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ACOUSTIC IDENTIFICATION OF MARINE SEDIMENTS BY STOCHASTIC METHODS

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by

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NOMENCLATURE

a	the damped natural frequency of compressional viscoelastic wave system,
A	the damped natural frequency of thermoviscoelastic medium,
b	the temporal attenuation of compressional wave system,
B	the temporal attenuation of thermoviscoelastic medium,
b_1	the absorption coefficient of compressional wave system,
B_1	the absorption coefficient of thermoviscoelastic medium,
c	the natural frequency of compressional wave system,
C	the natural frequency of thermoviscoelastic medium,
$e(t)$	envelope function,
$\alpha(t)$	noise function,
$\frac{b}{c}$	compressional wave system damping factor,
$\frac{B}{C}$	thermoviscoelastic medium damping factor,
β/b	is the relative comparison between the exponential decay coefficient of the noise correlation function and the decay coefficient associated with the compressional wave system of the viscoelastic medium,
a/Ω	is the relative comparison between the natural damped frequency of the compressional wave system and the frequency of the noise correlation function,
ct	number of response cycles of the compressional wave system of a viscoelastic medium,
Q	$\frac{1}{2(b/c)}$ is the quality factor of the compressional wave system,
μ'	is the shear modulus of the viscoelastic medium (Lamé parameter),
μ''	is the shear viscosity of the viscoelastic medium,

λ' is the compressional modulus of the viscoelastic medium (Lamé parameter),

λ'' is the compressional viscosity of the viscoelastic medium,

ρ is the density of medium,

k is the Fourier transform parameter (wave number),

β is the correlation function decay constant,

Ω is the harmonic frequency of the correlation function,

$r(t)$ is the response of the compressional wave system of the viscoelastic medium,

$G(t-t')$ is the retarded response Green's function for the compressional wave system of the viscoelastic medium in time domain,

$G'(t-t')$ is the real part (even) of $G(t-t')$,

$G''(t-t')$ is the imaginary (odd) part of $G(t-t')$,

$G(\omega)$ is the Green's function for the compressional wave system of the viscoelastic medium in frequency domain,

$G'(\omega)$ is the even (real) part of $G(\omega)$,

$G''(\omega)$ is the odd (imaginary) part of $G(\omega)$,

Subscripts

i Refers to initial valued problem,

L Refers to longitudinal wave,

T Refers to transverse wave,

t Refers to temperature dependent term,

ad Refers to adiabatic term (also superscript),

Superscripts

ad Refers to adiabatic term,

$\sim$ Refers to viscoelastic shear wave system.

ABSTRACT

ACOUSTIC IDENTIFICATION OF MARINE SEDIMENTS BY STOCHASTIC METHODS

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An attempt is made to understand the behavior of the quality factor Q of a viscoelastic compressional wave system modeled as a lightly damped harmonic oscillator excited by random acoustic inputs in an ocean environment so that the ocean subbottom soil sediments can be identified and thus classified.

After the introduction of fundamental differential equations of an elastic and viscoelastic medium with and without temperature effects the vector field equations are simplified by separating the field into longitudinal and transverse parts. The unit impulse response in the liquid is expressed as a Green's function due to point-source field excitation. Then the initial valued Green's function is determined using Kubo's formula.

The mean-square response of a lightly damped viscoelastic medium to a special type of non-stationary random excitation is determined. The excitation function is taken in the form of a product of a well-defined deterministic envelope function and a part which describes the statistical characteristic of the excitation. The latter is assumed white as well as correlated noise functions and both the unit step and rectangular step envelope functions are considered. By taking

into account this particular type of non-stationary input and the wave characteristics of the lightly damped viscoelastic medium, the mean-square response for various types of excitation and damping parameters is evaluated. Then the multi-layer problem is solved and the mean-square response of double layered viscoelastic medium to the correlated noise modulated by a rectangular step envelope function is analytically determined. Also, the initial value problem is solved and the mean-square response of the medium to non-stationary random inputs are also analytically determined.

To further understand the behavior of the quality factor in a viscoelastic medium, "Higher Order Autocorrelation" technique is introduced as a signal processing procedure. The first, second, third, and fourth order autocorrelations of the normalized rms of a visco-elastic medium to non-stationary random inputs are determined using Fast Fourier transform technique on the computer. It is shown that the system with high quality factor has a flat Gaussian envelope and the system with lower quality factor has a sharper Gaussian envelope function. This is a very useful technique to estimate the quality factor of a system by comparing them when digitized remote data are available.

The temperature effects are included in the medium and the compressional viscous Lamé parameter is expressed in terms of a temperature independent and a temperature dependent expressions and other thermo-dynamical and mechanical variables. Using this result the quality factor and thus damping of the medium and also bandwidth is expressed as a function of temperature.

The experimental data analysis shows that from the core data zones exist and for each zone there is a corresponding acoustical reflector peak in remotely obtained digitized data.

The extension of the single degree-of-freedom damped harmonic oscillator model of a single layer of viscoelastic medium in $(\vec{k};t)$ and $(\vec{k};\omega)$ domains to a multi degree-of-freedom damped harmonic oscillator model which prescribes the characteristics of the viscoelastic reflectors is validated when the extended model showed the presence of a peak due to second layer.

This thesis is dedicated to my parents, Dogan and Belkis, and to my brother, Osman. Without their unrelenting faith, support and firm belief in me and in education both in and out of school this thesis would have never been accomplished.

Such is fate of life:
One day you are gently placed on the saddle;
The next day the saddle is placed on your shoulders.

CHAPTER I

I. INTRODUCTION

Acoustic sensing methods have been used to identify and classify the sediments on the ocean floor. Researchers have developed analytical models and improved them to better understand acoustical reflectivity measurements from the ocean floor. The objective of this thesis will be to provide and develop an analytical model of the ocean subbottom which presents a deeper knowledge as well as a better understanding of acoustic reflections from the ocean floor in the form of digitized data. Furthermore, an appropriate acoustic signal processing technique is presented to aid in the understanding and interpretation of the acoustic data.

I-1 REVIEW OF PREVIOUS RESEARCH

Initially, extensive use of reflection profiling to study subsurface geology began in 1930's as a part of the search for petroleum (Dobrin, 1960). It should be noted that the low frequency seismic sources used in the exploration for oil have a wavelength of about 300 feet in the earth. Ewing and Ewing (1970) identified, in their CPS records in the Atlantic, a zone of reflectors bounded above and below by acoustically transparent layers which they call Horizon. The drilling of the DSDP has been unable to unequivocably establish the significance of Horizon A (Ewing and Hollister, 1972).

Ryan, et al. (1965) studied sediment cores in the Tyrrhenian plain. Correlations between layers of coarse grain, low porosity sediments embedded with finer grains and the seismic reflection records which were generated by both ship-mounted and near bottom 12 kHz sources were established.

Siva and Hollister (1973) examined a core taken in the shallow waters off the Gulf of Maine and made a tentative correlation between a zone of higher water content and a reflection observed in their 3.5 kHz records.

Hamilton (1965, 1969, 1971, 1972) has employed a relatively simple ray theory model for the identification of ocean subbottom soil sediments. This model was used to analyze the reflected acoustic pulses from the ocean subbottom. Here, the return signal is described by the Rayleigh reflection coefficient which has the acoustic impedence as the most important sediment parameter after accounting for the spherical spreading and dissipation of the wave.

Using the above model, Breslau (1964) developed an empirical relation between the subbottom reflection coefficient and sediment porosity. He was then able to classify ocean sediment types directly from observations of the reflected pulse since each sediment type can be closely related to its porosity.

Another important parameter of the subbottom soil is its rigidity which has been identified by the presence of shear waves. Taking this parameter into account the geometrical ray model of Breslau (1964) was modified to a field theoretical one. The ocean subbottom was then modeled as a Voigt viscoelastic solid. For a given acoustic inputs Magnuson (1972) analytically calculated the response for one subbottom soil layer. The computer results were obtained for grazing angles and a table for different material of subbottom sediments were determined. This then increased the ability to classify ocean sediment types from observations since each sediment type can be closely related to the magnitude of the return signal at each grazing angle.

Furthermore, since the ocean subbottom consists of more than one sediment layer, the model was extended to include multilayer effects by A. Yildiz (1972), Magnuson (1972) and Stewart (1975). In this model, the boundary conditions, namely, the continuity of stress field as well as the displacements at the interfaces are directly taken into consideration and the analysis was done in $(\vec{r};\omega)$, space-frequency domain.

I-2 RESULTS PRESENTED IN THESIS

Without losing the general features of the above model, it is shown that the theoretical model can be modified to a damped harmonic oscillator model by taking the spatial Fourier transformation of the (Voigt) viscoelastic differential equation.

Then the soil parameters, which are characterized by the viscoelastic parameters μ', μ'', λ', and λ'', are expressed in terms of the damped harmonic oscillator parameters such as the natural and damped oscillation frequency, and damping factor.

The analytical solution of the differential equation in wave-number and time domain, $(\vec{k};t)$, which describes the non-equilibrium behavior of the viscoelastic medium is presented. The absolute value of the solution in wave-number and frequency domain is used to determine the quality factor of the medium. The quality factor is related to the damping which gives a measure of rigidity and dissipation of energy in the viscoelastic body.

The quality factor becomes a very important parameter in the classification of ocean subbottom soil sediments since each sediment type can closely be related to its damping parameter. This will then increase the ability to classify ocean sediments since by comparing the quality factors of the input and reflected acoustic pulses an idea of the damping in the medium will be obtained. Thus, the subbottom soil can be classified according to their quality factors.

Furthermore, starting with the fundamental principles of elasticity and thermodynamics, the viscous Lamé parameters λ'' and μ'' are derived in terms of specific heat parameters and other relevant thermodynamical and mechanical variables. This gives more detailed information on the quality factor parameter and also on the width of resonance of acoustic pulses in wave-number and frequency domain.

Also, since the ocean subbottom consists of more than one sediment layer (see Figure II-3) the damped harmonic oscillator model is expanded to multidegree freedom one. The multidegree model is important in interpreting and understanding the peaks in the acoustic reflectivity measurements.

If these subbottom layers are assumed to be plane and thus forming zones, then they become simple set of reflectors to the acoustic input signals. This is indicated whenever a core measurement analysis indicates a presence of a zone there is a corresponding reflector peak in the return acoustic signal.

The input acoustic signals in the ocean are random. Hence, the acoustic response(s) of the ocean subbottom soil are theoretically calculated from the analytical model as the mean-square response of a viscoelastic medium to nonstationary random excitation (or random acoustic input).

Finally, the first, second, third and fourth order autocorrelations of mean-square response of a viscoelastic medium to nonstationary random inputs are determined. It is shown that the higher order autocorrelations of the response have an approximately a Gaussian envelope function. The system with low damping values of higher quality factor has a flat Gaussian envelope and the system with higher damping values of lower quality factor has a sharper Gaussian envelope function. This is a useful criteria to estimate the quality factor of a system by comparing them when digitized remote data are available.

All of the above will increase the present technology of identifying and classifying the ocean subbottom soil sediments by acoustic sensing.

II. THEORETICAL MODEL

A theoretical model which represents the physical situation of the field of measurement in the language of differential equations and their solutions both analytical and/or computer is the paramount need of the subbottom soil indentification for the following reasons:

1) To coordinate and exhange data and laboratory soil mechanics results with data and signal processing areas, and to adjust, readjust the model parameters of the system thus giving a more realistic system representation;

2) To assess and evaluate the results of both signal processors and soil mechanic investigators. The model of the sea and subbottom as a multidegree system is the only reference station where such assessments can be made;

3) To convey the necessary information to field measurement experimentalists in order that they may carry on the field experiments in the optimum way for the appropriate identification of subbottom soil. Examples of some vital information are:

a) The operation frequencies and acoustic power (and the wave-length regime) of the acoustic input have to be suggested to the field experimentalists,

b) The necessity of oblique incidence and the range of oblique reflection have to be suggested to the field experimentalists.

Furthermore, a theoretical model is also very important for the soil mechanics investigators:

1) A very intensive and close information exchange between the soil mechanics group and the model builders become obvious because of the need to know the following properties:

Density of the subbottom material,

Porosity of the subbottom material,

Liquid content of the subbottom material,

Granular size of the subbottom material, and

Heterogeneity of the subbottom material.

Besides processing this vital information which the model builders will use to simulate computer results with given acoustic input, the comparison and correlation of simulated results based on the model with the model independent of data and signal analysis, i.e., higher order autocorrelation techniques, are very important.

2) Also, the dynamical parameters such as longitudinal and transverse viscoelastic wave propogation velocities need to be measured:

$$C_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{II.1}$$

$$C_T = \sqrt{\frac{\mu}{\rho}} \tag{II.2}$$

where λ and μ are the two essential Lamé parameters of an isotropic viscoelastic medium. Furthermore, these parameters are complex functions (or numbers) in such a viscoelastic medium because of dissipative nature of the subbottom.

3) However, there exists with great certainty the need to know the following properties of the subbottom layers:

The anisotropy of the subbottom,

The existence of the Cosserat type properties of the subbottom, and

The degree of heterogeneity (or mixing) of the subbottom material.

4) The measurement of the viscoelastic parameters turn out to be probably the most sensitive measurement for the identification problem. Indeed, the Lamé parameters $\lambda = \lambda' + \lambda'' \frac{\partial}{\partial t}$, $\mu = \mu' + \mu'' \frac{\partial}{\partial t}$ have operator forms in the time domain whereas they have $\lambda = \lambda' + i\omega\lambda''$, $\mu = \mu' + i\omega\mu''$ complex forms in the frequency domain. These are further discussed in later sections.

There is also a very important effect which is the non-linear effects of the subbottom. However, this can be detected from frequency doubling and/or frequency tripling of the output acoustic signals. The cause of the nonlinearity is usually an interaction of the viscoelastic waves which are converted from acoustic waves from the liquid (ocean) - subbottom interface (boundary) couplings in the layer structure - (see Figure II-1).

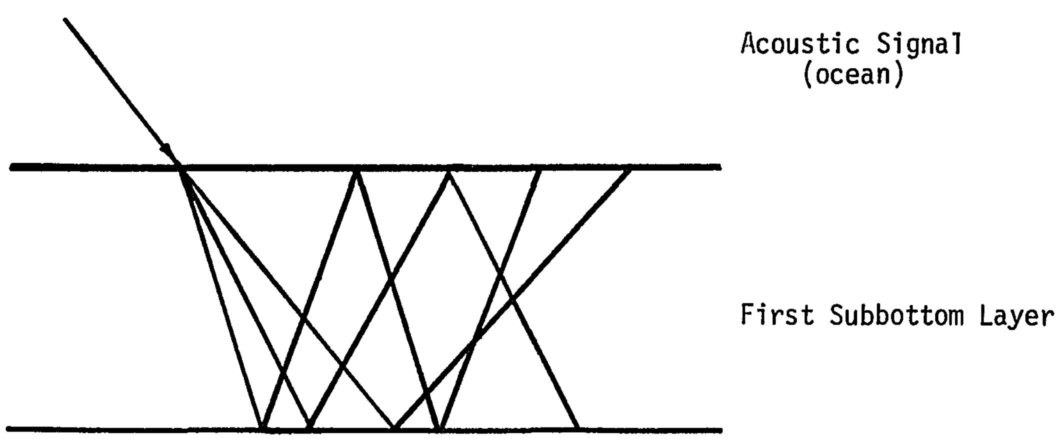


Figure II-1

On the other hand, the origin of the non-linearities can be attributed to the geomorphological reasons of the layers such as clustering of certain types of material such as clay in a random manner in the layer. (see Figure II-2)

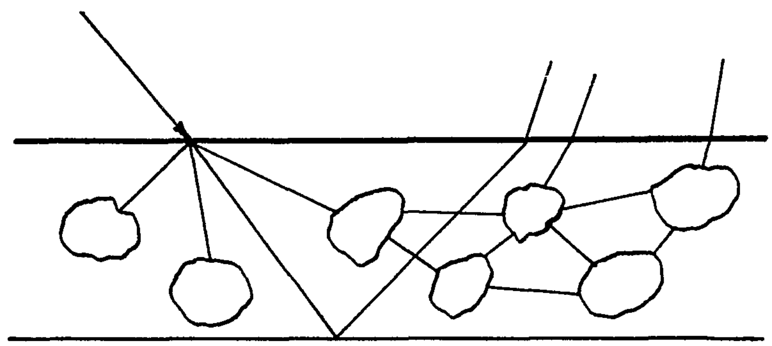


Figure II-2

Unless the frequency doubling and/or frequency tripling effects are observed in the output effects it can be concluded that the non-linearities are weak and the linear model will not need modifications to include non-linear behavior. In this investigation the non-linear behavior is not taken into consideration.

After these preliminary remarks, the description of the theoretical model of the ocean subbottom system is presented. The original model (Magnuson, 1972) was designed for the shallow water (up to 600 feet) conditions and the acoustic probing could be achieved by surface vehicle(s) (ship(s)). It is mainly a one liquid layer plus layered subbottom structure (see Fiugre II-3).

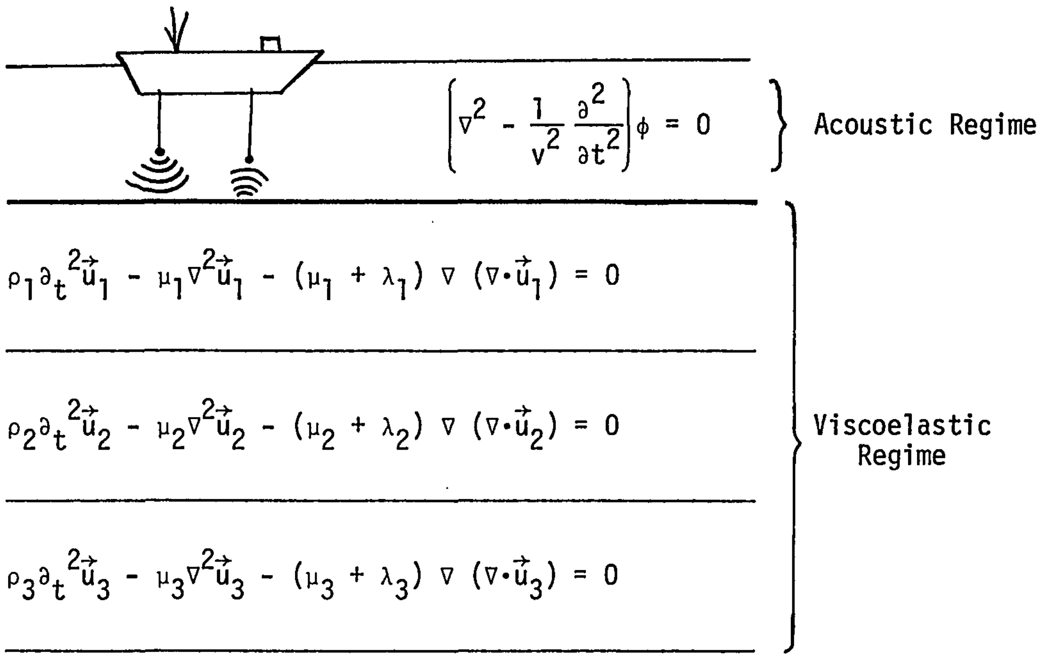


Figure II-3

The acoustic response from such a system with given acoustic inputs were first analytically calculated for one layer subbottom by A. Magnuson (1975) as mentioned earlier.

This model was the field theoretical version of the original Hersey-Breslau (geometric ray theory) model. The Hersey-Breslau model did not contain shear effects of the subbottom and was taken after the Rayleigh ray theory.

The analytical model used in this investigation is a damped harmonic oscillator model. The model is introduced from the fundamental field equations simply first going to the Fourier-domain in spatial coordinates. Briefly this will be shown.

The isotropic wave equation in one subbottom layer reads

$$\rho \partial_t^2 \vec{u} - \mu \nabla^2 \vec{u} - (\mu + \lambda)\, \nabla\, (\nabla \cdot \vec{u}) = 0 \tag{II.3}$$

where $\lambda = \lambda' + \lambda'' \partial_t$ and $\mu = \mu' + \mu'' \partial_t$ in the linear viscoelastic theory. The above equation then becomes

$$\rho \partial_t^2 \vec{u} - \mu' \nabla^2 \vec{u} - (\mu' + \lambda')\, \nabla\, (\nabla \cdot \vec{u}) - \mu'' \partial_t \nabla^2 \vec{u} - (\mu'' + \lambda'') \partial_t \nabla (\nabla \cdot \vec{u}) = 0 \tag{II.4}$$

where $\vec{u} = \vec{u}(\vec{r};t)$ and taking the spatial Fourier transform of the above differential equation leads to

$$\rho \partial_t^2 \vec{u} + \mu'' k^2 \partial_t \vec{u} + (\mu'' + \lambda'') \partial_t \vec{k}\, (\vec{k} \cdot \vec{u}) +$$

$$+ \mu' k^2 \vec{u} + (\mu' + \lambda')\, \vec{k}\, (\vec{k} \cdot \vec{u}) = 0 \tag{II.5}$$

where $\vec{u} = \vec{u}(\vec{k};t)$ which has essentially an analogous structure to the

$$\partial_t^2 \vec{u} + 2\zeta\omega_n \partial_t \vec{u} + \omega_n^2 \vec{u} = 0 \tag{II.6}$$

damped harmonic oscillator model. Now, the system (theoretical model) can be represented geometrically as in Figure II-4.

$$\left(\partial_t^{\,2} + \frac{k_0^{\,2}}{v^2}\right)\phi = 0$$

$$\rho_1 \partial_t^{\,2}\vec{u}_1 + \lambda_1 k_1^{\,2}\vec{u}_1 + (\mu_1 + \lambda_1)\vec{k}_1(\vec{k}_1 \cdot \vec{u}_1) = 0$$

$$\rho_2 \partial_t^{\,2}\vec{u}_2 + \lambda_2 k_2^{\,2}\vec{u}_2 + (\mu_2 + \lambda_2)\vec{k}_2(\vec{k}_2 \cdot \vec{u}_2) = 0$$

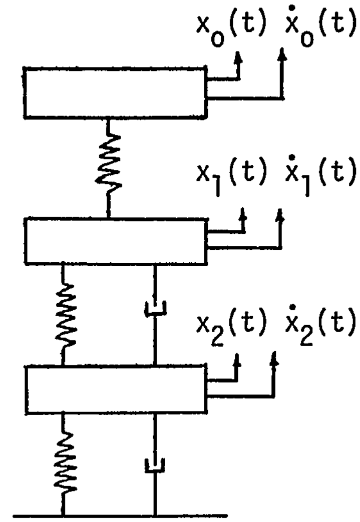


Figure II-4

Here, the boundary conditions, namely, the continuity of the stress field as well as the displacement vectors at the interfaces now have been substituted by harmonic oscillator initial conditions: $x(t)$ - displacement and $\dot{x}(t)$ - velocity conditions. A. Yildiz (1972), A. Magnuson (1972) and Stewart (1975) have considered these boundary conditions and determined the response of such a system in space-frequency domain.

In this investigation, the geometrical effects that is the boundary effects are taken into consideration in a different manner. In $(\vec{k};t)$ and $(\vec{k};\omega)$ domain the boundary effects are observed by noting changes in such parameters as the quality factor.

Continuing with the description of the theoretical model, it is important to note the advantages of this rather simple and effective model.

1) For one degree of freedom system one has a resonant frequency (see Figure II-5). This represents the most dominant frequency of the layer which also carries the characteristic information of the layer, namely

$$\omega_n = \omega_n(\mu,\lambda) \tag{II.7}$$

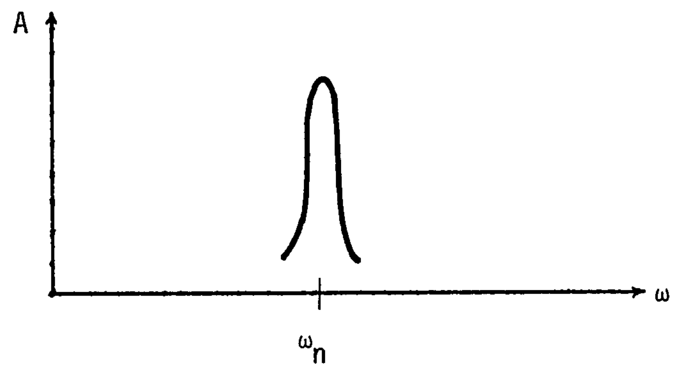


Figure II-5

The width of the peak is related to the imaginary parts of the Lamé parameters, namely μ" and λ".

2) The two degree of freedom will be represented by two resonant frequencies - see Figure II-6. Once again, the first resonant frequency is the characteristic frequency

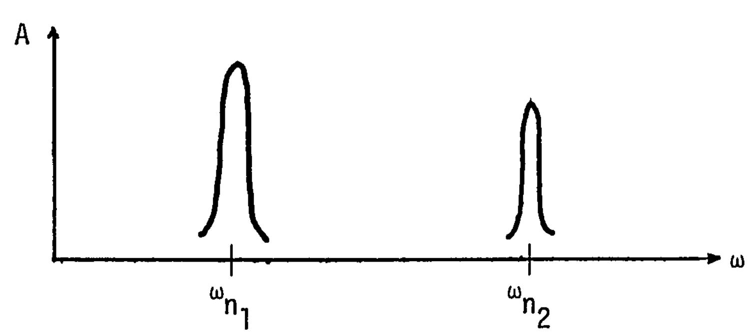


Figure II-6

of the first layer whereas the second resonant frequency is the characteristic frequency of the second layer. This can go on further for three, four degrees of freedom and determine their characteristic frequencies.

The motivation for choosing this model has come from the application of higher order correlation technique which the Theoretical and Applied Mechanics group has developed at the University of New Hampshire. Indeed, the raw acoustic data has been developed in the computer and the resonance peaks have been observed. The interpretation of the signal porcessing results leads one to adopt the damped harmonic oscillator model. As shall be discussed in the signal processing section, the virtue of the higher order autocorrelation technique is to make use of only the output signals with no reference or need of the input signals to the system whatsoever. This radical change from the usual signal processing is demanded because of considerable uncertainties in the input signals in the field experiments.

Again, one of the main contributions on the damped harmonic oscillator model is the determination of the width of the resonance peak in terms of thermodynamical and mechanical variables. Thus, the usual empirical description of the width now has given its place to an analytical expression. Therefore, the damping parameters of the sub-bottom soil can be determining independently by the model. The signal processing also yields an empirical width estimation. The comparison of these two values is the most sensitive probe for the general identification of the ocean subbottom soil.

III. LINEAR VISCOELASTIC THEORY

In this section, the general theory of viscoelasticity is presented. First, the linear elasticity theory is briefly outlined with emphasis on physics. The viscoelasticity is introduced from the fundamental elastic relations by introducing viscosity or energy dissipation mechanism into an elastic body. The differential equations which describes the elastic and viscoelastic compressional and shear wave mediums are obtained in $(\vec{r},t)$, $(\vec{r},\omega)$, $(\vec{k},t)$, and $(\vec{k},\omega)$ domains and their respective solutions are presented as a response to an unit impulse excitation, that is in terms of Green's functions. The dissipative and reactive parts and causality effect of Green's function are also presented. Finally, the initial value Green's function is evaluated using Kubo's formula.

III.1 LINEAR ELASTIC THEORY

The small deformation theory of elasticity has been established for many years and has been used for the solution of variety of problems (Love, 1927). The main concern here will be the mechanical aspects of the elastic medium which supports small deformations. The thermodynamical conditions can and will be subsequently built into the theory. For the purpose of these studies the equations of motion (or dynamical equations) and equations of state (or stress-strain relations) are the two primary groups of formalism which describes the mechanical behavior of an elastic body. Compatibility relations are nothing but geometric continuity equations which can also be brought into the picture whenever such a need arises.

Omitting the usual preludes, the dynamical equation of motion for an elastic body can be written as:

$$\partial_i \sigma_{ij} + f_j^{int} = F_j^{ext} \qquad (i,j = 1,2,3) \qquad \text{(III.1-1)}$$

where σ_{ij} is the stress tensor, $\partial_i = \frac{\partial}{\partial x_i}$ is the derivative operator, f_j^{int} is the internal force term or D'Alembertian and equals to $-\rho\partial_t^2 u_i$ with ρ is the density of the elastic body, u_i is the displacement vector, $\partial_t = \frac{\partial}{\partial t}$ is the time derivative and F_j^{ext} is the external force term. Thus, the equation of motion can be written as

$$\partial_i \sigma_{ij} - \rho\partial_t^2 u_j = F_j \qquad \text{(III.1-2)}$$

For small deformations a general relation between stress and strain must be developed and understood. To do this, a relation between the local strain at every point in an elastic body must be determined to the internal forces - the stresses in the body. For each small element of the body it is assumed that the Hooke's law holds true and the stresses are proportional to the strains. The stress tensor σ_{ij} is defined as the i th component of the force across a unit area perpendicular to the j-axis. Hooke's law states that each component of σ_{ij} is related to each of the components of strain $\varepsilon_{k\ell}$. Since σ_{ij} and $\varepsilon_{k\ell}$ each have nine components, there are eighty-one possible coefficients which describe the elastic properties of the body. These coefficients can be written as $C_{ijk\ell}$ and can be defined by the equation

$$\sigma_{ij} = C_{ijk\ell}\, \varepsilon_{k\ell} \qquad \text{(III.1-3)}$$

where $C_{ijk\ell}$ is called the "tensor of elasticity."

Before any further restrictions on the discussion of the stress-strain relations via tensor of elasticity to those bodies which are homogeneous and isotropic, a brief outline considering only infetisimal theory of elastic bodies which are not necessarily isotropic shall be discussed. Without loss of generality, it is assumed that the stress σ_{ij} and strain $\varepsilon_{k\ell}$ have the following symmetric properties:

$$\sigma_{ij} = \sigma_{ji}$$
$$\varepsilon_{k\ell} = \varepsilon_{\ell k} \qquad \text{(III.1-4)}$$

These dictate symmetry conditions on the $C_{ijk\ell}$ tensor:

$$C_{ijk\ell} = C_{jik\ell}, \quad C_{ijk\ell} = C_{ij\ell k}, \quad C_{ijk\ell} = C_{jik\ell} \qquad \text{(III.1-5)}$$

Because of the symmetric properties described in Equation (III.1-5) the tensor of elasticity becomes

$$C_{ijk\ell} = C_{jik\ell} \quad \text{(symmetry in i and j indices)} \qquad \text{(III.1-6a)}$$

$$C_{ijk\ell} = C_{ij\ell k} \quad \text{(symmetry in k and } \ell \text{ indices)} \qquad \text{(III.1-6b)}$$

$$C_{ijk\ell} = C_{ji\ell k} \quad \text{(symmetry in i and j, k and } \ell \text{ indices, simultaneously)} \qquad \text{(III.1-6c)}$$

and an additional property due to cartesian system of coordinates* is written as

$$C_{ijk\ell} = C_{k\ell ij} \qquad \text{(III.1-6d)}$$

---

*Actually the cartesian coordinate system ignores the difference between covariant and contravariant indices, i.e., $C^{ij}_{k\ell} = C^{k\ell}_{ij}$.

A propos, a few well known definitions can be made:

Elastically homogeneous: If the elastic coefficients $C_{ijk\ell}$ are constants, then the elastic body is homogeneous.

Isotropic: If the elastic coefficients do not depend on the spatial orientation, then the elastic body is an isotropic body.

Otherwise, the body is said to be aelotropic or anisotropic.

III.1.a ELASTIC COEFFICIENTS

In general, $C_{ijk\ell}$ forms a matrix of 9x9 with eighty-one coefficients. However, because of the symmetry of the stress ($\sigma_{ij} = \sigma_{ji}$) and the strain ($\varepsilon_{k\ell} = \varepsilon_{\ell k}$) tensors the tensor of elasticity $C_{ijk\ell}$ will be reduced to a 6x6 matrix or thirty-six coefficients. Again the symmetry of the indices (ij and/or kℓ) forces these thirty-six coefficients to be composed of twenty-one different coefficients. This is simply due to the symmetry of the off-diagonal terms in the coefficient matrix. Here the twenty-one elastic coefficients $C_{ijk\ell}$ will describe an aelotropic body.

From this general situation higher order symmetries of special cases shall be briefly reviewed. [Green and Zerna, 1954]

Symmetry with respect to a plane, say x_1-x_2 plane in cartesian coordinates, reduces the elastic coefficients from twenty-one to thirteen coefficients.

Symmetry with respect to two orthogonal planes, say $x_1 = 0$ and $x_3 = 0$, reduces the elastic coefficients from thirteen to nine coefficients. This type of symmetry is called orthotropy.

Symmetry with respect to rotation is obtained from an orthotropic body, say with rotation of axes in the form $x_1 \cos\theta + x_2 \sin\theta$, $-x_1 \sin\theta + x_2 \cos\theta$ and third axis remaining the same, reduces the elastic coefficients from nine to five coefficients.

Symmetry with respect to <u>all</u> possible changes to other rectangular cartesian systems of axes reduces the elastic coefficients from five to two coefficients. This type of a body is called an <u>isotropic</u> body. These two coefficients are given by C_{1111} and C_{1122} and the stress-strain relation can be written as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{1111}-C_{1122}) & 0 & 0 \\ & & & & \frac{1}{2}(C_{1111}-C_{1122}) & 0 \\ & & & & & \frac{1}{2}(C_{1111}-C_{1122}) \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \end{pmatrix}$$

(III.1-7)

Now these coefficients can be written in an alternate notation

$$\lambda = C_{1122} \quad , \quad \mu = \frac{1}{2}\left(C_{1111} - C_{1122}\right) \qquad \text{(III.1-8)}$$

where μ and λ are Lamé parameters.

Finally, from Equation (III.1-7) and from the symmetric properties shown in Equation (III.1-6) the elastic coefficients can be expressed in compact form as

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \qquad \text{(III.1-9)}$$

Now, the differential equation and its solution for an isotropic and homogeneous elastic body can be presented.

Using Equations (III.1-9) and (III.1-3) the stress relation becomes

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{\ell\ell} + \mu(\varepsilon_{ij} + \varepsilon_{ji})$$

or

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{\ell\ell} + 2\mu\varepsilon_{ij} \qquad \text{(III.1-10)}$$

since $\varepsilon_{ij} = \varepsilon_{ji}$.

