Spring 2018

Experimental investigation of turbulent wakes generated by wind turbine models in a large boundary layer wind tunnel

Gregory Gilbert Taylor-Power
University of New Hampshire, Durham

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EXPERIMENTAL INVESTIGATION OF TURBULENT WAKES GENERATED BY WIND TURBINE MODELS IN A LARGE BOUNDARY LAYER WIND TUNNEL

BY

GREG TAYLOR-POWER
BS, Mechanical Engineering, University of New Hampshire, 2016

THESIS

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Master of Science
in
Mechanical Engineering

September 2018
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>iii</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>x</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>xv</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Background and Motivation</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Outline of Thesis</td>
<td>6</td>
</tr>
<tr>
<td>2. THEORY</td>
<td>7</td>
</tr>
<tr>
<td>2.1 The Axisymmetric Turbulent Wake</td>
<td>7</td>
</tr>
<tr>
<td>2.2 The Axisymmetric Turbulent Wake with Swirl</td>
<td>9</td>
</tr>
<tr>
<td>2.3 Governing Equations</td>
<td>10</td>
</tr>
<tr>
<td>2.4 Streamwise and angular momentum conservation</td>
<td>12</td>
</tr>
<tr>
<td>2.4.1 Reynolds stress transport equations</td>
<td>12</td>
</tr>
<tr>
<td>2.5 Similarity Solution</td>
<td>14</td>
</tr>
<tr>
<td>2.6 Effect of a freestream pressure gradient on the Momentum Integral</td>
<td>17</td>
</tr>
<tr>
<td>3. EXPERIMENT FACILITY: FPF</td>
<td>19</td>
</tr>
<tr>
<td>3.1 FPF Description</td>
<td>19</td>
</tr>
<tr>
<td>3.2 Measurement challenges in the FPF</td>
<td>20</td>
</tr>
</tbody>
</table>
4. POROUS DISK EXPERIMENTAL SETUP ........................................... 23
   4.1 Setup ................................................................. 23
   4.2 Procedure .......................................................... 24

5. POROUS DISK WAKE RESULTS .............................................. 27
   5.1 Drag Measurements .................................................. 27
   5.2 Velocity Profiles .................................................... 28
   5.3 Similarity Scalings .................................................. 32

6. SCALE MODEL WIND TURBINE EXPERIMENTAL SETUP ............... 38
   6.1 Model Turbine ...................................................... 38
   6.2 Setup ............................................................... 39
   6.3 Procedure .......................................................... 42

7. SCALE MODEL WIND TURBINE WAKE RESULTS ......................... 46
   7.1 Hotwire Velocity Profiles ......................................... 47
   7.2 Comparison of Hotwire and pitot tube data ..................... 57
   7.3 Similarity Scalings ................................................ 60
   7.4 Comparison of wind turbine and disk wakes ................... 72

8. SUMMARY AND CONCLUSIONS ............................................. 75

BIBLIOGRAPHY ....................................................................... 77

APPENDICES

A. EXPANDED UNCERTAINTY ANALYSIS ...................................... 81
   A.1 Porous Disk Measurements - Pitot Tube ......................... 81
   A.2 Turbine Measurements - Hotwire Anemometer .................. 88

B. GOVERNING EQUATIONS .................................................. 96
   B.1 Continuity and Momentum Equations ............................. 96
   B.2 Reynolds Stress Transport Equations ............................. 99

C. SCALING OF THE GOVERNING EQUATIONS ............................ 103

D. DERIVATION OF THE SIMILARITY SOLUTION FOR THE
   AXISYMMETRIC WAKE WITH SWIRL .................................... 109
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>D.1</td>
<td>$\tau$-Momentum</td>
<td>109</td>
</tr>
<tr>
<td>D.2</td>
<td>Reynolds Stress Transport Equations</td>
<td>110</td>
</tr>
<tr>
<td>D.3</td>
<td>Conditions for the Existence of Similarity</td>
<td>112</td>
</tr>
<tr>
<td>D.4</td>
<td>Effects of Swirl</td>
<td>113</td>
</tr>
</tbody>
</table>
NOMENCLATURE

Symbols

\( \delta^* \)  \quad \text{Wake displacement thickness}

\( \delta_\frac{1}{2} \)  \quad \text{Wake half-width}

\( \mu \)  \quad \text{Dynamic viscosity}

\( \nu \)  \quad \text{Kinematic viscosity}

\( \theta \)  \quad \text{Azimuthal coordinate}

\( \theta^* \)  \quad \text{Wake momentum thickness}

\( D \)  \quad \text{Diameter (disk or turbine)}

\( R \)  \quad \text{Radius (disk or turbine)}

\( r \)  \quad \text{Radial coordinate}

\( Re_D \)  \quad \text{Reynolds number based on diameter (} Re_D = \frac{U_D}{\nu} \text{)}

\( Re_{\delta^*} \)  \quad \text{Local wake Reynolds number (} Re_D = \frac{U_{\delta^*}}{\nu} \text{)}

\( U \)  \quad \text{Mean streamwise velocity}

\( u \)  \quad \text{Fluctuating streamwise velocity}

\( u' \)  \quad \text{Streamwise RMS velocity}

\( U_o \)  \quad \text{Centerline velocity deficit}

\( U_\infty \)  \quad \text{Freestream velocity}

\( U_{CL} \)  \quad \text{Centerline velocity}

\( V \)  \quad \text{Mean radial velocity}

\( v \)  \quad \text{Fluctuating radial velocity}

\( W \)  \quad \text{Mean azimuthal velocity}

\( w \)  \quad \text{Fluctuating azimuthal velocity}
$w'$  Azimuthal RMS velocity
$x$  Downstream position
$y$  Vertical position
$z$  Horizontal position

**Abbreviations**

FPF  Flow Physics Facility
RMS  Root mean square
TSR  Tip-speed ratio
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Various disk wake studies with details of each experiment.</td>
</tr>
<tr>
<td>3.1</td>
<td>Boundary layer height based on downstream location in the UNH FPF as measured by Vincenti et al. [44]</td>
</tr>
<tr>
<td>5.1</td>
<td>Average temperature, freestream velocity, and Reynolds number based on disk diameter for each porous disk experiment</td>
</tr>
<tr>
<td>5.2</td>
<td>Porous disk drag measurements</td>
</tr>
<tr>
<td>7.1</td>
<td>Average temperature, freestream velocity, and Reynolds number based on turbine diameter for each turbine experiment</td>
</tr>
<tr>
<td>A.1</td>
<td>Sources of systematic error in measured quantities used to calculate pitot tube velocity</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>A naturally occuring visualization of the wake at the Horns Rev II wind farm off the West coast of Denmark [13]</td>
</tr>
<tr>
<td>1.2</td>
<td>Results from the Krogstad and Eriksen &quot;Blind Test&quot;, with numerical model predictions compared to actual data. Note the TKE plot is on a log scale. Figures taken from [24]</td>
</tr>
<tr>
<td>1.3</td>
<td>Excerpt from Phenomena Identification Ranking Table (PIRT) for modeling flow at the wind turbine scale [32]</td>
</tr>
<tr>
<td>2.1</td>
<td>Axisymmetric wake coordinates and definitions. $U_\infty$ is the free stream velocity. $U_{cl}$ is the wake centerline velocity. $U_o$ is the centerline wake deficit. $\delta$ is the wake width. $x$, $r$, and $\theta$ represent the streamwise, radial, and azimuthal coordinates of the wake with the origin at the center of the rotor. [17]</td>
</tr>
<tr>
<td>3.1</td>
<td>Section view of the FPF</td>
</tr>
<tr>
<td>3.2</td>
<td>Vertical mean inflow velocity profile 5 m downstream of the FPF inlet</td>
</tr>
<tr>
<td>4.1</td>
<td>Diagram of the porous disk tower setup</td>
</tr>
<tr>
<td>4.2</td>
<td>LSB302 load cell calibration curve</td>
</tr>
<tr>
<td>4.3</td>
<td>Porous disk setup in FPF, with downstream measurement locations labeled</td>
</tr>
<tr>
<td>4.4</td>
<td>Point spacing in FPF cross section with outline of the disk</td>
</tr>
<tr>
<td>5.1</td>
<td>Mean vertical and horizontal inflow profiles 3 m upstream of the porous disk</td>
</tr>
<tr>
<td>5.2</td>
<td>Raw velocity data for $x/D = 1:50$. Darker plot markers correspond to measurement locations farther downstream. $z = y = 0$ is the location of the disk centerpoint, positive $z$ is towards the right of the FPF in the flow direction, positive $y$ is towards the ceiling</td>
</tr>
<tr>
<td>5.3</td>
<td>Horizontal and vertical porous disk wake profiles plotted on the same figure</td>
</tr>
</tbody>
</table>
5.4 A visualization of the disk wake halfwidth evolution. The dashed line is the outline of the porous disk (D = 1m) and the rectangular plot boundaries represent the walls of the test section cross section (to scale). The circles are the actual half-width values, and the solid line is an elliptical fit to the 4 half-width points. ................................................................. 32

5.5 Mean velocity deficit profiles normalized by displacement thickness $\delta^*$ and centerline velocity deficit $U_o$ ................................................................. 33

5.6 Similarity profiles at downstream locations where there is reasonable collapse ...... 34

5.7 Velocity deficit normalized by freestream velocity at each downstream location normalized by momentum thickness $\theta^*$ ................................................................. 35

5.8 Displacement thickness for each horizontal and vertical profile at each downstream location, all normalized by momentum thickness $\theta^*$. $+z$ is the horizontal quarter wake profile in the direction of the right wall (in the direction of flow) $+y$ is wall normal quarter wake profile in the direction of the tunnel ceiling. ................................................................. 36

5.9 Streamwise momentum calculated with equation 5.3 and multiplied by air density. The left plot shows how the value of $M_o$ changes when integrated to different values of $z$. Right is the value of $M_o$ for all $x/D$ ................................................................. 37

6.1 Model turbine shown in FPF at 8 meters downstream of FPF inlet with traversing system in rear ................................................................. 38

6.2 Power (left) and thrust (right) coefficient vs TSR for the model turbine. Power coefficients have not been corrected for tare torque. [7] ................................................................. 39

6.3 Diagram of the model wind turbine with tunnel coordinates $x, y,$ and $z$ as well as wake coordinates $\theta$ and $r$. Also shown is an example wake profile with a streamwise velocity deficit profile and swirl profile. $\Omega$ is turbine rotation rate. ................................................................. 40

6.4 Traversing system shown with pitot tube (right) and X-wire (left) attached ......... 41

6.5 Hot-wire calibration unit which was designed and built at the University of New Hampshire specifically for calibration of multi-wire hot-wire sensors. The jet nozzle can be articulated $+/- 30^\circ$ in any direction with a fan controlled outlet velocity up to 15 m/s which is accurate to .01 m/s. ................................................................. 42

6.6 Hotwire measuring points shown in the tunnel cross section with the outline of the turbine ................................................................. 43
6.7 Hotwire calibration surfaces created using a non-linear least-squares surface fit to equation 6.1. Actual calibration points are shown as spherical points. ............ 44

6.8 Example of hotwire calibration interpolation at $x/D = 2$. Linear interpolation is applied between 3 reference calibration points to form the blue curve............. 45

7.1 Horizontal inflow profiles measured with a Dantec gold plated single wire anemometer. Left is mean velocity measured with the hotwire and pitot tube, and right is turbulence intensity $u'/U_\infty$. .......................................... 48

7.2 Horizontal inflow profiles measured with the homemade X-wire probe. Left is mean streamwise velocity measured with the hotwire and pitot tube, and right is azimuthal velocity............................................ 48

7.3 Mean streamwise velocity profiles in the wake of the model turbine obtained using the X-wire anemometer sensor. $z$ represents horizontal distance from the turbine centerline..................................................... 50

7.4 Mean azimuthal velocity profiles in the wake of the model turbine obtained using the X-wire anemometer sensor. $z$ represents horizontal distance from the turbine centerline. A solid line at $W=0$ is shown to help illustrate the shifting swirl component ................................................................. 50

7.5 Turbine wake mean streamwise and azimuthal velocity for $x/D = 1, 2, 4,$ and 6 ...... 52

7.6 Turbine wake mean streamwise and azimuthal velocity for $x/D = 8, 10, 15,$ and 20 ................................................................. 53

7.7 Streamwise RMS velocity profiles measured with the X-wire ....................... 54

7.8 Azimuthal RMS velocity profiles measured with the X-wire ....................... 54

7.9 Reynolds stress $uw$ measured with the X-wire........................................ 55

7.10 Turbine wake spectra at $z/R = 1$. Black dashed lines are multiples of rotor frequency. Red dashed line is the approximate Strouhal shedding frequency for a disk with the same diameter................................................. 56

7.11 Comparison of pitot tube and hotwire stream-wise velocity profiles ............. 58

7.12 Comparison of the difference between pitot tube and hotwire profiles and turbulence intensity at each point ............................................ 59

7.13 Local Reynolds number ($Re_{\delta^*} = \frac{U_o \delta^*}{\nu}$) vs downstream distance normalized by momentum thickness $\theta^*$ ................................................. 61
7.14 Sample streamwise spectra where $x/D = 1$ is shown in blue, $x/D = 2$ is shown in cyan, $x/D = 4$ is shown in green, and $x/D = 6$ is shown in red. The spectra follow a - 5/3 slope shown by the solid black line, which shows the existence of an inertial subrange.

7.15 Maximum streamwise velocity fluctuations at each downstream location divided by the corresponding centerline velocity deficit.

7.16 Maximum azimuthal velocity fluctuations at each downstream location divided by the corresponding centerline velocity deficit.

7.17 Streamwise velocity normalized by the freestream velocity ($U_\infty$). $R$ represents the turbine radius (0.5 m) and $z$ is the horizontal position.

7.18 Streamwise velocity normalized by scaling parameters $U_o$ and $\delta^*$. The plot on the right shows only the downstream locations where there is reasonable collapse.

7.19 Azimuthal velocity normalized by scaling parameters $U_o^{3/2}$ and $\delta^*$. The plot on the right shows only the downstream locations where there is reasonable collapse.

7.20 Streamwise velocity fluctuations normalized by scaling parameters $U_o$ and $\delta^*$.

7.21 Streamwise velocity fluctuations normalized by scaling parameters $U_o$ and $\delta^*$.

7.22 Centerline velocity deficit $U_o$ vs downstream distance normalized by momentum thickness $\theta^*$ on a linear scale (left) and log scale (right).

7.23 Displacement thickness $\delta^*$ vs downstream distance normalized by momentum thickness $\theta^*$ on a linear scale (left) and log scale (right).

7.24 Maximum azimuthal velocity vs downstream distance normalized by momentum thickness $\theta^*$ on a linear scale (left) and log scale (right).

7.25 Streamwise momentum of the turbine multiplied by air density. The left plot shows how the value of $M_o$ changes when integrated to different values of $z$. Right is the value of $M_o$ for all $x/D$.

7.26 Angular momentum of the turbine multiplied by air density. The left plot shows how the value of $G_o$ changes when integrated to different values of $z$. Right is the value of $G_o$ for all $x/D$.

7.27 Normalized mean velocity profiles of the turbine compared to the porous disk.
Velocity profiles of the wind turbine and porous disk normalized by similarity scaling variables $U_o$ and $\delta^*$ from $x/D=4 - 20$. Black triangles turbine points and blue crosses are disk points ........................................ 74

A.1 Autocorrelation of velocity time series at $x/D=1, z=0.5$ m ....................... 82

A.2 Porous disk horizontal mean velocity profiles with 95% confidence intervals for $x/D = 1 - 20$ ................................................................. 84

A.3 Porous disk horizontal mean velocity profiles with 95% confidence intervals for $x/D = 30 - 50$ ................................................................. 85

A.4 Porous disk vertical mean velocity profiles with 95% confidence intervals for $x/D = 1 - 20$ ................................................................. 86

A.5 Porous disk mean velocity profiles with 95% confidence intervals for $x/D = 30 - 50$ ................................................................. 87

A.6 Autocorrelation of hotwire velocity data at $x/D = 4, z/R = 1$ ....................... 89

A.7 Turbine wake mean streamwise and azimuthal velocity with 95% confidence error bars for $x/D = 1, 2, 4, \text{and} 6$ ........................................ 91

A.8 Turbine wake mean streamwise and azimuthal velocity with 95% confidence error bars for $x/D = 8, 10, 15, \text{and} 20$ ........................................ 92

A.9 Turbine wake $u_{rms}$ velocity with 95% confidence error bars for all $x/D$ .......... 93

A.10 Turbine wake $w_{rms}$ velocity with 95% confidence error bars for all $x/D$ ........ 94

A.11 Turbine wake $uw$ Reynolds stress with 95% confidence error bars for all $x/D$ .... 95
ABSTRACT
Experimental investigation of turbulent wakes generated by wind turbine models in a large boundary layer wind tunnel
by
Greg Taylor-Power
University of New Hampshire, September, 2018

Velocity profiles were measured in the wake of two wake generator models in the University of New Hampshire (UNH) Flow Physics Facility. Mean horizontal and vertical velocity profiles were measured with a pitot tube behind a 51% open area porous disk up to 50 diameters downstream. The disk wake is shown to elongate in the vertical direction and shift downward as it evolves downstream. Streamwise and azimuthal velocity profiles were measured in the wake of a 1 m diameter scale model wind turbine to 20 diameters downstream. The azimuthal velocity profile shifts horizontally with downstream distance but still adheres to a $W \sim x^{-1} \sim U_o^{3/2}$ similarity scale at select downstream locations; where $W$ is mean azimuthal velocity, $x$ is downstream distance, and $U_o$ is the wake centerline velocity deficit $U_o = U_\infty - U_{cl}$. Both disk and wind turbine wakes exhibit the classical high-Reynolds number scaling for axisymmetric turbulent wakes, where the mean velocity deficit decays as $U_o \sim x^{-2/3}$ and the wake grows as $\delta^* \sim x^{1/3}$ where $\delta^*$ is the wake displacement thickness.
CHAPTER 1
INTRODUCTION

1.1 Background and Motivation

The demand for renewable energy has increased dramatically in recent years, and society now faces the challenge of filling this demand with reliable, green sources. Though the end goal may involve many forms of energy technologies, wind energy presents one clean, sustainable alternative to fossil fuels, with a total resource large enough to power the entire United States [42]. The United States Department of Energy has shown its commitment to wind energy as a power source by publishing the study "20% Wind by 2030" in 2008. This set the goal of 20% of the nation’s electrical energy coming from wind by the year 2030, which is, with current capacity factors updated in the 2015 Wind Vision report, equivalent to an installed capacity of 224 GW [41, 42]. This will require the U.S. to install approximately 11 GW of new wind energy capacity per year for the next 13 years, necessitating the construction of many large wind farms.

Land based wind in the US has shown remarkable increase in recent years, with installed capacity growing from 2.5 GW in 2000 to 82 GW in 2016, 17 GW of that being from 2015 and 2016 alone [1]. The majority of large wind turbine arrays currently in use by the U.S. are located in the central plains. Due to power grid limitations and transmission losses, the wind farms cannot effectively meet the demands of the population near the coast. To combat this, new installations will have to include offshore wind farms. The "20% wind by 2030" scenario includes 22 GW from offshore wind as stated by the 2015 Wind Vision report [42]. Offshore wind provides an obstacle free upstream flow with high wind velocities, and gives the ability to provide power to coastal areas. Until very recently, the U.S. had no operational offshore wind farms. In December 2016, the 5 turbine, 30 MW Block Island Wind Farm became operational, the first offshore wind farm in
the United States [1]. For comparison, at the end of 2017 Europe had a total installed capacity of 15.8 GW across 11 countries, with 3.1 GW installed in 2017 alone [45].

Large wind turbine arrays typically do not generate the amount of power that would be expected by adding up the power rating of the individual turbines. Wind farm data shows that turbines located downstream of the front row (relative to wind direction) generate significantly less power than the turbines at the front. The wakes generated by upstream turbines create lower velocities for downstream turbines, and thus less power is produced. These energy losses, when averaged over a year for all wind directions, typically range from 5% to over 15% of rated capacity [30]. They have been measured as high as 20% in large offshore wind farms with regularly spaced wind turbines [2]. The corresponding losses in revenue are significant. For example, consider an offshore wind farm with a capacity factor of 0.4; income loss per year would be $3.67M (million) for each percent wake loss per installed GW capacity based on an average retail price of $0.1054 per kWh [43]. The Horns Rev wind farms in Denmark have produced iconic images of how the upstream wake affects the downstream turbines, as shown in Figure 1.1.

