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Comparative Spectral Analysis of Flexible Structure Models: the Euler-Bernoulli Beam model, the Rayleigh Beam model, and the Timoshenko Beam Model

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Contents

List of Figures

List of Tables

ABSTRACT

COMPARATIVE SPECTRAL ANALYSIS OF FLEXIBLE STRUCTURE MODELS: THE EULER-BERNOULLI BEAM MODEL, THE RAYLEIGH BEAM MODEL, AND THE TIMOSHENKO BEAM MODEL

By

Anhhong Rose Nguyen

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We derive herein approximate spectra for three different models of transversely vibrating beams. Each model consists of a system of partial differential equations (PDEs) with various boundary conditions. The three models that we consider are the Euler-Bernoulli model, the Rayleigh model, and the Timoshenko model. We first discuss a brief history of the models before delving into obtaining the spectral equations for each beam model under different boundary conditions. Lastly, we present asymptotic approximations of some of the various spectral equations we found from each model.

Introduction

We consider three models for a vibrating beam, all of which were developed by the mid-twentieth century. The Euler-Bernoulli beam model is one of the first mathematical descriptions of the motion of a vibrating beam; it was discovered by Jacob Bernoulli c. 1700. Attempting to improve upon the Euler-Bernoulli model, Lord Rayleigh introduced his model in 1877. Stephen Timoshenko developed the Timoshenko beam model in the early twentieth century, which added more observed physical effects to the Rayleigh model. Each of the beam models have four different boundary conditions depending on how the beam is attached (or not) to a boundary surface. These are hinged end, clamped end, free end, and sliding end conditions.

During the mid-1700s, there was considerable doubt among engineers regarding how applied mathematics presented in their field. During this time, the Euler-Bernoulli model (also know as classical beam theory) was developed. Thus, Leonhard Euler and Daniel Bernoulli formulated a useful and applicable theory. Daniel Bernoulli derived the differential equation governing the motion of a vibrating beam, while Leonhard Euler studied the shape of elastic beams under differential boundary conditions.

The Euler-Bernoulli beam model normally produces an overestimate of the natural frequencies of a vibrating beam. In 1877, in order to improve the model, Lord Rayleigh added the effect of rotational inertia of the cross-sectional area. This provides some improvement to the Euler-Bernoulli beam model, however, the natural frequencies are still overestimated.

The Timoshenko beam model was developed in the early 20th century by Stephen Timoshenko. Unlike the Euler-Bernoulli model, the Timoshenko Beam Model accounts for the effects of shear distortion and rotational inertia. In other words, the Timoshenko model adds rotational inertia to the shear model or adds the shear distortion to the Rayleigh model. Using the Timoshenko model, many authors have achieved the frequency equations and the mode shapes for different boundary conditions of a beam (see [1–5] and references therein.)

We present the results in two parts. The first part, which consists of Sections 2–4, contains the derivation of the so-called spectral equations for each beam model. To this end, we study the corresponding partial differential equation equipped with specific boundary conditions. In doing so, we consider different combinations of the standard boundary conditions for each model. The result being that we obtain a polynomial-exponential equation with respect to the spectral parameter, which is specific for each model. In some cases, the spectral equations are relatively simple, while in other cases these equations are quite complicated. This depends on the beam model and the choice of the end conditions. In the second part, we derive the solutions to the spectral equations. In some cases, such solutions can be given in closed form, while in other cases (when closed form solutions do not exist) we derive asymptotic approximations to the eigenvalues as the index of the set of eigenvalues tends to infinity (see [6-8] and references therein).

Derivation of the Spectral Equations for Three Beam Models

We now endeavor to find the spectral equations for various boundary conditions for the Euler-Bernoulli model, the Rayleigh model, and the Timoshenko model. For the Euler-Bernoulli model, we consider the four examples where (at least) one side is hinged. The remaining 12 scenarios are presented in Table 1. For the Rayleigh model, we derived the general solution. We then go on to consider symmetric conditions (e.g., clamped-clamped ends) explicitly. Lastly, for the Timoshenko model, we derive the general solution, which, along with it, comes a critical conditions that doubles our field of solutions. Like the Euler-Bernoulli model, we consider the four examples where (at least) one side is hinged. Thus, including the critical condition, we consider eight specific examples.

1 The Euler-Bernoulli Beam Model

In this section, we consider the sixteen combinations of the boundary conditions. However, only four pairs of different boundary conditions are presented in detail. Namely, these are hinged-hinged, hinged-clamped, hinged-free, and hinged-sliding boundary conditions [1,3,5].

The non-homogeneous partial differential equation for the Euler-Bernoulli beam model

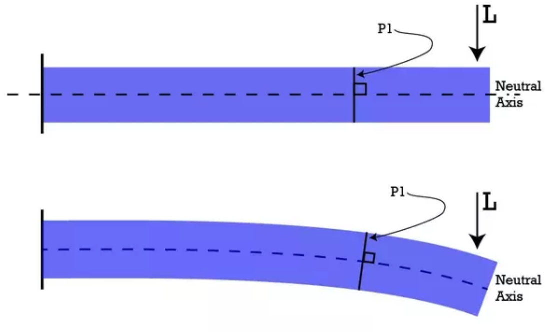

Figure 1: The Euler-Bernoulli beam model, where the it is fixed at one end and acted upon by load L at other, and $P1$ is perpendicular plane to the neutral axis.

is

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^4 v(x,t)}{\partial x^4} = f(x,t), \tag{1.1}$$

where $0 \leq x \leq L < \infty$ and $t \geq 0$, for some finite maximum beam length L, $v(x,t)$ is the vertical displacement at position x and time moment t, ρ is the density, A is the cross-sectional area, and $f(x,t)$ is the non-homogeneous forcing function of both space and time.

The following four boundary conditions are considered, where a is either 0 or L:

(a) Hinged end (b) Clamped end (c) Free end (d) Sliding end

(a) Hinged-end:

$$\frac{\partial^2 v(a,t)}{\partial x^2} = 0 \qquad v(a,t) = 0; \tag{1.2}$$

(b) Clamped-end:

$$\frac{\partial v(a,t)}{\partial x} = 0 \qquad v(a,t) = 0; \tag{1.3}$$

(c) Free-end:

$$\frac{\partial^2 v(a,t)}{\partial x^2} = 0 \qquad \frac{\partial^3 v(a,t)}{\partial x^3} = 0; \tag{1.4}$$

(d) Sliding-end:

$$\frac{\partial v(a,t)}{\partial x} = 0 \qquad \frac{\partial^3 v(a,t)}{\partial x^3} = 0. \tag{1.5}$$

In the above, we have the second derivative $\partial^2 v/\partial x^2$, which represents the moment of the beam, and the third derivative $\partial^3 v/\partial x^3$, which represents the shear of the beam.

In order to find eigenvalues and eigenfunctions, let us consider the homogeneous equation

$$\rho A \, \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^4 v(x,t)}{\partial x^4} = 0. \tag{1.6}$$

By applying separation of variables[1,2,6], we assume that $v(x,t) = W(x)T(t)$. We find the second derivative in time

$$\frac{\partial^2 v(x,t)}{\partial t^2} = W(x) \, \frac{d^2 T(t)}{dt^2} \tag{1.7}$$

and the fourth derivative in space

$$\frac{\partial^4 v(x,t)}{\partial x^4} = \frac{d^4 W(x)}{dx^4} \, T(t). \tag{1.8}$$

Substitute (1.7) and (1.8) into the homogeneous equation, and we achieve

$$\rho A \, \frac{T''(t)}{T(t)} = -\frac{W^{(4)}(x)}{W(x)} = \lambda^4, \tag{1.9}$$

where λ is the variable of separation. This is justified as equal functions of derivatives of distinct variables must equal a constant value. Since we are dealing with a conservative system, we assume that the variable of separation is real[2,3,4]. By separation of variables, (1.9) can be rewritten as two ordinary differential equations,

$$\rho A T''(t) - \lambda^4 T(t) = 0, \tag{1.10}$$

and

$$W^{(4)}(x) + \lambda^4 W(x) = 0. \tag{1.11}$$

To solve (1.11), we let $W(x) = e^{rx}$, which give us the characteristic equation,

$$r^4 + \lambda^4 = 0. \tag{1.12}$$

This equation has four different solutions. They are: $r_1 = \lambda$, $r_2 = -\lambda$, $r_3 = i\lambda$, and $r_4 = -i\lambda$. The general solution for $W(x)$ can be represented by the following linear combination:

$$W(x) = A(\lambda)\sin(\lambda x) + B(\lambda)\cos(\lambda x) + C(\lambda)\sinh(\lambda x) + D(\lambda)\cosh(\lambda x), \tag{1.13}$$

where $A(\lambda), B(\lambda), C(\lambda)$, and $D(\lambda)$ are arbitrary functions of λ .

Our goal is to find these values of the arbitrary parameters for which the function (1.13) satisfies the boundary conditions. To this end, we derive the spectral equation for each set of boundary conditions.

Remark 1. In what follows, we will derive the spectral equations for the different combinations of the boundary conditions. For the Euler-Bernoulli beam model some of the spectral equations are well-known and can be found in the literature (see, e.g.,[1,2,3]). For the Rayleigh and Timoshenko beam models, some spectral equations are well-known as well, while the others have been derived in the present work. The necessary remarks will appear in Sections 2 and 3 below. To keep the paper self-contained, we present the derivation of the spectral equations for all considered cases.

Let us differentiate $W(x)$:

$$W'(x) = A(\lambda)\lambda\cos(\lambda x) - B(\lambda)\lambda\sin(\lambda x) + C(\lambda)\lambda\cosh(\lambda x) + D(\lambda)\lambda\sinh(\lambda x) \tag{1.14}$$

$$W''(x) = -A(\lambda)\lambda^2\sin(\lambda x) - B(\lambda)\lambda^2\cos(\lambda x) + C(\lambda)\lambda^2\sinh(\lambda x) + D(\lambda)\lambda^2\cosh(\lambda x) \tag{1.15}$$

$$W'''(x) = -A(\lambda)\lambda^3\cos(\lambda x) + B(\lambda)\lambda^3\sin(\lambda x) + C(\lambda)\lambda^3\cosh(\lambda x) + D(\lambda)\lambda^3\sinh(\lambda x) \tag{1.16}$$

By applying all combinations of the boundary conditions, we obtain the spectral equations represented in Table 1. Here, let us consider the four cases where one end is hinged.

