STATIONARY AND NONSTATIONARY RANDOM VIBRATIONS OF CABLES, PLATES AND BEAMS WITH AN APPLICATION TO OCEAN STRUCTURES

FRANK PHILIP ALBERTI JR.

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STATIONARY AND NONSTATIONARY RANDOM VIBRATIONS OF CABLES, PLATES AND BEAMS WITH AN APPLICATION TO OCEAN STRUCTURES

Keywords
Applied Mechanics

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STATIONARY AND NONSTATIONARY RANDOM VIBRATIONS
OF CABLES, PLATES AND BEAMS WITH AN
APPLICATION TO OCEAN STRUCTURES

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A DISSERTATION

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August 25, 1975
Date
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**NOMENCLATURE**

\[ a_{mn} \] modal damping factor

\[ c \] cable damping constant and wave velocity

\[ \text{erf} \] error function

\[ e(t) \] envelope function

\[ f(\ ) \] function of ( )

\[ g \] gravity constant

\[ g(t-t') \] retarded response Green's function in time domain

\[ g'(t-t') \] real even part of \( g(t-t') \)

\[ g''(t-t') \] imaginary odd part of \( g(t-t') \)

\[ h \] plate thickness

\[ k \] spring constant

\[ l \] length and correlation length for moving source

\[ m \] mass per unit length or area

\[ p(\ ) \] probability density of ( ) and normal mode function of ( )

\[ r \] beam cross-sectional radius of gyration, space coordinate, and response function

\[ s_n \] beam eigenvalue

\[ t \] time coordinate

\[ u \] displacement in x-direction

\[ v \] flow velocity of source

\[ w \] displacement in z-direction

\[ x \] Cartesian coordinate or Re\([z]\)

\[ y \] Cartesian coordinate or Im\([z]\)

\[ z \] Cartesian coordinate or complex variable

\[ z = x + iy \]
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<td>source correlation area for plate</td>
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<td>B</td>
<td>flexural stiffness of plate ($B = \frac{Eh^3}{1-v^2}$)</td>
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<td>$C_L$</td>
<td>longitudinal plate sound velocity</td>
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<tr>
<td>$C_T$</td>
<td>transverse plate sound velocity</td>
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<tr>
<td>$C_P$</td>
<td>specific heat at constant pressure</td>
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<tr>
<td>$C_V$</td>
<td>specific heat at constant volume</td>
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<td>D</td>
<td>forcing function constant dependent on the strength of the source</td>
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<td>$E[\ ]$</td>
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<td>$F$</td>
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<td>$G$</td>
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<td>$I$</td>
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<td>$\text{Im}[\ ]$</td>
<td>imaginary part of a quantity</td>
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<td>$K_{zz}$</td>
<td>extensional plate stiffness</td>
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<td>$K$</td>
<td>covariance kernel of process ${z(t)}$ and plate bulk modulus</td>
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<tr>
<td>$L$</td>
<td>length of cable</td>
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<tr>
<td>$M$</td>
<td>total mass and bending moment</td>
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<tr>
<td>$N$</td>
<td>internal tensile or compressive, in-plane force</td>
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<td>$\text{Pr}[\ ]$</td>
<td>probability of $[\ ]$</td>
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<td>$P$</td>
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<td>quality factor ($1/2\xi$) and transverse shear</td>
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<td>$Q_n$</td>
<td>beam quality factor ($\omega_n/2\alpha_n$)</td>
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<td>$R_f(t)$</td>
<td>autocorrelation of source</td>
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NOMENCLATURE (cont.)