Figure 1.1: A naturally occurring visualization of the wake at the Horns Rev II wind farm off the West coast of Denmark [13]

Improved array spacing and overall wind farm design is necessary in order to reduce energy losses due to wake effects, and reduce turbulence loads on downstream turbines. For future wind
farm design, higher fidelity numerical simulations will be necessary to optimize array spacing. Current models used in industry for design and layout of wind farms are very basic. One example of this is the Jensen model, which uses a simple linear spreading parameter and does not capture the complex physics or turbulence in the wake [16]. Before these more complex models can be validated and used in industry, the processes driving wind turbine wake formation and decay need to be more well understood. One step in this process is providing high quality validation data for numerical modelers to reference.

![Graphs showing power coefficients vs tip speed ratio, mean velocity deficit profile, and turbulent kinetic energy profile.](image)

(a) Power Coefficients vs Tip Speed Ratio  (b) Mean Velocity Deficit Profile at x/D=1

(c) Turbulent Kinetic Energy Profile at x/D=1

**Figure 1.2:** Results from the Krogstad and Eriksen "Blind Test", with numerical model predictions compared to actual data. Note the TKE plot is on a log scale. Figures taken from [24].

A study by Krogstad and Eriksen in 2013 helps to illustrate this need for validation data [24]. In this study, eight different numerical modeling groups attempted to predict the performance and wake development for a model wind turbine that was designed and tested at the Norwegian
University of Science and Technology. Some of the results from this study can be seen in 1.2. It is clear that performance curves and near wake mean velocity profiles are predicted reasonably well, ±10%, and the main trends are resolved. The turbulent kinetic energy (TKE), which is important in determining rotor lifetime, shows orders of magnitude of difference between measurements and models. From these results it is clear that more work needs to be done to improve the accuracy of existing models. The models employed in this study are already considered "higher fidelity" models compared to what is used for wind farm design.

Validation data is needed across a spectrum of scales:

- Full scale, with rotor diameters on the order of 100 meters
  - Trujillo et al. 2011 measured the near wake of of a 116 m diameter 5 MW offshore turbine using lidar[39]
  - Kern et al. 2010 also used lidar to measure the wake of a 100 m diameter 2.5 MW land based turbine [22].

- The "SWiFT" scale, a Sandia National Lab project which uses Vestas V27 turbines with a rotor diameter of 27 meters (e.g. Maniaci 2015 [25].)

- The large aerodynamics wind tunnel scale, rotor diameter approximately 5-10 meters
  - Schreck 2002 performed experiments in the NASA Ames wind tunnel with a 10 m diameter turbine[31].
  - The MEXICO project (Snel 2007) tested a 4.5 m diameter turbine in the large scale facility of the DNW [34].

- The large boundary layer wind tunnel scale, where feasible rotor diameters are 1-2 meters
  - Krogstad 2012 (and many other NTNU studies) tested a 0.9 m turbine in their 1.9 m x 2.7 m wind tunnel [23].
The experiments presented in this thesis are working at this scale with a 1 m diameter turbine in a 6 m x 2.7 m wind tunnel.

Each of these scales presents advantages and disadvantages for collecting data. The full and "SWiFT" scales have the advantage of being subject to real world conditions experienced by wind turbines. The downside of this being that the turbines operate within the earth’s atmospheric boundary layer, where inflow conditions are highly variable and cannot be controlled. This makes it difficult to use full scale data to develop any generalized scaling laws or parameterizations. The aerodynamic wind tunnel scale has the advantage of controlled inflow, while still maintaining large scales, and consequently high Reynolds number. However, in these tunnels it is difficult to measure relatively far down stream due to the large turbine diameter. The boundary layer wind tunnel scale offers well controlled inflow conditions and the ability to study the wake far downstream of the turbine rotor, but may fail to achieve sufficiently high Reynolds number.

At the University of New Hampshire, we are fortunate to be home to the Flow Physics Facility (FPF), the world’s largest boundary layer wind tunnel [44]. This gives us the ability to perform wake experiments on model turbines on the order of 1 meter in diameter with moderate blockage ($A_T/A_{FPF} \sim 5\%$) to far downstream locations in both the freestream and boundary layer. Figure 1.3 shows a reduced version of the Phenomena Identification and Ranking Table for wind turbine/wind energy fluid dynamic phenomena developed by the National Renewable Energy Laboratory (NREL)[32]. Only the turbine scale phenomena which can be investigated and addressed by scale model turbine experiments in the UNH Flow Physics Facility were kept as entries (6 entries out of a total of 25 PIRT entries). It can be seen that most of the entries in Table 1 have high importance at the application level, but generally low model adequacy at in terms of physics, code trustworthiness and validation.
Figure 1.3: Excerpt from Phenomena Identification Ranking Table (PIRT) for modeling flow at the wind turbine scale [32]

### 1.2 Outline of Thesis

The study reported here involved two wake generator models: a 1 m diameter porous disk with an open area of 51%, and a 1 m diameter scale model 3-bladed wind turbine modeled after the NREL 5 MW offshore reference wind turbine [20].

The disk was chosen to have a drag approximately the same as the predicted drag of the model turbine. Mean streamwise velocity measurements were acquired with a pitot tube in the wake of the disk with the goal of quantifying the effect of the FPF aspect ratio on the wake a of an object with similar size, shape, drag and blockage ratio to the model wind turbine.

Streamwise and azimuthal velocities were measured with an X-wire hot-wire anemometer in the wake of the scale model wind turbine with the goal of characterizing the streamwise velocity recovery and the decay of the mean swirl induced by the rotor. This study aims to further the understanding of wind turbine dynamics, while examining the nature of the fundamental turbulent asymmetric wake with swirl.
CHAPTER 2
THEORY

2.1 The Axisymmetric Turbulent Wake

The axisymmetric wake is a form of turbulent wake created by any axisymmetric object moving through a fluid or fluid moving past an axisymmetric shape. Turbulent wakes are a class of free shear flow, meaning they evolve without the presence of a solid boundary [37]. Different flows of the same class tend to behave differently when presented with different initial conditions.

Axisymmetric wakes are generated by a variety of commonly occurring phenomena, both natural and industrial, giving this flow many applications in science and engineering. Any object propelling itself through a fluid creates a momentum-less wake, while a towed object will create a wake with momentum deficit. Stationary objects placed in a fluid flow (e.g. wind turbines) will also create a wake with a momentum deficit.

Wakes have proven to be difficult to measure experimentally, with experiments dating back to 1931 where a study by Marshall and Stanton [26] showed photographs of the wake of a circular disk in water. They used dye to visualize the flow and reported an unsteadiness when Reynolds number based on free stream velocity and disk diameter ($Re_D$) exceeded about 200. With the development of hot-wire anemometry, full wake profiles including turbulence terms could be measured. The first example of this was a study performed with an axisymmetric disk placed perpendicular to the flow by Carmody in 1964 [5]. Carmody measured mean velocity, turbulence intensity, Reynolds stress and wake growth at $Re_D$ of 70,000. This study proclaimed that the wake was self-similar 15 diameters downstream from the disk, meaning that the mean velocity profiles appeared to fall on the same curve when normalized by the centerline velocity deficit and a lateral length scale determined from the velocity profile itself. A similar study done by Hwang and Baldwin in 1966
measured the wake of a disk all the way back to 900 diameters downstream. Both the Carmody [5] and Hwang and Baldwin [15] studies show significant scatter, most likely due to the underdeveloped nature of hot-wire anemometers at the time.

Confidence with hot-wires improved in the 1970s, and a study by Bevilaqua and Lydoukis in 1978 [3] produced wake data of a porous disk and sphere with the same drag. They were able to show that both of these wakes became self-similar in terms of mean velocity and Reynolds stress profiles within 10 diameters of the sphere and 20 diameters of the disk, but they did not reach the same state of similarity. Another study that confirmed this difference was done by Cannon in 1991 [4], who measured the wake of a sphere, a solid disk, and 3 porous disks with different porosity. All of these objects had the same drag, and data was collected up to 125 diameters downstream. The wakes evolved at different rates, and reached different self-similar states, giving more evidence that geometry and initial conditions are a significant part of what dictates wake evolution. From these two studies, it is clear that drag alone does not determine how a wake evolves.

A more recent set of experiments performed by Johansson and George in 2006 [18] investigated the wake of an axisymmetric disk to 150 diameters downstream at a $Re_D$ of 26,400. They used a rake of hot-wires to capture the entire wake simultaneously, but due to the low magnitude of the velocity deficit and low turbulence intensity, the thermal drift of the hot-wires contributed significant uncertainty. After correcting for temperature, they were able to show that the mean streamwise velocity and streamwise turbulence intensity collapse well for $x/D > 30$.

The local Reynolds number of the axisymmetric wake, $Re_\delta = \frac{U_s \delta}{\nu} \sim x^{-\frac{1}{3}}$, decreases slowly, and the flow will eventually drop out of the high Reynolds number region where the classical scalings, $\delta^* \sim x^{\frac{1}{2}}$ and $U_s \sim x^{-\frac{2}{3}}$ are viable. George (1989) [11] proposes two different similarity solutions for the high and low Reynolds number regimes of the wake, and Johansson et al (2003) [19] was able to demonstrate that the wake will admit to these solutions. Table 2.1 shows a summary of various axisymmetric wake studies and their approximate regions of similarity.
Table 2.1: Various disk wake studies with details of each experiment.

<table>
<thead>
<tr>
<th>Disk</th>
<th>D (m)</th>
<th>Blockage (%)</th>
<th>Re_D</th>
<th>x/D</th>
<th>Similarity range</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid</td>
<td>0.051</td>
<td>0.36</td>
<td>70,000</td>
<td>15</td>
<td>x/D &gt; 15</td>
<td>Carmody 1964 [5]</td>
</tr>
<tr>
<td>Solid</td>
<td>0.019</td>
<td>8.6e-5</td>
<td>7600</td>
<td>900</td>
<td>x/D &gt; 1.00</td>
<td>Hwang 1966 [15]</td>
</tr>
<tr>
<td>Porous</td>
<td>0.0254</td>
<td>0.32</td>
<td>10,000</td>
<td>110</td>
<td>x/D &gt; 20</td>
<td>Bevilaqua 1978 [3]</td>
</tr>
<tr>
<td>Solid</td>
<td>0.028</td>
<td>0.11</td>
<td>13,000</td>
<td>130</td>
<td>-</td>
<td>Cannon 1991 [4]</td>
</tr>
<tr>
<td>Porous</td>
<td>0.031</td>
<td>0.14</td>
<td>13,000</td>
<td>130</td>
<td>-</td>
<td>Cannon 1991 [4]</td>
</tr>
<tr>
<td>Solid</td>
<td>0.02</td>
<td>0.03</td>
<td>26,400</td>
<td>150</td>
<td>x/D &gt; 30</td>
<td>Johansson 2006 [18]</td>
</tr>
<tr>
<td>Solid</td>
<td>0.028</td>
<td>0.027</td>
<td>28,000</td>
<td>50</td>
<td>-</td>
<td>Johansson 2002 [17]</td>
</tr>
<tr>
<td>Porous</td>
<td>1</td>
<td>4.8</td>
<td>700,000</td>
<td>50</td>
<td>x/D ≥ 4</td>
<td>Taylor-Power 2018</td>
</tr>
</tbody>
</table>

2.2 The Axisymmetric Turbulent Wake with Swirl

The turbulent axisymmetric wake with swirl is another important flow that exists in numerous natural and industrial areas. Any piece of rotating fluid machinery will produce a wake with some swirling component. Practical applications of the swirling wake can be divided into propulsion devices and power producing devices, but this flow is also important at a fundamental fluid mechanics level. Many practical studies involving this flow have been performed, but there still seems to be a lack of fundamental studies. Many wind turbine studies have been performed recently, but often the swirling component isn’t reported or even recorded. For example, Muhle 2017 studied the effect of rotation on downstream turbine performance, but the purpose of the study was not to examine the swirl decay and it was only reported up to x/D = 5.15 [27]. Schumann 2013 also reported the full swirl map behind a wind turbine, but only up to x/D = 3 [33]. Both of these studies report a shifting of the swirling velocity that is decoupled from the streamwise velocity.

Reynolds [29] correctly derived the traditional high-Reynolds number scaling for the non-swirling wake, but neglected to include the momentum integral and derived a solution for a swirl dominated flow where the swirl decays as $W \sim x^{-3/4}$. Steiger and Bloom [36] derived a linearized solution for swirling wakes which resulted in the swirl decaying exponentially. Wosnik and Dufresne 2013 [46] derived a scaling function for the mean swirl $W \sim x^{-1} \sim U_o^{3/2}$. They
measured the swirling wake of a model wind turbine to $x/D = 20$ and showed first evidence of the derived scaling function for the swirl.

The swirling wake is even more difficult than the non-swirling wake to measure accurately due to swirling velocity fluctuations on the same order as the mean swirling velocity. There is also the issue of ensuring the supporting structure of the wake generator has minimal effect on the swirling wake itself, which presents a variety of design difficulties. The goal of the experiments presented here was to examine a wake similar to that of Wosnik 2013, but with a more precisely designed wake generator device. The data set can then be used to look at similarity scalings and compare to wind turbine tests at similar facilities, and provide benchmark data for numerical models.

### 2.3 Governing Equations

The theory derived below will follow the approach of previous free shear flow investigation listed above (Johansson 2002, Shiri 2010), beginning with the Reynolds-averaged Navier-Stokes (RANS) equations of motion. The governing equations are derived in Appendix B, and an order of magnitude analysis is carried out in Appendix C. The axisymmetric wake coordinates used here are shown in Figure 2.1 as used in Johannson 2002 [17].

---

**Figure 2.1:** Axisymmetric wake coordinates and definitions. $U_\infty$ is the free stream velocity. $U_{cl}$ is the wake centerline velocity. $U_o$ is the centerline wake deficit. $\delta$ is the wake width. $x, r, \theta$ represent the streamwise, radial, and azimuthal coordinates of the wake with the origin at the center of the rotor. [17]
The reduced governing equations for the turbulent axisymmetric wake with swirl are as follows.

The mean continuity equation reduces to:

\[
\frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial (rV)}{\partial r} = 0
\] (2.1)

The streamwise (\(x\)) momentum equation reduces to:

\[
U_\infty \frac{\partial (U - U_\infty)}{\partial x} = -\frac{1}{r} \frac{\partial}{\partial r} (r uv) + \left\{ \frac{\partial}{\partial x} \left( \frac{u^2}{r} - \bar{u}^2 + \int_0^\infty \frac{1}{r} (w^2 - \bar{v}^2) + W^2 dr \right) \right\}
\] (2.2)

Where the terms in curly brackets in the \(x\)-momentum equation are of second order, but are kept for now to be able to investigate their contribution to the momentum integral. To first order, the reduced \(x\)-momentum consists of a balance between the leading order convection term and the leading order Reynolds stress. The radial (\(r\)) momentum equation becomes:

\[
\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{W^2}{r} - \frac{\partial v^2}{\partial r} + \frac{w^2 - \bar{v}^2}{r}
\] (2.3)

This can be integrated with respect to \(r\) to obtain the mean pressure distribution in the wake, which is then used to eliminate pressure in the \(x\)-momentum equation. The azimuthal (\(\theta\)) momentum equation becomes:

\[
U_\infty \frac{\partial W}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} (r \bar{v} \bar{w})
\] (2.4)

The reduced order \(\theta\) equation states that, to leading order, the change in streamwise transport of azimuthal momentum is equal to the radial transport of radial-azimuthal Reynolds stress \(\bar{v} \bar{w}\). This is the primary transport mechanism for redistributing azimuthal momentum as the swirling wake evolves downstream.
2.4 Streamwise and angular momentum conservation

From the governing equations, integral parameters can be derived. One of the two fundamental integrals of the RANS equations for the fully developed turbulent swirling wake is $M_x$, which is the total rate of transfer of kinematic linear momentum across any downstream plane, say at location $x$. At high Reynolds numbers this reduces to:

$$M_x(x) = M_o = 2\pi \int_0^\infty \left[ U_\infty(U - U_\infty) - \frac{W^2}{2} + \overline{u^2} - \frac{\overline{v^2 + w^2}}{2} \right] r dr$$

(2.5)

Since there are no net forces other than pressure, which is accounted for in the linear momentum equation acting on any control volume containing this plane and the wake plane, $M_x$ must remain equal to its source value, which is equal to the net drag imparted by the wake generator, for all downstream positions $x$.

The second fundamental parameter $G_\theta(x)$ is the rate which kinematic angular momentum is swept across any downstream plane. From the integration of the angular momentum equation (eqn. 2.4) with the same assumptions as above, this can be shown to reduce to:

$$G_\theta(x) = G_o = 2\pi \int_0^\infty [U_\infty W + \overline{uw}] r^2 dr.$$  

(2.6)

Like the linear momentum, $G_\theta(x)$ should remain constant at its source value, $G_o$, since in an infinite environment there are no torques acting on any control volume containing the source plane nor any plane that cuts perpendicularly through the wake axis.

2.4.1 Reynolds stress transport equations

The mean momentum and continuity equations are not sufficient to determine constraints on the similarity solution/scaling functions for all quantities of interest. Individual Reynolds stress transport equations, and a condition on the pressure-strain-rate terms from continuity (incompress-
ibility) also need to be considered. The transport equations for the Reynolds stress components to first order (after order of magnitude analysis) are as follows:

\( \overline{u^2} \) balance

\[
U_\infty \frac{\partial}{\partial x} \left( \frac{1}{2} \overline{u^2} \right) = -\overline{uv} \frac{\partial}{\partial r} (U - U_\infty) - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{1}{2} \overline{u^2} v \right) + \frac{\rho}{\rho} \frac{\partial u}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \overline{mu} + \nu \frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{r}{\partial r} \left( \frac{1}{2} \overline{u^2} \right) \right\} - \epsilon_u
\]  

(2.7)

\( \overline{v^2} \) balance

\[
U_\infty \frac{\partial}{\partial x} \left( \frac{1}{2} \overline{v^2} \right) = -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{1}{2} \overline{v^2} \right) + \frac{\overline{uw^2}}{r} + \frac{\rho}{\rho} \frac{\partial v}{\partial r} - \frac{1}{\rho} \frac{\partial}{\partial r} \overline{mv} + \nu \frac{1}{\rho} \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{2} \overline{v^2} \right) \right\} - \epsilon_v
\]  

(2.8)

\( \overline{w^2} \) balance

\[
U_\infty \frac{\partial}{\partial x} \left( \frac{1}{2} \overline{w^2} \right) = -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{1}{2} \overline{w^2} \right) - \frac{\overline{uw^2}}{r} + \nu \frac{1}{\rho} \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{1}{2} \overline{w^2} \right) \right\} - \epsilon_w
\]  

(2.9)

\( \overline{uw} \) balance

\[
U_\infty \frac{\partial}{\partial x} (\overline{uw}) = -\overline{v^2} \frac{\partial}{\partial r} (U - U_\infty) - \frac{1}{r} \frac{\partial}{\partial r} \left( \overline{uw^2} \right) + \frac{\overline{uw^2}}{r} + \frac{p}{\rho} \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial x} \right) - \frac{1}{\rho} \left( \frac{\partial}{\partial r} \overline{mu} + \frac{\partial}{\partial x} \overline{mv} \right) + \nu \frac{1}{\rho} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{1}{2} \overline{uw} \right) \right\} - \epsilon_{uw}
\]  

(2.10)
2.5 Similarity Solution

Similarity solutions are now substituted into the equations derived above. A more detailed, term by term derivation can be found in Appendix D. The similarity solutions are products of scaling functions which depend only on streamwise location, and similarity profiles which depend on a new similarity variable and possibly initial conditions. An important difference to the classical analysis of turbulent shear flows, e.g. [38], is that scaling functions are not assumed a priori, but will be determined from conditions on the existence of similarity solutions derived from the governing equations. The similarity functions have the following form:

\[ U - U_\infty = U_s(x)f(\eta, *); \quad -\overline{uv} = R_{s,uv}(x)g(\eta, *) \]

\[ W = W_s(x)h(\eta, *); \quad -\overline{uw} = R_{s,uw}(x)i(\eta, *) \]

\[ \frac{1}{2} \overline{u^2} = K_u(x)k_u(\eta, *); \quad \frac{1}{2} \overline{u^2v} = T_{uv}(x)\overline{t_{uv}}(\eta, *) \]

\[ \frac{p}{\rho} \frac{\delta u}{\delta x} = P_u(x)p_u(\eta, *); \quad \frac{1}{\rho} \overline{\rho u} = P^D_u(x)p^D_u(\eta, *) \]

\[ \varepsilon_u = D_u(x)d_u(\eta, *) \]

where \( \eta = r/\delta(x) \) and (*) denotes a dependence on initial conditions (wake generator, e.g., turbine type and operating condition). After substituting the similarity solutions and clearing terms, the \( x \)-momentum equation takes the following form

\[ \left[ \frac{\delta}{U_s} \frac{dU_s}{dx} \right] f - \left[ \frac{d\delta}{dx} \right] \eta f' = \left[ \frac{R_{s,uv}(x)}{U_\infty U_s} \right] \frac{(\eta g)'}{\eta} \]

where the terms in square brackets depend on downstream position \( x \) only, and the non-bracketed terms depend on the new similarity variable only. In order for a similarity solution to exist, all
bracketed terms must have the same $x$-dependence. This can be further simplified with the momentum integral and $\delta \equiv \delta^*$ and $U_s \equiv U_o = (U_\infty - U_{cl})$:

$$U_s\delta^2 \int_0^\infty f \eta d\eta = U_\infty \theta^2 \quad \rightarrow \quad \left[ \frac{U_s}{U_\infty} \right] \propto \left[ \frac{\theta^*}{\delta^*} \right]^2$$

(2.13)

This constraint is used later to derive the scaling functions. Substitution of similarity solutions in the Reynolds stress component equations leads to similar results as the mean momentum equations and gives more constraints on the existence of similarity.

