CONTRACTIONS WITH INFINITE DEFECT INDEX

KENNETH ROBERT WADLAND

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WADLAND, KENNETH ROBERT
CONTRACTIONS WITH INFINITE DEFECT INDEX,
UNIVERSITY OF NEW HAMPSHIRE, PH.D., 1978
CONTRACTIONS WITH INFINITE DEFECT INDEX

BY

KENNETH ROBERT WADLAND

A DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

December, 1978
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ACKNOWLEDGEMENTS

The author is deeply indebted to many people for their help and support while writing this paper. I would particularly like to thank my thesis advisor, Prof. Eric Nordgren, who has been a never ending source of suggestions and constructive criticism. I would also like to thank my typist, Yvette H. Damien of Speedy Secretarial Service, who tolerated all these special symbols and equations so well. But most of all, I would like to thank my wife, Vera, for her constant faith and reassurance without which I could never have finished.
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ABSTRACT

CONTRACTIONS WITH INFINITE DEFECT INDEX

by

KENNETH ROBERT WADLAND

UNIVERSITY OF NEW HAMPSHIRE, 1978

This paper investigates several questions related to \( C_0 \) contractions on complex, separable Hilbert spaces. The stepping off point is the recent extension, by Sz.-Nagy, of the Jordan model theory to \( C_0 \) contractions with one finite and one infinite defect index.

Section I states the various results obtained and introduces the definitions and notations which are used throughout.

In Section II, an equivalence relation, RP-equivalence, is defined on the class of analytic functions on the unit disk whose values are operators between two fixed Hilbert spaces. Several of its properties are developed, including its strength relative to quasi-equivalence (another equivalence relation on operator-valued analytic functions developed by Moore and Nordgren and extended by Sz.-Nagy). We prove that if \( T \) and \( T' \) are \( C_0 \) contractions whose characteristic operator functions are RP-equivalent, then \( T \) and \( T' \) are quasi-similar. The converse is given for several special cases.

In Section III, we prove that every (closed) subspace which is invariant under a \( C_0 \) contraction, \( T \), is also invariant under every operator in the double commutant of \( T \). This extends an earlier result by Wu which required that the defect indices of \( T \) be finite. As a corollary to this theorem, we give an alternate proof to Sz.-Nagy's theorem that every \( C_0 \) contraction is in Turner's class (dc), i.e. its
double commutant equals the smallest weakly-closed algebra with identity which contains it.

In Section IV, we prove that if \( T \) is a \( C_0 \) contraction whose defect indices are distinct and at least one of them is finite, then \( T \) is in class (dc) and the double commutant of \( T \) is interpolated by \( H^\infty \). (Here we are using Sarason's term, i.e. an algebra is interpolated by \( H^\infty \) if it is the set of all operators of the form \( h(T) \) where \( h \) is an \( H^\infty \)-function.) This is an extension of a result by Uchiyama which required that both defect indices be finite. We cite an example, due to Sz.-Nagy, showing that the other restriction (that the defect indices be distinct) cannot be removed.

Section V is an investigation of direct sums of operators with the unilateral shift, \( S_+ \). For sums of a \( C_0 \) contraction with \( S_+ \), the result of Section IV is shown to hold, namely: if \( T \) is any \( C_0 \) contraction then the direct sum of \( T \) with \( S_+ \) is in class (dc) and its double commutant is interpolated by \( H^\infty \). In contrast, we construct the double commutant of the direct sum of an isometry with the unilateral shift. Although such sums are still of class (dc), they are not interpolated by \( H^\infty \).
SECTION I

INTRODUCTION

Let T be an operator from H to K. By this, we shall always mean that H and K are complex, separable Hilbert spaces and that T is a bounded linear transformation from H into K. The operator T is called a contraction if for all \( h \in H \), \( \|Th\|_K \leq \|h\|_H \). If further the sequence \( (T^*h, T^2h, T^3h, \ldots) \) converges to 0, then T will be called a \( C_0 \) contraction. This paper will be concerned primarily with the class of \( C_0 \) contractions.

This class was first defined in 1963 (see[24]). Since then there has been considerable research activity on \( C_0 \) contractions. In fact, the text by Sz-Nagy and Foiaş (designated by [HA] in the bibliography) is essentially an exposition on contractions and subclasses of contractions. Their text serves as a basis for the notations and definitions in this paper and most of the current research in this area.

One of the questions which will be addressed here is the structure of the double commutant. For any operator T on H, the single commutant of T, written \( \{T\}' \), is the set of all operators which commute with T. The double commutant of T, written \( \{T\}'' \), is the set of all operators which commute with every operator which commutes with T.

If the underlying space, H, is finite dimensional, then the double commutant is just the double commutator and the classical result holds, namely: \( \{T\}'' \) is the smallest algebra (with identity) containing T. For normal operators the von Neumann double commutant theorem provides a
similar result: \( \{T\}'' \) is the smallest \( W^* \)-algebra (with identity) containing \( T \) (see eg. [6]). For algebraic operators, the corresponding result is: \( \{T\}'' = \alpha(T) \), where \( \alpha(T) \) is the smallest weakly-closed algebra with identity) containing \( T \) (see [30]). Following Turner's notation [29], we will refer to any operator for which its double commutant equals the weakly-closed algebra (with identity) which it generates as being of class (dc). For \( C_{00} \) contractions in general no such result is known.

One of the most useful tools for studying operators on finite dimensional spaces is the characteristic polynomial. One way of looking at it is to define \( \Theta(z) = T - zI \), where \( T \) is an operator on a finite dimensional space \( H \) and \( I \) is the identity on \( H \). Then, the determinant of any matrix representation of \( \Theta \) is the characteristic polynomial of \( T \).

Further, the characteristic values of \( T \) are precisely those \( z \) for which \( \Theta(z) \) fails to have an inverse. In a similar fashion it is also possible to obtain the characteristic vectors of \( T \), all of its invariant subspaces, and its Jordan model. However, there is no direct extension of these concepts to infinite dimensional spaces. (For example, the unilateral shift has no characteristic values or vectors.)

An analog of the function \( \Theta \) has been developed, though, which generalizes some of the properties of the finite dimensional case. For any \( C_{00} \) contraction, \( T \), there exists an analytic function, \( \Theta_T \), defined on the unit disk whose values are operators from \( E_T \) to \( E_T^{**} \), called the characteristic operator function of \( T \). The dimensions of \( E_T \) and \( E_T^{**} \) are called the defect indices of \( T \). (A detailed description of \( \Theta_T \) will be presented later in this section.) The general study of characteristic operator functions dates back to 1946 with the work of Livsic [10] and other Russian mathematicians. Work on the case of \( C_{00} \) contractions began in 1960 with papers of Rota [16], Rovnyak [17], and Helson [9].
The construction used in this paper is due to Sz.-Nagy and Foiaş [HA].

In order to make the best possible use of the characteristic operator function, it is important to know which properties of an operator can be deduced from its characteristic operator function and how they can be represented. For example, it is known that two operators are unitarily equivalent if and only if their characteristic operator functions "coincide" [HA, pg. 257]. No such result is known for quasi-similarity (the appropriate infinite dimensional analog of similarity).

Section II of this paper addresses this question. A property called "RP-equivalence" is defined and shown to be an equivalence relation on inner contractive analytic functions. The following theorem is proven: if $T$ and $T'$ are $C_0$ contractions whose characteristic operator functions are RP-equivalent, then they are quasi-similar. The converse is proven for several special cases. Whether or not the converse holds in general is still unknown. The strength of RP-equivalence relative to quasi-equivalence is presented. (Quasi-equivalence is another equivalence relation on contractive analytic functions first introduced by Moore and Nordgren [12 and 11] and later extended by Szücs [28] and Sz.-Nagy [23]. Finally, it is shown that quasi-similarity can be expressed entirely in terms of the characteristic operator function; however, the resulting equivalence relation is too unwieldy to be useful.

Section III investigates an important subclass of $C_0$ contractions called $C_0$ (those contractions which satisfy an analog of the Hamilton-Cayley theorem). Suppose that $T$ is a $C_0$ contraction and that $L$ is a (closed) subspace which is invariant under $T$, i.e. $TL \subseteq L$. We show that $L$ must also be invariant under every operator in the double commutant of $T$. This result extends an earlier result by Wu [33], which
required that the defect indices of $T$ be finite. As a corollary to this theorem, we give an alternate proof (cf. [27]) that all $C_0$ contractions are of class (dc), i.e. their double commutants are equal to the weakly-closed algebras (with identity) which they generate.

In Section IV we give a similar result for another class of $C_0$ contractions. In 1967 Sarason showed [18] that the unilateral shift is of class (dc) and that its double commutant is interpolated by $H^\infty$. (This will be defined later.) In a recent paper [32], Uchiyama extended this result to $C_0$ contractions whose defect indices are finite and distinct. We further extend this result to allow one of the defect indices to be infinite. In particular, we show that if $T$ is a $C_0$ contraction whose defect indices are not equal and at least one of them is finite, then $(T)^\prime\prime$ is equal to $\alpha(T)$ and is interpolated by $H^\infty$. We then present an example (due to Sz.-Nagy [22]) which shows that the restriction that the defect indices be distinct cannot be removed. The example is a $C_0$ contraction whose defect indices are both two and whose double commutant is not interpolated by $H^\infty$.

In examining the differences between the cases where the theorem does hold (distinct defect indices) and a case where the theorem does not hold (equal defect indices), the author noted that in the first case such operators are quasi-similar to operators of the form $T \otimes S_+$ where $T$ is a $C_0$ contraction and $S_+$ is the unilateral shift [23]; whereas, for operators with equal and finite defect indices this can never be true [11]. Will the result hold for any operator of this form? An affirmative answer is given in Section V, namely, if $T$ is any $C_0$ contraction and $S_+$ is the unilateral shift then $T \otimes S_+$ is in class (dc) and its double commutant is interpolated by $H^\infty$. The last result of this paper explicitly constructs the weakly-closed algebra generated by
operators of the form $T \oplus S_\uparrow$ where $T$ is an isometry and $S_\uparrow$ is the unilateral shift. We prove that although such operators are still of class (dc), their double commutants are not (in general) interpolated by $H^\infty$.

Having described the contents of this thesis, we will now present some of the notations and definitions which will be used throughout the remainder of this paper.

The symbol "\(=\)" is read "is defined to be" and will be used only on the first definition of a notation. The symbol "\(\subset\)" will denote set inclusion (possibly improper) and is read "is a subset of" or "which is a subset of", depending upon context. The symbols "\(\in\)" and "\(=\)" will also have this dual interpretation.

Suppose $H$ is a Hilbert space (note that all Hilbert spaces are assumed to be complex and separable). The norm of $f$ in $H$ will be denoted by $\|f\|_H$ and the inner product of $f$ with $g$ in $H$ will be denoted by $\langle f, g \rangle_H$. Whenever there is only one candidate for the underlying Hilbert space, the identifying subscript will be omitted.

The notation $F: X \to Y$ will denote a function, $F$, whose domain is $X$ and whose co-domain is $Y$. The words "range" and "image" will be used interchangeably to denote the set $F(X)$. If $A$ is a subset of $X$, then $G \equiv F \mid A$ will mean that $G$ is defined to be the restriction of the function $F$ to the set $A$.

For any probability measure, $\mu$, on the unit circle, $L^p(\mu)$ will denote the usual Lebesgue spaces (see eg. [7, pg. 23]). Thus the norm for $L^\infty(\mu)$ is the essential supremum norm. Actually, the elements of $L^p(\mu)$ are equivalence classes; but, we will abuse the notation by identifying the elements of each class and write "\(=\)" to mean equality almost everywhere (a.e.) with respect to $\mu$. If $\mu$ is normalized Lebesgue measure, we will write simply $L^p$. 
The Hardy spaces, $H^p$, consist of those $L^p$ functions whose negatively indexed Fourier coefficients vanish. For example,

$$H^\infty \equiv \left\{ f \in L^\infty : \int_0^{2\pi} f \chi_n dt = 0, \text{ for } n = 1, 2, 3, \ldots \right\}$$

where $\chi_n(z) \equiv z^n$.

The Hardy spaces can also be defined as spaces of analytic functions from the (open) unit disk into the complex plane satisfying certain growth conditions. That these two definitions are equivalent can be shown as follows: if $f$ satisfies the first definition then Poisson's formula (see eg. [HA, pg. 100]) uniquely determines an analytic function satisfying the second definition; conversely, if $f$ is an $H^p$ function then Fatou's theorem [15] establishes the existence of radial limits a.e. which satisfy the first definition. We will normally use the former definition. But, when convenient we will identify the two.

An $H^2$ function, $f$, is called "inner" if its absolute value (on the unit circle) is a.e. the constant function one. It is called "singular" if it can be represented in the form:

$$f(z) \equiv -\int_0^{2\pi} \frac{w + z}{w - z} d\nu(w)$$

where $\nu$ is a finite, non-negative measure which is singular with respect to Lebesgue measure. We will call an $H^2$ function, $f$, "outer" if $\{\chi_n f\}_{n=0}^\infty$ is dense in $H^2$. (Note: this is not the usual definition of outer but is equivalent to it by Beurling's theorem [3] and is the only property of it that we will use.) Beurling's factorization theorem [3] states that every $H^2$ function, $f$, can be factored into an inner part, $f^0$, and an outer part, $f^e$. And, the inner part can be further factored into a Blasche product and a singular function.
Since $H^2$ has no zero divisors, we can use the usual terminology for factorization. Thus, we will write $f \mid g$ for "$f$ divides $g$". Since the constant functions have inverses, the greatest common divisor (g.c.d.) is only unique up to a constant factor; and, two functions are called relatively prime if their g.c.d. is a constant.

The co-domain of functions in $L^2$ is the complex plane. We will frequently have need for an analog of $L^2$ whose functions are vector-valued. For any Hilbert space, $E$, we will denote by $L^2_E$ the class of all weakly measurable functions, $v$, with values in $E$ such that:

$$\|v\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \|v(e^{it})\|_E^2 \, dt < \infty$$

If we use this norm and identify functions which are a.e. equal, $L^2_E$ becomes a separable Hilbert space. There is a one-to-one correspondence between elements, $v$, of $L^2_E$ and sequences, $(a_k)_{k=-\infty}^\infty$, of elements of $E$ (called its Fourier coefficients) such that

$$v(e^{it}) = \sum_{k=-\infty}^{\infty} \chi_k(e^{it}) a_k$$

and

$$\|v\|_2^2 = \sum_{k=-\infty}^{\infty} \|a_k\|_E^2 < \infty$$

The class $H^2_E$ is then defined as the set of $L^2_E$ functions whose negatively-indexed Fourier coefficients vanish. Many of the classic results about $L^2_E$ also hold here. For example, every $L^2_E$ convergent sequence has a point-wise convergent subsequence [HA, pg. 183]. Further, Poisson's Formula and Fatou's Theorem hold [HA, pp. 185-186], allowing us to identify $H^2_E$ with analytic functions on the unit disk, as was done for $H^2$. 
By a bilateral shift we will mean an operator, $M_z$, acting on some $L^2_E$ by $M_z(v) = Xv$, where $X(z) = z$. Its restriction to $H^2_E$ will be called a unilateral shift and will also be written as $M_z$. In the event that the underlying space $E$ is simply the complex plane, then we will refer to $M_z$ as "the" bilateral or unilateral shift.