Hinged-Hinged Boundary Conditions

There are two sets of boundary conditions: At $x = 0$ we have

$$W''(0) = W(0) = 0; \tag{1.17}$$

and, at $x = L$, we have

$$W''(L) = W(L) = 0. \tag{1.18}$$

From the first set of boundary conditions (1.17), we notice that

$$W''(0) = -B(\lambda) + D(\lambda) = 0, \tag{1.19}$$

and

$$W(0) = B(\lambda) + D(\lambda) = 0. \tag{1.20}$$

Thus $B(\lambda) = D(\lambda) = 0$. The general solution reduces to the form

$$W(x) = A(\lambda)\sin(\lambda x) + C(\lambda)\sinh(\lambda x). \tag{1.21}$$

Let us apply the second set of boundary conditions (1.18):

$$W(L) = A(\lambda)\sin(\lambda L) + C(\lambda)\sinh(\lambda L) = 0, \tag{1.22}$$

and

$$W''(L) = -A(\lambda)\lambda^2\sin(\lambda L) + C(\lambda)\lambda^2\sinh(\lambda L) = 0. \tag{1.23}$$

where $A(\lambda)$ and $C(\lambda)$ are unknown functions. The homogeneous system has a solution if and only if its determinant is equal to zero, i.e.,

$$\det\begin{pmatrix} \sin(\lambda L) & \sinh(\lambda L) \\ -\sin(\lambda L) & \sinh(\lambda L) \end{pmatrix} = 0. \tag{1.24}$$

Thus, the spectral equation for the hinged-hinged boundary conditions is

$$\boxed{\sin(\lambda L)\sinh(\lambda L) = 0.} \tag{1.25}$$

Hinged-Clamped Boundary Conditions

The two sets of boundary conditions in this case are:

$$W''(0) = W(0) = 0, \tag{1.26}$$

and

$$W'(L) = W(L) = 0. \tag{1.27}$$

Similar to the hinged-hinged case, $W''(0) = W(0) = 0$ implies $B(\lambda) = D(\lambda) = 0$, and our general solution simplifies to

$$W(x) = A(\lambda)\sin(\lambda x) + C(\lambda)\sinh(\lambda x). \tag{1.28}$$

Our second set of the boundary conditions yields

$$W'(L) = A(\lambda)\lambda\cos(\lambda L) + C(\lambda)\lambda\cosh(\lambda L) = 0, \tag{1.29}$$

and

$$W(L) = A(\lambda)\sin(\lambda L) + C(\lambda)\sinh(\lambda L) = 0. \tag{1.30}$$

To find the solution for the homogeneous system we set its determinant equal to zero,

$$\det\begin{pmatrix}\cos(\lambda L) & \cosh(\lambda L)\\ \sin(\lambda L) & \sinh(\lambda L)\end{pmatrix} = 0. \tag{1.31}$$

Thus, the spectral equation for the hinged-clamped boundary condition is

$$\boxed{\cos(\lambda L)\sinh(\lambda L) - \sin(\lambda L)\cosh(\lambda L) = 0.} \tag{1.32}$$

Hinged-Free Boundary Conditions

The boundary conditions in this case are:

$$W''(0) = W(0) = 0, \tag{1.33}$$

and

$$W''(L) = W'''(L) = 0. \tag{1.34}$$

Again, $B(\lambda) = D(\lambda) = 0$ from the first set of boundary conditions, as in the previous cases. The general solution reduces to

$$W(x) = A(\lambda)\sin(\lambda x) + C(\lambda)\sinh(\lambda x). \tag{1.35}$$

Applying the second set of boundary conditions (1.34) yields

$$W''(L) = -A(\lambda)\lambda^2\sin(\lambda L) + C(\lambda)\lambda^2\sinh(\lambda L) = 0, \tag{1.36}$$

and

$$W'''(L) = -A(\lambda)\lambda^3\cos(\lambda L) + C(\lambda)\lambda^3\cosh(\lambda L) = 0. \tag{1.37}$$

The system has a solution if and only if its determinant is zero, i.e.,

$$\det\begin{pmatrix} -\sin(\lambda L) & \sinh(\lambda L) \\ -\cos(\lambda L) & \cosh(\lambda L) \end{pmatrix} = 0. \tag{1.38}$$

Thus, the spectral equation is

$$\boxed{\sinh(\lambda L)\cos(\lambda L) - \sin(\lambda L)\cosh(\lambda L) = 0.} \tag{1.39}$$

Hinged-Sliding Boundary Conditions

The two sets of boundary conditions in this case are:

$$W''(0) = W(0) = 0 \tag{1.40}$$

and

$$W'(L) = W'''(L) = 0. \tag{1.41}$$

As in the previous cases, after the first set of boundary conditions is applied, the general solution takes the form:

$$W(x) = A(\lambda)\sin(\lambda x) + C(\lambda)\sinh(\lambda x). \tag{1.42}$$

Applying the second set of boundary conditions, we have

$$W'(L) = A(\lambda)\lambda\cos(\lambda L) + C(\lambda)\lambda\cosh(\lambda L) = 0, \tag{1.43}$$

and

$$W'''(L) = -A(\lambda)\lambda^3\cos(\lambda x) + C(\lambda)\lambda^3\cosh(\lambda x) = 0. \tag{1.44}$$

The homogeneous system has a solution if and only if its determinant is zero, i.e.,

$$\det \begin{pmatrix} \cos(\lambda L) & \cosh(\lambda L) \\ -\cos(\lambda L) & \cosh(\lambda L) \end{pmatrix} = 0. \tag{1.45}$$

Thus, the spectral equation is

$$\boxed{\cos(\lambda L)\cosh(\lambda L) = 0.} \tag{1.46}$$

These four results, along with the twelve more options, are all collected in Table 1.

	Hinged end	Clamped end	Free end	Sliding end
Hinged end	$\sin(\lambda L)\sinh(\lambda L) = 0$	$\cos(\lambda L)\sinh(\lambda L)$ $-\sin(\lambda L)\cosh(\lambda L) = 0$	$\sinh(\lambda L)\cos(\lambda L)$ $-\sin(\lambda L)\cosh(\lambda L) = 0$	$\cosh(\lambda L)\cos(\lambda L) = 0$
Clamped end	$\cos(\lambda L)\sinh(\lambda L)$ $-\sin(\lambda L)\cosh(\lambda L) = 0$	$\cosh(\lambda L)\cos(\lambda L) = 1$	$\cosh(\lambda L)\cos(\lambda L) = -1$	$\cos(\lambda L)\sinh(\lambda L)$ $+\sin(\lambda L)\cosh(\lambda L) = 0$
Free end	$\sinh(\lambda L)\cos(\lambda L)$ $-\sin(\lambda L)\cosh(\lambda L) = 0$	$\cosh(\lambda L)\cos(\lambda L) = -1$	$\cosh(\lambda L)\cos(\lambda L) = 1$	$\sin(\lambda L)\cosh(\lambda L)$ $+\sinh(\lambda L)\cos(\lambda L) = 0$
Sliding end	$\cos(\lambda L)\cosh(\lambda L) = 0$	$\cos(\lambda L)\sinh(\lambda L)$ $+\sin(\lambda L)\cosh(\lambda L) = 0$	$\sin(\lambda L)\cosh(\lambda L)$ $+\sinh(\lambda L)\cos(\lambda L) = 0$	$\sin(\lambda L)\sinh(\lambda L) = 0$

Table 1: Spectral equations of different combinations of the boundary conditions for the Euler-Bernoulli model.

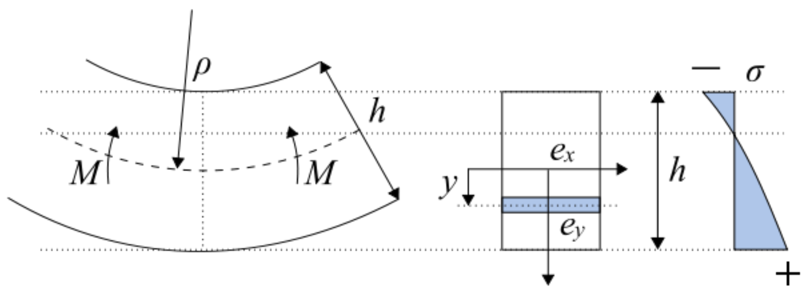

Figure 2: Bending deformation of Rayleigh beam model, where M is the bending moment, ρ is the local bending radius, h is the height, y is the position along the y-axis, and σ is the bending stress.

2 The Rayleigh Beam Model

In this section, we consider three different combinations of the boundary-value problem for the Rayleigh beam model. They are clamped-clamped boundary conditions, hinged-hinged boundary conditions, and free-free boundary conditions. For each set of boundary conditions we will show in detail the derivation of the spectral equation [1, 2, 5].

The Rayleigh beam model is governed by the partial differential equation.

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^4 v(x,t)}{\partial x^4} - \rho I \frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} = f(x,t), \tag{2.1}$$

where $0 \leq x \leq L < \infty$ and $t \geq 0$, for a beam of length L, $v(x,t)$ is the vertical displacement at position x and time moment t, ρ is the density, A is the cross-sectional area, I is the moment of inertia, and $f(x,t)$ is the non-homogeneous forcing function of both space and time.

The following boundary conditions will be considered, where a is at 0 or L:

(a) Hinged end (b) Clamped end (c) Free end (d) Sliding end

(a) Hinged-end:

$$\frac{\partial^2 v(a,t)}{\partial x^2} = 0 \qquad v(a,t) = 0; \tag{2.2}$$

(b) Clamped-end:

$$\frac{\partial v(a,t)}{\partial x} = 0 \qquad v(a,t) = 0; \tag{2.3}$$

(c) Free-end:

$$\frac{\partial^2 v(a,t)}{\partial x^2} = 0 \qquad \frac{\partial^3 v(a,t)}{\partial x^3} - \rho I \frac{\partial^3 v(a,t)}{\partial x \partial t^2} = 0; \tag{2.4}$$

(d) Sliding-end:

$$\frac{\partial v(a,t)}{\partial x} = 0 \qquad \frac{\partial^3 v(a,t)}{\partial x^3} - \rho I \frac{\partial^3 v(a,t)}{\partial x \partial t^2} = 0. \tag{2.5}$$

In order to find eigenvalues and eigenfunctions, let us consider the homogeneous equation:

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^4 v(x,t)}{\partial x^4} - \rho I \frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} = 0. \tag{2.6}$$

First, let us rescale our equation by letting $\widetilde{x} = kx$ for some k. For any function f we have

$$\frac{\partial f}{\partial x} = k \frac{\partial f}{\partial \widetilde{x}}, \qquad \frac{\partial^2 f}{\partial x^2} = k^2 \frac{\partial^2 f}{\partial \widetilde{x}^2}.$$

Rewriting (2.6) in term of $\widetilde{x}$, we have

$$\frac{\partial^2 v(\widetilde{x},t)}{\partial t^2} + \left(\frac{k^4}{\rho A}\right)\frac{\partial^4 v(\widetilde{x},t)}{\partial \widetilde{x}^4} - \left(\frac{Ik^2}{A}\right)\frac{\partial^2 v(\widetilde{x},t)}{\partial \widetilde{x}^2 \partial t^2} = 0. \tag{2.7}$$

We choose $k = \sqrt[4]{\rho A}$, then

$$\frac{I}{A}k^2 = \frac{I}{A}\sqrt{\rho A} = I\sqrt{\frac{\rho}{A}}. \tag{2.8}$$

Letting $c^2 = I\sqrt{\frac{\rho}{A}}$ and omitting the tildes from (2.7), we derive:

$$\frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^4 v(x,t)}{\partial x^4} - c^2\frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} = 0. \tag{2.9}$$

Solving (2.9) by separation of variables, we assume that $v(x,t) = W(x)T(t)$. Substituting $v(x,t)$ into the homogeneous equation, we attain

$$T''(t)W(x) + T(t)W^{(4)}(x) - c^2 T''(t)W''(x) = 0, \tag{2.10}$$

which simplifies to

$$W^{(4)}(x)T(t) + \left(W(x) - c^2 W''(x)\right)T''(t) = 0. \tag{2.11}$$

Assuming that $W^{(4)}(x) \neq 0$, we can rewrite (2.11) in the form

$$\frac{-\left(c^2 W''(x) - W(x)\right)}{W^{(4)}(x)} = -\frac{T(t)}{T''(t)} = \frac{1}{\lambda}. \tag{2.12}$$

The left-hand side of (2.12) is a function of space and the middle is a function of time. This means, as before, that each function must be a constant, which we have denoted by λ^{-1}. As such, we obtain the following system of two coupled ordinary differential equations:

$$T''(t) + \lambda T(t) = 0 \tag{2.13}$$

and

$$W^{(4)}(x) + \lambda\big(c^2 W''(x) - W(x)\big) = 0. \tag{2.14}$$

Remark 2. We will derive the spectral equations for the Rayleigh beam model with the following sets of the boundary conditions: (i) clamped-clamped conditions, (ii) hinged-hinged conditions, which are the same as for the model with the sliding-sliding conditions, (iii) free-free conditions. The spectral equations corresponding to different sets of the boundary conditions can be obtained using similar agreement. The results obtained follow are consistent with the results of [1], where some of the spectral equations are presented without any derivations.

