V}{R} \neq I_{rated} \)

12. Repeated sequences of a fixed number of steps to be executed in minimum time.

All of these situations can be explored by the phase plane method. To use the model, motor and load characteristics must be known so that the numbers which characterize motor behavior can be determined. That is, for the PM motor the following dimensionless parameters are needed:

\[
\begin{align*}
\bar{T}_L &= \frac{T_R}{K_T V}, & \bar{T}_f &= \frac{T_{fm}}{K_T V}, & \bar{T}_s &= \frac{T_{sm}}{K_T V} \\
\zeta &= \frac{B}{2} \left( \frac{R_T}{JK_T AV} \right)^{1/2}, & \omega_n &= \left( \frac{K_T VA}{JR_T} \right)^{1/2}, & \gamma &= K_b \left( \frac{K_T}{AVJR_T} \right)^{1/2}
\end{align*}
\]
As has been shown, these parameters can be measured for a
given motor alone or supplied by the manufacturer. For the given load situations:

\[ T_{fm} + T_{fL} = T_f \]
\[ T_{sm} + T_{sL} = T_s \]
\[ B_m + B_L = B \]
\[ J_m + J_L = J \]
\[ I_{\text{actual}} = \frac{V}{R_t} \]

Then the needed dimensionless parameters can be calculated for the
given application, and the phase planes developed to explore the
proposed step sequences. Thus all that is needed to describe the
motor is:

\[ T_{fm}, T_{sm}, B, I_{\text{rated}}, J, K_T, K_b \]

These could be supplied by the manufacturer or determined by experiment.
With these values and specified tolerances, most applications of the
motor could be explored by the phase plane method. This would have the
following advantages.

1. Motors could be tested, by analysis, in an application to see if they were suitable.
2. Potential problems could be detected and explored to alter specifications or modify difficult step sequences if necessary.
3. Marginal designs could be detected before hardware was purchased.
4. Development plans could be formulated before hardware was purchased.
5. The necessity for feedback could be determined ahead of time.

In this way, the phase plane method could be used as a design tool to explore a potential application of stepping motors. This approach could save a considerable amount of development time. It could be conducted before or while the system was being constructed rather than after the fact. It would seem to offer savings in experimental development time.

On the other hand, motor manufacturers are spending considerable experimental time testing their motors to provide customers with usable design data. Using their methods of data presentation and considering the wide variety of stepping motor applications, this is an almost impossible task. The published information is restricted to long step sequences with the motor subjected to friction loads and in some cases additional inertia. Thus the phase plane method could offer a much better way of presenting motor characteristics for design purposes.

Stepping motor manufacturers do not have standard ways of defining motor behavior. Most often the applications engineer will have difficulties because the specifications do not pertain to his application. The manufacturers specifications tend to be empirically developed, and often the experimental methods will not be defined or even consistent between manufacturers. For example, torque-speed curves may be developed from start-stop or from slewing data, with the method not being specified. To make things still more difficult, load torque, which is nearly always assumed to be coulomb friction, may be applied in either of two ways. In one method, the fixed friction load
is applied to the motor at rest, then the motor is run to determine the maximum attainable speed. In the other method, the motor is brought up to a given speed with no load, then a friction load is gradually applied until the motor fails. If the load has stiction or damping, and the resulting friction torque is measured at a standstill, then inaccurate values of torque will be assured. If the friction is applied and then the motor run, the curve is specified a "pull-in curve" and if the load is applied after the motor is running, it is a "pull-out curve." Often no distinction is made and the user is left to assume which of these approaches has been used to obtain the curves.

5-G Summary

A model has been developed which represents both the static and dynamic behavior of a stepping motor. It includes all of the major parameters which affect the behavior of the system and can be expanded to include other nonlinearities without changing the order of the model as long as the inductive time constant is small. The phase plane includes stable singularities, or detent positions as well as the unstable positions and provides a graphical method of determining failure. Separatrices divide the phase plane into regions, a trajectory in any region leads to only one stable detent position or singularly. There is also a curve in the plane called the zero slope isocline which along with the \( \theta \) axis encloses the region where the motor may accelerate, thus the peak of this zero slope isocline represents the maximum slew speed of the motor.
The model is easily solved by digital computer techniques and provides considerable insight into motor behavior. The theoretical slewing speed can be calculated from the equation for the zero work curve, when the inductance effects cannot be neglected. The other failures can be simulated using the computer or predicted by graphical analysis from the phase plane portrait.
CHAPTER VI

MODEL EXPERIMENTAL VERIFICATION

6-A Introduction

The purpose of any analytical model is to predict the response of a certain system to a given input. Therefore, it is important that actual motor responses be measured and compared to the expected analytical solution. With an experimental verification, it then seems reasonable to use the model to predict the behavior of a postulated system. Further, the model can serve as a framework which is filled in by measured motor parameters and then used to predict motor behavior. This section offers the experimental evidence that the model does indeed predict the motor performance. Most of the measurements were done at or below the start-stop limit, since this is the region in which the majority of stepping motors operate.

The model is developed for a total system, including the motor, load, and drive circuit. This is important, and means that the motor is only one part in an integrated system which must be designed and evaluated as a whole. It has already been shown that drive impedance and supply voltage and load friction, damping, and inertia all effect motor response.

The computer model was written in Fortran and utilizes Runge-Kutta integration. It has been designed for interactive use at a Tektronix graphics terminal and can be used on any computer system with appropriate modification, or it can be programmed in a simulation.
language such as CSMP.

6-B Experimental System

In order to verify the model, it is necessary to have an experimental system which includes a motor, driver, and appropriate control circuitry, as well as transducers to measure torque, velocity, and position. Motor velocity was measured with a tachometer, position with a film potentiometer, and torque with a strain gage force-transducer located at known radius.

The experimental system consisted of:

1) a four-phase, permanent magnet stepping motor with a 90° step angle
2) a driver circuit which will drive one phase at a time
3) a potentiometer to measure angular displacement
4) a tachometer to measure angular velocity
5) a variable frequency source
6) a controller for the frequency source
7) an oscilloscope to display the velocity and position signals

The details of each piece of equipment are given below.

Stepping Motor: Eastern Air Devices 90° stepper, model LD20ABE-1R, with four phases and double-ended shaft (for easier transducer mounting).

Driver: A basic single phase R/L drive is used as shown in Figure VI-1. Other schemes can be used to drive two or all four phases at a time. The driver resistance is 30 ohms, the motor resistance and average inductance are 12 ohms and 15 millihenries, respectively. The current rise time for this circuit (i.e., time
FIGURE VI-1
R/L Drive Circuit
constant) is the inductance divided by the total resistance, which in this case is $3.6 \times 10^{-4}$ seconds. If the instantaneous current shift assumption is valid for a rise time of less than one tenth of the total cycle time, the constant current assumption for this motor and drive is accurate up to 280 steps per second.

**Potentiometer:** A CIC potentiometer having better than 0.1% linearity. Model number 205.

**Tachometer:** A ServoTek type ST-721-7B tachometer with 0.1% linearity.

**Frequency Source:** An Exact model 707 variable frequency function generator is used to supply the reference frequency pulses. A counter was also employed to give an accurate measurement of stepping rate.

**Controller:** The step command pulse generator circuit is shown in Figure VI-2. It is a latching, presettable, up/down counter with trigger output to control the function generator. The desired number of steps is present in the controller and then the start switch is depressed, causing the trigger output to go high until that number of pulses from the frequency source has been received. This allows a selectable number of fixed period step commands to be gated to the system.

**Oscilloscope:** A Tektronix 5103/D15 storage oscilloscope was used to display an tachometer output versus potentiometer bridge output. Pictures of the display were recorded with a Tektronix C-12 Oscilloscope Camera.

The mechanical setup is shown in Figure VI-3. Inertia wheels can be added to the motor to simulate load inertia. An aluminum disc
Controller

Figure VI-2
Experimental Set-Up

Figure VI-3
and a permanent magnet which can be moved radially relative to the disc are used to provide adjustable load damping. The tachometer and potentiometer are mounted as shown.

Since the motor has a 90° step angle, the shaft makes one complete revolution for every four steps. As the motor has four separate phases, the torque-angle curve will complete one cycle for one complete rotation of the output shaft. In terms of the mathematical model parameters, this means that \( A = 1 \).

**6-C Experimental Methods of Parameter Evaluation**

There are five dimensionless parameters which must be evaluated before the model can be utilized. Each one has been developed analytically. In this section, methods for measuring or calculating these parameters will be suggested. These suggested methods are not the only way to evaluate the parameters, but are merely guidelines for the applications engineer.

**6-C.1 Torque Angle Curve**

The first characteristic which is needed for the model is the torque-angle curve. To measure the torque-angle curve, the motor shaft was supported on two bearing hangers as shown in Figure VI-4. This leaves the motor housing free to rotate. The potentiometer was connected to measure rotor angle relative to ground. The housing was connected to ground through a Celesco force transducer which uses semiconductor strain gages as the sensing elements. The transducer was mounted at a known radius so the radius times the force reading indicates the torque exerted by the stepping motor. One phase is then
TORQUE-ANGLE MEASURING SYSTEM

FiguRE IV-4
energized with a known current. The stepping motor shaft is driven slowly by a DC motor to cause an angular displacement between the stepper rotor and stator. The voltage from the force transducer strain gage bridge indicating motor torque and the potentiometer bridge indicating displacement are recorded by an X-Y plotter to produce the torque-angle curve. The torque-angle curve measured this way is shown in Figure VI-5. It can be noted that the shape is nearly sinusoidal so that the assumption made in the model is valid. The current in the winding can then be switched to a new phase, which will result in a shift of the torque-angle relationship by 90 degrees. There are some variations in curve magnitude when one of the other three windings are energized. This variation is shown in Figure VI-5 where the torque-angle curves for each of the four phases energized are shown. Variation in magnitude and shape can be accounted for by many reasons; winding resistance differences between phases, friction and alignment of test set up and manufacturing tolerances.

The torque produced by a permanent magnet stepper motor was postulated to be proportional to current, see equation VI-1. This is true until saturation occurs in the iron, at which point the torque output becomes nonlinear. This relationship can be found by the experimental set up, by fixing the rotor at a known angle, preferably at the peak torque output, and increasing winding current, a plot of torque versus current with result as shown in figure VI-6.

\[ T(\theta) = K_T i \sin A\theta \]  

Thus the torque constant, \( K_T \), may be measured by either the torque-angle curve, as long as the current is below the saturation level, or the torque-current curve in the linear region. It is important in
Figure VI-5 Torque-Angle Curves for Each Phase Energized
FIGURE VI-6

TORQUE-CURRENT CURVE
either case that one determines if the motor is operating in the linear region when using the model, or if not one should modify the model for the nonlinear behavior.

Another method for measuring the torque constant, which is perhaps the easiest to implement, is to measure the BEMF voltage and infer the torque constant. This can be realized by considering the mechanical-electrical power conversion relationships, see equation VI-2. In the absence of losses the power out equals the power in.

\[ T_0 \, \text{o}z \in \frac{\text{rad}}{\text{sec}} = VI \, \text{watts} = 141.6 \, VI \, \text{o}z \in \frac{\text{rad}}{\text{sec}} \]

however torque and voltage are functions of current, angle and speed hence:

\[ K_I \sin(A \theta) \dot{\theta} = 141.6 \, K_b \dot{\theta} \sin A \theta \, I \quad (VI-2) \]

the angle and current cancel hence:

\[ K_t \frac{\text{o}z \in}{\text{amp}} = 141.6 \, K_b \frac{\text{volts}}{\text{rad/sec}} \]

Thus if the BEMF constant is determined, then the torque constant may be found provided that the current used is below the saturation current. This method has great advantages for small angle steppers as the position transducer is not needed. All one needs to do is measure \( K_b \) by driving the motor at a known speed and determining the open circuit voltage.