In addition to the vector-valued $H^2_E$, we will occasionally need an operator-valued analog of $H^\infty$. Suppose $E$ and $F$ are Hilbert spaces. A function $\Theta$ on the unit disk whose values, $\Theta(z)$, are operators from $E$ to $F$ will be called analytic if it has a power series expansion:

$$\Theta(z) = \sum_{k=0}^{\infty} \Theta_k z^k$$

where each $\Theta_k$ is a bounded operator from $E$ to $F$ [HA, pg. 186]. It is called bounded if there is a bound for each $\Theta(z)$ which is independent of $z$. The set of all such bounded analytic functions will be denoted by $M(E, F)$.

Once again, Fatou's Theorem and Poisson's Formula hold [HA, pg. 186] and we can define $\Theta$ on the unit circle by taking radial limits. If, further, $\Theta(e^{it})$ is an isometry for almost every $t$, then $\Theta$ is called inner [HA, pg. 190]. An element, $\Theta$, of $M(E, F)$ can also be considered as an operator from $H^2_E$ to $H^2_F$ by defining $(\Theta f)(z) \equiv (\Theta(z))(f)$, for all $f \in H^2_E$. Still another representation of $\Theta$ is as a matrix, satisfying certain boundedness conditions [23], whose coefficients are in $H^\infty$ and whose dimensions are the dimensions of $F$ and $E$, respectively. (This last definition is made explicit in Section IV.) We will use whichever representation is most convenient, relying on the context to make it clear which one is being used.

For any inner $\Theta \in M(E, F)$, we define $H(\Theta)$ to be $H^2_F \circ \Theta H^2_E$, where
the symbol "e" denotes the orthogonal complement of $\mathcal{H}_E^2$ in $\mathcal{H}_F^2$. Define $S(\Theta)$ to be the compression to $H(\Theta)$ of the unilateral shift on $H_F^2$, i.e.

$$S(\Theta)v \equiv P(Xv) \mid H(\Theta)$$

where $P$ is the orthogonal projection of $H_F^2$ onto $H(\Theta)$.

We are now ready to give a formal definition of the characteristic operator function. Let $T$ be a contraction on a Hilbert space, $H$.

The defect operators of $T$ are defined by:

$$D_T \equiv (I - T^*T)^{1/2} \text{ and } D_{T^*} \equiv (I - TT^*)^{1/2}$$

The defect spaces, $E_T$ and $E_{T^*}$, are defined to be the closure of the ranges of $D_T$ and $D_{T^*}$, respectively. The dimensions, $d_T$ and $d_{T^*}$, of $E_T$ and $E_{T^*}$ are called the defect indices of $T$. For every $z$ in the unit disk, define $\Theta_T(z)$ from $E_T$ to $E_{T^*}$ by:

$$\Theta_T(z) \equiv -T + zD_{T^*}(I - zT)^{-1}D_T \mid E_T$$

Then $\Theta$ is a bounded analytic function in $M(E_T, E_{T^*})$ and is called the characteristic operator function of $T$ [HA, pg. 237].

The significance of these definitions is that they allow an arbitrary $C_\infty$ contraction to be represented as a compression of a unilateral shift. More precisely, $T$ is a $C_\infty$ contraction if and only if $\Theta_T$ is inner and $S(\Theta_T)$ is unitarily equivalent to $T$ [HA, pg. 248]. Using this representation, Sz.-Nagy [HA, pg. 258] extended a representation of the single commutant which Sarason [18] had first obtained for the unilateral shift. We will state a $C_\infty$ version of it here, since we will be referring to it frequently.
The Lifting Theorem

Suppose that $\Theta \in M(F, E)$ and $\Theta' \in M(F', E')$ are inner. (Thus, $S(\Theta)$ and $S(\Theta')$ are arbitrary $C_0$ contractions.) An operator $X$ intertwines $S(\Theta)$ and $S(\Theta')$, i.e. $S(\Theta)X = XS(\Theta')$, if and only if there exists a $Y \in M(E', E)$ such that $X = PY | H(\Theta')$ and $Y\Theta'H^2_F \subset \Theta'H^2_F$, where $P$ is the orthogonal projection of $H^2_E$ onto $H(\Theta)$.

Another important property of the characteristic operator function is the ease with which we can define the Nagy-Foiaş functional calculus for $C_0$ contractions [HA, pg. 114]. Let $T = S(\Theta)$ be a $C_0$ contraction where $\Theta \in M(E, F)$. For $u \in H^\infty$, define $u(T)$ on $H(\Theta)$ by $u(T)h = P(uh)$, where $P$ is the orthogonal projection of $H^2_F$ onto $H(\Theta)$. This agrees with the usual functional calculus for polynomials, i.e.

$$p(T) = \sum_{k=0}^{n} a_k T^k \quad \text{for} \quad p(z) = \sum_{k=0}^{n} a_k z^k$$

It also agrees with the usual functional calculus for normal operators (i.e. if $T$ is also normal, then $u(T)$ is the integral of $u(z)$ with respect to the spectral measure of $T$).

Another important class of contractions is $C_o$, which consists of those $C_0$ contractions, $T$, for which there exists a non-zero $H^\infty$-function, $u$, which annihilates $T$ (i.e. for which $u(T) = 0$). Every such operator has a minimal annihilator, in the sense that every annihilator is a multiple of it. It is called the minimal function of $T$ and is written $m_T$ [HA, pg. 123]. For any integer $N$, the subclass $C_o(N)$ consists of those $C_o$ contractions with one (and hence both) defect indices equal to $N$ [HA, pg. 350].

The last and largest class of contractions that we will consider
is the completely non-unitary (CNU) contractions. A contraction, $T$, is called CNU if it has no non-zero reducing subspaces to which its restriction is unitary. It is well known [HA, pg. 9] that every contraction has a canonical decomposition as the direct sum of a unitary part and a completely non-unitary part. Every $C_o$ (and hence every $C_0$) contraction is CNU. Much of the theory for $C_o$ contractions presented so far can be extended to CNU contractions; however, $\Theta_T$ is no longer inner and $S(\Theta)$ and $H(\Theta)$ become more complicated [HA, pg. 248]. It is known [13, pg. 180] that every CNU contraction is a compression of a unitary, called its unitary dilation, whose scalar spectral measure is absolutely continuous with respect to Lebesgue measure. Further, it is possible to define a functional calculus for any CNU contraction, $T$, as follows: for any $H^\infty$-function, $a$, define $a(T)$ to be the compression of $a(U)$ where $U$ is the unitary dilation of $T$ [HA, pg. 13]. The functional calculus defined in this way agrees with the Nagy-Foiaş functional calculus [HA, pg. 114].
SECTION II

RP-EQUIVALENCE AND JORDAN MODELS

In this section, the relation of RP-Equivalence is introduced and applied to the problem of finding Jordan models for quasi-similarity classes of \( C_0 \) contractions. In [25], Sz.-Nagy and Foiaş developed the theory of Jordan models for \( C_0 \) contractions. A later extension [1] proved that for a \( C_0 \) contraction, \( T \), on a separable Hilbert space, \( H \), there exists a unique operator \( S(M) \), which is quasi-similar to \( T \), where \( S(M) \) is called the Jordan model for \( T \) and has the following properties:

(i) \( M = \{m_j\}_{j=1}^\infty \), is a sequence of inner functions.

(ii) \( m_j \) divides \( m_{j+1} \) for all \( j \).

and (iii) \( S(M) \equiv S(m_1) \otimes S(m_2) \otimes S(m_3) \otimes \ldots \).

Moore and Nordgren have shown [11 and 12] that, for a \( C_0 \) contraction with finite defect indices, the terms of the sequence are the invariant factors of its characteristic operator function. Further, they introduced the notion of quasi-equivalence, which "lifts" quasi-similarity for operators with finite defect. That is, two \( C_0 \) contractions with finite defect indices are quasi-similar if and only if their characteristic operator functions, when viewed as matrices over \( H^\infty \), are quasi-equivalent. It was my intent to extend this relationship to a larger class of operators. Although the following section extends the relationship to include certain "diagonal" operators, the more general question remains unsolved.

Throughout the remainder of this section, let \( M \) be the algebra,
$M(E, E)$, of analytic, weakly-measurable, essentially-bounded functions from the unit disk to the algebra of bounded operators on a fixed, separable Hilbert space, $E$. Let $E$ have an orthonormal basis $\{e_j\}_{j=1}^{\infty}$. We will always consider $M$ as a matrix algebra over $H^\infty$.

**DEFINITION** For $\Theta, \Theta' \in M$, to say that $\Theta$ is RP-Equivalent to $\Theta'$ will mean that there exists $X, Y \in M$ and $\delta, \gamma \in H^\infty$, such that

1. $\gamma$ is a scalar multiple of both $\Theta$ and $\Theta'$,
2. $\delta$ is a scalar multiple of both $X$ and $Y$,
3. $X\Theta = \Theta'Y$,

and (4) $\delta$ are $\gamma$ relatively prime.

In order to justify using the word "equivalent", it should be an equivalence relation. Before proving that, we will need the following two lemmas:

**LEMMA 1** RP-Equivalence is symmetric on $M$.

Proof): Suppose $\Theta$ is RP-equivalent to $\Theta'$. Then, there exists $X, Y \in M$ and $\delta, \gamma \in H^\infty$ as above. Since $\delta$ is a scalar multiple of $X$, there exists an $X^{\text{adj}} \in M$, such that $(X^{\text{adj}})X = (\delta I)$, where $I$ is the identity on $M$. Similarly, there exists $Y^{\text{adj}} \in M$, such that $Y(Y^{\text{adj}}) = (\delta I)$. Multiplying (3) above by $X^{\text{adj}}$ on the right and by $Y^{\text{adj}}$ on the left, yields: $(X^{\text{adj}})X\Theta(Y^{\text{adj}}) = (X^{\text{adj}})\Theta(Y^{\text{adj}})$. This reduces to $(\delta I)\Theta(Y^{\text{adj}}) = (\delta I)(X^{\text{adj}})\Theta'$. We can cancel $(\delta I)$, since $\delta$ is relatively prime to $\gamma$ and hence non-zero.

So, we are left with $(X^{\text{adj}})\Theta' = \Theta(Y^{\text{adj}})$. But, this shows that $\Theta'$ is RP-equivalent to $\Theta$, since $X^{\text{adj}}$ and $Y^{\text{adj}}$ both have $\delta$ as a scalar multiple. Q.E.D.
Note that each element of $M$ has a minimal scalar multiple (also
called the characteristic scalar multiple [see 9]), since the greatest
common divisor of the set of all scalar multiples exists and is itself
a scalar multiple. A very useful property of RP-equivalence is

**Lemma 2.** RP-equivalence preserves minimal scalar multiples.

Proof: Let $\Theta_1$ be RP-equivalent to $\Theta_2$. Let $\gamma_1$ and $\gamma_2$ be the minimal
scalar multiples of $\Theta_1$ and $\Theta_2$, respectively. It will be sufficient to
show that $\gamma_1$ is a scalar multiple of $\Theta_2$, since this would imply that $\gamma_2$
divides $\gamma_1$ (since $\gamma_2$ is minimal). But by Lemma 1, we would then have
that $\gamma_1$ divides $\gamma_2$. Hence we would have $\gamma_1 = \gamma_2$ (up to a constant).

First, we will show that $\delta^2 \gamma_1$ is a scalar multiple of $\Theta_2$. As
before, there exists $\delta$, $\gamma$, $X$, $Y$, $X^{\text{adj}}$, and $Y^{\text{adj}}$ such that
$X \Theta_1 = \Theta_2 Y$,
$X X^{\text{adj}} = X^{\text{adj}} X = (\delta I)$,
$Y Y^{\text{adj}} = Y^{\text{adj}} Y = (\delta I)$, $\gamma$ is a multiple of $\gamma_1$ and $\gamma_2$, and
$\gamma$ is relatively prime to $\delta$. Further, since $\gamma_1$ is a scalar multiple of
$\Theta_1$, there exists a $\Theta_1^{\text{adj}} \in M$, such that $\Theta_1^{\text{adj}} = \Theta_1 \Theta_1 = (\gamma_1 I)$.

Direct calculations show that:

$$
\Theta_2 \left( (\delta I) Y \Theta_1^{\text{adj}} X^{\text{adj}} \right) = (\delta I) (\Theta_2 Y) \Theta_1^{\text{adj}} X^{\text{adj}}
= (\delta I) (X \Theta_1) \Theta_1^{\text{adj}} X^{\text{adj}}
= (\delta I) X (\gamma I) X^{\text{adj}} = (\delta \gamma I)(XX^{\text{adj}}) = (\delta^2 \gamma I)
$$

and

$$
\left( (\delta I) Y \Theta_1^{\text{adj}} X^{\text{adj}} \right)^{\Theta_2} = Y \Theta_1^{\text{adj}} X^{\text{adj}} \Theta_2
= Y \Theta_1^{\text{adj}} X^{\text{adj}} (\Theta_2 Y) X^{\text{adj}}
= Y \Theta_1^{\text{adj}} X^{\text{adj}} (X \Theta_1) Y^{\text{adj}}
= (\delta I) Y \Theta_1^{\text{adj}} \Theta_1 Y^{\text{adj}}
= (\delta \gamma I) YY^{\text{adj}} = (\delta^2 \gamma I)
$$

Therefore, $\Theta_2$ has scalar multiple of $\delta^2 \gamma$. 
But, we also know that $\gamma$ is a scalar multiple of $\Theta_2$. Therefore, their g.c.d. must also be a scalar multiple of $\Theta_2$. Since $\delta$ (and hence $\delta^2$) is relatively prime to $\gamma$ and since $\gamma_1$ divides $\gamma$, their g.c.d. must be $\gamma_1$.

Q.E.D.

With the lemmas completed, the reason for the term "RP-equality" can now be proven as

**Theorem 3** RP-equivalence is an equivalence relation on $M$.

**Proof:** Symmetry was proven as Lemma 1. To prove the reflexive property, let $\Theta \in M$ and observe that $I \Theta = \Theta I$. Let $\gamma$ be any scalar multiple of $\Theta$ (possibly zero). Then $\Theta$ is RP-equivalent to $\Theta$, since $1$ is a scalar multiple of $I$ and the g.c.d. of $1$ with $\gamma$ is $1$ (even if $\gamma = 0$).