We will now consider (2.14) with the clamped-clamped, hinged-hinged and free-free boundary conditions. First, we derive the basis for the linear space of solutions to (2.14). The characteristic equation is

$$r^4 + r^2 c^2 \lambda - \lambda = 0. \tag{2.15}$$

Substituting $r^2 = y$, we obtain the quadratic equation for y: $y^2 + \lambda c^2 y - \lambda = 0$, whose roots are

$$y_{1,2} = \frac{-\lambda c^2 \pm \sqrt{\lambda^2 c^4 + 4\lambda}}{2}. \tag{2.16}$$

Assuming that $\lambda > 0$, we get

$$y_1 = \frac{-\lambda c^2 - \sqrt{\lambda^2 c^4 + 4\lambda}}{2} < 0, \quad y_2 = \frac{-\lambda c^2 + \sqrt{\lambda^2 c^4 + 4\lambda}}{2} > 0. \tag{2.17}$$

Let us introduce the following notation:

$$r_{1,2} = \pm i\sqrt{|y_1|} \equiv \pm i\alpha, \quad r_{3,4} = \pm\sqrt{y_2} \equiv \pm\beta, \tag{2.18}$$

where

$$\alpha^2 = \frac{\lambda c^2 + \sqrt{\lambda^2 c^4 + 4\lambda}}{2} \text{ and } \beta^2 = \frac{\sqrt{\lambda^2 c^4 + 4\lambda} - \lambda c^2}{2}. \tag{2.19}$$

The basis for the space of solutions to (2.14) is $\{\sin(\alpha x), \cos(\alpha x), \sinh(\beta x), \cosh(\beta x)\}$. Thus the general solution of (2.14) can be written as

$$W(x) = A(\lambda)\cos(\alpha x) + B(\lambda)\sin(\alpha x) + C(\lambda)\cosh(\beta x) + D(\lambda)\sinh(\beta x), \tag{2.20}$$

where $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are arbitrary function of λ. Without misunderstanding, we use the same notation for the coefficients $(A(\lambda), B(\lambda), C(\lambda), D(\lambda))$ as we have used for the Euler-Bernoulli model in (1.13).

Clamped-Clamped Boundary Conditions

Let us consider the case when both ends are clamped. We have to find the solution to (2.14), that satisfies the following boundary conditions:

$$W'(0) = W(0) = 0, \tag{2.21}$$

and

$$W'(L) = W(L) = 0. \tag{2.22}$$

Without loss of generality for the rest of this section, we assume that $L = 1$. Applying the conditions (2.21) to to the function from (2.20), we get

$$A(\lambda) + C(\lambda) = 0, \qquad \alpha B(\lambda) + \beta D(\lambda) = 0. \tag{2.23}$$

The condition $W(1) = 0$ yields

$$A(\lambda)\cos(\alpha) + B(\lambda)\sin(\alpha) + C(\lambda)\cosh(\beta) + D(\lambda)\sinh(\beta) = 0. \tag{2.24}$$

The condition $W'(1) = 0$ yields

$$-\alpha A\sin(\alpha) + \alpha B\cos(\alpha) + \beta C\sinh(\beta) + \beta D\cosh(\beta) = 0. \tag{2.25}$$

Taking into account (2.23), we rewrite (2.24) and (2.25) in the form

$$A(\lambda)\left(\cos(\alpha) - \cosh(\beta)\right) + B(\lambda)\left(\sin(\alpha) - \frac{\alpha}{\beta}\sinh(\beta)\right) = 0, \tag{2.26}$$

$$-A(\lambda)\big(\alpha\sin(\alpha) + \beta\sinh(\beta)\big) + B(\lambda)\big(\alpha\cos(\alpha) - \alpha\cosh(\beta)\big) = 0. \tag{2.27}$$

This homogeneous system has a non-trivial solution if and only if it determinant is zero, i.e.,

$$\det\begin{pmatrix} \cos(\alpha) - \cosh(\beta) & \sin(\alpha) - \frac{\alpha}{\beta}\sinh(\beta) \\ -(\alpha\sin(\alpha) + \beta\sinh(\beta)) & \alpha\cos(\alpha) - \alpha\cosh(\beta) \end{pmatrix} = 0. \tag{2.28}$$

This is equivalent to

$$\big(\cos(\alpha) - \cosh(\beta)\big)\big(\alpha\cos(\alpha) - \alpha\cosh(\beta)\big)$$
$$+ \left(\sin(\alpha) - \frac{\alpha}{\beta}\sinh(\beta)\right)\big(\alpha\sin(\alpha) + \beta\sinh(\beta)\big) = 0.$$

Simplifying this equation we obtain the spectral equation for clamped-clamped model

$$\boxed{2\alpha\beta + (\beta^2 - \alpha^2)\sin(\alpha)\sinh(\beta) - 2\alpha\beta\cos(\alpha)\cosh(\beta) = 0.} \tag{2.29}$$

Hinged-Hinged Boundary Conditions

We have to find the spectral equation corresponding to the following boundary conditions:

$$W''(0) = W(0) = 0, \tag{2.30}$$

and

$$W''(L) = W(L) = 0. \tag{2.31}$$

Applying (2.30) to (2.20), we obtain

$$A(\lambda) = C(\lambda) = 0, \tag{2.32}$$

thus

$$W(x) = B(\lambda)\sin(\alpha x) + D(\lambda)\sinh(\beta x). \tag{2.33}$$

From $W''(1) = 0$, we have

$$-B(\lambda)\alpha^2 \sin(\alpha) + D(\lambda)\beta^2 \sinh(\beta) = 0. \tag{2.34}$$

From $W(1) = 0$, we have

$$B(\lambda)\alpha^2 \sin(\alpha) + D(\lambda)\beta^2 \sinh(\beta) = 0. \tag{2.35}$$

There exists a non-trivial solution for the system if and only if its determinant is zero, i.e.,

$$\det \begin{pmatrix} \sin(\alpha) & \sinh(\beta) \\ -\alpha^2 \sin(\alpha) & \beta^2 \sinh(\beta) \end{pmatrix} = 0, \tag{2.36}$$

which is equivalent to

$$(\beta^2 + \alpha^2) \sin\alpha \sinh\beta = 0. \tag{2.37}$$

Since $\alpha^2 + \beta^2 \neq 0$, we obtain the spectral equation for hinged-hinged model

$$\boxed{\sin\alpha \sinh\beta = 0.} \tag{2.38}$$

Free-Free Boundary Conditons

The boundary conditions for the free-free case are

$$W'''(0) + \eta W'(0) = 0, \qquad W''(0) = 0, \tag{2.39}$$

and

$$W'''(1) + \eta W'(1) = 0, \qquad W''(1) = 0, \tag{2.40}$$

where $\eta = \lambda c^2$, since we let $\widetilde{x} = kx$ and $k = \sqrt[4]{\rho A}$, where the differentiation is taken with respect to $\widetilde{x}$. Applying the first set of the boundary conditions to (2.20) we obtain that

$$D(\lambda) = \frac{\alpha(\alpha^2 - \eta)}{\beta(\beta^2 + \eta)} B(\lambda) \quad \text{and} \quad C(\lambda) = \frac{\alpha^2}{\beta^2} A(\lambda). \tag{2.41}$$

The second set of the boundary condition yields

$$\begin{aligned} &A(\lambda)\alpha^3 \sin(\alpha) + B(\lambda)\alpha^3 \cos(\alpha) + C(\lambda)\beta^3 \sinh(\beta) + D(\lambda)\beta^3 \cosh(\beta) \\ &+ \eta\big(- A(\lambda)\alpha \sin(\alpha) + B(\lambda)\alpha \cos(\alpha) + C(\lambda)\beta \sinh(\beta) + D(\lambda)\beta \cosh(\beta)\big) = 0, \end{aligned} \tag{2.42}$$

and

$$-A(\lambda)\alpha^2 \cos(\alpha) - B(\lambda)\alpha^2 \sin(\alpha) + C(\lambda)\beta^2 \cosh(\beta) + D(\lambda)\beta^2 \sinh(\beta) = 0. \tag{2.43}$$

These two equations can also be rewritten as the following system:

$$\begin{aligned} A(\lambda)\alpha(\alpha^2 - \eta) \sin(\alpha) - B(\lambda)\alpha(\alpha^2 - \eta) \cos(\alpha) \\ + C(\lambda)\beta(\beta^2 + \eta) \sinh(\beta) + D(\lambda)\beta(\beta^2 + \eta) \cosh(\beta) = 0, \end{aligned} \tag{2.44}$$

$$A(\lambda)\alpha^2 \cos(\alpha) + B(\lambda)\alpha^2 \sin(\alpha) - C(\lambda)\beta^2 \cosh(\beta) - D(\lambda)\beta^2 \sinh(\beta) = 0. \tag{2.45}$$

Where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ are functions of λ. Let us denote $\mathcal{T} = (\alpha^2 - \eta)$ and $\kappa = (\beta^2 + \eta)$. Taking into account (2.41), we eliminate $C(\lambda)$ and $D(\lambda)$ from the system to obtain

$$A(\lambda)\left(\alpha\mathcal{T}\sin(\alpha)+\frac{\alpha^2}{\beta}\kappa\sinh(\beta)\right)-B(\lambda)\left(\alpha\mathcal{T}\cos(\alpha)-\alpha\mathcal{T}\cosh(\beta)\right)=0, \tag{2.46}$$

$$A(\lambda)\left(\alpha^2\cos(\beta)-\alpha^2\cosh(\alpha)\right)-B(\lambda)\left(\alpha^2\sin(\alpha)+\alpha\beta\frac{\mathcal{T}}{\kappa}\sinh(\beta)\right)=0. \tag{2.47}$$

The system is solvable if and only if

$$\det\begin{pmatrix}\alpha\mathcal{T}\sin(\alpha)+\frac{\alpha^2}{\beta}\kappa\sinh(\beta) & -\left(\alpha\mathcal{T}\cos(\alpha)-\alpha\mathcal{T}\cosh(\beta)\right)\\ \alpha^2\cos(\beta)-\alpha^2\cosh(\alpha) & \alpha^2\sin(\alpha)+\alpha\beta\frac{\mathcal{T}}{\kappa}\sinh(\beta)\end{pmatrix}=0, \tag{2.48}$$

which is equivalent to

$$\begin{aligned}\alpha\mathcal{T}\sin^2(\alpha)-\alpha\mathcal{T}\sinh^2(\beta)-\beta\frac{\mathcal{T}^2}{\kappa}\sin(\alpha)\sinh(\beta)+\kappa\frac{\alpha^2}{\beta}\sin(\alpha)\sinh(\beta)\\ +\alpha\mathcal{T}\cos^2(\alpha)+\alpha\mathcal{T}\cosh^2(\beta)-2\alpha\mathcal{T}\cos(\alpha)\cosh(\beta)=0.\end{aligned} \tag{2.49}$$

Simplifying this equation, we obtain

$$2\alpha\beta\mathcal{T}\kappa-2\alpha\beta\mathcal{T}\kappa\cos(\alpha)\cosh(\beta)+\left(\alpha^2\kappa^2-\beta^2\mathcal{T}^2\right)\sin(\alpha)\sinh(\beta)=0. \tag{2.50}$$

Taking into account that $\alpha^2-\eta=\beta^2$ and $\beta^2+\eta=\alpha^2$, we obtain the spectral equation for free-free model

$$\boxed{2\alpha^3\beta^3-2\alpha^3\beta^3\cos(\alpha)\cosh(\beta)+(\alpha^6-\beta^6)\sin(\alpha)\sinh(\beta)=0.} \tag{2.51}$$

The spectral equations we have derive for Rayleigh Beam model are collected in Table 2.