6-C.2 BEMF

The BEMF effect is described by the parameter \( \gamma \) where:

\[ \gamma = \frac{\omega_n K_B}{AV} \quad (VI-3) \]
Once the natural frequency is known, and the total voltage applied to the system is known, it is only necessary to find the BEMF constant, $K_b$ (volts/rad/sec). This is easily accomplished by externally driving the shaft of the motor, with a DC motor a constant speed. With the stator unenergized the sinusoidal voltage which appears on one winding is displayed on an oscilloscope. $K_b$ is found as shown in Figure VI-7. The BEMF voltage is a sinusoidal function of angle with the same period as the torque-angle curve. Knowing $A$, the number of cycles that the torque-angle curve makes per revolution, allows the motor speed to be determined by the period $T$ of the BEMF voltage versus time curve. Thus from the peak voltage is $V_m$ at any speed, the effective BEMF constant $K_b$ is:

$$\text{speed} = \frac{2\pi}{TA} \text{ rad/sec (VI-4)}$$

$$K_b = \frac{V_m}{2\pi} \frac{TA}{\text{volts} \cdot \text{rad/sec}} \text{ (VI-5)}$$

An experimental measurement of the BEMF is shown in figure VI-8, where the open circuit stator voltage is display as a function of time on an oscilloscope.

From the BEMF constant $K_b$ the torque constant $K_T$ may be found by equation VI-2. This is much easier to implement and as will be seen this experimental setup can serve to make other system measurements.

In the case of a VR motor the BEMF depends on stator current. By supplying a stator phase with a current source and driving the motor at a fixed speed and measuring stator voltage, the BEMF expression can be found.

6-C.3 Friction Torque

$$\dot{T}_f = \frac{T_f}{K_r T}$$. (VI-6)
Calculating $K_b$

Figure VI-7

$$K_b = \frac{V_m T}{2\pi} \frac{\text{Volts}}{\text{RAD/SEC}}$$
Figure VI-8

BEMF Voltage Versus Time
The peak torque, $K_T I_R$, is once again in the equation and is known, $T_f$ can be measured in a number of ways, but it should not be confused with static friction or "stiction." $T_f$ is the friction when the system is actually in motion. Stiction $T_s$ can also be included in the model. Friction and stiction can be determined by measuring the "dead zone" which the motor will exhibit around its actual equilibrium position and using equation III-48. If the width of this dead zone (the maximum angle through which the rotor can be moved without it returning to its equilibrium position) is designated $L$, and $T_s$ is the dimensionless stiction.

$$\tilde{T}_s = \frac{T_s}{K_T I_R}$$ (VI-6)

Then,

$$T_f + \tilde{T}_s = \sin \frac{AL}{2}$$ (VI-7)

where $L$ is in radians

$T_f$, the cumb friction is measurable by determining the minimum torque necessary to keep the shaft turning once it is in motion. The stiction and friction can also be measured using a torque watch, by measuring the torque that just causes motion. This must be done with the motor windings open and then the detent torque of the motor must be accounted for. Usually in instrument applications $T_s$ is small and can be neglected.

For small angle motors this may be a difficult measurement because of the accuracy required by the position transducer. Another method that can be easily implemented is to use the DC motor employed
FIGURE VI-9

TORQUE VERSUS SPEED
D.C. MOTOR
in making the BEMF measurements. The torque produced by a DC motor is proportional to the armature current. If a plot of armature current versus speed for a DC motor is made, see figure VI-9, then the friction and damping of the DC motor may be found, by calibrating the current in terms of torque and knowing the speed. Once the damping and friction for the DC motor is known, any rotary device may be coupled to the DC motor, and the damping and friction may be found. An example of this measurement is given in figure VI-10, where the lower trace is the DC motor alone and the upper trace is the DC motor coupled with stepper. If the device being measured has a stiction component then that may be found also.

6-C.4 Damping Ratio

\[ \zeta = \frac{B \omega_n}{2AK,I_R} \]  

(VI-8)

The mechanical damping is a very important parameter. The motor mechanical damping is extremely small when compared to other forms of damping (i.e., air drag, viscous effects, etc.) which come from the load and not the motor itself. If the step size is large, and the friction is insignificant, the motor rotor can be deflected a small amount and the damping ratio determined by comparing the response to that of a second order system. This could be done using a position response of the form shown in Figure VI-11 using the log decrement method. However, the friction also causes the oscillation amplitude in Figure VI-11 to decrease. This obscures the effect of mechanical damping.

For small step sizes this is a difficult measurement as mentioned before and the DC motor allows a very fast and accurate measurement of
FIGURE VI-10

TORQUE VS SPEED
FOR
DC MOTOR-STEPPER AND
DC MOTOR ONLY
Finding $\omega_n$ from the Response After an Initial Displacement

Figure VI-11
both damping and friction as well as BEMF. Furthermore, the torque constant may be calculated from the BEMF, thus all but the rotor inertia and stator resistance and inductance can be found with one test setup.

This is not the only method to solve for $\zeta$, but it is the easiest to implement in a design application. In the application of the motor to a given stepping system, the load damping will augment the measured $\zeta$ for the motor alone. This load damping must be included in the model and will influence the response strongly since, typically, motor damping is so small. Thus load damping may completely determine motor behavior.

6-C.5 Natural Frequency

The natural frequency $\omega_n$ can be determined by one of two methods. From the earlier analysis the natural frequency is:

$$\omega_n = \sqrt{\frac{K_T I_R A}{J}}$$  \hspace{1cm} (VI-9)

The combined rotor and load inertia, $J$, is the total inertia of the system and may be either measured or calculated. The motor inertia is usually given in the manufacturer's catalog. It is generally possible to calculate the load inertia, and thus with the torque constant known the natural frequency may be calculated using equation VI-9.

It is also possible to measure $\omega_n$ by deflecting the rotor a small amount (it must be small because the system is nonlinear) and releasing it. The response will be that of a second order system and the natural frequency can be found by standard means. This is shown in Figure VI-11. This is somewhat difficult to do for motors with small step angles, but can be used as a check for the calculated value.
6-C.6 Load Torque

\[ \tilde{T}_L = \frac{T_L}{K_T I_R} \]  \hspace{1cm} (VI-10)

Thus \( \tilde{T}_L \) is simply the velocity independent load torque divided by the peak torque of the torque-angle curve. \( \tilde{T}_L \) will be zero unless a velocity independent load is used, such as a gravity load. Both the calculated and measured methods for such a load are straightforward.

6-D Step Sequence Comparisons

After all the parameters have been measured, the measured motor response must be compared to the response predicted by the computer simulation. The program which produces the phase planes and stepping sequences by shifting the trajectories to the right or left discontinuously is useful to graphically display failure, but it is hard to relate to the measured trajectories. For this reason, another program was written to reproduce the calculated response in the same format as would be observed on an oscilloscope face, thus making the comparisons easier to visualize. A number of different input sequences were applied to the system and the corresponding computer predicted responses were generated. Some of the results are displayed on the following pages. These results not only verify that the model represents motor behavior but illustrate several failure modes.

The experimental verification was made with a tachometer and a potentiometer attached to the motor with no additional inertia damping. With a given L/R ratio, and a fixed four phase voltage drive, driving one phase at a time, the peak torque, \( T_m \), can be found. This is often given in the manufacturers data, but was measured here by driving
the motor with a DC motor and measuring the BEMF, to calculate the torque constant. A force transducer can then be placed on the motor to measure the torque-current curve so that the motor will be operated in the linear torque constant range, as described earlier.

The damping and friction were found, using the DC motor from the current-speed curve. The natural frequency was measured from the single step response, by determining the period of the last few oscillations. It is important that when using this method for finding \( \omega_n \), that only the last oscillations are used; because of the non-linear nature of the system. With these parameters it is possible to calculate all of the dimensionless parameters for the model.

The measured single step response and the computer response determined by the measured parameters are shown in figure VI-12. The phase plane for the motor developed from the mathematical model based on the measured parameter values is shown in figure VI-13.

For step sequences, small changes in step period and variations between detent positions can cause large changes in system trajectories. A procedure was needed to obtain a quantitative measure of the model fit to the experimental data. Thus it was decided to predict the sequence response using the computer simulation. Then the measured response was determined by varying step period, \( T_s \), until the measured and predicted responses were almost identical. The difference between the period used in the simulation and the period used in the experimental measurements was then used as a measure of model fit.

Figure VI-14 shows the phase plane portrait for a four step sequence developed from the phase plane by shifting the trajectory to the left \( \frac{\pi}{2} \) at each step instant. The separatrix are also shown on the
Figure VI-12

Single Step Responses
ZETA = 0.0015 TL = 0.0000 TF = 0.0375 TS = 0.0000 GAMMA = 0.1260 DELT = 0.0100
MODEL # 0

FIGURE VI-19

PHASE PORTRAIT
ZETA = .8915  TF = .9375  GAMMA = .1260
UNT = 7.2000  NSTPS = 4  FNTIM = 49.760

FIGURE VI-14

PHASE PLANE STEPPING, UNT = 7.2
The stepping frequency is close to the natural frequency of the system, thus when step two is applied the motor has made one revolution in the phase plane and step three causes the sequence to cross the left separatrix. After the fourth step the motor settles at a detent position that is four steps behind the desired location. Since four forward steps were called for, the motor has returned to the same starting position. If a three step sequence had been executed the motor would have lost one step.

Note that the stepping period used in figure VI-14 is $WNT = 7.2$, where the small amplitude period is 6.28. Because of the nonlinear system the period for large amplitudes will be larger. There is a range of periods around the natural frequency that will cause step failure and in general this range will be larger for low damping and low friction systems. Figure VI-15 is the computer developed phase plane to convert the stepping sequences of the model into the form most useful for comparison of simulations to oscilloscope photographs. Since the step size is $90^\circ$, four steps will cause the rotor to turn $360^\circ$ or one complete revolution. This is seen in Figure VI-15 where the horizontal shifts occur to simulate the potentiometer shift every $2\pi$ radians. By counting these shifts, one can find the total travel, but since shifts can always occur in both directions care must be taken. To compare the measured and calculated phase planes, the period of the input step sequence was varied until the two response curves were similar. The periods, $T$, are indicated in figure VI-15.

The next well known failure mode is stepping above the start-stop rate. Figure VI-16 shows a four step sequence above the start-stop rate, where the stepping period is so small that the motor does
\[ T_m = 86.4 \text{ m sec} \]
\[ T_c = 84.8 \text{ m sec} \]

1.9% Error

\textbf{Figure VI-15}

Four Step Responses WNT = 7.2
ZETA = .0015  TF = .8575  GAMMA = .1269
UNT = .8800  NSTPS = 4  FVTIM = 49.750

FIGURE VI-16

PHASE PLANE STEPPING, UNT = .8
not have enough time to accelerate and the stepping trajectory crosses the separatrix after the four step resulting in a position error of four steps. Note that if a three step sequence would have been called for the motor could have executed it, although a large transient response would have occurred. In Figure VI-17, the computer and measured responses shows that the motor has not executed the four steps and remained at its original detent position.

Figure VI-18 shows a eight step sequence that was executed without the loss or gaining of steps, although a separatrix was crossed during the third step it was crossed back during the fifth step. The resulting measured and computer plots are shown in figure VI-19. Here the two horizontal shifts indicate two revolutions and thus the execution of the eight steps.

Figure VI-20 shows a four step sequence at the same stepping period. Since the fifth step was not executed to recross the separatrix, a position error of four steps results. Eight steps were executed instead of the four called for. This is also shown in figure VI-21. This type of failure mode occurs below the start-stop rate for short step sequences when the system has low damping and friction.