To prove transitivity, suppose $\Theta_1$ is RP-equivalent to $\Theta_2$ and $\Theta_2$ is RP-equivalent to $\Theta_3$. Applying the definition of RP-equivalence and the definition of scalar multiple, we have that there exists $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2 \in \mathbb{R}^\infty$ and $X_1$, $X_2$, $Y_1$, $Y_2$, $X_1^{adj}$, $X_2^{adj}$, $Y_1^{adj}$, $Y_2^{adj} \in M$, such that $X_1$, $X_2$, $Y_1$, $Y_2$, $X_1^{adj}$, $X_2^{adj}$, $Y_1^{adj}$, $Y_2^{adj} \in M$, such that

(a) $X_1 \Theta_1 = \Theta_2 Y_1$

(b) $X_2 \Theta_2 = \Theta_3 Y_2$

(c) $X_k^{adj} = X_k X_k^{adj} \in \mathbb{R}^\infty$, for $k = 1, 2$  

(d) $Y_k^{adj} = Y_k X_k^{adj} \in \mathbb{R}^\infty$, for $k = 1, 2$

(e) $\Theta_1$ and $\Theta_2$ have scalar multiple $\gamma_1$

(f) $\Theta_2$ and $\Theta_3$ have scalar multiple $\gamma_2$

and (g) $\gamma_k$ and $\delta_k$ are relatively prime, for $k = 1, 2$.

Setting $X_3 \equiv X_2 X_1$ and $Y_3 \equiv Y_2 Y_1$, we observe that
\[ x_3 \theta_1 = (x_2 x_1) \theta_1 = x_2 (\theta_2 y_1) = (\theta_3 y_2) y_1 = \theta_3 y_3 \]

and \[ x_3 (x_1^{\text{adj}} x_2^{\text{adj}}) = x_2 (x_1^{\text{adj}} x_2^{\text{adj}}) = (\delta_1 I)(x_2 x_2^{\text{adj}}) = \delta_1 \delta_2 I. \]

Similar calculations will show that \( x_3 \) and \( y_3 \) have scalar multiple \( \delta_1 \delta_2 \).

We would like to now claim that \( \delta_1 \delta_2 \) is relatively prime to \( y_1 y_2 \) (which is a scalar multiple of both \( \theta_1 \) and \( \theta_3 \)), in which case we would be done.

Unfortunately, this need not be true, since \( y_1 \) and \( \delta_2 \) might have a factor in common. To avoid this, let \( \gamma \) be the minimal scalar multiple of \( \theta_1 \).

By Lemma 2, \( \gamma \) is also the minimal scalar multiple of \( \theta_2 \) and hence of \( \theta_3 \), as well. By (g) above, \( \delta_k \) is relatively prime to \( \gamma_k \), for \( k = 1, 2 \). But \( \gamma_k \) is a multiple of \( \gamma \); hence, \( \delta_k \) is relatively prime to \( \gamma \), for \( k = 1, 2 \). Therefore, \( \delta_1 \delta_2 \) is relatively prime to \( \gamma \). So, we have shown that \( x_3 \) and \( y_3 \) have a common multiple \((\delta_1 \delta_2)\) which is relatively prime to a common multiple of \( \theta_1 \) and \( \theta_3 \), namely \( \gamma \). Q.E.D.

For the following definitions, let \( T_1 \) and \( T_2 \) be operators defined on Hilbert spaces \( H_1 \) and \( H_2 \), respectively; and, let \( V \) be an operator from \( H_1 \) to \( H_2 \). An operator \( V \) is called a quasi-affinity if it is one-to-one and has dense range. The operator \( T_1 \) is called a quasi-affine transform of \( T_2 \) if there exists a quasi-affinity, \( V \), such that \( VT_1 = T_2 V \). If \( T_1 \) and \( T_2 \) are both quasi-affine transforms of each other, then \( T_1 \) and \( T_2 \) are said to be quasi-similar [cf. HA, pg. 70].

The original motivation for the study of RP-equivalence was the observation that the previously mentioned result by Moore and Nordgren [11, Theorem 3] was based not on their definition of quasi-equivalence but on a property of it [12, Corollary 3.3]. Their result is modified and generalized to remove the finite defect index restriction after the following lemma.
**Lemma 4** Suppose $X$, $Y$, $\theta_1$, $\theta_2$ are elements of $M$ such that $X\theta_1 = \theta_2 Y$.

Further suppose that $S(\theta_k)$ is a $C_o$ contraction for $k=1,2$; and, $V \equiv PX \mid H(\theta_1)$, where $P$ is the orthogonal projection of $H^2_E$ onto $H(\theta_2)$.

(a) $V$ is one-to-one if and only if for all $f \in L^2_E$, $Yf \in H^2_E$ and $\theta_1 f \in H^2_E$ implies $f \in H^2_E$.

(b) $V$ has dense range if and only if the span of $XH^2_E$ and $

\theta_2 H^2_E$ is $H^2_E$.

(c) $V$ is onto if and only if $XH^2_E + \theta_2 H^2_E = H^2_E$.

Proof of a): By hypothesis, $S(\theta_k)$ is a $C_o$ contraction, for $k=1,2$. Therefore, $\theta_k(z)$ is an isometry for almost every $z$ in the open unit disk [see HA, pg. 190]. But, this implies that $\theta_k$ is an isometry on $H^2_E$ [see HA, pg. 190], and hence, $\theta_k$ is unitary on $L^2_E$, for $k=1,2$.

Suppose that $V$ is one-to-one, $f \in L^2_E$, and $Yf$, $\theta_1 f \in H^2_E$. Since $H^2_E = H(\theta_1) \oplus \theta_1 H^2_E$, there exists a (unique) $g \in H(\theta_1)$ and $h \in H^2_E$ such that $\theta_1 f = g + \theta_1 h$. But then, $Xg = X(\theta_1 f - \theta_1 h) = \theta_2(Yf - Yh)$. Since both $Yf$ and $Yh$ are in $H^2_E$, we have $Xg \in \theta_2 H^2_E$. Therefore, $PXg = 0$. But $PXg = Vg$ and $V$ was assumed to be one-to-one. Therefore, $g = 0$, and hence $\theta_1 f = \theta_1 h$.

Since $\theta_1$ is unitary, we have $f = h$ and hence $f \in H^2_E$.

Conversely, suppose that $u$ is in the kernel of $V$. Then, $u \in H(\theta_1)$ and $Xu \in \theta_2 H^2_E$. If $h = \theta_1 u$, then $\theta_1 h \in H^2_E$, since $\theta_1 h = \theta_1 (\theta_1 u) = u$. Also, $Yh \in H^2_E$, since $\theta_2 Yh = Xh = Xu \in \theta_2 H^2_E$. By hypothesis then, $h \in H^2_E$. But, this implies that $u \in H^2_E$ and also $u \in H(\theta_1)$. Therefore, $u = 0$, showing that $V$ is one-to-one.

Proof of b): Suppose $XH^2_E$ and $\theta_2 H^2_E$ do not span $H^2_E$. Then, there exists a non-zero $f \in H^2_E$ such that $f \perp Xg$ and $f = Pf$, for all $g \in H(\theta_1)$. But then,
\[ <f, Vg> = <f, PXg> = <Pf, Xg> = <f, Xg> = 0, \text{ for all } g \in H(\Theta). \text{ Therefore,} \\
f \perp VH(\Theta_1); \text{ and hence, } V \text{ does not have dense range.} \]

Conversely, suppose \( V \) does not have dense range. Then there exists a non-zero \( H_E^2 \) function, \( f \), such that \( f \perp \Theta_2 H_E^2 \) and \( f \perp VH(\Theta_1) \). If \( g \) is any \( H_E^2 \) function, then it can be expressed as \( g = g' + \Theta_2 h \), where \( g' \in H(\Theta_1) \) and \( h \in H_E^2 \). Then, \[ <f, Xg> = <f, Xg'> + <f, \Theta_2 h>. \]

But, \[ <f, \Theta_2 h> = <f, \Theta_2 Yh> = 0, \text{ since } f \perp \Theta_2 H_E^2. \]

Also, \[ <f, Xg'> = <Pf, Xg'> = <f, Vg'> = 0, \text{ since } f \in H(\Theta_2) \text{ and } f \perp VH(\Theta_1). \]

Therefore, \( f \perp XH_E^2 \). Hence, \( XH_E^2 \) and \( \Theta_2 H_E^2 \) do not span \( H_E^2 \).

Proof of c): Suppose \( XH_E^2 + \Theta_2 H_E^2 = H_E^2 \). In particular for \( f \in H(\Theta_2) \), there exists \( g, h \in H_E^2 \) such that \( f = Xg + \Theta_2 h \). Then, \( f = Pf = P(Xg + \Theta_2 g) = Vg + P\Theta_2 g = Vg \). Therefore, \( V \) is onto.

Conversely, suppose \( V \) is onto. For any \( H_E^2 \) function \( f \), there exists a (unique) \( g \in H(\Theta_2) \) and \( h \in H_E^2 \) such that \( f = g + \Theta_2 h \). Then, since \( V \) is onto, there exists a \( g' \in H(\Theta_1) \) such that \( g = PXg' \). But, \( PXg' = Xg' + \Theta_2 h' \), for some \( h' \in H_E^2 \). Thus,

\[ f = g + \Theta_2 h = (Xg' + \Theta_2 h') + \Theta_2 h = X(g') + \Theta_2 (h + h'). \]

Hence, \( f \in XH_E^2 + \Theta_2 H_E^2 \). Q.E.D.

With this done, the main theorem of this section can now be stated.

**Theorem 5** If \( T_1 \) and \( T_2 \) are \( C_0 \) contractions whose characteristic operator functions are RP-equivalent, then they are quasi-similar.

Proof): Let \( T_1 \) and \( T_2 \) be \( C_0 \) contractions whose characteristic operator functions \( \Theta_1 \) and \( \Theta_2 \) are RP-equivalent. Then, there exists \( \delta, \gamma \in H^\infty \) and \( X, Y, Xadj, Yadj, \Theta_1adj, \text{ and } \Theta_1adj \) in \( M \) such that:
(a) \( X_0 = \theta_2 Y \),
(b) \( XX^{adj} = X^{adj} X = \delta I \),
(c) \( YY^{adj} = Y^{adj} Y = \delta I \),
(d) \( \Theta_k \Theta_k^{adj} = \delta \Theta_k \), for \( k = 1, 2 \),

and (e) \( \delta \) and \( \gamma \) are relatively prime.

Define \( P_k \) to be the orthogonal projection of \( H_E^2 \) onto \( H(\Theta_k) \), for \( k = 1, 2 \).
Define \( V \) from \( H(\Theta_1) \) into \( H(\Theta_2) \) by \( Vf = P_2 Xf \). Observe that \( P_2 X \) is zero on \( \Theta_1 H_E^2 \), since \( (P_2 X) \Theta_1 = P_2 (\Theta_2 Y) = (P_2 \Theta_2) Y = 0 \). Therefore, \( P_2 X = P_2 X \).

Let \( U_+ \) be the unilateral shift on \( H_E^2 \). Since \( U_+ \) commutes with all multiplication operators (in particular \( X \)) and since \( \Theta_2 H_E^2 \) is invariant under \( U_+ \), we have, for all \( f \in H(\Theta_1) \),

\[
(P_2 X) U_+ f = P_2 (X U_+) f = P_2 U_+ (P_2 X) f.
\]

Thus, since \( V = P_2 X \), \( S(\Theta_1) = P_1 U_+ \), and \( S(\Theta_2) = P_2 U_+ \), we have

\[
VS(\Theta_1) = S(\Theta_2) V.
\]

To prove that \( V \) is one-to-one, suppose that \( h \in L_E^2 \) and that both \( Yh \) and \( \Theta_1 h \) are \( H_E^2 \) functions. Using (c), we have that \( \delta h = \gamma^{adj} Yh \in \gamma^{adj} H_E^2 \subset H_E^2 \).

Similarly by (d), we have \( \gamma h = \Theta_1^{adj} \Theta_1 h \in \Theta_1^{adj} H_E^2 \subset H_E^2 \). Let \( (h_j) \) be the vector representation of \( h \). By the above, \( \delta h_j \) and \( \gamma h_j \) are both in \( H_E^2 \) for all \( j \). But \( \delta \) and \( \gamma \) are relatively prime by (e); therefore, by a lemma of Sz.-Nagy [20], \( h_j \in H_E^2 \) for all \( j \). Therefore \( h \in H_E^2 \), and hence, by Lemma 4a, \( V \) is one-to-one.

To prove that \( V \) has dense range, suppose that \( w \in H(\Theta_2) \) and is perpendicular to \( XH_E^2 \). Let \( e_{ij} \) be a standard basis element for \( H_E^2 \), namely: \( z^i \) in the \( i \)th coordinate and zeroes elsewhere. By (b), \( \delta e_{ij} = XX^{adj} e_{ij} \in XH_E^2 \),
therefore, \( w \perp \delta e_{ij} \), for all \( i \) and \( j \). Similarly, by (d), \( \gamma e_{ij} = 0^2 \theta_2^\text{adj} e_{ij} e \theta_2^2 \) and therefore \( w \perp \gamma e_{ij} \), for all \( i \) and \( j \), since \( w \in H(\theta_2) \).

But by (e), \( \delta \) and \( \gamma \) are relatively prime; thus, by Beurling's theorem [3], \( w \perp e_{ij} \) for all \( i \) and \( j \). Therefore, \( w = 0 \), and hence by Lemma 4b, \( V \) has dense range.

So, we have produced a quasi-affinity, \( V \), such that \( VS(\theta_1) = S(\theta_2)V \); therefore, \( S(\theta_1) \) is a quasi-affine transform of \( S(\theta_2) \). By symmetry (Lemma 1), \( S(\theta_1) \) and \( S(\theta_2) \) are quasi-similar. But, \( T_k \) is unitarily equivalent to \( S(\theta_k) \), for \( k = 1,2 \) [see HA, page 248]; therefore, \( T_1 \) is quasi-similar to \( T_2 \). Q.E.D.

Now that the basic properties of RP-equivalence have been established, let us compare its strength relative to quasi-equivalence.

The original definition of quasi-equivalence [12] went as follows:

for any \( X \in M \), denote by \( \delta(X) \) the minimal scalar multiple of \( X \). Then a subset \( X \) of \( M \) is called a QUASI-UNIT if the set \( \{ \delta(X) : X \in X \} \) is relatively prime. If there exists quasi-units \( X \) and \( Y \) such that \( \{ X \theta_1 : X \in X \} = \{ \theta_2 Y : Y \in Y \} \), then \( \theta_1 \) is called QUASI-EQUIVALENT to \( \theta_2 \).

The definition of quasi-equivalence was later modified [28, 26, and 23] as follows: for any inner function \( \omega \), we say that \( \theta_1 \) is \( \omega \)-EQUIVALENT to \( \theta_2 \) if there exists \( X, Y \in M \) such that \( X \theta_1 = \theta_2 Y \) and \( \delta(X)\delta(Y) \) is relatively prime to \( \omega \). Then, \( \theta_1 \) is called QUASI-EQUIVALENT to \( \theta_2 \) if \( \theta_1 \) is \( \omega \)-equivalent to \( \theta_2 \) for every non-zero inner function \( \omega \). Their relative strengths are summed up as:

**Theorem 6** Suppose \( \theta_1, \theta_2 \in M \) and both have non-zero scalar multiples.

(a) If \( \theta_1 \) and \( \theta_2 \) are quasi-equivalent, in the latter sense, then they are RP-equivalent.