**Conditions for the existence of similarity**

From the equations above, the conditions for the existence of similarity solutions are, from $x$-momentum:

$$\left[ \frac{\delta}{U_s} \frac{dU_s}{dx} \right] \sim \left[ \frac{d\delta}{dx} \right] \sim \left[ \frac{R_{s,uv}}{U_\infty U_s} \right]$$

(2.14)

Here the symbol “$\sim$” means “has the same x-dependence as” Upon inspection, from the first and second terms, it can be seen that the Reynolds stress scaling function depends on the growth rate of the wake

$$R_{s,uv} \sim U_\infty U_s \frac{d\delta}{dx}$$

(2.15)

contrary to traditional wake analyses, e.g. [38] [37]. From the transport equation for the streamwise normal stresses we find the following conditions:

$$\frac{\delta}{K_u} \frac{dK_u}{dx} \sim \frac{d\delta}{dx} \sim \frac{T_u\delta}{U_\infty K_u} \sim \frac{D_u\delta}{U_\infty K_u}$$

(2.16)
A similarity solution, for large local Reynolds number, $U_\infty \delta / \nu$, (viscosity is identically equal to zero), is only possible if:

$$\frac{d\delta}{dx} \sim \frac{D_u \delta}{U_\infty K_u}$$  \hspace{1cm} (2.17)

From the other Reynolds stress transport equations, the remaining constraints are found:

$$K_u \sim K_v \sim K_w \sim U_s^2$$
$$D_u \sim D_v \sim D_w \sim \frac{U_s^3}{\delta}$$  \hspace{1cm} (2.18)

When considering all of the constraints above, it can be shown that the mean flow has similarity solution with scaling functions of the following form:

$$\frac{\delta^*}{\theta^*} = a \left[ \frac{x - x_o}{\theta^*} \right]^{\frac{1}{3}}$$
$$\frac{U_s}{U_\infty} = b \left[ \frac{x - x_o}{\theta^*} \right]^{-\frac{2}{3}}$$  \hspace{1cm} (2.19)

These are the same as in the classical solution e.g. Johansson et al. (2002) [17], but the scaling functions for higher moments (Reynolds stresses etc) are shown to be more complicated. Also, since the axisymmetric turbulent wake is a flow with diminishing local Reynolds number ($U_s \delta_s / \nu$), as can be seen from the scaling function above, the flow will eventually “fall out” of this infinite Reynolds number, viscosity-independent similarity solution, but may arrive at another, viscous-dominated low-Reynolds number similarity solution [19].

**Effects of Swirl**

To investigate the behavior of the swirling component of mean velocity, we now also consider the $W, <uw>, <vw>$ equations. The rate at which kinematic angular momentum is swept downstream (from integrated angular momentum equation) can be written as

$$G_{\theta}(x) = G_{\phi} = 2\pi \int_0^\infty [UW + \bar{u}\bar{w}] r^2 dr$$  \hspace{1cm} (2.20)
Note that the extraction of linear momentum and the addition of angular momentum impose an additional length scale (from the source conditions) as $L_*=G_o/M_o$. We can neglect the $<uw>$-term and substitute the similarity solutions.

$$G_x = [U_\infty W_{max} \delta^*]^3 \pi \int_0^\infty f \eta^2 d\eta$$  \hspace{1cm} (2.21)

With $\delta^* \sim x^{\frac{1}{3}}$ and $U_\infty = \text{constant}$, the azimuthal velocity has to decay as

$$\frac{W_s}{U_\infty} = \left[\frac{x-x_o}{\theta^*}\right]^{-1}$$  \hspace{1cm} (2.22)

In general, it should be noted that properly normalized mean velocity profiles always collapse, and the source-dependent differences will show up in the wake spreading rate and the higher turbulent moments. If a numerical model for the axisymmetric, turbulent, swirling wake cannot reproduce scaling behavior predicted by an equilibrium similarity solution, then it is not capturing the essential wake physics.

### 2.6 Effect of a freestream pressure gradient on the Momentum Integral

In the scaling of the governing equations, the streamwise pressure gradient is substituted in the x-momentum equation by integrating the radial pressure gradient in the r-momentum equation. During this, the freestream pressure gradient $\frac{dP_\infty}{dx}$ is set equal to zero. In flows with non-negligible blockage, like the wake flow in this study, there is significant flow speedup in the freestream near the location of the wake generator, and likewise a pressure gradient that relaxes as one moves farther from the wake origin. The simplified x-momentum equation including a freestream pressure gradient is as follows

$$U_\infty \frac{\partial(U - U_\infty)}{\partial x} = -\frac{1}{r} \frac{\partial}{\partial r} (r \bar{uv}) + \left\{ \frac{\partial}{\partial x} \left( \bar{v^2} - \bar{u^2} - \frac{P_\infty}{\rho} + \int_0^\infty \frac{1}{r} (\bar{w^2} - \bar{v^2}) + W^2 dr \right) \right\}$$  \hspace{1cm} (2.23)
Flow outside the wake is governed by the Euler equation:

\[ U_\infty \frac{dU_\infty}{dx} = -\frac{1}{\rho} \frac{dP_\infty}{dx} \]  

(2.24)

After substitution and integration across the flow, the momentum integral including pressure gradient becomes

\[ M_x(x) = M_o + 2\pi \int_0^\infty \left[ \frac{1}{2} \left( U_{\infty_x}^2 - U_{\infty_o}^2 \right) r dr = \right. \]

\[ 2\pi \int_0^\infty \left[ U_{\infty_x} (U - U_{\infty_x}) - \frac{W^2}{2} + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} \right] r dr \]  

(2.25)

Where \( U_{\infty_x} \) is the local freestream velocity at a given \( x \) location and \( U_{\infty_o} \) is the freestream velocity upstream of the wake generator. This equation shows that a freestream pressure gradient will cause the momentum integral to vary with downstream distance until the freestream velocity gradient decays to zero, at which point the momentum integral will reach its constant value.
CHAPTER 3

EXPERIMENTAL FACILITY: FPF

3.1 FPF Description

All experiments were performed in the UNH Flow Physics Facility (FPF). The FPF is, to the author’s knowledge, the largest boundary layer wind tunnel in the world. The FPF test section has a width of 6 m, a height of 2.7 m, and a length of 72 m. The test section height increases with downstream distance to account for boundary layer growth on all four walls and to maintain a zero pressure gradient in the core of the tunnel, and therefore a zero pressure gradient turbulent boundary layer on all walls, if the test section is empty. It is currently in phase 1 of construction which consists of an open air circuit test section. It can achieve velocities of up to 14 m/s with free stream turbulence intensities of less than 0.5% [44]. The FPF was designed to investigate high Reynolds number turbulent boundary layers with adequate spatial and temporal resolution. It can achieve Reynolds numbers, expressed as scale ratios of \( \delta^+ = \frac{L_{outer}}{L_{inner}} = \frac{\delta u_{*}}{\nu} = 20,000. \)

Figure 3.1: Section view of the FPF
Table 3.1: Boundary layer height based on downstream location in the UNH FPF as measured by Vincenti et al. [44]

<table>
<thead>
<tr>
<th>Downstream Location</th>
<th>Boundary Layer Height</th>
<th>Test Section Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 ( (m) )</td>
<td>.08 ( (m) )</td>
<td>7 ( (m/s) )</td>
</tr>
<tr>
<td>8 ( (m) )</td>
<td>.14 ( (m) )</td>
<td>7 ( (m/s) )</td>
</tr>
<tr>
<td>16 ( (m) )</td>
<td>.24 ( (m) )</td>
<td>7 ( (m/s) )</td>
</tr>
<tr>
<td>32 ( (m) )</td>
<td>.43 ( (m) )</td>
<td>7 ( (m/s) )</td>
</tr>
<tr>
<td>66 ( (m) )</td>
<td>.73 ( (m) )</td>
<td>7 ( (m/s) )</td>
</tr>
</tbody>
</table>

While the FPF was designed to investigate turbulent boundary layers, its large size also allows model wind turbine studies, particularly wake studies. The test section of the FPF allows for the use of a turbine rotor diameter of up to 1 m while maintaining a blockage ratio under 5%, where ratios larger than this will affect the power and thrust coefficients when running near optimal tip speed ratio [6]. The large scale models permitted by this facility allow for higher resolution wake measurements, using a larger wake generator inherently increases the Kolmogrov length scales (smallest scales of turbulent motion) in the wake [10]. This improves the spatial resolution of traditional measurement techniques. The larger scale of the turbine models also allows for large Reynolds number based on blade chord, which makes for a more realistic turbine models.

3.2 Measurement challenges in the FPF

The FPF is a one of a kind facility and it provides unique opportunities for a variety of fluid dynamics studies, but it in its current open return phase it also presents a few unique challenges that make experiments difficult. The large test section size makes traversing downstream an issue along with traversing large horizontal distances, as there is currently no built-in traversing system. The most notable issue, however, is that the open return system subjects the tunnel flow to uncontrollable atmospheric conditions. Properties like atmospheric pressure, humidity, and temperature are changing constantly outside. Changes in atmospheric conditions affect important fluid parameters such as density, viscosity, and subsequently any Reynolds number. The weather also limits exper-
iment timing, as the FPF cannot be run during any precipitation else significant moisture will be pulled into the test section, potentially damaging instrumentation. Air temperature is arguably the fluid property with the biggest effect on experiments, and it is also the most variable. Temperature not only affects flow conditions, but also any instrumentation that is sensitive to flow temperature. For example, the hotwire anemometer is one of the most popular devices for measuring turbulent flows, but it is extremely sensitive to changes in fluid temperature. Effects of temperature can be somewhat mitigated by running during hours without significant temperature drift with help from weather predictions, but this doesn’t account for small time-scale changes in temperature due to unpredictable things like passing clouds.

The original characterization of the FPF boundary layer done by Vincenti et al states that outside wind has no noticeable effect on the mean flow but shows up as a higher freestream turbulence intensity at speeds lower than $\sim 7$ m/s [44]. This may have been true at the far downstream locations that were measured in that study, but there are definite effects from wind on the mean flow near the tunnel inlet. Strong cross-winds create horizontal flow gradients near the inlet and can cause lower overall mean velocity readings. These findings are qualitative, but are noticeable if one watches measurement readouts on a windy day. There are also day-to-day variations in freestream velocity reported in both studies in this thesis that are not correlated with air temperature. Evidence of this can be seen in Tables 5.1 and 7.1 which report average experiment freestream velocities and temperatures.

The FPF’s intended purpose is to study turbulent boundary layers, and thus no studies performed in the facility report data far into the freestream of the empty tunnel. Figure 3.2 shows the vertical inflow profile of the FPF at 5 m downstream of the inlet and turbulence management section. This profile was measured with a pitot tube at the horizontal center of the tunnel, 3 m from either side wall, with the FPF fans running at 900 RPM. It is clear from this plot that the flow does not reach a true freestream at the top of the boundary layer, and the velocity profile starts to curl back. This overshoot near the walls is likely due in part to the lack of contraction on the FPF inlet. There is a bellmouth structure on the FPF outside, but no full contraction section. The signif-
significant kink in the profile at $\sim 1.7$ m is at the same vertical location as an overlap in the turbulence management screens, i.e. 2 screens are connected at this location and there is a $\sim 1$ cm section of increased screen solidity which could be causing this velocity deficit.

![Graph showing vertical mean inflow velocity profile 5 m downstream of the FPF inlet](image)

**Figure 3.2:** Vertical mean inflow velocity profile 5 m downstream of the FPF inlet

The tunnel also produces significant noise which disturbs local residents when operating above 600 RPM, which limits all high RPM experiments to daylight hours. Most of these issues would be solved if the facility were upgraded to the phase 2 design which includes a complete tunnel enclosure, re-circulation, and temperature control. For now, we can only report these issues and do our best to work around them.
CHAPTER 4
POROUS DISK EXPERIMENTAL SETUP

4.1 Setup

The disk used was made from 1/8” Aluminum Type 3003-H14 with 3/4” holes at 1” center-to-center spacing to create 51% open area. The disk was mounted to a 4x4” fir wood block of similar dimensions as the UNH 1m research wind turbine nacelle described in chapter 6 and attached to the top of the three-tier cylindrical aluminum tower and force balance designed and built to hold the model turbine [8, 7]. The force balance located at the tower base contained a 50 lb FUTEK LSB302 load cell used to measure drag. The tower was positioned such that the plane of the disk was perpendicular to the direction of flow and located 8 meters downstream of the FPF inlet and turbulent management section. This downstream location was chosen to ensure sufficient decay of grid turbulence generated by the turbulence management section section at the inlet, but also remains close enough to the tunnel inlet to avoid any wake interactions with the naturally grown wall-bounded flows in the region near the disk. The disk was centered in both the horizontal and vertical directions, positioning the disk center 3 m from each side wall, and 1.35 m from the floor and ceiling. The load cell was calibrated using calibration weights from 10 to 40 lb (to determine sensitivity).
4.2 Procedure

Velocity measurements were acquired with 2 pitot tubes simultaneously. A reference free-stream pitot tube was positioned 3 m downstream of the FPF inlet, at 1.35 m height and 1 m from the right wall (looking in the direction of flow). The 2nd pitot tube was traversed through the flow at multiple downstream locations to create velocity profiles. The streamlined 2-D traversing system was produced by Hambleton Instruments and measures 2.4 m tall x 2.4 m wide and has an approximate traversing distance of 1.6 m in each direction. The stepper motors attached to the linear rail system are controlled through Phidget motion controllers. To move to new streamwise
locations, the entire traversing system was positioned in the streamwise direction. The traversing system and probes were positioned and aligned using laser levels and marks of known position on the tunnel floor. The alignment marks were positioned using a reference centerline created by running a string down the center of the entire tunnel. To reach desired measurement locations outside of the normal range of the traverse, the pitot tube was attached to a 0.7 m sting that could be flipped in any vertical/horizontal direction, adding another 1.4 m in addition to the traversing distance that can be traveled by the pitot tube. Pressure samples were acquired from an MKS Baratron 698A Differential Pressure Transducer. Temperature measurements were acquired simultaneously at a location 3 m behind the traverse using a k type thermocouple being amplified by an AD595C thermocouple amplifier.

![Diagram](image1.png)

(a) Front view

![Diagram](image2.png)

(b) Side view

**Figure 4.3:** Porous disk setup in FPF, with downstream measurement locations labeled
Velocity profiles were measured along the horizontal and vertical centerlines of the disk at downstream locations of $x/D=1, 2, 4, 6, 8, 10, 15, 20, 30, 40, \text{ and } 50$. The $x/D$ locations were spaced more finely near the disk location to capture the steeper streamwise gradients present in this region. The pitot tube was traversed from the floor to the ceiling for the vertical profiles, and to 1.5 meters on either side of the disk for the horizontal profiles. Inflow profiles of the same shape were recorded at 3 m upstream of the disk plane. The reference pitot tube recorded free-stream velocity concurrently with profile measurements. The FPF fans were run at a constant 900 RPM for all experiments which gives a freestream velocity of $\sim 10 \text{ m/s}$ and a Reynolds number based on disk diameter of $Re_D \approx 750,000$ (in the winter), when the experiments were conducted.

A point spacing of 10 cm and sampling time of 2 minutes per point were chosen to keep total experiment time under 2 hrs per profile while maintaining sufficient spatial resolution and acceptable uncertainty levels. The unevenly spaced points near the outside of the crosshair are due to limitations in sting placement for points exceeding the traversing distance.

**Figure 4.4:** Point spacing in FPF cross section with outline of the disk
CHAPTER 5
POROUS DISK WAKE RESULTS

Velocity measurements were performed in the wake of the 1 m diameter porous disk at 1, 2, 4, 6, 8, 10, 15, 20, 30, 40, and 50 diameters downstream of the disk plane. All experiments were performed with the tunnel fans running at a constant 900 RPM, which provides a freestream velocity of roughly 10 m/s. Table 5.1 shows the average temperature and Reynolds number based on disk diameter for each profile. Average freestream velocity and Reynolds number for all tests were 9.78 m/s and 730,000 respectively. Velocity at each point was calculated by assuming the differential pressure measurements recorded with the MKS Baratron were equal to the dynamic pressure in the flow. Density was calculated separately for each point in every profile using the average temperature over the point sampling time.

\[
U = \sqrt{\frac{2 \cdot \Delta P}{\rho}} = \sqrt{\frac{2 \cdot \Delta P}{P_{\text{atm}} \frac{R}{T}}}
\]

(5.1)

where \(P_{\text{atm}}\) is atmospheric pressure, \(R\) is the gas constant for air equal to 287.06 \(\frac{J}{\text{kgK}}\), and \(T\) is temperature measured with the thermocouple.

5.1 Drag Measurements

A drag measurement for the porous disk was needed to determine the total momentum removed from the flow, and to compare to the turbine. To isolate drag due to the disk, drag force was measured for two cases: the full configuration with disk attached, and for the tower only. Table 5.2 shows the drag experiment results. Subtracting the tower drag from the drag of disk and tower combined gives about 46 N, resulting in a drag coefficient for the disk of \(C_D = \frac{F_{\text{drag disk}} - F_{\text{drag tower}}}{\frac{1}{2} \rho A_{\text{disk}} U_{\infty}} \approx 0.97\).
Table 5.1: Average temperature, freestream velocity, and Reynolds number based on disk diameter for each porous disk experiment

<table>
<thead>
<tr>
<th>Date</th>
<th>x/D</th>
<th>Profile direction</th>
<th>Avg temp (°C)</th>
<th>Avg $U_\infty$ (m/s)</th>
<th>Avg $Re_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>29JAN18</td>
<td>01</td>
<td>Horizontal</td>
<td>0</td>
<td>9.49</td>
<td>704000</td>
</tr>
<tr>
<td>30NOV17</td>
<td>01</td>
<td>Vertical</td>
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<td>9.83</td>
<td>705000</td>
</tr>
<tr>
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<td>Horizontal</td>
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<td>Vertical</td>
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</tr>
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<td>721000</td>
</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
<tr>
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<td>9.80</td>
<td>735000</td>
</tr>
<tr>
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<td>Vertical</td>
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</tr>
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<td>731000</td>
</tr>
<tr>
<td>19JAN18</td>
<td>50</td>
<td>Vertical</td>
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<td>9.85</td>
<td>733000</td>
</tr>
</tbody>
</table>

Table 5.2: Porous disk drag measurements

<table>
<thead>
<tr>
<th>Test</th>
<th>$U_\infty$ (m/s)</th>
<th>$Re_D$</th>
<th>$F_D$ (N)</th>
<th>$C_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disk &amp; tower</td>
<td>9.77</td>
<td>717000</td>
<td>49.7</td>
<td>1.04</td>
</tr>
<tr>
<td>Tower only</td>
<td>9.67</td>
<td>710000</td>
<td>4.2</td>
<td>0.715</td>
</tr>
<tr>
<td>Disk – Tower</td>
<td>—</td>
<td>—</td>
<td>45.5</td>
<td>0.965</td>
</tr>
</tbody>
</table>

5.2 Velocity Profiles

The pitot tube, although not able to measure high frequency velocity fluctuations, can give insight into how the mean flow of the disk wake behaves. Figure 5.1 shows the mean inflow velocity 3
m upstream of the disk. The jagged shape of the vertical inflow is discussed in Chapter 3. Figure 5.2 shows raw mean velocity measured by the pitot tube at each downstream location. Velocity overshoot caused by blockage is apparent at the locations close to the disk in both vertical and horizontal profiles. A simple 1-D continuity calculation tells us that, for 5% blockage at 9.8 the velocity increase is

\[ V_2 = \frac{A_1}{A_2} V_1 = \frac{9.8\,m/s}{0.95} = 10.3\,m/s \]  

or, more generally, about a 5% increase in freestream velocity. The boundary layers on the floor and ceiling can be seen in the vertical profiles, and become more apparent with increasing downstream distance. The vertical wake is almost symmetric near the disk, but becomes less symmetric farther downstream due to interactions with the boundary layers and the tower wake. The horizontal profiles remain symmetric at all downstream locations, since the wake does not interact with the boundary layers growing on the vertical wall due to the much larger extent of the FPF test section in this direction.