Figure VI-22 shows a stepping period that can be executed for any number of steps, since it never crosses a separatrix it will never gain or lose steps. The measured and computer response are shown in Figure VI-23, clearly the four step sequence was executed correctly. If for both this and the previous example, the motor was allowed to continue for a large number of steps a small closed path, a limit cycle, will be formed. This limit cycle will always be below the separatrix thus for a large number of steps at the start-stop rate or below, the step sequence will be executed correctly.
Figure VI-17

Four Step Responses, WNT = .8
FIGURE VI-18

PHASE PLANE STEPPING, VNT = 1.2
\[ T_m = 14.189 \text{ m sec} \]
\[ T_c = 14.134 \text{ m sec} \]

.39% Error

Figure VI-19

Eight Step Responses, WNT = 1.2
FIGURE VI-29

PHASE PLANE STEPPING, \( \text{UNT}=1.2 \)
\[ T_m = 14.208 \text{ m sec} \]
\[ T_c = 14.134 \text{ m sec} \]

\[ \text{.52% Error} \]

Figure VI-21

Four Step Responses, WNT = 1.2
\[
\text{ZETA} = 0.8815 \quad \text{TF} = 0.0375 \quad \text{GAMMA} = 0.1260
\]
\[
\text{UNT} = 1.5000 \quad \text{NSTPS} = 4 \quad \text{FINTIN} = 48.840
\]

**Figure VI-22**

Phase Plane Stepping, UNT = 1.5
\[ T_m = 17.685 \text{ m sec} \]
\[ T_c = 17.668 \text{ m sec} \]

1% Error

Zeta = 0.2315  \quad WNT = 1.5833  \quad Gain = 0.1293

Figure VI-23

Four Step Responses, WNT = 1.5
CHAPTER VII

OPTIMUM STEPPING SEQUENCES

7-A Introduction

It is possible to find stepping sequences which have minimum overshoot and minimum response time by the use of the phase portraits. By determining the minimum response time of a system for a given number of steps, it is possible to optimize the system for a given repetitive command sequence, and in doing so, a general method for finding optimum sequences may be discerned. It will be noted that this is possible only if the motor, drive, and load are all known and can be characterized by the five dimensionless parameters.

If the motor is to exhibit no overshoot to a step sequence consisting of a fixed number of steps, the last step must occur when the motor trajectory has zero velocity and has a dimensionless position of one step away from the origin. For stepping in the phase plane, the motor trajectory will normally be shifted discontinuously to the left by an amount equal to the step size, for a four phase motor full stepping $\pi/2$. This is equivalent to saying that when the final step command is given, the position of the motor in the phase plane will be one step to the right of its final stable equilibrium point. If this is true there will be no oscillation, as shown in Figure VII-1, for a three step sequence.

This is easily plotted on a phase portrait, but the required times for the step commands to be initialized are hard to establish
Final Position for no Overshoot

Figure VII-1
graphically. For this reason, the computer is best used to determine the stepping times. In order to utilize this type of approach, one must develop some kind of strategy to accelerate and decelerate the motor, at the appropriate times, as to achieve minimum execution time and minimum overshoot. One such strategy, that could be used, is based on the zero slope isocline. Since the area under the zero slope isocline is a region in which the motor accelerates, and outside that area the motor must decelerate.

This development will be followed by other strategies that try to optimize the acceleration and deceleration to achieve maximum execution times.

7-B Zero Slope Isocline Switching

To find the period between steps, a computer program is utilized which allows the motor to accelerate to the zero slope isocline and then executes another step. In order to efficiently calculate the deceleration steps, the motor is started at the desired final position and time is run backwards until the number of desired deceleration steps have been executed. A final step is found such that the acceleration ramp and the deceleration ramp are connected by one step at a given velocity.

Thus an optimum switching sequence is based on the following approach.

1. If the position of the motor is one step away from the desired equilibrium point and at zero velocity at the instant of the final step command then the motor will certainly have no overshoot.

2. If the motor follows a trajectory until the acceleration is zero (zero slope on the trajectory) then it is certainly not losing
speed in any step.

3. During deceleration if the motor is again switched so that after the step, it is located on the zero slope-isocline it decelerates without slipping ahead to gain steps.

If the motor is stepped when it reaches zero acceleration (on the edge of the zero slope isocline), the motor will always be accelerating. This is useful for ramping the motor as shown in Figure VII-2. Thus the motor is stepped when its trajectory reaches the zero slope isocline. To slow down as fast as possible, the motor is stepped down along the outside of this isocline so that it always decelerates between steps. This slowdown strategy is also shown in Figure VII-2.

To combine the ramping or slewing strategy with the minimum overshoot strategy provides a method of obtaining fast response for a sequence of fixed number of steps. The only drawback to solving this problem for a given set of parameters is the step at the top of the stepping sequence, step 4 in Figure VII-3.

Since all steps except for one at the top of the sequence have one end on the zero slope isocline, and this is the curve where the slope of the trajectories is equal to zero, construction of all except the top step is easily accomplished. At the start of the sequence, the first step will cause the motor to have position-velocity co­ordinates of $-\frac{\pi}{2}$, 0. That trajectory is followed as it accelerates until the slope equals zero, and another step is then executed. Likewise, the position and velocity of the motor immediately before the last step are known to be $+\frac{\pi}{2}$ and 0, respectively. By integrating with negative time increments, it is possible to work backwards from this point, constructing the last steps in the same manner as the
Stepping on the Zero Slope Isocline

Figure VII-2
Minimum Stepping Sequence

Figure VII-3
first, always stepping when the slope is zero.

When all of the desired number of steps have been constructed except for one, the last step must be found. To do this, the trajectories from both sets of steps, one accelerating, one decelerating, are extended, and at a given point they will both have the same velocity and their positions will be a distance equal to the step size \(\frac{T}{2}\) apart. The last step is drawn in between these two points.

To develop a program to do this on a digital computer, it is difficult to determine how many acceleration and deceleration steps must be used in order to complete step sequence. In general a system will need more acceleration steps to achieve a given velocity then deceleration steps to stop from that velocity. A program has been developed which works by inputing the number of acceleration and deceleration steps needed, but more research must be devoted to this to perfect the method. A trial and error procedure is used, where the number of acceleration and deacceleration steps are inputs. Different combinations can be tried until the sequence is completed for the desired total number of steps. When this is satisfied the sequence and times between steps is outputed, see figure VII-4.

The optimum sequence as mentioned above has some very unique properties. The optimum sequence consists of a number of step commands spaced in time at unequal intervals which cause the motor to accelerate and decelerate rapidly. By causing the last step to occur at the correct time, the motor will come to rest with no overshoot. This sequence is valuable in applications where the motor is repeatedly asked to execute a fixed number of steps. Since there is no settling time, the system will be stationary immediately after the last step is
OPTIMUM STEPPING SEQUENCE OF 4 NSTPS
FORWARD STEPS- 2  BACKWARD STEPS- 1

ZETA= 0.00100
GAMMA= 0.00000
FRIC= 0.04270
LOAD= 0.00000

THESE ARE YOUR TIMES BETWEEN EACH STEP

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<th>PERIOD IN SECONDS</th>
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</table>

FIGURE VII-4
4 STEP OPTIMUM SEQUENCE
executed allowing another operation to occur in any overall system and eliminating any settling time. In order to do this effectively the motor and load friction and damping and load torques must be constant. Changes in these parameters will alter the trajectories and will make the chosen sequence suboptimal.

A very useful result from this process, is that once a particular timing sequence is selected, the system parameters may be varied in order to determine when the system will no longer execute the sequence correctly or within the system specifications, see figure VII-5. Here a system was designed with an optimum sequence for given values of friction and damping. These values can then be varied independently to find the effects on settling time. The system performance can now be predicted for changing system parameters.

It is interesting to note that feedback methods used to get high slew rates are based on a strategy similar to that shown in Figure VII-2. An encoder is attached to the motor which emits a pulse when motor position equals zero or some other fixed angle on the phase plane. If motor acceleration were detected and switching accomplished at nearly zero acceleration a more effective switching strategy would be accomplished for acceleration. The deceleration strategy is more difficult as it is desired to switch to the zero acceleration point which occurs when the next phase excitation is energized.

7-C Maximum Acceleration Switching

Another strategy for obtaining fast acceleration times would be to maximize the average acceleration over one step cycle. The maximum acceleration will occur when the maximum torque over that cycle occurs.
FIGURE VII-6
EFFECT OF FRICTION AND DAMPING ON EXECUTION TIME
This can be found in a similar manner as was the zero work curve.

Figure VII-6 is a plot of the static torque-angle curve for a four phase motor. The maximum work for this curve would occur for a step centered around $-\frac{\pi}{2}$, since that is the maximum area. At any constant speed the dynamic torque angle curve may be found, see figure VII-7. As the speed is increased, the lag angle necessary to maximize the work will increased, thus a dynamic analysis must be made to develop a strategy for optimum time response. Furthermore, the $L/R$ time constant will effect the dynamic torque which will be considered later.

The development of the maximum work curve will be based on the phase plane model which neglects the inductance but includes the effects of BEMF. If the motor is assumed to be traveling at a constant velocity, then equation VII-1 describes the relationship between the motor position and velocity as a function of time.

$$\theta = \theta_0 t$$  \hspace{1cm} (VII-1)

In the steady state the mechanical equation of motion reduces to equation VII-2, where $T_A$ is the torque available to accelerate the motor. Solving equation VII-2 for the acceleration torque and normalizing the equation as before results in equation VII-3. The voltage equation for the stator phase will result in the normalized equation VII-4, if the $L/R$ time constant is neglected. Thus the resulting normalized acceleration torque equation is given by VII-5. Since this is an instantaneous torque at a particular angle and speed, to get a measure of the torque applied to the rotor by the stator when a given phase is on some form of averaging or integrating must be used. The work done during one step period is an integral function and it can be
FIGURE VII-6

STATIC TORQUE-ANGLE CURVE
FIGURE VII-7

DYNAMIC TORQUE ANGLE CURVE
found by integrating equation VII-5 over the step angle, where $\theta_0$ is the lead angle at the beginning of the step, see equation VII-6. For some lead angle, $\theta_0$, a maximum for equation VII-6 can be found for any given constant speed.

$$T_A = -K_T i \sin \alpha - \beta_0 - T_f$$  \hspace{1cm} (VII-2)

let

$$T_A = \frac{T_f}{K_T I_R}$$

$$\zeta = \frac{B_0 n}{2AK_T I_R}$$

$$T_f = \frac{T_f}{K_T I_R}$$

$$\bar{I} = \frac{i}{I_R}$$

$$\bar{T}_A = -\bar{I} \sin \theta - 2\bar{\zeta} \theta - \bar{T}_f$$  \hspace{1cm} (VII-3)

$$\bar{I} = 1 + \gamma \theta \sin \theta$$  \hspace{1cm} (VII-4)

where $\gamma = \frac{K_B \omega n}{A V_R}$

$$\bar{T}_A = -\sin \theta - \gamma \theta \sin^2 \theta - 2\bar{\zeta} \theta - \bar{T}_f$$  \hspace{1cm} (VII-5)

$$\theta_0 + \pi/2$$

$$\text{WORK} = \int_{\theta_0}^{\theta_0 + \pi/2} \bar{T}_A \, d\theta$$  \hspace{1cm} (VII-6)

A computer program was developed to simulate the above equations. The lead angle, $\theta_0$, for a given velocity, $\dot{\theta}$, that resulted in the maximum work over that step, was stored and plotted in the phase plane, see figure VII-8. Also plotted on figure VII-8 is the zero slope isocline, which are the points of zero acceleration. A more useful way to represent the maximum work curve is to plot the switching angle to achieve maximum acceleration. This can be done easily by adding the step size ($\pi/2$) to the lead angle in figure VII-8, which results in figure VII-9. Note that a direct comparison between switching on the zero slope isocline and the maximum work curve can now be made.

It appears from figure VII-9 that a better switching strategy
FIGURE VII-8

LEAD ANGLE FOR MAXIMUM WORK CURVE
FIGURE VII-9

SWITCH ANGLE FOR MAXIMUM WORK CURVE
than the zero slope isocline can be found. The maximum work curve could be directly implemented in developing an optimum step sequence, by finding a maximum negative work curve, which would result in maximum deacceleration. However for every step executed, the step angle would have to be varied to find the maximum work done. Since in developing a step sequence the mechanical equations of motion must be integrated versus time this would become a very lengthy process. Thus an approximate method for stepping on the maximum work curve will be used.