(b) If \( \theta_1 \) and \( \theta_2 \) are RP-equivalent, then they are quasi-equivalent, in the former sense.
(c) If the underlying space $E$ is finite dimensional, then all three definitions coincide.

Proof: To prove part (a), suppose that $\Theta_1$ is $\omega$-equivalent to $\Theta_2$ for every non-zero inner function $\omega$. In particular, they are $\omega$-equivalent for $\omega \equiv \delta(\Theta_1)\delta(\Theta_2)$, since by hypothesis, this is non-zero. By definition, there exists $X, Y \in \mathcal{M}$ such that $X\Theta_1 = \Theta_2 Y$ and $\delta(X)\delta(Y)$ is relatively prime to $\omega$. If $\delta \equiv \delta(X)\delta(Y)$ and $\gamma \equiv \omega$, then it follows that $\Theta_1$ is RP-equivalent to $\Theta_2$, since $\delta$ is a scalar multiple of $X$ and $Y$ and $\gamma$ is a scalar multiple of $\Theta_1$ and $\Theta_2$.

To prove part (b), suppose that $\Theta_1$ is RP-equivalent to $\Theta_2$. Define $\delta, \gamma, X$, and $Y$ as in the definition of RP-equivalence. If $X \equiv \{X, \Theta_2\}$ and $Y \equiv \{Y, \Theta_1\}$, then $X\Theta_1, \Theta_2 \Theta_1 = \{\Theta_2 Y, \Theta_2 \Theta_1\}$. Also, $X$ is a quasi-unit since $\delta(X)$ divides $\delta$ and $\delta(\Theta_1)$ divides $\gamma$, and hence, their g.c.d. divides both. Similarly, $Y$ is a quasi-unit.

To prove part (c), it remains only to show that for finite dimensional $E$, the former definition of quasi-equivalence implies the latter. That this is so follows directly from a result of Nordgren [12, Theorem 3.1]. Q.E.D.

A converse of Theorem 5 was proven, in the case that the underlying space $E$ is finite dimensional, by Moore and Nordgren [11] for quasi-equivalence, and hence, for RP-equivalence. However, no converse is known for any larger class. But, it is possible, as the following theorem shows, to express the definition of quasi-affine transform entirely in terms of the characteristic operator functions.
THEOREM 7 Suppose that $T_1$ and $T_2$ are $C_0$ contractions with characteristic operator functions $\Theta_1$ and $\Theta_2$, respectively. Then, $T_1$ is a quasi-affine transform of $T_2$ if and only if there exist $X, Y \in M$ such that:

1. $X\Theta_1 = \Theta_2 Y$,
2. the span of $XH^2_E$ with $\Theta_2 H^2_E$ is $H^2_E$, and
3. $yf \in H^2_E$ and $\Theta_1 f \in H^2_E$ implies $f \in H^2_E$, for all $f \in L^2_E$.

Proof: Let $P_k$ be the orthogonal projection of $H^2_E$ onto $H(\Theta_k)$, for $k=1,2$; and, let $U_+$ be the unilateral shift on $H^2_E$.

Suppose there exists $X, Y \in M$ satisfying (i)-(iii). As in Theorem 5, if $V = P_2 X \upharpoonright H(\Theta_1)$, then $VS(\Theta_1) = S(\Theta_2)V$. By Lemma 4, (ii) and (iii) imply that $V$ is quasi-affine. Therefore, $S(\Theta_1)$ is a quasi-affine transform of $S(\Theta_2)$; and hence, $T_1$ is a quasi-affine transform of $T_2$.

Conversely, suppose that $T_1$ is a quasi-affine transform of $T_2$. Then, $S(\Theta_1)$ is a quasi-affine transform of $S(\Theta_2)$; hence, there exists a quasi-affinity, $V$, from $H(\Theta_1)$ into $H(\Theta_2)$ such that $VS(\Theta_1) = S(\Theta_2)V$.

By the Lifting Theorem [HA, pg. 258], $V$ can be expressed in the form $P_2 X \upharpoonright H(\Theta_1)$ for some $X \in M$, satisfying $X\Theta_2 H^2_E \subseteq H^2_E$. But, this implies that there exists a $Y \in M$ such that $X\Theta_1 = \Theta_2 Y$, since $X$ commutes with $U_+$ [HA, pg. 195]. By Lemma 4, $X$ and $Y$ satisfy (ii) and (iii). Q.E.D.

The remainder of this section will be concerned primarily with "diagonal" elements of $M$. The notation $\Theta = \operatorname{diag}(f_1, f_2, \ldots)$ will mean that $\Theta \in M$ is the DIAGONAL MATRIX defined by $\Theta e_i = f_i e_i$, for all $i$, where $(e_i)$ is a (fixed) basis for $E$ and $(f_i)$ is a sequence of $H^\infty$-functions.
As was stated earlier, the existence of a Jordan model for any $C_0$ contraction was established [1]. A recent result [2] directly related the Jordan model to the characteristic operator function. However, there is still no equivalence theory to determine when two characteristic operator functions possess the same Jordan model, or even what that model is. The following simple example illustrates a particular case where RP-equivalence can be used to verify that what appears to be the Jordan model for a diagonal matrix is indeed its model.

**Example 8** Let $(f_j)$ be a sequence of pairwise relatively prime Blaschke products whose product $\prod_{j=1}^{\infty} f_j$ exists. If $\Theta = \text{diag}(f_1, f_2, \ldots)$ then it is RP-equivalent to its Jordan model which is $\text{diag}(\prod_{j=1}^{\infty} f_j, 1, 1, \ldots)$.

To begin, let us make the following definitions:

$$F_n = \prod_{k=1}^{n} f_k'$$

$$F = \prod_{k=1}^{\infty} f_k = \lim_{k \to \infty} F_k$$

$$\Theta = \text{diag}(f_1, f_2, \ldots), \quad J = \text{diag}(F, 1, 1, \ldots).$$

$$X = \begin{pmatrix}
F_{/4f_1} & F_{/4f_2} & F_{/8f_3} & \cdots & F_{/2^n f_n} & \cdots \\
1/4 & 1 & 0 & \cdots & 0 & \cdots \\
1/8 & 0 & 1 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1/2^n & 0 & 0 & \cdots & 1 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots
\end{pmatrix}$$
Observe that $X$ and $Y$ are elements of $M$, since they are both the sum of three simple matrices. Further, direct multiplication will show that:

$$X0 = JY = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \ldots & \frac{1}{2^n} & \ldots \\
\frac{f_1}{4} & f_2 & 0 & \ldots & 0 & \ldots \\
\frac{f_1}{8} & 0 & f_3 & \ldots & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{f_1}{2^n} & 0 & 0 & \ldots & f_n & \ldots
\end{pmatrix}$$

If $0^{adj} \equiv \text{diag}(\frac{F}{f_1}, \frac{F}{f_2}, \ldots)$ and $J^{adj} \equiv \text{diag}(1, F, F, \ldots)$, then it is a simple matter to show that $0$ and $J$ have $F$ as a scalar multiple. Therefore, all that remains to be shown is that $X$ and $Y$ have a scalar multiple which is relatively prime to $F$.

Define $X_n$ to be the $n \times n$ upper left-hand corner of $X$, or more precisely, $X_n(z) = P_n X(z) \mid E_n$, where $E_n$ is the subspace spanned by $\{e_1, e_2, \ldots, e_n\}$ and $P_n$ is the orthogonal projection of $E$ onto $E_n$. 
Then, $X_n$ is a finite dimensional matrix and its determinant can be calculated by expanding $X_n$ on its first row as follows:

$$
\det X_n = \frac{F}{4f_1} - \frac{F}{4f_2} \cdot \frac{1}{4} + \frac{F}{8f_3} \cdot \frac{1}{8} - \cdots \pm \frac{F}{2^n f_n} \cdot \frac{1}{2^n}
$$

$$
= \sum_{k=1}^{n} (-1)^{k+1} \frac{F}{2^{2k} f_k}
$$

Now, define $X_n^{(n)} \in M$ to be $X_n \otimes I_n$, where $I_n(z)$ is the identity on $E \otimes E_n$. If $W_n^{(n)} \equiv X_n^{\text{adj}} \otimes (\det X_n) I_n$, where $X_n^{\text{adj}}$ is the classical adjoint of $X_n$, then $W_n^{(n)} X_n^{(n)} = X_n^{(n)} W_n^{(n)} = (\det X_n) I_n$. Therefore, $X^{(n)}$ has scalar multiple $(\det X_n)$.

The limit of the partial sums, $(\det X_n)$, exists since the infinite sum is absolutely convergent, call the sum $\delta$. Also, $(\det X_n)(z)$ is bounded in absolute value by 1 and $W_n^{(n)}$ is bounded in norm by 1. Therefore, by the Vitali-Montel theorem [as in HA, pg. 326], there exists a $W \in M$ which is the weak limit of the sequence $\{W_n^{(n)}\}$. Taking limits, we have $WX = XW = (\delta I)$. Therefore, $X$ has $\delta$ as a scalar multiple.

Similarly, to show that $\delta$ is a scalar multiple of $Y_n$ defined by $Y_n^{(n)} \equiv P_n Y(z) \mid E_n$. Expanding $Y_n^{(n)}$ on the first column yields:

$$
\det Y_n = \frac{1}{4} \left(f_2 f_3 \ldots f_n\right) - \frac{1}{4} \left(\frac{1}{4} f_3 \ldots f_n + \frac{1}{8} f_2 \frac{1}{8} f_4 \ldots f_n - \cdots
$$

$$
= \sum_{k=1}^{n} (-1)^{k+1} \frac{F}{2^{2k} f_k}
$$
By direct calculation, we have, for any $n$:

$$\left| \delta - (\det Y_n) \right| \leq \left| \sum_{k=1}^{\infty} (-1)^{k+1} \frac{F}{2^k f_k} \right| - \left| \sum_{k=1}^{\infty} (-1)^{k+1} \frac{F}{2^k f_k} \right| + \left| \sum_{k=1}^{n} (-1)^{k+1} \frac{F}{2^k f_k} \right|.$$

But, both of these terms converge to zero (since $F_n$ is a partial product of $F$). Therefore, $(\det Y_n)$ converges to $\delta$. As in the case for $X$, $Y^{(n)} = Y_n + I_n$ and $W^{(n)} = Y_n^{adj} + (\det Y_n) I_n$. Again applying the Vitali-Montel theorem, we obtain a $W \in M$, which shows $\delta$ is a scalar multiple of $Y$.

Finally, to show that $\delta$ is relatively prime to $F$, suppose that there is a non-constant inner function which divides both. Since $F$ is a Blaschke product, we need only consider zeroes of $F$. Suppose that $z$ is a zero of one of the terms $f_j$. Then $z$ is not a zero of any of the other $f_k$'s, since by hypothesis they are pairwise relatively prime. Therefore, $z$ is a zero of $F/f_k$ if and only if $k$ is distinct from $j$. Thus, $\delta(z)$ has exactly one non-zero term. So, we have shown that $\delta$ and $F$ have no zeroes in common.

Q.E.D.

A recent Lemma by Sz.-Nagy [23] states that if $\{f_{ij}\}_{i,j=1}^{\infty}$ is a bounded set of $H^\infty$ functions and $\{\omega_i\}_{i=1}^{\infty}$ is a set of inner functions such
that \( \{\omega_i, f_{i1}, f_{i2}, f_{i3}, \ldots \} \) is relatively prime for all \( i \), then there exists a numerical sequence \( (a_i)_{i=1}^{\infty} \) such that

(i) \( a_1 = 1 \),

(ii) \( \sum_{i=2}^{\infty} |a_i| \) is arbitrarily small, and

(iii) \( \omega_i \) is relatively prime to \( \sum_{j=1}^{\infty} a_j f_{ij} \), for all \( i \).

Using this lemma, the preceding example can be generalized to:

**Theorem 9** If \( (f_j) \) is a sequence of pairwise relatively prime \( H^\infty \) functions whose product, \( F \), exists, then \( \text{diag}(f_1, f_2, \ldots) \) is \( \text{RP-equivalent} \) to its Jordan model which is \( \text{diag}(F, 1, 1, \ldots) \).

**Proof:** For each \( j \), define \( F_j \) to be \( F/f_j \). To show that \( \{F_1, F_2, \ldots\} \) is relatively prime, assume that the g.c.d. of \( \{F_1, F_2, \ldots, F_{n-1}\} \) is \( \prod_{j=n}^{\infty} (f_j) \), since this clearly holds for \( n=2 \). Since we have assumed that the set \( \{f_j\} \) is pairwise relatively prime, we have that the g.c.d. of \( \prod_{j=n}^{\infty} (f_j) \) with \( \prod_{j \neq n}^{\infty} (f_j) \) is \( \prod_{j=n+1}^{\infty} (f_j) \). Therefore, the g.c.d. of \( \{F_1, \ldots, F_n\} \) is \( \prod_{j=n+1}^{\infty} (f_j) \). Hence, by induction, the g.c.d. of \( \{F_1, F_2, \ldots\} \) is the limit of these partial products, which is one.

Therefore, we can apply Sz.-Nagy’s Lemma to obtain a sequence of constants \( (a_j)_{j=1}^{\infty} \) such that \( F \) is relatively prime to \( \sum a_j F_j \) and \( \sum |a_j| < 2 \).

Let \( b_j \) be the square root of \( a_j \), for all \( j \). Define \( \Theta, J, X, Y \in M \) as follows:
By direct multiplication, we have $X \Theta = JY$. Clearly $F$ is a scalar multiple of both $\Theta$ and $J$. So, all that remains is to show that $X$ and $Y$ have a scalar multiple which is relatively prime to $F$.

Define $X_n$ by $X_n(z) \equiv P_n X(z)|_{E_n}$, where $E_n$ is the subspace spanned by $\{e_1, \ldots, e_n\}$ and $P_n$ is the orthogonal projection of $E$ onto $E_n$. Then, expanding the finite dimensional matrix $X_n$ on its first column, we have

$$
\det X_n = b_1^2 (f_2 \ldots f_n) - (-b_2)(b_2 f_1)(f_3 \ldots f_n) + b_3 f_2(b_3 f_1)(f_4 \ldots f_n) \\
= \sum_{j=1}^{n} a_j F_j
$$
The limit of these partial sums exists (since it's absolutely summable), call it \( \delta \). As in Example 8, we can use the Vitali-Montel theorem to show that \( \delta \) is a scalar multiple of \( X \).

Similarly, \( \delta \) is a scalar multiple of \( Y \). Finally, since the sequence \(( a_j )\) was constructed such that \( \sum a_j F_j \) is relatively prime to \( F \), we have shown that \( \Theta \) is \( R\Pi \)-equivalent to \( J \). Q.E.D.

The following technical Lemma is given only to simplify the Theorem which follows it.

**Lemma 10** Let \(( g_i )_{i=1}^\infty \) and \(( h_i )_{i=1}^\infty \) be sequences of \( H^\infty \)-functions. If their products \( G \) and \( H \), respectively, are also in \( H^\infty \) and if the set \( \{ h_1 G/g_1, h_2 G/g_2, \ldots \} \) is relatively prime, then \( \text{diag}(g_1 h_1, g_2 h_2, \ldots) \) is \( \omega \)-equivalent to \( \text{diag}(G, h_1, h_2, \ldots) \), for any inner function \( \omega \).