The differential equation becomes

$$-\rho\partial_t{}^2u_j + \lambda\partial_j\varepsilon_{\ell\ell} + 2\mu\partial_i\varepsilon_{ij} = F_j \qquad \text{(III.1-11)}$$

where the strain tensor is defined as

$$\varepsilon_{\ell\ell} = \partial_\ell u_\ell \quad , \quad \varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \qquad \text{(III.1-12)}$$

Hence, Equation (III.1-11) becomes

$$\rho\,\partial_t{}^2u_j - (\lambda + \mu)\partial_j\partial_i u_i - \mu\partial^2 u_j = F_j \qquad \text{(III.1-13)}$$

in tensor notation; in vector notation it becomes

$$\rho\partial_t^2\vec{u} - (\lambda + \mu)\,\nabla\,(\nabla\cdot\vec{u}) - \mu\nabla^2\vec{u} = 0 \qquad \text{(III.1-14)}$$

for no external force. This gives the differential equation for an homogeneous and isotropic elastic bodies. Noting that this is a vector equation which combines three types of polarizations: longitudinal -L, vertical shear -VS and horizontal shear -HS (modes), the equation can be separated into longitudinal or shear (transverse) parts with the following definitions:

$$\vec{u} = \vec{u}_L + \vec{u}_T$$

$$\nabla \times \vec{u}_L = 0 \qquad \text{(III.1-15)}$$

$$\nabla \cdot \vec{u}_T = 0$$

Thus, the equation results in the well-known wave equation form

$$\left(\nabla^2 - \frac{1}{C_T^2}\,\partial_t^2\right)\vec{u}_T = 0 \qquad \text{(III.1-16a)}$$

$$\left(\nabla^2 - \frac{1}{C_L^2}\,\partial_t^2\right)\vec{u}_L = 0 \qquad \text{(III.1-16b)}$$

where $C_T = \sqrt{\mu/\rho}$ and $C_L = \sqrt{(\lambda + 2\mu)/\rho}$ are the transverse and longitudinal velocities, respectively. (Landau, Lifshitz, 1970)

The above set of equations can be appropriately solved by the Green's function method. In Green's function formalism the equations become

$$\left(\nabla^2 - \frac{1}{C_{T,L}^{\,2}}\,\partial_t^{\,2}\right) G_{T,L}(\vec{r}-\vec{r}\,';t-t') = \delta(\vec{r}-\vec{r}\,')\delta(t-t') \qquad \text{(III.1-17)}$$

where $C_{T,L}$ and $G_{T,L}$ are the appropriate velocity and Green's function for the desired polarization, respectively. Using the Fourier transformations

$$G_{T,L}(\vec{r}-\vec{r}\,';t-t') = \int\frac{d\omega}{2\pi}\iiint\frac{d^3k}{(2\pi)^3}\,G_{T,L}(\vec{k},\omega)e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\,e^{\,i\omega(t-t')} \qquad \text{(III.1-18a)}$$

$$\delta(\vec{r}-\vec{r}\,') = \iiint\frac{d^3k}{(2\pi)^3}\;e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \qquad \text{(III.1-18b)}$$

$$\delta(t-t') = \int\frac{d\omega}{2\pi}\;e^{\,i\omega(t-t')} \qquad \text{(III.1-18c)}$$

Equation (III.1-17) becomes in $(\vec{k};\omega)$ domain

$$\left(-k^2 + \frac{\omega^2}{C_{T,L}^{\,2}}\right) G_{T,L}(\vec{k};\omega) = 1 \quad ; \qquad \text{(III.1-19a)}$$

in $(\vec{k};t)$ domain

$$\left(-k^2 - \frac{1}{C_{T,L}^{\,2}}\,\partial_t^{\,2}\right) G_{T,L}(\vec{k};t-t') = \delta(t-t') \quad ;$$

or

$$\left(\partial_t^{\,2} + C_{T,L}^{\,2}\, k^2\right) G_{T,L}(\vec{k};t-t') = -\delta(t-t') \qquad \text{(III.1-19b)}$$

which is in the undamped harmonic oscillator form;

in $(\vec{r};\omega)$ domain

$$\left(\nabla^2 + \frac{\omega^2}{C_{T,L}^{\,2}}\right) G_{T,L}(\vec{r}-\vec{r}\,';\omega) = \delta(\vec{r}-\vec{r}\,') \qquad \text{(III.1-19c)}$$

which is in the Helmholtz equation form. The solutions to these set of equations are well-known and are given by:

in $(\vec{r};\omega)$ domain

$$G_{T,L}(\vec{r}-\vec{r}\,';\omega) = \frac{e^{ik_{T,L}|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \qquad \text{(III.1-20a)}$$

where $k_{T,L} = \omega/C_{T,L}$;

in $(\vec{k};t)$ domain

$$G_{T,L}(\vec{k};t-t') = \frac{e^{ikC_{T,L}(t-t')}}{k\,C_{T,L}} \quad ; \qquad \text{(III.1-20b)}$$

in $(\vec{k};\omega)$ domain

$$G_{T,L}(\vec{k};\omega) = \frac{1}{\left(\frac{\omega}{C_{T,L}}\right)^2 - k^2} \quad ; \qquad \text{(III.1-20c)}$$

and finally, in $(\vec{r};t)$ domain

$$G_{T,L}(\vec{r}-\vec{r}';t-t') = \frac{\delta\left[(t-t') - \frac{|\vec{r}-\vec{r}'|}{C_{T,L}}\right]}{4\pi|\vec{r}-\vec{r}'|} \qquad \text{(III.1-20d)}$$

Thus, the solution to the differential equation of an isotropic and homogeneous elastic body in Green's function formalism was presented.

III.2. LINEAR VISCOELASTICITY THEORY

In discussion of motion in elastic bodies, it was assumed that the deformation, that is the exhibition of solid bodies to change in shape and volume due to external applied forces, is reversible. In reality, if the motion occurs with infitesimal speed only then the process is thermodynamically reversible. However, an actual motion has finite velocities, hence the body is not in equilibrium at every instant and therefore processes will take place in it which tend to return it to the equilibrium position. These processes has the result that the motion is irreversible and the mechanical energy, which is the sum of the kinetic energy of the macroscopic motion in the elastic body and its elastic potential energy arising from the deformation, is dissipated as heat.

There are two types of dissipation of energy which may occur. First, if the temperature at different points in the body is different, then irreversible processes of thermal conduction can take place in it. Second, if any internal motion occurs in the body, then there are irreversible processes arising from the finite velocity of the applied motion. This type of energy dissipation is referred to as <u>viscosity</u>.

Elastic bodies possessing these mechanical properties are called (damped) viscoelastic solids.

In many cases the velocity of macroscopic motions in the body is so small that the energy dissipation is not considerable. Hence, a state of "almost irreversible" processes exist. If a mechanical system whose motion involves the dissipation of energy, then this motion can be described by the ordinary equations of motion (see Equation (III.1-14)) with the external forces acting on the system increased by the dissipative forces or frictional forces, which are linear functions of the velocities of the applied motion. Hence, for viscoelastic bodies the first time derivitive of the strain will give rise to frictional forces which is linear function of velocity. In general, therefore, the modified Hooke's law can be written as

$$\sigma_{ij} = \sum_{n=0}^{\infty} C_{ijk\ell}^{(n)} \frac{\partial^{(n)}}{\partial t^{(n)}} \varepsilon_{k\ell} \qquad \text{(III.2-1a)}$$

or

$$\sigma_{ij} = C_{ijk\ell}^{(0)} \varepsilon_{k\ell} + C_{ijk\ell}^{(1)} \frac{\partial^{(1)}}{\partial t^{(1)}} \varepsilon_{k\ell} + \cdots \qquad \text{(III.2-1b)}$$

For a body of this type, which is known as a Voigt solid, the stress is the sum of two terms, one proportional to the strain and the other proportional to the rate of strain. Hence, stress can be expressed in the following form:

$$\sigma_{ij} = \sigma'_{ij} + \sigma''_{ij} \qquad \text{(III.2-2)}$$

where

$$\sigma'_{ij} = C^{(0)}_{ijk\ell} \varepsilon_{k\ell}$$

$$\sigma'_{ij} = \lambda'\delta_{ij}\varepsilon_{\ell\ell} + 2\mu'\varepsilon_{ij} \qquad \text{(III.2-2a)}$$

the stress is proportional to the strain, and

$$\sigma''_{ij} = C^{(1)}_{ijk\ell} \frac{\partial^{(1)}}{\partial t^{(1)}} \varepsilon_{k\ell}$$

$$\sigma''_{ij} = \lambda''\delta_{ij} \frac{\partial}{\partial t} \varepsilon_{\ell\ell} + 2\mu'' \frac{\partial}{\partial t} \varepsilon_{ij} \qquad \text{(III.2-2b)}$$

the stress is proportional to the rate of strain. In other words, this expression takes into consideration the dissipation of energy in the elastic body. It should be noted that the equation of motion for a viscoelastic medium can be obtained from relations similar to those obtained for an elastic body, but the Lamé parameters are modified to become operators in the following manner:

$$\lambda = \lambda' + \lambda'' \frac{\partial}{\partial t} \qquad \text{(III.2-3a)}$$

$$\mu = \mu' + \mu'' \frac{\partial}{\partial t} \qquad \text{(III.2-3b)}$$

where μ', μ'' are the shear modulus and shear viscosity, and λ', λ'' are the compressional modulus and compressional viscosity of the viscoelastic medium.

Hence, the differential equation for a viscoelastic medium can be written as

$$\partial_i \sigma'_{ij} + \partial_i \sigma''_{ij} - \rho \partial_t^{2} u_j = 0 \qquad \text{(III.2-4a)}$$

or

$$\lambda' \partial_j \varepsilon_{\ell\ell} + 2\mu' \partial_i \varepsilon_{ij} + \lambda'' \partial_t \partial_j \varepsilon_{\ell\ell} + 2\mu'' \partial_t \partial_i \varepsilon_{ij}$$

$$-\rho \partial_t^{2} u_j = 0 \qquad \text{(III.2-4b)}$$

or using Equation (III.2-3) and (III.1-14)

$$\rho \partial_t^{2} \vec{u} - (\lambda' + \lambda'' \partial_t + \mu' + \mu'' \partial_t) \nabla(\nabla \cdot \vec{u}) - (\mu' + \mu'' \partial_t) \nabla^2 \vec{u} = 0$$

$$\text{(III.2-4c)}$$

Once again, the Green's function appropriate for viscoelastic compressional and shear waves shall be developed.

i) Green's Function for Viscoelastic Compressional Waves

The Green's function for compressional waves can be obtained according to the procedures outlined in Section (III.1). Thus, the Green's function for $\vec{u}_L$ is obtained in $(\vec{r};t)$ domain as

$$[\partial_t^{2} - \left(\frac{\lambda' + 2\lambda'}{\rho}\right) \nabla^2 - \left(\frac{\lambda'' + 2\mu''}{\rho}\right) \nabla^2 \partial_t] G(\vec{r}-\vec{r}';t-t') = \delta(\vec{r}-\vec{r}')\delta(t-t') .$$

$$\text{(III.2-5)}$$

Using time and space Fourier transformation Equation (III.2-5) becomes:

in $(\vec{k};\omega)$ domain

$$[-\omega^2 + \left(\frac{\lambda' + 2\mu'}{\rho}\right)k^2 + i\left(\frac{\lambda'' + 2\mu''}{\rho}\right)k^2\omega]G(k;\omega) = 1 \quad ; \qquad \text{(III.2-6a)}$$

in $(\vec{k};t)$ domain

$$[\partial_t^2 + [\left(\frac{\lambda'' + 2\mu''}{\rho}\right)k^2]\partial_t + \left(\frac{\lambda' + 2\mu'}{\rho}\right)k^2]G(k;t-t') = \delta(t-t') \quad ; \qquad \text{(III.2-6b)}$$

which is in the damped harmonic oscillator form

$$(\partial_t^2 + 2\zeta\omega_n\partial_t + \omega_n^2)\, f(t) = 0 \qquad \text{(III.2-6c)}$$

where ω_n and ζ are the natural frequency and damping factor of the system. Equation (III.2-6b) states that G(k;t-t') is the Green's function for the ordinary differential equation describing the non-equilibrium behavior of the system. Specifically, it gives the response to a unit impulsive external force at time t'.

In $(\vec{r};\omega)$ domain

$$\left[-\omega^2 - [\frac{\lambda' + 2\mu'}{\rho} - i\left(\frac{\lambda'' + 2\mu''}{\rho}\right)\omega]\nabla^2\right] G(\vec{r}-\vec{r}\,';\omega) = \delta(\vec{r}-\vec{r}\,') \qquad \text{(III.2-6d)}$$

The Fourier transformation of G(k;t-t') can be written in the following manner.

$$G(k;t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G(k;\omega)$$

and

$$G(k;\omega) = \int_{-\infty}^{\infty} d(t-t') \, e^{-i\omega(t-t')} G(k;t-t')$$

$$= \int_{0}^{\infty} d(t-t') \, e^{-i\omega(t-t')} G(k;t-t')$$

where the last transformation reflects the causal nature of $G(k;t-t')$.

The solution to the above equations are given below. In $(\vec{k};\omega)$ domain

$$G(k;\omega) = \frac{1}{-\omega^2 + c^2 + i\omega 2b} \qquad \text{(III.2-7a)}$$

where

$$c = \omega_n = \left(\frac{\lambda' + 2\mu'}{\rho}\right)^{1/2} k \qquad \text{(III.2-7b)}$$

is the natural frequency of the viscoelastic body,

$$b = \frac{(\lambda'' + 2\mu'')k^2}{2\rho} \qquad \text{(III.2-7c)}$$

is the temporal attenuation of the viscoelastic body,

$$\frac{b}{c} = \zeta = \frac{(\lambda'' + 2\mu'')k}{2\rho^{1/2}(\lambda' + 2\mu')^{1/2}} \qquad \text{(III.2-7d)}$$

is the damping factor of the viscoelastic body; in $(\vec{k};t)$ domain

$$G(k;t-t') = \eta(t-t')\, e^{-b[t-t']}\, \frac{\sin[a(t-t')]}{a} \qquad \text{(III.2-8a)}$$

where

$$a = \omega_d = \omega_n\sqrt{1-\zeta^2} = \left(\frac{\lambda' + 2\mu'}{\rho}\right)^{1/2} k\left[1 - \frac{(\lambda'' + 2\mu'')^2 k^2}{4\rho(\lambda' + 2\mu')}\right] \qquad \text{(III.2-8b)}$$

is the damped natural frequency of the viscoelastic body;

in $(\vec{r};\omega)$ domain

$$G(\vec{r}-\vec{r}';\omega) = \frac{1}{4\pi|\vec{r}-\vec{r}'|}\, e^{-b_1|\vec{r}-\vec{r}'|}\, e^{+ib_2|\vec{r}-\vec{r}'|} \qquad \text{(III.2-9a)}$$

where

$$b_1 = \frac{\omega}{C_L'}\left[\frac{D(\omega)^2 - 1}{2\, D(\omega)^4}\right]^{1/2} \qquad \text{(III.2-9b)}$$

is the absorption coefficient for viscoelastic compressional waves,

$$b_2 = \frac{\omega}{C_L'}\left[\frac{D(\omega)^2 + 1}{2\, D(\omega)^4}\right]^{1/2}, \qquad \text{(III.2-9c)}$$

$$D(\omega) = \left[1 + \omega^2 \left(\frac{C_L''}{C_L'}\right)^4\right]^{1/4}, \qquad \text{(III.2-9d)}$$

$$C_L' = \sqrt{\frac{\lambda' + 2\mu'}{\rho}} \qquad \text{(III.2-9e)}$$

is the compressional wave velocity,

$$C_L'' = \sqrt{\frac{\lambda'' + 2\mu''}{\rho}} \qquad \text{(III.2-9f)}$$

For real ω, the Green's function is usually divided into two parts: a dissipative and a reactive part. In this case, these are given respectively by the real and imaginary parts of $G(\omega)$, and are denoted as $G''(\omega)$ and $G'(\omega)$ as illustrated in Figures III-2 and III-3. Defining $G(\omega) = G'(\omega) + i\, G''(\omega)$, then

$$G'(k;\omega) = \frac{c^2 - \omega^2}{[c^2 - \omega^2]^2 + [\omega 2b]^2} \qquad \text{(III.2-10a)}$$

is the dissipative part and

$$G''(k;\omega) = \frac{-2b\omega}{[c^2 - \omega^2]^2 + [2\omega b]^2} \qquad \text{(III.2-11a)}$$

as the reactive part. Taking the Fourier transformation of the above equations, the real even and imaginary odd functions of time are obtained as

$$G'(k;t-t') = e^{-b|t-t'|} \frac{\sin [a|t-t'|]}{2a} \qquad \text{(III.2-10b)}$$

$$G''(k;t-t') = e^{-b|t-t'|} \frac{\sin [a(t-t')]}{2a} \qquad \text{(III.2-11b)}$$

which are illustrated in Figures III-4 and III-5.

Since the response is causal, the real and imaginary parts of $G(k;\omega)$ are related by Hilbert transform according to the relations

$$G'(k;\omega) = P \int \frac{d\omega'}{\pi} \frac{G''(k;\omega)}{\omega'-\omega} \tag{III.2-12a}$$

$$G''(k;\omega) = P \int \frac{d\omega'}{\pi} \frac{G'(k;\omega)}{\omega'-\omega} \tag{III.2-12b}$$

where "P" implies principal value integral, that is, an integral symmetrical with respect to the singularity.

Keeping in mind that the Green's functions obtained above already contain the initial conditions, that is the system is initially at rest in its equilibrium position, an initial value Green's function shall now be presented where the initial displacement is not equal to zero.

In order to achieve this the viscoelastic compressional wave system is written for $\vec{v}_L$ via Equation (III.2-4c).

$$\rho \partial_t^2 \vec{v}_L - \left(\frac{\lambda'' + 2\mu''}{\rho}\right) \nabla^2 \partial_t \vec{v}_L - \left(\frac{\lambda' + 2\mu'}{\rho}\right) \nabla^2 \vec{v}_L = 0 \tag{III.2-13}$$

Noting that here $\vec{v}_L$ represents a deviation of the velocity from the uniform time independent equilibrium value, the relaxation of a deviation $\vec{v}_L(\vec{r};t)$ can be computed in terms of the initial value of the displacement and its derivitive.

In order to accomplish this a Fourier transformation

$$\vec{v}_L(k;\omega) = \int_0^\infty dt\, e^{-i\omega t} \int d^3r\, e^{i\vec{k}\cdot\vec{r}}\, \vec{v}_L(\vec{r};t) \tag{III.2-14}$$

is used to obtain

$$(-\omega^2 + c^2 + i\omega 2b)\ \vec{v}_L\ (k;\omega) \ = \ (-i\omega + 2b)\ \vec{v}_L\ (k;t=0) + \partial_t\ \vec{v}_L\ (k;t=0) \tag{III.2-15}$$

Using the relationship between $\partial_t \vec{v}_L(k;t=0)$ and $\nabla\cdot\vec{\vec{\sigma}}(k;t=0)$ the previous equation becomes (see Appendix-2)

$$\frac{v_L(k;\omega)}{i\omega v_L(k;t=0) + c^2 u_L(k;t=0)} = \frac{-1}{-\omega^2 + c^2 - i\omega 2b} \tag{III.2-16}$$

The initial valued Green's function can be obtained by using Kubo's formula

$$\frac{R(k;\omega)}{F(k;t=0)} = G_i(k;\omega) - G_i(k;0) \tag{III.2-17}$$

to be

$$G_i(k;\omega) = \frac{c^2 - i\omega 2b}{-\omega^2 + c^2 - i\omega 2b} \tag{III.2-18a}$$

$$G_i(k;0) = 1 \ . \tag{III.2-18b}$$

In $(\vec{k};t)$ domain

$$G_i(k;t-t') = \eta(t-t')\ [c^2\ G(k;t-t') + 2b\ \partial_t\ G(k;t-t')] \tag{III.2-19a}$$

$$= \eta(t-t')\ e^{-b[t-t']} \left\{ \left(\frac{c^2 - 2b^2}{a}\right) \sin\,[a(t-t')] + 2b\cos\,[a(t-t')] \right\} \tag{III.2-19b}$$

and in $(\vec{r};\omega)$ domain

$$G_i(\vec{r}-\vec{r}\,';\omega) = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|}\ (b_2 + ib_1)^2\ e^{-b_1|\vec{r}-\vec{r}\,'|}\ e^{ib_2|\vec{r}-\vec{r}\,'|} \tag{III.2-20a}$$

are the appropriate initial valued Green's functions. Here, the subscript i indicates initial value is being considered.

ii) Green's Function Appropriate for Viscoelastic Shear Wave System

The Green's function for viscoelastic shear wave system can be obtained by taking the curl of Equation (III.2-4c). Hence, for $\vec{u}_T$ the differential equation in $(\vec{r};t)$ domain is obtained as

$$\left(\partial_t^{\,2} - \frac{\mu'}{\rho}\,\nabla^2 - \frac{\mu''}{\rho}\,\partial_t\right)\vec{u}_T = 0 \qquad \text{(III.2-21)}$$

This equation is in the form as the one for the compressional waves. The mathematical form of the differential equations and their solutions of the shear waves are identical for the compressional waves except now the compressional modulus λ' and viscosity λ'' are neglected. Hence, the solutions for the viscoelastic shear wave system is given by:
in $(\vec{k};\omega)$ domain

$$\tilde{G}(k;\omega) = \frac{1}{-\omega^2 + \tilde{c}^2 + i\omega 2\tilde{b}} \qquad \text{(III.2-21)}$$

where

$$\tilde{c} = \left(\frac{\mu'}{\rho}\right)^{1/2} k \qquad \text{(III.2-21a)}$$

$$\tilde{b} = \left(\frac{\mu''}{2\rho}\right) k^2 \qquad \text{(III.2-21b)}$$

$$\frac{\tilde{b}}{\tilde{c}} = \tilde{\zeta} = \frac{\mu''\,k}{2\rho^{1/2}(\mu')^{1/2}} \qquad \text{(III.2-21c)}$$

are the natural frequency, the temporal attenuation and the damping factor for the viscoelastic shear wave system, respectively;

in $(\vec{k};t)$ domain

$$\tilde{G}(k;t-t') = \eta(t-t')e^{-\tilde{b}[t-t']} \frac{\sin[\tilde{a}(t-t')]}{\tilde{a}} \quad \text{(III.2-22)}$$

where

$$\tilde{a} = \tilde{c}\sqrt{1-\tilde{\zeta}^2}$$

is the damped natural frequency of the viscoelastic shear wave system;

in $(\vec{r};\omega)$ domain

$$\tilde{G}(\vec{r}-\vec{r}';\omega) = \frac{1}{4\pi|\vec{r}-\vec{r}'|} e^{-\tilde{b}_1|\vec{r}-\vec{r}|} e^{i\tilde{b}_2|\vec{r}-\vec{r}'|} \quad \text{(III.2-23)}$$

where

$$\tilde{b}_1 = \frac{\omega}{C_T'} \left[\frac{\tilde{D}(\omega)^2 - 1}{2\tilde{D}(\omega)^4} \right]^{1/2} \quad \text{(III.2-23a)}$$

is the absorption coefficient for transverse waves,

$$\tilde{b}_2 = \frac{\omega}{C_T'} \left[\frac{\tilde{D}(\omega)^2 + 1}{2\tilde{D}(\omega)^4} \right]^{1/2} \quad \text{(III.2-23b)}$$

$$\tilde{D}(\omega) = \left[1 + \omega^2 \frac{(C_T'')^4}{(C_T')^4} \right]^{1/4} \quad \text{(III.2-23c)}$$

$$C_T' = \sqrt{\frac{\mu'}{\rho}} \qquad \text{(III.2-23d)}$$

is the transverse wave velocity,

$$C_T'' = \sqrt{\frac{\mu''}{\rho}} \quad . \qquad \text{(III.2-23e)}$$

The initial valued Green's function for viscoelastic shear wave system can be determined analogous to that of compressional waves. The results can be immediately written as

$$\tilde{G}_i(k;\omega) = \frac{\tilde{c}^2 - i\omega 2\tilde{b}}{-\omega^2 + \tilde{c}^2 - i\omega 2\tilde{b}} \quad ; \qquad \text{(III.2-24)}$$

$$\tilde{G}_i(k;t-t') = \eta(t-t')\, e^{-\tilde{b}[t-t']} \left\{ \left(\frac{\tilde{c}^2 - 2\tilde{b}^2}{\tilde{a}}\right) \sin\left[\tilde{a}(t-t')\right] + 2\tilde{b} \cos\left[\tilde{a}(t-t')\right] \right\} \quad ; \qquad \text{(III.2-25)}$$

$$\tilde{G}_i(\vec{r}-\vec{r}\,';\omega) = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|} (\tilde{b}_1 + i\tilde{b}_2)^2\, e^{-\tilde{b}_1|\vec{r}-\vec{r}\,'|}\, e^{i\tilde{b}_2|\vec{r}-\vec{r}\,'|}$$

(III.2-26)

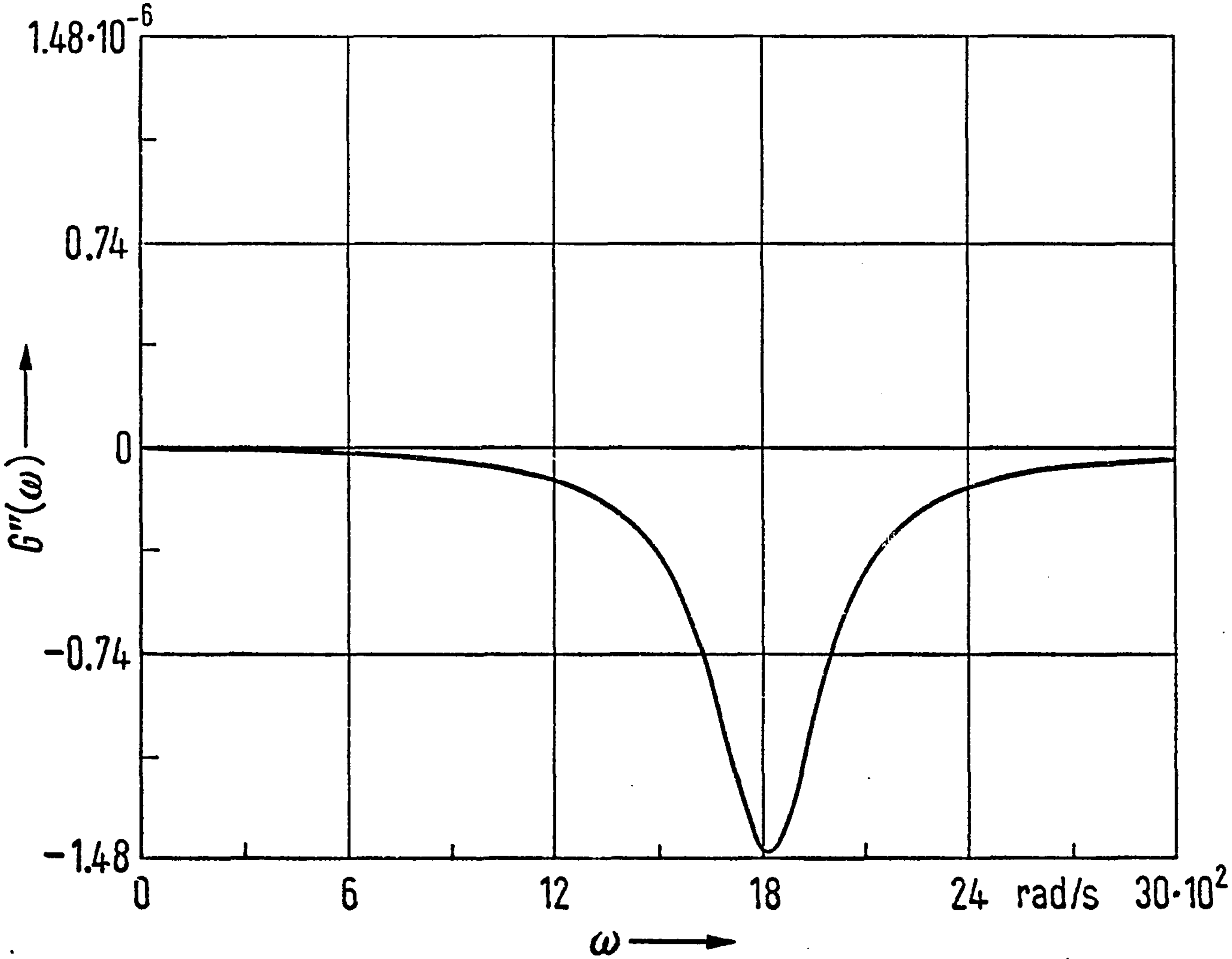


Figure III-1 Plot of G"($\vec{k}$;ω)