![Figure 5.1: Mean vertical and horizontal inflow profiles 3 m upstream of the porous disk.](image)

The difference in horizontal and vertical profiles can be seen more clearly in Figure 5.3 where the profiles are plotted on the same axis. The horizontal and vertical profiles look identical excluding the boundary layers up to x/D=4 where they start to diverge. From \( x/D = 6-10 \) the velocity
Figure 5.2: Raw velocity data for $x/D = 1:50$. Darker plot markers correspond to measurement locations farther downstream. $z = y = 0$ is the location of the disk centerpoint, positive $z$ is towards the right of the FPF in the flow direction, positive $y$ is towards the ceiling.

deficit in the vertical profiles is greater than that of the horizontal. The deficit in the negative $y$ direction, closer to the floor, is more substantial than the positive which is likely due to the influence of the tower wake. The centerline velocity, being measured at the same physical point in space in both cases, is the same for both vertical and horizontal profiles. From $x/D = 15$ onward, the profiles look similar except for the boundary layers growing into the vertical wake profiles, a presence which grows larger with increasing downstream distance. At $x/D = 50$, the centerline velocity has recovered to 90% of the freestream velocity.

The wake half-width is a useful quantity for quantifying wake shape and spreading. The half-width, $\delta_{1/2}$, is defined as the radial location at which the velocity deficit is half of the centerline velocity deficit, $U_o$. Figure 5.4 shows an elliptical fit to the horizontal and vertical wake half-widths. This helps to visualize the shape of the wake and how it evolves downstream. The vertical wake does not reach half of the centerline velocity deficit past $x/D = 20$, so there are no half-width plots for further downstream locations. The general trend of the half-width plots is an elongation of in the vertical direction and an overall downward shift. The vertical elongation is somewhat counter intuitive. One would think the wake would spread in the horizontal direction where it has
more room to grow. The actual shape is due to the boundary layers having an impeding effect on the wake, causing the velocity to be slower overall in the vertical profiles. The half-width then grows outward faster than the unimpeded horizontal profiles which gives the appearance of wake spreading and leads to the shape shown in Figure 5.4.

**Figure 5.3:** Horizontal and vertical porous disk wake profiles plotted on the same figure
Figure 5.4: A visualization of the disk wake halfwidth evolution. The dashed line is the outline of the porous disk (D = 1m) and the rectangular plot boundaries represent the walls of the test section cross section (to scale). The circles are the actual half-width values, and the solid line is an elliptical fit to the 4 half-width points.

5.3 Similarity Scalings

The similarity solutions discussed in Chapter 2 can be applied to the mean velocity profiles measured downstream of the porous disk. Figure 5.5 shows the horizontal and vertical velocity profiles
normalized by the similarity variables $\delta^*$, the wake displacement thickness, and $U_o$, the centerline velocity deficit. The horizontal profiles appear to collapse well for $x/D = 4$ and on. Other studies of the axisymmetric turbulent wake, discussed in Chapter 2, have reported similarity in the mean velocity profiles at much further downstream locations [3, 5, 15, 17]. The fact that wake profiles in this study collapse from $x/D = 4$ on is attributed mainly to the much higher Reynolds number (see Table 2.1). The vertical wake also appears to start collapsing well after $x/D = 4$ for the same reasons as the horizontal wake. However, the profiles start to collapse less well as the boundary layers start to grow into the wake. The similarity scaling fails to make the vertical profiles collapse after about $x/D = 20$.

**Figure 5.5:** Mean velocity deficit profiles normalized by displacement thickness $\delta^*$ and centerline velocity deficit $U_o$

Figure 5.6 shows the same profiles but only for the downstream location where they collapse reasonably well, i.e., $x/D = 4 - 50$ in the horizontal direction and $x/D = 4 - 50$ in the vertical direction. From this result we conclude that any measurements past $x/D \approx 20$ are not representative of the unconstrained axisymmetric wake due to the interference of the boundary layers on the floor and ceiling.
(a) Horizontal Similarity Profiles for $x/D = 4 - 50$  
(b) Vertical Similarity Profiles for $x/D = 4 - 20$

**Figure 5.6:** Similarity profiles at downstream locations where there is reasonable collapse

The derivation of the similarity solution for the axisymmetric wake indicates that certain quantities should grow or decay with downstream distance at known rates. Specifically, the centerline velocity deficit $U_o$ should decay with $x^{-2/3}$ and the displacement thickness $\delta^*$ should grow with $x^{1/3}$ where $x$ is downstream distance. Figure 5.7 shows the evolution of the mean velocity deficit with downstream distance from the disk. The centerline velocity deficit appears to decay faster than the $x^{-2/3}$ decay rate up until $x/D = 8$ where the slope starts to match the expected rate.

Figure 5.8 shows the displacement thickness $\delta^*$, which is representative of wake width, plotted against downstream distance for all four sides of the wake that were measured. None of the $\delta^*$ values appear to follow the $x^{1/3}$ slope until $x/D = 4$. The $\delta^*$ values calculated from horizontal profiles appear to follow the expected trend for all downstream locations past $x/D = 4$, while the vertical $\delta^*$ values start to diverge beyond $x/D = 20$. This agrees with the regions of collapse for the similarity velocity profiles presented earlier.

The theory discussed in Chapter 2 also states that the total rate of transfer of kinematic linear momentum, $M_o$, should be equal to its source value which is equal to the net drag imparted by the wake generator. The pitot tube cannot measure azimuthal velocity or any turbulent velocity fluctuations, so the momentum integral was only calculated to first order which reduces the momentum
Figure 5.7: Velocity deficit normalized by freestream velocity at each downstream location normalized by momentum thickness $\theta^*$. Integral equation to

$$M_o = 2\pi \int_0^\infty U_\infty (U - U_\infty) r dr$$

Figure 5.9 (a) shows how the value of the momentum integral changes with how far out the integration is performed in the $z$ direction. It is clear that the integral reaches a constant value outside of the wake, so the furthest $z$ position measured was used as the integration limit. Figure 5.9 (b) shows the value of $M_o$ for each downstream location, multiplied by density to give it units of force. The locations close to the disk show a steep gradient with increasing downstream distance, contrary to the theory which states it should be constant at any $x$ position. This is most likely due to the significant streamwise pressure gradient present in this region. The momentum appears to be leveling out and approaching a constant value which is lower than the drag imparted by the disk which was measured to be $\sim 45 \text{ N}$. One reason for this is the lack of turbulence quantities in the integral. Although these are second order, they become more important at further $x/D$ locations as the mean velocity deficit decays. Another reason for the discrepancies far downstream is that the wake is not completely axisymmetric, and becomes less axisymmetric with increasing
Figure 5.8: Displacement thickness for each horizontal and vertical profile at each downstream location, all normalized by momentum thickness $\theta^*$. $+z$ is the horizontal quarter wake profile in the direction of the right wall (in the direction of flow) $+y$ is wall normal quarter wake profile in the direction of the tunnel ceiling.

downstream distance. This momentum integral assumes a circular wake which is not the case at these far downstream locations.
(a) Dependence of $M_0$ on integration limit

(b) Streamwise momentum of the porous disk at each $x/D$ location

**Figure 5.9:** Streamwise momentum calculated with equation 5.3 and multiplied by air density. The left plot shows how the value of $M_0$ changes when integrated to different values of $z$. Right is the value of $M_0$ for all $x/D$
CHAPTER 6

SCALE MODEL WIND TURBINE EXPERIMENTAL SETUP

6.1 Model Turbine

The model turbine used for wake measurements was a 1m diameter scale model research wind turbine designed and built at UNH [8, 7]. The turbine was scaled from the NREL 5 MW reference turbine [20]. It was designed for an optimal tip-speed ratio (TSR) of 7 using the NREL S801 low Reynolds number airfoil. The chord was then stretched by a factor of 1.35 to increase $Re_C$ which brought down the optimal TSR to about 6. The blockage ratio based on swept area of the turbine was 4.8% which is below the suggested limit of 5% for wind turbine studies, above which the power and thrust coefficients are affected when running near optimal TSR [6].

Figure 6.1: Model turbine shown in FPF at 8 meters downstream of FPF inlet with traversing system in rear
Figure 6.2 shows the power and thrust coefficients of the model turbine vs tip-speed ratio. These performance curves are preliminary and do not take any bearing torque into consideration 1.

Figure 6.2: Power (left) and thrust (right) coefficient vs TSR for the model turbine. Power coefficients have not been corrected for tare torque. [7]

6.2 Setup

The wind turbine was placed in the same location as the porous disk, at 8 meters downstream of the FPF inlet and turbulent management section, centered in the vertical and horizontal directions. This gave the turbine a hub height of 1.35 meters, and positioned the center of the nacelle 3 meters from either side wall. Figure 6.3 shows a diagram of the turbine with coordinates defined and an example ideal wake profile. The Cartesian $x, y, z$ coordinates represent location in the FPF test section with the origin being at the center of the turbine hub. The wake cylindrical coordinates $r, \theta$ are shown in the example wake diagram. For the results in this study, $z$ was used as the horizontal coordinate when plotting wake profiles as the profiles pass through zero. The turbine rotates clockwise when looking in flow direction.

1 waiting for replacement torque sensor
Figure 6.3: Diagram of the model wind turbine with tunnel coordinates \( x, y, \) and \( z \) as well as wake coordinates \( \theta \) and \( r \). Also shown is an example wake profile with a streamwise velocity deficit profile and swirl profile. \( \Omega \) is turbine rotation rate.

Velocity profiles were measured using a pitot tube and 2-D X-wire anemometer simultaneously. The pitot tube and hot-wire were placed on separate stings so that they were at the same height and streamwise location, but separated by 20 cm horizontally. The X-wire and pitot tube stings were mounted to the 2.4 m x 2.4 m traversing system as shown in figure 6.4. This setup was chosen to help validate hot-wire velocity measurements with a pitot tube, and to allow for mid-profile freestream re-calibration of the X-wire. A reference pitot tube was also placed 3 m upstream of the wind turbine, 1.35 m off the floor and 1 m from the left wall of the test section (when looking in the direction of the flow). Temperature measurements were acquired simultaneously at a location 3 m behind the traverse using a k type thermocouple being amplified by an AD595C thermocouple amplifier. Pitot tube pressure samples were acquired from a MKS Baratron 698A Differential Pressure Transducer. All data was acquired through a Data Translation DT936 A/D DAQ device via BNC connectors.

X-wire probes used in the wind turbine experiments were constructed in house using 5 \( \mu \)m diameter tungsten wire. Ends of the wire are coated with copper so that they can be soldered to
Figure 6.4: Traversing system shown with pitot tube (right) and X-wire (left) attached

the probe prongs. The copper plating leaves a 1 mm active sensing region for each wire. Two Dantec 54T42 MiciCTAs set to an overheat ratio of 0.7 were used for X-wire measurements. The overheat ratio is defined by Dantec as \( \frac{R_W - R_o}{R_o} \) where \( R_W \) is the wire hot resistance and \( R_o \) is the wire resistance at ambient temperature. The hot-wires were calibrated before each profile using an articulating jet, which has the ability to adjust the mean flow +/- 30° in any direction. The calibration unit is capable of adjusting the jet velocity in increments of 0.01 m/s up to 15 m/s. The sensor used here was calibrated at velocities of 2, 4, 6, 8, 10, and 11 m/s and inflow angles of +30° to -30° in 15° increments. This covers the range of speeds and inflow angles that the sensor will encounter during the experiment. The calibration jet can be seen in Figure 6.5.

Hot-wire calibrations were applied using the method described in Henbest 2016 [14]. Pre-experiment calibrations were performed using the calibration jet which creates calibration surfaces for each wire. Subsequent calibration points, as described in section 6.3, were acquired by travers-
ing the hotwire and pitot tube into the freestream with the turbine stopped and stepping through a range of freestream velocities. This method helps to correct for hot-wire drift due to changes in tunnel temperature and other potential sources without having to perform the lengthy jet calibration. The largest difference in experiment start and end temperatures was 3 °C, all experiment temperatures can be seen in table 7.1.

Figure 6.5: Hot-wire calibration unit which was designed and built at the University of New Hampshire specifically for calibration of multi-wire hot-wire sensors. The jet nozzle can be articulated +/- 30° in any direction with a fan controlled outlet velocity up to 15 m/s which is accurate to .01 m/s.

6.3 Procedure

The size of the traversing system limited horizontal travel to about 1.6 m without manually moving the system itself. With this limitation it was only feasible to measure half of the horizontal wake at each location. Point spacing was reduced to 5 cm and sampling time was increased to 200
seconds per point to increase spatial resolution and reduce uncertainty in the mean and turbulence quantities. Point spacing can be seen in figure 6.6. At x/D locations 15 and 20, the traversing system was manually moved and realigned mid profile to make sure the hotwire profiles reached the freestream. The X-wire was re calibrated upon moving to the new location. Samples were recorded at 5 kHz with the built in Dantec MiniCTA 3 kHz filter applied to help reduce high frequency electronic noise.

**Figure 6.6:** Hotwire measuring points shown in the tunnel cross section with the outline of the turbine

The X-wire was calibrated using the 2-D jet before all experiments. For x/D=1-10, the X-wire was re-calibrated in the freestream using the pitot tube before, midway through, and after the profiles. This corresponded to calibrations times of $t = 0, 60, \text{ and } 120 \text{ minutes.}$ For x/D=15 and 20, the X-wire was re-calibrated in the freestream before the profile, before moving the traverse to a further radial location, and after moving the traverse. The freestream calibrations consisted of 8 freestream velocities corresponding to 8 fan RPMs. The RPM values used in this experiment were 0, 200, 400, 550, 700, 900, 1000, and 1100 RPM, corresponding to nominal velocities of 0, 2.2, 4.5, 5.6, 7.8, 10, 11.2, and 12.3 m/s. These freestream speeds provided calibration data above and below the range of voltages that the hotwire would experience during experiments. Care was taken to ensure the tunnel reached a true 0 velocity before starting the re-calibration. The mid-profile calibration used in x/D 1-10 was performed in an attempt to minimize the effect of thermal drift on
the hotwires, but for most profiles it was found to be unneeded as the before and after calibrations
provided sufficient correction. Calibrations were applied using the method described in Henbest 2016 [14]. The method first uses the full 2-D jet calibration to create a non-linear least-squares
surface fit to the following equation

$$U_\infty = \frac{a_0 + a_1 E + a_2 E^2 + a_3 E^3}{\sqrt{\cos^2(\Psi_e + \theta) + k^2 \sin^2(\Psi_e + \theta)}}$$  \hspace{1cm} (6.1)

where $U_\infty$ is calibration velocity magnitude, $E$ is hotwire voltage, $a_{0-3}$ are cubic coefficients, $\Psi_e$ is the effective wire angle, $k$ is a longitudinal cooling coefficient, and $\theta$ is the calibration velocity angle. This equation is derived by setting the effective cooling velocity equal to a physically based mathematical model (denominator of equation 6.1) and a 3rd order polynomial with the hotwire voltage as the dependent variable (numerator).

![Figure 6.7: Hotwire calibration surfaces created using a non-linear least-squares surface fit to equation 6.1. Actual calibration points are shown as spherical points.](image)

Once a calibration surface is created for each wire, the subsequent calibrations are used to shift the surface by refitting the 3rd order polynomial, but leaving the other variables, $\Psi_e$ and $k$ the same as they are physical properties of the wire. Calibrations are then applied to the data by using the analytical solution to the system equations given by the individual cooling velocities of each wire.

$$U_{e1} = U_m \sqrt{\cos^2(\Psi_{e1} + \theta) + k_1^2 \sin^2(\Psi_{e1} + \theta)}$$  \hspace{1cm} (6.2)
\begin{equation}
U_{e2} = U_m \sqrt{\cos^2(\Psi_{e2} + \theta) + k_2^2 \sin^2(\Psi_{e2} + \theta)}
\end{equation}

Solving these equations with the velocity magnitude \(U_m\) and angle \(\theta\) as the unknowns allows the equations to be solved with a known analytical solution. Calibrations at the beginning, middle, and end of each profile were then used as points for a linear interpolation. Figure 6.8 shows an example of how the calibration points were interpolated. The interpolation flattens out the freestream of the wake to a constant velocity which is a good check that the process is working correctly.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6.8.png}
\caption{Example of hotwire calibration interpolation at \(x/D = 2\). Linear interpolation is applied between 3 reference calibration points to form the blue curve.}
\end{figure}
CHAPTER 7
SCALE MODEL WIND TURBINE WAKE RESULTS

Velocity measurements were performed in the wake of the model wind turbine at 1, 2, 4, 6, 8, 10, 15, and 20 diameters downstream of the rotor plane. All experiments were performed with a tunnel free stream velocity of nominally 10 m/s. The tunnel fans were run at a constant 900 RPM which is controlled by the variable frequency drives (VFDs). Table 7.1 summarizes the conditions of each experiment. An active TSR control was attempted, but never functioned reliably due to noise in velocity data caused by the turbine motor drive. For this reason, all measurements were performed with the turbine running at a constant RPM corresponding to a tip-speed ratio of 6.1, which was found during performance tests to be the optimal tip-speed ratio. This tip-speed ratio corresponded to a power coefficient $C_P = 0.33$ (before applying any tare torque correction) and a thrust coefficient $C_T = 1.05$. Maximum variations in freestream velocity were on the order of $\pm 0.15$ m/s over the entire length of a profile, which corresponds to an accuracy in the TSR over a profile of $\lambda = 6.1 \pm 0.1$. Performance tests also determined that the turbine reached a Reynolds number independent region at speeds greater than 9 m/s [7].
Table 7.1: Average temperature, freestream velocity, and Reynolds number based on turbine diameter for each turbine experiment

<table>
<thead>
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<th>Date</th>
<th>x/D</th>
<th>T&lt;sub&gt;start&lt;/sub&gt; (°C)</th>
<th>T&lt;sub&gt;end&lt;/sub&gt; (°C)</th>
<th>Avg temp (°C)</th>
<th>Avg U&lt;sub&gt;∞&lt;/sub&gt; (m/s)</th>
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<td>4.6</td>
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<td>718000</td>
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<td>06</td>
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<td>8.2</td>
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<td>9.62</td>
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</tr>
</tbody>
</table>

7.1 Hotwire Velocity Profiles

Horizontal inflow was measured with the homemade X-wire and a dantec gold plated single wire probe. Figure 7.1 shows the mean velocity and turbulence intensity measured with the single wire. The pitot tube was also attached during these measurements, and the mean velocity agrees well between the two profiles. Figure 7.2 shows the mean inflow measured with the X-wire. The streamwise velocity agrees well with between pitot tube and X-wire, and the azimuthal velocity is zero at all points within 1% of the freestream velocity. Mean streamwise velocity data is shown in Figure 7.3. In all figures, the blade tips extend to 0.5 m. The streamwise profiles appear to remain symmetric about the rotor center-line z = 0. The profile at x/D = 1 shows a noticeable kink near the center of the wake, potentially due to the space near the blade root where the blade transitions from a circular cross-section to an airfoil. There is also likely an effect from the motor/nacelle assembly which protrudes 0.45 m behind the rotor plane. At x/D = 2, the center-line velocity deficit has not changed significantly, but the profile has smoothed out and the expected wake shape starts to form. Profiles from x/D = 1-4 show significant speedup in the freestream due to blockage, but this decays quickly with downstream distance. Note that although the blockage in this experiment is sufficiently low for wind turbine experiments, it is much higher than other fundamental studies of the axisymmetric wake. For example, Cannon 1991 had a blockage ratio of 0.15%, and Johansson

47
2003 & 2006 had blockage ratios of 0.03% and 0.027% respectively [4, 19, 18]. The 4.8% blockage in these experiments is orders of magnitude higher than these past studies, so there is a non-negligible streamwise pressure gradient near the rotor plane location. Azimuthal velocity profiles

(a) Mean velocity

(b) Turbulence intensity (%)

Figure 7.1: Horizontal inflow profiles measured with a Dantec gold plated single wire anemometer. Left is mean velocity measured with the hotwire and pitot tube, and right is turbulence intensity $u'/U_\infty$.