**Proof**: If \( \Theta_1 = \text{diag}(g_1 h_1, g_2 h_2, \ldots) \) and \( \Theta_2 = \text{diag}(1, g_1 h_1, g_2 h_2, \ldots) \), then \( S(\Theta_2) = 0 \oplus S(\Theta_1) \); therefore, \( S(\Theta_1) \) is unitarily equivalent to \( S(\Theta_2) \). But, this implies that \( \Theta_1 \) and \( \Theta_2 \) "coincide", i.e. that there exists an invertible \( Z \in M \) such that \( Z \Theta_1 = \Theta_2 Z \) [see HA, pg. 257]. In particular then \( \Theta_1 \) is \( \omega \)-equivalent to \( \Theta_2 \), for any inner function \( \omega \).

If \( G_n = G/g_n \), then by hypothesis, \( \{ G, h_1 G_1, h_2 G_2, \ldots \} \) is relatively prime; hence, by Sz.-Nagy's Lemma, there exists a numerical sequence \(( a_i )_{i=1}^\infty \) such that \( G + \sum_{i=2}^\infty (a_i h_i G_i) \) is relatively prime to \( \omega \), for any inner fixed function \( \omega \), and \( \sum |a_i| < 2 \). Define \( b_i \) to be a square root of \( a_i \), for all \( i \). If
\[ \Theta_3 \equiv \text{diag} \left( G, h_1, h_2, \ldots \right), \]

\[
X \equiv \begin{pmatrix}
G & b_1G & b_2G & b_3G & \ldots \\
-b_1h_1 & 1 & 0 & 0 & \ldots \\
b_2h_2 & 0 & 1 & 0 & \ldots \\
-b_3h_3 & 0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

and \( Y \equiv \begin{pmatrix}
1 & b_1h & b_2h_2 & \ldots \\
-b_1 & g_1 & 0 & \ldots \\
b_2 & 0 & g_2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \)

then \( X\Theta_2 = \Theta_3Y \). As in Example 8, we can show that the limit of the determinants of the leading rows and columns of \( X \) and \( Y \) is a scalar multiple of both \( X \) and \( Y \). In this case, the limit works out to be \( G + \sum_{1=2}^\infty (a_1h_1G_1) \), call the limit \( \delta \). But, by the construction of the sequence \((a_1)\), \( \delta \) is relatively prime to every inner function. Thus, \( \Theta_2 \) is \( \omega \)-equivalent to \( \Theta_3 \). By the transitivity of \( \omega \)-equivalence (the proof of which is straightforward), we have shown that \( \Theta_1 \) is \( \omega \)-equivalent to \( \Theta_3 \). Q.E.D.

The final Theorem of this section essentially extends the converse of Theorem 5 from finite defect indices to finite multiplicity, for the limited class stated.
THEOREM 11 If \((f_i)_{i=1}^\infty\) is a sequence of \(H^\infty\) functions whose product, \(F\), is a Blaschke product none of whose zeroes has multiplicity greater than some fixed \(N\), then \(\text{diag}(f_1, f_2, \ldots)\) is RP-equivalent to its Jordan model.

Proof: Let \(\{z_j\}\) be the set of all zeroes of \(F\). Denote by \(m(f, z_j)\) the multiplicity of \(z_j\) as a zero of \(f\). If \(z\) is not a zero of \(f\), we will write \(m(f, z) = 0\). For example, the hypothesis states that \(m(F, z_j) < N\), for every \(j\).

Let \(G\) be the least common multiple of \(\{f_i\}\). We would like to split \(G\) into "maximal", pairwise relatively prime components. To do this, define an index function, \(t\), such that \(m(f_t(j), z_j)\) is the maximum of the set \(\{m(f_i, z_j)\}_{i=1}^\infty\). Now, for each \(i\), define \(g_i\) such that \(m(g_i, z_j) = m(f_i, z_j)\) if \(j\) is in the image of \(t\), and \(m(g_i, z_j) = 0\) otherwise. Then, each zero of \(F\) is in exactly one \(g_i\). So, \(G = \prod g_i\) and the set \(\{G/g_i\}\) is relatively prime.

Now, since \(g_i\) divides \(f_i\), we can define \(f'_i\) to be \(f_i/g_i\), for all \(i\). Since \(z_j\) is not a zero of \(f'_t(j)G/g_t(j)\), for all \(j\), we have that the set \(\{f'_t(j)G/g_t(j)\}_{j=1}^\infty\) is relatively prime. So, Lemma 10 applies, i.e. \(\text{diag}(f_1, f_2, \ldots)\) is RP-equivalent to \(\text{diag}(G, f'_1, f'_2, \ldots)\).

Repeating this procedure, let \(F'\) be the product of the terms \(f'_i\) and let \(G'\) be their l.c.m. Observe that \(m(F', z_j) < N-1\), for all \(j\). As before, we can define sequences \((g'_i)\) and \((f''_i)\) such that \(f'_i = g'_i f''_i\), \(G' = \prod g'_i\), and the set \(\{f''_i G'/g'_i\}\) is relatively prime. Again applying the Lemma, we have \(\text{diag}(f'_1, f'_2, \ldots)\) is \(F\)-equivalent to \(\text{diag}(G', f''_1, f''_2, \ldots)\). That is, there exists \(X\) and \(Y\) in \(M\) such that \((X)\text{diag}(f'_1, f'_2, \ldots) = \text{diag}(G', f''_1, f''_2, \ldots)(Y)\) and \(X\) and \(Y\) have a scalar multiple, \(\delta\), which is relatively prime to \(F\). Since \(I \not\subset X\) and \(I \not\subset Y\) also have scalar multiple \(\delta\), we have shown that \(\text{diag}(G, f'_1, f'_2, \ldots)\) is RP-equivalent to
diag \( (G, G', f_1'', f_2'', \ldots) \).

Observe that \( m(F'', z_j) < N-2 \), for all \( j \), where \( F'' = \Pi f''_i \). So, this process must eventually reach the constant sequence one. By transitivity of RP-equivalence (Theorem 3), we have shown that diag\( (f_1, f_2, \ldots) \) is RP-equivalent to diag\( (G, G', G'', \ldots, G^{(N)}, 1, 1, \ldots) \). That this latter matrix is a Jordan model follows from the way in which the \( G \)'s are constructed, namely: each is the l.c.m. of itself and all following terms.

Q.E.D.
SECTION III

BI-INVARIENCE AND C₀

The intent of this section is to extend to the class of all C₀ contractions a result by Wu [33], for the class of C₀(n) contractions.

This result is:

THEOREM 1 Every invariant subspace of a C₀ contraction is bi-invariant

(Ie. is invariant under every operator in the double commutant).

Before beginning the proof let us construct the representations which the proof will use. First, without loss of generality, we can assume that the given C₀ contraction, T, is of the form S(θ) operating on H = H(θ) for some contractive analytic function, θ, since any CNU contraction is unitarily equivalent to the compression S(θ) where θ is its characteristic operator function [cf. HA pg. 345]. We will consider θ as acting on $H_{E}^2$ for some (possibly finite dimensional) Hilbert space, E.

By a well known representation theorem [21], every invariant subspace, K, of S(θ) corresponds to a (regular) factorization, $\theta = \theta_2 \theta_1'$, such that $K = \theta_2 H_{E}^2 \ominus H_{E}^2$.

Further, the double commutant of S(θ) is known to be the set of all operators of the form $\phi(T)$ where $\phi \in N_{T}$ [see 1]. Here $N_{T}$ is considered to be the space of all $\phi$ which are meromorphic on the open disk and have a representation $u/v$ where $u$ and $v$ are $H^\infty$-functions and $v(T)^{-1}$ has dense domain. In which case, $\phi(T)$ is defined to be $v^{-1}(T) \cdot u(T)$, which can be shown to be independent of the choices of $u$ and $v$ [see HA pg. 155].
Proof of theorem: Let $T = S(\theta)$ be a $C_0$ contraction on $H = H(\theta)$. Let $K = S_2 H^2_E \circ \theta H^2_E$ be a subspace of $H$ which is invariant under $T$. Let $\phi(T)$ be an element of the double commutant of $T$, where $\phi = u/v$. Let $f \in S_2 H^2_E$ and define $h = \phi(T)f$. Clearly $h \in H^2_E \circ \theta H^2_E$, so the problem is to show that $h \in S_2 H^2_E$ since this would imply that $h \in K$ and hence $K$ is invariant under $\phi(T)$.

Consider the equality of $h = \phi(T)f$. Applying the fact that $u(T) = v(T)\phi(T)$, yields $v(T)h = u(T)f$. Further, since $f = \theta_2g$ for some $g \in H^2_E$, we have $v(T)h = u(T)\theta_2g$. But, $v(T)h = P(\theta_2g)$ and $u(T)\theta_2g = P(\theta_2g)$, where $P$ is the orthogonal projection of $H^2_E$ onto $H$ and $I$ is the identity on $H^2_E$. Combining these we have $P(\theta_2g) = P(\theta_2g)$, which implies that $(\theta_2g)h \in \theta_2 H^2_E$, since $\theta_2 H^2_E \subset \theta_2 H^2_E$. Therefore, there exists a $y \in H^2_E$ such that $(\theta_2g)h = \theta_2y$.

Let $m_T$ be the minimal factor of $T$. Since $v(T)$ is quasi-affine, $v$ is relatively prime to $m_T$ [see HA, pg. 125]. The minimal function is in particular a scalar multiple of $\theta$ and hence of $\theta_2$; therefore, there exists an analytic function, $W$, such that $W\theta_2 = m_TI$. Thus, $W(\theta_2h) = W\theta_2y$, which implies that $(\theta_2h)W = (m_TI)y$. But, this implies that $v$ divides $y$.

To show this, let $(w_{ij})$ be the (possibly infinite dimensional) matrix representation of $W$ with respect to some fixed basis for $E$ and let $(h_j)$ and $(y_j)$ be the corresponding vector representations of $h$ and $y$, respectively. Then, for all $j$,

$$m_Ty_j = (m_TI)y_j = (\theta_2h) W = \sum w_{ij}h_i.$$ 

Thus, $v$ divides $m_Ty_j$ for all $j$, which implies that $v$ divides $y_j$ for all $j$, since $v$ is relatively prime to $m_T$. This shows that $v$ divides $y$ or, equivalently, that there exists an $x \in H^2_E$ such that $y = vx$. 
Thus, \((vI)h = \Theta_2 y = \Theta_2 (vx) = (vI) \Theta_2 x\). Finally, since \(v\) is not identically zero, we have \(h \in \mathcal{G}_2^H\).

Q.E.D.

The following result is not new [cf. 27]; however, the proof used here is an extension of the method used by Wu [33] for the case of finite defect indices and is quite different from the technique used by Sz.-Nagy and Foiaş when they proved this result. It is included here to show an application of Theorem 1.

**COROLLARY 2** The double commutant of any \(C_0\) contraction is equal to the weakly-closed algebra it generates.

**Proof:** Let \(T\) be a \(C_0\) contraction. Let \(\mathcal{A}\) be the weakly-closed algebra generated by \(I\) and \(T\). For any operator \(A\) and any positive integer \(n\), denote by \(A^{(n)}\) the operator

\[
A \oplus A \oplus A \oplus \ldots \oplus A
\]

\(n\) times

Let \(S\) be any operator in \(\{T\}''\). For any \(n\), \(S^{(n)} \in \{T^{(n)}\}''\), since, for any \(A \in \{T\}'\), \(S^{(n)} A^{(n)} = (SA)^{(n)} = A^{(n)} S^{(n)}\). Further, \(T^{(n)}\) is a \(C_0\) contraction, since it has the same minimal function as \(T\). Therefore, by Theorem 1, every invariant subspace of \(T^{(n)}\) is also invariant for \(S^{(n)}\). But, this implies that \(S \in \mathcal{A}\) [see 35, Theorem 7.1]. Therefore, \(\{T\}'' \subset \mathcal{A}\). Since \(\mathcal{A} \subset \{T\}''\) for any operator \(T\), we have \(\mathcal{A} = \{T\}''\).

Q.E.D.
SECTION IV

DOUBLE COMMUTANTS OF C₀ CONTRACTIONS

The intent of this section is to extend a recent result by Uchiyama [32], namely: if T is a C₀ contraction with distinct, finite defect indices, then its double commutant is interpolated by H∞, i.e. 
\{T\}'' = \{a(T) : a \in H∞\}. In this section we will remove the restriction that both defect indices be finite and require that only one of them be finite.

Throughout this section, let E and F be fixed (complex, separable) Hilbert spaces. Assume that F is finite dimensional and has an orthonormal basis \{f_i\}_{i=1}^m. Let \{e_j\}_{j \in J} be an orthonormal basis for E, where J is a finite or countably infinite index set with at least m+1 elements. Define E', J', F', m', e'_j, and f'_j analogously.

Before proceeding, we need to develop some notation for dealing with semi-finite matrices. Let A be an analytic operator-valued function in \(M(E, E')\). As in section I, it is possible to consider A as an operator from \(H^2_E\) to \(H^2_{E'}\). Define \(\langle Ae, e' \rangle\) to be the \(H^\infty\)-function \(\langle A(z)e, e' \rangle\), for every \(e \in E\) and \(e' \in E'\), where \(\langle \cdot , \cdot \rangle\) denotes the inner product in \(E'\). Using this notation, we can consider A as a (possibly infinite) matrix over \(H^\infty\) (see eg. [HA, pg. 323]). In fact, this gives rise to the matrix representation of A when considered as a linear transformation from \(H^2_E\) to \(H^2_{E'}\), since

\[\langle Af, e' \rangle = \sum_{j \in J} \langle Ae_j, e' \rangle \langle f, e_j \rangle\] for \(f \in H^2_E\) and \(e' \in E'\)
Dealing with minors will present somewhat of a difficulty, since the classical cofactor definition will not lead to an appropriate definition of a determinant. The method we will use follows that described by Flanders [8].

For any integer $n$, denote by $\Lambda^n E$ the space of all $n$-vectors on $E$. For $n=1$, this is simply $E$. For $n=2$, this space is defined as follows: an element of $\Lambda^2 E$ is a finite sum of terms of the form $a \wedge b$ where $a$ and $b$ are vectors in $E$. The symbol $\wedge$ is the (unique) bi-linear, anticommutative function from the cartesian product of $E$ with itself into the quotient space, $\Lambda^2 E$, such that $e_i \wedge e_j$ is zero if and only if $i = j.$ The set $\{e_i \wedge e_j\}, i, j \in J$ forms an orthonormal basis for $\Lambda^2 E$.