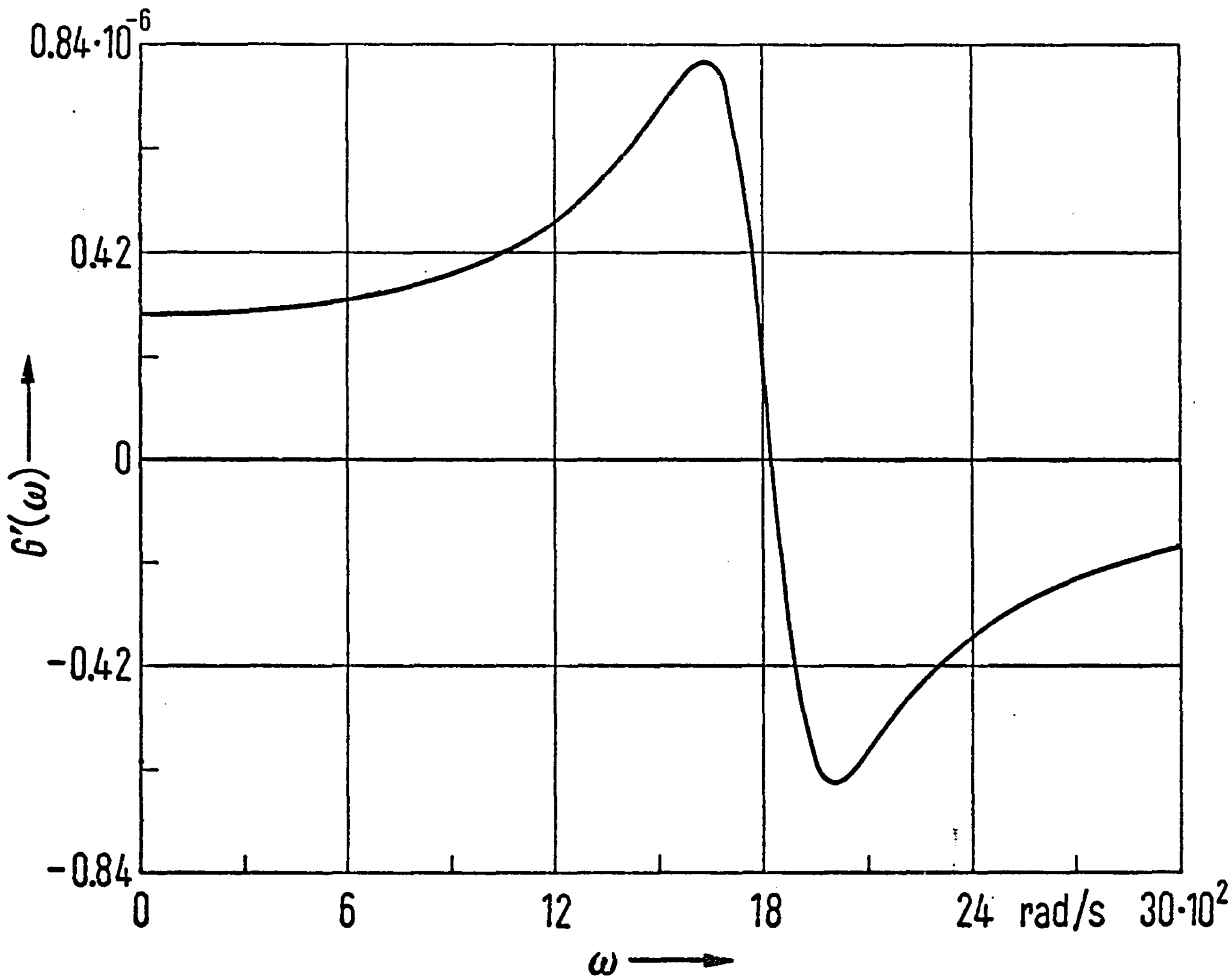


Figure III-2 Plot of $G'(\vec{k};\omega)$

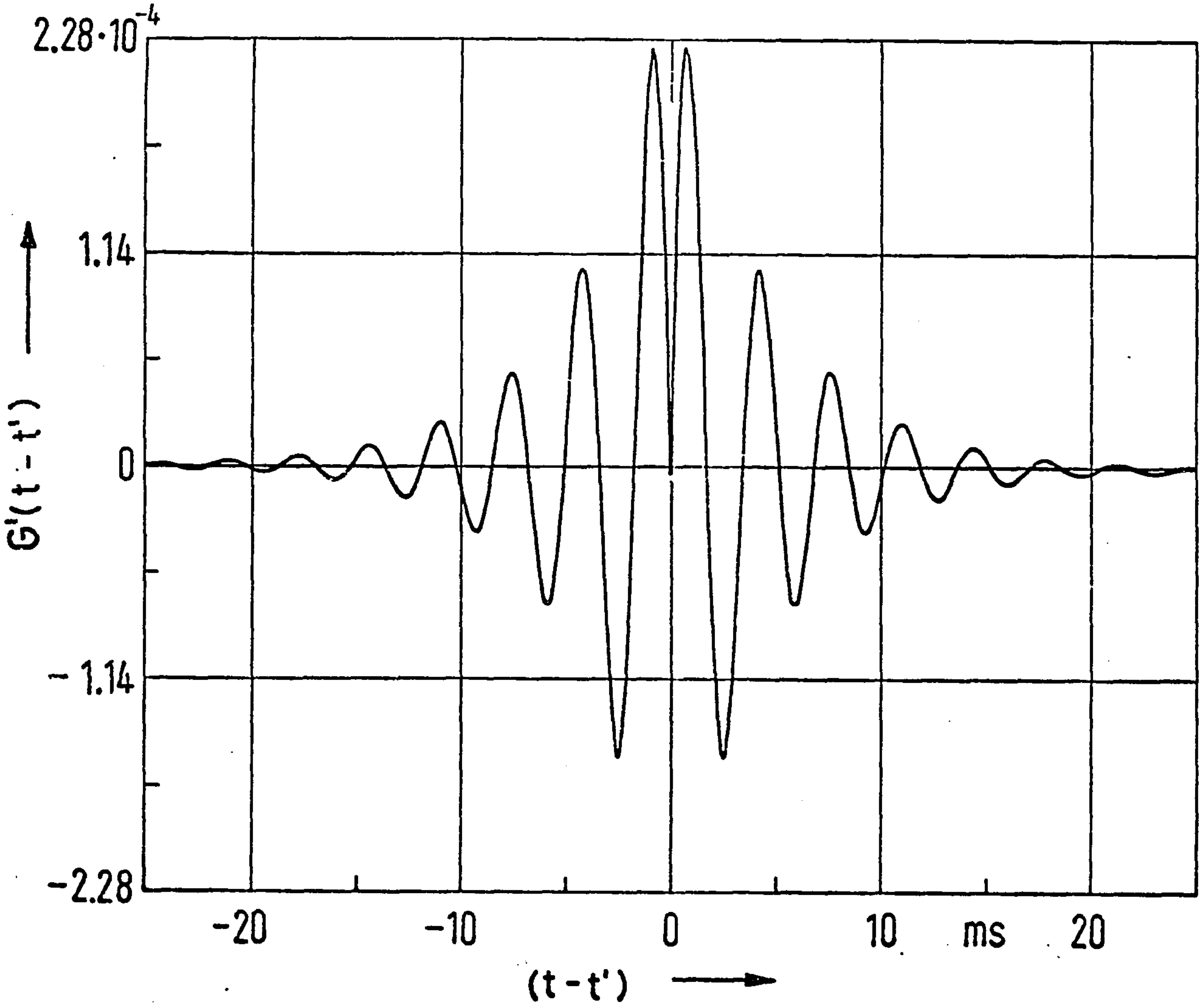


Figure III-3 Plot of $G(\vec{k};t-t')$

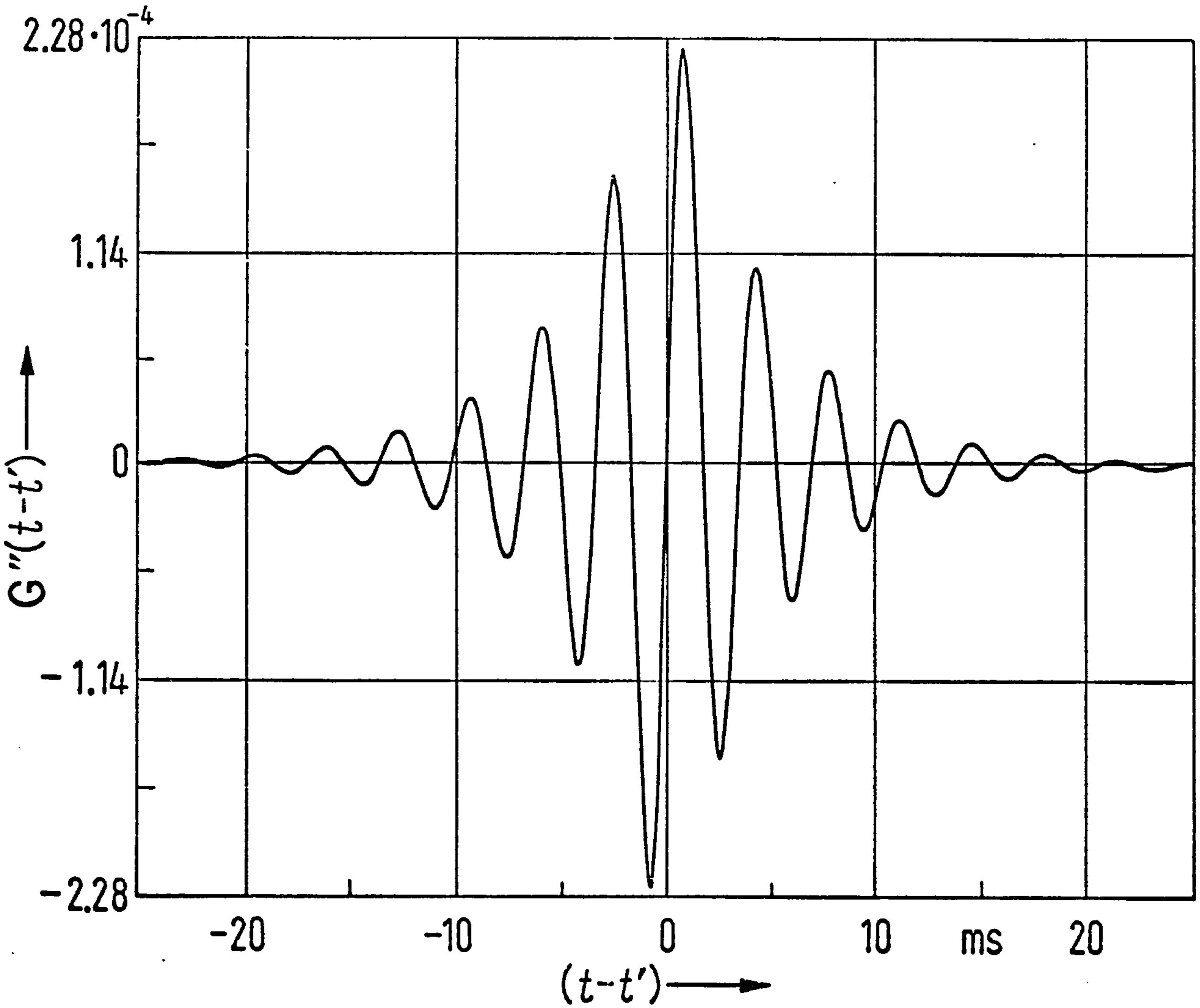


Figure III-4 Plot of $G''(\vec{k};t-t')$

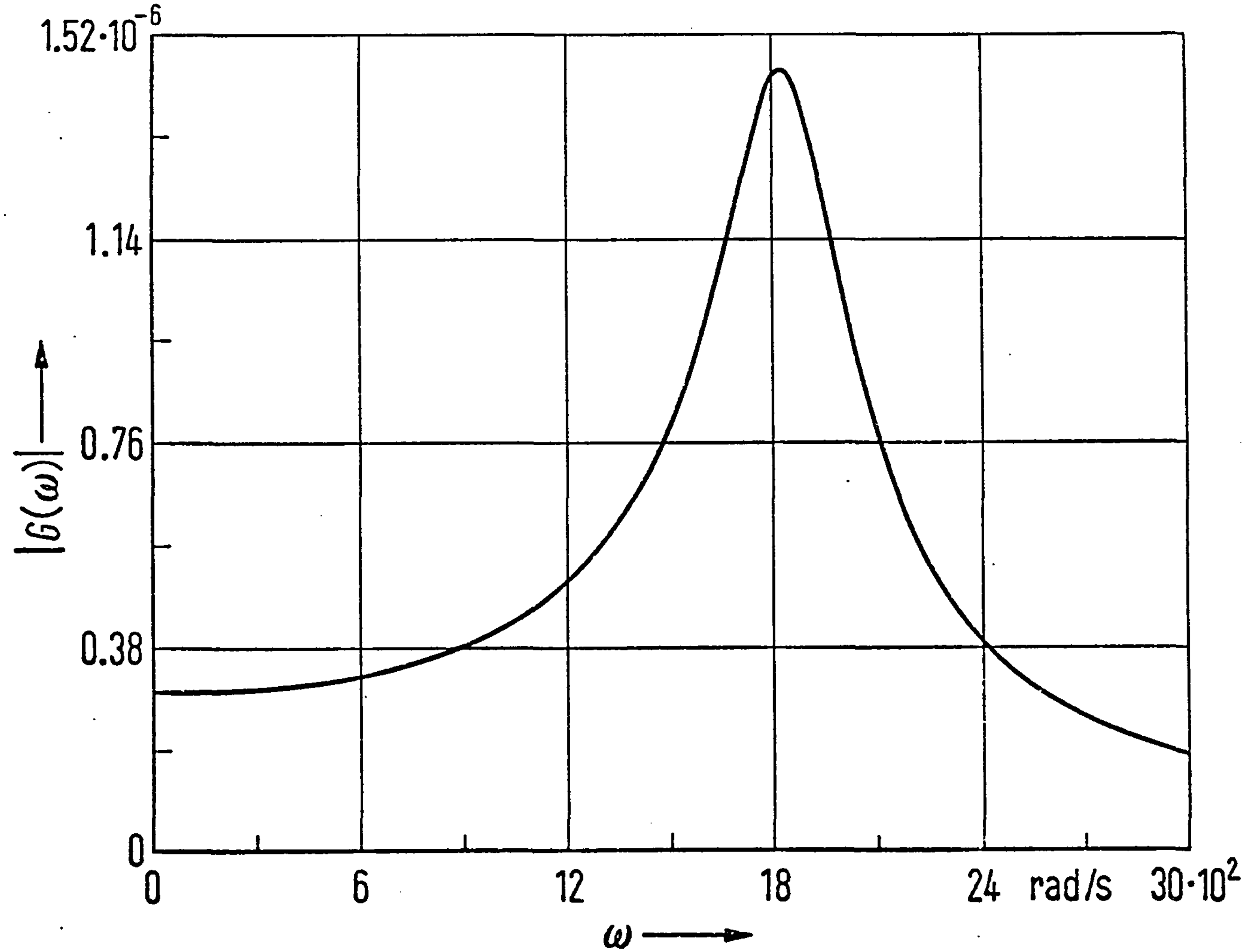


Figure III-5 Plot of |G(ω)|

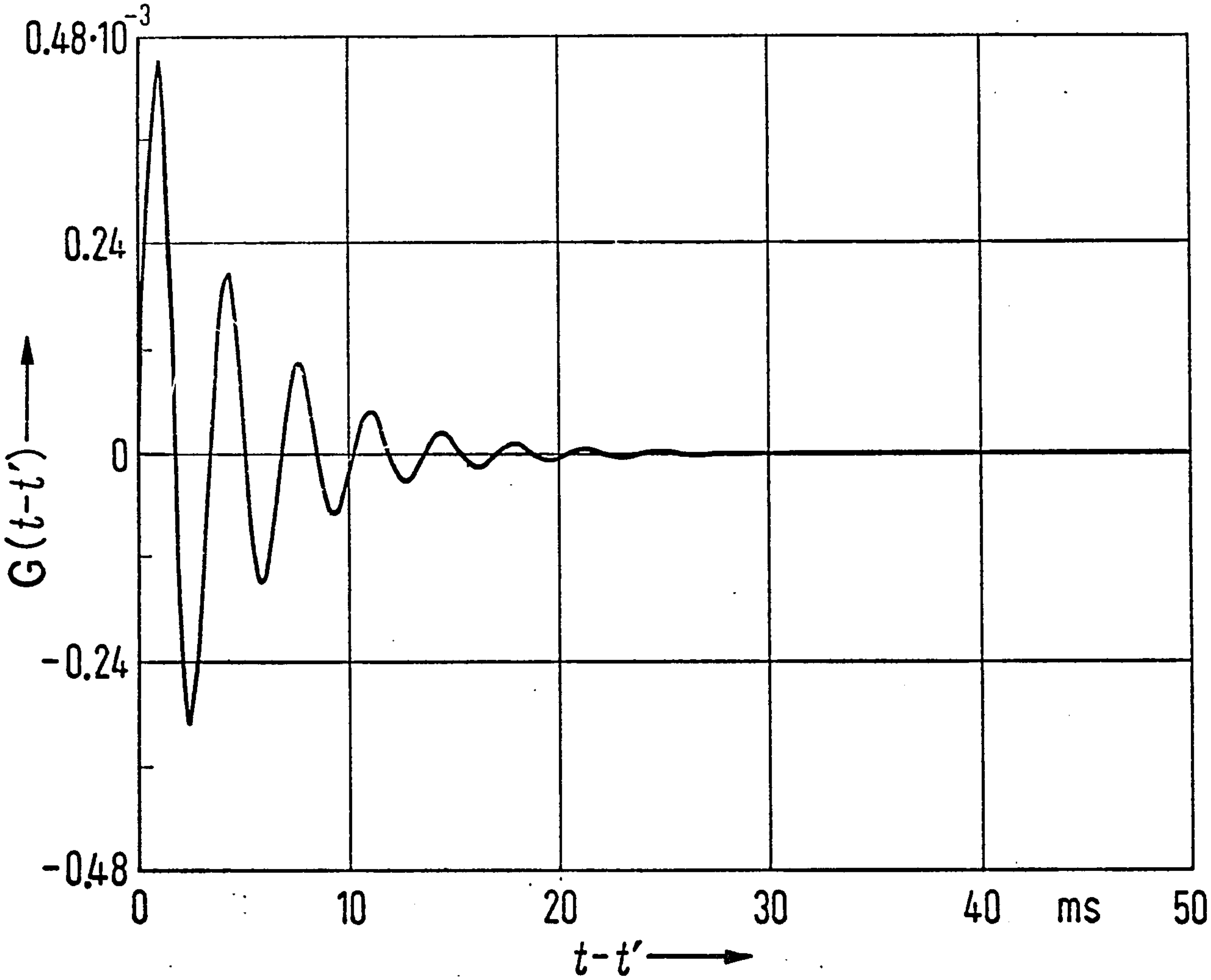


Figure III-6 Plot of $G(\vec{k};t-t')$

IV. LINEAR VISCOELASTIC THEORY WITH TEMPERATURE EFFECTS

In this section, the linear viscoelastic system with temperature effects are presented. The differential equations for the system are developed and their solutions are again presented in terms of Green's functions. The compressional viscous parameter expression is obtained in terms of thermodynamical and mechanical variables. This result is finally related to the resonance peak width in a damped harmonic oscillator model.

IV.1 DEFORMATIONS IN VISCOELASTIC MEDIUM WITH TEMPERATURE EFFECTS

In this section, the deformations which are accompanied by a change in the temperature of the viscoelastic body will be considered. This change in the body can be the result of the deformation process itself or from external causes.

Let the undeformed state of the body in the absence of external forces be at some given temperature T_o. If the body is at a temperature T different from T_o, then, even if there are no external forces, due to thermal expansion, it will be deformed. Since only the sum of the diagonal terms of the strain tensor is the relative volume change of the body, the thermal expansion due to temperature difference will only affect the coefficient of $u_{\ell\ell}$ in the stress tensor given in Equation (III.1-10). Hence, of the Lamé parameters only the compressional modulus λ will be altered. On the other hand, the Lamé shear parameter μ will not be affected by temperature.

In order to present the affects of temperature change on a body it will be useful to rewrite the strain tensor in an appropriate fashion. Since any deformation can be represented as a sum of a pure shear and a hydrostatic compression, the strain can be written as

$$\varepsilon_{ik} = (\varepsilon_{ik} - \frac{1}{3}\delta_{ik}\varepsilon_{\ell\ell}) + \frac{1}{3}\delta_{ik}\varepsilon_{\ell\ell} \qquad \text{(IV.1-1)}$$

and hence, the stress tensor is

$$\sigma_{ik} = K\varepsilon_{\ell\ell}\delta_{ik} + 2\mu(\varepsilon_{ik} - \frac{1}{3}\delta_{ik}\varepsilon_{\ell\ell}) \qquad \text{(IV.1-2)}$$

for an elastic homogeneous isotropic body. Here, K is modulus of compression and is related to the Lamé coefficients by

$$K = \lambda + \frac{2}{3}\mu . \qquad \text{(IV.1-3)}$$

The above expressions can be easily extended to viscoelastic body by using Equations (III.2-3a) and (III.2-3b). The stress tensor then becomes

$$\sigma_{ik} = K'\varepsilon_{\ell\ell}\delta_{ik} + 2\mu'(\varepsilon_{ik} - \frac{1}{3}\delta_{ik}\varepsilon_{\ell\ell})$$
$$+ K''\delta_{ik}\partial_t\varepsilon_{\ell\ell} + 2\mu''\partial_t(\varepsilon_{ik} - \frac{1}{3}\delta_{ik}\varepsilon_{\ell\ell}) \qquad \text{(IV.1-4)}$$

where

$$K' = \lambda' + \frac{2}{3}\mu'$$
$$K'' = \lambda'' + \frac{2}{3}\mu'' \qquad \text{(IV.1-5)}$$

Now, among various types of thermodynamic deformations, isothermal and adiabatic deformations are very important. In isothermal deformations, the temperature of the body does not change. Therefore, expression $T = T_o$ is valid and Equations (IV.1-4) and (IV.1-5) hold true for isothermal deformations. The coefficients K', K'', μ', and μ'' can then be called isothermal moduli.

If an adiabatic deformation occurs, then there is no exchange of heat between the various parts of the body. In this case the entropy S remains constant and the change of temperature $T-T_o$ due to deformation is proportional to ε_{ii}. An expression for the stress tensor is obtained in the usual manner as

$$\sigma_{ik}^{ad} = K_{ad}\, \varepsilon_{\ell\ell}\, \delta_{ik} + 2\mu(\varepsilon_{ik} - \frac{1}{3}\, \delta_{ik}\, \varepsilon_{\ell\ell}) \qquad \text{(IV.1-6)}$$

for an isotropic homogeneous elastic body.

From now on an elastic body possessing energy dissipation mechanism and adiabatic thermal properties shall be referred to as a <u>thermo-viscoelastic</u> body (or medium).

The relation between the adiabatic modulus K_{ad} and the ordinary isothermal modulus K can be found for an elastic and homogeneous body by using the Maxwell relation with the appropriate Jacobian transformations. Since deformations due to temperature change results in volume change of the body, the derivitives $\partial V/\partial T$ and $\partial V/\partial P$ which give the relative volume changes in heating and compression respectively, become important. Therefore, the thermodynamic definitions

$$\left.\frac{\partial V}{\partial P}\right)_S = -\frac{1}{K_{ad}} \qquad \text{(IV.1-7)}$$

which is the adiabatic compressibility coefficient,

$$\left.\frac{\partial V}{\partial T}\right)_P = \alpha \qquad \text{(IV.1-8)}$$

is the coefficient of thermal expansion,

$$T\left(\frac{\partial S}{\partial T}\right)_P = C_P \qquad \text{(IV.1-9a)}$$

is the specific heat per unit volume at constant pressure, and

$$\left.\frac{\partial V}{\partial P}\right)_T = -\frac{1}{K} \qquad \text{(IV.1-9b)}$$

is the isothermal compressibility coefficient, are used to find a relationship between K_{ad} and K. Using the Jacobian transformations on K_{ad} as

$$\frac{1}{K_{ad}} = -\left.\frac{\partial V}{\partial P}\right)_S = -\frac{\partial(V,S)}{\partial(P,S)} = \frac{\partial(V,S)}{\partial(S,P)} \qquad \text{(IV.1-10a)}$$

and since the changes in temperature are desired, the above expression can be written as

$$\frac{1}{K_{ad}} = +\frac{\partial(V,S)}{\partial(S,P)}\;\frac{\partial(T,P)}{\partial(T,P)}$$

$$= \frac{1}{\dfrac{\partial(S,P)}{\partial(T,P)}} \cdot \frac{\partial(V,S)}{\partial(T,P)}$$

$$= \frac{1}{\left.\frac{\partial S}{\partial T}\right)_P} \cdot \left[\left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial S}{\partial P}\right)_T - \left(\frac{\partial S}{\partial T}\right)_P \left(\frac{\partial V}{\partial P}\right)_T\right] \qquad \text{(IV.1-10b)}$$

which results in

$$\frac{1}{K_{ad}} = \frac{T\ \alpha}{C_P} \left(\frac{\partial S}{\partial P}\right)_T + \frac{1}{K} \qquad \text{(IV.1-10c)}$$

from the thermodynamic definitions. But from Maxwell's relation

$$\left.\frac{\partial S}{\partial P}\right)_T = -\left.\frac{\partial V}{\partial T}\right)_P$$

$$= -\ \alpha \qquad \text{(IV.1-10d)}$$

the final form of the relation between K_{ad} and K can now be written down as

$$\frac{1}{K_{ad}} = \frac{1}{K} - \frac{T\ \alpha^2}{C_P} \quad , \qquad \text{(IV.1-11a)}$$

$$\mu_{ad} = \mu\ . \qquad \text{(IV.1-11b)}$$

IV.2 ELASTIC WAVES IN THERMOVISCOELASTIC MEDIUM

In this section the differential equation appropriate for a thermoviscoelastic medium shall be determined.

In general, when motion occurs in a deformed body, then its temperature will vary in both time and space. This obviously will complicate the exact equations of motion in the general case of arbitary motions.

However, the situation can be simplified if the assumption is made in the transfer of heat from one part of the body to another, by thermal conduction, occurs very slowly. Also, if the heat exchange in the body during the period of the oscillatory motions is negligible, then any part of the body can be regarded as insulated, that is, the motion is adiabatic. Therefore, the stress tensor is the equation of motion will be simply given in its adiabatic form. The equation of motion is then

$$\rho\, \partial_t^2\, u_i \;-\; \partial_k\, \sigma_{ik}^{ad} \;=\; f_i \qquad \text{(IV.2-1a)}$$

and using Equation (IV.1-4) and the definition of the strain tensor it becomes

$$\rho\, \partial_t^2\, u_i - (\mu' + \mu''\partial_t)\partial^2\, u_i - [K'_{ad} + \tfrac{1}{3}\,\mu' + (K''_{ad} + \tfrac{1}{3}\,\mu'')\partial_t]\partial_i\partial_\ell u_\ell = f_i\;; \qquad \text{(IV.2-1b)}$$

and in vector form it becomes

$$\rho\, \partial_t^2\, \vec{u} - (\mu' + \mu''\partial_t)\nabla^2\, \vec{u} - [K'_{ad} + \tfrac{1}{3}\,\mu' + (K''_{ad} + \tfrac{1}{3}\,\mu'')\partial_t]\nabla(\nabla\cdot\vec{u}) = \vec{f} \qquad \text{(IV.2-1c)}$$

Noting that $K''_{ad} + \frac{1}{3}\,\mu'' \;=\; \lambda''_{ad} + \mu''$ the final form of the equation of motion for a thermoviscoelastic medium with an external forcing term can be written as

$$\rho\partial_t^2 \vec{u} - (\mu' + \mu''\partial_t)\nabla^2\vec{u} - [K'_{ad} + \frac{1}{3}\mu' + (\lambda''_{ad} + \mu'')\partial_t]\nabla(\nabla\cdot\vec{u}) = \vec{f}. \quad \text{(IV.2-2)}$$

The compressional and transverse wave equations for a thermovisco-elastic medium can be obtained in the same manner outlined in Section-I. Thus, the compressional wave equation becomes

$$[\rho\partial_t^2 - (\lambda''_{ad} + 2\mu'')\partial_t\nabla^2 - (K'_{ad} + \frac{4}{3}\mu')\nabla^2]\vec{u}_L = 0 \quad \text{(IV.2-3)}$$

and the transverse wave equation becomes

$$(\rho\partial_t^2 - \mu''\partial_t\nabla^2 - \mu'\nabla^2)\vec{u}_T = 0 \quad \text{(IV.2-4)}$$

which is same as Equation (III.2-21), that is the isothermal case. Since the solution of this equation was already presented it shall not be further discussed.

i) Green's Function for Thermoviscoelastic Compressional Waves

The Green's function for Equation (IV.2-3) can be obtained by applying the appropriate Fourier transformations. In $(\vec{k};\omega)$ domain the expression

$$[\partial_t^2 + (\frac{(\lambda''_{ad} + 2\mu'')}{\rho})\, k^2\partial_t + (\frac{K'_{ad} + 4\mu'/3}{\rho})k^2]G_{ad}(k;t-t') = \delta(t-t') \quad \text{(IV.2-5)}$$

is obtained; and in $(\vec{k};\omega)$ the expression is

$$G_{ad}(k;\omega) = \frac{1}{-\omega^2 + C^2 + i2\omega B} \tag{IV.2-6}$$

where

$$C = \omega_n^{ad} = \left(\frac{K'_{ad} + 4\mu'/3}{\rho}\right)^{1/2} k \tag{IV.2-7}$$

is the natural frequency,

$$B = \left(\frac{\lambda''_{ad} + 2\mu''}{2\rho}\right) k^2 \tag{IV.2-8}$$

is the temporal attenuation of the system, and

$$\frac{B}{C} = \zeta_{ad} = \frac{(\lambda''_{ad} + 2\mu'')k}{2\rho^{1/2}(K'_{ad} + \frac{4}{3}\mu')^{1/2}} \tag{IV.2-9}$$

is the damping factor of the thermoviscoelastic compressional wave system, respectively.

It should be noted that the above two equations once again are in the form of damped harmonic oscillator model in $(\vec{k};t)$ and $(\vec{k};\omega)$ domains.

The solutions to the above equations are the following: in $(\vec{k};t)$ domain

$$G_{ad}(k;t-t') = \eta(t-t')e^{-B[t-t']} \frac{\sin[A(t-t')]}{A} \tag{IV.2-10}$$

where

$$A = \omega_d^{ad} = \omega_n^{ad}(1 - \zeta_{ad}^2)^{1/2} \tag{IV.2-11}$$

is the damped natural frequency of the system; in $(\vec{r};\omega)$ domain

$$G_{ad}(\vec{r}-\vec{r}';\omega) = \frac{1}{4\pi|\vec{r}-\vec{r}'|} e^{-B_1|\vec{r}-\vec{r}'|} e^{iB_2|\vec{r}-\vec{r}'|} \qquad (IV.2\text{-}12)$$

where

$$B_1 = \frac{\omega}{(C_L')_{ad}} \left[\frac{D(\omega)_{ad}^2 - 1}{2\, D(\omega)_{ad}^4} \right]^{1/2} \qquad (IV.2\text{-}13)$$

is the absorption coefficient for thermoviscoelastic compressional waves;

$$B_2 = \frac{\omega}{(C_L')_{ad}} \left[\frac{D(\omega)_{ad}^2 + 1}{2\, D(\omega)_{ad}^2} \right]^{1/2} \quad ; \qquad (IV.2\text{-}14)$$

$$D(\omega)_{ad} = \left[1 + \omega^2 \left(\frac{C_L''}{C_L'}\right)_{ad}^4 \right]^{1/4} \quad ; \qquad (IV.2\text{-}15)$$

$$(C_L')_{ad} = \sqrt{\frac{K'_{ad} + 4\mu'/3}{\rho}} \qquad (IV.2\text{-}16)$$

is the adiabatic wave speed for an elastic body; and

$$(C_L'')_{ad} = \sqrt{\frac{\lambda''_{ad} + 2\mu''}{\rho}} \quad . \qquad (IV.2\text{-}17)$$

A relationship between the viscous compressional Lamé parameter λ''_{ad} and temperature for an isotropic viscoelastic body can be determined by using the definition of temporal attenuation

$$B = \frac{|E[\dot{\Phi}_{mech}]|}{2E[\Phi]} = \frac{k^2}{2\rho} (\lambda''_{ad} + 2\mu'') \qquad (IV.2\text{-}18)$$

where $\Phi, \dot{\Phi}$ is the mechanical energy of the body and time rate of change of that energy, respectively. Here, E[ ] is the expected value or the mean value of the function in [ ].

Expressing the thermal conduction part of the energy dissipation as

$$\dot{\Phi}_t = - \frac{\kappa}{T} \int (\nabla T)^2 dV \tag{IV.2-19}$$

and on account of viscosity, the energy viscosity term as

$$\dot{\Phi}_v = -2\mu'' \int (\dot{\varepsilon}_{ik} - \frac{1}{3} \delta_{ik} \dot{\varepsilon}_{\ell\ell})^2 dV - (\lambda'' + \frac{2}{3} \mu'') \int \dot{\varepsilon}_{\ell\ell}^{\,2} \, dV \tag{IV.2-20}$$

the total energy dissipation term can be written as

$$\dot{\Phi}_{mech} = \dot{\Phi}_t + \dot{\Phi}_v \tag{IV.2.21}$$

Using the definition of temporal attenuation in Equation (IV.2-18) - see Appendix I - the following relation is obtained

$$B = \frac{k^2}{2\rho} [\lambda'' + 2\mu'' + \frac{\kappa \, T \, \alpha^2 \, \rho^2}{C_P^{\,2}} (C_L')_{ad}^{\,2} \left(1 - \frac{4(C_T')_{ad}^{\,2}}{3(C_L')_{ad}^{\,2}}\right)^2] \tag{IV.2-22}$$

where κ is thermal conductivity. Rewriting this in a compact form

$$B = \frac{k^2}{2\rho} [\lambda'' + \lambda_t'' + 2\mu''] \tag{IV.2-23}$$

where

$$\lambda_t'' = \frac{K\,T\,\alpha^2\,\rho^2}{C_p^2}\,(C_L')_{ad}^2\left[1 - \frac{4(C_T')_{ad}^2}{3(C_L')_{ad}^2}\right]^2 , \qquad \text{(IV.2-24)}$$

a relationship between λ''_{ad} and T can now be written as

$$\lambda''_{ad} = \lambda'' + \lambda''_t. \qquad \text{(IV.2-25)}$$

Thus, an explicit expression for the Lamé compressional parameter has been obtained in terms of specific heat parameter C_p and other relevant thermodynamical and mechanical variables.

Using the above result an analytical expression for the resonance peak width can be shown. In forced vibration the quality factor Q of the system is related to damping which is a measure of the sharpness of resonance. The relation between the quality factor and the damping factor is given by

$$Q = \frac{1}{2\zeta_{ad}} \qquad \text{(IV.2-26)}$$

and to the resonance width as

$$Q = \frac{\omega_n^{ad}}{\Delta\omega} \qquad \text{(IV.2-27)}$$

where $\Delta\omega$ is the bandwidth. See Figure IV-1.

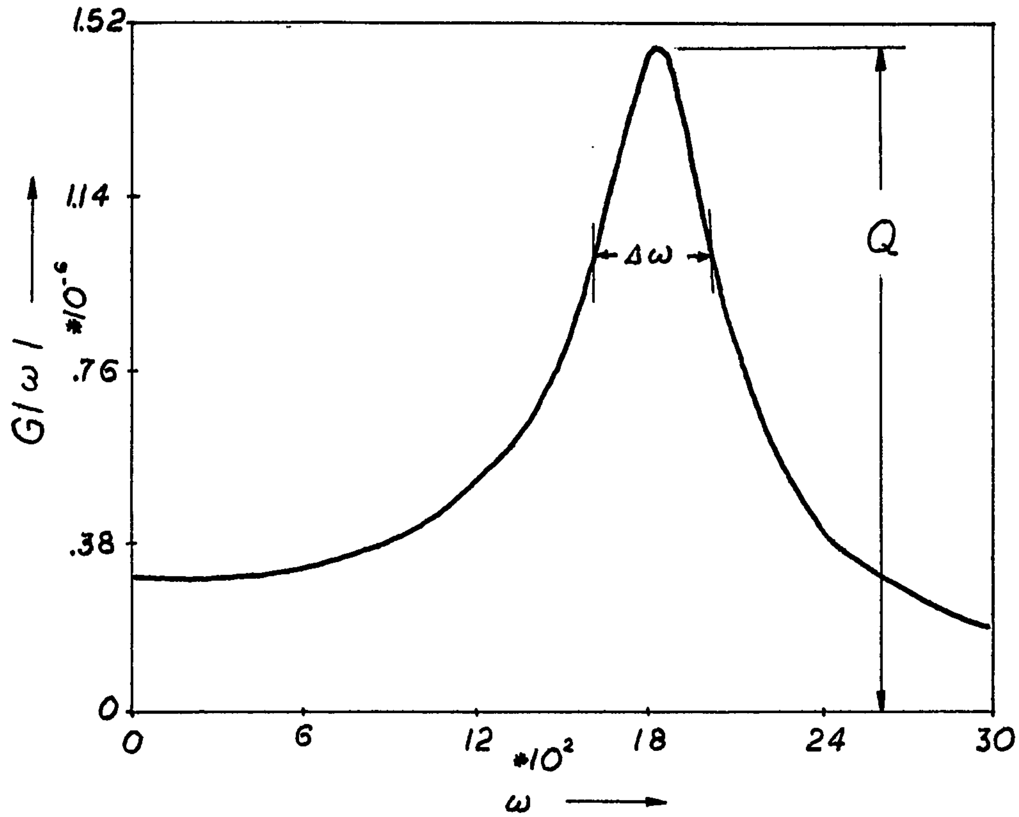


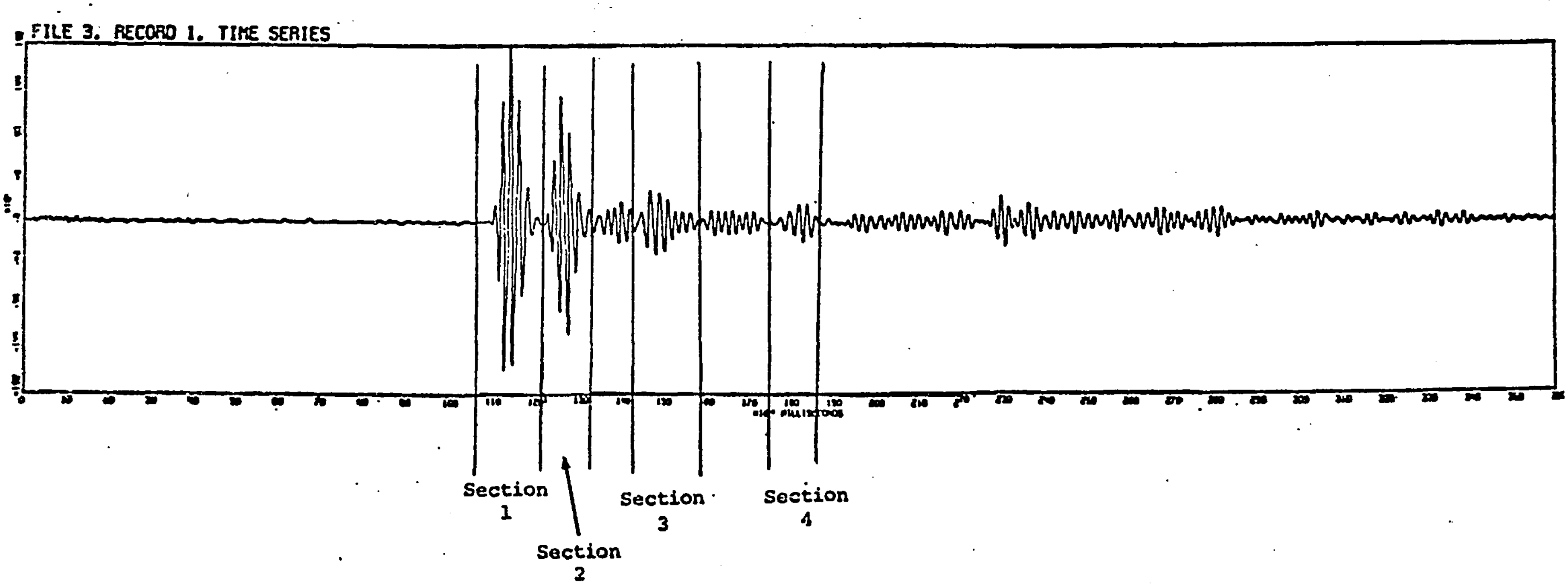
Figure IV-1

Using Equations (IV.2-26), (IV.2-27), (IV.2-7) and (IV.2-9) the resonance width expression becomes

$$\Delta\omega = \frac{2}{\omega_n^{ad}} \zeta_{ad}$$

$$\Delta\omega = \frac{\lambda''_{ad} + 2\mu''}{K'_{ad} + \frac{4}{3}\mu'} \qquad (IV.2\text{-}27)$$

where λ''_{ad} and K'_{ad} are given by Equations (IV.2-25) and (IV.1-11a), respectively.



Section 1: Input signal in the ocean
Section 2: First return signal from the ocean subbottom
Section 3: Second return signal from second ocean subbottom layer
Section 4: Third return signal from third ocean subbottom layer

Figure V-1 Acoustic Reflection Signatures from Ocean Subbottom Soil

V. RANDOM EXCITATION OF VISCOELASTIC MEDIUM

A typical set of acoustic input to the ocean and output (reflection) from the ocean subbottom are exhibited by Figure V-1. This set of acoustic signatures were remotely obtained by the Raytheon Company, 1972. It is observed from these signatures that the subbottom soil behaves as a vibrating mechanical system since its parts fluctuate in time. In studying such time series records it is natural to look for some kind of regularity in order to characterize the vibration in a simple manner. When there is no obvious pattern in a vibration record it is sometimes called a random vibration. The characteristic of such random function is that its instantaneous value cannot be predicted in a deterministic manner. Randomness involves the notion that in addition to the given record one should consider the totality of possible records that might equally well have been produced under the same conditions. If the identical experiment is performed many times and the records obtained are always alike, the process is said to be deterministic. However, if all the conditions (under the control of experimenter) are maintained to be same, the records continually differ from each other, the process is said to be random. In this case, a single record is not as meaningful as a statistical description of totality of possible records. [Crandall, Mark, 1963].