(a) Streamwise velocity

(b) Azimuthal velocity

Figure 7.2: Horizontal inflow profiles measured with the homemade X-wire probe. Left is mean streamwise velocity measured with the hotwire and pitot tube, and right is azimuthal velocity.
are shown in Figure 7.4. High peaks at x/D = 1 & 2 are likely due to vortices formed at the rotor hub and blade tip. These peaks decay quickly, however, and a mean swirl distribution forms. Mean swirl is induced as the oncoming flow is deflected by the rotor blades and rotates opposite of the rotor. There is also a significant shift of the center of swirl. In a true axisymmetric swirling wake, the zero crossing of the swirl velocity would occur at z = 0, the center of the turbine. In this data this is a clear shift of the zero crossing in the + z direction. By x/D = 6 we can see the negative swirl that would, in an ideal situation, be present only on the negative z side. At x/D = 20, the swirl zero crossing has shifted to approximately z = 0.75m, or z/R = 1.5. Potential sources for the shifting swirl include a small yaw angle of the turbine, tower effects, and boundary layer effects. It is unlikely that a yaw angle is responsible as the turbine was laser aligned to zero yaw within \( \pm 0.6^\circ \). It is possible that the tower is generating swirl through vortex shedding, or that it is somehow impeding the swirl generation in one direction and causing this shift. Note that even at peak values, the swirl is an order of magnitude smaller than the mean stream-wise velocity. Figures 7.5 and 7.6 show the streamwise and azimuthal velocity profiles plotted individually.

Stream-wise and azimuthal velocity fluctuation (RMS) profiles are shown in figures 7.7 and 7.8 respectively. There are clear peaks in both fluctuation components at a z location just outside the blade radius at x/D = 1 & 2 due to the tip vortex. The \( u_{rms} \) peak decreases in magnitude from x/D = 1 - 2, while the \( w_{rms} \) peak increases. As downstream distance increases, the turbulence spreads in the expected manner and by x/D = 8 both components have the same shape and similar magnitude. The azimuthal fluctuations are of the same order as the mean azimuthal velocity. The Reynolds stress \( \overline{uw} \) is shown in figure 7.9. The tip and hub vortices are visible at x/D = 1 & 2, but subsequent downstream locations show a mean negative value, indicating that momentum is being transported downward. This could be an effect of the overall downward shift of the wake due to the tower wake. Similar to the mean azimuthal velocity, in a truly axisymmetric swirling wake there should be a zero crossing of the Reynolds stress \( \overline{uw} \) at z = 0. This is present at x/D = 1 & 2, but is not seen in the further downstream locations, and there is a mean negative \( \overline{uw} \) profile that spans the whole measurement range except where it declines to zero in the freestream.
**Figure 7.3:** Mean streamwise velocity profiles in the wake of the model turbine obtained using the X-wire anemometer sensor. \( z \) represents horizontal distance from the turbine centerline.

**Figure 7.4:** Mean azimuthal velocity profiles in the wake of the model turbine obtained using the X-wire anemometer sensor. \( z \) represents horizontal distance from the turbine centerline. A solid line at \( W=0 \) is shown to help illustrate the shifting swirl component.
Figure 7.10 shows the power spectral density of the turbine wake at the blade tip \((z/R = 1)\) for each downstream distance. Spectra were computed using Welch’s method with a Hanning window. The rotor rotational frequency and multiples of it are seen clearly at \(x/D = 1 - 2\) due to tip vortices being shed at frequencies associated with blade passage, but seem to completely die down by \(x/D = 4\). From this location onward, there seems to be a low frequency peak that arises around 1.5-2 Hz. This frequency is close to the vortex shedding frequency of a disk at the same size. If the Strouhal number were approximately 0.18, the shedding frequency for a 1 m disk at a freestream velocity of 9.8 m/s would be 1.76 Hz, which is where this peak appears to form. The data is not completely conclusive, but if this is the case, one could reason that the wake forgets its origins and begins to behave like that of a typical bluff body.
Figure 7.5: Turbine wake mean streamwise and azimuthal velocity for $x/D = 1$, 2, 4, and 6
Figure 7.6: Turbine wake mean streamwise and azimuthal velocity for $x/D = 8, 10, 15, \text{ and } 20$
Figure 7.7: Streamwise RMS velocity profiles measured with the X-wire

Figure 7.8: Azimuthal RMS velocity profiles measured with the X-wire
Figure 7.9: Reynolds stress $\overline{uw}$ measured with the X-wire
Figure 7.10: Turbine wake spectra at z/R = 1. Black dashed lines are multiples of rotor frequency. Red dashed line is the approximate Strouhal shedding frequency for a disk with the same diameter.
### 7.2 Comparison of Hotwire and Pitot Tube Data

A comparison of hotwire and pitot tube velocity profiles is shown in Figure 7.11. There are significant differences in velocity magnitude near the center of the wake at $x/D = 2$ & 4. The profiles match well from $x/D=6$ on. The discrepancy at the locations close to the turbine are likely due to high turbulence intensity values near the center of the wake where the mean velocity is low. Figure 7.12 shows the turbulence intensity of $x/D = 1 - 4$ compared with the percent difference in pitot tube and hotwire velocity at the same points. There is a definite correlation between turbulence intensity and pitot/hotwire discrepancy. The pitot tube is also less able to measure steep flow gradients which are present at locations closest to the rotor plane. Further downstream, where the velocity deficit has recovered and the turbulence has decayed, the turbulence intensity values are much lower and there is minimal difference in the pitot tube and hotwire profiles.
Figure 7.11: Comparison of pitot tube and hotwire stream-wise velocity profiles
Figure 7.12: Comparison of the difference between pitot tube and hotwire profiles and turbulence intensity at each point
7.3 Similarity Scalings

George 1995 determined that high Reynolds number similarity solution for free shear flows only applies if a clear inertial subrange appears in the power spectrum. This is evident for local Reynolds numbers \( Re_{\delta^*} = \frac{U_0 \delta^*}{\nu} \) greater than 1600 [12]. When \( Re_{\delta^*} = 400 \) the existence of the inertial subrange becomes questionable and by \( Re_{\delta^*} = 200 \) it no longer exists [19] [12]. It is clear from Figure 7.13 that the local Reynolds number is much higher than the threshold of 1600. Figure 7.14 shows that sample spectra taken at multiple radial locations appear to follow a -5/3 slope which indicates the existence of at least 2 decades of inertial subrange. Along with this, it must be shown that the turbulence intensity ratios \( \frac{u'}{U_o} \) and \( \frac{w'}{U_o} \) reach a constant value in order for the similarity theory to be valid. This is difficult to conclude due to the fact that these experiments only span downstream distances which are comparable to the beginning of the traditional "far wake" region. Turbulence intensity ratios are shown in Figure 7.15 and Figure 7.16 and do appear to be trending toward a constant value, but it is unclear if that value is reached in these experiments. This implies that while similarity for the first order moments (mean velocity) is observed a short distance downstream of the wake generator, it will take until further downstream for higher order moments to achieve similarity.

Figure 7.17 shows the streamwise velocity profiles normalized by the freestream velocity. This gives a visual representation of the mean velocity deficit in the wake of the rotor. Figure 7.18 shows the same data normalized by the velocity difference between freestream and centerline \( (U_o) \) and \( \delta^* \), the scaling parameters derived from the similarity solution in 2. The profiles appear to collapse well from \( x/D = 4 \) and on, which is similar to the results of the porous disk measurements. All similarity profiles are shown in Figure 7.18 (a) and only those with good collapse are shown in Figure 7.18 (b). The similarity scaling shows more scatter at further downstream locations, but this is a result of significantly smaller centerline velocity deficit which emphasizes small variations when scaled with this parameter.
Figure 7.13: Local Reynolds number \((Re_{\delta^*} = \frac{U_o \delta^*}{\nu})\) vs downstream distance normalized by momentum thickness \(\theta^*\)

Figure 7.19 shows the mean azimuthal velocity normalized by \(U_o^{3/2}\) which should decay as \(x^{-1}\), the same decay rate predicted for the mean swirl. From \(x/D = 4\) and on, the similarity scaling causes the mean swirl less than zero to collapse well, but the positive swirl peak does not collapse after \(x/D = 10\). Even though the wake is not actually axisymmetric in the azimuthal velocity, there is still evidence that the mean swirl decays as predicted. Note that the governing equations do not preclude the streamwise and swirling wake becoming decoupled to first order, as shown in Appendix B. To show that the mean swirl collapses quite well in an intermediate region even though it is no longer axisymmetric in the defined geometry, the scaled \(W\) profiles for \(x/D = 6, 8, 10\) only are plotted in Figure 7.19 (b). The collapse is quite remarkable especially when comparing to figure 7.4, providing further evidence that the \(W \sim x^{-1}\) scaling works. To study the true axisymmetric wake with rotation, a study must be performed where there is no interference with the swirling wake from things like the tower or boundary layers.

Streamwise velocity fluctuations scaled with \(U_o\) are shown in Figure 7.20. The profiles do not appear to collapse in this region of \(x/D\), except for the last 2 locations. This result makes
sense given that the maximum streamwise velocity fluctuations normalized by centerline velocity
deficit have not yet reached a constant value, as shown in Figure 7.15. This result is not surprising
considering the disk wake of Johansson 2006 did not show collapse in the velocity fluctuations
until about x/D = 40, which is also approximately the distance at which the normalized turbulence
intensity reached a constant value. [18]. The normalized azimuthal fluctuations shown in figure
7.21 show virtually the same trend as the streamwise values, with no clear collapse until x/D=15
and on.
Figure 7.14: Sample streamwise spectra where $x/D = 1$ is shown in blue, $x/D = 2$ is shown in cyan, $x/D = 4$ is shown in green, and $x/D = 6$ is shown in red. The spectra follow a $-5/3$ slope shown by the solid black line, which shows the existence of an inertial subrange.
Figure 7.15: Maximum streamwise velocity fluctuations at each downstream location divided by the corresponding centerline velocity deficit

Figure 7.16: Maximum azimuthal velocity fluctuations at each downstream location divided by the corresponding centerline velocity deficit
Figure 7.17: Streamwise velocity normalized by the freestream velocity ($U_\infty$). $R$ represents the turbine radius (0.5 m) and $z$ is the horizontal position.

(a) $x/D = 1-20$

(b) $x/D = 4-20$

Figure 7.18: Streamwise velocity normalized by scaling parameters $U_0$ and $\delta^*$. The plot on the right shows only the downstream locations where there is reasonable collapse.
Figure 7.19: Azimuthal velocity normalized by scaling parameters $U_o^{3/2}$ and $\delta^*$. The plot on the right shows only the downstream locations where there is reasonable collapse.

Figure 7.20: Streamwise velocity fluctuations normalized by scaling parameters $U_o$ and $\delta^*$
Figure 7.21: Streamwise velocity fluctuations normalized by scaling parameters $U_o$ and $\delta^*$
Figure 7.22 shows the centerline velocity deficit vs downstream distance. The similarity theory predicts that this should decay as $x^{-2/3}$ and from this plot it looks like the velocity deficit is reaching this decay rate after some initial rearranging of the wake. Measurements at farther downstream distances would help to confirm this, but as was shown with the porous disk experiments, the boundary layers begin to significantly affect the wake past $x/D = 20$ so we cannot be completely confident that the wake at these further locations is representative of the undisturbed axisymmetric wake.

The downstream evolution of the displacement thickness $\delta^*$ is shown in Figure 7.23. This quantity should scale with $x^{1/3}$, and again it appears that this growth rate is found after $x/D = 4$. It is possible that the blockage in this experiment is affecting these integral quantities at locations close to the turbine rotor, as they reference a free stream velocity $U_\infty$ which is not constant in this region. All quantities were calculated using the local freestream velocity, as opposed to the upstream reference velocity.

Figure 7.24 shows the maximum swirl vs downstream distance and its predicted decay rate, $x^{-1}$. There does seem to be some agreement between the curves, which is reassuring considering the swirl shifted off-axis significantly during these experiments.

The decay rates presented are derived from the infinite Reynolds number (inviscid) similarity solution of the reduced order equations. The similarity solution only becomes valid as the wake becomes independent of its origins. This eliminates the use of any Reynolds number associated with the wake generator in the verification of the inviscid assumption. The local Reynolds number was calculated at various downstream locations and is shown in Figure 7.13.

The theory for the axisymmetric wake with swirl discussed in Chapter 2 states that the total rate of transfer of kinematic linear momentum, $M_o$, should be equal to its source value which is equal to the net drag imparted by the wake generator, and the rate at which kinematic angular momentum is swept across any downstream plane, $G_o$, should remain constant as well. The full momentum integral was used to calculate $M_o$ for the turbine wake, with the exception of the $\overline{v^2}$ term which was not measured. Figure 7.25 (a) shows how the value of the momentum integral changes with how
far out the integration is performed in the $z$ direction. It is clear that the integral reaches a constant value outside of the wake, so, like the disk wake, the furthest $z$ position measured was used as the integration limit for all profiles. Figure 7.25 (b) shows the value of $M_o$ for each downstream location, multiplied by density to give it units of force. Like the disk measurements, there is a significant effect of blockage at locations close to the turbine, but in this case the reaches a clear constant value of about 40 $\text{N}$, just under the drag measured for the turbine which was $\sim 47 \text{ N}$.
Figure 7.24: Maximum azimuthal velocity vs downstream distance normalized by momentum thickness $\theta^*$ on a linear scale (left) and log scale (right).

Figure 7.26 (a) shows how the value of the swirl integral changes with how far out the integration is performed in the $z$ direction. The integration start point was chosen as the $z$ location of the zero crossing in the mean azimuthal velocity, which, as shown in Figures 7.5 & 7.6, is not always the center of the streamwise velocity profile. Figure 7.26 (b) shows the value of $G_o$ for each downstream location, multiplied by density to give it units of torque. Like the streamwise momentum, the values of $G_o$ close to the turbine change significantly with downstream distance, but $G_o$ appears to reach a relatively constant value by $x/D = 6$. 
Figure 7.25: Streamwise momentum of the turbine multiplied by air density. The left plot shows how the value of $M_o$ changes when integrated to different values of $z$. Right is the value of $M_o$ for all $x/D$.

Figure 7.26: Angular momentum of the turbine multiplied by air density. The left plot shows how the value of $G_o$ changes when integrated to different values of $z$. Right is the value of $G_o$ for all $x/D$. 

7.4 Comparison of wind turbine and disk wakes

The dimensionless mean velocity profiles of the wind turbine wake and porous disk wake are compared in Figure 7.27. Though we expect the wakes to have different growth rates due to different initial conditions, the wake generators are the same size and have similar drag coefficients so we there should be some similarities. The first two downstream locations are significantly different in shape and magnitude, but by \( x/D = 4 \) they have smoothed out to a similar shape. From \( x/D = 6 \) and on they are almost identical in shape.

The comparison in wake shape and evolution can also be seen in Figure 7.28 where the velocity profiles are normalized by similarity variables \( U_o \) and \( \delta^* \). Profiles are only plotted from \( x/D = 4-20 \), the locations of good collapse determined earlier. The disk and turbine wakes actually collapse well which indicates that the wakes are indeed growing at a similar rate. More data at further downstream locations would be needed to see if this agreement continues. This comparison does, however, show that a this disk models the mean wake of this model turbine quite well, and could be used as an experimental wake generator model replacement for a turbine. This is especially true for \( x/D \) locations of 6-10 which covers the range of typical wind farm spacing. The wakes are expected to be somewhat similar as the generators have very similar drag coefficients; 0.97 for the porous disk and 1.05 for the turbine, but even objects with the same drag have been shown to have wakes that evolve differently, like in Cannon 1991 [4].
Figure 7.27: Normalized mean velocity profiles of the turbine compared to the porous disk
Figure 7.28: Velocity profiles of the wind turbine and porous disk normalized by similarity scaling variables $U_o$ and $\delta^*$ from $x/D = 4 - 20$. Black triangles turbine points and blue crosses are disk points.

This data produced in this study is useful for wind turbine validation purposes and provides some insight into the swirling wake behavior, but fails to capture the true swirling axisymmetric wake due to tower interference and boundary layer growth on the wind tunnel floor and ceiling. To study the fundamental axisymmetric wake with swirl, the tower and boundary layer effects would have to be mitigated. To do this in the FPF, a wake generator of slightly smaller diameter would allow for much farther downstream measurements without boundary layer effects. This model would have to be supported by a structure with minimal effect on the flow, potentially a very thin, streamlined tower. With these conditions met, a sufficiently high Reynolds number could be reached and the far wake could be measured accurately.
High quality wind turbine wake validation data is needed across a range of scales, from full scale to the wind tunnel scale. The wind tunnel scale has the advantage of controlled inflow conditions which are essential for creating general models. The goal is to create models of full wind farms, but to get there a model must first be able to accurately recreate the physics present in the wake of a single wind turbine. The data collected here can be used as benchmark data for wind turbine simulations. The wake of a 1 meter diameter scale wind turbine modeled after the NREL 5 MW offshore reference turbine was experimentally characterized using an X-wire hot-wire anemometer up to 20 diameters downstream. The wake of a 1 m diameter porous disk was also measured in both the horizontal and vertical directions up to 50 diameters downstream with the goal of observing the effect of a rectangular cross section on a circular wake. Both experiments were conducted in the test section of the UNH Flow Physics Facility with a 6 m x 2.7 m cross section and 72 m long test section.

The disk wake was shown to elongate in the vertical direction and shift downward. The vertical elongation is thought to be an effect of the boundary layers growing on the floor and ceiling which impede the recovery of the wake in that direction. The downward shift is likely due to the wake of the tower supporting the disk merging with the wake of the disk itself. The disk wake adheres to similarity solutions in the horizontal direction from $x/D = 4-50$, while the solutions only apply to the vertical direction from $x/D = 4-20$ at which point the boundary layers begin to have significant effects on the vertical wake. The disk wake guided how far downstream velocity profiles should be measured for the turbine.

The turbine wake was shown to stay axisymmetric in the streamwise velocity, but the azimuthal velocity shifts in horizontal direction. Streamwise velocity profiles collapsed well in classical
similarity coordinates, and azimuthal profiles showed reasonable collapse with the $W \sim x^{-1} \sim U_o^{2/3}$ scaling at $x/D$ locations of 6, 8, and 10 even with the horizontal shift. The experiments performed here can help guide future wind turbine experiments and the data can be used to aid in the improvement of wind farm design.


APPENDIX A

EXPANDED UNCERTAINTY ANALYSIS

The uncertainty analysis carried out here will follow the Taylor series method for propagation of uncertainties which is defined in Coleman and Steele 2009 [9]. For the case where the result variable $r$ is a function of $J$ variables, the standard uncertainty $u_r$ is given as

$$u_r^2 = \sum_{i=1}^{J} \left( \frac{\partial r}{\partial X_i} \right)^2 b_{X_i}^2 + \sum_{i=1}^{J} \left( \frac{\partial r}{\partial X_i} \right)^2 s_{X_i}^2$$  \hspace{1cm} (A.1)

Where $b_{X_i}$ is systematic error and $s_{X_i}$ is random error. Using defined systematic and random uncertainties, the standard uncertainty is

$$u_r = (b_r^2 + s_r^2)^{\frac{1}{2}}$$  \hspace{1cm} (A.2)

The expanded uncertainty is then

$$U_r = t\% (b_r^2 + s_r^2)^{\frac{1}{2}}$$  \hspace{1cm} (A.3)

Where $t\%$ is the student t-value for the corresponding % uncertainty. For this analysis a t-value corresponding to 95% confidence of $t_{95} \approx 2$ is used.