A computer program was developed which found the maximum acceleration at each step and then executed the next step at a fixed angle from the maximum acceleration point. This tries to achieve a maximum average acceleration over one step. Note that for the static torque-angle curve, maximum acceleration occurs at $-\pi/2$ and one would want to switch at $-\pi/4$ to achieve maximum torque. In order to increase the flexibility of the program, the switch angle, after maximum acceleration occurs, is an input to the program, so that it may be varied to optimize the stepping profile.

The results of such a optimum sequence are shown in figure VII-10, where a ten step sequence for a given system is executed. Note that the sequence steps on the switching line predicted by the MWC, see figure VII-9.

7-D Experimental Comparison

In order to verify the optimum sequence programs and compare the two different ramping techniques, an experiment test set up was used. A ninety degree stepper and a tachometer, to measure rotor velocity, was used with a variable pulse length generator. System
OPTIMUM STEPPING SEQUENCE OF 10 NSTPS
FORWARD STEPS— 6  BACKWARD STEPS— 3

ZETA = 0.00100  GAMMA = 0.12800  FRIC = 0.01000
LOAD = 0.00000  XLEAD = 0.85000

FIGURE VII-10

TEN STEP OPTIMUM SEQUENCE
FOR MAXIMUM ACCELERATION
parameters were measured with the techniques discussed earlier and the two computer programs were used to find the optimum step times for each method. Figure VII-11 shows the step sequence given by the zero slope isocline method. In order to compare these results with the experiment data the motor velocity versus dimensionless time was also plotted by the computer program. This is shown in figure VII-12, note that between steps there is a discontinuity in velocity. Figure VII-13 is a listing of the time between pulses for each step. A total execution time of 82.8 msec is predicted by the model.

The maximum acceleration technique was then used to develop a ten step sequence for the same system. Figure VII-14 is the phase plane stepping sequence for the maximum acceleration technique. Figure VII-15 shows the velocity profile for the same sequence, note that in this method the velocity profile is smooth between steps, except for the top fitting step. Figure VII-16 is the listing the stepping periods, where the total execution time has been reduced to 78.4 msec.

Both sequences were programmed into the experimental set up and the velocity profiles were displayed on a storage scope. Figure VII-17 is a photograph of the velocity profile and the pulse sequence used, shown as a series of dots in the upper part of the trace. The lower photograph is the ZSI method response, where the total execution time is approximately 80 msec less the small amount of overshoot. The upper picture is the velocity profile given by the maximum acceleration method, where the execution time is reduced to less than 80 msec, but the overshoot on the last step is considerably greater. This should be easily reduced by changing the timing of the transition pulse. By changing the seventh pulse, the velocity profile was modified to that
OPTIMUM STEPPING SEQUENCE OF 18 STEPS
FORWARD STEPS - 7  BACKWARD STEPS - 2
ZETA = 0.86188 GAMMA = 0.24788 FRICT = 0.00588 LOAD = 0.00088

FIGURE VII-11

ZERO SLOPE ISOCLINE
TEN STEP OPTIMUM SEQUENCE
FIGURE VII-12

VELOCITY PROFILE
OPTIMUM STEPPING SEQUENCE OF 10 NSTPS
FORWARD STEPS- 7 BACKWARD STEPS- 2

ZETA= 0.00100 GAMMA= 0.24700 FRIC= 0.06500 LOAD= 0.00000

THESE ARE YOUR TIMES BETWEEN EACH STEP

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FIGURE VII-13

STEPPING PERIODS FOR SEQUENCE
OPTIMUM STEPPING SEQUENCE OF 10 NSTPS
FORWARD STEPS—7 BACKWARD STEPS—2
ZETA= 0.08100 GAMMA= 0.24700 FRIC= 0.06500
LOAD= 0.00800 XLEAD= 0.98000

FIGURE VII-14
MAXIMUM ACCELERATION FOR
TEN STEP SEQUENCE
FIGURE VII-15

VELOCITY PROFILE
OPTIMUM STEPPING SEQUENCE OF 10 NSTPS
FORWARD STEPS—7 BACKWARD STEPS—2

ZETA = 0.00100 GAMMA = 0.24700 FRIC = 0.00500
LOAD = 0.00000 XLEAD = 0.99999

THESE ARE YOUR TIMES BETWEEN EACH STEP

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<td>STEP# 10 = 1.45000</td>
<td>0.01250</td>
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| TOTAL                | 9.10000           | 0.07845

FIGURE VII-16
STEPPING PERIOD FOR SEQUENCE
FIGURE VII-17
EXPERIMENTAL VELOCITY PROFILES
shown in figure VII-18. The execution time is now 72 msec and the overshoot is minimal.

Two methods for developing optimum sequences are given and experimentally verified. The maximum acceleration method yielded better performance but some modification of the stepping periods was needed. In any system there will be inductance which has been neglected in this model, but could be added to it. A more complex model should be used when the system being designed, is going to be operated in the slew range of the driver-motor and load. Although the resulting model for including inductance will be third order it is still possible to plot the trajectories in the phase plane, but they might cross each other. The real power of the phase plane is the development of step sequence and their display.
FIGURE VII-18

MODIFIED MAXIMUM ACCELERATION SEQUENCE PROFILE
CHAPTER VIII

APPLICATION AND DESIGN METHODS

8-A Introduction

The applications of stepping motors have been extended to many areas; computer peripherals, process controls and machine tools. Almost every system design using stepping motors has different requirements. Driving large inertias, lifting weights, short start-stop sequences, slewing speeds requiring position accuracy and repeated step sequences are just a few examples. Since most manufacturers' data supplied is obtained from a motor with a given type of load, driver and sequence, one must be able to extrapolate from the data to predict how the motor will perform for the system being designed.

One of the most difficult choices in the application of stepping motors is the initial selection of what size motor to use. One can always pick a larger size motor than is necessary, but that results in extra cost, a larger driver and power supply and more space may be needed for a larger frame size motor. Many different schemes have been used to select motors, based on the start-stop curve, slew curve, or acceleration capability. The problem with these approaches is that the drive and load used by the manufacturer is often not specified and in any case will not be the same as the system that is to be designed. In acceleration methods using the motor and load torque to inertia ratio, one must be careful to use the correct torque. The manufacturer's value for peak torque is usually specified.
with two phases energized. If single phase excitation is to be used this value must be divided by $\sqrt{2}$ for a 4 phase motor.

A method for selecting a motor will be described which is based on the peak torque and the total friction of the system. The reason for this choice is that the static accuracy specification can be satisfied and then the phase plane model can be used to predict if the selected motor will meet the speed and time requirements. One drawback to this method is that the friction of the proposed system must be known, thus a proto-type or a previous system must be constructed to predict the friction values.

8-B System Requirements and Motor Selection

A proposed stepping motor application will be presented to select a motor from the manufacturer's data. The phase plane model will then be used to predict and optimize the driving sequence to be used in the designed system. The application is carriage line feed system for a computer terminal. The desired system has to line feed one to six copies of fifteen-inch wide paper that is continuously fed through the machine. Line feeds are expected to be $1/6 \pm .005$ inches apart on the paper and have to be executed within .020 seconds. Further, special control characters allow eight-line and fifty-line feeds as fast as possible. The motor is expected to operate from the existing 12-volt power source with a maximum current of 1.5 amps.

Initial data needed for the system is the required step angle and load friction. The stepping motor is to be coupled to the load by gears, see figure VIII-I. The load is represented here by a cylinder, but actually consists of a set of tractors on a shaft. This system
FIGURE VIII-1
CARRIAGE LINE FEED SYSTEM
allows two degrees of freedom in the design. First the motor step angle is governed by the gear ratios and the required motion of the carriage. The gear ratio is specified by $R_1$ and $R_2$ as shown in Figure VIII-I. Since a four phase motor will be used, a twelve step sequence could be used to move the required 1/6 of an inch. This has the advantage in that, the same phase will be energized during printing to minimize error differences between phases. The relation between step angle and linear displacement is given in equation VIII-1.

$$\theta = 12\theta_s = \left(\frac{R_2}{R_1 R_3}\right) \left(\frac{360}{2\pi}\right) X$$

(VIII-1)

- $\theta$ - absolute angle in degrees
- $\theta_s$ - step angle in degrees
- $X$ - linear displacement
- $R_3$ - paper tractor driver radius

Secondly the gearing can reduce the reflected inertia, $I_{EQ}$, of the load by the square of the gear ratio, see equation VIII-2. By varying tractor drive radius $R_3$ different gear ratios can be used. Added inertia will lower the maximum stepping rates of a given motor, so it is often desirable to reduce the reflected inertia to a minimum where system speed is important. Increasing inertia will decrease the system damping ratio to cause longer transients and will decrease the system's natural frequency to cause slower speeds. The calculated inertia, $I_2$, of the carriage is given in equation VIII-3.

$$I_{EQ} = \left(\frac{R_1}{R_2}\right)^2 I_2$$

(VIII-2)

$$I_2 = .01 \text{ oz in sec}^2$$

(VIII-3)
Eastern Air Devices, Inc.
PERMANENT MAGNET STEPPING MOTOR
1.8°-5°-7.5°

SIZE 23
2-7/32" Diameter

FIGURE VIII-2

EAD Size 23 permanent magnet DC stepping motors are precision bi-directional devices with position accuracy of ± 3% noncumulative. Motors are totally enclosed with permanently lubricated ball bearings. Standard motors have 6 leads. Motors with 5 or 8 leads can be furnished to meet existing applications.

PERFORMANCE DATA - 4 PHASE STEPPING MOTOR

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<tr>
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<th>STEP ANGLE °</th>
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<th>CURRENT PER PHASE AMP</th>
<th>RESISTANCE OHMS</th>
<th>INDUCTANCE MILLIHENRIES</th>
<th>NOMINAL RATED TORQUE OZ.-IN.</th>
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<th>ROTOR INERTIA GM-CM²</th>
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PERIODIC PROPORTIONAL CHARACTERISTICS (START-STOP)

SPEED (STEPS SEC.)

CONNECTION DIAGRAMS

4 PHASE STEPPING MOTOR

2 PHASE STEPPING MOTOR
An estimated maximum friction for the system is 10 oz-in, based on previous designs. The stepping rate requirement is twelve steps in .020 seconds or a average of 600 steps per second if there is no transient on the last step. For a given gear ratio a motor can now be selected from figure VIII-2 and tested to see if it meets the system requirement. The motor has to be selected so its current rating is within the required value of 1.5 amps and with a series resistance it can be driven from the 12-volt available source. First picking a motor step angle of 1.8 degrees and given a commercial tractor set that has a displacement of four inches per revolution, \( R_3 = \frac{2/\pi \text{ in}}{\text{Rad}} \) the ratio \( R_2/R_1 \) can be found from equation VIII-1, \( R_2/R_1 = 1.44 \).