The field of probability and statistics must be utilized to acquire sediment parameters and is a convenience which cannot be overlooked. The statistical analysis of the return echoes gives results which predict the soil characteristics in a consistent manner.

The vibration record as seen in Figure V-1 is characterized by its amplitude and frequency. In random vibrations, the initial conditions and the phase have little meaning, and are therefore ignored. The main concern is with the average energy, which can be associated with the mean-square value of the response of the system. Here, the response of the system is the amplitude of the reflected acoustic signal from the ocean floor. This vibration record can be sufficiently described by an average amplitude and by a decomposition in frequency. The average amplitude can be determined by the calculation of root-mean square value. The frequency decomposition is indicated by the mean square spectral density. Another statistical parameter, the quality factor Q, can be obtained to provide a more complete picture of the record. It is the latter parameter Q which is the most important to obtain from the analysis. From Q the attenuation constant for the ocean subbottom soil can easily be determined.

V-1 NONSTATIONARY RANDOM PROCESSES

This is a very important section in analyzing the acoustic response signatures from the ocean floor. The data obtained from the experiment indicates that the process is non-stationary since the statistical properties of the data change with time. The time varying properties of the data can be determined only by performing instanteous averages over the ensemble of sample responses forming the process. In actual analysis, it is sometimes not necessary to obtain a huge number of sample records to allow an accurate measurement of soil properties by ensemble averaging. The non-stationary random data produced by this actual physical phenomena can be classified into a special category of non-stationarity. This particular experiment results in a random data which is a result of a non-stationary random process input $\{s(t)\}$ where each sample input function is given by $s(t) = e(t)\alpha(t)$. Here, $e(t)$ is well-defined, deterministic envelope function and $\alpha(t)$ is a Gaussian broadband noise function from a stationary random process $\{\alpha(t)\}$. This excitation is a nonstationary processes created by the multiplication of sample functions from a stationary process and the deterministic function $e(t)$. Since the nonstationary random data from the experiment is a result of this particular input, ensemble averaging is not always needed to describe the data. The various desired properties can then be estimated from a single sample signature, as is valid for ergodic stationary data.

An adequate methodology does not seem to exist for the analysis of all types of stationary data. This is mainly due to the fact that a non-stationary conclusion is generally a negative statement stating the lack of stationarity properties, rather than a positive statement defining the precise nature of nonstationarity. Hence, it follows that special techniques must be developed for non-stationary data which only apply to limited classes of data. Some examples of different types of non-stationary data are: (1) time-varying mean value, (2) time-varying mean square value, and (3) time-varying frequency value. (Carrier, Tugal, and Yildiz, 1974).

V.2 INPUT-OUTPUT RELATIONS FOR NONSTATIONARY DATA

Assume a sample input function $s(t)$ belonging to a non-stationary random process $\{s(t)\}$ excites a constant parameter linear system with Green's function $G(t)$ and frequency response function $G(\omega)$. For an arbitary input sample $s(t)$, the sample response (output) function $r(t)$ belonging to $\{r(t)\}$ is given by

$$r(t) = \int_{-\infty}^{\infty} G(t')s(t-t')dt' \qquad (V.2\text{-}1)$$

or

$$r(t) = \int_{-\infty}^{\infty} G(\omega)S(\omega)e^{i\omega t}\,\frac{d\omega}{2\pi} \qquad (V.2\text{-}2)$$

where $S(\omega)$ is a Fourier transform of $s(t)$.

The autocorrelation function of the system to nonstationary input excitation is given by

$$R_r(t_1,t_2) = E[r(t_1)r(t_2)] \qquad (V.2\text{-}3)$$

Upon substitution of Equation (V.2-2) into Equation (V.2-3)

$$R_r(t_1,t_2) = \iint P_r(\omega_1,\omega_2)e^{-i(\omega_1 t_1-\omega_2 t_2)}d\omega_1 d\omega_2 \qquad (V.2\text{-}4)$$

where

$$P_r(\omega_1,\omega_2) = G^*(\omega_1)\ G(\omega_2)\ P_s(\omega_1,\omega_2) \qquad (V.2\text{-}5)$$

with the spectrum of the input excitation given by

$$P_s(\omega_1,\omega_2) = \frac{1}{(2\pi)^2} E[S^*(\omega_1)S(\omega_2)] \tag{V.2-6}$$

Now the mean-square response is

$$E[r^2(t)] = R_r(t,t) \tag{V.2-7}$$

and from Equation (V.2-4)

$$E[r^2(t)] = \iint G^*(\omega_1)\, G(\omega_2)\, P_s(\omega_1,\omega_2)\, e^{i(\omega_1-\omega_2)t}\, d\omega_1 d\omega_2 \tag{V.2-8}$$

Since the generalized spectrum of the input excitation can be written as

$$P_s(\omega_1,\omega_2) = \iint R_s(t_1,t_2)\, e^{i(\omega_1 t_1-\omega_2 t_2)}\, \frac{dt_1\, dt_2}{(2\pi)^2} \tag{V.2-9}$$

where

$$R_s(t_1,t_2) = e(t_1)\, e(t_2)\, R_\alpha(\tau) \tag{V.2-10}$$

and $R_\alpha(\tau)$ has the Fourier transform $P_\alpha(\omega)$, the final form of Equation (V.2-9) becomes

$$P_s(\omega_1,\omega_2) = \int \frac{d\omega}{(2\pi)^2}\, P_\alpha(\omega)\, S_e(\omega-\omega_1)\, S_e(\omega_2-\omega) \tag{V.2-11}$$

with the envelope transformation functions given by

$$S_e(\omega-\omega_1) = \int \frac{dt_1}{2\pi}\, e(t_1)\, e^{-i(\omega-\omega_1)t_1}$$

$$S_e(\omega_2-\omega) = \int \frac{dt_2}{2\pi} e(t_2) e^{-i(\omega_2-\omega)t_2} \quad \text{(V.2-12)}$$

Noting the functions in (V.2-12) to be conjugate pairs when $\omega_1 = \omega_2$, the substitution of Equation (V.2-11) into Equation (V.2-8) gives the mean-square response of the system

$$E[r^2(t)] = \int_{-\infty}^{\infty} P_\alpha(\omega) \, |\Lambda(t,\omega)|^2 \, d\omega \quad \text{(V.2-13)}$$

where

$$\Lambda(t,\omega) = \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} G(\omega_2) S_e(\omega_2-\omega) e^{i\omega_2 t} \quad \text{(V.2-14)}$$

The desired general information for inputs of amplitude modulated stationary noise is given by Equation (V.2-13).

Finally, the initial or non-initial Green's functions of visco-elastic compressional (or shear) wave systems in $(\vec{k};\omega)$ domain can be used in Equations (V.2-13) and (V.2-14) to obtain the mean-square response of viscoelastic and even thermoviscoelastic mediums to this particular type of non-stationary random input.

V.3 MEAN SQUARE RESPONSE OF VISCOELASTIC COMPRESSIONAL WAVE SYSTEM TO NONSTATIONARY RANDOM EXCITATION (NON-INITIAL VALUED)

In this section, the mean square response of viscoelastic compressional wave medium to a nonstationary random process input sample given by $s(t) = e(t)\,\alpha(t)$ is determined. The Green's function is non-initial valued

$$G(\omega) = \frac{1}{-\omega^2 + c^2 + i2\omega b} \qquad \text{(V.3-1)}$$

where the parameter k is suppressed. It should be noted that the calculations and the results obtained here can easily be applied to viscoelastic shear wave medium as well as to thermoviscoelastic mediums with the appropriate modifications of the damped harmonic oscillator parameters.

The mean-square response $E[r^2(t)]$ of the system when e(t) is a unit and a rectangular step function and $\alpha(t)$ has the correlation functions

$$R_\alpha(\tau) = 2\pi K_0 \delta(\tau) \qquad \text{(V.3-2)}$$

for white noise, and

$$R_\alpha(\tau) = K_0 \, e^{-\beta|\tau|} \cos \Omega \tau \qquad \text{(V.3-3)}$$

for the correlated noise are determined. The τ is the time difference $t_2 - t_1$.

V.3.a UNIT STEP ENVELOPE FUNCTION

If a unit step envelope function $e(t) = \eta(t) = 1,\ t \geq 0$ and zero elsewhere then the frequency shifted unit step envelope transformation function becomes

$$S_e(\omega_2-\omega) = \int e^{-i\omega_2 t} \eta(t) e^{i\omega t} \, dt \, , \qquad \text{(V.3-3)}$$

$$S_e(\omega_2-\omega) = \pi\delta(\omega_2-\omega) + \frac{1}{i(\omega_2-\omega)} \; .$$

Substitution of Equation (V.3-3) into Equation (V.2-14) and the evaluation of the resultant integral gives

$$|\Lambda(t,\omega)|^2 = |G(\omega)|^2 \, M(t,\omega) \qquad \text{(V.3-4)}$$

where

$$M(t,\omega) = 1 + \Gamma_1(t) + \Gamma_2(t) \left[\frac{b^2 - a^2 + \omega^2}{a^2}\right] -$$

$$- 2\Gamma_3(t) \cos \omega t - 2\Gamma_4(t) \frac{\omega}{a} \sin \omega t \qquad \text{(V.3-5)}$$

with

$$\Gamma_1(t) = e^{-2bt} \left[1 + \frac{b}{a} \sin 2at\right] \, ,$$

$$\Gamma_2(t) = e^{-2bt} \sin^2 at \; ,$$

$$\Gamma_3(t) = e^{-bt} \left[\cos at + \frac{b}{a} \sin at\right] \; ,$$

$$\Gamma_4(t) = e^{-bt} \sin at \; . \qquad \text{(V.3-6)}$$

Hence, the mean-square response via Equation (V.2-13) becomes

$$E[r^2(t)] = \int |G(\omega)|^2 M(t,\omega) P_\alpha(\omega) d\omega \qquad (V.3-7)$$

It should be noted that in Equation (V.3-7), in the limit as $t \to \infty$, $M(t,\omega) \to 1$, so the last expression reduces to the mean- square response formulation for stationary inputs.

V.2.a(i) WHITE NOISE INPUTS

If the input noise is assumed white, then the spectral density function $P_\alpha(\omega)$ becomes a constant P_0. So, the mean-square response becomes

$$E[r^2(t)] = P_0 \int_{-\infty}^{\infty} |G(\omega)|^2 M(t,\omega) d\omega \qquad (V.3-8)$$

The result of the last expression is:

$$E[r^2(t)] = \frac{\pi P_0}{2bc^2} [1 - e^{-2bt} x$$

$$x \left(1 + \frac{b}{a} \sin 2 at + 2 \frac{b^2}{a^2} \sin^2 at\right)] \qquad (V.3-9)$$

A normalized plot of Equation (V.3-9) is shown in Figure (V-2).

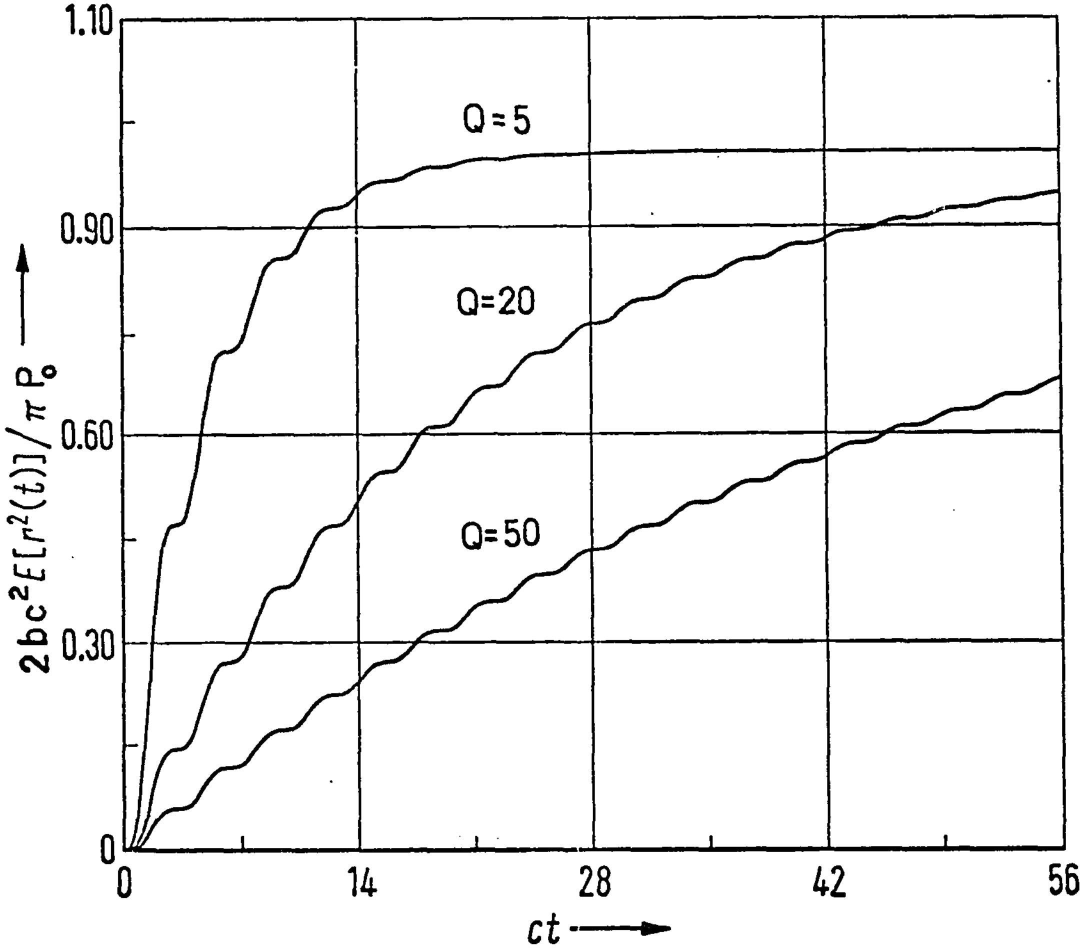


Figure V-2 Normalized rms Response to White Noise Modulated by a Unit Step Function

V.3.a(ii) CORRELATED INPUT EXCITATION

If the input excitation is assumed correlated as stated in Equation (V.3-3), then the spectral density becomes, Figure (V-3),

$$P_\alpha(\omega) = \frac{K_0}{\pi} \frac{\beta(\beta^2 + \Omega^2 + \omega^2)}{(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \qquad (V.3\text{-}10)$$

where $\omega_3 = \Omega + i\beta$ and $\omega_4 = -\Omega + i\beta$. It should be noted that for white noise $P_0 = \lim_{\beta\to\infty} \beta\, P_\alpha(\omega) = K_0/\pi$ and this expression is useful for checking the consistency of results, as it will be shown later.

Upon substitution of the spectral density for correlated noise in Equation (V.3-10) into expression (V.3-7), the mean square becomes

$$E[r^2(t)] = K_0[R_1T_1(t) + I_1T_2(t) + R_3T_3(t) - I_3T_4(t)] \qquad (V.3\text{-}11)$$

where

$$T_1(t) = \frac{a}{2b}[1 - \Gamma_1(t)], \; T_2(t) = \Gamma_2(t),$$

$$T_3(t) = \Big\{ 1 + \Gamma_1(t) + \frac{b^2 - a^2 - \Omega^2 - \beta^2}{a^2} \times$$

$$\times\, \Gamma_2(t) - 2\,[\Gamma_3(t) + \frac{\beta}{a}\Gamma_4(t)]\, e^{-\beta t} \cos \Omega t -$$

$$- 2\,\frac{\Omega}{a}\,\Gamma_4(t)\, e^{-\beta t} \sin \Omega t \Big\}$$

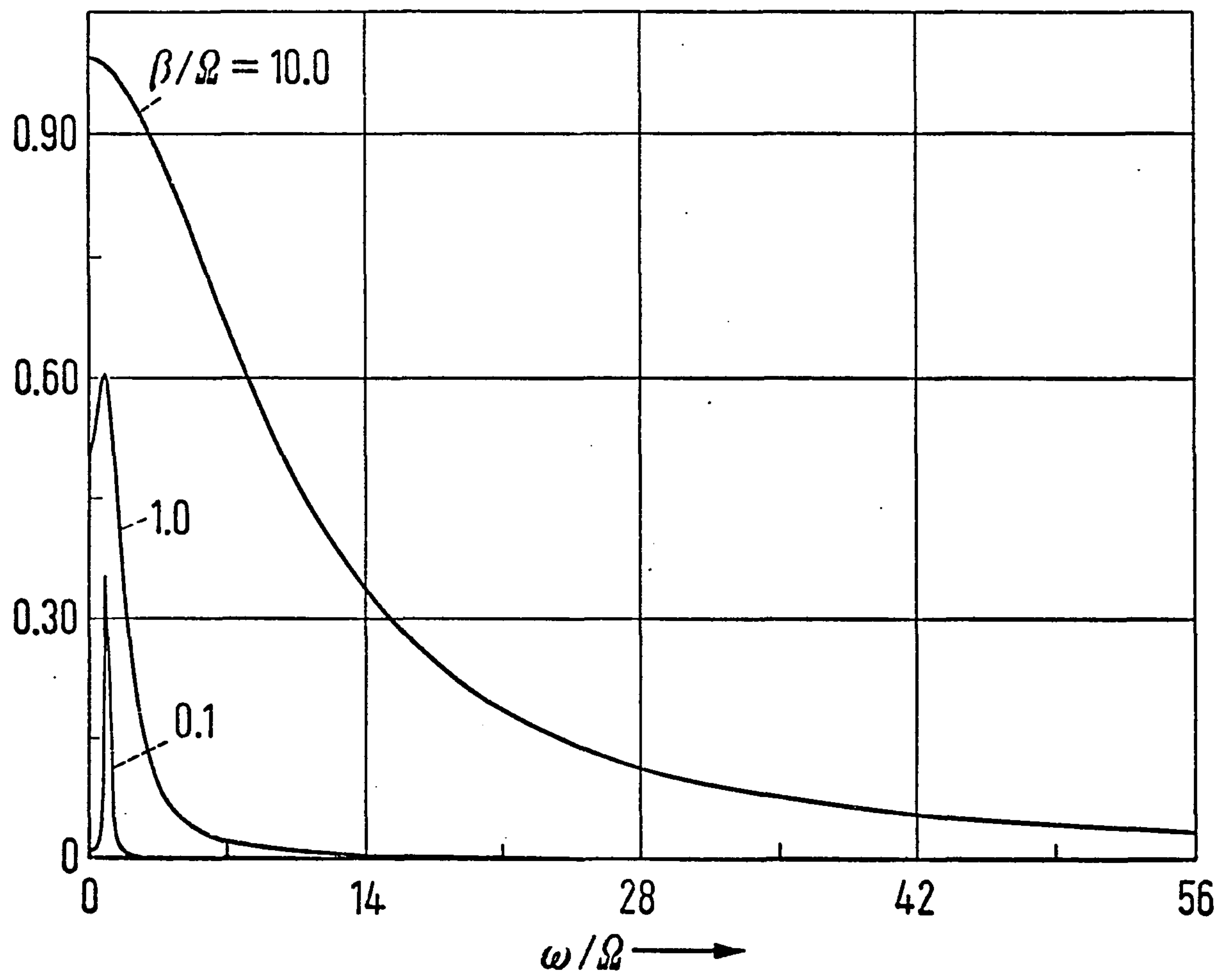


Figure V-3 Normalized Spectral Density $\beta P_{\alpha}(\omega)$

$$T_4(t) = \Big\{2 \frac{\beta\Omega}{a^2} \Gamma_2(t) - 2 \left[\Gamma_3(t) + \frac{\beta}{a} \Gamma_4(t)\right] x$$

$$x\ e^{-\beta t} \sin \Omega t + 2 \frac{\Omega}{a} \Gamma_4(t)\ e^{-\beta t} \cos \Omega t \Big\} \qquad \text{(V.3-12)}$$

and

$$R_1 = Re \left[\frac{\Omega^2 + \beta^2 + \omega_1^2}{\omega_1(\omega_1^2 - \omega_3^2)(\omega_1^2 - \omega_2^2)}\right] \frac{\beta}{a^2}$$

$$R_3 = Re \left[\frac{1}{(\omega_3^2 - \omega_1^2)(\omega_3^2 - \omega_2^2)}\right]$$

$$I_1 = Im \left[\frac{\Omega^2 + \beta^2 + \omega_1^2}{\omega_1(\omega_1^2 - \omega_3^2)(\omega_1^2 - \omega_2^2)}\right] \frac{\beta}{a^2}$$

$$I_3 = Im \left[\frac{1}{(\omega_3^2 - \omega_1^2)(\omega_3^2 - \omega_2^2)}\right] \qquad \text{(V.3-13)}$$

With little algebra, it can be shown from the limiting process $\lim_{\beta\to\infty} \beta\, E[r^2(t)]$ the mean-square response for a correlated noise given in Equation (V.3-11) reduces to the one for a white noise expression given in Equation (V.3-9).

The above expression, (V.3-11), indicates the compressional wave system's response is dependent upon variables which involve a Lamé parameter λ, that is λ' and λ'', the shear modulus μ', the shear velocity μ'', the viscoelastic soil medium density ρ, the wave number k, the correlation function decay constant β, and the correlation frequency Ω. Further note that for a large number of response cycles ct the exponential decay terms in the correlated noise in Equation (V.3-11) tend to zero and the mean-square response reduces to the stationary value

$$E[r^2(t)] \Big|_{\beta\to\infty} = K_0[(a/2b)R_1 + R_3]$$

V.3.b RECTANGULAR STEP ENVELOPE FUNCTION

For a rectangular step envelope function of duration t',

$$e(t) = \eta(t) - \eta(t-t')$$

and upon substitution into Equation (V.2-12) the rectangular step envelope transformation function becomes

$$S_e(\omega_2 - \omega) = [1 - e^{-i(\omega_2-\omega)t'}] \times$$

$$\times \; [\pi\delta(\omega_2 - \omega) + \frac{1}{i(\omega_2 - \omega)}] \qquad \text{(V.3-14)}$$

Substitution of the last expression into Equation (V.2-14) one obtains

$$|\Lambda(t,\omega)|^2 = |G(\omega)|^2 \Big\{ M(t,\omega)\eta(t) +$$

$$+ (\Gamma_1(t) - M(t,\omega) + \Gamma_1(t-t') +$$

$$+ [\frac{b^2 - a^2 + \omega^2}{a^2}] [\Gamma_2(t) + \Gamma_2(t-t')] -$$

$$- 2 [\Gamma_3(t) \Gamma_3(t-t') + \frac{\omega^2}{a^2} \Gamma_4(t) \Gamma_4(t-t')] \times$$

$$\times \cos \omega t' + 2 \frac{\omega}{a} [\Gamma_4(t-t') \Gamma_3(t) -$$

$$- \Gamma_3(t-t') \Gamma_4(t)] \sin \omega t') \eta(t-t') \Big\} \quad \text{(V.3-15)}$$

Hence, from Equation (V.2-14) the mean-square response becomes

$$E[r^2(t)] = \int_{-\infty}^{\infty} |G(\omega)^2| P_\alpha(\omega) M(t,\omega) \quad \text{for} \quad 0 \leq t \leq t'$$

$$E[r^2(t)] = \int_{-\infty}^{\infty} |G(\omega)|^2 P_\alpha(\omega) M_r(t,\omega) \quad \text{for} \quad t \geq t' \quad \text{(V.3-16)}$$

where $M(t,\omega)$ is given by Equation (V.3-5) and

$$M_r(t,\omega) = \Gamma_1(t) + \Gamma_1(t-t') +$$

$$+ \frac{b^2 - a^2 + \omega^2}{a^2} [\Gamma_2(t) + \Gamma_2(t-t')] -$$

$$- 2\ [\Gamma_3(t)\ \Gamma_3(t-t') + \frac{\omega^2}{a^2}\ \Gamma_4(t-t')]\ x$$

$$x\ \cos\ \omega t' + 2\ \frac{\omega}{a}\ [\Gamma_3(t)\ \Gamma_4(t-t') -$$

$$- \Gamma_3(t-t')\ \Gamma_4(t)]\ \sin\ \omega t' \qquad (V.3\text{-}17)$$

V.3.b(i) WHITE NOISE INPUT

If input excitation is assumed white, then

$$E[r^2(t)] = P_0 \int_{-\infty}^{\infty} |G(\omega)|^2\ M(t,\omega)d\omega \quad \text{for} \quad 0 \leqq t \leqq t'$$

$$E[r^2(t)] = P_0 \int_{-\infty}^{\infty} |G(\omega)|^2\ M_r(t,\omega) \quad \text{for} \quad t \geqq t' \qquad (V.3\text{-}18)$$

The first integral is exactly Equation (V.3-9) and the second integral is

$$E[r^2(t)] = \frac{\pi P_0}{2bc^2} \Big\{ \Gamma_1(t) + \Gamma_1(t-t') +$$

$$+ 2\ \frac{b^2}{a^2}\ [\Gamma_2(t) - \Gamma_2(t-t')] - 2\ [\Gamma_3(t)\ \Gamma_3(t') +$$

$$+ \frac{c^2}{a^2}\ \Gamma_4(t)\ \Gamma_4(t')]\ \Gamma_3(t-t') +$$

$$+ 2\ \frac{c^2}{a^2}\ [2\ \frac{b}{a}\ \Gamma_4(t)\ \Gamma_4(t') - \Gamma_4(t)\ \Gamma_3(t') +$$

$$+ \Gamma_3(t)\, \Gamma_4(t')\,]\, \Gamma_4(t-t') \Big\} \text{ for } t \geq t' \qquad (V.3\text{-}19)$$

No plots are done for the first integral in Equation (V.3-18), since it is exactly the same as that for a unit step envelope function. A normalized plot of Equation (V.3-19) is shown in Figure (V-6). In the graph, the duration of the rectangular step function t' was taken to be 10/c. It is observed from the graphs, that the response is a square of an exponentially decaying harmonic function.

V.3.b(ii) CORRELATED NOISE INPUT

If the unput excitation is assumed correlated as in Equation (V.3-3), then $P_\alpha(\omega)$ is given by Equation (V.3-10). Upon substitution of Equation (V.3-10) into Equation (V.3-16) and the evaluation of the resultant integral, the mean-square response is obtained

$$E[r^2(t)] = K_0[R_1T_1(t) + I_1T_2(t) +$$

$$+ R_3T_3(t) - I_3T_4(t)] \quad \text{for } 0 \leq t \leq t'$$

$$E[r^2(t)] = K_0[R_1T_{11}(t) - I_1T_{22}(t) +$$

$$+ R_3T_{33}(t) - I_3T_{44}(t)] \quad \text{for } t \geq t' \qquad (V.3\text{-}20)$$

where

$$T_{11}(t) = \frac{a}{2b} \left\{[\Gamma_1(t) + \Gamma_1(t-t')] - 2\left[\left(\Gamma_3(t) + \frac{b}{a}\Gamma_4(t)\right)\Gamma_3(t-t') -\right.\right.$$

$$-\left(\frac{b}{a}\Gamma_3(t) + \frac{b^2-a^2}{a^2}\Gamma_4(t)\right)\Gamma_4(t-t')\Big]\Gamma_3(t') + 2\left[\left(\frac{b}{a}\Gamma_3(t) + \frac{b^2-a^2}{a^2}\Gamma_4(t)\right)\Gamma_3(t-t') -\right.$$

$$\left.\left. - \left(\frac{b^2-a^2}{a^2}\Gamma_3(t) + \frac{b(b^2-3a^2)}{a^3}\Gamma_4(t)\right)\Gamma_4(t-t')\right]\Gamma_4(t')\right\}$$

$$T_{22}(t) = \frac{a}{b}\left\{\frac{b}{a}[\Gamma_2(t)+\Gamma_2(t-t')] + \left[\Gamma_4(t)\Gamma_3(t-t') - \left(\Gamma_3(t) + \frac{2b}{a}\Gamma_4(t)\right)\Gamma_4(t-t')\right]\Gamma_3(t') -\right.$$

$$\left. -\left[\left(\Gamma_3(t) + \frac{2b}{a}\Gamma_4(t)\right)\Gamma_3(t-t') - \left(\frac{2b}{a}\Gamma_3(t) + \frac{3b^2-a^2}{a^2}\Gamma_4(t)\right)\Gamma_4(t-t')\right]\Gamma_4(t')\right\}$$

$$T_{33}(t) = \Gamma_1(t) + \Gamma_1(t-t') + \left(\frac{b^2-a^2+\Omega^2-\beta^2}{a^2}\right)[\Gamma_2(t)+\Gamma_2(t-t')] - 2\left[\left(\Gamma_3(t) + \frac{\beta}{a}\Gamma_4(t)\right) \times\right.$$

$$\times\, \Gamma_3(t-t') - \left(\frac{\beta}{a}\Gamma_3(t) + \frac{\beta^2-\Omega^2}{a^2}\Gamma_4(t)\right)\Gamma_4(t-t')\Big]e^{-\beta t'}\cos\Omega t' - \frac{2\Omega}{a}\Big[\Gamma_4(t)\Gamma_3(t-t') -$$

$$- \left(\Gamma_3(t) + \frac{2\beta}{a}\Gamma_4(t)\right)\Gamma_4(t-t')\Big]e^{-\beta t'}\sin\Omega t'$$

$$T_{44}(t) = 2\left\{\frac{\beta\Omega}{a^2}[\Gamma_2(t) + \Gamma_2(t-t')] - \left[\left(\Gamma_3(t) + \frac{\beta}{a}\Gamma_4(t)\right)\Gamma_3(t-t') -\right.\right.$$

$$- \left(\frac{\beta}{a}\Gamma_3(t) + \frac{\beta^2-\Omega^2}{a^2}\Gamma_4(t)\right)\Gamma_4(t-t')\Big]e^{-\beta t'}\sin\Omega t' +$$

$$\left. + \frac{\Omega}{\beta}\left[\Gamma_4(t)\Gamma_3(t-t') - \left(\Gamma_3(t) + \frac{2\beta}{a}\Gamma_4(t)\right)\Gamma_4(t-t')\right]e^{-\beta t'}\cos\Omega t'\right\}$$

V.4 MEAN-SQUARE RESPONSE OF COMPRESSIONAL VISCOELASTIC SYSTEM TO NONSTATIONARY RANDOM EXCITATION. (INITIAL VALUED)

The initial valued Green's function was derived to be, in $(\vec{k};\omega)$ domain

$$G_i(\omega) = \frac{c^2 - i\omega 2b}{-\omega^2 - c^2 + i\omega 2b} \quad . \tag{V.4-1}$$

Thus, Equation (V.2-14) becomes

$$\Lambda(t,\omega) = \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \; G_i(\omega_2) S_e(\omega_2 - \omega) e^{i\omega_2 t} \tag{V.4-2}$$

or

$$\Lambda(t,\omega) = \left(c^2 + 2b \frac{\partial}{\partial t}\right) \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \; G(\omega_2) S_e(\omega_2 - \omega) e^{i\omega_2 t} \tag{V.4-3}$$

where $G(\omega)$ is given by (V.3-1). To determine the mean-square response of this particular system the same procedure as in the previous section is followed. The results are given in Appendix 3.

V.5 MEAN-SQUARE RESPONSE OF MULTILAYERED VISCOELASTIC COMPRESSIONAL WAVE SYSTEM TO NONSTATIONARY RANDOM EXCITATION

Here, the single damped harmonic oscillator model of a viscoelastic medium in $(\vec{k};t)$ and $(\vec{k};\omega)$ domain representations is extended to a multidegree freedom damped harmonic oscillator model which prescribes the characteristics of viscoelastic reflectors in the ocean subbottom.

The equation of motion for the j th viscoelastic layer is given by

$$\rho_j \partial_t^2 \vec{u}_j - [(\lambda'_j + \mu'_j) + (\lambda''_j + \mu''_j)\partial_t]\nabla \nabla\cdot\vec{u}_j$$

$$- (\mu'_j + \mu''_j \partial_t)\nabla^2 \vec{u}_j = \vec{f}_j \qquad (V.5\text{-}1)$$

and the compressional viscoelastic layer equation in terms of Green's function formalism is given by

$$[\partial_t^2 - \left(\frac{\lambda'_j + 2\mu'_j}{\rho_j}\right) \nabla^2 - \left(\frac{\lambda''_j + 2\mu''_j}{\rho_j}\right) \nabla^2\partial_t]G_j(\vec{r}-\vec{r}';t-t') =$$

$$= \delta(t-t')\ \delta(\vec{r}-\vec{r}')\cdot \qquad (V.5\text{-}2)$$

For the j th layer the Green's function is

$$G_j(\omega) = \frac{1}{-\omega^2 + c_j^2 + i2\omega b_j} \qquad (V.5\text{-}2a)$$

$$G_j(t-t') = \eta(t-t')e^{-b_j(t-t')}\ \frac{\sin [a_j(t-t')]}{a_j}\ . \qquad (V.5\text{-}2b)$$

Defining the input excitation as in the previous sections, then the j th viscoelastic compressional wave medium layer response can be written as

$$r_j(t) = \int s(t')\ G_j(t-t')dt' = \int S(\omega)\ G_j(\omega)e^{i\omega t}\ \frac{d\omega}{2\pi} \qquad (V.5\text{-}3)$$

and the total response of n-layers is given by

$$r(t) = \sum_{j=1}^{n} r_j(t) \; . \qquad \text{(V.5-4)}$$

Here the mean-square response of two layers to correlated noise modulated by a rectangular step envelope function shall be presented in order to present the procedures to determine the total response from n-layers.

The autocorrelation function for n-layers when the input force is nonstationary is expressed as

$$E[r(t_1)r(t_2)] = \iint \sum_{j=1}^{n} G^*(\omega_1) \sum_{\ell=1}^{n} G_\ell(\omega_2)\; P_s(\omega_1,\omega_2) e^{-i(\omega_1 t_1 - \omega_2 t_2)} \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi}$$

$$\text{(V.5-5)}$$

where $P_s(\omega_1,\omega_2)$ is given by Equation (V.2-9). Using the same procedure as in Section (V.2), the total mean-square response with Equation (V.5-5) is given to be

$$E[r^2(t)] = \int P_\alpha(\omega) \sum_{j,\ell=1}^{n} \Lambda^*{}_j(t,\omega)\; \Lambda_\ell(t,\omega) d\omega \qquad \text{(V.5-6)}$$

$$\Lambda_j(t,\omega) = \int G_j(\omega_2)\; S_e(\omega_2 - \omega) e^{i\omega_2 t} \frac{d\omega_2}{2\pi} \qquad \text{(V.5-7)}$$

The analytical results for two viscoelastic compressional wave mediums are obtained by letting n = 2 in Equation (V.5-6).

$$E[r^2(t)] = \int P_\alpha(\omega) \, [|\Lambda_1(t,\omega)|^2 + |\Lambda_2(t,\omega)|^2 + \Lambda_1^*(t,\omega) \, \Lambda_2(t,\omega)$$

$$+ \Lambda_1(t,\omega) \, \Lambda_2^*(t,\omega)] d\omega \qquad \text{(V.5-8)}$$

where the first two expressions on the right hand side represents the response of first and second mediums acting as infinite mediums and the last two represents the cross-terms between the layers.

The total mean-square response of two layers to correlated noise modulated by a rectangular step envelope function is given in Appendix 4.