Boundary Conditions	Spectral Equations
Clamped-clamped	$2\alpha\beta + (\beta^2 - \alpha^2)\sin(\alpha)\sinh(\beta) - 2\alpha\beta\cos(\alpha)\cosh(\beta) = 0$
Hinged-hinged	$\sin\alpha\sinh\beta = 0$
Free-free	$2\alpha^3\beta^3 - 2\alpha^3\beta^3\cos(\alpha)\cosh(\beta) + (\alpha^6 - \beta^6)\sin(\alpha)\sinh(\beta) = 0$

Table 2: Spectral equations of different combinations of the boundary conditions for the Rayleigh beam model.

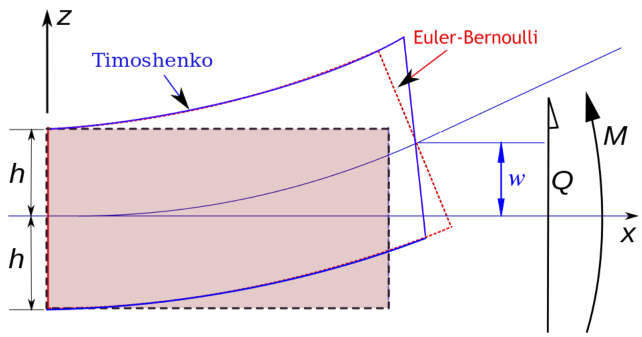

Figure 3: Bending deformation of the Timoshenko beam model, where h is the length of the beam, w is the displacement, M is the bending moment, and Q is the shear force.

3 The Timoshenko Beam Model

We consider the same four different pairs of boundary conditions as we did for the Euler-Bernoulli model: hinged-hinged, hinged-clamped, hinged-sliding, and hinged-free boundary condition [1, 2, 4].

The equations of the Timoshenko beam model are given by

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} - k' G A \left(\frac{\partial^2 v(x,t)}{\partial x^2} - \frac{\partial \alpha(x,t)}{\partial x} \right) = f(x,t), \tag{3.1}$$

and

$$\rho I \frac{\partial^2 \alpha(x,t)}{\partial t^2} - \frac{\partial^2 \alpha(x,t)}{\partial x^2} - k' G A \left(\frac{\partial v(x,t)}{\partial x} - \alpha(x,t) \right) = 0, \tag{3.2}$$

where $0 \leq x \leq L < \infty$ and $t \geq 0$, for some finite maximum beam length L, $v(x,t)$ is the transverse displacement at position x and time moment t, ρ is the density, A is the cross-sectional area, I is the moment of inertia, $f(x,t)$ is the non-homogeneous forcing function of both space and time, $\alpha(x,t)$ is the angle of rotation of the cross-section due to the bending moment at position x and at time t, and G and k' are miscellaneous physical quantities.

The boundary conditions are given by, where a is either 0 or L:

(a) Hinged end (b) Clamped end (c) Free end (d) Sliding end

(a) Hinged-end:

$$\frac{\partial \alpha(a,t)}{\partial x} = 0 \qquad v(a,t) = 0; \tag{3.3}$$

(b) Clamped-end:

$$\alpha(a,t) = 0 \qquad v(a,t) = 0; \tag{3.4}$$

(c) Free-end:

$$\frac{\partial \alpha(a,t)}{\partial x} = 0 \qquad k'GA\left(\frac{\partial v(a,t)}{\partial x} - \alpha(a,t)\right) = 0; \tag{3.5}$$

(d) Sliding-end:

$$\alpha(a,t) = 0 \qquad \left(\frac{\partial v(a,t)}{\partial x} - \alpha(a,t)\right) = 0. \tag{3.6}$$

Remark 3. We will derive the spectral equations for the Timoshenko beam model with the following sets of the boundary conditions: (i) hinged-hinged conditions, (ii) hinged-clamped conditions, (iii) hinged-sliding conditions, (iv) hinged-free conditions. Any other combination of the boundary conditions can be treated in a similar fashion. In our approach below we rewrite the Timoshenko system (see (3.7) and (3.8) below) as a matrix equation (see (3.9) below) and study the spectral properties of the matrix differential operator $\mathcal{A}$ appearing at the right-hand side of (3.9) as shown below, the spectral equations for $\mathcal{A}$ depend whether the spectral parameter less than the critical value denoted by ω_c or greater than ω_c. We derive the spectral equations corresponding to both cases. One case (the hinged-hinged problem) coincides with the result found in the literature (see [1,2,3]) and the other cases have been derived in the present work. The method used here is a modification of the method used in paper [7].

Let us consider the homogeneous problem by setting the forcing function $f(x,t)$ in (3.1) to zero, and we assume that the cross-sectional area $A = 1$. Thus,

$$v_{tt}(x,t) - \frac{k'G}{\rho}\left(v_{xx}(xt) - \alpha_x(x,t)\right) = 0, \tag{3.7}$$

and

$$\alpha_{tt}(x,t) - \frac{1}{\rho I}\alpha_{xx}(x,t) - \frac{Gk'}{\rho I}\left(v_x(x,t) - \alpha(x,t)\right) = 0. \tag{3.8}$$

The corresponding homogeneous problem is

$$W_{tt}(x,t) = (\mathcal{A}W)(x,t), \tag{3.9}$$

where

$$W(x,t) = \begin{pmatrix} v(x,t) \\ \alpha(x,t) \end{pmatrix}, \tag{3.10}$$

and

$$\mathcal{A} = \begin{pmatrix} \frac{k'G}{\rho}\frac{d^2}{dx^2} & -\frac{k'G}{\rho}\frac{d}{dx} \\ \frac{k'G}{\rho I}\frac{d}{dx} & \frac{1}{\rho I}\frac{d^2}{dx^2} - \frac{k'G}{\rho I} \end{pmatrix}. \tag{3.11}$$

The eigenvalue-eigenfunction equation for the operator $\mathcal{A}$ can be presented as

$$\mathcal{A}W = \lambda W, \tag{3.12}$$

where, $W(x)$ is given in (3.10). Applying the method of separation of variables, we obtain the system of spatial equations

$$\frac{k'G}{\rho}v''(x) - \frac{k'G}{\rho}\alpha'(x) = \lambda v(x), \tag{3.13}$$

$$\frac{1}{\rho I}\alpha''(x) + \frac{k'G}{\rho I}v'(x) - \frac{k'G}{\rho I}\alpha(x) = \lambda\alpha(x). \tag{3.14}$$

Let us eliminate $\alpha(x)$ from the system by differentiating (3.14) once with respect to x:

$$\frac{1}{\rho I}\alpha'''(x) + \frac{k'G}{\rho I}v''(x) - \frac{k'G}{\rho I}\alpha'(x) = \lambda\alpha'(x). \tag{3.15}$$

From (3.13) we get

$$\alpha'(x) = v''(x) - \frac{\lambda\rho}{k'G}v(x). \tag{3.16}$$

Let us differentiate (3.16) two more times to get

$$\alpha'''(x) = v^{(4)}(x) - \frac{\lambda\rho}{k'G}v''(x). \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), we obtain

$$\frac{1}{\rho I}v^{(4)}(x) - \left(\frac{\lambda}{Ik'G} + \lambda\right)v''(x) + \left(\frac{\lambda}{I} + \frac{\lambda^2\rho}{k'G}\right)v(x) = 0. \tag{3.18}$$

Now, we derive a basis for the linear space of solutions of (3.18). The characteristic equation can be written as

$$\frac{1}{\rho I}r^4 - \left(\frac{\lambda}{Ik'G} + \lambda\right)r^2 + \left(\frac{\lambda}{I} + \frac{\lambda^2\rho}{k'G}\right) = 0. \tag{3.19}$$

Substituting $r^2 = y$, we obtain a quadratic equation for y:

$$y^2 - \left(\frac{\lambda\rho}{k'G} + \lambda\rho I\right) y + \left(\lambda\rho + \frac{\lambda^2\rho^2 I}{k'G}\right) = 0. \tag{3.20}$$

The solutions of this quadratic equation are given by

$$y_{1,2} = \frac{1}{2}\left(\frac{\lambda\rho}{k'G} + \lambda\rho I\right) \pm \sqrt{\frac{1}{4}\left(\frac{\lambda\rho}{k'G} + \lambda\rho I\right)^2 - \left(\lambda\rho + \frac{\lambda^2\rho^2 I}{k'G}\right)}, \tag{3.21}$$

which implies

$$r_{1,2} = \pm\sqrt{y_{1,2}} = \pm\sqrt{\frac{\lambda\rho}{2}\left(\frac{1}{k'G} + I\right) \pm \sqrt{\frac{\lambda^2\rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 - \lambda\rho}}. \tag{3.22}$$

Let us introduce a new variable ω where $\lambda = -\omega^2$. Substituting ω into (3.22), we obtain

$$r_{1,2} = \pm\sqrt{-\frac{\omega^2\rho}{2}\left(\frac{1}{k'G} + I\right) \pm \sqrt{\frac{\omega^4\rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 + \omega^2\rho}}. \tag{3.23}$$

There exist four different roots to (3.23). The two purely imaginary roots are given by

$$r_{1,2} = \pm i\sqrt{\frac{\omega^2\rho}{2}\left(\frac{1}{k'G} + I\right) + \sqrt{\frac{\omega^4\rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 + \omega^2\rho}}. \tag{3.24}$$

Let us denote $r_{1,2} = \pm i\gamma$. The other two roots are given by

$$r_{3,4} = \pm\sqrt{-\frac{\omega^2\rho}{2}\left(\frac{1}{k'G} + I\right) + \sqrt{\frac{\omega^4\rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 + \omega^2\rho}}. \tag{3.25}$$

These two roots are either both real or both imaginary depending on ω. If ω is such that

$r_{3,4}^2 \geq 0$, then $r_{3,4}$ are real. However, if ω is such that $r_{3,4}^2 < 0$, then $r_{3,4}$ are purely imaginary.