The angular error has two components, one due to friction and the other due to manufacturing tolerances. Because of the measuring method used by the manufacturer, the tolerance is made up of mechanical tolerances and the motor friction. From Equation VIII-1 the total error may be calculated assuming a 5% manufacturing error. Since, the desired system error is given, and friction is known, equation VIII-4 may be rearranged to solve for the peak torque, \( T_m \), needed to meet the system requirements, see equation VIII-5. From equation VIII-5 and the system specifications that \( X = .005 \) the angular error would be, \( \theta_{\text{error}} = .51 \) degrees. Thus a minimum peak torque of \( T_m = 28 \) oz-in is needed to insure the static accuracy of the system.

\[
\theta_{\text{Error}} = \frac{1}{A} \sin^{-1} \left( \frac{T_f}{T_m} \right) + .05 \theta_s \quad (\text{VIII-4})
\]

\[
T_m = \frac{T_f}{\sin(A(\theta_{\text{Error}} .05 \theta_s))} \quad (\text{VIII-5})
\]
The positioning time requirement for the system may be tested by selecting a motor from figure VIII-2 and running a simulation for each motor that meets the static accuracy requirement. Another method is to estimate the torque needed to accelerate the given load in the required time. An approximate method for this would be to assume a constant acceleration and deceleration for the step sequence. For this assumption, the velocity and positions are given by equations VIII-6 and VIII-7. If it is assumed that the velocity profile is triangular with an equal number of acceleration and deceleration steps, then the acceleration may be found from equation VIII-7 for six acceleration steps in 10 msec, which is half the execution time.

\[ V = at \quad \text{(VIII-6)} \]
\[ x = \frac{1}{2} at^2 \quad \text{(VIII-7)} \]

Substituting position and time into equation VIII-7 yields the acceleration. The peak velocity may be found from equation VIII-6. If damping BEMF and friction are neglected in the mechanical equation of motion, the acceleration can be related to the torque to inertia ratio, see equation VIII-8. Table VIII-1 is a listing of the possible motors that are applicable to this design. In comparing the torque to inertia ratios, from Table VIII-1, all three motors appear to be about the same and satisfy the static accuracy and acceleration requirements. This same method could be used for two phase or four phase on at a time operation as long as the motor was being full stepped. For half stepping, the peak torque will vary every other step, so the peak torque choice is not obvious.
### TABLE VIII-1

Motor Selection Chart

<table>
<thead>
<tr>
<th>Motor</th>
<th>Voltage</th>
<th>Current</th>
<th>( k_b ) (Volts/\text{Rad/sec})</th>
<th>Peak Torque One phase on</th>
<th>Rotor Inertia (oz in sec(^2))</th>
<th>Torque to Inertia Ratio</th>
<th>( \frac{V}{k_b} ) (steps/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LA23ECK-4</td>
<td>5.1</td>
<td>1.0</td>
<td>0.265</td>
<td>37.5</td>
<td>( 1.24 \times 10^{-3} )</td>
<td>18,500</td>
<td>1441</td>
</tr>
<tr>
<td>LA23ACK-1</td>
<td>6</td>
<td>0.88</td>
<td>0.397</td>
<td>49.5</td>
<td>( 1.7 \times 10^{-3} )</td>
<td>19,880</td>
<td>962</td>
</tr>
<tr>
<td>LA23BCK-1</td>
<td>5.4</td>
<td>1.5</td>
<td>0.299</td>
<td>63.6</td>
<td>( 2.85 \times 10^{-3} )</td>
<td>17,472</td>
<td>1277</td>
</tr>
</tbody>
</table>
\[ a = 3770 \text{ Rad/sec}^2 \]
\[ V = 37.7 \text{ Rad/sec} = 1200 \text{ steps/sec} \]
\[ T/J = a = 3770 \text{ Rad/sec}^2 \] (VIII-8)

The problem with selecting a motor by the torque to inertia ratio is that the motor can not provide a constant torque, that is, the torque angle curve is sinusoidal and therefore dependent on angle. Furthermore, its magnitude is dependent on the current through that winding. For a voltage drive, the BEMF voltage will limit the high speed behavior of the motor. Equation VIII-9 is the electrical equation, neglecting inductance, for an equivalent stator phase circuit. For a lag angle of \(-\pi/2\), the BEMF voltage will be a maximum, thus this will be the worst case. Equation VIII-9 can now be solved for the velocity at which the current is zero, see equation VIII-10, for a particular motor (LA23ECK-4). The maximum speed predicted by equation VIII-10 is given in Table VIII-1 for the three motors considered.

\[ V_S = R_i - K_b \dot{\theta} \sin \alpha \theta \] (VIII-9)
\[ \dot{\theta} = \frac{V_S}{K_b} = 45.28 \text{ Rad/sec} = 1441 \text{ step/sec} \] (VIII-10)

Note that the maximum speed values are determined by supply voltage \( V_S \) and motor BEMF. Since these values are very close to that given by equation VIII-8, and because of friction and damping the system will probably not be able to attain these speeds.

A better way of representing the maximum speed is to plot the zero slope isocline and zero work curve for each of the motors. Using the predicted values of friction and damping, the dimensionless parameters for the three motors are given in Table VIII-2. Using
these numbers, the zero slope isoclines for the three motors may be found, see figures VIII-3 through VIII-5.

Since all three motors have approximately the same speed capabilities, the lowest torque motor will be used (LA23ECK-4). A twelve volt L/R Drive, energizing one phase at a time, with constant stepping periods, will be utilized for the twelve step sequence. Using a computer simulation, the stepping period was varied to minimize the execution time. Figure VIII-6 is the result of the predicted minimum time response given by a dimensionless stepping period of $\omega_{NT} = 1.3$, this corresponds to 740 step/sec. The execution time, $FNTIM = 20.26$, is 21 msec which is one msec longer than that desired. For these system parameters an optimum sequence, as discussed before, is not possible as the combination of friction and BEMF is large enough to cause the first backward step to go higher than the maximum velocity of the system. This can be seen by looking at the phase plane step sequence for this motor-drive-load system, see figure VIII-7. Since the system can not accelerate to this velocity needed to land on the trajectory that would produce minimum overshoot, this strategy will not work for this system. Other methods, such as putting a backward step at the appropriate time, could work for a system like this.

One way to get better performance from this system is to build a chopper drive to reduce the effects of the BEMF voltage. If a large enough drive voltage is used for the speed range that the motor will be driven in, then the BEMF parameter gamma can be assumed to be zero.

Figure VIII-8 shows the resulting phase plane optimum step sequence for a current drive. The execution time has been reduced to
FIGURE VIII-9
ZSI AND ZUC FOR LA29ECK-4
FIGURE VIII-4

ZVC AND ZSI FOR LAZACK-1
FIGURE VIII-6
ZIC AND ZSI FOR LA23BCK-1
TABLE VIII-2

Dimensionless Motor Parameters

<table>
<thead>
<tr>
<th>Motor</th>
<th>$\zeta$</th>
<th>$T_f$</th>
<th>$\gamma$</th>
<th>$\omega_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LA23ECK-4</td>
<td>.001</td>
<td>.27</td>
<td>.424</td>
<td>961</td>
</tr>
<tr>
<td>LA23ACK-1</td>
<td>.001</td>
<td>.20</td>
<td>.658</td>
<td>995</td>
</tr>
<tr>
<td>LA23BCK-1</td>
<td>.001</td>
<td>.157</td>
<td>.466</td>
<td>935</td>
</tr>
</tbody>
</table>
FIGURE VIII-6
MINIMUM FIXED PERIOD STEP SEQUENCE
OPTIMUM STEPPING SEQUENCE OF 13 NSTPS
FORWARD STEPS= 18  BACKWARD STEPS= 2

ZETA= 0.89100 GAMMA= 0.42400 FRIC= 0.27000
LOAD= 0.68900 XLEAD= 1.29000

FIGURE VIII-7
PHASE PLANE STEP SEQUENCE
OPTIMUM STEPPING SEQUENCE OF 12 NSTPS
FORWARD STEPS— 7 BACKWARD STEPS— 4
ZETA= 0.80888 GAMMA= 0.80888 FRIC= 0.27888
LOAD= 0.80888 XLEAD= 0.80888

FIGURE VIII-8
OPTIMUM STEP SEQUENCE
USING A CURRENT DRIVE
9.9 msec, shown by the velocity profile, see figure VIII-9. Thus the system speed has been increased by a factor of two. The drive voltage necessary to provide constant current at maximum speed can be found using equation VIII-10. The maximum speed from figure VIII-9 is 2450 steps/sec, being conservative a maximum value of 5000 steps per second will be used. Thus the drive voltage should be 42 volts, see equation VIII-11.

\[ V = K_b \dot{\theta} = (.265 \text{ Volts/Rad/sec})(15.9 \frac{\text{Rad}}{\text{sec}}) \]

\[ V = 42 \text{ volts} \]

8-C Summary

A process of motor selection and implementation has been presented. Many methods can be used to select a motor, but since most ignore the stator circuit equations and the dynamic performance, an approach based on static accuracy should be sufficient for a preliminary motor selection. Once the peak torque is selected for static accuracy, simulation of the proposed system can predict peak velocity and execution times for given sequences. Furthermore system friction, damping and inertia may be varied to investigated changes in system performance.

Based on the simulation results, system changes or drive changes may be made to insure the desired design goals. These methods can help to eliminate marginal designs and improve system performance. Most importantly they can reduce design time and proto-type costs as the hardware trial and error procedure is eliminated.
FIGURE VIII-9

VELOCITY PROFILE
CHAPTER IX

CONCLUSIONS

9-A Summary

The basic stepping motor has remained almost unchanged since its conception, with but minor variations in design. Since the advent of the transistor, the market for steppers has grown considerably. Steppers are used in applications ranging from computer terminals to x-y plotter axes drives. The motors are ideal digital to displacement converters.

A major problem with steppers has been the lack of knowledge about the methods of motor stepping failure. This has hampered design efforts in the past and restricted the use of steppers to relatively low technology, non-critical applications. In critical applications, it has led to extensive experimental development programs.

Many have attempted to model the stepper, but the majority of mathematical models now available are too complex to apply easily to a practical system. The model presented herein, on the other hand, is a second order model which can be completely characterized on the phase plane. The model is constructed from basic physics and a knowledge of the shape of torque-angle curve. It has been developed for current drives, assuming a small L/R time constant. It may be made considerably more complex to describe nonsinusoidal torque-angle curves and nonlinear relationships between torque-angle curves and phase currents. However,
often this is not necessary and the basic model given here will be sufficient for most applications.

By plotting solutions to this model on a phase plane it is possible to obtain a global picture of stepping motor behavior. Modes of failure, optimum stepping sequences, and general trends can be easily identified and utilized by use of this solution display technique.

Sepatrices, trajectories which divide the phase plane up into regions, pass through every unstable equilibrium point. These curves form the boundaries for operation without failure, and any failure of the stepper to execute the desired number of steps is directly attributed to crossing a sepatrix. During stepping, the motor shifts in displacement in the phase plane, an amount equal to the step size. This is the mechanism of sepatrix crossing.

The model has been shown to fit actual data with acceptable accuracy for all areas of operation where a constant current drive assumption is valid. The model predicts slewing speed, as well, until the L/R time constant approaches phase "on" time. Since the model predicts the behavior of a stepper, it can be used to determine other properties of the motor.

Optimum stepping sequences may be constructed for any motor and are a direct result of the phase plane analysis. These sequences have minimum response time and no overshoot or oscillation at the completion of the sequence. They are useful in many applications where a fixed number of steps is repeatedly being executed. An optimum sequence may be found for any number of steps.

This model provides considerable insight into all facets of stepping motor behavior and explains failure as the crossing of the
sepatrix. The model has been experimentally verified for a permanent magnet stepper and can be used for any PM stepping motor when the torque-angle curve is provided. By using the phase plane it is possible to determine global motor behavior which leads to a better understanding of stepping motor behavior as a whole.

More work must be done on this subject, not only in determining better application techniques using this model, but also in the design and development of the "optimum stepping motor". Present current drives are better than voltage drives, however, optimum drive schemes must be developed for use in high performance systems. The zero work curve, which includes inductive effects, provides insight into drive effectiveness and a means for comparing driver performance. There are many aspects of stepper design and application which must be re-evaluated in light of this extension of the phase plane. Designers can save time and money by using the model and the phase plane approach for design evaluation instead of the present methods of trial and error.