V.6 SUMMARY

The mean-square response of ocean subbottom modeled as a linear single and multi-degree-of-freedom damped harmonic oscillator to a particular nonstationary input is calculated. The general response formulation is presented in terms of the Green's function of the visco-elastic compressional wave medium and the spectrum of the input excitation. A unit step modulation and a rectangular step modulation are considered in conjunction with both correlated and uncorrelated noise of zero mean.

This time-varying response is dependent upon the viscoelastic medium damping and natural frequency, the shape of the modulation function, and the parameters of the noise correlation function. For white noise modulated by a unit step function the time-varying mean-square response does not exceed the stationary mean-square response to white noise (see Figure (V-2)). For a correlated noise modulated by a unit step function the mean-square response shows how the quality factors influence the magnitude of the stationary value and affect how quickly the stationarity is achieved. The lower quality factors of larger damping values result in lower stationary values and the response becomes stationary in a shorter duration. See Figures (V-4) and (V-5). For a white noise modulated by a rectangular step function as the quality factor decreases, the mean-square response increases. See Figure (V-6). For a correlated noise modulated by a rectangular step function as the quality factor decreases the mean-square response damps out quicker. See Figures (V-7) and (V-8). This is also true for the mean-square response of two layered medium for a given pair of quality factors. See Figures (V-9) and (V-10).

Furthermore, it should be noted that the cross-terms of the mean-square response of two layered medium are the terms that give rise to the second peak due to the second layer. Compare Figures (V-11) and (V-12).

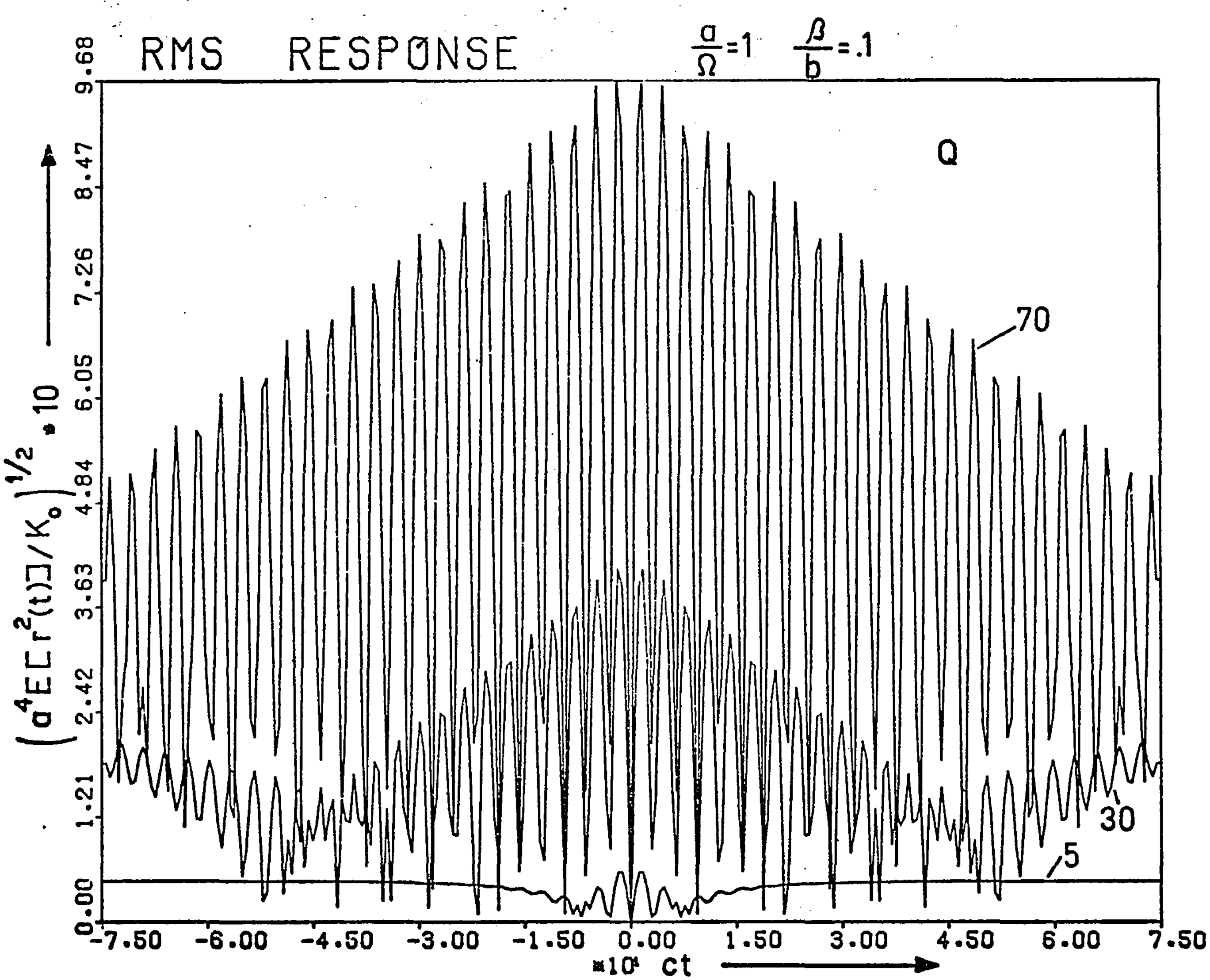


Figure V-4: Normalized rms Response to Correlated Noise Modulated by a Unit Step Function

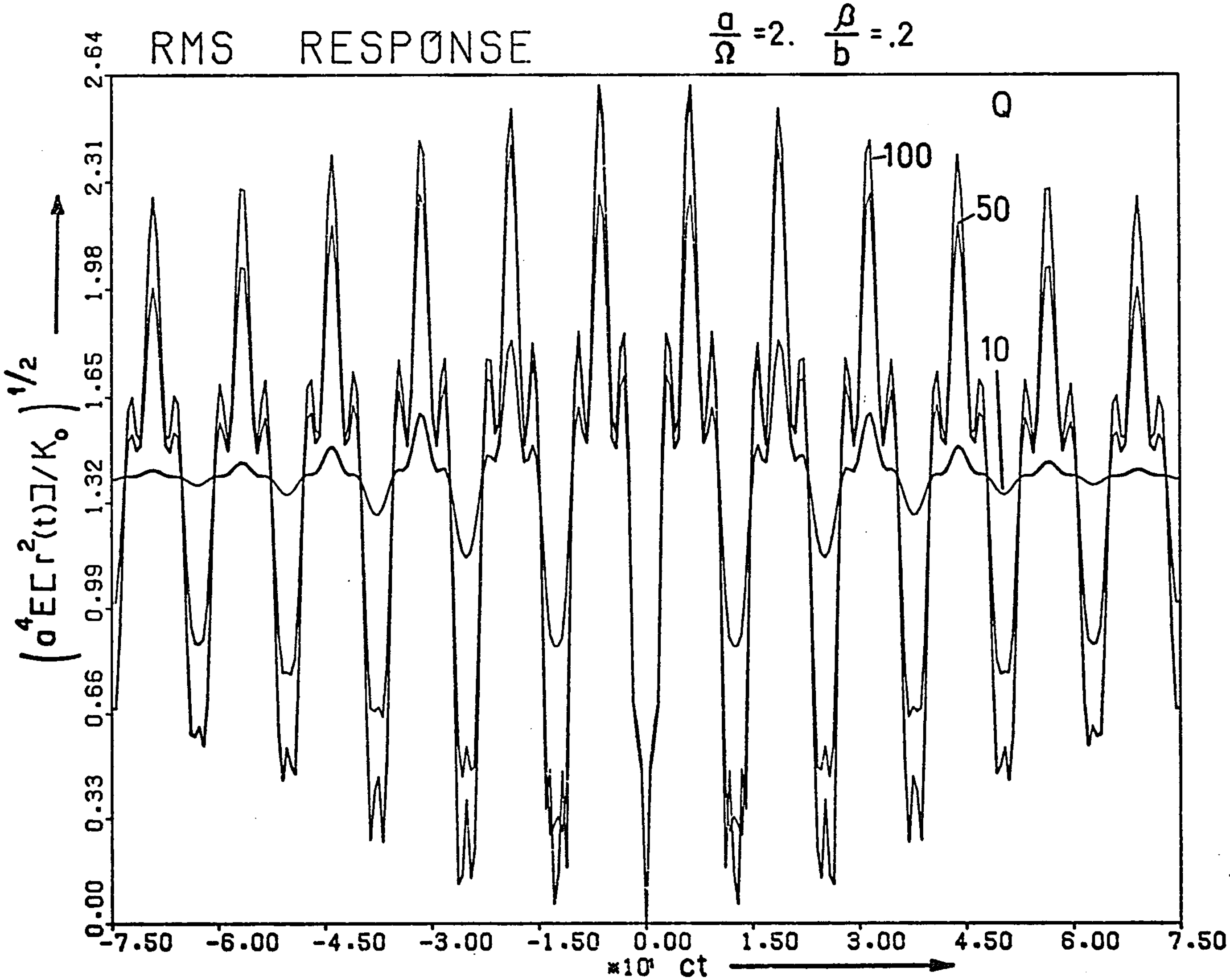


Figure V-5: Normalized rms Response to Correlated Noise Modulated by a Unit Step Function

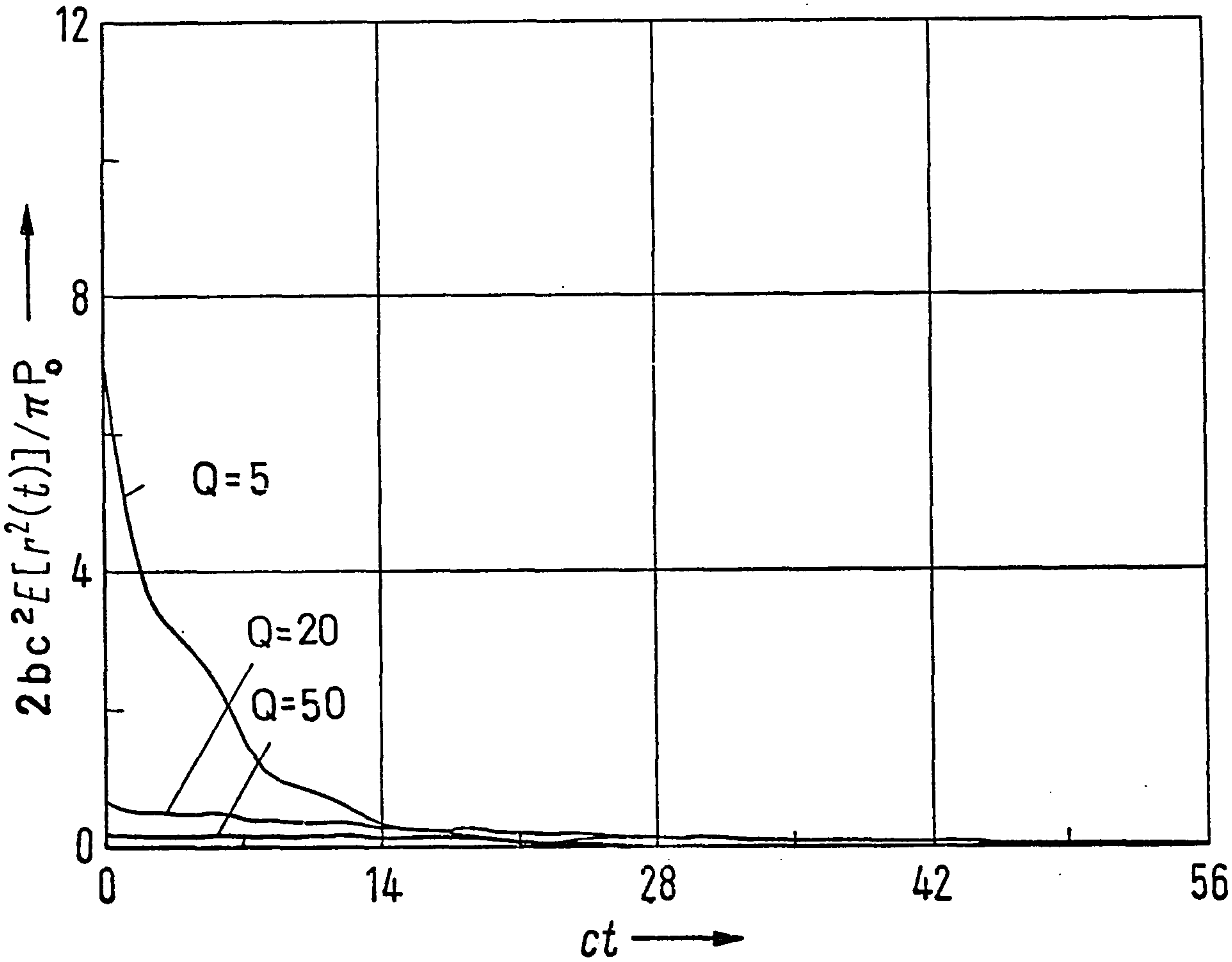


Figure V-6: Normalized rms Response to White Noise Modulated by a Rectangular Step Function

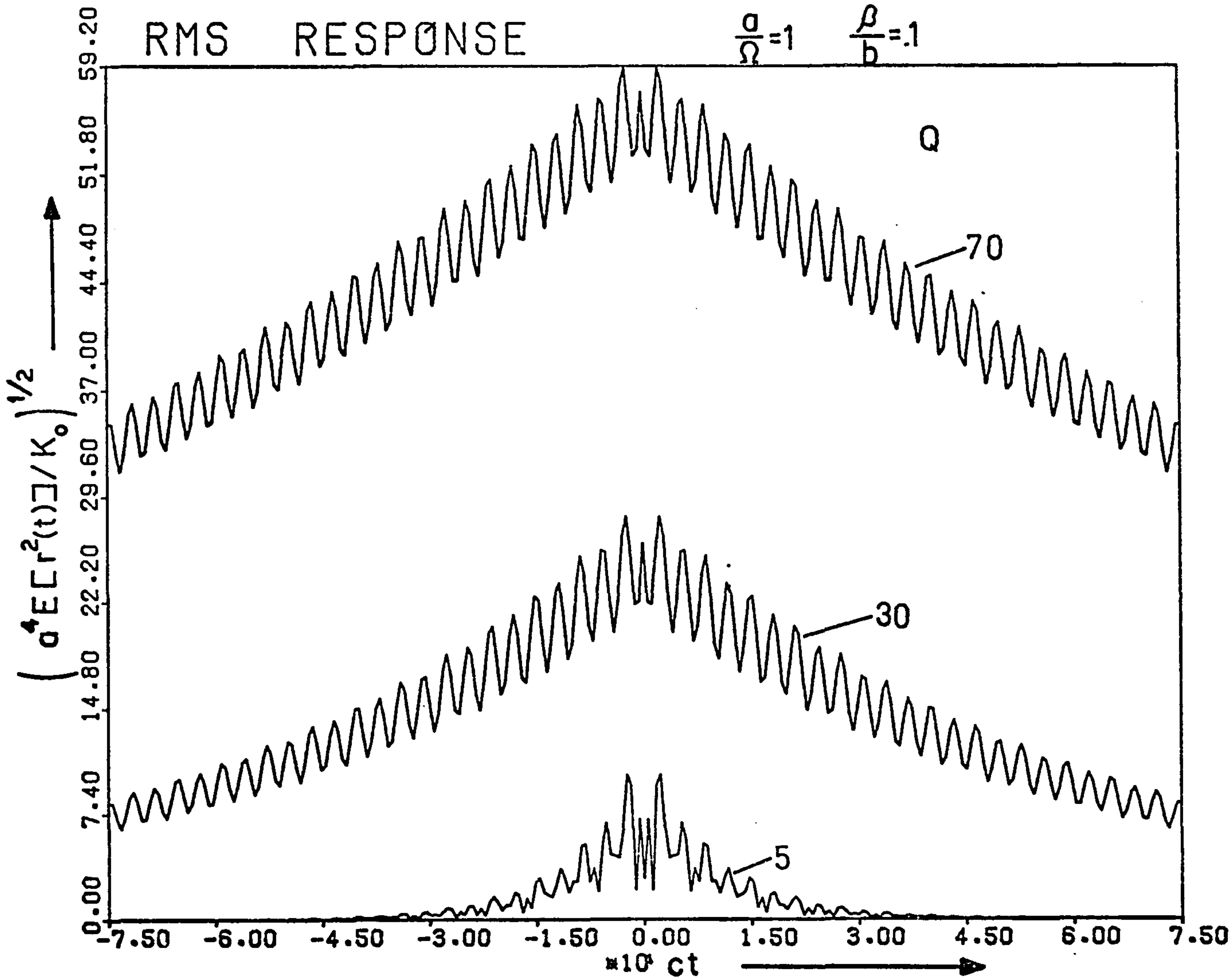


Figure V-7: Normalized rms Response to Correlated Noise Modulated by a Rectangular Step Function

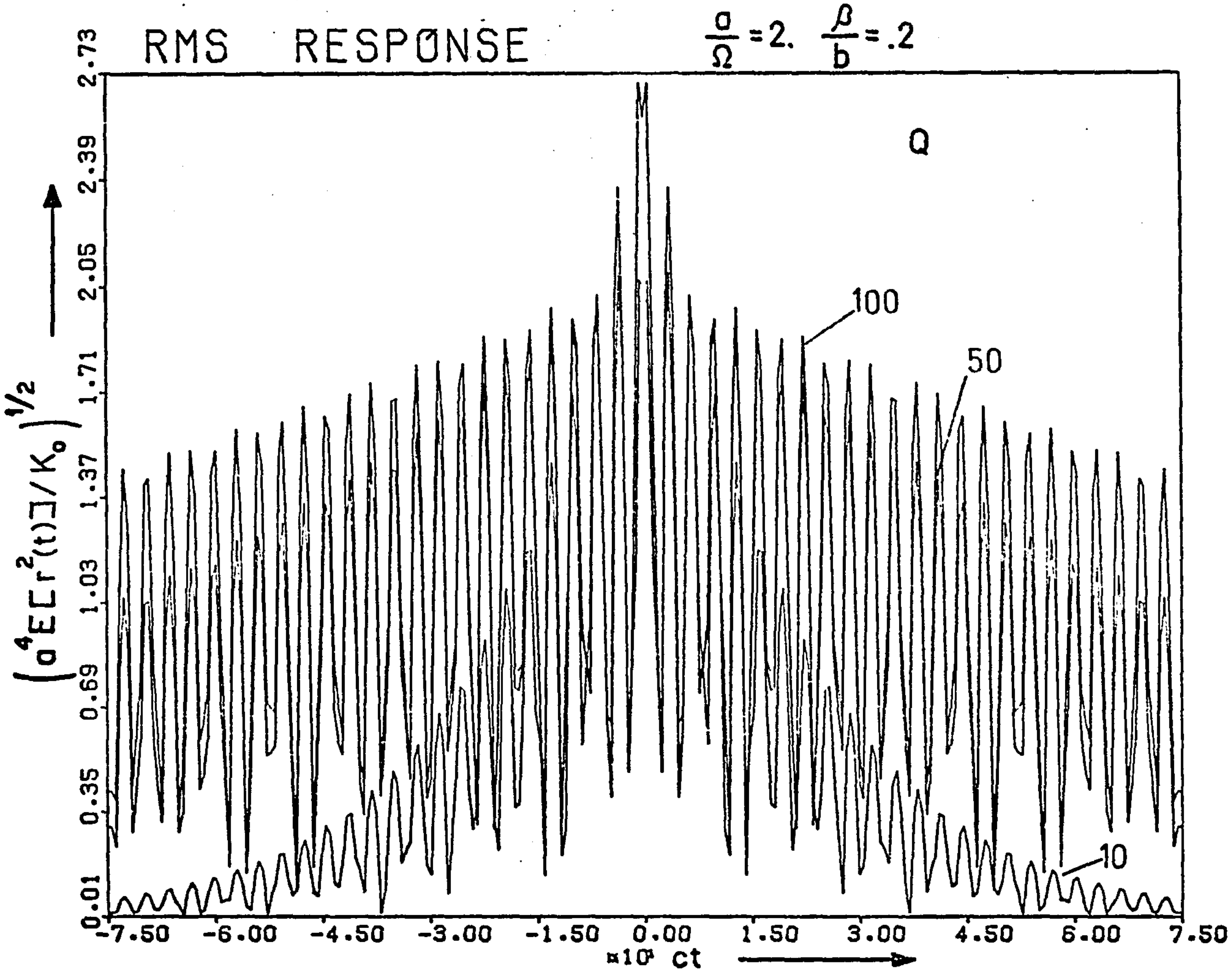


Figure V-8: Normalized rms Response to Correlated Noise Modulated by a Rectangular Step Function

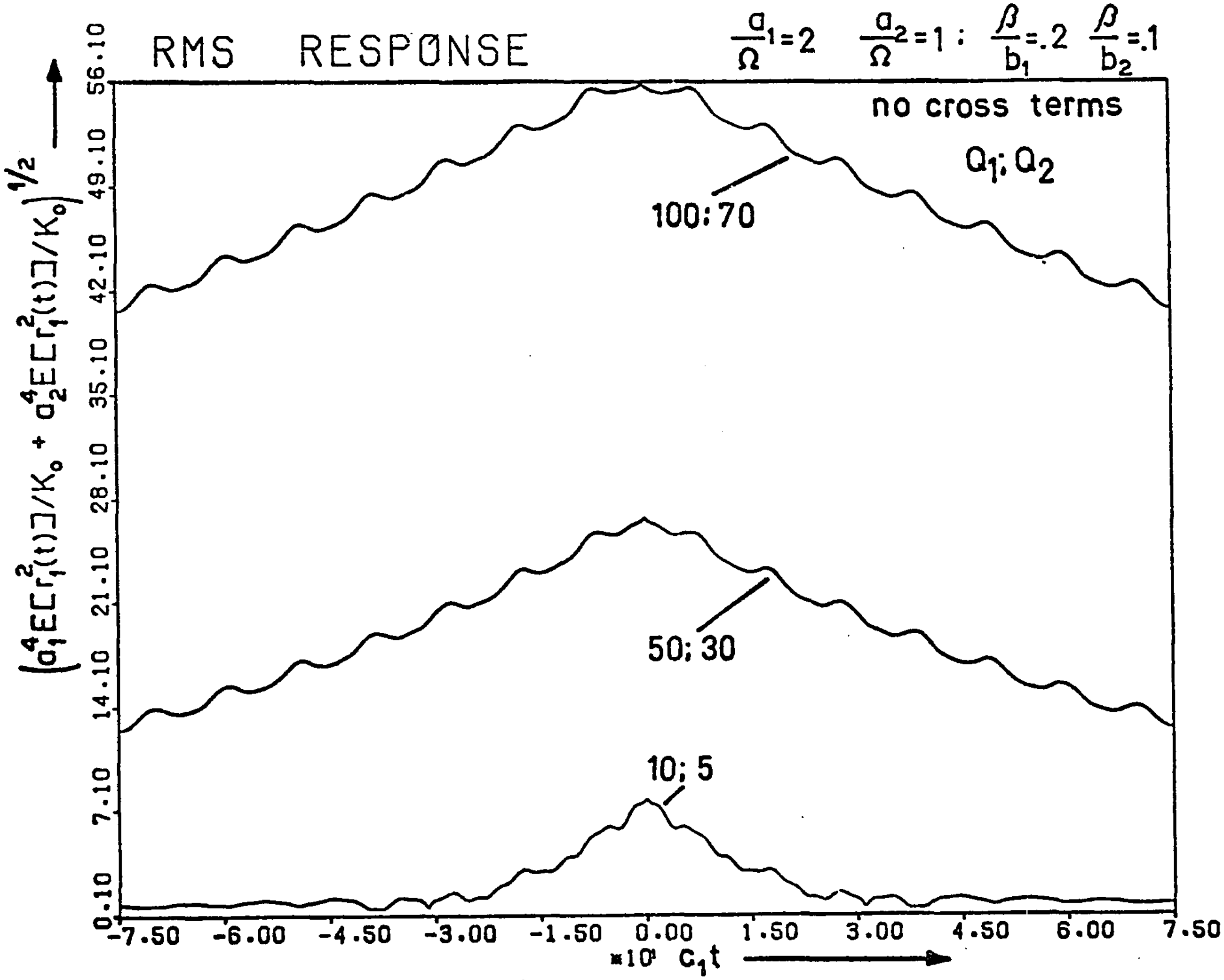


Figure V-9: Normalized rms Response of Double Layer to Correlated Noise Modulated by a Rectangular Step Function

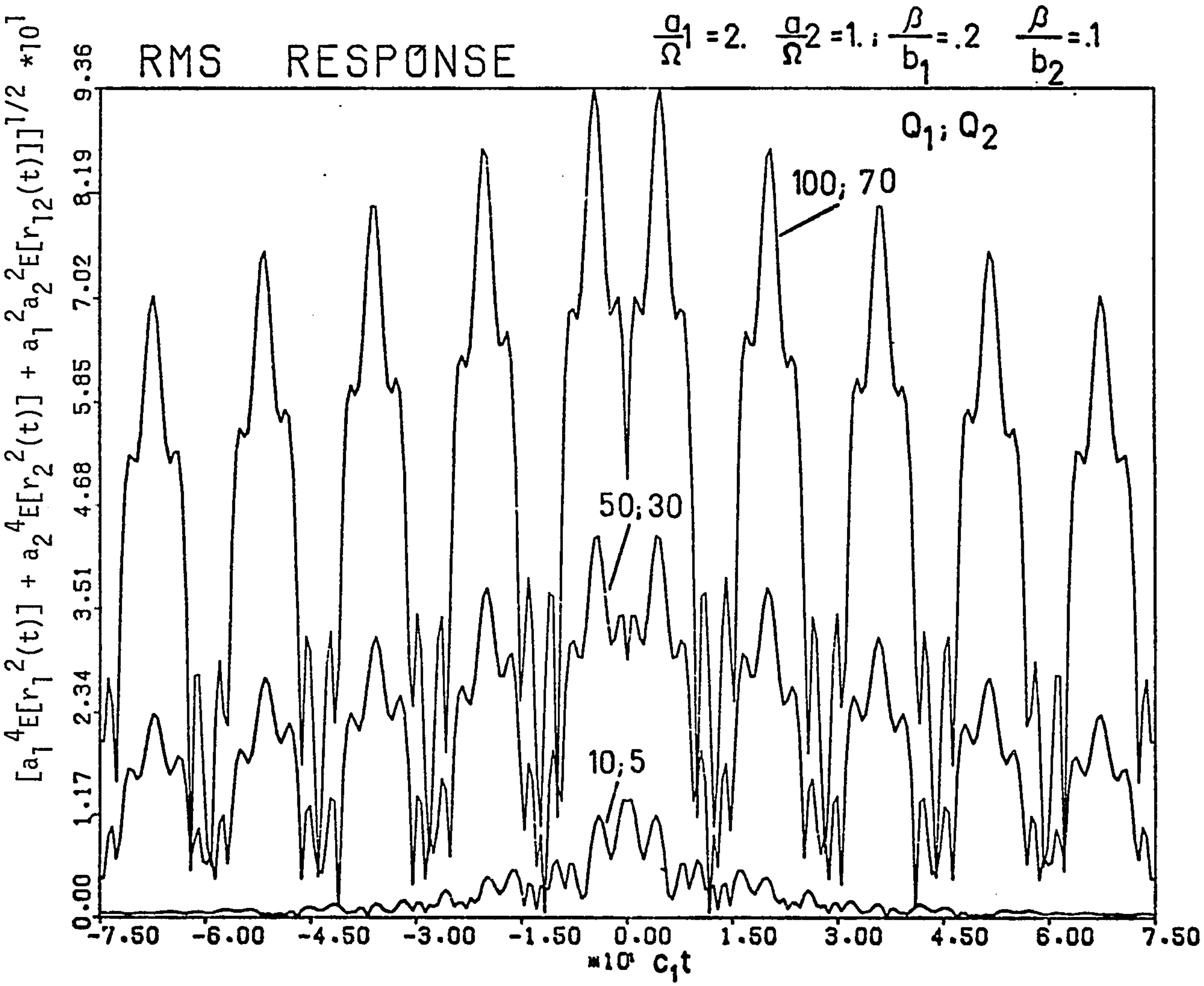


Figure V-10: Normalized rms Response of Double Layer to Correlated Noise Modulated by a Rectangular Step Function (with Cross Terms)

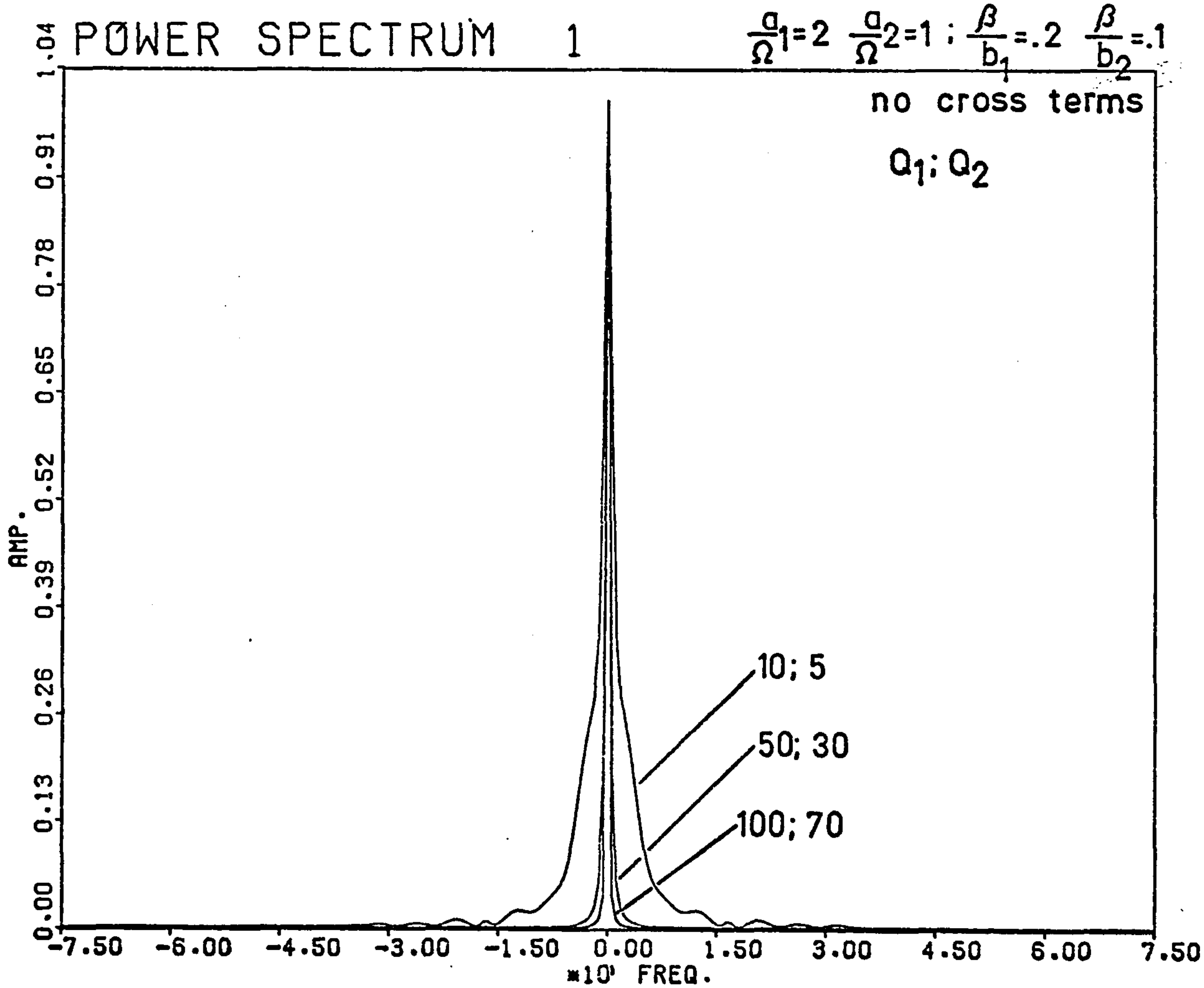


Figure V-11: Power Spectrum of Figure V-9

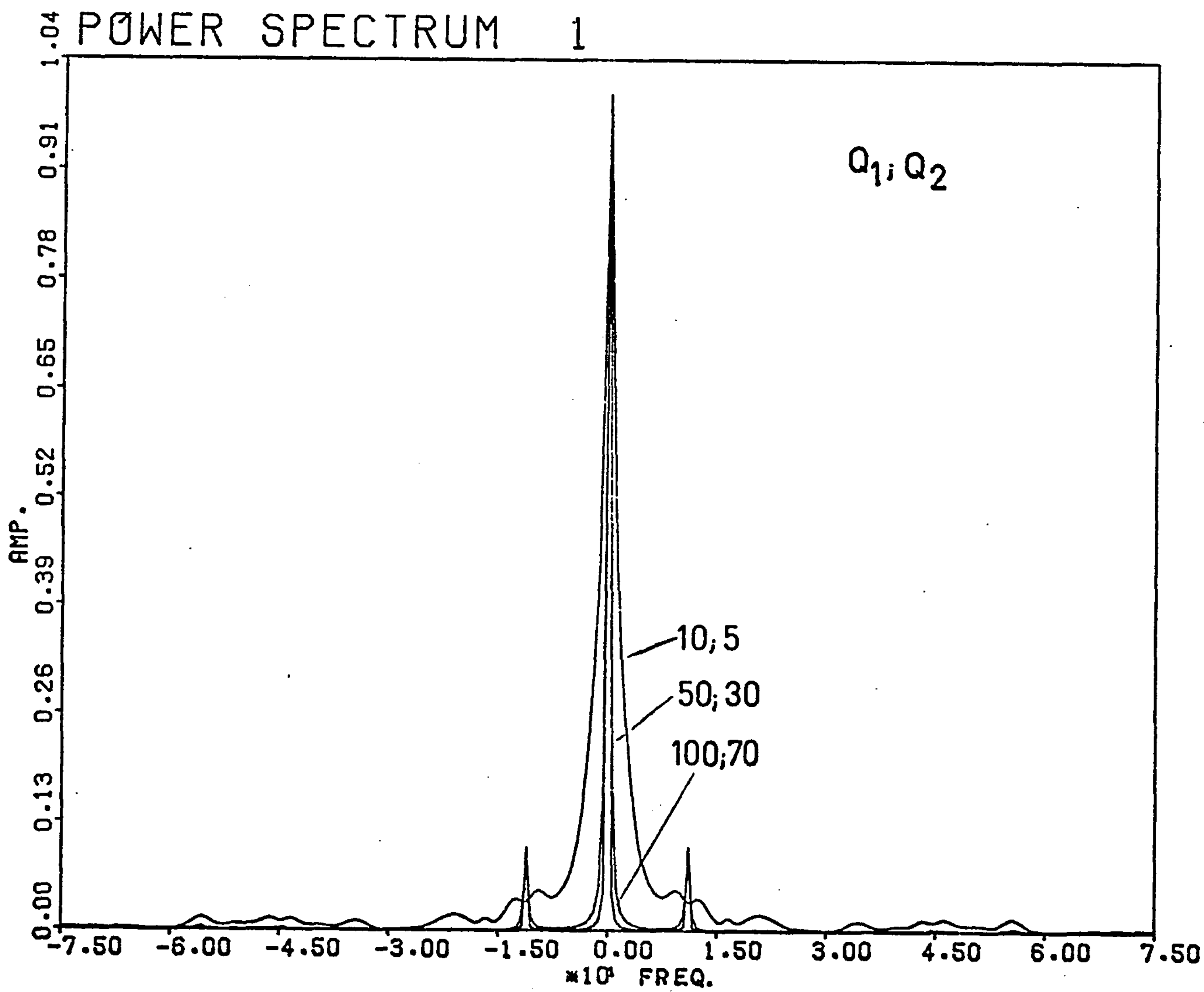


Figure V-12: Power Spectrum of Figure V-10 (with Cross Terms)

VI. ESTIMATION OF THE QUALITY FACTOR BY HIGHER-ORDER CORRELATIONS

In the previous section it was shown that for a given quality factor of the system and holding other system parameters constant a specific mean-square response was obtained. However, the quality factor is the desired parameter to investigate from the acoustic reflectivity measurements of the ocean floor in order to identify and classify ocean subbottom soil.