Let us find ω_c (the critical value of ω) for which expression (3.25) is equal to zero. We have

$$\left(I + \frac{1}{k'G}\right)\frac{\rho\omega^2}{2} = \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega^4}{4} + \rho\omega^2}, \tag{3.26}$$

which yields

$$\omega_c = \sqrt{\frac{k'G}{\rho I}}. \tag{3.27}$$

If $\omega > \omega_c$, then

$$0 > -\left(I + \frac{1}{kG}\right)\frac{\rho\omega^2}{2} + \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega^4}{4} + \rho\omega^2}. \tag{3.28}$$

This implies that

$$r_{3,4} = \pm i\sqrt{\left(I + \frac{1}{kG}\right)\frac{\rho\omega^2}{2} - \sqrt{\left(I - \frac{1}{kG}\right)\frac{\rho^2\omega^4}{4} + \rho\omega^2}}. \tag{3.29}$$

Let us denote $r_{3,4} = \pm i\widetilde{\beta}$. However, if $\omega < \omega_c$, then

$$0 < -\left(I + \frac{1}{kG}\right)\frac{\rho\omega^2}{2} + \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega^4}{4} + \rho\omega^2}. \tag{3.30}$$

In other words,

$$r_{3,4} = \pm\sqrt{-\left(I + \frac{1}{kG}\right)\frac{\rho\omega^2}{2} + \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega^4}{4} + \rho\omega^2}}. \tag{3.31}$$

In this scenario, $r_{3,4} = \pm\beta$. Then the general solutions for the two spatial equation cases can be written in the following form:

1. For $\omega > \omega_c$, we have

$$v(x) = \widetilde{A}(\omega)\sin(\gamma x) + \widetilde{B}(\omega)\cos(\gamma x) + \widetilde{C}(\omega)\sin(\widetilde{\beta}x) + \widetilde{D}(\omega)\cos(\widetilde{\beta}x). \tag{3.32}$$

2. For $\omega < \omega_c$, we have

$$v(x) = A(\omega)\sin(\gamma x) + B(\omega)\cos(\gamma x) + C(\omega)\sinh(\beta x) + D(\omega)\cosh(\beta x). \tag{3.33}$$

The Boundary Condition

Let us rewrite the boundary condition for the Timoshenko model in terms of $v(x)$ only. From (3.16), we know

$$\alpha'(x) = v''(x) + \frac{\rho\omega^2}{k'G}v(x), \tag{3.34}$$

thus,

$$\alpha''(x) = v'''(x) + \frac{\rho\omega^2}{k'G}v'(x). \tag{3.35}$$

By substituting $\alpha''(x)$ into (3.14), we can represent α as

$$\alpha''(x) + k'Gv'(x) - k'G\alpha(x) = -\omega^2\rho I\alpha(x), \tag{3.36}$$

and obtain a formula for α in terms of $v(x)$

$$\alpha(x) = \frac{1}{k'G - \omega^2\rho I}v'''(x) + \frac{k^2G^2 + \omega^2\rho}{k'G(k'G - \omega^2\rho I)}v'(x). \tag{3.37}$$

Let us assume the left end of beam model is hinged, and use different boundary conditions for the right end. Using (3.3), we represent the left end conditions in terms of v

$$v''(0) - \frac{\rho\omega^2}{k'G}v(0) = 0, \qquad v(0) = 0. \tag{3.38}$$

Case 1: $\omega > \omega_c$

The general solution is given in the form (3.32)

$$v(x) = \widetilde{A}(\omega)\sin(\gamma x) + \widetilde{B}(\omega)\cos(\gamma x) + \widetilde{C}(\omega)\sin(\widetilde{\beta}x) + \widetilde{D}(\omega)\cos(\widetilde{\beta}x). \tag{3.39}$$

where from (3.24)

$$\gamma = \sqrt{\frac{\omega^2\rho}{2}\left(\frac{1}{k'G} + I\right) + \sqrt{\frac{\omega^4\rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 + \omega^2\rho}}, \tag{3.40}$$

and from (3.29)

$$\widetilde{\beta} = \sqrt{\left(I + \frac{1}{k'G}\right)\frac{\rho\omega^2}{2} - \sqrt{\left(I - \frac{1}{k'G}\right)^2\frac{\rho^2\omega^4}{4} + \rho\omega^2}}. \tag{3.41}$$

Applying the boundary conditions, we get

$$\widetilde{B}(\omega) + \widetilde{C}(\omega) = 0, \qquad \gamma^2\widetilde{B}(\omega) + \widetilde{\beta}^2\widetilde{D}(\omega) = 0. \tag{3.42}$$

The determinant of the system is

$$\begin{aligned}\widetilde{\beta}^2-\gamma^2=&\frac{\omega^2\rho}{2}\left(\frac{1}{k'G}+I\right)-\sqrt{\frac{\omega^4\rho^2}{4}\left(I-\frac{1}{k'G}\right)^2+\omega^2\rho}\\&-\frac{\omega^2\rho}{2}\left(\frac{1}{k'G}+I\right)-\sqrt{\frac{\omega^4\rho^2}{4}\left(I-\frac{1}{k'G}\right)^2+\omega^2\rho}\\=&\,2\sqrt{\frac{\omega^4\rho^2}{4}\left(I-\frac{1}{k'G}\right)^2+\omega^2\rho}\neq 0.\end{aligned}\tag{3.43}$$

Thus, $\widetilde{B}(\omega)=\widetilde{D}(\omega)=0$. Therefore, the general solution is

$$v(x)=\widetilde{A}(\omega)\sin(\gamma x)+\widetilde{C}(\omega)\sin(\widetilde{\beta}x).\tag{3.44}$$

Hinged-Hinged Boundary Conditions

Applying the hinged conditions at the right end, we obtain a system for the coefficients:

$$\widetilde{A}(\omega)\sin(\gamma)+\widetilde{C}(\omega)\sin(\widetilde{\beta})=0,\tag{3.45}$$

$$\gamma^2\widetilde{A}(\omega)\sin(\gamma x)+\beta^2\widetilde{C}(\omega)\sin(\widetilde{\beta}x)=0.\tag{3.46}$$

This system is solvable if and only if its determinant is zero, i.e.,

$$\det\begin{pmatrix}\sin(\gamma) & \sin(\widetilde{\beta})\\ \gamma^2\sin(\gamma) & \widetilde{\beta^2}\sin(\widetilde{\beta})\end{pmatrix}=(\widetilde{\beta}^2-\gamma^2)\sin(\widetilde{\beta})\sin(\alpha)=0.\tag{3.47}$$

Since $(\beta^2 - \gamma^2)$ is non-zero, the spectral equation for hinged-hinged model is

$$\boxed{\sin(\widetilde{\beta})\sin(\gamma) = 0.} \tag{3.48}$$

Hinged-Clamped Boundary Conditions

The clamped-end conditions are given by (3.4).

Eliminating α, we obtain the following conditions:

$$v'''(1) + \frac{k^2G^2 + \omega^2\rho}{k'G}v'(1) = 0, \qquad v(1) = 0. \tag{3.49}$$

From the hinged-end conditions, we have the general solution in the form (3.44). Applying the clamped end boundary conditions to the right hand side we get

$$\widetilde{A}(\omega)\sin(\gamma) + \widetilde{C}(\omega)\sin(\widetilde{\beta}) = 0, \tag{3.50}$$

$$\widetilde{A}(\omega)\gamma\zeta\cos(\gamma) + \widetilde{C}(\omega)\widetilde{\beta}\widetilde{\wp}\cos(\widetilde{\beta}) = 0. \tag{3.51}$$

where

$$\widetilde{\wp} = \left(-\widetilde{\beta}^2 + k'G + \frac{\rho\omega^2}{k'G}\right) \text{ and } \zeta = \left(-\gamma^2 + k'G + \frac{\rho\omega^2}{k'G}\right). \tag{3.52}$$

System (3.50) and (3.51) is solvable if and only if its determinant is zero, i.e.,

$$\det\begin{pmatrix} \sin(\gamma) & \sin(\widetilde{\beta}) \\ \gamma\zeta\cos(\gamma) & \widetilde{\beta}\widetilde{\wp}\cos(\widetilde{\beta}) \end{pmatrix} = \widetilde{\beta}\widetilde{\wp}\cos(\widetilde{\beta})\sin(\gamma) - \gamma\zeta\cos(\gamma)\sin(\widetilde{\beta}) = 0. \tag{3.53}$$

Therefore, we can conclude that the spectral equation for the hinged-clamped model is

$$\boxed{\widetilde{\beta}\widetilde{\wp}\cos(\widetilde{\beta})\sin(\gamma) - \gamma\zeta\cos(\gamma)\sin(\widetilde{\beta}) = 0.} \tag{3.54}$$

Hinged-Sliding Boundary Conditions

The general solution is given by (3.4), and

the sliding boundary condition can be written as:

$$\alpha(1) = 0, \qquad v'(1) = 0. \tag{3.55}$$

Let us rewrite these boundary conditions in term of $v(x)$:

$$v'''(1) + \frac{k^2G^2 + \omega^2\rho}{k'G}v'(1) = 0, \qquad v'(1) = 0. \tag{3.56}$$

Applying the boundary conditions, we obtain the following system:

$$\gamma\zeta\widetilde{A}(\omega)\cos(\gamma) + \widetilde{\beta}\widetilde{\wp}\widetilde{C}(\omega)\cos(\widetilde{\beta}) = 0, \tag{3.57}$$

$$\gamma\widetilde{A}(\omega)\cos(\gamma) + \widetilde{\beta}\widetilde{C}(\omega)\cos(\widetilde{\beta}) = 0, \tag{3.58}$$

where ζ and $\widetilde{\wp}$ are defined in (3.52). The determinant for the system is:

$$\det\begin{pmatrix} \gamma\zeta\cos(\gamma) & \widetilde{\beta}\widetilde{\wp}\cos(\widetilde{\beta}) \\ \gamma\cos(\gamma) & \widetilde{\beta}\cos(\widetilde{\beta}) \end{pmatrix} = \widetilde{\beta}\gamma\zeta\cos(\gamma)\cos(\widetilde{\beta}) - \widetilde{\beta}\gamma\widetilde{\wp}\cos(\gamma)\cos(\widetilde{\beta}) = 0. \tag{3.59}$$

Therefore the spectral equation for the hinged-sliding beam model is

$$\boxed{\cos(\widetilde{\beta})\cos(\gamma) = 0.} \tag{3.60}$$

Hinged-Free Boundary Conditions

Finally, we consider the hinged-free model and derive the spectral equation for this case. The conditions are given by

$$\alpha'(1) = 0 \qquad k'G(v'(1) - \alpha(1)) = 0. \tag{3.61}$$

These conditions can be written in terms of $v(x)$ as follows:

$$\begin{gathered} v''(1) + \frac{\omega^2 \rho}{k'G} v(1) = 0, \\ \frac{1}{k'G - \omega^2 \rho I} v'''(1) + \left(\frac{k^2 G^2 + \omega^2 \rho}{k'G} - 1 \right) v'(1) = 0. \end{gathered} \tag{3.62}$$

The general solution is given by (3.4). Applying the free end conditions, we obtain the system whose determinant can be presented as:

$$\det \begin{pmatrix} \left(\frac{\rho\omega^2}{k'G} - \gamma^2\right)\sin(\gamma) & \left(\frac{\rho\omega^2}{k'G} - \widetilde{\beta^2}\right)\sin(\widetilde{\beta}) \\ \gamma \Upsilon \cos(\gamma) & \widetilde{\beta}\widetilde{\Psi}\cos(\widetilde{\beta}) \end{pmatrix}, \tag{3.63}$$

where

$$\begin{aligned} \Upsilon &= \left(\frac{-\gamma^2}{k'G - \omega^2 \rho I} + k'G + \frac{\rho\omega^2}{k'G} - 1 \right) \\ \widetilde{\Psi} &= \left(\frac{-\widetilde{\beta}^2}{k'G - \omega^2 \rho I} + k'G + \frac{\rho\omega^2}{k'G} - 1 \right). \end{aligned} \tag{3.64}$$

The spectral equation for the hinged-free beam model can be represented in the form:

$$\boxed{\widetilde{\beta}\widetilde{\Psi} \left(\frac{\rho\omega^2}{k'G} - \gamma^2 \right) \sin(\gamma)\cos(\widetilde{\beta}) - \gamma\Upsilon \left(\frac{\rho\omega^2}{k'G} - \widetilde{\beta^2} \right) \sin(\widetilde{\beta})\cos(\gamma) = 0.} \tag{3.65}$$

All of the result for this case are collected in Table 3.