The stepping motor will never replace the DC servo system in many applications, nor should it, but it will always remain an inexpensive, reliable instrument of incremental motion control. When very high performance systems are to be designed, that is, motor speeds that are near the maximum slewing capability of the driver, then it is necessary to include the voltage equations in the model, although this makes the model third order, the result may still be represented graphically on the phase plane using the zero work curves. Furthermore, the optimum sequences are still valid for this case and are a very useful design tool.
CHAPTER X

FUTURE WORK

In stepping motor systems where the inductance effects become significant, the second order model developed is not accurate enough to represent the stepping process. As was seen in Chapter IV one can model the driver electronics with a first order equation, for the voltage drive. When using computer simulation techniques the same equations can be modified slightly to represent the current or chopper driver. As was shown in Appendix B, the mechanical equation of motion and the electrical equation can be solved simultaneous to yield a third order system response. The response to any step sequence for this third order system may be projected on the phase plane, which will yield results similar to that previously shown. In Chapter IV a step sequence was plotted in the phase plane by defining a new state variable, which was the angle of the rotor minus the angle of the flux density vector in the air gap. When a step command was given to the motor, since the current rise time was assumed instantaneous, the flux density vector shifted instantaneously by one step, thus the trajectory in the phase plane shifted instantaneously to the right at each input positive step command, or to the left for a negative step command, see figure X-1.

For a large number of drives flyback diodes and high voltage transistors are used without the shut off time for the current in any phase is almost instantaneous. The penalty paid for this instantaneous shut-off is that there is a very large voltage spike developed across
ZETA = .6015  TF = .8975  GAMMA = .1268
WNT =1.5000  NSTPS = 4  FNTIM = 48.840

FIGURE X-1
SECOND ORDER MODEL
FOUR STEP SEQUENCE
the power transistor. This is why such drives require transistors with high voltage ratings, which generally have higher costs. For instantaneous current shut-off and a given current rise time, the flux density vector direction changes instantaneously because the current is shut off suddenly, but the flux density vector magnitude increases proportionately with the current in the on winding. The current waveforms for each of the four stator phases during a four step sequence, driving one phase on at a time, are shown in figure X-2. The corresponding four step sequence projected in the phase plane is shown in figure X-3 (note that the trajectory is shifted back instantaneously at each step because of the sudden turnoff in the winding current).

In order to avoid damage to power transistors, because of high voltage shut-off spikes, some drives incorporate flyback diodes around the stator windings, see figure X-4. For this type of drive, when the transistor is turned off, the current in the winding decays through the flyback diode. Thus, for this case, the flux density vector direction does not change instantaneously, but is dependent on the vector sum of the currents in the winding being turned off and the winding being turned on. Furthermore the BEMF voltage generated in the windings causes a current to flow in the windings through the diodes, when the transistor is off. The current waveforms for each of the four stator phases during a four step sequence using flyback diodes across the windings, are shown in figure X-5. When the step sequence for this drive is projected onto the phase plane the trajectory no longer shifts instantaneously at input step commands, but follows a path dictated by the change in direction of the flux density vector, see figure X-6.

By projecting the solutions of the third order model for the
ZETA = 0.8815  TF = 0.9375  GAMMA = 0.1260  RLVN = 0.8988
WNT = 1.5000  NSTPS = 4  FNTIM = 41.850

Figure X-2
Current waveforms for a voltage drive
ZETA = .8015  TF = .8375  GAMMA = .1260  RLIM = .8380
VNT = 1.5000  NSTS = 4  FNTIM = 41.850

FIGURE X-3
FOUR STEP SEQUENCE
FOR A VOLTAGE DRIVE
FIGURE X-4

VOLTAGE DRIVE WITH FLYBACK DIODES
ZETA = 0.8015  TF = 0.8375  GAMMA = 0.1260  RLWN = 0.8300  
UNT = 1.5000  NSTPS = 4  FNTIM = 26.060  

FIGURE X-5  
CURRENT WAVEFORMS  
L \frac{W_n}{R} T = 0.83
ZETA = .0015  TF = .0375  GAMMA = .1260  RLMN = .0388
UNT =1.5000  NSTPS = 4  FNTIN = 26.066

FIGURE X-6
FOUR STEP SEQUENCE
L \, W_{N}/R_{T} = .03
stepping motor, which includes the inductive effects, onto the phase plane one can use the graphic output to determine the system response to a input command sequence. At low speeds the current transients are over quickly and the projection from the three space is nearly the same as the phase plane. At high speeds the inductance effects significantly change the form of the response. This is also true for drive systems where the current rise time due to the L/R ratio is a significant portion of the stepping period. Figure X-7 shows the four phase currents where the L/R time constant is ten times that used to generate figure X-5. Figure X-8 is the resulting step sequence in the phase plane, where the trajectories are significantly different from that shown in figure X-6. By organizing the third order solutions in this phase plane one can, as before, identify step sequence failure modes and ways of preventing such failures. One such failure mode occurs at the start-stop rate for a few steps, see figure X-9. Here the motor input step commands are stopped when the system is above a sepatrix and as was shown in the phase plane model, the motor will gain four steps for this situation. By included inductance in the stepping motor model, it may now be possible to determine the high speed failure modes, which could not be predicted by the previous model.

As before one can design optimum step sequences by using one of the given acceleration-deceleration schemes from Chapter Seven. Some additional computation will be necessary for optimum sequences since the flux density vector direction does not change instantaneously. A different matching technique will have to be used between the acceleration and deceleration steps, since backwards integration of the current is not practical. One might pick the optimum sequence by match-
ZETA = .0015  TF = .0375  GAMMA = .1268  RLIN = .3000
WNT =1.5000  NSTPS = 4  FNTIM = 25.960

FIGURE X-7
CURRENT WAVEFORMS
L VN/RT = .3
ZETA = .0010  TF = .0500  GAMMA = .0500  RUN = .1000  
WNT = 1.2000 NSTPS = 4  FNTIM = 99.851

FIGURE X-9
FAILURE MODE AT THE 
START-STOP RATE
ing the last two steps, by trial and error, to produce minimum overshoot.

By plotting the velocity of the rotor versus the position of the rotor, the position of the flux density vector in the phase plane, for a third order model, one can expand the model to explain phenomena that occur due to current buildup in the windings. This is increasingly important in systems where the current rise time and decay time is a significant portion of the stepping period. For most high performance stepping motor drives this does not occur until very high stepping rates are reached. To include the effects of inductance it is necessary to build a more complex model, but by projecting the solutions onto the phase plane, the simplicity of graphical output is retained.

In order to verify any model one should experimentally measure a known system and compare the measured results to the predicted results. Because of the measurement system limitations it would be difficult to find the flux density vector direction. For the special case that includes flyback diodes, the phase plane could be generated by plotting the position of the rotor minus the input commanded position, which will be instantaneously shifted at each step input command. This method is not as informative as using the flux density vector direction, but it would be suitable for experimental verification of the model. If the current decay in the windings during shut off is instantaneous then the results will be the same for the two different state variables.
REFERENCES


Appendix A

Phase Plane Model for the Variable Reluctance Motor

It can be shown that the torque versus current relationship for a VR motor tends to be nonlinear. Not only does the torque angle curve depend upon the sine of the rotor angle and the number of phases energized simultaneously, but it also tends to be proportional to the square of the phase current. If the winding is saturated then this relationship can be somewhat simplified to depend on the absolute value of the phase current. Furthermore, if the L/R time constant for the motor can be neglected, then it is possible to write an expression which relates the phase voltage on the stator phases to the current through the stator phases. One difficulty, however, is that the back BEMF voltage on a stator phase, unlike the PM motor, tends to depend on the current flowing in that phase as well as the velocity and the angle. Consequently the BEMF term is considerably more nonlinear than it was for the PM motor. This is shown below for the \( K \)th phase assuming that the rotor and stator iron is not saturated.

\[
T = \sum_{K=1}^{n} K_i^2 a^2 \sin(A \theta - \frac{2\pi(K-1)}{n}) \quad A-1
\]

\[
V_k - i_k (R_s + R_d) + K_b i_k \sin(A \theta - \frac{2\pi k}{n}) = 0 \quad A-2
\]

for \( \frac{L}{R_s + R_d} \frac{di_k}{dt} \) small
This expression can be solved for the phase current as shown below.

\[ i_K = \frac{V_k}{R_s + R_d + K_b \sin(A \theta - \frac{2\pi k}{n})} \]

As long as the BEMF voltage is very small, which is possible for many VR motors, since many VR have laminated rotors, then the phase current can be written quite simply. However, if this is not the case then the BEMF depends on current and the relationship shown above for the phase current must be used. This phase current can then be applied to the torque equation. If it is assumed that the motor is unsaturated then the resulting torque equation can be written entirely in terms of motor speed and motor angle and phase voltage, as shown below.

\[ T = \sum_{k=1}^{n} K_T \left( \frac{V_k}{R_s + R_d + K_b \sin(A \theta - \frac{2\pi k}{n})} \right)^2 \sin(A \theta - \frac{2\pi k-1}{n}) \]

Thus the torque-angle curve is a more complex function of a angle and speed then was the curve for the PM motor. This can still be accounted for within the framework of the phase plane.

Consider the torque-angle curve which results for each of the energized phases. The total torque-angle relation is:

\[ T = \sum_{k=1}^{n} K_T i_k^2 \sin(A \theta - \frac{2\pi k-1}{n}) \]

for a 4 phase motor with all windings energized:

\[ T = K_T i_1^2 \sin[A\theta] + K_T i_2^2 \sin[A \theta - \frac{\pi}{2}] + K_T i_3^2 \sin[A \theta - \pi] + K_T i_4^2 \sin[A \theta - \frac{3\pi}{2}] \]

for the \( k^{th} \) phase
\[ V_K - i_K(R_s + R_d) + K_b i_K \delta \sin(A \theta - \frac{2\pi(K-1)}{4}) = 0 \]

Then, if \( V_K \) is the same for each phase, for each of the phases energized, by itself, a torque-angle curve results \( T_K \) which is angle and speed dependent.

\[
T_1 = \frac{K_T V^2 \sin A \theta}{(R_s + R_d - K_b \delta \sin(A \theta))^2} \tag{A-8}
\]

\[
T_2 = \frac{K_T V^2 \sin(A \theta - \frac{\pi}{2})}{[R_s + R_d - K_b \delta \sin(A \theta - \frac{\pi}{2})]^2} \tag{A-9}
\]

\[
T_3 = \frac{K_T V^2 \sin(A \theta - \pi)}{[R_s + R_d - K_b \delta \sin(A \theta - \pi)]^2} \tag{A-10}
\]

\[
T_4 = \frac{K_T V^2 \sin(A \theta - \frac{3\pi}{2})^2}{[R_s + R_d - K_b \delta \sin(A \theta - \frac{3\pi}{2})]^2} \tag{A-11}
\]

These four torque-angle speed functions are identical in form. They are merely shifted relative to each other by an angle of \( \frac{\pi}{2} \). Thus even though current-torque relationship is nonlinear and BEMF depends on current and speed, the stepping process is just causing a shift in the torque-angle-speed function of \( \theta_s \) which in this case is \( \frac{\pi}{2} \).