Therefore, keeping in tune with the theme of the thesis, that is to obtain an understanding of the behavior of the quality factor of a viscoelastic compressional wave system modeled as a lightly damped harmonic oscillator excited by random acoustic inputs in an ocean environment so the ocean subbottom soil sediments can be identified and thus classified, a procedure must be developed to acquire a behavior of the quality factor of acoustic return signals from the ocean floor. Thus, in this section the fundamental ideas behind the "Higher Order Autocorrelations" technique shall be presented.

VI.1 HIGHER ORDER AUTOCORRELATION FUNCTIONS

The first autocorrelation of the time-varying mean-square response, $E[r^2(t)] = F(t)$, is

$$R_1(\tau) = E[F(t)F(t+\tau)] \qquad \text{(VI.1-1)}$$

Taking the Fourier transformation of $r(t)$ and $r(t+\tau)$ and substituting into Equation (VI-1), remembering the Fourier transform of the delta function and $r(-\omega) = r(\omega)^*$ Equation (V.1-1) becomes

$$R_1(\tau) = \int |F(\omega)|^2 e^{-i\omega t} \frac{d\omega}{2\pi} \quad \text{(VI.1-2)}$$

The second-order autocorrelation function $R_2(\tau)$ is determined from $R_1(\tau)$ and is given by

$$R_2(\tau) = E[R_1(t)R_1(t+\tau)] \quad \text{(VI.1-3)}$$

Using Equation (VI-2), it becomes

$$R_2(\tau) = \int |F(\omega)|^4 e^{-i\omega\tau} \frac{d\omega}{2\pi} \quad \text{(VI.1-4)}$$

Following the same procedure the third-order autocorrelation function is given by

$$R_3(\tau) = \int |F(\omega)|^8 e^{-i\omega\tau} \frac{d\omega}{2\pi} \quad \text{(VI.1-5)}$$

and in general, the n th order autocorrelation function is given by

$$R_n(\tau) = \int |F(\omega)|^{2^n} e^{-i\omega\tau} \frac{d\omega}{2\pi} \quad \text{(VI.1-6)}$$

In normalized form, Equation (VI-6) can be written as

$$\frac{R_n(\tau)}{R_n(0)} = \frac{\int |F(\omega)|^{2^n} e^{-i\omega\tau} d\omega}{\int |F(\omega)|^{2^n} d\omega} . \qquad \text{(VI.1-7)}$$

In order to estimate the quality factor of a lightly damped viscoelastic compressional wave system via higher order autocorrelations the following procedures are employed.

(1) The mean-square response of the system to a specific non-stationary random input is analytically determined. The input is taken to be correlated noise modulated by a rectangular step envelope function. See Equation (V.3-20).

(2) For different values of Q and for specific values of a/Ω and β/b the normalized root-mean-square response of the system is plotted. See Figure (VI-1).

(3) The higher order autocorrelation functions are plotted from the results of procedures (1) and (2) by using the Fast Fourier transform technique.

These plots, Figures (VI-2), (VI-3), (VI-4), and (VI-5), indicates how the higher order autocorrelation functions effect the behavior of the quality factor of the system.

VI.2 SUMMARY

The first, second, third, and fourth order autocorrelation functions of a response of a lightly damped viscoelastic compressional wave system is determined. It is found that the higher order autocorrelations of the response have an approximately Gaussian envelope function. The system with low damping values of higher quality factor has a flat Gaussian envelope and the system with higher damping values of lower quality factor has a sharper Gaussian envelope function. This would be a very useful criteria to estimate the parameter Q of the system by comparing them when digitized remote data are available.

The higher order autocorrelation procedures seem to provide the necessary means to distinguish between the lightly and highly damped systems. Indeed, the first autocorrelation function does not distinguish clearly between lightly and highly damped systems. See Figure (VI-2).

The distinction between lightly and highly damped systems becomes clearly identifiable when the order of autocorrelation function is increased as shown sequentially in Figures (VI-3), (VI-4), and (VI-5).

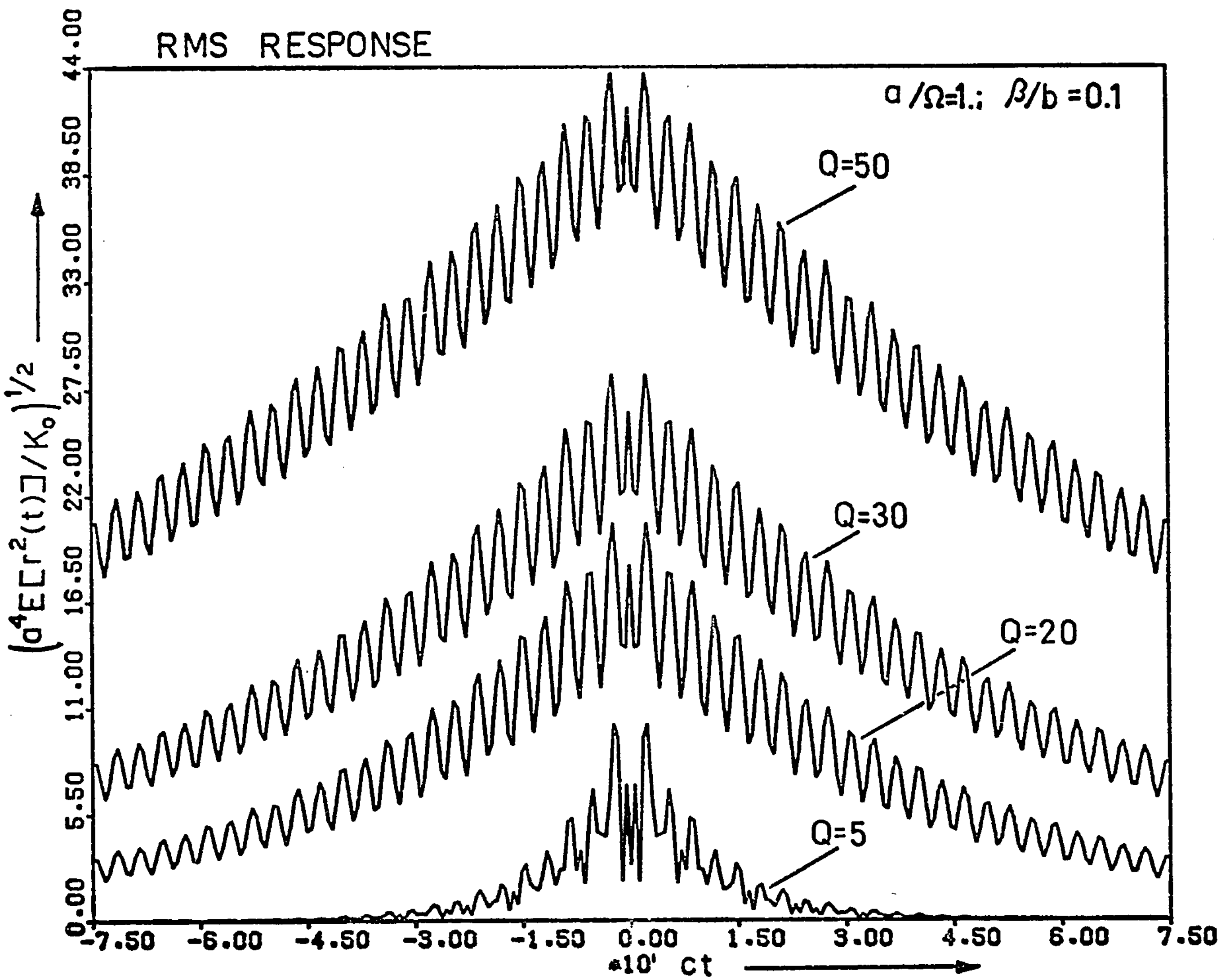


Figure VI-1: Normalized rms Response to Correlated Noise Modulated by a Rectangular Step Function

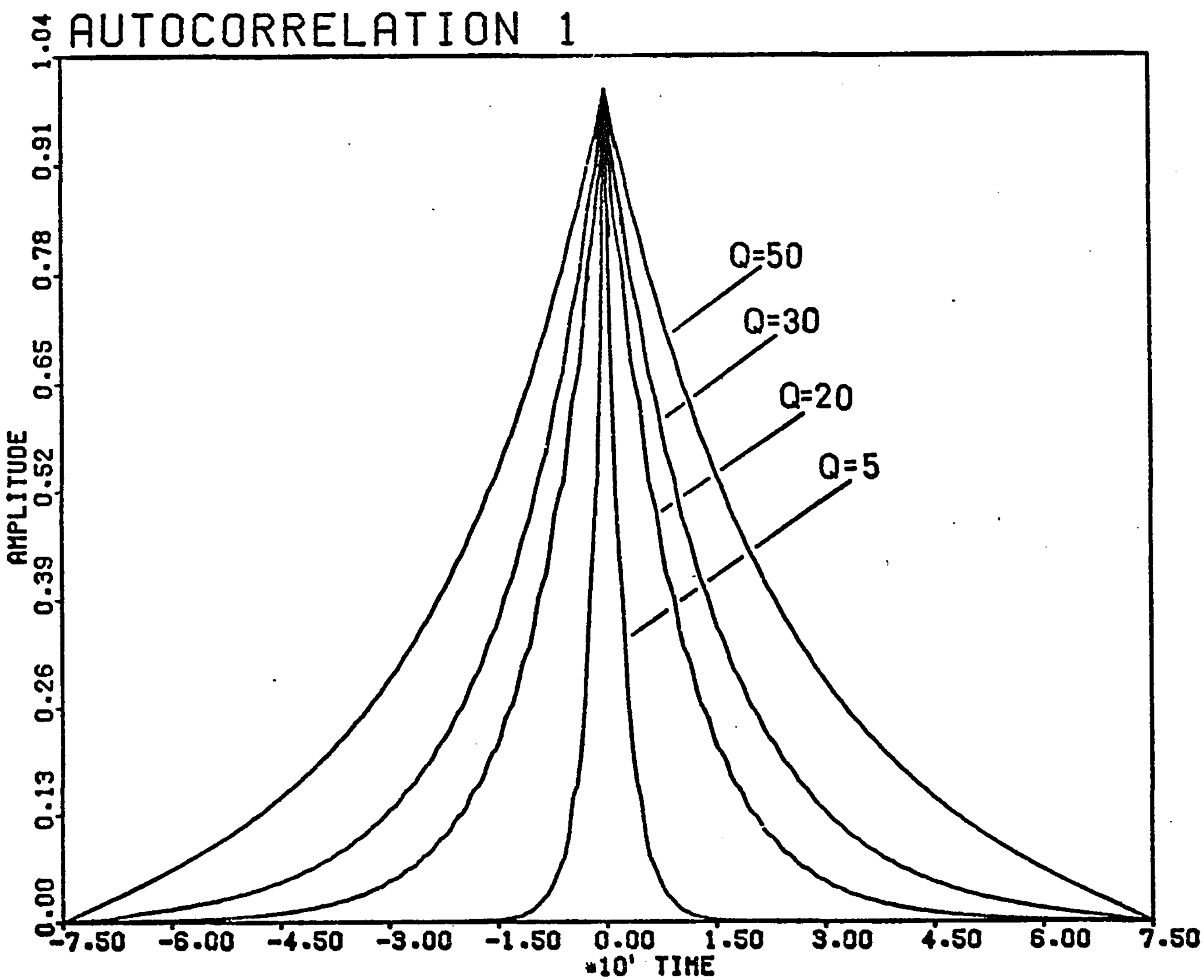


Figure VI-2: First Autocorrelation of Figure VI-1

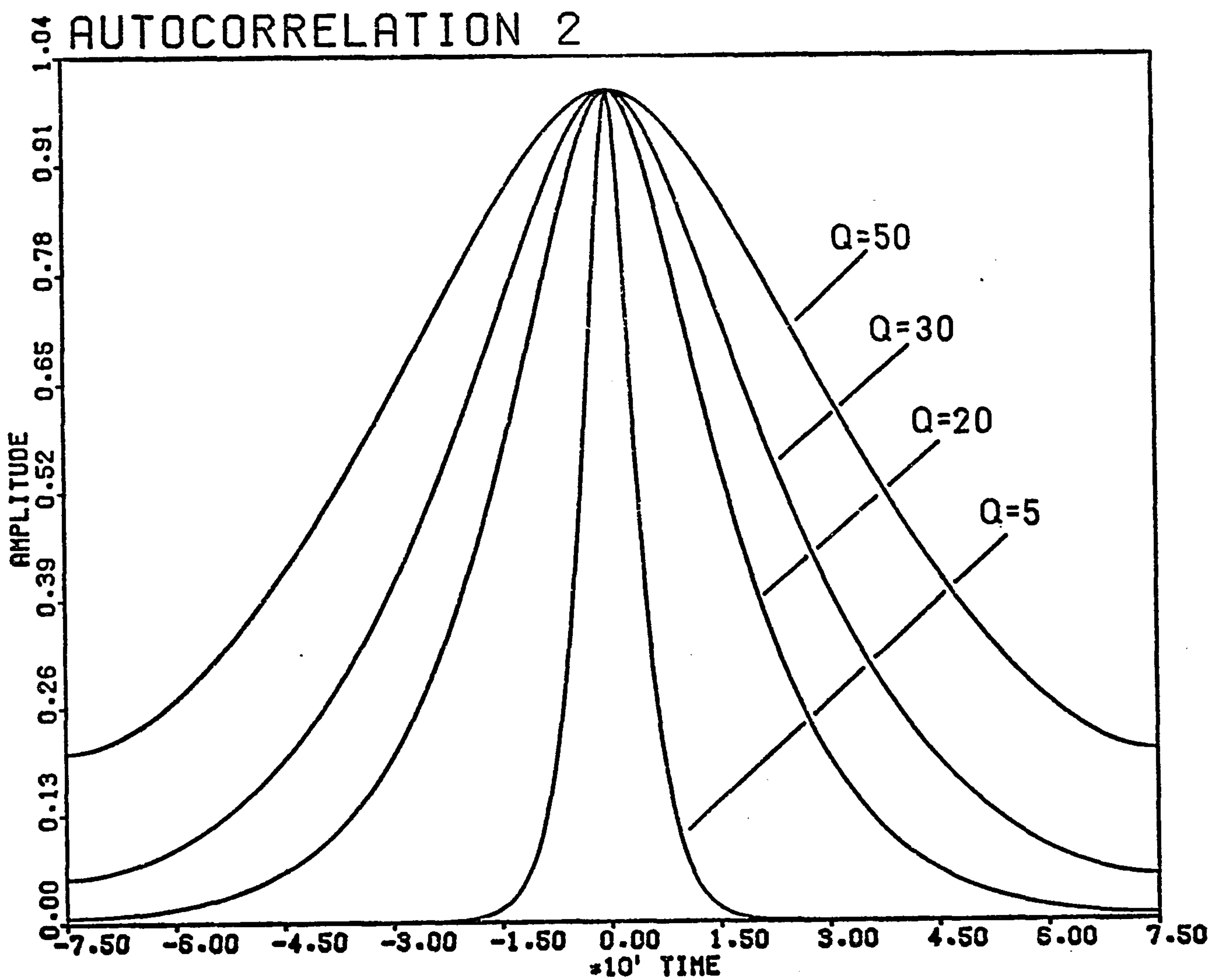


Figure VI-3: Second Autocorrelation of Figure VI-1

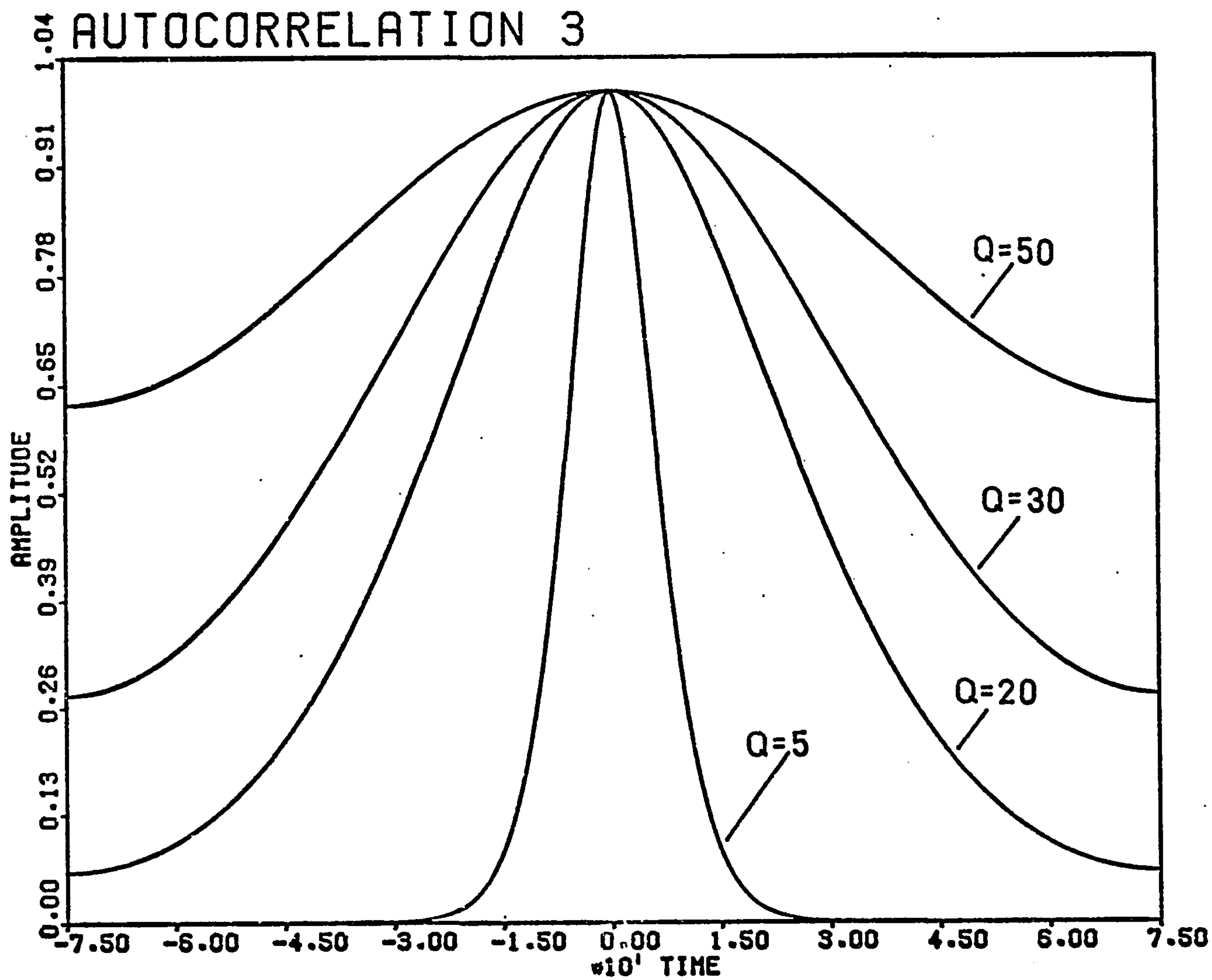


Figure VI-4: Third Autocorrelation of Figure VI-1

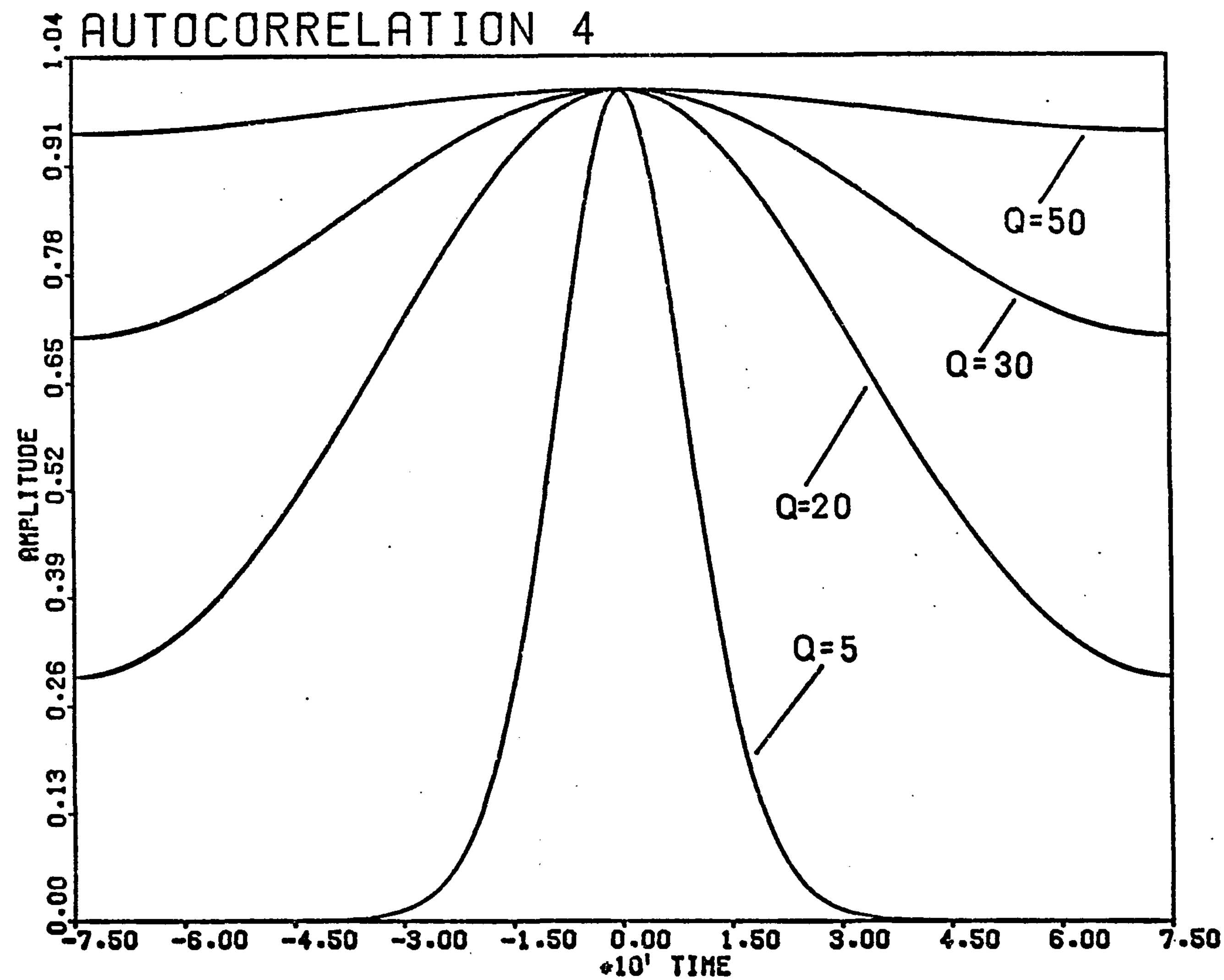


Figure VI-5: Fourth Autocorrelation of Figure VI-1

VII. SIGNAL PROCESSING

In this section the importance of higher order autocorrelation technique developed in the previous section shall be further discussed in relation to standard field theory methods. Before this is done, it is appropriate to describe the standard field theory methods.

VII.1 FILTER THEORIES

In this approach, the signal processing consists of finding the third unknown when two of the total three parts of a system are given by successive methods. See Figure (VII-1). In the present experimental

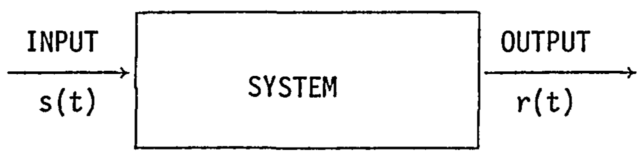


Figure (VII-1)

situation, the input s(t) is simply the acoustic signal generated by the field experimentalists and the system is the liquid and the sub-bottom layers.

Usually, one assumes a model and then describes a transfer function or Green's function - G(t) for the system, thus, being able to write the familiar response relation

$$r(t) = \int G(t,t')\, s(t')dt'.$$

For such an expression statistical data can be used as inputs to obtain again output informations. From this form the autocorrelation techniques and statistical procedures are derivable as have been presented in the previous section.

It must be remembered that the identification problem is to find the system describing the Green's function $G(t,t')$ when the input $s(t)$ and the output $r(t)$ signals are known.

The damped harmonic oscillator model is simple, easy to work with and has the proper generalities of the field theoretical model introduced earlier. Using a model-based approach with given input signals in the field measurement to determine output signals, have already been presented analytically in previous sections.

VII.2 SIGNAL PROCESSING METHOD-I (HIGHER ORDER AUTOCORRELATION TECHNIQUES)

One of the best descriptions of higher order autocorrelation techniques would be the situation where only the output signal $r(t)$ and no, or very little, information of the input and the system $G(t,t')$ is known. Although this is stretching the situation in the present field experimental conditions, knowing that the input signal information in the ocean is rather hazy and claiming no knowledge to the sub-bottom structure with this method one can start with known to discover the unknowns. When the field experimental situation improves and becomes controllable and the first generation of core analysis provides some information of the subbottom geology, then the results of higher autocorrelation techniques will be very rewarding.

Nevertheless, the technique itself requires several outputs $r_1(t)$, $r_2(t)$, $r_3(t)$, ... in the same area and constructing autocorrelation techniques of the individual signals from these outputs:

$$E[r_n(t)\ r_n(t + \tau)] = R_{nn}^{(2)}(\tau) \quad ; \quad n = 1,2,3,\ldots$$

If the first autocorrelation results are not satisfactory or clear, then it is necessary to take higher order autocorrelations

$$E[R_n^{(2)}(t)\ R_n^{(2)}(t + \tau)] = R_{nn}^{(4)}(\tau)$$

If again the results cannot be interpreted, then the same operation is repeated once more

$$E[R_n^{(4)}(t)\ R_n^{(4)}(t + \tau)] = R_{nn}^{(8)}(\tau)$$

and so on. Actual calculations and estimations are done in the frequency domain by using the Fast Fourier transform (FFT) procedures.

Finally, a spectrum which clearly looks like (Figure VII-2) where the individual peaks represent the characteristic frequency responses of the first ω_1, second ω_2, third ω_3 layers and so on.

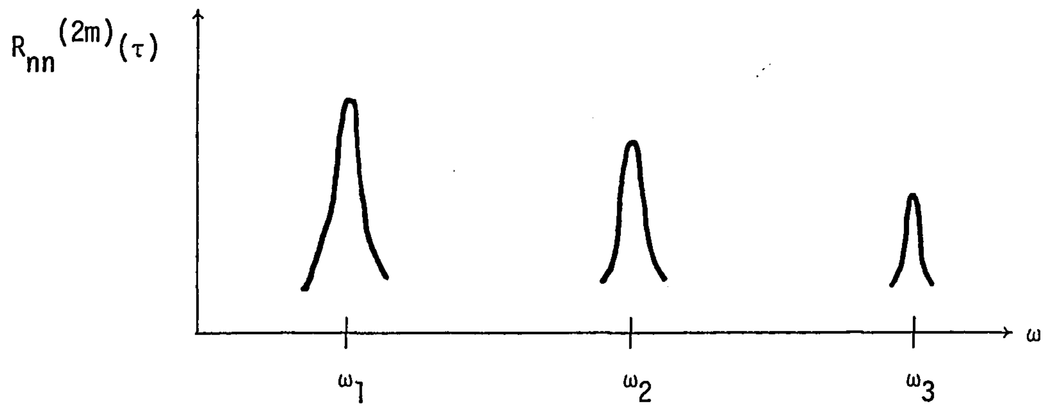


Figure - (VII-2)

Categorically, the advantages of this method are the following:

1. Only output signals are effectively used.

2. There is no need for modeling or input signals for the first round estimations.

3. Autocorrelations or cross-correlation procedures will effectively eliminate statistical fluctuations and noise background.

4. It is a method which provides the first hand information for the subbottom structure and is the starting method which dictates strategies for the signal processing Method II (with model description included) with the help of soil mechanics results.

5. If the results of the autocorrelation procedures do not give clear signals (output frequencies) due to ambient and other noise background, then cross-correlation techniques are to be used,

$$E[r_1(t)\ r_2(t + \tau)] = R_{12}^{(2)}(\tau)$$

instead of $R_{11}^{(2)}(\tau)$ estimations. Here, $r_1(t)$ and $r_2(t)$ are frequency modulated output signals in the same location.

VII.3 SIGNAL PROCESSING METHOD-II (FILTER THEORIES)

This is a well understood method provided that a knowledge of the two parts of the entire system are known:

1) Input signals;

2) A realistic theoretical model of the system. Since the experimental measurements $r_{exp}(t)$ for a reliable and a realistic theoretical model of the system must satisfy the condition

$$|r(t) - r_{exp}(t)|^2 \leq \varepsilon$$

and assuming that the input signals are controllable

$$\int dt'[G(t-t') - G_\delta(t-t')]s(t')|^2 \leq \varepsilon$$

where ε is an arbitary small number which sets the scale for percentage of accuracy desired in the model. It should be remembered that $G(t-t')$ is the response function which contains the parameters such as damping, Lamé parameters, density, and others in the subbottom soil. Thus, $G_\delta(t-t')$ is the desired and parameter corrected response function. For instance, in the harmonic oscillator model

$$G_\delta(t-t') = \eta(t-t') \exp \{-t(t-t')[\lambda'' + \delta\lambda'' + 2(\mu'' + \delta\mu'')]k^2/\rho+\delta\rho \quad \cdot$$

$$\cdot \frac{\sin \{(t-\tau)k[\frac{\lambda' + \delta\lambda' + 2(\mu'+\lambda\mu')}{(\rho+\delta\rho)}]^{1/2} [1 - \frac{k^2(\ell+\delta\ell)}{\rho+\delta\rho}]^{1/2}\}}{k[\frac{\lambda' + \delta\lambda + 2(\mu' + \delta\mu')}{\rho+\delta\rho}]^{1/2} [1 - \frac{k^2(\ell+\delta\ell)}{\rho+\delta\rho}]^{1/2}}$$

where $\ell^2 = \frac{(\lambda'' + 2\mu'')^2}{4(\lambda' + 2\mu')}$.

The small variations will provide the adjustments in the parameters. To do this it is also stated that

$$G_\delta(t-t') = G(t-t') + \frac{\partial G_\delta(t-t')}{\partial\lambda'}\bigg|_{\partial\lambda'=0} \partial\lambda' + \frac{\partial G_\delta(t-t')}{\partial\mu'}\bigg|_{\partial\mu'=0} \partial\mu'$$
$$+ \frac{\partial G_\delta(t-t')}{\partial\rho}\bigg|_{\delta\rho=0} \delta\rho + \quad \ldots$$

is the realistic Taylor series expansion and the adjustments and re-adjustments can be made accordingly with the help of soil mechanices results and the information from Method-I results.

The Method-II provides the following advantageous properties:

1. It suggests improvements in the theoretical model (parameter corrections);

2. It coordinates the soil mechanices results with the Method-I results;

3. It provides the necessary suggestions to the field experimentalists; such as location of acoustic receivers in order to get shear deformation information of the subbottom soil sediments.

VIII. APPLICATION TO FIELD EXPERIMENTAL RESULTS

The previously developed procedures, "Higher Order Autocorrelation" techniques and signal processing, are applied in the analysis of acoustic reflectivity measurements in the form of digitized data and are compared with the core measurement analysis.

VIII.1 EXPERIMENTAL DATA ANALYSIS

Here the data obtained by core measurements and by analyzing remotely obtained digitized data in time series shall be presented. In order to determine any correlations between the core measurements and digitized data three assumptions are made: (1) the ocean subbottom soil is made up of a system of horizontal layers (see Figure VIII-1); (2) each layer is a simple reflector to acoustic pulses; and (3) the cross-core reflection coefficients predict the presence of a zone rather than the axial reflection coefficients. Since digitized data is obtained by utilizing the normal incident acoustical signals, the axial velocity of the core measurements are used in the analysis.

VIII.1a CORE MEASUREMENT ANALYSIS

The core measurement analysis is done according to the information given by Woods Hole Oceanographic Institute as shown in Table (VIII-1). From the core measurements, the positions of the second and thrid zones, indicated by an order-of-magnitude change in the cross-core reflection coefficients, can easily be determined as shown in Table (VIII-2). In these measurements

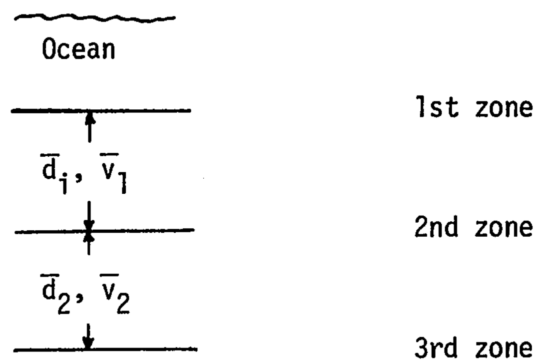


Figure (VIII-1)

i	$\overline{d}_i$(m)	$\overline{v}_i$(m/sec)	t_i(sec)	ν_i(kHz)
1	4.1 ± .2	1513 ± 9	0.00027	0.37 ± .01
2	4.2 ± .4	1502 ± 18	0.000279	0.36 ± .02

Table (VIII-2)

the soil density is taken to be a constant. In Table (VIII-2), $\overline{d}_i$ is the distance between the core zones, $\overline{v}_i$ is the core axial velocity between the zones, t_i is the time it takes for an acoustic signal to travel between a given set of zones and $\nu_i = (t_i)^{-1}$.

The core measurement analysis predicts the acoustical signature due to the second zone to be 0.37 $\pm$ .01 kHz later than the first zone signature in the frequency domain.

VIII.1b DIGITIZED DATA ANALYSIS

Figure (VIII-2) shows remotely obtained digitized data in time series. Figures (VIII-3), (VIII-4) and (VIII-5) represent the normalized first, second, and third higher order power spectrum of the digitized data in frequency domain. From the second order power spectrum, Figure (VIII-4), note that the highest amplitude peaks at 11.20 $\pm$ .08 kHz, the second highest amplitude peaks at 11.55 $\pm$ .05 kHz and the third amplitude which can correspond to the third zone peaks at 11.88 $\pm$ .07 kHz.

Therefore, the digitized data analysis predicts the first reflector to be at 11.20 $\pm$ .08 kHz, the second reflector to be at 11.55 $\pm$ .05 kHz, and the third reflector to be at 11.88 $\pm$ .07 kHz.