Boundary conditions	Spectral Equations
Hinged-hinged	$\sin(\widetilde{\beta})\sin(\gamma) = 0$
Hinged-clamped	$\widetilde{\beta}\widetilde{\wp}\cos(\widetilde{\beta})\sin(\gamma) - \gamma\zeta\cos(\gamma)\sin(\widetilde{\beta}) = 0$
Hinged-sliding	$\cos(\widetilde{\beta})\cos(\gamma) = 0$
Hinged-free	$\widetilde{\beta}\widetilde{\Psi}\left(\frac{\rho\omega^2}{k'G} - \gamma^2\right)\sin(\gamma)\cos(\widetilde{\beta}) - \gamma\Upsilon\left(\frac{\rho\omega^2}{k'G} - \widetilde{\beta^2}\right)\sin(\widetilde{\beta})\cos(\gamma) = 0$

Table 3: Spectral equations of different combinations of the boundary conditions for the Timoshenko model for case 1 where $\omega > \omega_c$.

Case 2 : $\omega < \omega_c$

We will find the spectral equations for the boundary conditions when $\omega < \omega_c$. In this case, the general solution is presented as in (3.33), i.e.,

$$v(x) = A(\omega)\sin(\gamma x) + B(\omega)\cos(\gamma x) + C(\omega)\sinh(\beta x) + D(\omega)\cosh(\beta x),$$

where

$$\beta = \sqrt{-\left(I + \frac{1}{kG}\right)\frac{\rho\omega^2}{2} + \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega^4}{4} + \rho\omega^2}}, \tag{3.66}$$

and γ are defined in (3.40). We study the same four sets of the boundary conditions: hinged-hinged, hinged-clamped, hinged-sliding, and hinged-free. Using (3.34) and (3.37), in (3.38), we rewrite the left end (hinged-end) boundary conditions in term of $v(x)$, and have

$$v''(0) - \frac{\rho\omega^2}{k'G}v(0) = 0 \qquad v(0) = 0. \tag{3.67}$$

Applying the boundary conditions we get

$$B(\omega) + D(\omega) = 0, \qquad -B(\omega)\gamma^2 + D(\omega)\beta^2 = 0. \tag{3.68}$$

Which implied that $B(\omega) = 0$ and $D(\omega) = 0$. Therefore, the general solution is

$$v(x) = A(\omega)\sin(\gamma x) + C(\omega)\sinh(\beta x). \tag{3.69}$$

Hinged-Hinged Boundary Conditions

Applying the hinged conditions at the right end, we obtain a system for the coefficients:

$$A(\omega)\sin(\gamma) + C(\omega)\sinh(\beta) = 0, \tag{3.70}$$

$$-A(\omega)\left(\gamma^2 - \frac{\rho\omega^2}{k'G}\right)\sin(\gamma) + C(\omega)\left(\beta^2 - \frac{\rho\omega^2}{k'G}\right)\sinh(\beta) = 0. \tag{3.71}$$

This system is solvable if and only if it determinant is equal to zero, i.e.,

$$\det\begin{pmatrix} \sin(\gamma) & \sinh(\beta) \\ \left(-\gamma^2 - \frac{\rho\omega^2}{k'G}\right)\sin(\gamma) & \left(\beta^2 - \frac{\rho\omega^2}{k'G}\right)\sinh(\beta) \end{pmatrix} = (\beta^2+\gamma^2)\sin(\gamma)\sinh(\beta) = 0. \tag{3.72}$$

Thus, the spectral equation for the hinged-hinged model is

$$\boxed{\sin(\gamma)\sinh(\beta) = 0.} \tag{3.73}$$

Hinged-Clamped Boundary Conditions

The clamped-end conditions are given in (3.49). From the hinged-end conditions, we have the general solution in the form (3.69). Applying the clamped-end boundary conditions, we obtain the system

$$A(\omega)\sin(\gamma) + C(\omega)\sinh(\beta) = 0, \tag{3.74}$$

$$A(\omega)\gamma\left(-\gamma^2 + k'g + \frac{\rho\omega^2}{k'G}\right)\cos(\gamma) + C(\omega)\beta\left(\beta^2 + k'g + \frac{\rho\omega^2}{k'G}\right)\cosh(\beta) = 0. \tag{3.75}$$

The system is solvable if and only if its determinant is equal to zero, i.e.,

$$\det\begin{pmatrix} \sin(\gamma) & \sinh(\beta) \\ \gamma\zeta\cos(\gamma) & \beta\wp\cosh(\beta) \end{pmatrix} = \beta\wp\sin(\gamma)\cosh(\beta) - \gamma\zeta\cos(\gamma)\sinh(\beta) = 0, \tag{3.76}$$

where ζ is the same as in (3.52) and repeated here,

$$\zeta = \left(-\gamma^2 + k'g + \frac{\rho\omega^2}{k'G}\right), \qquad \wp = \left(\beta^2 + k'g + \frac{\rho\omega^2}{k'G}\right). \tag{3.77}$$

Thus, the spectral equation for the hinged-clamped boundary conditions model is

$$\boxed{\beta\wp\sin(\gamma)\cosh(\beta) - \gamma\zeta\cos(\gamma)\sinh(\beta) = 0.} \tag{3.78}$$

Hinged-Sliding Boundary Conditions

The general solution is given in (3.69). The sliding boundary conditions written in term of $v(x)$ are given by (3.56). Applying the boundary conditions, we obtain the following system of equations:

$$A(\omega)\gamma\zeta\cos(\gamma) + C(\omega)\beta\wp\cosh(\beta) = 0, \tag{3.79}$$

$$A(\omega)\gamma\cos(\gamma) + C(\omega)\beta\cosh(\beta) = 0, \tag{3.80}$$

where ζ and $\wp$ are defined in (3.77). The determinant for the system is.

$$\det\begin{pmatrix} \gamma\zeta\cos(\gamma) & \beta\wp\cosh(\beta) \\ \gamma\cos(\gamma) & \beta\cosh(\beta) \end{pmatrix} = \beta\gamma\zeta\cos(\gamma)\cosh(\beta) - \beta\wp\gamma\cos(\gamma)\cosh(\beta) = 0. \tag{3.81}$$

Therefore the spectral equation for the hinged-clamped beam model is

$$\boxed{\cosh(\beta)\cos(\gamma) = 0.} \tag{3.82}$$

Hinged-Free Boundary Conditions

Finally, we consider the hinged-free model. The conditions are given by (3.62). The general solution is given by (3.69). Applying the free-end conditions, we obtain the system which has the determinant

$$\det \begin{pmatrix} \left(-\gamma^2 + \frac{\rho\omega^2}{k'G}\right)\sin(\gamma) & \left(\beta^2 + \frac{\omega^2\rho}{k'G}\right)\sinh(\beta) \\ \gamma\Upsilon\cos(\gamma) & \beta\Psi\cosh(\beta) \end{pmatrix} = 0, \tag{3.83}$$

where Υ is the same as in (3.64), and

$$\Psi = \left(\frac{\beta^2}{k'G - \omega^2\rho I} + k'G + \frac{\omega^2\rho}{k'G} - 1\right).$$

The spectral equation for the hinged-free beam model can be given as follows:

$$\boxed{\beta\Psi\left(-\gamma^2 + \frac{\rho\omega^2}{k'G}\right)\sin(\gamma)\cosh(\beta) - \gamma\Upsilon\left(\beta^2 + \frac{\omega^2\rho}{k'G}\right)\sinh(\beta)\cos(\gamma) = 0.} \tag{3.84}$$

All of the result for when $\omega_c < \omega$ are collected in Table 4.

Boundary conditions	Spectral Equations
Hinged-hinged	$\sin(\gamma)\sinh(\beta) = 0$
Hinged-clamped	$\beta\wp\sin(\gamma)\cosh(\beta) - \gamma\zeta\cos(\gamma)\sinh(\beta) = 0$
Hinged-sliding	$\cos(\gamma)\cosh(\beta) = 0$
Hinged-free	$\beta\Psi\left(-\gamma^2 + \frac{\rho\omega^2}{k'G}\right)\sin(\gamma)\cosh(\beta) - \gamma\Upsilon\left(\beta^2 + \frac{\omega^2\rho}{k'G}\right)\sinh(\beta)\cos(\gamma) = 0.$

Table 4: Spectral equations of different combinations of the boundary conditions for the Timoshenko model for case 2 where $\omega < \omega_c$.

Asymptotic Approximations for the Eigenvalues of the Boundary Value Problems

In this section, we derive the asymptotic approximations [6 - 8] for the eigenvalues of the beam models. These beam models include the Euler-Bernoulli beam model, the Rayleigh beam model, and the Timoshenko beam model. For the Euler-Bernoulli beam model, we obtain the formulas for the eigenvalues corresponding to hinged-hinged, hinged-sliding and clamped-clamped boundary conditions. For the Rayleigh beam model, we obtain the formulas for the eigenvalues corresponding to the hinged-hinged and clamped-clamped boundary conditions. For the Timoshenko beam model, we obtain the formulas for the eigenvalues corresponding to hinged-hinged and hinged-sliding boundary conditions for the two different cases, i.e., $\omega > \omega_c$ and $\omega < \omega_c$.

4 Euler-Bernoulli Model

Since we consider only conservative boundary conditions, the variable of separation in (1.9) must be real and positive.