\[ R_x(\cdot) \] autocorrelation of response
\[ \text{Res}(\cdot) \] residue of (\cdot)
\[ \text{Re} [ \cdot ] \] real part of a quantity
\[ S_f \] power spectral density of source
\[ S_x \] power spectral density of response
\[ S \] surface of plate and entropy
\[ T \] temperature, time, and torque
\[ U \] unit step function
\[ V \] strain energy in bending, volume and shear force
\[ W \] weight
\[ \alpha \] coefficient of thermal expansion and ratio of flow to wave velocity
\[ \alpha_m \] convection frequency (\( \alpha_m = m\pi v/L \))
\[ \alpha_n \] decay rate of beam eigenvalue
\[ \alpha(t) \] noise function
\[ \beta \] decay constant of the correlated noise or damping factor
\[ \gamma \] damping parameter and time difference coordinate
\[ \delta \] attenuation constant (\( \delta = 2\omega_n \zeta \))
\[ \Gamma_{mn} \] eigenvalues of plate response
\[ \delta_{ij} \] Kronecker delta
\[ \kappa \] thermal conductivity
\[ \varepsilon_{ij} \] strain tensor
\[ \varepsilon_{1,2} \] internal beam damping coefficients
\[ \lambda \] wavelength and length ratio (\( \lambda/L \))
\[ \lambda_n \] mode number (\( n\pi/L \))
NOMENCLATURE (cont.)

\( \Omega \)  \hspace{1cm} \text{harmonic frequency of the correlated noise}

\( \rho \)  \hspace{1cm} \text{density per unit length or per unit area}

\( \sigma \)  \hspace{1cm} \text{standard deviation (} \sigma^2 \text{ is the variance)}

\( \sigma_{ij} \)  \hspace{1cm} \text{stress tensor}

\( \zeta \)  \hspace{1cm} \text{standard damping factor (} C/C_R = 1/(2\omega_n) \text{)}

\( \varphi \)  \hspace{1cm} \text{phase angle}

\( \Theta \)  \hspace{1cm} \text{temporal autocorrelation function}

\( \theta \)  \hspace{1cm} \text{argument of complex variable (} z = \text{Re}^i\theta, \text{ or time difference coordinate)}

\( \tau \)  \hspace{1cm} \text{time difference or shear stress}

\( \omega \)  \hspace{1cm} \text{frequency}

\( \omega_n \)  \hspace{1cm} \text{natural frequency}

\( \omega_{nd} \)  \hspace{1cm} \text{damped natural frequency (also } \omega_n, \omega_{1,n} \text{ and } \omega_n')

\( u_i \)  \hspace{1cm} \text{displacement vector}

\( \xi \)  \hspace{1cm} \text{displacement and space difference coordinate}

\( \langle \rangle \)  \hspace{1cm} \text{temporal mean value}

\( \psi \)  \hspace{1cm} \text{slope of beam}

\( \nabla \)  \hspace{1cm} \text{del operator}

\( \eta(\cdot) \)  \hspace{1cm} \text{unit step function and viscous parameter}

\( | | \)  \hspace{1cm} \text{absolute value}

\( \Lambda \)  \hspace{1cm} \text{nonstationary expression}

\( S_{eq} \)  \hspace{1cm} \text{Fourier transform of the envelope function}

\( \varphi_{mn} \)  \hspace{1cm} \text{plate normal mode function}

\( \mu \)  \hspace{1cm} \text{time difference coordinate}

\( \nu \)  \hspace{1cm} \text{Poisson's ratio}

\( \zeta \)  \hspace{1cm} \text{transverse plate displacement in } z\text{-direction}
ABSTRACT

The mean square responses of lightly damped finite viscoelastic cables and simply supported plates and beams (assuming Voigt viscoelastic solid models) to a special type of nonstationary random excitation are determined. The excitation function is taken in the form of a product of a well defined, slowly varying envelope function, and a part which prescribes the statistical characteristics of the excitation. The latter is assumed to be white or correlated as a wide band process. By taking into consideration the slow variation of the envelope function and the wave characteristics of the lightly damped viscoelastic medium, the mean square nonstationary responses (for the various types of excitation and damping parameters) are evaluated for these structural elements.

The stationary mean square responses of the structural elements mentioned above are also determined for the case of a correlated moving force field. Source correlations for the force fields deal with the expected loss of correlation (due to flow viscous loss and the introduction of new random components) by assuming a "modified" exponentially decaying correlation function.