VIII.1c SUMMARY

From the above results it is concluded that if the digitized data indicates the first reflector to be at 11.20 $\pm$ .08 kHz, then the core measurement analysis predicts the second zone to be at 0.37 $\pm$ .01 kHz later than the first, i.e., a peak at 11.57 $\pm$ .08 kHz. Furthermore, the core measurement analysis predicts the third zone to be at 0.36 $\pm$ .02 kHz later than the second, i.e. a peak at 11.93 $\pm$ .08 kHz. Thus, when a core measurement analysis indicates a presence of a zone there is a corresponding reflector peak in digitized data.

VIII.2 CONCLUSIONS

The experimental data analysis shows that from core data zones exist and for each zone there is a corresponding acoustical reflector peak in digitized data in frequency domain.

The extension of the single damped harmonic oscillator model of a thermoviscoelastic medium in $(\vec{k};t)$ and $(\vec{k};\omega)$ domains, to a multi-degree freedom damped harmonic oscillator model which prescribes the characteristics of the viscoelastic reflectors is a valid one since the extended model shows the presence of peaks due to second layer. See Figure (V-12).

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TABLE VIII-1

DEPTH CM	VELOCITY LONG	VELOCITY TRANS	(AXIAL) REFL COEF	(CROSS-CORE) REFL COEF	
41	15Ø9.6	1472.7	.ØØ3	-.ØØ9	1st. ZONE
51	1518 8	15Ø8 2	- ØØ3	Ø12	
61	1497 4	1481 4	- ØØ7	- ØØ9	
71	15Ø6 5	1485 8	ØØ3	ØØ1	
81	1488 5	15Ø8 2	- ØØ6	ØØ7	
98	159Ø 1	1468 4	Ø33	- Ø13	
1Ø7	16ØØ 4	162Ø 7	ØØ3	Ø49	
117	1479 6	1459 8	- Ø39	- Ø52	
126	1494 4	1526 7	ØØ5	Ø22	
137	1485 5	1459 9	- ØØ3	- Ø22	
147	1491 4	1472 7	ØØ2	ØØ4	
157	1497 4	1472 7	ØØ2	ØØØ	
167	1497 4	1451 3	ØØØ	- ØØ7	
177	15ØØ 5	1472 7	ØØ1	ØØ7	
182	Ø	1468 4	-1 ØØØ	- ØØ1	
2Ø5	15Ø9 6	1464 1	1 ØØØ	- ØØ1	
215	15Ø3 5	1472 7	- ØØ2	ØØ3	
225	15Ø3 5	1464 1	ØØØ	- ØØ3	
235	15Ø3 5	1472 7	ØØØ	ØØ3	
245	15Ø9 6	149Ø 3	ØØ2	ØØ6	
255	1512 6	1464 1	ØØ1	- ØØ9	
265	15Ø6 5	1468 3	- ØØ2	ØØ1	
275	1497 4	1459 8	- ØØ3	- ØØ3	
34Ø	15ØØ 5	1481 4	ØØ1	ØØ7	
35Ø	1512 6	1464 1	ØØ4	- ØØ6	
36Ø	1515 7	1459 9	ØØ1	- ØØ1	
37Ø	1525 Ø	1459 8	ØØ3	- ØØØ	
3Ø3	1512 6	15Ø8 2	- ØØ4	Ø16	
3Ø8	15Ø3 5	1451 3	- ØØ3	- Ø19	
4Ø8	1521 9	1499 2	ØØ6	Ø16	2nd. ZONE
432	1521 9	154Ø 8	ØØØ	Ø14	
442	15Ø3 5	1455 6	- ØØ6	- Ø28	
452	15Ø9 6	1472 7	ØØ2	ØØ6	
477	1624 8	Ø	Ø37	-1 ØØØ	
487	1485 8	1477 1	- Ø45	1 ØØØ	
497	1491 6	1451 3	ØØ2	- ØØ9	
51Ø	1591 6	Ø	Ø32	-1 ØØØ	
52Ø	1471 4	1464 1	- Ø39	1 ØØØ	
53Ø	1471 4	1468 4	ØØØ	ØØ1	
54Ø	1497 5	1512 8	ØØ9	Ø15	
55Ø	148Ø Ø	1517 4	- ØØ6	ØØ2	
56Ø	1491 Ø	1438 8	ØØ4	- Ø27	

56Ø	1491.6	1438.8	.ØØ4	-.Ø27	
57Ø	1457 3	143Ø 6	- Ø12	- ØØ3	
58Ø	1491 6	1443 1	Ø12	ØØ4	
635	15Ø9 4	1373 Ø	ØØ6	- Ø25	
644	1479 2	1451 3	- Ø1Ø	Ø28	
659	1471 4	1438 8	- ØØ3	- ØØ4	
675	1494 5	1439 7	ØØ8	ØØØ	
685	1512 4	1443 Ø	ØØ6	ØØ1	
785	1488 7	1469 1	- ØØ8	ØØ9	
715	15Ø6 9	1477 1	ØØ6	ØØ3	
725	1482 9	1455 6	- ØØ8	- ØØ7	
75Ø	1494 5	1451 3	ØØ4	- ØØ1	
76Ø	1491 6	149Ø 3	- ØØ1	Ø13	
776	15ØØ 4	15Ø3 7	ØØ3	ØØ4	
791	1549 4	1564 9	Ø16	Ø2Ø	
8Ø1	1477 1	1438 8	- Ø24	- Ø42	
816	1512 4	1485 8	Ø12	Ø16	
826	15ØØ 4	1536 Ø	- ØØ4	Ø17	3rd ZONE
836	15Ø3 4	143Ø 6	ØØ1	- Ø36	
848	1497 5	1536 Ø	- ØØ2	Ø36	
858	1491 6	1598 2	- ØØ2	- ØØ9	
885	1494 5	1438 8	ØØ1	- Ø24	
895	15Ø8 9	1476 5	ØØ5	Ø13	
9Ø5	1494 7	1455 6	- ØØ5	- ØØ7	
915	1488 7	1451 3	- ØØ2	- ØØ1	
925	1488 7	1468 9	ØØØ	ØØ6	
935	15Ø6 4	1455 6	ØØ6	- ØØ5	
945	1491 6	1447 1	- ØØ5	- ØØ3	
935	1497 5	1455 6	ØØ2	ØØ3	
97Ø	1555 7	1531 3	Ø19	Ø25	
981	15ØØ 4	1517 4	- Ø18	- ØØ5	
993	1509 4	1443 Ø	ØØ3	- Ø25	
1ØØ5	1468 5	1438 8	- Ø14	- ØØ1	
1Ø34	1522 8	143Ø 6	Ø18	- ØØ3	
1Ø44	1426 6	141Ø 4	- Ø12	- ØØ7	
1Ø54	155Ø 8	1434 7	Ø24	ØØ9	
1Ø64	15Ø4 5	1438 8	- Ø18	ØØ1	
1Ø74	1543 5	143Ø 6	ØØ3	- ØØ3	

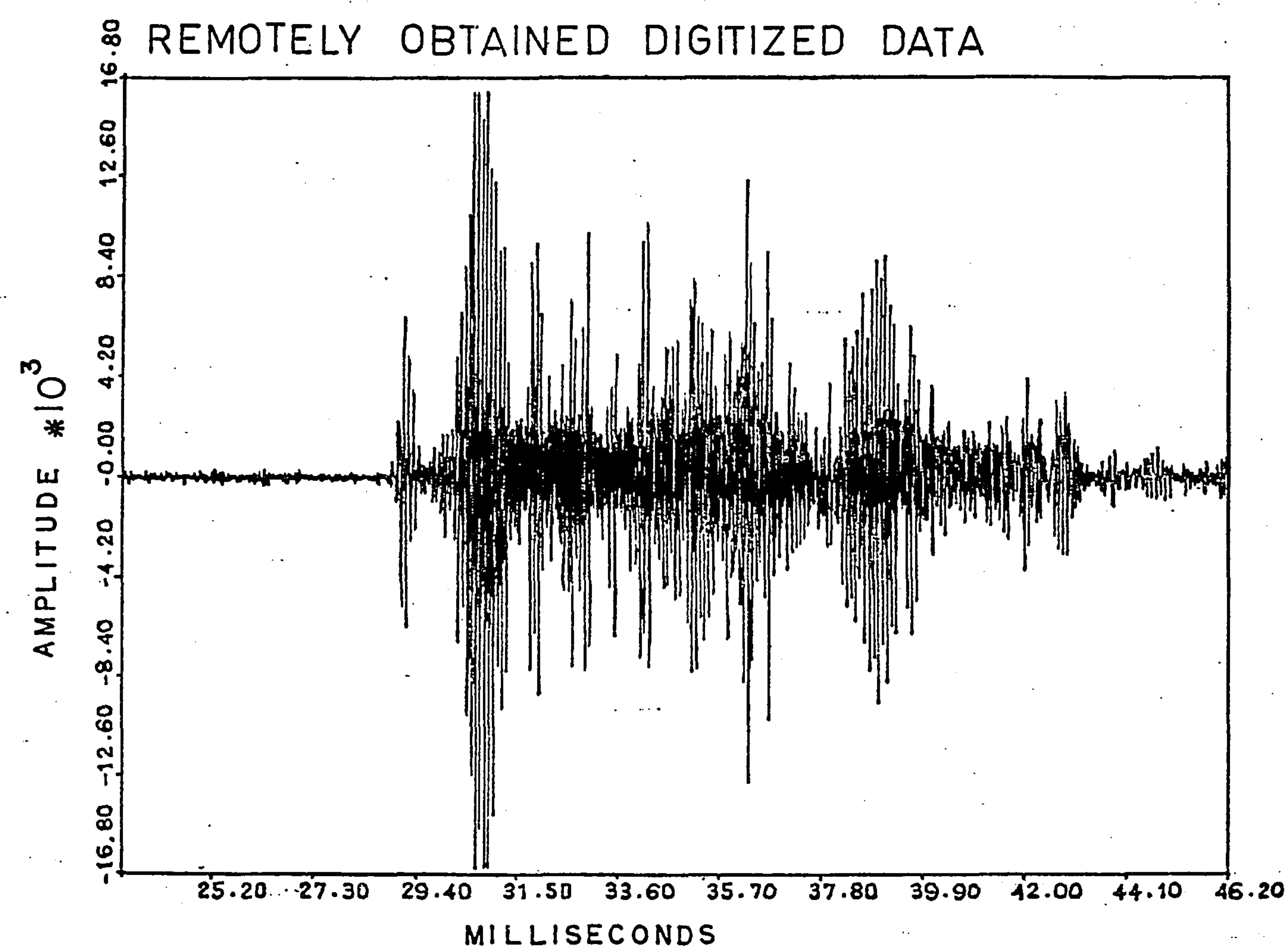

Figure VIII-2: Acoustic Reflections from Ocean Subbottom Soil - WHOI, 1976

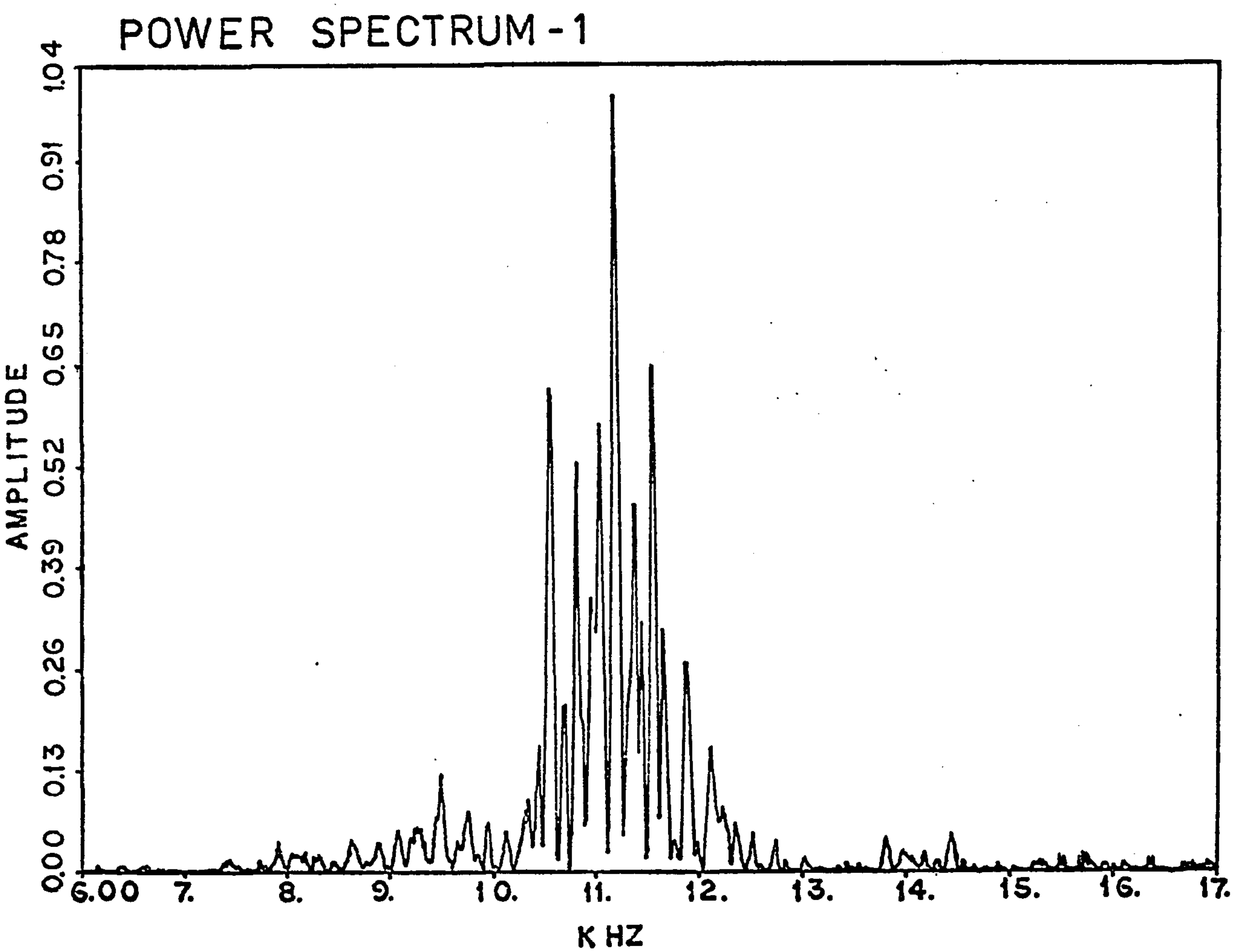

Figure VIII-3: First Order Power Spectrum of Digitized Data

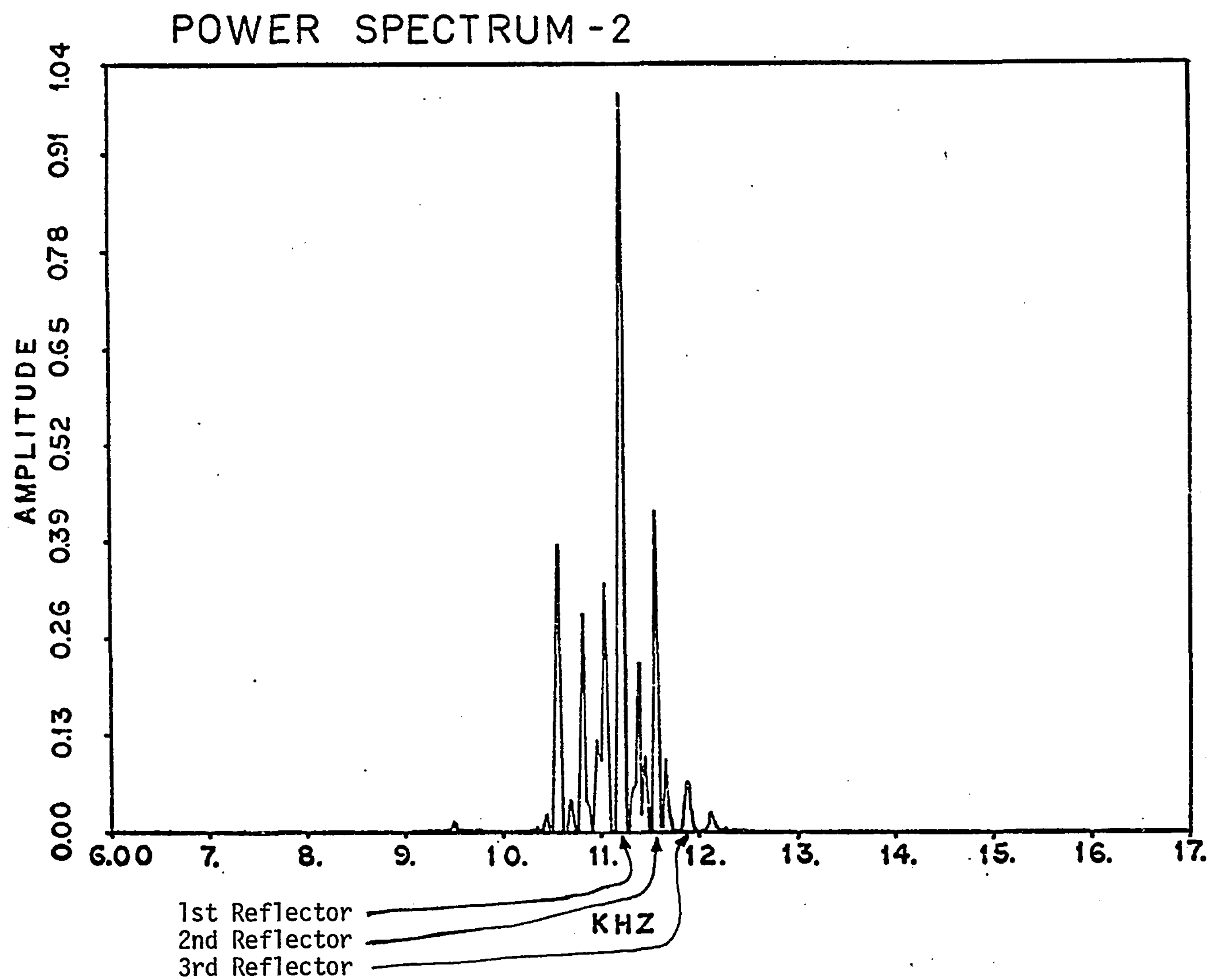

Figure VIII-4: Second Order Power Spectrum of Digitized Data

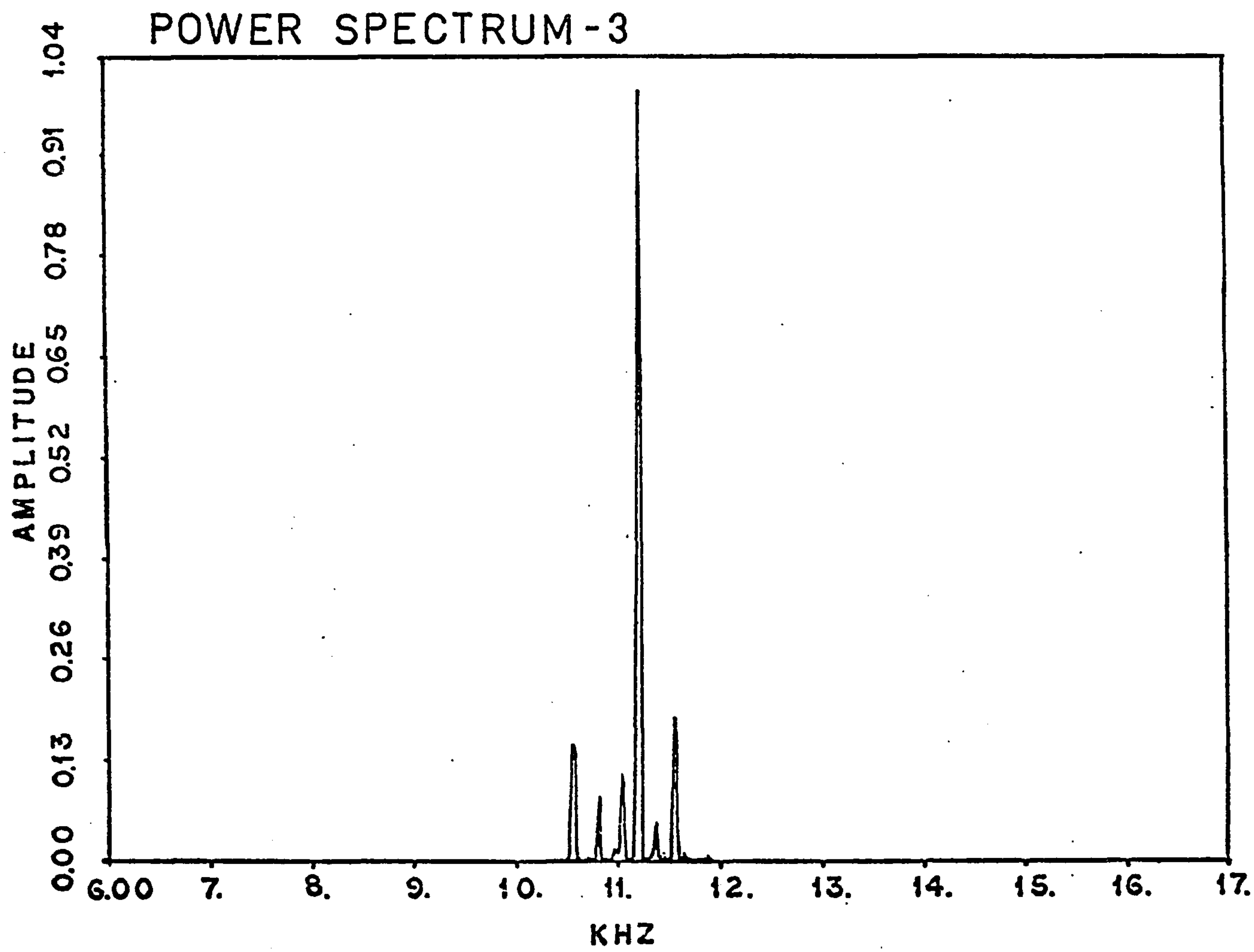

Figure VIII-5: Third Order Power Spectrum of Digitized Data

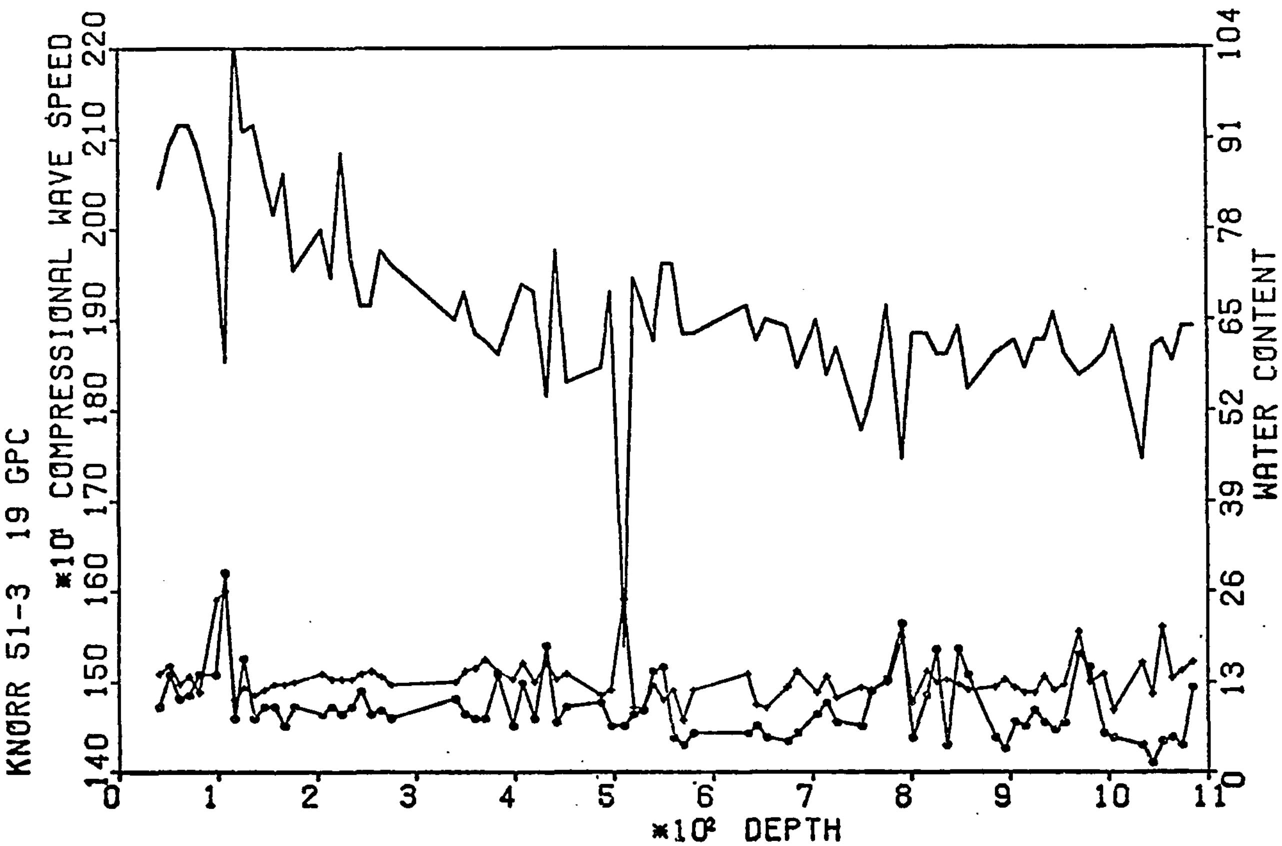

Figure VIII-6: Graph of Core Data Given by WHOI in Table VIII-1

IX. SUMMARY AND RECOMMENDATIONS

IX.1 SUMMARY

After an introduction to the thesis problem in Chapter I, a theoretical model of the ocean subbottom as a viscoelastic medium is presented in Chapter II. The fundamental idea of this chapter is that the viscoelastic medium has a damped harmonic oscillator structure in spatial coordinates introduced by the Fourier transformation. This gives simple resonant peaks of each subbottom layer modeled as a multi-degree freedom damped harmonic oscillator system. These peaks have been observed in acoustic reflections from the ocean floor. The determination of the height and width of the resonance peaks becomes paramount in the identification of subbottom soil sediments. These quantities which are related to the quality factor must be therefore determined analytically to understand the nature of acoustic reflection peaks from the ocean floor.

The understanding of the quality factor and thus the damping of a system is begun in Chapter III by presenting the viscoelastic theory from the elastic one by introducing Voigt type of damping.

In Chapter IV, the thermodynamics are introduced into the viscoelastic medium and it is shown that the viscous compressional Lamé parameter λ'' is made up of two parts - temperature independent and temperature dependent terms:

$$\lambda_{ad}'' = \lambda'' + \lambda_t''$$

where

$$\lambda_t'' = \frac{\kappa\, T\, \alpha^2 (C_L')_{ad}^{\,2}}{C_P^{\,2}} \left[1 - \frac{4}{3}\frac{(C_T')_{ad}^{\,2}}{(C_L')_{ad}^{\,2}}\right]^2 .$$

The temporal attenuation then becomes appropriately

$$b = \left(\frac{\lambda_{ad}'' + 2\mu''}{2\rho}\right) k^2$$

and hence the resonance peak given by the quality factor becomes

$$Q = \frac{c}{2b} = \frac{(\lambda' + 2\mu')^{1/2}\sqrt{\rho}}{(\lambda_{ad}'' + 2\mu'')\, k}$$

and the width of the resonance peak, $\Delta\omega$, becomes

$$\begin{aligned} \Delta\omega &= 2b \\ &= \left(\frac{\lambda_{ad}'' + 2\mu''}{\rho}\right) k^2 . \end{aligned}$$

Thus, an analytical expressions are obtained for temporal attenuation constant, the quality factor and the resonance width in terms of thermodynamical and mechanical variables.

Furthermore, the solution to the differential equations of elastic, viscoelastic and thermoviscoelastic medium is given in terms of Green's functions. This type of solution becomes very useful in obtaining the mean-square response of (continuous) viscoelastic medium to non-stationary random excitation which is discussed in Chapter V. The non-stationary random processes become important in analyzing the acoustic reflections from the ocean floor since its parts fluctuate in time. The normalized rms response of the system in terms of system parameters are shown graphically and the behavior of the system to different quality factor values are observed. These results are obtained for single layer medium and later are extended to multi-layered medium.

In Chapters VI and VII, higher order autocorrelation technique is introduced to further understand the behavior of the quality factor in a viscoelastic medium. This technique is able to distinguish between a lightly and highly damped system, and thus, increases the ability of researchers to estimate the quality factor of the system by comparing them when digitized remote data is available from the ocean floor.

Finally, in Chapter VIII the above technique is applied to acoustic reflections from ocean floor in digitized data form. A direct relation between the reflected peaks and zones predicted by core analysis is obtained.

IX.2 RECOMMENDATIONS

Obviously, the activities of soil mechanics, model building, signal processing, and field experimental activities must be coordinated to identify and classify ocean subbottom sediments. Some of these activities can be stated as follows:

1) The output data should be processed according to Method-I

at the computer center by the signal processing group. Main purpose of this effort should be to measure the width of the resonance peaks.

2) The soil mechanics instrumentation presently does not have the capability to measure accurately the value of $\Delta\omega$. An attenuameter should be built to measure the width value from the core samples. Although the output signals are frequency modulated, the signal envelope does not necessarily give the correct damping parameter information of the subbottom sediment.

3) Meanwhile, the theoretical estimations of the imaginary parts of the Lame parameters will be extremely helpful to check the width values obtained by Method-I. This is approximately possible since other relevant information from soil mechanics results can be obtained, namely

$$\Delta\omega = \Delta\omega(\lambda', \mu', \lambda'', \mu'')$$

4) After the comparison and the exchange of results obtained from the signal processing, Method-II should provide suggestions to the field experimentalists.

5) Finally, some theoretical soil mechanics efforts such as the estimation of the effective stress distribution in the subbottom, anisotropy and heterogeneity of the subbottom should prove to be very important.

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APPENDIX-1

The derivation of the attenuation constant, b, for a thermo-viscoelastic medium shall be outlined. [Yildiz, 1974]

Using the definition of attenuation constant

$$b = \frac{\left|E[\dot{\Phi}_{mech}]\right|}{2E[\Phi]} \qquad \text{(A.1-1)}$$

the terms $\dot{\Phi}_{mech}$ and Φ must be determined.

Now, $\dot{\Phi}_{mech} = -T_0 \dot{S} = -T_0 \frac{d}{dt} \int \rho s \, dV$ where s is entropy S per unit mass and T_0 is the undeformed state temperature. The rate of increase of entropy is given by [Landau, Lifshitz, 1959]

$$\frac{d}{dt} \int \rho s \, dV = \kappa \int \frac{(\nabla T)^2}{T^2} dV + \int \sigma_{ik} \, \partial_k \, v_i \frac{dV}{T} \qquad \text{(A.1-2)}$$

where κ is thermal conductivity and

$$\sigma_{ik} = \mu(\partial_k u_i + \partial_i u_k - \frac{2}{3} \delta_{ik} \partial_k u_k) + (\lambda + \frac{2}{3} \mu)\delta_{ik} \partial_k u_k \qquad \text{(A.1-3)}$$

For the viscoelastic medium the Lamé parameters become, in the first order approximation, $\mu = \mu' + \mu'' \partial_t$ and $\lambda = \lambda' + \lambda'' \partial_t$. Substituting the latter expressions into Equation (A.1-3) and later in to Equation (A.1-2) with temperature gradient given by [Landau, Lifshitz, 1970]

$$\nabla T = \frac{T_0 \alpha \rho}{C_p} [(C_L')_{ad}^2 - \frac{4}{3} (C_T')_{ad}^2] \nabla u_{ii} \tag{A.1-4}$$

the rate of entropy increase for a thermoviscoelastic medium becomes

$$\frac{d}{dt} \int \rho s \, dV = \kappa \int \left(\frac{T_0 \alpha \rho}{C_p}\right)^2 \left[(C_L')_{ad}^2 - \frac{4}{3} (C_T')_{ad}^2\right]^2 (\nabla u_{ii})^2 \frac{dV}{T}$$

$$+ \frac{1}{T} \mu' \int (\partial_k u_i \partial_k \dot{u}_i + \partial_i u_k \partial_k \dot{u}_i - \frac{2}{3} \partial_k \dot{u}_k \partial_k u_k) dV$$

$$+ \frac{1}{T} (\lambda' + \frac{2}{3} \mu') \int \partial_k u_k \partial_k \dot{u}_k \; dV$$

$$+ \frac{1}{T} \mu'' \int [(\partial_k \dot{u}_i)^2 + \partial_i \dot{u}_k \partial_k \dot{u}_i - \frac{2}{3} (\partial_k \dot{u}_k)^2] dV$$

$$+ \frac{1}{T} (\lambda'' + \frac{2}{3} \mu'') \int (\partial_k \dot{u}_k)^2 \, dV \tag{A.1-5}$$

For longitudinal sound wave one can assume

$$u_x = u_0 \cos (kx - \omega t) \quad , \quad u_y = u_z = 0$$

$$u_{xx} = \partial_x u_x = -ku_0 \sin (kx - \omega t)$$

$$\dot{u}_{xx} = \partial_x \dot{u}_x = k \, \omega \, u_0 \cos (kx - \omega t) \tag{A.1-6}$$

and also that the temperature varies slightly in the medium and differs from T_0 slightly then T can be substituted for T_0 and taken to be a constant.

Substituting expression (A.1-6) into (A.1-5) and taking its expected value one obtains

$$E[\dot{\Phi}_{mech}] = -\frac{T\kappa\alpha^2\rho^2}{C_p^2}(C_L')_{ad}^2\left\{1 - \frac{4}{3}\frac{(C_T')_{ad}^2}{(C_L')_{ad}^2}\right\}^2 k^2 u_0^2 \frac{V_0}{2}\omega^2$$

$$- (\lambda'' + 2\mu'')k^2 u_0^2 \frac{V_0}{2} \omega^2 \qquad \text{(A.1-7)}$$

and for the total energy one obtains

$$E[\Phi] = \frac{1}{2}\rho u_0^2 \omega^2 V_0 \qquad \text{(A.1-8)}$$

Finally, the attenuation constant for the thermoviscoelastic medium becomes via (A.1-1)

$$b = \frac{k^2}{2\rho}\left\{\lambda'' + 2\mu'' + \frac{\kappa T\alpha^2\rho^2(C_L')_{ad}^2}{C_p^2}\left\{1 - \frac{4(C_T')_{ad}^2}{3(C_L')_{ad}^2}\right\}^2\right\} \qquad \text{(A.1-9)}$$

The last expression in the parantheses can be re-written using the definitions

$$(C_L')_{ad}^2 = \left(\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}\right)_{ad} \quad ; \quad (C_T')_{ad} = \left(\frac{E}{2\rho(1+\sigma)}\right)_{ad}$$

as

$$\frac{\kappa\, T\, \alpha^2\, \rho\, (3K_{ad}' + 4\mu')}{27\, C_p^2}\;\frac{(1+\sigma)_{ad}^2}{(1-\sigma)_{ad}^2}$$

where $K_{ad}' = \lambda_{ad}' + \frac{2}{3}\mu'$ and E and σ are the Young's modulus and Poisson's ratio, respectively.