Spectral Asymptotics for Hinged-Hinged Conditions

The spectral equation corresponding to (1.25) is given by

$$\sin(\lambda L)\sinh(\lambda L) = 0.$$

This equation means that either $\sin(\lambda L) = 0$ or $\sinh(\lambda L) = 0$. Since $\sinh(\lambda L) = 0$ at only one point, when $\lambda = 0$, then $\sin(\lambda L) = 0$. This implies the following formula for the eigenvalues:

$$\boxed{\lambda_n = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots} \tag{4.1}$$

Spectral Asymptotics for Hinged-Sliding Conditions

The spectral equation for the hinged-sliding conditions (1.46) can be represented as

$$\cosh(\lambda L)\cos(\lambda L) = 0.$$

This equation means that either $\cosh(\lambda L) = 0$ or $\cos(\lambda L) = 0$. However, $\cosh(\lambda L)$ can never be equal to zero; therefore, the solutions for (1.46) is given by $\cos(\lambda L) = 0$, which yields

$$\boxed{\lambda_n = \frac{(2n+1)\pi}{2L}, \qquad n = 1, 2, \ldots} \tag{4.2}$$

Remark 4. As it can be easily seem from Tables 1 - 4, there are two spectral equations in Table 1 and one spectral equation in Table 2 that allows closed from solution since in each case, the spectrum coincides with the set of roots of a certain trigonometric function. In the remaining cases of Tables 1 and 2 and in all cases of Tables 3 and 4, we have to use the methods of asymptotic analysis (see, e.g.[8]) to derive the asymptotic approximations of the eigenvalues as the number of an eigenvalue tends to ∞,

Spectral Asymptotics for Clamped-Free Conditions

Let us consider the spectral equation for the clamped-free boundary conditions. The spectral equation is as shown in Table 1:

$$\cosh(\lambda L)\cos(\lambda L) = -1. \tag{4.3}$$

Let us find the asymptotic distribution of the roots of (4.3). Let $\lambda > 0$, then using the following definitions, we have

$$\cosh(\lambda L) = \frac{1}{2}\left(e^{\lambda L} + e^{-\lambda L}\right), \qquad \cos(\lambda L) = \frac{1}{2}\left(e^{i\lambda L} + e^{-i\lambda L}\right). \tag{4.4}$$

When $\lambda \to \infty$, we have

$$\cosh(\lambda L) = \frac{1}{2}e^{\lambda L} + \mathcal{O}(e^{-\lambda L}), \tag{4.5}$$

where $\mathcal{O}(e^{-\lambda L})$ is the remainder term, which decays exponentially fast as $\lambda \to \infty$. Let us rewrite (4.3) using the above definitions:

$$\begin{aligned} \cos(\lambda L) &= -\frac{1}{\cosh(\lambda L)} \\ &= -\frac{1}{\frac{1}{2}e^{\lambda L}\left(1 + \mathcal{O}(e^{-2\lambda L})\right)} \\ &= -\frac{2e^{-\lambda L}}{1 + \mathcal{O}(e^{-2\lambda L})}. \end{aligned} \tag{4.6}$$

Consider the term $\frac{1}{1+\mathcal{O}(e^{-2\lambda L})}$; using the formula for geometric series we can represent it as

$$\begin{aligned}\frac{1}{1+\mathcal{O}(e^{-2\lambda L})} &= 1+\mathcal{O}(e^{-2\lambda L})+\mathcal{O}(e^{-4\lambda L})+\cdots \\ &= 1+\mathcal{O}(e^{-2\lambda L}).\end{aligned} \tag{4.7}$$

Therefore,

$$\cos\lambda L = -2(e^{-2\lambda L})\left(1+\mathcal{O}(e^{-2\lambda L})\right). \tag{4.8}$$

We can see that $\cos(\lambda L)$ will approach zero as $\lambda \to \infty$. Therefore, we can model this behavior using the *model equation*, defined as

$$\cos(\lambda L) = 0, \tag{4.9}$$

whose solutions, denoted as λ_n^0, are

$$\lambda_n^0 = \frac{(2n+1)\pi}{2L}, \qquad n = 0, 1, \ldots \tag{4.10}$$

The solution of (4.8) can be given in the form

$$\lambda_n = \frac{(2n+1)\pi}{2L} + \mathcal{O}\left(e^{-2\lambda_n L}\right). \tag{4.11}$$

Notice,

$$e^{-2\lambda_n L} \approx e^{-\frac{(2n+1)\pi}{2}} = e^{-n}e^{-\frac{\pi}{2}}.$$

The final formula for the asymptotic approximations of the eigenvalues as $n \to \infty$ is

$$\boxed{\lambda_n = \frac{(2n+1)\pi}{2L} + \mathcal{O}\left(e^{-n}\right).} \tag{4.12}$$

5 Rayleigh Beam Model

Spectral Asymptotic for Hinged-Hinged Condition

Let us derive the asymptotic approximations for the eigenvalues corresponding to hinged-hinged condition (2.38). The spectral equation is

$$\sin\alpha \sinh\beta = 0, \tag{5.1}$$

where α and β are given by

$$\alpha^2 = \frac{\lambda c^2 + \sqrt{\lambda^2 c^4 + 4\lambda}}{2} \quad \text{and} \quad \beta^2 = \frac{\sqrt{\lambda^2 c^4 + 4\lambda} - \lambda c^2}{2}. \tag{5.2}$$

This implies that either $\sinh(\beta) = 0$, or $\sin(\alpha) = 0$. For the former, $\sinh\beta = 0$ if and only if $\beta = 0$, and we obtain $\lambda = 0$. For the latter, $\sin(\alpha) = 0$ we obtain the infinite sequence of solutions.

$$\alpha_n = n\pi, \qquad n = 1, 2, 3 \cdots. \tag{5.3}$$

Let us solve for λ_n by substituting (5.3) into α in (5.2). We obtain

$$2(n\pi)^2 = \lambda_n c^2 + \sqrt{\lambda_n^2 c^4 + 4\lambda_n},$$

which reduces to

$$\lambda_n^2 c^4 = \left(2(n\pi)^2 - \lambda_n c^2\right)^2.$$

The solutions to which are

$$\boxed{\lambda_n = \frac{(n\pi)^4}{1+(n\pi c)^2}; \qquad n = 1, 2, \ldots} \tag{5.4}$$

Spectral Asymptotics for Clamped-Clamped Conditions

Let us derive the spectral asymptotics for the clamped-clamped model. The spectral equation from (2.29) is

$$2\alpha\beta + (\beta^2 - \alpha^2)\sin(\alpha)\sinh(\beta) - 2\alpha\beta\cos(\alpha)\cosh(\beta) = 0, \tag{5.5}$$

where α and β are given in (5.2). Let $\lambda \to \infty$. Using the binomial formula, we have

$$\begin{aligned}
\sqrt{\lambda^2 c^4 + 4\lambda} &= \lambda c^2 \left(1 + \frac{4}{\lambda c^4}\right)^{\frac{1}{2}} \\
&= \lambda c^2 \left(1 + \frac{2}{\lambda c^4} - \frac{1}{8}\left(\frac{4}{\lambda c^4}\right)^2\right) + \mathcal{O}\left(\frac{1}{\lambda^3}\right) \\
&= \lambda c^2 + \frac{2}{c^2} - \frac{2\lambda c^2}{\lambda^2 c^8} + \mathcal{O}\left(\frac{1}{\lambda}\right) \\
&= \lambda c^2 + \frac{2}{c^2} + \mathcal{O}\left(\frac{1}{\lambda}\right).
\end{aligned} \tag{5.6}$$

From (5.2) we get

$$\beta^2 - \alpha^2 = -\lambda c^2. \tag{5.7}$$

Using (5.6) we obtain

$$\beta^2 = \frac{1}{c^2} + \mathcal{O}\left(\frac{1}{\lambda}\right), \tag{5.8}$$

$$\alpha^2 = \lambda c^2 + \frac{1}{c^2} + \mathcal{O}\left(\frac{1}{\lambda}\right), \tag{5.9}$$

By (5.8), we have

$$\beta = \frac{1}{c} + \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{5.10}$$

By (5.9), we have

$$\begin{aligned}
\alpha &= \left(\lambda c^2 + \frac{1}{c^2} + \mathcal{O}\left(\frac{1}{\lambda}\right)\right)^{\frac{1}{2}} \\
&= \sqrt{\lambda} c \left(1 + \frac{1}{\lambda c^4} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right)^{\frac{1}{2}} \\
&= \sqrt{\lambda} c + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right).
\end{aligned} \tag{5.11}$$

Thus, (5.10) and (5.11) yield,

$$\alpha\beta = \sqrt{\lambda}. \tag{5.12}$$

Next, let us evaluate the asymptotic approximation for $\sin(\alpha)$, $\cos(\alpha)$, $\sinh(\beta)$, and $\cosh(\beta)$. By using the approximation for α from (5.11), we obtain

$$\begin{aligned}
\sin(\alpha) &= \sin\left(\sqrt{\lambda} c + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) \\
&= \sin(\sqrt{\lambda} c) \cos\left(\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) + \cos(\sqrt{\lambda} c) \sin\left(\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right),
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}\cos(\alpha) &= \cos\left(\sqrt{\lambda}c + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) \\ &= \cos(\sqrt{\lambda}c)\cos\left(\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) - \sin(\sqrt{\lambda}c)\sin\left(\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right). \end{aligned} \tag{5.14}$$

When $\lambda \to \infty$, we can use the Taylor series to obtain

$$\begin{aligned}\sin\left(\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) &= \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \\ \cos\left(\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) &= 1 + \mathcal{O}\left(\frac{1}{\lambda}\right). \end{aligned} \tag{5.15}$$

Substituting (5.15) into (5.13) and (5.14), we can write:

$$\sin(\alpha) = \sin(\sqrt{\lambda}c) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \tag{5.16}$$

$$\cos(\alpha) = \cos(\sqrt{\lambda}c) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{5.17}$$

From (5.10) we get

$$\begin{aligned}\sinh(\beta) &= \sinh\left(\frac{1}{c} + \mathcal{O}\left(\frac{1}{\lambda}\right)\right) \\ &= \sinh\left(\frac{1}{c}\right) + \mathcal{O}\left(\frac{1}{\lambda}\right), \end{aligned} \tag{5.18}$$

and

$$\cosh(\beta) = \cosh\left(\frac{1}{c}\right) + \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{5.19}$$

Let us multiply (5.16) and (5.18) to obtain

$$\begin{aligned}\sin(\alpha)\sinh(\beta) &= \left(\sin(\sqrt{\lambda}c) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right)\left(\sinh\left(\frac{1}{c}\right) + \mathcal{O}\left(\frac{1}{\lambda}\right)\right) \\ &= \sin(\sqrt{\lambda}c)\sinh\left(\frac{1}{c}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned} \tag{5.20}$$

Also, multiplying (5.17) and (5.19), we have

$$\begin{aligned}\cos(\alpha)\cosh(\beta) &= \left(\cos(\sqrt{\lambda}c) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right)\left(\cosh\left(\frac{1}{c}\right) \mathcal{O}\left(\frac{1}{\lambda}\right)\right) \\ &= \cosh(\sqrt{\lambda}c)\cosh\left(\frac{1}{c}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \end{aligned} \tag{5.21}$$

Substituting (5.20) and (5.21) into (5.5) yields

$$2\sqrt{\lambda} - \lambda c^2\left(\sin(\sqrt{\lambda}c)\sinh\left(\frac{1}{c}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) - 2\lambda\left(\cosh(\sqrt{\lambda}c)\cosh\left(\frac{1}{c}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) = 0. \tag{5.22}$$

Let us divide (5.22) by 2λ to obtain

$$\frac{c^2}{2}\sin(\sqrt{\lambda}c)\sinh\left(\frac{1}{c}\right) + \cosh(\sqrt{\lambda}c)\cosh\left(\frac{1}{c}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{5.23}$$

Let

$$\mathcal{A} = \frac{c^2}{2}\sinh\left(\frac{1}{c}\right), \qquad \mathcal{B} = \cosh\left(\frac{1}{c}\right). \tag{5.24}$$

Then (5.23) can be written in the form

$$\mathcal{A}\sin(\sqrt{\lambda}c) + \mathcal{B}\cos(\sqrt{\lambda}c) = \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{5.25}$$

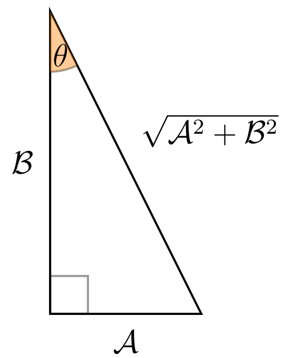

Figure 4: The triangle induced by (5.23) and (5.24).