The dependence of the motor BEMF upon velocity and current can be verified by driving the motor at a fixed speed with a large series resistor and several stator current values. The resulting voltage versus angle curve should be recorded noting the supply voltage and current and measuring the phase resistance and velocity. From this information it should be possible to find a BEMF constant \( K_T \) and
determine the magnitude of the BEMF term. If the magnitude of the BEMF term is much smaller than the IR drop in the winding than it can be neglected. However, there will always be a maximum speed for which this assumption is valid. Incidentally preliminary experimental measurements indicate that $K_p$ decreases with speed and amplitude of stator current when the stator is beyond saturation which would explain why high slewing speeds are possible with a VR motor.
APPENDIX B

Phase Plane Model for a Voltage Drive

If the motor is driven by a voltage source, the basic equations are the same and equations B-1 and B-2 are the dimensional equations as developed earlier.

\[
J\ddot{\theta} + B\dot{\theta} + K_I \sin\theta = -T_f \frac{\dot{\theta}}{|\dot{\theta}|} - T_L \quad B-1
\]

\[
\dot{I} = \frac{1}{L} (V - IR_m + K_b \dot{\theta} \sin A\theta) \quad B-2
\]

where \(R_m\) is the total circuit resistance. In order to make equation D-1 dimensionless, the same substitutions were made. Dividing both sides of A-1 by \(\frac{K_T V}{R_m}\) and letting \(\theta = A\theta\) gives equation B-3.

\[
\frac{JR_m}{AK_TV} \frac{d^2\theta}{dt^2} + \frac{BR_m}{AK_TV} \frac{d\theta}{dt} + \frac{IR_m}{V} \sin\theta = -\frac{T_f}{|\dot{\theta}|} \ddot{\theta} - T_L \quad B-3
\]

By defining the following terms, the equation becomes dimensionless.

\[
\omega_n = \sqrt{\frac{AK_TV}{JR_m}} \quad B-4
\]

\[
\zeta = \frac{B}{2} \sqrt{\frac{R_m}{AK_TVJ}} \quad B-5
\]

\[
\ddot{T}_L = \frac{T_L}{R_m} \quad B-6
\]

\[
\ddot{T}_f = \frac{T_f}{R_m} \quad B-7
\]
The last term, equation B-8, did not appear in the previous model because it was assumed \( I \frac{R_m}{V} = 1 \); however, this is not acceptable for a voltage drive. Likewise, by manipulating equation B-2, it is possible to make the entire two equations dimensionless.

\[
\frac{d^2 \phi}{d\tau^2} + 2\zeta \frac{d\phi}{d\tau} + I \sin \phi = -\tau_f \frac{\dot{\phi}}{|\dot{\phi}|} - \tau_L
\]

The defining equations become

\[
\omega_n L \frac{dI}{d\tau} = 1 - \frac{R_m I}{V} + \frac{K_B \omega_n}{AV} \frac{d\phi}{d\tau} \sin \phi
\]

or by defining

\[
\beta = \frac{R_n}{\omega_n L}, \gamma = \frac{K_B \omega_n}{AV}
\]

\[
\frac{dI}{d\tau} = \beta - \beta I + \beta \gamma \sin \phi \frac{d\phi}{d\tau}
\]

The defining equations become

\[
\frac{d^2 \phi}{d\tau^2} + 2\zeta \frac{d\phi}{d\tau} + I \sin \phi = -\tau_L - \tau_f \frac{\dot{\phi}}{|\dot{\phi}|}
\]

\[
\frac{dI}{d\tau} = \beta - \beta I + \beta \gamma \sin \phi \frac{d\phi}{d\tau}
\]

These equations are in the form to be solved directly by computer. However, the set is third order and the system solutions must be plotted in a 3 dimensional space to prevent trajectories from crossing. Thus to get a global picture of system response under these conditions,
three dimensional plots are needed. Any given step sequence can be projected onto the phase plane, so that although the model is third order it is possible to find optimum sequences and plot their trajectories.
APPENDIX C

A Programmable Calculator Program for Phase Plane Analysis

1. Phase plane data can be obtained from a programmable calculator and plotted by hand. Given \( T_f, \zeta, \gamma, T_L \), the calculator can get \( \theta, \dot{\theta} \) as function of time and by plotting these one can find:

a. phase plane trajectories, separatrix
b. sequences of any time spacing and length
c. start-stop sequences
d. optimum sequences

2. To illustrate, the system differential equation is:

a. \[
\ddot{\theta} + (2\zeta + \gamma \sin^2 \theta) \dot{\theta} + \sin \theta = -\dot{T}_f \frac{\dot{\theta}}{|\dot{\theta}|} - T_L
\] (C-1)

b. The definition of the derivative yields:

\[
\ddot{\theta} = \frac{d\dot{\theta}}{d\tau} \quad \text{or} \quad d\dot{\theta} = \ddot{\theta} d\tau
\]

if \( d\tau \) small then \( d\tau \rightarrow \Delta \tau \) and \( \Delta \theta = \ddot{\theta}(\tau) \Delta \tau \)

Thus if start at \( \tau = 0 \) with \( \dot{\theta}(0), \ddot{\theta}(0) \)

\( \ddot{\theta}(\Delta \tau) = \ddot{\theta}(0) + \ddot{\theta}(0)\Delta \tau \), this is called rectangular integration and one can continue the process so velocity can be found for all time.

\[
\ddot{\theta}(2\Delta \tau) = \ddot{\theta}(\Delta \tau) + \ddot{\theta}(\Delta \tau)\Delta \tau
\]

\ldots

\[
\ddot{\theta}(n\Delta \tau) = \ddot{\theta}((n-1)\Delta \tau) + \ddot{\theta}[(n-1)\Delta \tau]\Delta \tau
\]

In this way, starting with \( \ddot{\theta}(0), \ddot{\theta}(0) \), the program can develop \( \ddot{\theta}(n\Delta \tau) \) in a step by step process.

c. The calculator can do this by computing \( \ddot{\theta}(n\Delta \tau), \ddot{\theta}(n\Delta \tau) \) to
get \(\dot{\phi}[n+1]\Delta t\) and outputing \(n\Delta t\) and \(\dot{\phi}[n+1]\Delta t\) at each step.

d. also \(\frac{d\phi}{dt} = \dot{\phi}\) or \(d\phi = \dot{\phi}dt\) but \(\dot{\phi}\) is changing because of acceleration so one might use average velocity i.e.

\[\ddot{\phi}(n\Delta t) + \frac{\ddot{\phi}(n\Delta t)}{2} = \ddot{\phi}\text{ average from }\tau = n\Delta t \text{ to } (n+1)\Delta t \text{ thus }\]

\[\Delta t(n\Delta t) = \Delta t[\ddot{\phi}(n\Delta t) + \frac{\ddot{\phi}(n\Delta t)}{2}]\]

which is a more precise method of integration than that used for velocity.

e. This yields 2 equations which can be solved step by step for position and velocity

\[\dot{\phi}[n+1]\Delta t = \dot{\phi}(n\Delta t)\Delta t + \ddot{\phi}(n\Delta t)\]

\[\phi[n+1]\Delta t = \dot{\phi}(n\Delta t)\Delta t + \ddot{\phi}(n\Delta t)\frac{\Delta t^2}{2} + \phi(n\Delta t)\]

to solve them let \(n=0\) then need \(\dot{\phi}(0), \phi(0)\) and \(\Delta t\) so that the differential equation (C-1) can be solved to get, \(\ddot{\phi}(\tau)\)

\[\ddot{\phi}(\tau) = -T_F \frac{\dot{\phi}(\tau)}{|\dot{\phi}(\tau)|} - T_L - \sin \phi(\tau) - [2\zeta + \gamma \sin^2 \phi(\tau)] \dot{\phi}(\tau)\]

or

\[\ddot{\phi}(n\Delta t) = -T_F \frac{\dot{\phi}(n\Delta t)}{|\dot{\phi}(n\Delta t)|} - T_L - \sin \phi(n\Delta t) - [2\zeta + \gamma \sin^2 \phi(n\Delta t)] \dot{\phi}(n\Delta t)\]

at \(\tau = 0\) or \(n = 0\):

\[\dot{\phi}(0) = -T_F \frac{\dot{\phi}(0)}{|\dot{\phi}(0)|} - T_L - \sin \phi(0) - [2\zeta + \gamma \sin^2 \phi(0)] \dot{\phi}(0)\]

notice the first term is indeterminant if \(\dot{\phi}(0) = 0\) so make \(\dot{\phi}(0)\)

small + or -thus need \(\dot{\phi}(0), \phi(0)\) and \(\Delta t, T_F, T_L, \zeta, \gamma\) then the equations can be solved in a step by step manner.

a. specify \(\dot{\phi}(0) \neq 0, \phi(0), T_F, T_L, \zeta, \gamma\)

b. specify a small \(\Delta t\) usually .02

c. use equations to compute and display \(\dot{\phi}(\Delta t), \phi(\Delta t), \tau = \Delta t\) so can plot or write down these values.
d. Then repeat to get $\dot{\theta}(2\Delta t)$, $\theta(2\Delta t)$, $\tau = 2\Delta t$ and again to get $\ddot{\theta}(3\Delta t)$ etc.

e. Often select $\Delta t$ much smaller for accuracy than desired for plotting. As $\Delta t$ gets smaller get better accuracy. So output only the values needed for plotting, that is, every $m$th value.

3. Specific uses

a. Phase Planes

1. Start at several $\dot{\theta}(0)$, $\theta(0)$ to get data for phase plane trajectories.

2. Start at $\dot{\theta}(0)$ small and $\theta(0) = \pm \pi K$ to get sepatratrix by plotting time backwards. ($\Delta t$ negative)

3. Plot data as it is computed to allow selection of trajectories that are useful. Remember plot repeats every $\pm K\pi$.

b. Step Sequence - start at $\dot{\theta}(0)$ small, $\theta(0) = -\frac{\pi}{2}$ for 4 phase stepper and compute until $n\Delta t = \text{time of 2nd step } T_2$ (select $\Delta t$ so that $T_2/\Delta t = \text{integer}$). Stop program at $T_2$ and subtract $\pi/2$ from $\theta(T_2)$ and restart. Continue. (Can change $\Delta t$ so that $\frac{T_{n-1}}{\Delta t}$ is always integer).

c. Optimum Sequences - can compute $\ddot{\theta}(n\Delta t)$ when $\dot{\theta} = \theta(n\Delta t) - \frac{\pi}{2}$ and find acceleration of trajectory to which will step if stepped at $n\Delta t$. Can step to equal acceleration or maximum acceleration by this method or switch on zero acceleration.

4. Program

a. Store $T_F$, $2\zeta, \gamma; \Delta t, m; \theta(n\Delta t), \dot{\theta}(n\Delta t), \ddot{\theta}(n\Delta t), n\Delta t$

b. To start input $T_F$, $2\zeta, \gamma, \Delta t, \text{in } \theta(0), \dot{\theta}(0), \ddot{\theta}(0) = \tau = 0$

c. program computes $\ddot{\theta}(n\Delta t)$, $\dot{\theta}(n\Delta t)$, $\theta(n\Delta t)$, $n\Delta t$ and overwrites the values in memory so contents are latest values. Displays every $m$th value and waits until restart.
d. Can display every value, \( m = 1 \) (useful when nearing a switch point or looking for zero accelerate) or can display every \( m \) values so that retain integration accuracy and only get enough values to plot. This is much faster than displaying every value. Do this in the program by presetting a counter to value \( m \) and counting it down everytime it goes through a \( \theta[(n+1)\Delta t] \) and \( \dot{\theta}[(n+1)\Delta t] \) computation. When the counter is zero then display \( \theta(n+1)\Delta t \) and reset it.

e. The program stops after displaying \( (n+1)\Delta t \), \( \dot{\theta}[(n+1)\Delta t] \) and \( \theta[(n+1)\Delta t] \). This allows the following actions

a. \( \Delta t \) can be made larger or smaller to find a particular value of \( \theta \), \( \dot{\theta} \) or \( \ddot{\theta} \), zero acceleration for example.

b. \( m \) can be changed to display more or less frequently

c. \( \theta[(n+1)\Delta t] \) can be changed to account for a step.

d. \( \Delta t \) can be made negative to move backwards in time.

e. \( \xi, \gamma, \bar{T}_F, \bar{T}_L \) can be changed to account for a max flux density change in the airgap such as would occur when half stepping.

f. The data may be plotted as it is produced. It is usually better, however, to also record the \( \theta, \dot{\theta} \) values.