APPENDIX-2

The compressional waves obey the following expression given in (A.2-1). By using Kubo's formula the Green's function appropriate to the compressional propogation in an isotropic viscoelastic medium is derived.

The equation is

$$\left(\partial_t^2 - \left(\frac{\lambda'' + 2\mu''}{\rho}\right)\nabla^2\partial_t - \left(\frac{\lambda' + 2\mu'}{\rho}\right)\nabla^2\right) v_L = 0 \qquad \text{(A.2-1)}$$

Apply Laplace in time and Fourier in space transform

$$L[\ddot{F}(t)] = -\omega^2 L[F(t)] + i\omega F(0) - \dot{F}(0) \qquad \text{(A.2-2a)}$$

$$L[\dot{F}(t)] = -i\omega L[F(t)] - F(0) \qquad \text{(A.2-2b)}$$

to obtain

$$[-\omega^2 - i(\frac{\lambda'' + 2\mu''}{\rho})k^2\omega + (\frac{\lambda' + 2\mu''}{\rho})k^2]v(k;\omega) +$$

$$v_L(k;0)\,[i\omega - \frac{\lambda'' + 2\mu''}{\rho}k^2] - \frac{\partial}{\partial t}v(k;0) = 0 \qquad \text{(A.2-3)}$$

Using the continuity equation $\frac{\partial}{\partial t}\rho - \nabla\cdot\rho\,\vec{v} = 0$ the relationship between $v(k;0)$ and $\frac{\partial}{\partial t}v(k;0)$ is determined.

Equation of motion is given by

$$\rho\ \partial_t\ v_i - \partial_j\ \sigma_{ji} = 0 \qquad \text{(A.2-4)}$$

where

$$\sigma_{ji} = C'_{ijkl}\ \partial_k\ u_l + C''_{ijkl}\ \partial_t\ \partial_k\ u_l \qquad \text{(A.2-5a)}$$

$$C_{ijkl} = \lambda\ \delta_{ij}\ \delta_{kl} + \mu(\delta_{ik}\ \delta_{jl} + \delta_{il}\ \delta_{jk}) \qquad \text{(A.2-5b)}$$

Substituting the last expression into (A.2-4) and using the fact that ρ is a constant thus $\nabla\cdot\vec{v} = 0$ by the continuity equation, at $t = 0$ the relation

$$\frac{\partial}{\partial t}\ v(k;0) = -\ k^2\ \left(\frac{\lambda' + 2\mu'}{\rho}\right)\ u(k;0) - k^2\ \left(\frac{\lambda'' + 2\mu''}{\rho}\right)\ v(k;0) \qquad \text{(A.2-6)}$$

is obtained. The substitution of this expression into (A.2-3) results in

$$[-\omega^2 + c^2 - i2\omega b]v(k;\omega) + i\omega v(k;0) + c^2u(k;0) = 0$$

where the Lamé parameters are substituted by letters defined earlier. This can be re-written in Kubo's formula form

$$i\omega \frac{R(k;\omega)}{F(k;\omega)} = G(k;\omega) - G(k;0) \qquad \text{(A.2-7)}$$

as

$$\frac{v(k;\omega)}{v(k;0) + \frac{2b}{i\omega} u(k;0)} = \frac{c^2 - i2\omega b}{-\omega^2 + c^2 - i2\omega b} - 1 \qquad \text{(A.2-8)}$$

Thus, the initial valued Green's function is given by

$$G(k;\omega) = \frac{c^2 - i2\omega b}{c^2 - \omega^2 - i2\omega b} \qquad \text{(A.2-9a)}$$

$$G(k;0) = 1. \qquad \text{(A.2-9b)}$$

APPENDIX-3

The mean-square response of initial valued viscoelastic compressional wave system is determined. The initial valued Green's function is given by Equation (V.4-1) and $\Lambda(t,\omega)$ by Equation (V.4-3).

3.1 UNIT STEP ENVELOPE FUNCTION

The substitution of equation (V.3-3) into equation (V.2-14) and the evaluation of the resultant integral gives for the initial value problem

$$|\Lambda(t,\omega)|^2 = c^4 |G(k;\omega)|^2 \, M_i(t,\omega) \qquad \text{(A.3-1)}$$

where

$$\begin{aligned} M_i(t,\omega) = {} & 1 + \Gamma_1(t) + \frac{b^2-a^2+\omega^2}{a^2}\,\Gamma_2(t) - 2\Gamma_3 \cos \omega t \\ & - 2\,\frac{\omega}{a} \sin \omega t \; \Gamma_4(t) - 4\,\frac{\omega^2 \cos \omega t}{c^2}\left[2\,\frac{b^2}{c^2}\,\Gamma_3(t) - \left(1-4\,\frac{b^2}{c^2}\right)\frac{b}{a}\,\Gamma_4(t)\right] \\ & + 4\,\frac{b}{a}\,\Gamma_4(t) \cos \omega t + 4\,\frac{b^2}{c^2}\,\Gamma_4(t)\,\frac{\omega}{a} \sin \omega t \\ & + 4\,\frac{\omega^2}{c^2}\left[\frac{b^2}{c^2}\,\Gamma_1(t) + 2\,\frac{b^2}{a^2}\left(3\,\frac{b^2}{c^2} - 2\right)\Gamma_2(t) + \right. \\ & \left. + \frac{b}{a}\left(1 - 4\,\frac{b^2}{c^2}\right)\Gamma_4(t)\,\Gamma_3(t)\right] + 4\,\frac{b^2}{c^2}\,\Gamma_2(t) - 4\,\frac{b}{a}\,\Gamma_4(t)\,\Gamma_3(t) \end{aligned}$$

(A.3-1)

Hence, the mean-square response via equation (V.2-13) becomes

$$E[r^2(t)] = \int |G(k;\omega)|^2 P_\alpha(\omega)[M_i(t,\omega)]\, d\omega \qquad (A.3\text{-}2)$$

White Noise Input: If the input noise is assumed white, then the spectral density function $P_\alpha(\omega)$ becomes a constant P_0. So, the mean-square response for the initial value problem becomes

$$E[r^2(t)] = \frac{\pi P_0 c^2}{2b} \left\{1-e^{-2bt}[1+4\,\frac{b^2}{c^2} + \frac{b}{a}\,(1-4\,\frac{b^2}{c^2})\,\sin^2 at - 2(\frac{b}{a})^2(3-4\frac{b^2}{c^2})\sin^2 at]\right\}$$

$$(A.3\text{-}3)$$

Correlated Noise Input Excitation: Upon substitution of the spectral density for correlated noise into expression (A.3-2) the mean-square response for initial-valve problem becomes

$$E[r^2(t)] = c^4 K_0[F_1 L_1{}^i(t) + G_1 L_2{}^i(t) + F_3 L_3{}^i(t) + G_3 L_4{}^i(t)] \quad (A.3\text{-}4)$$

where

$$L_1{}^i(t) = \frac{a}{2b}\left\{1-e^{-2bt}[1+4\,\frac{b^2}{c^2}\,(1-2\,\frac{b^2}{c^2}) + \frac{b}{a}[1-2(5-4\,\frac{b^2}{c^2})]\,\sin^2 at\right.$$

$$\left. -\,4\,\frac{b^2}{a^2}\,[2-\,\frac{b^2}{c^2}\,(5-4\,\frac{b^2}{c^2})]\,\sin^2 at]\right\}$$

$$L_2{}^i(t) = e^{-2bt}\left\{4\,\frac{a^2b^2}{c^2} + \frac{ab}{c^4}\,(3-4\,\frac{b^2}{c^2})\,\sin^2 at + \right.$$

$$\left. + [1-2\,\frac{b^2}{c^2}\,(3-4\,\frac{b^2}{c^2})]\,\sin^2 at\right\}$$

$$L_3{}^i(t) = 1+4\,\frac{b^2}{c^2}\,\Gamma_2(t) - 4\,\frac{b}{a}\,\Gamma_4(t)\Gamma_3(t) + [1+4\,\frac{b^2}{c^2}\,\frac{\Omega^2-\beta^2}{c^2}]\Gamma_1(t)$$

$$+ [\frac{b^2-a^2+\Omega^2-\beta^2}{a^2} + 8\,\frac{b^2}{c^2}\,\frac{\Omega^2-\beta^2}{a^2}\,(7\,\frac{b^2}{c^2} - 2)]\Gamma_2(t)$$

$$+ 4\,\frac{b}{a}\,\frac{\Omega^2-\beta^2}{c^2}\,(1 - 4\,\frac{b^2}{c^2})\,\Gamma_4(t)\Gamma_3(t)$$

$$+ 2\left\{[2\,\frac{b}{a}(1 - \frac{\Omega^2-\beta^2}{c^2}\,(1-4\,\frac{b^2}{c^2})) - \frac{\beta}{a}\,(1-2\,\frac{b^2}{c^2})]\Gamma_4(t)\right.$$

$$\left. - (1 + 4\,\frac{b^2}{c^2}\,\frac{\Omega^2-\beta^2}{c^2})\Gamma_3(t)\right\} e^{-\beta t}\,\cos\Omega t$$

$$+ 2\left\{[4\,\frac{\Omega\beta}{c^2}\,\frac{b}{a}\,(1 - 4\,\frac{b^2}{c^2}) - \frac{\Omega}{a}\,(1 - 2\,\frac{b^2}{c^2})]\Gamma_4(t)\right.$$

$$\left. + 8\,\frac{\Omega\beta}{c^2}\,\frac{b^2}{c^2}\,\Gamma_3(t)\right\} e^{-\beta t}\,\sin\Omega t$$

$$L_4{}^i(t) = -2\,\frac{\Omega\beta}{a^2}\,\Gamma_2(t)-8\,\frac{\Omega\beta}{c^2}\,[\frac{b^2}{c^2}\,\Gamma_1(t) + 2\,\frac{b^2}{c^2}\,(3\,\frac{b^2}{c^2} - 2)\Gamma_2(t)$$

$$+ \frac{b}{a}\,(1-4\,\frac{b^2}{c^2})\Gamma_4(t)\Gamma_3(t)] + \left\{2(1+4\,\frac{b^2}{c^2}\,\frac{\Omega^2-\beta^2}{c^2})\Gamma_3(t)\right.$$

$$- 2[2 \frac{b}{a} [1- \frac{\Omega^2-\beta^2}{c^2} (1-4 \frac{b^2}{c^2})] - \frac{\beta}{a} (1 - 2 \frac{b^2}{c^2})]\Gamma_4(t)\Big\} \text{ x}$$

$$\text{x } e^{-\beta t} \sin \Omega t + 2 \Big\{8 \frac{\Omega\beta}{c^2} \frac{b^2}{c^2} \Gamma_3(t) -$$

$$- 2[\frac{\Omega}{a} (1-2 \frac{b^2}{c^2}) - 4 \frac{b}{a} \frac{\Omega\beta}{c^2} (1-4 \frac{b^2}{c^2})]\Gamma_4(t)\Big\} \; e^{-\beta t} \cos \Omega t$$

(A.3-5)

3.2 RECTANGULAR STEP ENVELOPE FUNCTION

Substituting the Equation (V.3-14) into equation (V.2-14), we obtain for the initial value problem

$$|\Lambda(t,\omega)|^2 = c^4|G_0(k;\omega)|^2 \Big\{M_i(t,\omega)\eta(t) + [-M_i(t,\omega) + M_r{}^i(t,\omega)]\eta(t-t')\Big\}$$

(A.3-6)

where

$$M_r{}^i(t,\omega) = M_r(t,\omega) + \frac{4}{c^4} \Big\{2b^2\omega^2 + \frac{b^2}{a^2} [c^2(3b^2-a^2) + \omega^2(a^2-b^2)] \text{ x}$$

$$\text{x } [\Gamma_2(t) + \Gamma_2(t-t')] + \frac{b}{a} [\omega^2(a^2-b^2)-2b^2c^2][\Gamma_4(t)\Gamma_3(t)$$

$$+ \Gamma_4(t-t')\Gamma_3(t-t')] + [\frac{b}{a} [c^4 + \omega^2(3b^2-a^2)][\Gamma_4(t)\Gamma_3(t-t')$$

$$+ \Gamma_3(t)\Gamma_4(t-t')] - 2\,\frac{b^2}{a^2}\,[c^4 - 2(a^2-b^2)\omega^2]\Gamma_4(t)\Gamma_4(t-t')$$

$$- 2b^2\omega^2\,\Gamma_3(t)\,\Gamma_3(t-t')]\cos\omega t' \Big\} \qquad \text{(A.3-7)}$$

Hence, from Equation (V.2-13) the mean-square response becomes

$$E[r^2(t)] = \int_{-\infty}^{\infty} |G_0(k;\omega)|^2\, P_\alpha(\omega)\, \{M_i(t,\omega)\}\, d\omega \quad \text{for} \quad 0 \leq t \leq t'$$

$$E[r^2(t)] = \int_{-\infty}^{\infty} |G_0(k;\omega)|^2\, P_\alpha(\omega)\, \{M_r{}^i(t,\omega)\}\, d\omega \quad \text{for} \quad t \geq t'$$

(A.3-8)

<u>White Noise Input</u>: If the input excitation is assumed white, then

$$E[r^2(t)] = P_0 \int_{-\infty}^{\infty} |G_0(k;\omega)|^2\, \{M_i(t,\omega) \text{ or } M(t,\omega)\}\, d\omega \quad \text{for} \quad 0 \leq t \leq t'$$

$$E[r^2(t)] = P_0 \int_{-\infty}^{\infty} |G_0(k;\omega)|^2\, \{M_r{}^i(t,\omega) \text{ or } M_r(t,\omega)\}\, d\omega \quad \text{for} \quad t \geq t'$$

(A.3-9)

The first integrals are given by Equation (A.3-3) and the second integral for the inital value problem is given by

$$E[r^2(t)] = \frac{\pi P_0 c^2}{2b} \Big\{ 8\,\frac{b^2}{c^2} + \Gamma_1(t) + \Gamma_2(t-t')$$

$$+ 2\frac{b^2}{a^2}\left[\frac{a^2+5b^2}{c^2}\ \Gamma_2(t) + \frac{3b^2-a^2}{c^2}\ \Gamma_2(t-t')\right] +$$

$$+ 2\left[(8\frac{b}{a}\frac{b^2}{c^2}\Gamma_4(t) - (1 + 4\frac{b^2}{c^2})\Gamma_3(t))\Gamma_3(t')\right.$$

$$+ \left(\frac{b}{a}\frac{b^2}{c^2}\Gamma_3(t) - \left(\frac{a^2+2b^2}{a^2}\right)\Gamma_4(t)\right)\Gamma_4(t')]\Gamma_3(t-t')$$

$$+ 2\left[\left(\left(\frac{4b^2(a^2-3b^2)}{c^2a^2} - \frac{c^2}{a^2}\right)\Gamma_4(t) + \frac{8b}{a}\frac{b^2}{a^2}\Gamma_3(t)\right)\Gamma_3(t')\right.$$

$$+ \left(\left(\frac{4b^2(a^2-3b^2)}{c^2a^2} + \frac{c^2}{a^2}\right)\Gamma_3(t) + \frac{2b}{a}\left(\frac{b^3(3b+2a)+a^3(a-2b)}{c^2a^2}\right)\Gamma_4(t)\right)\Gamma_4(t')]$$

$$\times\ \Gamma_4(t-t') + \frac{4b}{a}\ \frac{(a^2-3b^2)}{c^2}[\Gamma_4(t)\Gamma_3(t) + \Gamma_4(t-t')\Gamma_3(t-t')]\Big\}\quad \text{for } t \geq t'$$

(A.3-10)

<u>Correlated Input Excitation</u>: For correlated input excitation, the mean-square response is:

$$E[r^2(t)] = c^4K_0[F_1L_1{}^i(t) - G_1L_2{}^i(t) + F_3L_3{}^i(t) + G_3L_4{}^i(t)] \quad \text{for } 0 \leq t \leq t'$$

$$E[r^2(t)] = c^4K_0[F_1L_{11}{}^i(t) - G_1L_{22}{}^i(t) + F_3L_{33}{}^i(t) - G_3L_{44}{}^i(t)] \quad \text{for } t \geq t'$$

(A.3-11)

where the first expression is given by Equation (A.3-4) and the second expression is given by

$$L_{11}{}^{i}(t) = \frac{a}{2b}\Big\{ 8\frac{b^2}{c^2}\frac{(a^2-b^2)}{c^2} + \Gamma_1(t) + \Gamma_1(t-t') + 4\frac{b^2}{c^2}\frac{(a^2-3b^2)}{c^2}[\Gamma_2(t)+\Gamma_2(t-t')]$$

$$+ 8\frac{ab}{c^2}\frac{(a^2-3b^2)}{c^2}[\Gamma_3(t)\Gamma_4(t) + \Gamma_4(t-t')\Gamma_3(t-t')]$$

$$+ 2\Gamma_3(t')[[\left(\frac{b}{a} + \frac{4b^2(3a^2-b^2)}{c^4}\right)\Gamma_3(t) - \left(\frac{a^4(a^2-3b^2)+b^4(23a^2-5b^2)}{a^2c^2}\right)\Gamma_4(t)]\Gamma_4(t-t')$$

$$-[\frac{b}{a}\left(1 - \frac{4b^2(2a^2-b^2)}{c^4}\right)\Gamma_4(t) + \left(1 + \frac{4b^2(a^2-b^2)}{a^2c^2}\right)\Gamma_3(t)]\Gamma_3(t-t')]$$

$$+2\Gamma_4(t')[[\left(\frac{b^4(5b^2-23a^2)+a^4(a^2+2b^2)}{c^4a^2}\right)\Gamma_4(t)+ \frac{b}{a}\left(1 + \frac{4b^2(3a^2-b^2)}{c^4}\right)\Gamma_3(t)]\Gamma_3(t-t')$$

$$+\Gamma_4(t-t')[\frac{b}{a}\left(\frac{a^4(b^2+3a^2)+5b^4(5a^2-b^2)}{c^4a^2}\right)\Gamma_4(t)+\left(\frac{a^4(a^2-4b^2)+b^4(3b^2-17a^2)}{c^4a^2}\right)\Gamma_3(t)]\Big\}$$

$$L_{22}{}^{i}(t) = \frac{a}{b}\Big\{8\frac{ab^3}{c^4} + \frac{b}{a}\left(1 + \frac{4b^2(a^2-b^2)}{c^4}\right)(\Gamma_2(t) + \Gamma_2(t-t'))$$

$$+ \frac{4b^2(a^2-b^2)}{c^4}(\Gamma_3(t)\Gamma_4(t) + \Gamma_3(t-t')\Gamma_4(t-t'))$$

$$+ \Gamma_3(t')[[(1 + \frac{8b^2(3b^2-a^2)}{c^4})\Gamma_4(t) - 16\frac{ab^3}{c^4}\Gamma_3(t)]\Gamma_3(t-t')$$

$$-\left[\left(1-\frac{8b^2(3b^2+a^2)}{c^4}\right)\Gamma_3(t)+\frac{2b}{a}\left(1-\frac{16b^2(a^2-b^2)}{c^4}\right)\Gamma_4(t)\right]\Gamma_4(t-t')]$$

$$+\Gamma_4(t')[[\frac{11b^2-a^2}{a^2}\Gamma_4(t)+\frac{2b}{a}\left(1+\frac{16b^2(a^2-b^2)}{c^4}\right)\Gamma_3(t)]\Gamma_4(t-t')$$

$$-[\frac{2b}{a}\left(1-\frac{16b^2(a^2-b^2)}{c^4}\right)\Gamma_4(t)+\left(1+\frac{8b^3(a^2-3b^2)}{ac^4}\right)\Gamma_3(t)]\Gamma_3(t-t')]\Big\}$$

$$L_{33}{}^{i}(t)=\frac{8b^2(\Omega^2-\beta^2)}{c^4}+\Gamma_1(t)+\Gamma_1(t-t')+\left(\frac{\Omega^2-\beta^2}{a^2}-\frac{(c^2+3b^2)}{c^2}+\frac{9b^4}{a^2c^2}\right)\times$$

$$\times[\Gamma_2(t)+\Gamma_2(t-t')]+\frac{4b}{a}\frac{(a^2-b^2)(\Omega^2-\beta^2)}{c^4}-\frac{b^2}{c^2}[\Gamma_4(t)\Gamma_3(t)+\Gamma_4(t-t')\Gamma_3(t-t')]$$

$$+2\,e^{-\beta t'}\cos\Omega t'\Big\{[\left(\frac{2b-\beta}{a}+\frac{4b^2(\Omega^2-\beta^2)}{c^4}\right)\Gamma_4(t)$$

$$-\left(1+\frac{4b^2(\Omega^2-\beta^2)}{c^4}\right)\Gamma_3(t)]\Gamma_3(t-t')$$

$$+[\left(\frac{2b+\beta}{a}+\frac{4b^3(\Omega^2-\beta^2)}{ac^4}\right)\Gamma_3(t)$$

$$+\left(\frac{4b^2-(\Omega^2-\beta^2)}{a^2}+\frac{4b^3(\Omega^2-\beta^2)}{ac^4}\right)\Gamma_4(t)]\Gamma_4(t-t')\Big\}$$

$$-2\,e^{-\beta t'}\sin\Omega t'\Big\{[\left(\frac{8\Omega\beta b^2}{c^4}-\frac{\Omega}{a}\right)\Gamma_4(t)-\frac{8\Omega\beta b^2}{c^4}\Gamma_3(t)]\Gamma_3(t-t')$$

$$-[\frac{4\Omega\beta}{a^2}\left(1 - \frac{2b^2(a^2-b^2)}{c^4}\right)\Gamma_4(t) - \left(\frac{\Omega b}{a^2} + \frac{8\Omega\beta b^2}{c^4}\right)\Gamma_3(t)]\Gamma_4(t-t')\ \Big\}$$

$$L_{44}{}^{i}(t) = \frac{8\Omega\beta}{c^4}\left\{4b^2 + \frac{b}{a}(a^2-b^2)[\Gamma_4(t-t')\Gamma_3(t-t') + \Gamma_4(t)\Gamma_3(t)]\right\}$$

$$+ \frac{2\Omega\beta}{a^2}\left(1 + \frac{12b^2(a^2-b^2)}{c^4}\right)[\Gamma_2(t) + \Gamma_2(t-t')]$$

$$+ e^{-\beta t'}\sin\Omega t'\Big\{ [\left(\frac{\beta}{a} + \frac{4b}{a}\left(1 + \frac{(b^2-a^2)(\Omega^2-\beta^2)}{c^4}\right)\right)\Gamma_3(t)$$

$$- \left(\frac{\Omega^2-\beta^2}{a^2} + \frac{8b^2}{a^2}\left(1 + \frac{(b^2-a^2)(\Omega^2-\beta^2)}{c^4}\right)\right)\Gamma_4(t)]\Gamma_4(t-t')$$

$$+ [\left(\frac{4b}{a}\left(1 + \frac{(b^2-a^2)(\Omega^2-\beta^2)}{c^4}\right) - \frac{\beta}{a}\right)\Gamma_4(t) - \Gamma_3(t)]\Gamma_3(t-t')\ \Big\}$$

$$+ e^{-\beta t'}\cos\Omega t'\ \Big\{[2\left(\frac{16\Omega\beta b^2(a^2-b^2)}{a^2c^4} - \frac{\Omega}{a}\right)\Gamma_4(t)$$

$$+ \left(\frac{8\Omega\beta b(3b^2-a^2)}{ac^4} - \frac{\Omega}{\beta}\right)\Gamma_3(t)]\Gamma_4(t-t')$$

$$+ [\left(\frac{8\Omega\beta(3b^2-a^2)}{ac^4} + \frac{\Omega}{\beta}\right)\Gamma_4(t) - \frac{16\Omega\beta b^2}{c^4}\Gamma_3(t)]\Gamma_3(t-t')\ \Big\}$$

(A.3-12)

APPENDIX-4

The mean-square response of two viscoelastic compressional wave mediums to the correlated noise modulated by a rectangular step envelope function is:

$$E[r^2(t)]=K_{o}\sum_{i=1}^{2}\{H_7^i(t)(F_1^i-F_3^i) + \frac{\Gamma_2^i(t)+\Gamma_2^i(t-t')}{h_i^2(7)}(F_1^i h_i(1)-G_1^i h_i(2)+F_3^i h_3(1)-G_3^i h_2(2))$$

$$-2\Gamma_3^i(t)\Gamma_3^i(t-t')(F_1^i S_i(t') - G_1^i T_i(t) + F_3^i S_3(t') - G^i{}_3 T_3(t'))$$

$$-\frac{2\Gamma_4^i(t)\Gamma_4^i(t-t')}{h_i^2(7)}[(F_1^i h_i(1) - G_1^i h_i(2))S_i(t') - (F_1^i h_i(2) + G_1^i h_i(1))T_i(t')$$

$$+ (F_3^i h_3(1) - G_3^i h_3(2))S_3(t') - (F_3^i h_3(2) + G_3^i h_3(1))T_3(t')]$$

$$+ 2H_3^i(t)[(F_1^i h_i(6) + G_1^i h^i(7))S_i(t') + (F_1^i h_i(7) - G_1^i h_i(6))T_i(t')$$

$$+ (F_3^i h_3(6) + G_3^i h_3(7))S_3(t') + (F_3^i h_3(7) - G_3^i h_3(6))T_3(t')]\}$$

$$+ 2K_{o}\sum_{i=1}^{3}[U_i L_i^c(t) + V_i M_i^c(t)] \qquad \text{(A.4-1)}$$

where

$$L_i^c(t) = NH_1(t) + h_i(1)[N_1H_1(t) - 2N_2H_4(t) + H_2(t)(N + h_i^2(1) - 3h_i^2(2))]$$

$$+ (h_i^2(1) - h_i^2(2))[H_1(t) + N_1H_2(t) + 2N_3H_9(t)]$$

$$+ \{h_i(1)[2N_2H_{10}(t) - N_1H_5^+(t) - H_8(t)(h_i^2(1) + 2N_1H_4(t)$$

$$- 2N_2H_6(t) - N_3H_5^-(t)) + 2h_i^2(6)h_i(4)]$$

$$- (h_i^2(1) - h_i^2(2))[H_5^+(t) + N_1H_8(t) + 2N_3H_{10}(t)] - NH_5^+(t)\} S_i(t')$$

$$+ h_i(2)[2(N_3h_i(1) - N_2)H_{10}(t) + (N_1 + 2h_i(1))H_5^+(t)$$

$$+(3h_i^2(1) - h_i^2(2) - N + 2N_1h_i(1))H_8(t) + h_i(5)(2NH_4(t) + N_2H_5^-(t))$$

$$+\{h_i(6)[h_i(4)[2H_4(t)(N_1 + h_i(1)) + 2H_6(t)(N_3h_i(1) - N_2) - N_3H_5^-(t)]$$

$$- 4h_i^2(5)h_i(3)(H_4(t) + N_3H_6(t))]\} T_i(t') ; \qquad \text{(A.4-2a)}$$

$$M_i^c(t) = -h_i(2)[2H_9(t)(N_2 + N_3h_i(1)) + (N_1 + 2h_i(1))H_1(t)$$

$$+ (N + N_1H_1(t) - 2h_i(2) + 3h_i(1))H_2(t)]$$

$$+ \{h_i(2)[4(N_2 + h_i(1))H_{10}(t) + 2(h_i(1) - N_2)H_5^+(t)$$

$$+ (2N_1h_i(1) - 2N + 3h_i^2(1) - h_i^2(2))H_8(t)]$$

$$+ 2h_i(6)[h_i(4)[(N_1 + h_i(1))H_4(t) + (h_i(1)N_3 - N_2)H_6(t) - N_3H_5^-(t)]$$

$$- 2h_i^2(5)h_i(3)(H_4(t) + N_3H_6(t))]\}S_i(t')$$

$$+ \{(h_i^2(1) - h_i^2(2))(H_5^+(t) + N_1H_8(t) + 2N_3H_{10}(t)) - h_i(1)[2N_2H_{10}(t)$$

$$- N_1H_5^+(t) + (h_i^2(1) - 3h_i^2(2) - N)H_8(t)] + h_i(5)[2NH_4(t) +$$

$$+ N_2H_5^-(t) - h_i(3)[2H_4(t)(N_1 + h_i(1)) + 2(N_3h_i(1) - N_3H_5^-(t))]]\} T_i(t') \qquad \text{(A.4-2b)}$$

where

$$H_1(t) = \Gamma_3^{(1)}(t-t')\Gamma_3^{(2)}(t-t') + \Gamma_3^{(1)}(t)\Gamma_3^{(2)}(t);$$

$$H_2(t) = [\Gamma_4^{(1)}(t-t')\Gamma_4^{(2)}(t-t') + \Gamma_4^{(1)}(t)\Gamma_4^{(2)}(t)]/A_1A_2$$

$$H_3^i(t) = [\Gamma_3^{(i)}(t)\Gamma_4^{(i)}(t-t') - \Gamma_3^{(i)}(t-t')\Gamma_4^{(i)}(t)]/h_i(6);$$

$$H_4(t) = [\Gamma_4^{(2)}(t-t')\Gamma_3^{(1)}(t) - \Gamma_4^{(2)}(t)\Gamma_3^{(1)}(t-t')]/A_2$$

$$+ [\Gamma_4^{(1)}(t-t')\Gamma_3^{(2)}(t) - \Gamma_4^{(1)}(t)\Gamma_3^{(1)}(t-t')]/A_1;$$

$$H_5^{\pm}(t) = \Gamma_3^{(2)}(t)\Gamma_3^{(1)}(t-t') \pm \Gamma_3^{(2)}(t-t')\Gamma_3^{(1)}(t);$$

$$H_6(t) = [\Gamma_4^{(2)}(t-t')\Gamma_4^{(1)}(t) - \Gamma_4^{(2)}(t)\Gamma_4^{(1)}(t-t')]/A_1A_2$$

$$H_7^i(t) = \Gamma_1^{(i)}(t) + \Gamma_1^{(i)}(t-t') + (\Gamma_2^{(i)}(t) + \Gamma_2^{(i)}(t-t'))h_i(1)/h_i^2(6);$$

$$H_8(t) = [\Gamma_4^{(2)}(t)\Gamma_4^{(1)}(t-t') + \Gamma_4^{(2)}(t-t')\Gamma_4^{(1)}(t)]/A_1A_2;$$

$$H_9(t) = [\Gamma_3^{(2)}(t)\Gamma_4^{(1)}(t) + \Gamma_4^{(1)}(t-t')\Gamma_3^{(2)}(t-t')]/A_1$$

$$- [\Gamma_4^{(2)}(t)\Gamma_3^{(1)}(t-t') + \Gamma_4^{(2)}(t-t')\Gamma_3^{(1)}(t)]/A_2;$$

$$H_{10}(t) = [\Gamma_4^{(1)}(t-t')\Gamma_3^{(2)}(t) + \Gamma_4^{(1)}(t)\Gamma_3^{(2)}(t-t')]/A_1$$

$$- [\Gamma_4^{(2)}(t)\Gamma_3^{(1)}(t-t') + \Gamma_4^{(2)}(t-t')\Gamma_3^{(1)}(t)]/A_2;$$

$h_i(1) = A_i^2 - B_i^2$; $h_i(2) = 2A_iB_i$; $h_i(3) = 3A_i^2 = B_i^2$; $h_i(4) = A_i^2 = 3B_i^2$; $h_i(5) = B_i$;

$h_i(6) = A_i$, $S_i(t') = e^{-B_it'} \cos A_it'$ $T_i(t') = \Gamma_4^{(i)}(t')$; for $i = 1, 2$;

$h_3(1) = \Omega^2 - \beta^2$, $h_3(2) = 2\Omega\beta$, $h_3(3) = 3\Omega^2 - \beta^2$, $h_3(4) = \Omega^2 - 3\beta^2$,

$h_3(5) = \beta$, $h_3(6) = \Omega$, $S_3(t') = e^{-\beta t'}\cos\Omega t'$, $T_3(t') = e^{-\beta t'}\sin\Omega t'$, for $i=3$;

$$F_1^i = \mathrm{Re}[\frac{A_i}{2B_i} \frac{\beta(\beta^2 + \Omega^2 + \omega_i^2)}{A_i^2\omega_i(\omega_i^2 - \omega_3^2)(\omega_i^2 - \omega_4^2)}] \; ; \quad R_1 = \mathrm{Re}[\frac{1}{(\omega_1^2 - \omega_{21}^2)(\omega_1^2 - \omega_2^2)}]$$

$$G_1^i = \mathrm{Im}[\frac{A_i}{2B_i} \frac{\beta(\beta^2 + \Omega^2 + \omega_i^2)}{A_i^2\omega_i(\omega_i^2 - \omega_3^2)(\omega_i^2 - \omega_4^2)}]; \quad I_1 = \mathrm{Im}[\frac{1}{(\omega_1^2 - \omega_{21}^2)(\omega_1^2 - \omega_2^2)}]$$

$$F_3^1 = \mathrm{Re}\left[\frac{1}{(\omega_3^2 - \omega_{12}^2)(\omega_3^2 - \omega_1^2)}\right] \; ; \qquad R_2 = \mathrm{Re}\left[\frac{1}{(\omega_2^2 - \omega_{12}^2)(\omega_2^2 - \omega_1^2)}\right];$$

$$G_3^1 = \mathrm{Im}\left[\frac{1}{(\omega_3^2 - \omega_{12}^2)(\omega_3^2 - \omega_1^2)}\right] \; ; \qquad I_2 = \mathrm{Im}\left[\frac{1}{(\omega_2^2 - \omega_{12}^2)(\omega_2^2 - \omega_1^2)}\right];$$

$$F_3^2 = \mathrm{Re}\left[\frac{1}{(\omega_3^2 - \omega_{21}^2)(\omega_3^2 - \omega_2^2)}\right]; \qquad G_3^2 = \mathrm{Im}\left[\frac{1}{(\omega_3^2 - \omega_{21}^2)(\omega_3^2 - \omega_2^2)}\right];$$

$$U_i = F_1^i R_i - I_i G_1^i, \qquad V_i = G_1^i R_i + I_i F_1^i \qquad \text{for } i = 1, 2;$$

$$U_3 = F_3^1 F_3^2 - G_3^1 G_3^2, \qquad V_3 = G_3^1 F_3^2 + G_3^2 F_3^1 \qquad \text{for } i = 3;$$

$$\omega_i = A_i + iB_i, \qquad \omega_{12} = -A_1 + iB_1, \qquad \omega_{21} = -A_2 + iB_2,$$

$$N = C_1^2 C_2^2, \qquad N_1 = 4B_1B_2 - C_1^2 - C_2^2, \qquad N_2 = C_1^2 B_2 - C_2^2 B_1, \qquad N_3 = B_2 - B_1;$$

$$\Gamma_1^{(i)}(t) = e^{-2B_i t}\left[1 + \frac{B_i}{A_i} \sin 2 A_i t\right], \qquad \Gamma_2^{(i)}(t) = e^{-2B_i t} \sin^2 A_i t,$$

$$\Gamma_3^{(i)}(t) = e^{-B_i t}\left[\cos A_i t + \frac{B_i}{A_i} \sin A_i t\right], \qquad \Gamma_4^{(i)}(t) = e^{-B_i t} \sin A_i t.$$

"Better is the end of a thing than the beginning thereof."

Ecclesiastes