By multipling (5.25) by $\frac{1}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}$, we get

$$\frac{\mathcal{A}}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}\sin(\sqrt{\lambda}c)+\frac{\mathcal{B}}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}\cos(\sqrt{\lambda}c)=\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{5.26}$$

Let us denote $\mathcal{D}=\frac{1}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}$, where$\mathcal{D}>0$. From Figure 4, we see that

$$\frac{\mathcal{A}}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}=\sin\theta,$$
$$\frac{\mathcal{B}}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}=\cos\theta.$$

Setting $\theta=\arctan\left(\frac{\mathcal{A}}{\mathcal{B}}\right)$, one can see that the model equation corresponding to (5.26) can be written as

$$\sin(\sqrt{\lambda}c)\sin(\theta)+\cos(\sqrt{\lambda}c)\cos(\theta)=\cos(\sqrt{\lambda}c-\theta)=0. \tag{5.27}$$

From (5.27) we derive that

$$\cos(\sqrt{\lambda}c-\theta)=0 \tag{5.28}$$

and

$$\sqrt{\lambda_n}c=\frac{(2n+1)\pi}{2}+\theta, \qquad n=0,1,2\cdots. \tag{5.29}$$

for $n = 0, 1, \ldots$. Thus, the asymptotic approximation [7,8] for the eigenvalues of the clamped-clamped model is

$$\boxed{\lambda_n = \left(\frac{2\pi n + \theta}{c}\right)^2 + \mathcal{O}\left(\frac{1}{n}\right), \qquad n \to \infty.} \tag{5.30}$$

6 Timoshenko Beam Model

Spectral Asymptotics for Hinged-Hinged Conditions when $\omega > \omega_c$

The corresponding spectral equation (see (3.48)) has the form

$$\sin(\widetilde{\beta})\sin(\gamma) = 0,$$

where $\widetilde{\beta}$ and γ are given by formulas (3.40) and (3.41). This implies that either $\sin(\widetilde{\beta}) = 0$ or $\sin(\gamma) = 0$. Equation $\sin(\widetilde{\beta}) = 0$, implies

$$\widetilde{\beta}_n = n\pi, \qquad n = 1, 2, \ldots \tag{6.1}$$

Substituting (6.1) into (3.41), we obtain

$$\sqrt{\left(I + \frac{1}{k'G}\right)\frac{\rho\omega_n^2}{2} - \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega_n^4}{4} + \rho\omega_n^2}} = n\pi, \tag{6.2}$$

or

$$\left(\left(I + \frac{1}{k'G}\right)\frac{\rho\omega_n^2}{2} - (n\pi)^2\right)^2 = \frac{\omega_n^4\rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 + \omega_n^2\rho. \tag{6.3}$$

Since $\lambda_n = -\omega_n^2$, (6.3) can be simplified to the form of the following quadratic equation:

$$\lambda_n^2 \frac{I\rho^2}{k'G} + \lambda_n \rho \left(1 + (n\pi)^2 \left(I + \frac{1}{k'G}\right)\right) + (n\pi)^4 = 0. \tag{6.4}$$

Let us denote the solutions for $\sin(\widetilde{\beta}) = 0$ by $\widetilde{\lambda_n}$. We obtain the following formula:

$$\boxed{\widetilde{\lambda_n} = \frac{-\rho\left(1 + (n\pi)^2\left(I + \frac{1}{k'G}\right)\right) \pm \sqrt{\rho^2\left(1 + (n\pi)^2\left(I + \frac{1}{k'G}\right)\right)^2 - 4\frac{I\rho^2}{k'G}(n\pi)^4}}{2\frac{\rho^2 I}{k'G}},} \tag{6.5}$$

for $n = 1, 2, \ldots$. Let us simplify the expression under the square root:

$$\sqrt{\rho^2\left(1 + 2(\pi n)^2\left(I + \frac{1}{k'G}\right) + (\pi n)^4\left(I + \frac{1}{k'G}\right)^2\right) - \frac{4I\rho^2}{k'G}(\pi n)^4}$$
$$= \rho\sqrt{1 + 2(\pi n)^2\left(I + \frac{1}{k'G}\right) + (\pi n)^4\left(I - \frac{1}{k'G}\right)^2}. \tag{6.6}$$

Thus, we see that the expression under the square root is always positive. Now, we consider the equation $\sin(\gamma) = 0$, and have

$$\gamma = m\pi; \qquad m = 1, 2, \ldots \tag{6.7}$$

which means that

$$(m\pi)^2 = \frac{\omega_m^2 \rho}{2}\left(\frac{1}{k'G} + I\right) + \sqrt{\frac{\omega_m^4 \rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 + \omega_m^2 \rho}. \tag{6.8}$$

Let us denote λ_m to be the solution for $\sin(\gamma) = 0$. After simplifying (6.8), we obtain λ_m is similar to $\widetilde{\lambda_n}$ in (6.5).

$$\boxed{\lambda_m = \frac{-\rho\left(1 + (m\pi)^2\left(I + \frac{1}{k'G}\right)\right) \pm \sqrt{\rho^2\left(1 + (m\pi)^2\left(I + \frac{1}{k'G}\right)\right)^2 - 4\frac{I\rho^2}{k'G}(m\pi)^4}}{2\frac{\rho^2 I}{k'G}},} \tag{6.9}$$

where $m = 1, 2, \ldots$

Spectral Asymptotics for Hinged-Sliding Conditions when $\omega > \omega_c$:

Let us find the asymptotic approximation for the solutions of (3.60), which is

$$\cos(\widetilde{\beta})\cos(\gamma) = 0.$$

This implies that

$$\cos(\widetilde{\beta}) = 0 \quad \text{and} \quad \cos(\gamma) = 0. \tag{6.10}$$

From (6.10), we obtain

$$\widetilde{\beta}_n = \frac{(2n+1)\pi}{2} \quad \text{and} \quad \gamma_n = \frac{(2m+1)\pi}{2}; \quad m, n = 0, 1, \ldots \tag{6.11}$$

Substituting $\widetilde{\beta}_n$ from (6.11) into (3.41), we get

$$\sqrt{\left(I + \frac{1}{k'G}\right)\frac{\rho\omega_n^2}{2} - \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega_n^4}{4} + \rho\omega_n^2}} = \frac{(2n+1)\pi}{2}. \tag{6.12}$$

For $\widetilde{\lambda_n} = -\omega_n^2$, we can simplify (6.12) and obtain the formula for the solutions of the equation $\cos(\widetilde{\beta}) = 0$, which is

$$\boxed{\widetilde{\lambda_n} = \frac{-\rho\left(1+\left(\frac{(2n+1)\pi}{2}\right)^2\left(I+\frac{1}{k'G}\right)\right) \pm \sqrt{\rho^2\left(1+\left(\frac{(2n+1)\pi}{2}\right)^2\left(I+\frac{1}{k'G}\right)\right)^2 - 4\frac{I\rho^2}{k'G}\left(\frac{(2n+1)\pi}{2}\right)^4}}{2\frac{\rho^2 I}{k'G}},} \tag{6.13}$$

where $n = 0, 1, \ldots$. Similarly, by substituting γ_n from (6.11) into (3.40), we obtain

$$\sqrt{\left(I+\frac{1}{k'G}\right)\frac{\rho\omega_m^2}{2} - \sqrt{\left(I-\frac{1}{k'G}\right)^2\frac{\rho^2\omega_m^4}{4} + \rho\omega_n^2}} = \frac{(2m+1)\pi}{2}, \tag{6.14}$$

Since $\lambda_m = -\omega_m^2$, we can simplify (6.14) and obtain the formula for the spectrum, which is

$$\boxed{\lambda_m = \frac{-\rho\left(1+\left(\frac{(2m+1)\pi}{2}\right)^2\left(I+\frac{1}{k'G}\right)\right) \pm \sqrt{\rho^2\left(1+\left(\frac{(2m+1)\pi}{2}\right)^2\left(I+\frac{1}{k'G}\right)\right)^2 - 4\frac{I\rho^2}{k'G}\left(\frac{(2m+1)\pi}{2}\right)^4}}{2\frac{\rho^2 I}{k'G}},} \tag{6.15}$$

where $m = 0, 1, \ldots$

Spectral Asymptotics For Hinged-Hinged Conditions when $\omega < \omega_c$:

The spectral equation corresponding to this case (see (3.73)) is

$$\sin(\gamma)\sinh(\beta) = 0.$$

This implies that either $\sinh(\beta) = 0$ or $\sin(\gamma) = 0$, where β and γ are defined in (3.66) and (3.40). For the former, $\sinh(\beta) = 0$ if and only if $\beta = 0$. Therefore,

$$\left(I + \frac{1}{k'G}\right)\frac{\rho\omega^2}{2} = \sqrt{\left(I - \frac{1}{k'G}\right)^2 \frac{\rho^2\omega^4}{4} + \rho\omega^2},$$

which simplifies to

$$\frac{\omega^4\rho^2 I}{k'G} - \rho\omega^2 = 0. \tag{6.16}$$

Let us denote the solutions for $\sinh(\beta) = 0$ to be $\widetilde{\lambda}$. Since $\widetilde{\lambda} = -\omega^2$, we obtain

$$\boxed{\widetilde{\lambda} = 0, \qquad \text{or} \qquad \widetilde{\lambda} = -\frac{k'G}{\rho I}.} \tag{6.17}$$

When $\sin(\gamma) = 0$, we have

$$\gamma = n\pi; \qquad n = 1, 2, \ldots \tag{6.18}$$

which implies that

$$(n\pi)^2 = \frac{\omega^2\rho}{2}\left(\frac{1}{k'G} + I\right) + \sqrt{\frac{\omega^4\rho^2}{4}\left(I - \frac{1}{k'G}\right)^2 + \omega^2\rho}. \tag{6.19}$$

Let us denote λ_n to be the solution in this case. After simplifying (6.19) we obtain

$$\boxed{\lambda_n = \frac{-\rho\left(1 + (n\pi)^2\left(I + \frac{1}{k'G}\right)\right) \pm \sqrt{\rho^2\left(1 + (n\pi)^2\left(I + \frac{1}{k'G}\right)\right)^2 - 4\frac{I\rho^2}{k'G}(n\pi)^4}}{2\frac{\rho^2 I}{k'G}},} \tag{6.20}$$

where $n = 1, 2, \ldots$

Spectral Asymptotics For Hinged-Sliding Conditions when $\omega < \omega_c$:

The spectral equation for (3.82) can be represented as

$$\cos(\gamma)\cosh(\beta) = 0.$$

Since $\cosh(\beta) \neq 0$, we have the spectral equation of $\cos(\gamma) = 0$. The set of solutions can be given in the form

$$\lambda_m = \frac{(2n+1)\pi}{2}, \qquad m = 0, 1, \ldots \tag{6.21}$$

Substituting (6.21) into (3.40), we obtain

$$\boxed{\lambda_m = \frac{-\rho\left(1+\left(\frac{(2m+1)\pi}{2}\right)^2\left(I+\frac{1}{k'G}\right)\right) \pm \sqrt{\rho^2\left(1+\left(\frac{(2m+1)\pi}{2}\right)^2\left(I+\frac{1}{k'G}\right)\right)^2 - 4\frac{I\rho^2}{k'G}\left(\frac{(2m+1)\pi}{2}\right)^4}}{2\frac{\rho^2 I}{k'G}}} \tag{6.22}$$

where $m = 0, 1, \ldots$

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