5. The basic method described here can be used to employ a hand-held programmable calculator to explore any of the analytical methods described in the stepping motor course, such as:

a. phase planes

b. zero slope isoclines

c. stepping sequences

d. optimum sequences

e. zero work curves
f. maximum work curves

g. high speed current waveforms.
HP67 Phase Plane Program

9/20/79

\[ T\!_F \rightarrow 1, \ \dot{\theta} \rightarrow 2, \ \ddot{\theta} \rightarrow 3, \ \Delta t \rightarrow 4, \ 2\zeta \rightarrow 5, \ \tau \rightarrow 6, \ M \rightarrow 7, \ \dot{\theta} \rightarrow 8, \ \gamma \rightarrow 9 \]

Preset registers 1-8 as above:

<table>
<thead>
<tr>
<th>Line No.</th>
<th>Command</th>
<th>Keystroke</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 1</td>
<td>2 5 1 1</td>
<td>fLBLA</td>
</tr>
<tr>
<td></td>
<td>3 4 0 7</td>
<td>RCL7</td>
<td>get number of time increments m between display</td>
</tr>
<tr>
<td></td>
<td>3 5 3 3</td>
<td>hSTI</td>
<td>preset I register counter</td>
</tr>
<tr>
<td></td>
<td>3 5 4 2</td>
<td>hRAD</td>
<td>read angles in radians</td>
</tr>
<tr>
<td>5</td>
<td>3 1</td>
<td>2 5 1 3</td>
<td>fLBDL</td>
</tr>
<tr>
<td></td>
<td>3 4 0 2</td>
<td>RCL2</td>
<td>get ( \theta(n\Delta t) ) from location 2</td>
</tr>
<tr>
<td></td>
<td>3 1 6 2</td>
<td>fSIN</td>
<td>( \sin \theta(n\Delta t) )</td>
</tr>
<tr>
<td></td>
<td>4 2</td>
<td>CHS</td>
<td>( -\sin \theta(n\Delta t) )</td>
</tr>
<tr>
<td></td>
<td>3 4 0 3</td>
<td>RCL3</td>
<td>get ( \dot{\theta}(n\Delta t) ) from location 3</td>
</tr>
<tr>
<td>10</td>
<td>3 1 7 1</td>
<td>fX&lt;0</td>
<td>if ( \dot{\theta}(n\Delta t)&gt;0 ) skip next instruction</td>
</tr>
<tr>
<td></td>
<td>2 2 1 2</td>
<td>GTOB</td>
<td>go to label B</td>
</tr>
<tr>
<td></td>
<td>4 4</td>
<td>CLX</td>
<td>clear ( \dot{\theta}(n\Delta t) )</td>
</tr>
<tr>
<td></td>
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<td>get ( T_F ) from location 1</td>
</tr>
<tr>
<td></td>
<td>4 2</td>
<td>CHS</td>
<td>( -T_F )</td>
</tr>
<tr>
<td>15</td>
<td>6 1</td>
<td>+</td>
<td>( -\sin \theta(n\Delta t) - T_F )</td>
</tr>
<tr>
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<td>2 5 1 4</td>
<td>fLBLD</td>
<td>label D</td>
</tr>
<tr>
<td></td>
<td>3 4 0 2</td>
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<td>get ( \theta(n\Delta t) ) from location 2</td>
</tr>
<tr>
<td></td>
<td>3 1 6 2</td>
<td>fSIN</td>
<td>( \sin \theta(n\Delta t) )</td>
</tr>
<tr>
<td></td>
<td>3 2 5 4</td>
<td>g X^2</td>
<td>( \sin^2 \theta(n\Delta t) )</td>
</tr>
<tr>
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<td>3 4 0 9</td>
<td>RCL9</td>
<td>get ( \gamma ) from location 9</td>
</tr>
<tr>
<td></td>
<td>7 1</td>
<td>X</td>
<td>( \gamma \sin^2 \theta(n\Delta t) )</td>
</tr>
<tr>
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<td>get ( 2\zeta ) from location 5</td>
</tr>
<tr>
<td></td>
<td>6 1</td>
<td>+</td>
<td>( 2\zeta + \gamma \sin^2 \theta )</td>
</tr>
<tr>
<td></td>
<td>3 4 0 3</td>
<td>RCL3</td>
<td>get ( \dot{\theta}(n\Delta t) ) from location 3</td>
</tr>
<tr>
<td>25</td>
<td>7 1</td>
<td>X</td>
<td>( \dot{\theta}(n\Delta t)(2\zeta + \gamma \sin^2 \theta) )</td>
</tr>
<tr>
<td></td>
<td>5 1</td>
<td>-</td>
<td>( \ddot{\theta}(n\Delta t) = -\sin \theta(n\Delta t) - T_F - (2\zeta + \gamma \sin^2 \theta) \dot{\theta}(n\Delta t) )</td>
</tr>
<tr>
<td></td>
<td>3 3 0 8</td>
<td>STO8</td>
<td>store ( \dot{\theta}(n\Delta t) ) in location 8</td>
</tr>
<tr>
<td></td>
<td>3 4 0 4</td>
<td>RCL4</td>
<td>get ( \Delta t ) from location 4</td>
</tr>
</tbody>
</table>
\begin{align*}
32 & \quad 54 \quad g X^2 \quad (\Delta t)^2 \\
30 & \quad 0 \quad 2 \quad 2 \\
81 & \quad \div \quad (\Delta t)^2/2 \\
71 & \quad \times \quad \theta(n\Delta t) \Delta t^2/2 \\
34 & \quad 0 \quad 3 \quad \text{RCL3} \quad \text{get } \theta(n\Delta t) \text{ from location 3} \\
34 & \quad 0 \quad 4 \quad \text{RCL4} \quad \text{get } \Delta t \text{ from location 4} \\
35 & \quad 7 \quad 1 \quad \times \quad \theta(n\Delta t) \Delta t \\
61 & \quad + \quad \theta(n\Delta t) \Delta t + \theta(n\Delta t) \Delta t^2/2 \\
34 & \quad 0 \quad 2 \quad \text{RCL2} \quad \text{get } \theta(n\Delta t) \text{ from location 2} \\
61 & \quad + \quad \theta[(n+1)\Delta t]=\theta(n\Delta t)+\theta(n\Delta t)\Delta t+\theta(n\Delta t)\Delta t^2/2 \\
33 & \quad 0 \quad 2 \quad \text{ST02} \quad \text{store } \theta[(n+1)\Delta t] \text{ in location 2} \\
40 & \quad 3 \quad 4 \quad 0 \quad 8 \quad \text{RCL8} \quad \text{get } \theta(n\Delta t) \text{ from location 8} \\
34 & \quad 0 \quad 4 \quad \text{RCL4} \quad \text{get } \Delta t \text{ from location 4} \\
71 & \quad \times \quad \theta(n\Delta t) \Delta t \\
34 & \quad 0 \quad 3 \quad \text{RCL3} \quad \text{get } \theta(n\Delta t) \text{ from location 3} \\
61 & \quad + \quad \theta[(n+1)\Delta t]=\theta(n\Delta t)+\theta(n\Delta t)\Delta t \\
45 & \quad 3 \quad 3 \quad 0 \quad 3 \quad \text{ST03} \quad \text{store } \theta[(n+1)\Delta t] \text{ in 3} \\
34 & \quad 0 \quad 6 \quad \text{RCL6} \quad \text{get } n\Delta t \text{ from location 6} \\
34 & \quad 0 \quad 4 \quad \text{RCL4} \quad \text{get } \Delta t \text{ from location 4} \\
61 & \quad + \quad (n+1)\Delta t \\
33 & \quad 0 \quad 6 \quad \text{ST06} \quad \text{store } (n+1)\Delta t=\tau \text{ in 6} \\
50 & \quad 3 \quad 1 \quad 3 \quad 3 \quad \text{fDSZ} \quad \text{count I register down 1} \\
51 & \quad 2 \quad 2 \quad 1 \quad 3 \quad \text{GTOC} \quad \text{unless I contents zero go to C} \\
34 & \quad 0 \quad 6 \quad \text{RCL6} \quad \text{get } (n+1)\Delta t \text{ from location 6} \\
31 & \quad 8 \quad 4 \quad \text{f-X-} \quad \text{display } (n+1)\Delta t \\
34 & \quad 0 \quad 3 \quad \text{RCL3} \quad \text{get } \theta[(n+1)\Delta t] \text{ from location 3} \\
55 & \quad 3 \quad 1 \quad 8 \quad 4 \quad \text{f-X-} \quad \text{display } \theta[(n+1)\Delta t] \\
34 & \quad 0 \quad 2 \quad \text{RCL2} \quad \text{get } \theta[(n+1)\Delta t] \text{ from location 2} \\
31 & \quad 8 \quad 4 \quad \text{f-X-} \quad \text{display } \theta[(n+1)\Delta t] \\
84 & \quad \text{R/S} \quad \text{stop until push R/S to start} \\
22 & \quad 1 \quad 1 \quad \text{GTOA} \quad \text{go to A} \\
60 & \quad 3 \quad 1 \quad 2 \quad 5 \quad \text{fLBLB} \quad \text{label B} \\
44 & \quad \text{CLX} \quad \text{clear } \theta(n\Delta t) \\
34 & \quad 0 \quad 1 \quad \text{RCL1} \quad \text{get } T_F \text{ from location 1} \\
61 & \quad + \quad -\sin \theta(n\Delta t) + T_F \\
22 & \quad 1 \quad 4 \quad \text{GTOD} \quad \text{go to D} 
\end{align*}
Example 1. Phase plane for $\bar{t} = .1$, $\xi = .05$, $\gamma = 0$, let $\theta(0) = +\pi$, $\dot{\theta}(0) = +.001$

$\Delta t = .02$, $m = 5$, $\theta(0) = 0 = r$, set registers $1 + 1$, $\pi + 2$, $.001 + 3$, $-.02 + 4$, $1.1 + 5$, $0 + 6 + 8$, $5 + 7$ this will yield a sepaatrix. Run until $\theta = -\pi$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\theta$</th>
<th>$\dot{\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>-.5</td>
<td>3.1281</td>
<td>.0544</td>
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- change $m$ to 25 (25-ST07) display every 1/2 second
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<th>$\dot{\theta}$</th>
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<tbody>
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<td>-1</td>
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<td>.3999</td>
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<tr>
<td>-2.5</td>
<td>2.5801</td>
<td>.6761</td>
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</table>

- change $\Delta t$ to -.1 , $m$ to 5
<table>
<thead>
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<th>$\dot{\theta}$</th>
</tr>
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<tbody>
<tr>
<td>-3</td>
<td>2.1444</td>
<td>1.0947</td>
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<td>-3.5</td>
<td>1.4567</td>
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<td>.4677</td>
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<td>-4.5</td>
<td>-.7211</td>
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<td>-1.8696</td>
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<tr>
<td>-6</td>
<td>-3.8736</td>
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</tr>
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</table>

A trajectory starting at $\theta = -\pi$ $\dot{\theta} = 1.5$ $-\pi + 2$ $1.5 + 3$ $\Delta t \rightarrow 8, 6$ $1 +$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\theta$</th>
<th>$\dot{\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-\pi$</td>
<td>1.5</td>
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<tr>
<td>.5</td>
<td>-2.4009</td>
<td>1.5185</td>
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<tr>
<td>1</td>
<td>-1.5755</td>
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<td>2</td>
<td>.4696</td>
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<tr>
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- $\Delta t \rightarrow 8, 6$ $1 +$
A trajectory starting at \( \Theta = -\pi \)
\( \dot{\Theta} = 2.5 \)

<table>
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<th>( \Theta )</th>
<th>( \dot{\Theta} )</th>
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</thead>
<tbody>
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<td>-1.9003</td>
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<tr>
<td>4</td>
<td></td>
<td></td>
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</tbody>
</table>

A trajectory starting at \( \Theta = -\pi \)
\( \dot{\Theta} = .5 \)

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \Theta )</th>
<th>( \dot{\Theta} )</th>
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A trajectory starting at \( \Theta = \pi \)
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A separatrix starting at $\theta = 3\pi$, $\dot\theta = .01$

$3\pi + 2$, $.01 \rightarrow 3$, $-1 \rightarrow 4$ $0 \rightarrow 6, 8$ $5 \rightarrow 7$

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Note: since trajectories repeat could have started at $\theta=2\pi-2.8756$ and $\dot\theta=1.9306$ and gotten this separatrix.
A step sequence which steps at $\tau = 0, 1, 2, 3, 6.2$

\[ \Delta \frac{\pi}{2} = 2, \ T_s = 1, \ \Delta \rightarrow 3, \ .01 \rightarrow 4, \ .1 \rightarrow 0, \ 0 \rightarrow 6, 8, 9 \ 5 \rightarrow 7 \]

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Figure C.1 Phase plane obtained from calculator