For $n > 2$, we proceed as follows: Let $\Delta_n$ be the set of all order preserving maps from $\{1, 2, 3, \ldots, n\}$ into $J$. (We will use symbols such as $a$ and $\tau$ to denote elements of $\Delta_n$.) For $\sigma \in \Delta_n$ denote by $e_\sigma$ the vector $e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge e_{\sigma(3)} \wedge \cdots \wedge e_{\sigma(n)}$. Using this notation the set $\{e_\sigma\}_{\sigma \in \Delta_n}$ is an orthonormal basis for $\Lambda^n E$. The inner product of $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ with $b_1 \wedge b_2 \wedge \cdots \wedge b_n$ is the determinant of

$$
\begin{pmatrix}
<a_1, b_1> & <a_2, b_1> & \cdots & <a_n, b_1> \\
<a_1, b_2> & <a_2, b_2> & \cdots & <a_n, b_2> \\
\vdots & & & \\
<a_1, b_n> & <a_2, b_n> & \cdots & <a_n, b_n>
\end{pmatrix}
$$

Note that the dimension of $\Lambda^n F$ is $\binom{m}{n}$, the number of combinations of $m$ things taken $n$ at a time. In particular, $\Lambda^m F$ has dimension 1.

For any operator $T$ from $E$ to $E'$ we can define $T^\wedge E$ from $\Lambda^n E$ to $\Lambda^n E'$ by $T^\wedge (e_\sigma(1) \wedge \cdots \wedge e_\sigma(n)) = Te_\sigma(1) \wedge \cdots \wedge Te_\sigma(n).$
And, for \( \Theta \in M(E, E') \) we can define \( \Theta^{n} \) pointwise, i.e.
\[
(\Theta^{n}(z))(e_{\sigma}) = (\Theta(z))^{n}(e_{\sigma}).
\]
Then, \( \langle \Theta^{n}e_{\sigma}, e'_{\tau} \rangle \) is the determinant of
\[
\begin{pmatrix}
\langle \Theta e_{\sigma(1)}, e'_{\tau(1)} \rangle & \langle \Theta e_{\sigma(2)}, e'_{\tau(1)} \rangle & \cdots & \langle \Theta e_{\sigma(n)}, e'_{\tau(1)} \rangle \\
\langle \Theta e_{\sigma(1)}, e'_{\tau(n)} \rangle & \langle \Theta e_{\sigma(2)}, e'_{\tau(n)} \rangle & \cdots & \langle \Theta e_{\sigma(n)}, e'_{\tau(n)} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \Theta e_{\sigma(1)}, e'_{\tau(n)} \rangle & \langle \Theta e_{\sigma(2)}, e'_{\tau(n)} \rangle & \cdots & \langle \Theta e_{\sigma(n)}, e'_{\tau(n)} \rangle
\end{pmatrix}
\]
and hence, is one of the \( n \)th order minors of \( \Theta \).

This notation for minors allows us to represent expansion by minors in a very concise form; namely: \( \langle \Theta^{n}e_{\sigma}, e'_{\tau} \rangle \) can be expanded along the \( i' \)th row as
\[
\sum_{j=1}^{n} (-1)^{i+j} \langle \Theta^{n-1}e_{\sigma_{j}}, e'_{\tau_{i}} \rangle \langle \Theta e_{j}, e'_{i} \rangle
\]
or along the \( j' \)th column as
\[
\sum_{i=1}^{n} (-1)^{i+j} \langle \Theta^{n-1}e_{\sigma_{j}}, e'_{\tau_{i}} \rangle \langle \Theta e_{j}, e'_{i} \rangle
\]
where \( \sigma_{j}(k) = \begin{cases} 
\sigma(k), & \text{for } k < j \\
\sigma(k+1), & \text{for } k > j 
\end{cases} \) and \( \tau_{i}(k) = \begin{cases} 
\tau(k), & \text{for } k < i \\
\tau(k+1), & \text{for } k > i 
\end{cases} \)

For a semi-finite matrix \( \Theta \in M(F, E) \) which is an isometry, it can be shown [see 26] that
\[
\sum_{\sigma \in \Delta_{n}} \langle \Theta f_{\tau}, e_{\sigma} \rangle^{2} = 1
\]
where \( \tau \) is the identity on \{1, 2, \ldots, m\}. 
An immediate consequence of this fact is that for every inner function in \( M(F, E) \), the g.c.d. of its \( m' \)th order minors exists and is non-zero. To see this, recall that by the hypothesis of this section the index set, \( J \), has more than \( m \) elements; so, \( m' \)th order minors must exist. Further, by the above result, the sum of their absolute-values squared is one; therefore, at least one of them must be non-zero.

Having described the notational devices which will be used, we are now ready to prove the first Lemma of this section.

**Lemma 1** Let \( \theta \) be an inner function in \( M(F, E) \). Let \( \omega \) be the g.c.d. of the \( m' \)th order minors of \( \theta \). If \( \gamma \) is an \( H^\infty \)-function which is relatively prime to \( \omega \), then \( \gamma(S(\theta)) \) is injective.

**Proof:** Suppose that \( f \in H(\theta) \) such that \( \gamma(S(\theta))f = 0 \). Then, \( f \in H^2_E \) and \( \gamma f \in \theta H^2_F \). Pick a \( \sigma \in \Delta_m \) and define \( E' \) to be the subspace spanned by \( \{e_\sigma(1), e_\sigma(2), \ldots, e_\sigma(m)\} \). Let \( P \) be the orthogonal projection of \( E \) onto \( E' \). Define \( \Theta_\sigma \) to be the \( m \) by \( m \) submatrix of \( \theta \) determined by \( \sigma \), namely:

\[ P_\sigma \theta. \]

Since \( \Theta_\sigma \) is finite dimensional, its classical adjoint, \( \Theta_\sigma^{\text{adj}} \) exists such that \( \Theta_\sigma \Theta_\sigma^{\text{adj}} = \xi_\sigma I_E \), and \( \Theta_\sigma^{\text{adj}} \Theta_\sigma = \xi_\sigma I_F \), where \( I_E \) and \( I_F \) are the identities on \( E' \) and \( F \), respectively and \( \xi_\sigma \) is the classical determinant of \( \Theta_\sigma \).

Since \( \gamma f \in \Theta H^2_F \), there must exist a \( g \in H^2_F \) such that \( \gamma f = \Theta g \). But, this implies that \( \Theta_\sigma (\gamma f) = \Theta_\sigma (\Theta g) \) and hence \( \gamma (\Theta_\sigma f) = \Theta_\sigma g \). Multiplying on the left by \( \Theta_\sigma^{\text{adj}} \), we have \( \gamma (\Theta_\sigma^{\text{adj}} \Theta_\sigma f) = \xi_\sigma g \).

Thus \( \gamma \) divides \( \xi_\sigma g_\sigma \) for all \( i \), where \( (g_\sigma)_i \) is the vector representation of \( g \). But, \( \sigma \) was arbitrary; so, \( \gamma \) must divide the g.c.d. over all \( \sigma \) of \( \xi_\sigma g_\sigma \) which is \( \omega g \). By hypothesis, \( \omega \) and \( \gamma \) are relatively prime. Therefore, \( \gamma | g_i \) for all \( i \), and hence, \( (g/\gamma) \in H^2_F \). So, we have shown
that \( f = \Theta(g/\gamma) \in \Theta_{PE}^2 \). Since \( f \) is also in \( \mathcal{H}(\Theta) \), we must have \( f = 0 \).

Therefore, \( \gamma(S(\Theta)) \) is injective. Q.E.D.

The next lemma is an extension of a result by Uchiyama [32, Lemma 3] with the removal of the restrictions that \( \Theta \) be a Jordan model and that \( E \) be finite dimensional. We can think of this lemma as stating that for certain \( C_{\infty} \) contractions, elements of their double commutant are "almost" interpolated by \( \mathcal{H}_0 \). Then, Theorem 5 will remove this "almost" restriction. We now state the precise description of this result.

**Lemma 2** Let \( T \) and \( T' \) be \( C_{\infty} \) contractions with characteristic operator functions \( \Theta \in M(F, E) \) and \( \Theta' \in M(F', E') \), respectively. Let \( \omega \) be any inner function and let \( A^0 \) be in the double commutant of \( T' \).

If \( \Theta \) and \( \Theta' \) are quasi-equivalent and if for some \( e \in E \) the image of \( \Theta(z) \) is perpendicular to \( e \) for all \( z \), then there exist relatively prime \( \mathcal{H}_0 \)-functions \( \alpha \) and \( \beta \) which are both relatively prime to \( \omega \) such that \( \alpha(T') = \beta(T')A^0 \).

**Proof:** Suppose that \( T, T', \Theta, \Theta', A^0 \) and \( \omega \) are as stated above. Let \( \mathcal{H} \equiv \mathcal{H}(\Theta) \) and \( \mathcal{H}' \equiv \mathcal{H}(\Theta) \). Let \( P \) be the orthogonal projection of \( \mathcal{H}_E^2 \) onto \( \mathcal{H} \) and let \( P' \) be the orthogonal projection of \( \mathcal{H}_E'^2 \) onto \( \mathcal{H}' \). We will assume that \( T = S(\Theta) \) and \( T' = S(\Theta') \), since they are always at least unitarily equivalent. By the Lifting Theorem [HA, pg. 258], there must exist an \( A \in M(F', E') \) such that \( A^0 = P'A \mid \mathcal{H}' \), since in particular \( A^0 \in \{ T' \}' \).

Let \( \omega' \) be the g.c.d. of all \( m \)'th order minors of \( \omega \). As before, \( \omega' \) is non-zero, since at least one of the minors is non-zero. Let \( \omega'' = \omega' \omega \).

Without loss of generality we can assume that the image of \( \Theta(z) \) is perpendicular to \( e_1 \), for all \( z \), since we know that it is perpendicular to
some $e_i e_j$ and a new basis could be chosen such that this element is $e_1$.

Fix a subscript, $r$, and define $R \in M(E,E)$ by $R e_i = \delta_{i,1} e_r$, where $\delta$ is the Kronecker delta (i.e. $\delta_{i,j} = 1$ if $i=j$ and $0$ otherwise). Then, $R \Theta = 0$,
since $e_i$ is in the kernel of $R$ for all $i > 1$. In particular, $R \Theta H^2_F \subset H^2_E$.

By the Lifting Theorem, $R \Theta$ is in the commutant of $T$, where $R \Theta = PR | H$.

By hypothesis, $\Theta$ and $\Theta'$ are quasi-equivalent. Sz.-Nagy has shown [23] that this implies the existence of $X \in M(E',E)$, $Y \in M(E,E')$, and $Z \in M(E,E')$
with scalar multiples $\gamma_X$, $\gamma_Y$, and $\gamma_Z$, respectively, such that:

(a) $\gamma_X$, $\gamma_Y$, and $\gamma_Z$ are each relatively prime to $\omega''$.
(b) $X^0$ is injective,
(c) $Y^0 H$ and $Z^0 H$ span $H'$,
(d) $X^0 T' = T X^0$,
(e) $Y^0 T = T' Y^0$, and
(f) $Z^0 T = T' Z^0$,

where $X^0 \equiv P X | H'$, $Y^0 \equiv P' Y | H$, and $Z^0 \equiv P' Z | H$.

Having completed the constructions, we can now show that $P(XY)R(XAY) = P(XAY)R(XY)$. By (d), we have $(Y^0 R X^0) T' = Y^0 R (T X^0)$.

But, this equals $Y^0 (T R^0) X$, since $R^0 \in \{T\}'$. And, by (e), that equals $T' (Y^0 R X^0)$. So, we have shown that $Y^0 R X^0 \in \{T\}'$. Since $A^0$ commutes with everything in $\{T\}'$, we have in particular, $Y^0 R X^0 A^0 = A^0 Y^0 R X^0$.

Multiplying on the left by $X^0$ and on the right by $Y^0$ yields:

$X^0 Y^0 R X^0 A^0 Y^0 = X^0 A^0 Y^0 R X^0 Y^0$. By direct replacement we get:

$(P X)(P' Y)(P' A)(P' Y) = (P X)(P' A)(P' Y)(P R)(P X)(P' Y)$. Most of these projections are unnecessary. For example, by (e), $Y^0 T = T' Y^0$; so, by the Lifting Theorem, $Y \Theta H^2_F \subset \Theta' H^2_F$. Hence, $P' Y P = P' Y$; therefore, $(P' Y)(P R) = P' (Y R)$. Similarly, we can eliminate the other projections, leaving: $P(XY)R(XAY) = P(XAY)R(XY)$. 

Consider the left hand side of this equation. If we let it act
on some fixed basis element \( e_s \in E \) and expand it we get:

\[
P(XY)R(XAY)e_s = P \sum_i <XRYXAYe_s, e_i> e_i
\]

\[
= P \sum_{i,j,k} <XYe_s, e_i> <Re_k, e_j> <XAYe_s, e_k> e_i
\]

\[
= P \sum_{i,j,k} <XYe_s, e_i> \delta_{k,1} \delta_{j,r} <XAYe_s, e_k> e_i
\]

\[
= P \sum_i <XYe_s, e_i> <XAYe_s, e_i> e_i
\]

\[
= P \sum_i <X(XAYe_s, e_i I')Ye_r, e_i> e_i
\]

\[
= P X(XAYe_s, e_i I')Ye_r
\]

where \( I' \) is the identity on \( H_E' \).

Similarly, expanding the right side we get:

\[
P XAYRXYe_s = P \sum_{i,j,k} <XAYe_s, e_i> <Re_k, e_j> <XYe_s, e_k> e_i
\]

\[
= P \sum_{i,j,k} <XAYe_s, e_i> \delta_{k,1} \delta_{j,r} <XYe_s, e_k> e_i
\]

\[
= P \sum_i <XAYe_s, e_i> <XYe_s, e_i> e_i
\]

\[
= P X(XAYe_s, e_i I')AYe_r
\]

Combining these we have \( P X(XAYe_s, e_i I')Ye_r = P X(XAYe_s, e_i I')AYe_r \).

But, since \( r \) was arbitrary, we have shown that

\[
(P X)(XAYe_s, e_i I')Y = (P X)(XAYe_s, e_i I')AY.
\]

Now, using the same method which we used earlier to remove pro-
jections, we can introduce some here, namely:

\[
(PX) P'(XAYe_s, e_i I') (P'Y) = (PX) P' (XAYe_s, e_i I') (P'A) (P'Y)
\]

which by our earlier notation is:
\[ X^0 <XAYe_s, e_1>(T') Y^0 = X^0 <XYe_s, e_1>(T') A^0 Y^0 \]

But, by (b), \( X^0 \) is injective, so we can cancel it, giving:

\[ <XAYe_s, e_1>(T') Y^0 = <XYe_s, e_1>(T') A^0 Y^0 \]

We know that \( X \) and \( Y \) have scalar multiples \( \gamma_X \) and \( \gamma_Y \); so, there exists \( X^{\text{adj}} e M(E,E') \) and \( Y^{\text{adj}} e M(E',E) \) such that \( XX^{\text{adj}} = \gamma_X I \) and \( YY^{\text{adj}} = \gamma_Y I' \), where \( I \) is the identity on \( H_2^E \). Multiplying our equation on the left by \( <Y^{\text{adj}} X^{\text{adj}} e_1, e_s>(T') \) we get:

\[ <XAYe_s, e_1> <Y^{\text{adj}} X^{\text{adj}} e_1, e_s>(T') Y^0 = <XYe_s, e_1> <Y^{\text{adj}} X^{\text{adj}} e_1, e_s>(T') A^0 Y^0 \]

Summing this over \( s \) yields:

\[ <XAY^{\text{adj}} X^{\text{adj}} e_1, e_1>(T') Y^0 = <XY^{\text{adj}} X^{\text{adj}} e_1, e_1>(T') A^0 Y^0 \]

If \( \alpha' = <XAX^{\text{adj}} e_1, e_1> \) and \( \beta' = \gamma_X \), then this reduces to

\[ (\gamma_Y \alpha')(T') Y^0 = (\gamma_Y \beta')(T') A^0 Y^0, \]

or equivalently,

\[ \gamma_Y(T') \alpha'(T') Y^0 = \gamma_Y(T') \beta'(T') A^0 Y^0. \]

By Lemma 1, \( \gamma_Y(T') \) is injective, since \( \gamma_Y \) is relatively prime to \( \omega' \) by (a). So, we can cancel the term \( \gamma_Y(T') \) in the previous equation, giving: \( \alpha'(T') Y^0 = \beta'(T') A^0 Y^0 \). But, we could have used \( Z \) instead of \( Y \) throughout all these arguments, since neither \( \alpha' \) nor \( \beta' \) depend on \( Y \).