A.1 Porous Disk Measurements - Pitot Tube

The uncertainty analysis for the porous disk pitot tube measurements will follow a similar study performed by Turner 2017 [40] which calculates uncertainty for porous disk measurements performed in the same facility. When using a pitot tube, velocity is calculated using the measured dynamic pressure in the flow which can be expanded using the ideal gas law as shown below,

$$V = \sqrt{\frac{2P}{\rho}} = \sqrt{\frac{2PRT}{P_{atm}}}$$  \hspace{1cm} (A.4)

Where $V$ is air velocity, $\rho$ is air density, $P$ is dynamic pressure, $R$ is the air gas constant, $T$ is air temperature, and $P_{atm}$ is atmospheric pressure. Inserting this equation for velocity into equation A.1, dividing by $U_V^2$ and simplifying gives us the following equation for standard relative uncertainty.

$$\frac{U_V^2}{V^2} = \frac{1}{4} \left( \frac{U_P}{P} \right)^2 + \frac{1}{4} \left( \frac{U_T}{T} \right)^2 + \frac{1}{4} \left( \frac{U_{P_{atm}}}{P_{atm}} \right)^2$$  \hspace{1cm} (A.5)

Where $U$ represents the standard uncertainty for each variable. The uncertainty for each variable is now calculated. Table A.1 shows sources of systematic error for each quantity used to calculate velocity with the pitot tube. Random error in the sample can then be determined using the standard
Table A.1: Sources of systematic error in measured quantities used to calculate pitot tube velocity

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</tr>
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</tr>
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</tr>
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<td>Temperature</td>
<td>Thermocouple gain error</td>
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</tr>
<tr>
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<td>Digitization</td>
<td>±0.15 °C</td>
</tr>
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<td>Barometric shift</td>
<td>±0.5 kPa</td>
</tr>
</tbody>
</table>

deviation and number of independent samples in a measurement point. The uncertainty in the mean to 95% confidence is written as:

$$U_{m_{95\%}} = \frac{t_{95} \sigma}{\sqrt{N}} = \frac{2\sigma}{\sqrt{N}}$$

(A.6)

Where $N$ is the number of independent samples for a given data point and $\sigma$ is the standard deviation. An in situ autocorrelation of a representative data point can be used to determine the independent sample time. Figure A.1 shows that the first zero crossing in the autocorrelation of a representative data point is at 0.27 seconds, which will be used as the independent sample time for the pitot tube measurements. Using this sample time and standard deviation the random uncertainty in mean velocity is approximately 1.3%.

Figure A.1: Autocorrelation of velocity time series at $x/D=1$, $z=0.5$ m
Combining the sources of systematic and random uncertainty gives us a total expanded relative uncertainty of $U_V/V = 1.63\%$. This uncertainty is shown applied to mean data in Figures A.2, A.3, A.4, and A.5 show the porous disk velocity profiles with 95% confidence interval error bars.
Figure A.2: Porous disk horizontal mean velocity profiles with 95% confidence intervals for x/D = 1 - 20
Figure A.3: Porous disk horizontal mean velocity profiles with 95% confidence intervals for x/D = 30 - 50
Figure A.4: Porous disk vertical mean velocity profiles with 95% confidence intervals for x/D = 1
Figure A.5: Porous disk mean velocity profiles with 95% confidence intervals for $x/D = 30 - 50$
A.2 Turbine Measurements - Hotwire Anemometer

A guide on hot-wire anemometers produced by Dantec lists the main sources of error and their typical magnitudes. The common sources of error considered here are from the calibrator, calibration fit linearization, A/D board resolution, temperature variations, density variations, and ambient pressure variations. Each of these sources and their relative standard uncertainty values are listed in Table A.2. Values used in this table to calculate uncertainties are: $T_w - T_0 = 200^\circ C$, $U = 10m/s$, $A = 1.396$, $B = 0.895$, $\partial U/\partial E = 46.5m/s/volt$.

**Table A.2:** Uncertainty sources in hot-wire measurements. From *A guide to measuring turbulence with hot-wire anemometers* [21]

<table>
<thead>
<tr>
<th>Uncertainty source</th>
<th>Input variance</th>
<th>Typical value</th>
<th>Relative output variance</th>
<th>Typical value</th>
<th>Coverage factor</th>
<th>Relative standard uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibrator</td>
<td>$\Delta U_{cal}$</td>
<td>1%</td>
<td>2·STDV(100·$\Delta U_{cal}$)</td>
<td>0.02</td>
<td>2</td>
<td>0.01</td>
</tr>
<tr>
<td>Linearization</td>
<td>$\Delta U_{fit}$</td>
<td>0.5%</td>
<td>2·STDV(100·$\Delta U_{fit}$)</td>
<td>0.01</td>
<td>2</td>
<td>0.005</td>
</tr>
<tr>
<td>A/D resolution</td>
<td>$E_{A/D}$</td>
<td>10 V, 16 bit</td>
<td>$\frac{1}{U} \frac{E_{A/D}}{2^{15}} \frac{\partial U}{\partial E}$</td>
<td>0.0005</td>
<td>$\sqrt{3}$</td>
<td>0.0003</td>
</tr>
<tr>
<td>Temperature variations 1</td>
<td>$\Delta T$</td>
<td>1 °C</td>
<td>$\frac{1}{U} \frac{\Delta T}{T_w - T_0} \left( \frac{A}{B} U^{-0.5} + 1 \right)$</td>
<td>0.012</td>
<td>$\sqrt{3}$</td>
<td>0.007</td>
</tr>
<tr>
<td>Temperature variations 2</td>
<td>$\Delta T$</td>
<td>1 °C</td>
<td>$\frac{\Delta T}{273}$</td>
<td>0.004</td>
<td>$\sqrt{3}$</td>
<td>0.002</td>
</tr>
<tr>
<td>Ambient pressure</td>
<td>$\Delta P$</td>
<td>1 kPa</td>
<td>1 - $\frac{P_0}{P_0 + \Delta P}$</td>
<td>0.01</td>
<td>$\sqrt{3}$</td>
<td>0.006</td>
</tr>
</tbody>
</table>

With the typical values in Table A.2, the relative expanded uncertainty of a typical data point is

$$U(U_{sample}) = 2 \cdot \sqrt{\sum \left( \frac{1}{k} \frac{1}{U} \Delta y_i \right)^2} = 0.03 = 3\%$$ (A.7)

Where $k$ is the coverage factor, similar to the t-value but more general. This calculation accounts for the systematic error present in the measurements, but there is also random error in the data due to a finite sampling time. The uncertainty in the mean to 95% confidence is written as:

$$U_{m_{95\%}} = \frac{t_{05} \sigma}{\sqrt{N}} = \frac{2 \sigma}{\sqrt{N}}$$ (A.8)

Where $N$ is the number of independent samples for a given data point and $\sigma$ is the standard deviation, which, in the case of a velocity measurement, is just the $u_{rms}$ value at that point. The number of independent samples can be estimated using an auto correlation of the velocity sample to determine at which time the sample is no longer correlated with itself. Figure A.6 shows an
autocorrelation of hotwire velocity data at $x/D = 4$, at the horizontal location of the blade tip, $z/R = 1$. This shows a first zero crossing at $t \approx 0.14\,s$. An independent sample time of 0.2 s was used as a conservative estimate for all random uncertainty measurements. For an independent sample time of $\tau = 0.2\,s$, the number of independent samples in a 200 s sample duration is $N = T/\tau = 1000$. The expanded random uncertainty in the mean for a point with $u_{\text{rms}} = 1\,m/s$ is then:

$$U_{m_{95\%}} = \frac{(2) \cdot (1\,m/s)}{\sqrt{1000}} = 0.06\,m/s$$ (A.9)

which is 0.6% of a 10 m/s sample. Systematic and random error were calculated separately for each data point using the corresponding mean velocity and RMS velocity. Figures A.7 and A.8 show the mean streamwise and azimuthal velocity profiles with errorbars. Overall uncertainty is around ±3% for most points, but exceeds ±10% for some of the swirl velocity points with magnitudes very close to zero.

Uncertainty in the variance, or, in the case of a turbulence measurement, the RMS velocity, is determined using the $X^2$ distribution, which is asymmetric. The expanded uncertainty for a variance value is calculated with:

$$\frac{N\sigma^2}{X^2}$$ (A.10)

Values of the $X^2$ distribution for measurements with independent samples $N > 120$ are determined using the following equations for the lower and upper limits on the variance respectively.

$$X_{0.975}^2 = N \cdot \left(1 - \frac{2}{9N} + t_{95}\left[\frac{2}{9N}\right]^{\frac{1}{2}}\right)^3$$ (A.11)
\[ X_{0.025}^2 = N \cdot \left( 1 - \frac{2}{9N} - t_{0.05}\left[\frac{2}{9N}\right]^\frac{1}{2} \right)^3 \]  \hspace{1cm} (A.12)

For sample with \( \sigma = u_{\text{rms}} = 1 \text{ m/s} \) the upper and lower error values are 0.958 m/s and 1.045 m/s respectively. These correspond to % error values of 4.2% and 4.59%. The error in the Reynolds stress \( \overline{u'w'} \) corresponds to the error in the variance, or the square of the error in the standard deviation, and will be higher than the error in the RMS velocities. Figures A.9 A.10 and A.11 show the RMS velocity and \( \overline{u'w'} \) Reynolds stress profiles with 95% confidence error bars.
Figure A.7: Turbine wake mean streamwise and azimuthal velocity with 95% confidence error bars for x/D = 1, 2, 4, and 6
Figure A.8: Turbine wake mean streamwise and azimuthal velocity with 95% confidence error bars for $x/D = 8, 10, 15,$ and $20$
Figure A.9: Turbine wake $u_{rms}$ velocity with 95% confidence error bars for all $x/D$.
Figure A.10: Turbine wake $w_{\text{rms}}$ velocity with 95% confidence error bars for all $x/D$
Figure A.11: Turbine wake $\overline{uw}$ Reynolds stress with 95% confidence error bars for all $x/D$
APPENDIX B
GOVERNING EQUATIONS

The governing equations for the turbulent axisymmetric wake with swirl are derived in cylindrical coordinates \((x, r, \theta)\). The instantaneous velocity components in the in axial, radial and azimuthal directions are \(\tilde{u}, \tilde{v}\) and \(\tilde{w}\), respectively. Gravitational forces are neglected. The flow is assumed to be steady state (really: stationary in the mean), hence temporal derivatives \((\partial/\partial t)\) are also neglected.

B.1 Continuity and Momentum Equations

The continuity and momentum equations for a Newtonian fluid in cylindrical coordinates are given by (e.g., \([35, 28]\)):

**Continuity Equation:**

\[
\frac{\partial}{\partial x} (\rho \tilde{u}) + \frac{1}{r} \frac{\partial}{\partial r} (\rho r \tilde{v}) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho \tilde{w}) = 0 \quad (B.1)
\]

**Momentum Equation in Streamwise (Axial) Direction:**

\[
\tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial r} + \tilde{w} \left(\frac{1}{r} \frac{\partial \tilde{u}}{\partial \theta} - \tilde{w} \right) = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x} + \frac{2}{\rho} \nabla \cdot (\mu \tilde{S}_x) \quad (B.2)
\]

**Momentum Equation in Radial Direction:**

\[
\tilde{u} \frac{\partial \tilde{v}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial r} + \tilde{w} \left(\frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} - \tilde{v} \right) = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial r} + \frac{2}{\rho} \nabla \cdot (\mu \tilde{S}_r) \quad (B.3)
\]

**Momentum Equation in Tangential Direction:**

\[
\tilde{u} \frac{\partial \tilde{w}}{\partial x} + \tilde{v} \frac{\partial \tilde{w}}{\partial r} + \tilde{w} \left(\frac{1}{r} \frac{\partial \tilde{w}}{\partial \theta} + \frac{\tilde{v}}{r} \right) = -\frac{1}{\rho} \left(\frac{1}{r} \frac{\partial \tilde{p}}{\partial \theta} \right) + \frac{2}{\rho} \nabla \cdot (\mu \tilde{S}_\theta) \quad (B.4)
\]

The viscous terms on the right hand side of the component momentum equations are generally defined as follows (neglecting the 2nd viscosity):

\[
2 \nabla \cdot (\mu \tilde{S}_x) \equiv 2 \frac{\partial}{\partial x} \left( \mu \frac{\partial \tilde{u}}{\partial x} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \mu \left( \frac{\partial \tilde{u}}{\partial r} + \frac{\partial \tilde{v}}{\partial x} \right) \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \mu \left( \frac{\partial \tilde{w}}{\partial x} + \frac{1}{r} \frac{\partial \tilde{u}}{\partial \theta} \right) \right) \quad (B.5)
\]

\[
2 \nabla \cdot (\mu \tilde{S}_r) \equiv \frac{\partial}{\partial x} \left( \mu \left( \frac{\partial \tilde{u}}{\partial r} + \frac{\partial \tilde{v}}{\partial x} \right) \right) + 2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \mu \frac{\partial \tilde{v}}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \mu \left( \frac{\partial \tilde{w}}{\partial r} + \frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} \right) \right) \]

\[
- 2 \mu \left( \frac{1}{r^2} \frac{\partial \tilde{w}}{\partial \theta} + \frac{\tilde{v}}{r^2} \right) \quad (B.6)
\]
\[ 2 \nabla \cdot (\mu \tilde{S}_\theta) = \frac{\partial}{\partial x} \left( \mu \left( \frac{\partial \tilde{w}}{\partial x} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right) \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( \mu r^2 \left( \frac{\partial}{\partial r} \left( \frac{\tilde{w}}{r} \right) + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \right) \]
\[ + \frac{1}{r} \frac{\partial}{\partial \theta} \left( 2 \mu \left( \frac{1}{r} \frac{\partial \tilde{w}}{\partial \theta} + \frac{\tilde{v}}{r} \right) \right) \]  \hspace{1cm} (B.7)

The assumption of incompressibility is reasonable for wakes with rotation generated by wind turbines. Viscosity is a function of temperature, however, it can be assumed to be constant here (If variation of viscosity with temperature is of concern, then derivations of governing equations should be carried out by keeping viscosity in inside the divergence \((\nabla \cdot (\mu S_{ij}))\). Note that decomposing viscosity(\(\mu\)) into mean and fluctuating parts would add viscosity-velocity correlations and thereby significant complexity!). Reynolds decomposition is now applied to divide the instantaneous components in the equations above into a mean (uppercase) and fluctuating (lowercase) part:

Streamwise (axial) velocity \((x)\): \( \tilde{u} = U + u \)
Radial velocity \((r)\): \( \tilde{v} = V + v \)
Tangential Velocity \((\theta)\): \( \tilde{w} = W + w \)
Pressure: \( \tilde{p} = P + p \)

These terms are substituted into the continuity and momentum equations, which are then averaged. The mean continuity equation becomes:

**Averaged Continuity Equation:**

\[ \frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial (rV)}{\partial r} + \frac{1}{r} \frac{\partial W}{\partial \theta} = 0 \]  \hspace{1cm} (B.8)

The mean momentum equations become:

**Averaged Momentum Equation in Streamwise (Axial) Direction:**

\[ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial r} + W \frac{\partial U}{\partial \theta} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left[ \frac{\partial^2 U}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} \right] \]
\[ - \left( \frac{\partial \tilde{u}^2}{\partial x} + \frac{\partial \tilde{w} \tilde{v}}{\partial r} + \frac{1}{r} \frac{\partial \tilde{w} \tilde{v}}{\partial \theta} + \tilde{w} \tilde{v} \right) \]  \hspace{1cm} (B.9)

**Averaged Momentum Equation in Radial Direction:**

\[ U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial r} + W \frac{\partial V}{\partial \theta} - W^2 \frac{\partial}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} V \right) \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial W}{\partial \theta} \right] \]
\[ - \left( \frac{\partial \tilde{w} \tilde{v}}{\partial x} + \frac{\partial \tilde{v}^2}{\partial r} + \frac{1}{r} \frac{\partial \tilde{w} \tilde{v}}{\partial \theta} - \frac{w^2 - \tilde{v}^2}{r} \right) \]  \hspace{1cm} (B.10)
\[ \frac{U}{r} \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial r} + \frac{W}{r} \frac{\partial W}{\partial \theta} + \frac{V W}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial \theta} + \nu \left[ \frac{\partial^2 W}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rW)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial V}{\partial \theta} \right] \]

\[ - \left( \frac{\partial u' w'}{\partial x} + \frac{\partial v' w'}{\partial r} + \frac{1}{r} \frac{\partial w'^2}{\partial r} + \frac{v' w'}{r} \right) \] (B.11)

The averages of non-linear occurrences of fluctuating velocities in the averaged momentum equations are generally non-zero. The resulting terms were re-written with the help of the fluctuating continuity equation and moved to the right-hand side of the momentum equations since they act as apparent stresses, also referred to as "Reynolds stresses" (cf. [37])

Since the flow is axisymmetric, there must be symmetry with respect to the \( \theta \)-direction, i.e, \( \partial / \partial \theta = 0 \). In other words, the flow is statistically homogeneous flow in tangential direction. However, there is expected to be a mean swirl component, and \( W \neq 0 \). The averaged continuity and momentum equations thus further simplify:

**Averaged Continuity Equation:**
\[ \frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial (rV)}{\partial r} = 0 \] (B.12)

The mean momentum equations become:

**Averaged Momentum Equation in Streamwise (Axial) Direction:**
\[ \frac{U}{r} \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left[ \frac{\partial^2 U}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rU)}{\partial r} \right) \right] \]

\[ - \left( \frac{\partial u' u'}{\partial x} + \frac{1}{r} \frac{\partial (r u' w')}{\partial r} \right) \] (B.13)

**Averaged Momentum Equation in Radial Direction:**
\[ \frac{U}{r} \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial r} - \frac{W^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[ \frac{\partial^2 V}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rV)}{\partial r} \right) \right] \]

\[ - \left( \frac{\partial u' v'}{\partial x} + \frac{\partial v' v'}{\partial r} - \frac{w'^2}{r} \right) \] (B.14)

**Averaged Momentum Equation in Tangential Direction:**
\[ \frac{U}{r} \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial r} + \frac{V W}{r} = \nu \left[ \frac{\partial^2 W}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rW)}{\partial r} \right) \right] \]

\[ - \left( \frac{\partial u' w'}{\partial x} + \frac{\partial v' w'}{\partial r} + \frac{v' w'}{r} \right) \] (B.15)
B.2 Reynolds Stress Transport Equations

To calculate the transport equations for the Reynolds stresses in an incompressible, high-Reynolds number flow, we subtract instantaneous momentum equations (B.2, B.3 and B.4) from averaged momentum equations (B.13, B.14 and B.15):

\[ \begin{align*}
U \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial r} + W \frac{\partial u}{r \partial \theta} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \Bigg\{ \left( \frac{u}{\partial x} + v \right) \left( \frac{-\partial u}{\partial r} + w \frac{\partial u}{r \partial \theta} \right) \Bigg\} \\
&\quad - \left\{ \left( \frac{u}{\partial x} + v \frac{\partial u}{r \partial \theta} + w \frac{\partial u}{r \partial \theta} - \frac{u^2}{r} \right) \right\} \tag{B.16}
\end{align*} \]

\[ \begin{align*}
U \frac{\partial v}{\partial x} + V \frac{\partial v}{\partial r} + W \frac{\partial v}{r \partial \theta} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} \Bigg\{ \left( \frac{v}{\partial x} + w \frac{\partial v}{r \partial \theta} \right) \left( \frac{w}{\partial r} - \frac{W}{r} \right) \Bigg\} \\
&\quad - \left\{ \left( \frac{v}{\partial x} + w \frac{\partial v}{r \partial \theta} \right) \left( \frac{w}{r} \right) \right\} \tag{B.17}
\end{align*} \]

\[ \begin{align*}
U \frac{\partial w}{\partial x} + V \frac{\partial w}{\partial r} + W \frac{\partial w}{r \partial \theta} &= -\frac{1}{\rho} \frac{\partial p}{r \partial \theta} \Bigg\{ \left( \frac{w}{\partial x} + \frac{v}{\partial r} \right) \left( \frac{w}{r} \right) \Bigg\} \\
&\quad - \left\{ \left( \frac{w}{\partial x} + \frac{v}{\partial r} \right) \left( \frac{w}{r} \right) \right\} \tag{B.18}
\end{align*} \]

where \( s' \) refers to fluctuating viscous stress terms. Now each equation above is multiplied with each fluctuating velocity, and averaged. Combinations of the multiplied/averaged equations are then added together to form the Reynolds stress component transport equations as follow:

- **Average \((u \times \text{Eqn B.16}) + \text{Average} \,(u \times \text{Eqn B.16}):**

\[ \begin{align*}
U \frac{\partial u^2}{\partial x} + V \frac{\partial u^2}{\partial r} + W \frac{\partial u^2}{r \partial \theta} &= -2 \left\{ \frac{\partial p}{\partial x} - \left( \frac{\partial u}{\partial x} \right) \right\} + \frac{2}{\rho} \left\{ 2 \nabla \cdot (s'_x) \right\} \\
&\quad - 2 \left\{ \frac{u^2}{\partial x} + \frac{v^2}{\partial r} \right\} \left( \frac{u}{\partial x} + \frac{v}{\partial r} \right) \left( \frac{w^2}{r} \right) \tag{B.19}
\end{align*} \]

- **Aaverage\((v \times \text{Eqn.B.17}) + \text{Average}(v \times \text{Eqn.B.17}):**

\[ \begin{align*}
U \frac{\partial v^2}{\partial x} + V \frac{\partial v^2}{\partial r} + W \frac{\partial v^2}{r \partial \theta} &= -2 \left\{ \frac{\partial p}{\partial r} - \left( \frac{\partial v}{\partial r} \right) \right\} + \frac{2}{\rho} \left\{ 2 \nabla \cdot (s'_y) \right\} \\
&\quad - 2 \left\{ \frac{v^2}{\partial x} + \frac{w^2}{\partial r} \right\} \left( \frac{v}{\partial x} + \frac{w}{\partial r} \right) \left( \frac{w^2}{r} \right) \tag{B.20}
\end{align*} \]
• Average($w \times$ Eqn.B.18) + Average($w \times$ Eqn.B.18):

$$
U \frac{\partial w^2}{\partial x} + V \frac{\partial w^2}{\partial r} + W \frac{\partial w^2}{\partial \theta} + 2Vw^2 = -2 \frac{1}{r} \left\{ \frac{\partial pw}{\partial \theta} - <p \frac{\partial w}{\partial \theta}> \right\} + 2 \frac{\rho}{r} <w \nabla \cdot (\mu s'_x)> \\
-2 \left\{ \frac{uw}{r} \frac{\partial W}{\partial x} + \frac{vw}{r} \frac{\partial W}{\partial r} + \frac{w^2}{r} \frac{\partial W}{\partial \theta} + \frac{vw}{r} \frac{W}{r} \right\} - \left\{ \frac{\partial uw^2}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \frac{uw^2}{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \frac{uw^2}{r} + \frac{2}{r} \frac{vw^2}{r} \right\}
$$

(B.21)

• Average($v \times$ Eqn.B.16) + Average($u \times$ Eqn.B.17)

$$
U \frac{\partial uv}{\partial x} + V \frac{\partial uv}{\partial r} + W \frac{\partial uv}{\partial \theta} - W \frac{uw}{r} = -\frac{1}{\rho} \left\{ \frac{\partial pv}{\partial x} + \frac{\partial pu}{\partial r} \right\}
\\
+ \frac{1}{\rho} \left\{ <p \frac{\partial v}{\partial x} > + <p \frac{\partial u}{\partial r} > \right\} + 2 \frac{\rho}{\rho} \left\{ <v \nabla \cdot (\mu s'_x) > + <u \nabla \cdot (\mu s'_y) > \right\}
\\
- \left\{ \frac{uw}{r} \frac{\partial U}{\partial x} + \frac{vw}{r} \frac{\partial U}{\partial r} + \frac{w^2}{r} \frac{\partial U}{\partial \theta} + \frac{uw}{r} \frac{U}{r} + \frac{w}{r} \frac{\partial V}{\partial \theta} + \frac{uw}{r} \frac{V}{r} - \frac{uw}{r} \frac{W}{r} \right\}
\\
- \left\{ \frac{\partial u^2v}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \frac{u^2v}{r} + \frac{1}{r} \frac{\partial uwv}{\partial \theta} - \frac{uwv}{r} \right\}
$$

(B.22)

• Average($w \times$ Eqn.B.16) + Average($u \times$ Eqn.B.18)

$$
U \frac{\partial uw}{\partial x} + V \frac{\partial uw}{\partial r} + W \frac{\partial uw}{\partial \theta} + V \frac{uw}{r} = -\frac{1}{\rho} \left\{ \frac{\partial pw}{\partial x} + \frac{1}{r} \frac{\partial}{\partial \theta} \frac{uw}{r} \right\}
\\
+ \frac{1}{\rho} \left\{ <p \frac{\partial w}{\partial x} > + \frac{1}{r} <p \frac{\partial u}{\partial \theta} > \right\} + 2 \frac{\rho}{\rho} \left\{ <w \nabla \cdot (\mu s'_x) > + <u \nabla \cdot (\mu s'_y) > \right\}
\\
- \left\{ \frac{uw}{r} \frac{\partial U}{\partial x} + \frac{vw}{r} \frac{\partial U}{\partial r} + \frac{w^2}{r} \frac{\partial U}{\partial \theta} + \frac{uw}{r} \frac{U}{r} + \frac{w}{r} \frac{\partial W}{\partial \theta} + \frac{uw}{r} \frac{W}{r} \right\}
\\
- \left\{ \frac{\partial u^2w}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \frac{u^2w}{r} + \frac{1}{r} \frac{\partial uw^2}{\partial \theta} + \frac{uw^2}{r} \right\}
$$

(B.23)

• Average($w \times$ Eqn.B.17) + Average($v \times$ Eqn.B.18)
\[
U \frac{\partial \overline{uv}}{\partial x} + V \frac{\partial \overline{uv}}{\partial r} + \frac{W}{r} \frac{\partial \overline{uv}}{\partial \theta} - \frac{W}{r} \overline{w^2} + V \frac{\overline{vw}}{r} = - \frac{1}{\rho} \left\{ \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial pv}{\partial \theta} \right\} \\
+ \frac{1}{\rho} \left\{ <p \frac{\partial w}{\partial r}> + \frac{1}{r} <p \frac{\partial v}{\partial \theta}> \right\} + \frac{2}{\rho} \left\{ <v \nabla \cdot (\mu_s' \theta)> + <w \nabla \cdot (\mu_s' r)> \right\} \\
- \left\{ \frac{uw}{r} \frac{\partial V}{\partial x} + \frac{vw}{r} \frac{\partial V}{\partial r} + \frac{w^2}{r} \frac{\partial V}{\partial \theta} + \frac{uv}{r} \frac{\partial W}{\partial x} + \frac{v^2}{r} \frac{\partial W}{\partial r} + \frac{vw}{r} \frac{\partial W}{\partial \theta} + (\overline{v^2} - \overline{w^2}) \frac{W}{r} \right\} \\
- \left\{ \frac{\partial \overline{uw}w}{\partial x} + \frac{1}{r} \frac{\partial \overline{vw}w}{\partial r} + \frac{1}{r} \frac{\partial \overline{vw}w}{\partial \theta} + \frac{\overline{v^2}w}{r} - \frac{\overline{w^3}}{r} \right\} \quad (B.24)
\]

**Turbulent kinetic energy equation:**

The equation for the turbulent kinetic energy in cylindrical coordinate can be obtained by adding the normal Reynolds stress equations and defining \( k \), the average fluctuating kinetic energy per unit mass, as:

\[
k \equiv \frac{1}{2} \overline{q^2} = \frac{1}{2} [\overline{u^2} + \overline{v^2} + \overline{w^2}] \quad \text{(B.25)}
\]

**Turbulent kinetic energy equation:**

\[
U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial r} + \frac{W}{r} \frac{\partial k}{\partial \theta} - \frac{W}{r} \overline{w^2} + V \frac{\overline{vw}}{r} = \\
- \frac{1}{\rho} \left\{ \frac{\partial p}{\partial x} + \frac{1}{r} \frac{(r \overline{pv})}{\partial r} + \frac{1}{r} \frac{\partial pv}{\partial \theta} \right\} - \frac{1}{2} \left\{ \frac{\partial \overline{u^2}}{\partial x} + \frac{1}{r} \frac{\partial \overline{r(v^2)}}{\partial r} + \frac{1}{r} \frac{\partial \overline{v^2}}{\partial \theta} \right\} + \frac{2}{\rho} \nabla.\mu \overline{s_{ij}} > \\
- \left\{ \overline{u^2} \frac{\partial U}{\partial x} + \frac{uw}{r} \frac{\partial U}{\partial r} + \frac{uw}{r} \frac{\partial U}{\partial \theta} + \frac{uw}{r} \frac{\partial V}{\partial x} + \frac{v^2}{r} \frac{\partial V}{\partial r} + \frac{v^2}{r} \frac{\partial V}{\partial \theta} + \frac{uv}{r} \frac{\partial W}{\partial x} + \frac{vw}{r} \frac{\partial W}{\partial r} + \frac{w^2}{r} \frac{\partial W}{\partial \theta} \right\} \\
- \frac{2}{\rho} \mu < s_{ij} s_{ij} > \quad (B.26)
\]

Note that the incompressibility condition for the fluctuating continuity was used to eliminate the pressure-strain rate term.

In the turbulent kinetic energy equation, the viscous terms are split into the turbulence transport (or divergence) terms and dissipations, shown below:

- **Transport of kinetic energy due to viscous stresses:**
\[
\frac{2}{\rho} \nabla \cdot \mu < \bar{v}_s \bar{s}_{ij} > \equiv \frac{\partial}{\partial x} \rho \left\{ \frac{\partial k}{\partial x} + \frac{\partial u^2}{\partial x} + \frac{1}{r} \frac{\partial rw}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right\} \\
\quad + \frac{1}{r} \frac{\partial}{\partial r} \rho \left\{ \frac{\partial k}{\partial r} + \frac{\partial rw}{\partial x} + \frac{1}{r} \frac{\partial rv^2}{\partial r} + \frac{1}{r} \frac{\partial vw}{\partial \theta} - \frac{w^2}{r} \right\} \\
\quad + \frac{1}{r} \frac{\partial}{\partial \theta} \rho \left\{ \frac{\partial k}{\partial \theta} + \frac{\partial vw}{\partial x} + \frac{1}{r} \frac{\partial rv}{\partial r} + \frac{1}{r} \frac{\partial w^2}{\partial \theta} + \frac{vw}{r} \right\} \tag{B.27}
\]

- **Rate of dissipation of turbulence kinetic energy:**

\[
- \epsilon \equiv -\frac{2}{\rho} \mu < \bar{s}_i \bar{s}_j >= -\frac{\mu}{\rho} \left\{ \frac{\partial v}{\partial x} \right\}^2 + \frac{2}{\rho} \left\{ \frac{\partial u}{\partial r} \right\}^2 - \frac{2}{\rho} \frac{v^2}{r^2} + \frac{2}{\rho} \left\{ \frac{1}{r} \frac{\partial w}{\partial \theta} \right\}^2 \\
\left\{ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial v}{\partial \theta} + r \frac{\partial w}{\partial r} \left( \frac{w}{r} \right)^2 + \left( \frac{\partial w}{\partial x} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 \right\} \tag{B.28}
\]
APPENDIX C

SCALING OF THE GOVERNING EQUATIONS

An Order of Magnitude Analysis is conducted here to identify leading order terms in the governing equations. The swirling turbulent wake is a free shear layer flow, and the so-called "thin shear layer hypothesis"

$$\frac{\partial}{\partial x} \sim \frac{1}{L} \ll \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$  \hspace{1cm} (C.1)

states that changes in the streamwise direction occur much more gradually than changes in the cross-stream direction. Here $L$ is a streamwise length scale and $\delta$ is cross-stream length scale, both to be defined more precisely at a later stage.

A scale for the streamwise velocity in the wake is defined as

$$U \sim U_s \sim (U - U_\infty)$$  \hspace{1cm} (C.2)

Note that the wake is characterized by the difference of the mean velocity from the free stream velocity, $(U - U_\infty)$, which is small compared to the free stream velocity if one moves sufficiently far downstream. Therefore

$$U_s \ll U_\infty$$  \hspace{1cm} (C.3)

The scaling of the streamwise velocity in the governing equations has two distinct cases:

- $U \sim U_s$ when $U$ occurs inside a derivative, i.e., is inside the wake.
- $U \sim U_\infty$ when $U$ is a convective velocity.

This scaling is unique to the turbulent wake and distinguishes the results from the order of magnitude from other free turbulent shear flows such as the jet.

$$\overline{u^2}, \overline{v^2}, \overline{w^2}, \overline{uv} \sim u_s^2$$

$$\partial/\partial x \sim 1/L$$

$$\partial/\partial r \sim 1/\delta$$

$$\partial/\partial \theta = 0$$

$$r \sim \delta$$  \hspace{1cm} (C.4)

The order of magnitude scaling is demonstrated in detail for continuity and the three momentum equations. Any equation governing a turbulent shear flow within a "thin shear layer hypothesis",
such as turbulent kinetic energy equation and Reynolds stress component equations for the axisymmetric wake with swirl discussed here, can can be scaled using the same procedure. The averaged continuity equation within the assumptions of incompressibility and axisymmetry is

\[
\frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial (rV)}{\partial r} = 0
\]  

(C.5)

and the terms occurring in it scale as follows

\[
\frac{U_s}{L} + \frac{1}{r} \frac{rV_s}{\delta} \sim 0
\]  

(C.6)

The scale for the mean radial velocity \( V_s \) is then found as

\[
V_s \sim U_s \frac{\delta}{L}
\]  

(C.7)

**x-Momentum:**
The x-momentum equation will be considered first. Within the assumptions of incompressibility, constant viscosity and axisymmetry, and Reynolds decomposition and averaging the x-momentum equation is

\[
U \frac{\partial (U - U_\infty)}{\partial x} + V \frac{\partial (U - U_\infty)}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 U}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) \right) - \left( \frac{\partial u^2}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \nu \tau) \right)
\]  

(C.8)

Upon substituting the order of magnitude scaling from equation C.4 above the x-momentum equation becomes

\[
\frac{U_\infty}{L} \frac{U_s}{L} + U_s \frac{\delta}{L} \frac{U_s}{\delta} \sim \ ? + \frac{U_s}{L^2} + \nu \frac{1}{r} \frac{1}{\delta} \frac{U_s}{\delta} + \frac{u^2}{L} + \frac{1}{r} \frac{1}{\delta} \frac{ru^2}{L}
\]  

(C.9)

where the symbol “ \sim ” is used instead of an equal sign to mean “order of magnitude”. Equation C.9 is then multiplied through by \( L/U_\infty U_s \) to non-dimensionalize and help group/clear terms

\[
1 + \frac{U_s}{U_\infty} \sim \ ? + \nu \frac{1}{LU_\infty} + \nu \frac{L^2}{U_\infty} \frac{\left( \frac{L^2}{\delta^2} \right)}{U_\infty U_s} + \frac{u^2}{U_\infty U_s} \frac{u^2}{L} \frac{\left( \frac{L}{\delta} \right)}{U_\infty U_s}
\]  

(C.10)

The first fraction in the two viscous terms is identified as the inverse of the Reynolds number, \( 1/Re_L = \nu / U_\infty L \). The two velocity scalings for the turbulent wake outlined at the beginning of this appendix lead to the second convection term being an order of magnitude smaller than the first convection term. On the right-hand side of the equation, the pressure gradient term is of unknown magnitude at this point in the order of magnitude scaling, hence the “ ? ”. The second term on
the right-hand side (first viscous term) is small compared to the third term (second viscous term), and the fourth term (first turbulence term) is small compared to the fifth term (second turbulence term). The larger of the two viscous terms can be made as small as desired by increasing Reynolds number, in fact a Reynolds number $Re_L > (L/\delta)^3$ would suffice to make it a second order term in this equation. It is easy to see how turbulent free shear flows such as this wake can develop without viscous terms, as they do not have to satisfy a no-slip condition at a solid wall.

$$1 + \frac{U}{U_\infty} \sim ? + \frac{1}{Re_L} + \frac{1}{Re_L} \left( \frac{L^2}{\delta} \right) + \frac{u^2}{U_\infty U_s} + \frac{u^2}{U_\infty U_s} \left( \frac{L}{\delta} \right)$$

(C.11)

For the larger of the two turbulence terms to remain in the equation, it is required that

$$\frac{u^2}{U_\infty U_s} \left( \frac{L}{\delta} \right) \rightarrow 1$$

(C.12)

For now, one convection term, the pressure gradient and one turbulence term remain. The magnitude of the pressure gradient term will be derived next.

$$U \frac{\partial(U - U_\infty)}{\partial x} \sim -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \bar{uv} \right)$$

(C.13)

**r-Momentum:**

Now conservation of radial momentum is considered. Within the assumptions of incompressibility, constant viscosity and axisymmetry, and Reynolds decomposition and averaging the r-momentum equation is

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial r} - \frac{W^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (rV) \right) \right)$$

$$- \left( \frac{\partial \bar{uv}}{\partial x} + \frac{\partial v^2}{\partial r} - \frac{w^2 - v^2}{r} \right)$$

(C.14)

Upon substituting the order of magnitude scaling from equation C.4 above the r-momentum equation becomes

$$U_\infty U_s \frac{\delta}{L^2} + U_s^2 \frac{\delta^2}{L^2} \frac{1}{r} - \frac{W_s^2}{r} \sim ? + \nu \frac{U_s \delta}{L^2} + \nu \frac{1 \delta}{r \delta} \frac{U_s}{L} \frac{u^2}{L} \frac{u^2}{\delta} \frac{u^2}{r}$$

(C.15)

where the mean azimuthal (swirl) velocity was given its own order of magnitude scaling $W \sim W_s$. The radial coordinate $r$ is assumed to scale with the cross-stream length scale, $r \sim \delta$. Equa-
tion C.15 is then multiplied through by $\delta/U_\infty U_s$ to non-dimensionalize and help group and clear terms

$$
\left( \frac{\delta}{L} \right)^2 + \frac{U_s}{U_\infty} \left( \frac{\delta}{L} \right)^2 + \frac{W_s^2}{U_\infty U_s} \sim ? + \frac{1}{Re_L} \left( \frac{\delta}{L} \right)^2 + \frac{1}{Re_L} + \frac{u^2}{U_\infty U_s} \left( \frac{\delta}{L} \right) + \frac{u^2}{U_\infty U_s} + \frac{u^2}{U_\infty U_s}
$$

(C.16)

The second term on the left-hand side is small compared to the first term. If $(\delta/L)^2 \ll W_s^2/U_\infty U_s$, then both the first and second terms on the left-hand side are small compared to the third term. On the right-hand side of the equation, the pressure gradient term is of unknown magnitude. The second term on the right-hand side (first viscous term) is small compared to the third term (second viscous term), and the fourth term (first turbulence term) is small compared to the fifth and sixth terms (second and third turbulence terms). The larger of the two viscous terms scales as $\sim 1/Re_L$ and can be made arbitrarily small by increasing Reynolds number.

$$
\left( \frac{\delta}{L} \right)^2 + \frac{U_s}{U_\infty} \left( \frac{\delta}{L} \right)^2 + \frac{W_s^2}{U_\infty U_s} \sim ? + \frac{1}{Re_L} \left( \frac{\delta}{L} \right)^2 + \frac{1}{Re_L} + \frac{u^2}{U_\infty U_s} \left( \frac{\delta}{L} \right) + \frac{u^2}{U_\infty U_s} + \frac{u^2}{U_\infty U_s}
$$

(C.17)

We can now compare the scaling of the azimuthal velocity $W_s$ to the largest turbulence terms: For $W_s^2/U_\infty U_s$ to be of second order compared to the fifth and sixth terms on the right-hand side of equation C.17 would require that

$$
\frac{W_s^2}{U_\infty U_s} \ll \frac{u^2}{U_\infty U_s} \quad \text{or: } \quad W_s^2 \ll u^2
$$