In this case we would have shown that \( \alpha'(T') Z^0 = \beta'(T') A^0 Z^0 \). By (c), the images of \( Y^0 \) and \( Z^0 \) span \( H' \); therefore, \( \alpha'(T') = \beta'(T') A^0 \).

If we define \( \gamma \) to be the g.c.d. of \( \alpha' \) with \( \beta' \), then there exists relatively prime \( H_0 \)-functions \( \alpha \) and \( \beta \) such that \( \alpha' = \alpha \gamma \) and \( \beta' = \beta \gamma \).

Applying these to our equations yields: \( \gamma(T') \alpha(T') = \gamma(T') \beta(T') A^0 \).

Since \( \gamma \) is relatively prime to \( \omega' \), we can apply Lemma 1 again; hence \( \gamma(T') \) is injective. Therefore, we can cancel it giving the desired result:

\[ \alpha(T') = \beta(T') A^0. \]

Q.E.D.
For $\sigma \in \Delta_m$, denote by $\xi(\sigma)$ the $n$'th order minor of $\Theta$ whose rows are selected by $\sigma$, namely: $\langle \hat{\Theta}_{\eta}^{n}, e_{\sigma} \rangle$, where $\eta$ is the identity on $\{1, 2, \ldots, m\}$. For $\sigma \in \Delta_m$, define $\ell_{\sigma}$ as follows: if $J$ is finite then let $\ell_{\sigma}$ be the smallest element of $J$ which is not in the image of $\sigma$ (This always exists since, by the hypothesis of the section, $J$ has more than $m$ elements.); otherwise, let $\ell_{\sigma}$ be the sum $\sum_{j=1}^{m} \sigma(j)$.

The next lemma is used to transform $\xi(\sigma)$, which is a minor not involving $\ell_{\sigma}$, into a sum of minors which do involve $\ell_{\sigma}$.

**Lemma 3** If $\Theta \in M(F,E)$ and $\sigma \in \Delta_m$, then for any $f \in F$,

$$
\xi(\sigma) \langle \Theta f, e_{\sigma} \rangle = \sum_{i=1}^{m} (-1)^{i+m} \xi(\tau_i) \langle \Theta f, e_{\sigma(i)} \rangle
$$

where $\tau_i(k) = \sigma(k+1)$, for $i \leq k < m$

and $\ell_{\sigma}$, for $k = m$

Note: Technically, $\tau_i$ is not a $\Delta_m$ function, since it need not be order preserving. In this case, however, it is not a problem. If $E$ is infinite dimensional, then $\tau_i$ is order preserving. If $E$ is finite dimensional, then we would need only introduce a factor of $\pm 1$. Further, for $E$ finite dimensional, a simpler proof exists [32].

**Proof:** Let $\eta$ be the identity on $\{1, 2, \ldots, m\}$ and define $\eta_j$ and $\sigma_i$ by

$$
\eta_j(k) = \begin{cases} 
k, \text{ for } k < j \\
k+1, \text{ for } k \geq j
\end{cases}
$$

and

$$
\sigma_i(k) = \begin{cases} 
s(k), \text{ for } k < i \\
s(k+1), \text{ for } i \geq k
\end{cases}
$$

where $j, i$ and $k$ are as above.
Consider the equation given in the statement of this lemma. Its left hand side can be expanded using the Fourier series expansion of $f$, $f = \sum x_j f_j$, giving:

$$\xi(\sigma) \langle \Theta f, e_\sigma \rangle = \sum_j x_j \xi(\sigma) \langle \Theta f_j, e_\sigma \rangle$$

Observe that expansion by minors on the $j$'th row can be written as:

$$\delta_j k \xi(\sigma) = \sum_i (-1)^{i+j} \langle \Theta^{m-1} f, e_\sigma \rangle < \Theta f_k, e_\sigma(i) >$$

where $\delta$ is the Kronecker delta, since for $j \neq k$, the term on the right is the determinant of a matrix having two rows which are equal. We can introduce $\delta$ into our equation by summing over $k$. Thus we have:

$$\xi(\sigma) \langle \Theta f, e_\sigma \rangle = \sum_j \sum_k x_k \delta_j, k \xi(\sigma) \langle \Theta f_j, e_\sigma \rangle$$

Since all of these sums are finite, we can rearrange the terms, giving:

$$\sum_j \sum_k x_k (-1)^{i+m} \langle \Theta f_k, e_\sigma(i) \rangle \sum_j (-1)^{j+m} \langle \Theta^{m-1} f, e_\sigma \rangle \langle \Theta f_j, e_\sigma \rangle$$

Compressing along the $j$'th row, yields:

$$\sum_i \sum_k x_k (-1)^{i+m} \langle \Theta f_k, e_\sigma(i) \rangle < \Theta^m e_\sigma, e_\tau_i >$$

Finally, reversing the Fourier series expansion back to $f$, gives the right hand side of the equation in the statement of the lemma, namely:

$$\sum_i (-1)^{i+m} \xi(\tau_i) \langle \Theta f, e_\sigma(i) \rangle$$

Q.E.D.

The last lemma is later used to show that the $\beta$ produced in Lemma 1 is actually the constant function one.
LEMMA 4 Suppose that $\Theta \in \mathcal{M}(F, E)$, $Z \in \mathcal{M}(E, F)$, $A \in \mathcal{M}(E, E)$, and $\omega$ is the g.c.d. of the $m$th order minors of $\Theta$. If $\alpha$ and $\beta$ are relatively prime $H^m$ functions both of which are relatively prime to $\omega$ and if $\alpha I - \beta A = \Theta Z$, where $I$ is the identity on $H^2_E$, then $\beta = 1$.

Proof): Assume that $\Theta$, $Z$, $A$, $I$, $\alpha$, $\beta$, and $\omega$ are as stated above. Pick $\sigma \in A_m$ and define $\tau_i$ as in Lemma 3. Abbreviate $\xi$ as simply $\xi$. If we multiply the equation $\alpha I - \beta A = \Theta Z$ by $\xi(\sigma)$ and apply it to $e_\lambda$, we have:

$$\xi(\sigma) \langle (\alpha I - \beta A)e_\lambda, e_\lambda \rangle = \xi(\sigma) \langle \Theta Z e_\lambda, e_\lambda \rangle$$

Using the linearity of inner products on the left side gives us

$$\langle \xi(\sigma) \alpha e_\lambda, e_\lambda \rangle - \langle \xi(\sigma) \beta A e_\lambda, e_\lambda \rangle,$$

which is equal to $\xi(\sigma) \alpha - \xi(\sigma) \beta \langle A e_\lambda, e_\lambda \rangle$. If we take the right hand side, we can use Lemma 3 to give us:

$$\sum_{i=1}^{m} (-1)^{i+m} \xi(\tau_i) \langle \Theta Z e_\lambda, e_{\sigma(i)} \rangle$$

But, $\langle \Theta Z e_\lambda, e_{\sigma(i)} \rangle = \langle (\alpha I - \beta A)e_\lambda, e_{\sigma(i)} \rangle$ which is equal to

$$\langle \alpha e_\lambda, e_{\sigma(i)} \rangle - \beta \langle A e_\lambda, e_{\sigma(i)} \rangle.$$ Since $\lambda \neq \sigma(i)$ for any $i$, we have

$$\langle \alpha e_\lambda, e_{\sigma(i)} \rangle = 0.$$ Therefore, $\langle \Theta Z e_\lambda, e_{\sigma(i)} \rangle = -\beta \langle A e_\lambda, e_{\sigma(i)} \rangle$.

Recombining the left and right halves, we have shown that

$$\xi(\sigma) \alpha - \xi(\sigma) \beta \langle A e_\lambda, e_\lambda \rangle = -\beta \sum_{i=1}^{m} (-1)^{i+m} \xi(\tau_i) \langle A e_\lambda, e_{\sigma(i)} \rangle$$

This can be rewritten as $\alpha \xi(\sigma) = \beta \zeta(\sigma)$ where

$$\zeta(\sigma) \equiv \xi(\sigma) \langle A e_\lambda, e_\lambda \rangle - \sum_{i=1}^{m} (-1)^{i+m} \xi(\tau_i) \langle A e_\lambda, e_{\sigma(i)} \rangle$$

Written this way, it is obvious that $\beta$ divides $\alpha \xi(\sigma)$. But, by hypothesis, $\beta$ is relatively prime to $\alpha$. Thus the inner factor of $\beta$, call
it $\beta'$, must divide $\xi(\sigma)$ for all $\sigma \in \Delta_m$. Hence $\beta'$ divides their g.c.d.
which is $\omega$. But, $\beta$ is also relatively prime to $\omega$. Therefore, $\beta'=1$,
i.e. $\beta$ is outer.

Since $\alpha \xi(\sigma) = \beta \xi(\sigma)$ for all $\sigma \in \Delta_m$, we can take their absolute
squared sum, giving:

$$ |\alpha|^2 \sum_{\sigma} |\xi(\sigma)|^2 = |\beta|^2 \sum_{\sigma} |\xi(\sigma)|^2 $$

As was stated at the beginning of this section, $\sum |\xi(\sigma)|^2 = 1$. If we
could show that $\sum |\xi(\sigma)|^2$ was bounded, by say $M$, then we would have
shown that $|\alpha|^2 \geq M |\beta|^2$, and thus that the $H^2$ function $\alpha/M$ is bounded
in norm by the outer function $\beta$. By a well known property of $H^2$
[see eg. 7, pg. 158], $\beta$ would divide $\alpha$. But, $\alpha$ and $\beta$ are relatively
prime; therefore, we would have shown that $\beta=1$.

So, all that remains is to show that $\sum_{\sigma} |\xi(\sigma)|^2$ is bounded. By
the Minkowski inequality, we have:

$$ \left( \sum_{\sigma} |\xi(\sigma) < A_{\xi_{\sigma}}, e_{\xi_{\sigma}} > - \sum_{i=1}^m (-1)^{i+m} \xi(\tau_i) < A_{\xi_{\mathbf{1}}}, e_{\xi(\mathbf{1})} > \right)^{1/2} \leq \left( \sum_{\sigma} |\xi(\sigma)|^2 |< A_{\xi_{\sigma}}, e_{\xi_{\sigma}} > |^2 \right)^{1/2} + \left( \sum_{i=1}^m |\xi(\tau_i)|^2 |< A_{\xi_{\mathbf{1}}}, e_{\xi(\mathbf{1})} > |^2 \right)^{1/2} $$

Since $|< A_{\mathbf{1}}, e_{\mathbf{1}} | \leq ||A||$ for all $i$ and $j$ and since $\sum |\xi(\sigma)|^2 = 1$, the above
is bounded by

$$ ||A|| \left( 1 + \sum_{i=1}^m \sum_{\sigma \in \Delta_m} |\xi(\tau_i)|^2 \right) $$

Therefore, all that remains is to show that $\sum_{\sigma} |\xi(\tau_i)|^2$ is bounded. For
finite $J$ this is easy since there is only a finite number of minors. So,
suppose that $J$ is infinite and that $\sigma, \sigma' \in \Delta_m$ are distinct. Let $\tau_{\mathbf{1}}$ and $\ell$
correspond to $\sigma$ as before and let $\tau_{\mathbf{1}}'$ and $\ell'$ correspond to $\sigma'$; and, suppose
that $\tau_1 = \tau_1'$. Then $a(k) = a'(k)$, for $k < i$, since

$$a(k) = \tau_1(k) = \tau_1'(k) = a'(k).$$

Also $a(k) = a'(k)$, for $k > i$, since

$$a(k) = \tau_1(k-1) = \tau_1'(k-1) = a(k).$$

Therefore, $a(k) = a'(k)$ if and only if $k=i$. But, $\ell = \tau_1'(m) = \tau_1'(m) = \ell'$; hence $\sum a(k) = \sum a'(k)$, which is a contradiction. Therefore, distinct $\sigma$'s give rise to distinct $\tau_1$'s.

Thus, $\sum |\xi(\tau_1)|^2$ is a subsum of the sum of the squares of all minors, which is one. In conclusion, this shows that $\sum |\xi(\sigma)|^2$ is bounded by $\|A\| (1+m)$; and hence, $\beta=1$.

Q.E.D.

Having completed the lemmas, we are now ready to prove the main result of this section, namely:

**THEOREM 5** If $T$ is a $C_0^*$ contraction whose defect indices are distinct and at least one of them is finite, then the double commutant of $T$ is interpolated by $H^\infty$.

**Proof**: Let $\Theta \in \mathcal{M}(F,E)$ be the characteristic operator function of $T$. Since $T$ is $C_0^*$, $\Theta(z)$ is an isometry for almost every $z$ [see HA, pg. 190]. Therefore, the dimension of $E$ is greater than or equal to that of $F$. But, by hypothesis their dimensions are not equal and one of them is finite. So, we have shown that $E$ and $F$ satisfy the hypothesis of this section, namely: $\dim F = m$ and $\dim E > m$.

Sz.-Nagy has extended the Jordan model to this case [23]. He proved that every matrix in $\mathcal{M}(F,E)$ is quasi-equivalent to some matrix, $J$, of the form:
where \( \{h_i\} \) are the invariant factors of \( \Theta \). So, their product, call it \( \omega \), is also the g.c.d. of the \( m \)'th order minors of \( \Theta \).