(C.18)

$$
\frac{W_s^2}{u_s^2} \rightarrow 0
$$

While $W_s^2 \ll u^2$ is in principal possible for the far, far wake, it is not the expected scaling behavior for the far wind turbine wake, which in this study is defined as $10 \leq x/D \leq 20$. Hence the third term on the right-hand side of equations C.14,C.17 remains, and the reduced $r$-momentum equation for the turbulent axisymmetric wake becomes

$$
\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{W_s^2}{r} - \frac{\partial u_s^2}{\partial r} + \frac{u_s^2 - v_s^2}{r}
$$

(C.19)
Back to $\partial p/\partial x$ and $x$-Momentum:

Returning to the question of order of magnitude of the pressure gradient in the $x-$momentum equation: Equation C.19 can be integrated from $r$ to $\infty$

$$
\int_{r}^{\infty} \frac{1}{\rho} dp = \frac{1}{\rho} (p_\infty - p) = \int_{r}^{\infty} \frac{W^2}{r} dr - \int_{r}^{\infty} \frac{\partial v^2}{\partial r} dr + \int_{r}^{\infty} \frac{\overline{w^2} - \overline{v^2}}{r} dr \tag{C.20}
$$

which results in

$$
- \frac{1}{\rho} (p - p_\infty) = \int_{r}^{\infty} \frac{W^2}{r} dr + -\overline{v^2}\bigg|_{r}^{\infty} + \int_{r}^{\infty} \frac{\overline{w^2} - \overline{v^2}}{r} dr \tag{C.21}
$$

where

$$
-\overline{v^2}\bigg|_{r}^{\infty} = -(0 - \overline{v^2}) \tag{C.22}
$$

Equation C.21 can be differentiated with respect to $x$ to obtain an expression for the streamwise pressure gradient as

$$
-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial \overline{v^2}}{\partial x} + \int_{r}^{\infty} \frac{1}{r} \left( \frac{\partial \overline{w^2}}{\partial x} - \frac{\partial \overline{v^2}}{\partial x} \right) dr + \int_{r}^{\infty} \frac{1}{r} \frac{dW^2}{dx} dr \tag{C.23}
$$

A quick check of the orders of magnitudes of the terms in the $\partial p/\partial x$ expression gives

$$
\left( \frac{1}{\rho} \frac{\partial p}{\partial x} \right) \sim \frac{u^2}{L} + \frac{1}{r} \left( \frac{u^2}{L} \right) r + \frac{W^2}{L} \tag{C.24}
$$

and multiplying equation C.24 by $L/U_\infty U_s$ yields

$$
\left( \frac{L}{U_\infty U_s} \right) \left( \frac{1}{\rho} \frac{\partial p}{\partial x} \right) \sim \frac{u^2}{U_\infty U_s} + \frac{u^2}{U_\infty U_s} + \frac{W^2}{U_\infty U_s} \left( \frac{W}{U_\infty} \right) \left( \frac{W}{U_s} \right) \tag{C.25}
$$

It is evident that both the turbulence terms, which scale as $u^2_s/U_\infty U_s$, are second order terms when compared to the leading order term in equation C.11. For now we can assume that $W_s/U_\infty \sim 1$, and since $W_s/U_\infty$ is small in comparison to the leading order convective term, the third term in equation C.23 is also of second order.

Equation C.23 can now be substituted into equation C.13 to yield the reduced $x$-momentum equation for the turbulent axisymmetric swirling wake

$$
U_\infty \frac{\partial (U - U_\infty)}{\partial x} = -\frac{1}{r} \frac{\partial}{\partial r} (r \overline{uv}) + \left\{ \frac{\partial}{\partial x} \left( \overline{v^2} - \overline{u^2} + \int_{0}^{\infty} \frac{1}{r} (\overline{w^2} - \overline{v^2}) + W^2 \right) \right\} \tag{C.26}
$$

The terms in curly brackets are of second order. To first order the reduced $x$-momentum consists of a balance between the leading order convection term and the leading order Reynolds stress.

**Azimuthal ($\theta$)-Momentum:**

Finally conservation of azimuthal momentum is considered. Within the assumptions of incompressibility, constant viscosity and axisymmetry, and Reynolds decomposition and averaging the
azimuthal \((\theta)\)-momentum equation is

\[
U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial r} + \frac{VW}{r} = \nu \left[ \frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial W}{\partial r} (rW) \right) \right] - \left( \frac{\partial vu}{\partial x} + \frac{\partial vu}{\partial r} + \frac{vu}{r} \right)
\]

(C.27)

The mean pressure gradient in the azimuthal direction is zero due to axisymmetry. Upon substituting the order of magnitude scaling from equation C.4 above into the \(\theta\)-momentum equation it becomes

\[
\frac{U_\infty W_s}{L} + \frac{U_s \delta W_s}{\delta \delta} + \frac{U_s \delta W_s}{\delta \delta} \sim \nu \frac{W_s}{L^2} + \nu \frac{1}{\delta^2} W_s + \frac{u^2}{L} + \frac{u^2}{\delta} + \frac{u^2}{\delta}
\]

(C.28)

Equation C.28 is then multiplied through by \(L/U_\infty U_s\) to non-dimensionalize and help group and clear terms:

\[
1 + \frac{U_\infty W_s}{U_\infty W_s} \sim \frac{1}{Re_L} + \frac{1}{Re_L} \left( \frac{L}{\delta} \right)^2 + \frac{u^2}{U_\infty W_s} + \frac{u^2}{U_\infty W_s} \left( \frac{L}{\delta} \right) + \frac{u^2}{U_\infty W_s} \left( \frac{L}{\delta} \right)
\]

(C.29)

The second and third convection term on the left-hand side are small compared to the first term, which scaled with \(U \sim U_\infty\) as convection velocity. The first viscous term on the right-hand side is small compared to the second viscous term, but the larger of the two viscous terms can be made arbitrarily small by increasing Reynolds number. The third term (first turbulence term) is small compared to the fourth and fifth terms (second and third turbulence terms). This yields the reduced \(\theta\)-momentum equation for the turbulent axisymmetric swirling wake

\[
U_\infty \frac{\partial W}{\partial x} \approx - \left( \frac{\partial}{\partial r} \frac{VW}{r} + \frac{vu}{r} \right) = \frac{1}{r} \frac{\partial}{\partial r} (rW)
\]

(C.30)

where the two leading order turbulence terms can be combined into a single term. The reduced \(\theta\)-equations states that, to leading order, the change in streamwise transport of azimuthal momentum is equal to the radial transport of radial-azimuthal Reynolds stress \(\overline{vw}\). This is the primary transport mechanism for redistributing azimuthal momentum as the swirling wake evolves downstream.
APPENDIX D

DERIVATION OF THE SIMILARITY SOLUTION FOR THE AXISYMMETRIC WAKE WITH SWIRL

The similarity variables are defined as follows:

\[ U - U_\infty = U_s(x)f(\eta, *) ; \quad -\overline{uv} = R_{s,uv}(x)g(\eta, *) \]
\[ W = W_s(x)h(\eta, *) ; \quad -\overline{uw} = R_{s,uw}(x)i(\eta, *) \]
\[ \frac{1}{2}u^2 = K_u(x)k_u(\eta, *) ; \quad \frac{1}{2}v^2 = K_v(x)k_v(\eta, *) \]
\[ \frac{1}{2}w^2 = K_w(x)k_w(\eta, *) ; \quad \frac{1}{2}u^2v = T_{u^2v}(x)T_{u^2v}(\eta, *) \]
\[ \frac{1}{2}\rho \frac{\delta u}{\delta x} = P_u(x)p_u(\eta, *) ; \quad \frac{1}{\rho} \frac{\delta p_u}{\delta x} = P_{uD}(x)p_{uD}(\eta, *) \]
\[ \epsilon_u = D_u(x)d_u(\eta, *) \]

where \( \eta = r/\delta(x) \) and (*) denotes a dependence on initial conditions (wake generator, e.g., turbine type and operating condition).

D.1 \( x \)-Momentum

The reduced \( x \) momentum equation:

\[ U_\infty \frac{\partial (U - U_\infty)}{\partial x} = -\frac{1}{r} \frac{\partial}{\partial r} (r \overline{uv}) + \left\{ \frac{\partial}{\partial x} \left( \overline{v^2 - u^2} + \int_0^\infty \frac{1}{r}(w^2 - \overline{w^2}) + W^2 \right) \right\} \]

(D.2)

The terms in curly brackets are second order and will be neglected for this analysis. Integrating this equation over the wake cross section yields an integral constraint for the conservation of momentum:

\[ U_\infty \int_0^\infty (U_\infty - U) 2\pi r \, dr \cong \pi \theta^2 U_\infty^2 \]

(D.3)

Substituting the similarity variables into the momentum integral and simplifying:

\[ \int_0^\infty 2(U_s f)(\eta \delta) d(\eta \delta) \cong \theta^2 U_\infty \]
\[ U_s \delta^2 \int_0^\infty 2f \, d\eta \cong \theta^2 U_\infty \]

(D.4)

It follows that if \( \delta \equiv \delta_* \) and \( U_s \equiv U_o = (U_\infty - U_c) \):
\[
\frac{U_s}{U_\infty} \sim \left[ \frac{\theta}{\delta_s} \right]^2
\]  

(D.5)

where the "\( \sim \)" symbol means "has the same \( x \) dependence as". Substituting the similarity variables into \( x \) momentum:

\[
U_\infty \frac{\partial U_s f}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left( r R_{s,uv} g \right)
\]  

(D.6)

Simplifying the first term using the chain rule:

\[
U_\infty \frac{\partial U_s f}{\partial x} = U_\infty \left( \frac{U_s df}{d\eta} \frac{d\eta}{d\delta} \frac{d\delta}{dx} + f \frac{dU_s}{dx} \right) = U_\infty \left( -\frac{r U_s d\delta}{\delta^2} \frac{d\delta}{dx} f' + f \frac{dU_s}{dx} \right)
\]  

(D.7)

The second term:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r R_{s,uv} g \right) = \frac{R_{s,uv}}{r} \left( g + r \frac{dg}{d\eta} \frac{d\eta}{dr} \right) = \frac{R_{s,uv}}{r} (g + \eta g') = \frac{R_{s,uv}}{\delta} \frac{(\eta g')'}{\eta}
\]  

(D.8)

After some manipulation, the \( x \) momentum equation becomes:

\[
\left[ \frac{\delta}{U_s \frac{dx}{dx}} \right] f - \left[ \frac{d\delta}{dx} \right] \eta f' = \left[ \frac{R_{s,uv}}{U_\infty U_s} \right] \frac{(\eta g')'}{\eta}
\]  

(D.9)

where the terms in square brackets depend on downstream position \( x \) only, and the non-bracketed terms depend on the new similarity variable only.

**D.2 Reynolds Stress Transport Equations**

The individual Reynolds Stress transport equations are necessary to determine the constraints on the similarity solutions/scaling functions. The reduced transport equation for \( \overline{w^2} \) is as follows:

\[
U_\infty \frac{\partial}{\partial x} \left( \frac{1}{2} \overline{u^2} \right) = -\overline{uv} \frac{\partial}{\partial r} (U - U_\infty) - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{2} \overline{u^2 v} \right) + \frac{p}{\rho} \frac{\partial}{\partial x} - \frac{1}{\rho} \frac{\partial}{\partial x} p u + \nu \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left( \frac{1}{2} \overline{u^2} \right) \right\} - \epsilon_u
\]  

(D.10)

Substituting the similarity variables:

\[
U_\infty \frac{\partial}{\partial x} (K_u k_u) = R_{s,uv} g \frac{\partial}{\partial r} (U_s f) - \frac{1}{r} \frac{\partial}{\partial r} \left( r T_{uv} u^2 v \right) + P_u p_u - \frac{\partial}{\partial x} \left( P_u^D p_u^D \right) + \nu \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} (K_u k_u) \right\} - D_u d_u
\]  

(D.11)

Simplifying the first term using the chain rule:
\[ U_\infty \frac{\partial}{\partial x} (K_u k_u) = U_\infty \left( k_u \frac{dK_u}{dx} + K_u \frac{dk_u}{d\eta} \frac{\partial \eta}{\partial x} \right) = U_\infty \left( k_u \frac{dK_u}{dx} - K_u \frac{d\delta}{dx} \frac{\eta k_u'}{\delta} \right) \quad (D.12) \]

The second term:
\[ R_{s,uv} g \frac{\partial}{\partial r} (U_s f) = R_{s,uv} U_s \frac{df}{d\eta} \frac{d\eta}{d\delta} \frac{d\delta}{dx} = \frac{R_{s,uv} U_s}{\delta} f' g \quad (D.13) \]

The third term:
\[ \frac{1}{r} \frac{\partial}{\partial r} (T_u^2 t_{u^2 v}) = \frac{T_u^2 v}{r} \left( t_{u^2 v} + r \frac{dt_{u^2 v}}{d\eta} \frac{d\eta}{d\delta} \frac{d\delta}{dx} \right) = \frac{T_u^2 v}{r} \left( t_{u^2 v} + \frac{\eta t_{u^2 v}'}{\delta} \right) \quad (D.14) \]

The fourth term requires no simplification. The fifth term is simplified as follows:
\[ \frac{\partial}{\partial x} (P_u^D p_u^D) = p_u^D \frac{dP_u^D}{dx} + P_u^D \frac{dp_u}{d\eta} \frac{d\eta}{d\delta} \frac{d\delta}{dx} = p_u^D \frac{dP_u^D}{dx} - (p_u^D)' \frac{d\delta}{dx} \frac{\eta}{\delta} \quad (D.15) \]

The sixth term:
\[ \frac{\nu}{r} \frac{\partial}{\partial r} \left( \frac{r}{\delta} \frac{\partial}{\partial r} (K_u k_u) \right) = \frac{\nu K_u}{r} \left( \frac{r K_u}{d\eta} \frac{d\eta}{d\delta} \frac{d\delta}{dx} \right) = \frac{\nu K_u}{r} \frac{\partial}{\partial r} (\eta k_u') \quad (D.16) \]

The seventh term requires no simplification. After manipulation, the streamline normal stress transport equation takes this form:
\[ \left[ U_\infty \frac{dK_u}{dx} \right] k_u - \left[ \frac{K_u U_\infty \frac{d\delta}{dx}}{\delta} \right] \eta k_u' = \left[ R_{s,uv} U_s \frac{df}{d\eta} \frac{d\eta}{d\delta} \frac{d\delta}{dx} \right] f' g - \left[ \frac{T_u^2 v}{\delta} \right] \left( \eta t_{u^2 v}' \right) \frac{\eta}{\delta} \quad (D.17) \]

The other component equations can be manipulated in a similar fashion to produce the following equations:

Radial normal stress:
\[ \left[ U_\infty \frac{dK_v}{dx} \right] k_v - \left[ \frac{K_v U_\infty \frac{d\delta}{dx}}{\delta} \right] \eta k_v' = - \left[ \frac{T_v^3}{\delta} \right] \left( \eta t_{v^3}' \right) \frac{\eta}{\delta} + \left[ \frac{T_{v^2 v}}{\delta} \right] \left( \eta t_{v^2 v}' \right) \frac{\eta}{\delta} \quad (D.18) \]
Angular normal stress:

\[
U_\infty \frac{dK_w}{dx} k_w - \left[ \frac{U_\infty K_w}{\delta} \frac{d\delta}{dx} \right] \eta k'_w = - \left[ \frac{T_{uw^2}}{\delta} \right] \left( \eta t_{uw^2} \right)' \nonumber \\
- \left[ \frac{T_{uw^2}}{\delta} \right] t_{uw^2} + \left[ \nu K_w \right] \frac{\eta k'_w}{\eta} - [D_w]d_w \tag{D.19}
\]

(uv) Reynolds shear stress:

\[- \left[ U_\infty \frac{dR_{s,uv}}{dx} \right] g - \left[ \frac{U_\infty R_{s,uv}}{\delta} \frac{d\delta}{dx} \right] \eta g' = \left[ \frac{K_u U_s}{\delta} \right] f' k_v - \left[ \frac{T_{uv^2}}{\delta} \right] \left( \eta t_{uv^2} \right)' \\
+ \left[ \frac{T_{uw^2}}{\delta} \right] t_{uw^2} + \left[ P_{uw} \right] p_{uv} - \left[ \frac{P_{uv}^D}{\eta} \right] (p_{uv}^D)' - \left[ \frac{dP_{uv}^D}{dx} \right] (p_{uv}^D)' \\
- \left[ \frac{P_{uv}^D}{\eta} \frac{d\delta}{dx} \right] \eta (p_{uv}^D)' - \left[ \frac{\nu R_{s,uv}}{\delta^2} \right] \left( \eta g' \right)' - [D_{uw}]d_{uw} \tag{D.20}
\]

D.3 Conditions for the Existence of Similarity

From the x-momentum equation we have that, in order for similarity, these terms must have the same x-dependence:

\[
\left[ \frac{\delta}{U_s} \frac{dU_s}{dx} \right] \sim \left[ \frac{d\delta}{dx} \right] \sim \left[ \frac{R_{s,uv}}{U_\infty U_s} \right] \tag{D.21}
\]

From this, we can obtain the proper scale for the Reynolds stress:

\[ R_{s,uv} \sim U_\infty U_s \frac{d\delta}{dx} \tag{D.22} \]

Inserting this into the streamwise normal stress equation (D.17), we have:

\[ U_\infty \frac{dK_u}{dx} \sim \frac{K_u U_\infty \frac{d\delta}{dx}}{K_u} \sim \frac{U_\infty^2 U_\infty \frac{d\delta}{dx}}{U_\infty K_u} \sim P_u \sim D_u \sim \frac{\nu K_u}{\delta^2} \tag{D.23} \]

Multiplying by \( \frac{\delta}{U_\infty K_u} \):

\[ \frac{\delta}{K_u} \frac{dK_u}{dx} \sim \frac{d\delta}{dx} \sim \frac{U_\infty^2 K_u}{U_\infty K_u} \sim \frac{D_u \frac{\delta}{U_\infty K_u}}{\nu} \tag{D.24} \]

A similarity solution for large Reynolds number is only possible if:

\[ \frac{d\delta}{dx} \sim \frac{D_u \frac{\delta}{U_\infty K_u}}{\nu} \tag{D.25} \]

From the Reynolds stress transport equations combined with (D.22) we find the remaining constraints:
Using these constraints, D.25 can now be written as:

\[
\frac{d\delta}{dx} \sim \frac{U_s}{U_\infty}
\]

Combining this with the result of the momentum integral (D.5), we have:

\[
\frac{d\delta}{dx} \sim \left( \frac{\theta}{\delta} \right)^2
\]

Integrating gives us the scaling function for \( \delta^* \):

\[
\frac{\delta^*}{\theta} = a \left[ \frac{x - x_o}{\theta} \right]^{\frac{1}{3}}
\]

Inserting this result into (D.5) gives us the scaling function for the velocity:

\[
\frac{U_s}{U_\infty} = b \left[ \frac{x - x_o}{\theta} \right]^{-\frac{2}{3}}
\]

These are the same as in the classical solution, e.g. Johansson et al. (2003) [19], but the scaling functions for higher moments (Reynolds stresses etc) are shown to be more complicated by Johansson et al. 2003. Also, since the axisymmetric turbulent wake is a flow with diminishing local Reynolds number \( \frac{U_s \delta^*}{\nu} \), as can be seen from the scaling function above, the flow will eventually fall out of this infinite Reynolds number, viscosity-independent similarity solution, but may arrive at another, viscous-dominated low-Reynolds number similarity solution [19].

### D.4 Effects of Swirl

To investigate the behavior of the swirling component of mean velocity, we now also consider the \( W, <uw>, <vw> \) equations. The rate at which kinematic angular momentum is swept downstream (from integrated angular momentum equation) can be written as

\[
G_\theta = G_o = 2\pi \int_0^\infty [U_\infty W + \overline{uw}] r^2 dr
\]

Knowing that \( \delta^* \sim x^{\frac{1}{3}} \) and \( U_\infty = \text{constant} \), the azimuthal velocity has to decay as

\[
\frac{W_s}{U_\infty} = \left[ \frac{x - x_o}{\theta} \right]^{-1}
\]

In general, it should be noted that properly normalized mean velocity profiles always collapse, and the source-dependent differences will show up in the wake spreading rate and the higher turbulent moments. If a numerical model for the axisymmetric, turbulent, swirling wake cannot reproduce
scaling behavior predicted by an equilibrium similarity solution, then it is not capturing the essential wake physics.