Let \( A^0 \) be in the double commutant of \( S(\Theta) \). As in Lemma 2, we have by the Lifting Theorem, \( A^0 = PA \mid H(\Theta) \), where \( P \) is the orthogonal projection of \( H_E^2 \) onto \( H(\Theta) \) and \( AcM(E,E) \). We have now satisfied the hypotheses of Lemma 2; so, there exist relatively prime \( H^\infty \)-functions \( \alpha \) and \( \beta \) which are relatively prime to \( \omega \) such that \( \alpha(S(\Theta)) = \beta(S(\Theta))A^0 \). In matrix form this means that \( P(\alpha I) = P(\beta I)(PA) \). As before, we can remove the last \( P \), giving: \( P(\alpha I - \beta A) = 0 \), or equivalently, \( (\alpha I - \beta A)H_F^2 \subset H_E^2 \). Since \( \alpha I - \beta A \) commutes with the unilateral shift, we have [see eg. HA, pg. 195] that there exists a \( ZeM(E,F) \) such that \( \alpha I - \beta A = \Theta Z \). By Lemma 4, we have \( \beta = 1 \); therefore, \( \alpha(S(\Theta)) = A \). Finally, since \( S(\Theta) \) is unitarily equivalent to \( T \), we have that \( \{T\}'' = \{\alpha(T) : \alpha \in H^\infty\} \). Q.E.D.

In Theorem 5, we assumed that the defect indices were distinct. The following example (due to Sz.-Nagy [22]) shows that this restriction cannot be removed. Let \( \alpha \) and \( \beta \) be relatively prime inner functions for which \( ax + \beta y \) is outer for all \( x \) and \( y \) in \( H^\infty \). Define \( T = S(\alpha) \Theta S(\beta) \) and consider \( \phi(T) \) where \( \phi = (\alpha+\beta) / (\alpha-\beta) \). Then \( T \) is a \( C_0 \) contraction (in fact, a \( C_0 \) contraction) and \( \phi \) is in its double commutant [HA, pg. 155]. If a generalization of Theorem 5 were to hold then we would have to have
\[ \phi(T) = \omega(T) \text{ for some } \omega \in \mathcal{H}^0. \text{ But, Sz.-Nagy has shown [22] that this would imply that } \alpha x + \beta y = 1 \text{ for some } x \text{ and } y \in \mathcal{H}^0, \text{ which is a contradiction.} \]

Therefore, there exists a \( C_0 \) contraction with at least one finite defect index whose double commutant is not interpolated by \( \mathcal{H}^0 \).
SECTION V

CONTRACTIONS CONTAINING THE SHIFT

In this section we will consider contractions of the form \( T \otimes S_+ \), where \( S_+ \) is the unilateral shift. The unilateral shift, \( S_+ \), will be considered as \( M_2 \) on \( \mathbb{H}^2 \), where \( M_2(f) \equiv z \cdot f \). Explicit representations will be given for both the double commutant and the weakly-closed algebra generated by \( T \otimes S \), in the two special cases where \( T \) is a \( C_c \) contraction and where \( T \) is an isometry.

For operators \( T \) on \( \mathbb{H} \) and \( T' \) on \( \mathbb{H}' \), denote the set of operators which intertwine \( T \) with \( T' \), i.e. \( \{ X \mid XT = T'X \} \), by \( \Phi(T, T') \). For any operator, \( T \), define \( \mathcal{A}(T) \) to be the weakly-closed algebra (with identity) generated by \( T \). If for some operator, \( T \), the set \( \{T\}' \) (or \( \mathcal{A}(T) \)) is of the form \( \{a(T) \mid a \in H^\infty \} \), then we will say that \( \{T\}' \) (respectively, \( \mathcal{A}(T) \)) is interpolated by \( H^\infty \).

An operator \( T \) is said to be of class \( (dc) \) if \( \mathcal{A}(T) = \{T\}' \). Many well-known operators have been shown to be in this class, including: the unilateral shift [4], all algebraic operators [30], every weighted unilateral shift [19], and every non-unitary isometry [31]. In this section we will show that \( T \otimes S_+ \) is in this class for every \( C_c \) contraction \( T \).

The first Lemma of this section is probably not new. It is included here because it is expressed in the notation which will be used throughout this section and greatly simplifies the proofs which follow it.
**Lemma 1** Let \( J \) be any index set (finite or countably infinite). For each \( j \in J \), suppose that \( T_j \) is an operator on \( H_j \). If \( h = \bigoplus H_j \) and if \( T = \bigoplus T_j \), then \( \{T\}'' \) is the set of all operators of the form \( \bigoplus A_j \) such that:

1. \( A_j \) is an operator on \( H_j \), for all \( j \in J \)
2. \( \phi(T_i, T_j) \subseteq \phi(A_i, A_j) \), for all \( i, j \in J \).

**Proof:** Let \( A \in \{T\}'' \). To show that \( A \) is of the form \( \bigoplus A_j \), we need only show that for each \( k \in J \), \( H_k \) reduces \( A \) (when considered as a subspace of \( H \)). Let \( P_k \) be the orthogonal projection of \( H \) onto \( H_k \). Then, \( P_k T = T P_k \), since \( H_k \) reduces \( T \). But, this implies that \( P_k A = A P_k \), since \( P_k \in \{T\}' \).

Hence, \( H_k \) reduces \( A \). Therefore, \( A = \bigoplus A_j \), where \( A_j \equiv P_j A \mid H_j \).

Fix \( i \) and \( j \) in \( J \) and suppose that \( X \in \phi(T_i, T_j) \). In order to show that \( X \) satisfies property (ii), we need only show that \( X A_j X = A_j X \).

Define an operator \( B \) from \( H_j \) to \( H_j \) by \( B(u) = uX \). Extend \( B \) to \( H \) by \( B(u) = 0 \), for all \( u \) perpendicular to \( H_j \). Then, \( B T = B(\bigoplus T_k = XT_i \). Similarly, \( T B = (\bigoplus T_k) B = T_j X \). Therefore, \( B \in \{T\}' \subseteq \{A\}' \). Hence, \( B A = A B \).

But, \( B A = B(\bigoplus A_k) = X A_j \) and \( A B = (\bigoplus A_k) B = A_j X \). Therefore, \( X \in \phi(A_i, A_j) \).

Conversely, suppose \( A = \bigoplus A_j \) satisfies properties (i) and (ii).

Let \( B \in \{T\}' \) and fix \( i \) and \( j \) in \( J \). If \( X = P_j B \mid H_i \), then:

\[
XT_i = (P_j B \mid H_i) T_i = (P_j BT) \mid H_i = (P_j TB) \mid H_i = T_j (P_j B \mid H_i) = T_j X
\]

Therefore, \( X \in \phi(T_i, T_j) \subseteq \phi(A_i, A_j) \). But, this implies that:

\[
P_j(BA) \mid H_i = (P_j B \mid H_i)A_j = X A_j X = A_j X = A_j (P_j B \mid H_i) = P_j (AB) \mid H_i
\]

Finally, since this holds for all \( i \) and \( j \), we have shown that \( B A = A B \), i.e., \( A \in \{T\}'' \).

Q.E.D.
COROLLARY 2 If $T$ is an operator on $H$ and $T'$ is an operator on $H'$, then

$\{T \oplus T'\}''$ is the set of all operators of the form $A \oplus A'$ such that:

(i) $A \in \{T\}''$

(ii) $A' \in \{T'\}''$

(iii) $\phi(T, T') \subseteq \phi(A, A')$

(iv) $\phi(T', T) \subseteq \phi(A', A)$

The following result gives an explicit representation for the weakly-closed algebra generated by the types of operators considered in this section. (The author wishes to thank Prof. D. W. Hadwin for the proof outline.)

LEMMA 3 Suppose $V = T \oplus U \oplus S^+$, where $T$ is a CNU contraction and $U$ is a unitary whose scalar spectral measure is absolutely continuous with respect to Lebesgue measure. Then, $\alpha(V)$ is interpolated by $H^\omega$.

Proof): Define $\pi_1$ from $H^\omega$ to $\alpha(T)$ by $\pi_1(u) = u(T)$. Similarly, define $\pi_3$ from $H^\omega$ to $\alpha(S^+)$ by $\pi_3(u) = u(S^+)$. These are both continuous homomorphisms because $T$ and $S^+$ are CNU contractions [HA, pg. 114]. Define $\pi_2$ from $H^\omega$ to $\alpha(U)$ by $\pi_2(u) = u(U)$. This is a continuous homomorphism by the spectral theorem (see eg. [7, pg. 113]).

Suppose that $X \in \alpha(V)$, then $X$ must be of the form $A \oplus B \oplus C$ where $A \in \alpha(T)$, $B \in \alpha(U)$, and $C \in \alpha(S^+)$, since any reducing space of $V$ must reduce any limit of polynomials in $V$. But, $\alpha(S^+)$ is interpolated by $H^\omega$[18]; hence, there exists an $H^\omega$-function, $\gamma$, such that
\[ C = \gamma(S_+) \]. Therefore, \((A - \gamma(T)) \oplus (B - \gamma(U)) \oplus 0\) must also be in \(\alpha(V)\).

So, it must also be a weak limit of some net, \((p_i)\), of polynomials. If we let "\(\text{wlim}\)" stand for the weak limit of the net, we have:

\[
\begin{align*}
\text{wlim } p_i(T) &= A - \gamma(T) \\
\text{wlim } p_i(U) &= B - \gamma(U) \\
\text{and wlim } p_i(S+) &= 0
\end{align*}
\]

Expressed in terms of \(\pi_3\), the last equation says: \(\text{wlim } \pi_3(p_i) = 0\). But, this implies \(p_i\) converges weak* to zero, since \(\pi_3\) is continuous. Conversely, \(p_i\) converging weak* to zero implies that \(\pi_1(p_i)\) converges weakly to zero, since \(\pi_1\) is also continuous. It also implies that \(\pi_2(p_i)\) converges weakly to zero, by the functional calculus for normal operators [7, pg. 113]. Therefore, \(A = \gamma(T)\) and \(B = \gamma(U)\). So, we have shown that \(X = \gamma(T) \oplus \gamma(U) \oplus \gamma(S+) = \gamma(V)\). Q.E.D.

In Section IV, we proved that for certain \(C\) contractions their double commutant is interpolated by \(H^\infty\). The technique used depended heavily on the existence of a contractive analytic function, \(\Theta\), for which the union of ranges, \(\Theta(z)H^2_F\), was perpendicular to some fixed vector. Another case where this occurs is in summands of the unilateral shift. This observation prompted the following theorem:

**THEOREM 4** If \(T\) is any \(C\) contraction, then \([T \oplus S+]''\) is interpolated by \(H^\infty\).

Proof: Without loss of generality we can assume that \(T = S(\Theta)\) for some contractive analytic function \(\Theta \in \mathcal{H}(F, E)\) where \(E\) and \(F\) are Hilbert spaces
(since every \( C \) contraction is unitarily equivalent to some such operator). By Corollary 2, a typical element of \( \{ T \circ S_+ \}^\infty \) is of the form \( A \circ A' \) where \( A \in \{ T \}' \), \( A' \in \{ S_+ \}' \), and \( \Phi(S_+, T) \subset \Phi(A, A') \). But, \( \{ S_+ \}' \) is known to be interpolated by \( H^\infty [18] \); hence, \( A' = \alpha(S_+) \) for some \( \alpha \in H^\infty \).

All that remains is to show that \( \alpha(T) = A \).

By the Lifting Theorem [HA, pg. 258], there exists a \( \Delta \in M(F, E) \) such that \( A = PA \mid H \) and \( \Delta H^2_F \subset \Omega H^2_F \) where \( P \) is the orthogonal projection of \( H^2_E \) onto \( H(\Theta) \) and \( H = H(\Theta) \). Let \( \{ e_1, e_2, \ldots \} \) be a (possibly finite) orthonormal basis for \( E \). Fix a subscript, \( k \), and define an operator, \( B \), from \( H^2 \) to \( H^2_E \) by \( B(u) = u \cdot e_k \). Then, for any \( u \in H^2 \) we have

\[
(PBS_+)u = pB(Mu) = PM'_z(u \cdot e_k),
\]

where \( M_z \) is multiplication by \( z \) on \( H^2 \) and \( M'_z \) is multiplication by \( z \) on \( H^2_E \). Similarly, \( (TPB)u = TP(u \cdot e_k) = PM'_z(p \cdot e_k) = PM'_z(u \cdot e_k) \), since \( H \) is invariant under \( M'_z \). Therefore, \( (PB)S_+ = T(PB) \).

Thus, \( PB \in \Phi(S_+, T) \subset \Phi(\alpha(S_+), A) \). In other words, \( PB(\alpha(S_+)) = APB \).

In particular, \( PB(\alpha(S_+))(1) = APB(1) \), where \( 1 \) is the constant function one. But, \( PB(\alpha(S_+))(1) = PB(\alpha(l))(1) = P(\alpha l)B(l) = P(\alpha l)e_k \). On the other hand, \( APB(1) = (PA)PB(1) = PAB(l) = PAe_k \), since \( \Delta \theta H^2_F \subset \Omega H^2_F \).

Therefore, \( P(\alpha l)e_k = PAe_k \), for all \( k \).

So, we have \( P(\alpha l) = PA \). But, this is equivalent to \( \alpha(T) = A \).

Q.E.D.

The main result of this section now becomes a simple corollary to Lemma 3 and Theorem 4, namely:

**Corollary 5** If \( T \) is any \( C \) contraction, then \( T \circ S_+ \) is in class (dc).

Can Theorem 4 be extended to general contractions? The next result answers this question in the negative. It turns out that for
more general operators their internal structure must be considered.

Let T be an arbitrary contraction. As was mentioned in Section I, the canonical decomposition [HA, pg. 9] allows to write T as the direct sum of its unitary part, $T_u$, and its completely non-unitary (CNU) part, $T_c$. The unitary part can be further decomposed into its singular part, $T_s$, and its absolutely continuous part, $T_a$, where this means that the spectral measure of $T_s$ (or $T_a$) is singular (respectively, absolutely continuous) with respect to Lebesgue measure.

In Theorem 4 we considered operators of the form $T \otimes S_+$ where T is a C∞ contraction. The next result investigates the opposite extreme, unitary operators. In fact, it is somewhat more general considering isometries instead of simply unitaries.

**Theorem 6** If T is any isometry and $S_+$ is the unilateral shift, then $\{T \otimes S_+\}''$ is equal to $\mathcal{A}(T \otimes S_+)$ and both are equal to
\[
\{\alpha(T_s) \otimes \beta(T_a \otimes T_c \otimes S_+) \mid \alpha \in L^\infty(\mu) \text{ and } \beta \in H^\infty\}
\]
where $T = T_s \otimes T_a \otimes T_c$ as above and $\mu$ is the scalar spectral measure of $T_s$.

**Proof:** Turner has shown [31] that all isometries are in class (dc). Since $T \otimes S_+$ is itself an isometry, we have $\{T \otimes S_+\}'' = \mathcal{A}(T \otimes S_+)$. Since $T_a \otimes T_c \otimes S_+$ is the compression of a unitary whose scalar spectral measure is absolutely continuous with respect to Lebesgue measure, and hence is singular with respect to $\mu$, $\mathcal{A}(T \otimes S_+)$ must split [5], i.e.,
\[
(T \otimes S_+) = \mathcal{A}(T_s) \otimes \mathcal{A}(T_a \otimes T_c \otimes S_+)
\]
By the spectral theorem (see eg. [7, pg. 93]), the former term of this sum is interpolated by $L^\infty(\mu)$. By Lemma 3, the latter is interpolated by $H^\infty$. Q.E.D.
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