Coincidence effects are found to appear between the flow velocity of the forcing field and the velocity of the response waves. The larger the damping, however, the more
diminished is the magnitude of the coincidence effects which appear. Exact results are calculated not only for the one-dimensional cable and beam but also for the two-dimensional plate (previously simply approximated). All results are shown to be consistent with earlier results.
INTRODUCTION

A number of recent papers (1, 2, 3, 8, 10) have considered the response of dynamic systems to random excitation. However, the appropriate theory is well-known (4, 5) for calculating the mean square response of linear systems to both stationary and nonstationary random excitation. We consider, here, the mean square response of waves of viscoelastic structural elements to both stationary and nonstationary random excitation.

The nonstationary random excitation is of the form:

\[ s(t) = e(t) \alpha(t) \]

where \( e(t) \) is a well-defined envelope function and \( \alpha(t) \) is the Gaussian narrow band stationary statistical part of the excitation which has zero mean. The nonstationary process is generated by multiplying the sample functions from a stationary process \( \alpha(t) \) and the deterministic function \( e(t) \).

The mean square response is developed in terms of the viscoelastic medium frequency response function (or Green's function in frequency domain) and generalized spectral density function of the input excitation. Both white noise and noise with an exponentially decaying harmonic correlation function are extended to include a rectangular step envelope function.

Further discussion into the nature and need for more complete expressions for both stationary and nonstationary
response functions for ocean structures is set forth in Section 1, together with the nature by which the Green's function method can be coupled with Fourier analysis to yield the desired results.
SECTION I

THE OCEAN STRUCTURE AS A SINGLE-DEGREE-OF-FREEDOM SYSTEM

A. Introduction

In our investigation of ocean structures, we will be investigating structures similar to that shown in Fig. 1.1, consisting of a platform, supported by several cylindrical legs. Such structures, called "Texas Towers," are used extensively for offshore oil drilling, ocean exploration and weather stations. Crossbracing between supports is kept to a minimum in order to minimize the wave and current forces on the structure.

Figure 1.1

(a) Typical ocean structure.

(b) Single-degree-of-freedom spring-mass system.

(c) Free body.
Such a structure is constantly battered by ocean waves arriving with frequency and severity. The larger the number of waves with frequencies spanning the bending frequencies of the structure, the higher the probability is that the structure will undergo extreme vibrations, with possible resonance and failure.

The first mode shape, corresponding to the lowest or natural bending mode of oscillation is usually the most important mode for ocean structures since higher modes are usually quickly dissipated. We will therefore take as our model for these ocean structures the single-degree-of-freedom system. The mode shape is also shown in the Figure. The displacement $x$ of the platform, assumed to remain horizontal during motion, is the same as the displacement for the equivalent spring-mass system shown in Fig. 1b. The undamped natural frequency of the structure and of the equivalent spring-mass system in water is:

$$\omega_n = \left(\frac{k}{m}\right)^{\frac{1}{2}}$$

(1.1)

where $k$ is the spring constant determined from the structural stiffness. The mass $m$ is a composite of the platform mass, the leg mass, and a portion of the water mass which is moved by the legs during vibrations. The value of $m$ can be calculated from Eq. (1.1) if $k$ and $\omega_n$ are known.

The value of $k$ can be found by employing the approximate method discussed by Timoshenko (1956) (14). The elastic deflection curve for one leg, in terms of the symbols in Fig. 1.1, can be approximated by:
where:
\[
\alpha = \frac{Wl^3}{\pi^2EI}
\]
\[E = \text{Young's Modulus} ;
I = \text{leg moment of inertia} ;
3F = \text{total horizontal static force on the platform along } x ;
3W = \text{total platform weight} .
\]

The value of \( k \) is determined from Eq. (1.2) using the relationship
\[
k = \frac{F}{x}
\]
where \( x \) is the value of \( \xi \) at \( z = \ell \).

It follows that:
\[
k = \frac{12EI}{\ell^3} \left(1 - \frac{Wl^2}{\pi^2EI}\right)
\] (1.3)
which is the spring constant for the whole structure as well as for one leg, as long as the stiffening effects of the crossbracing between platform legs can be assumed negligible. Note that \( k \) decreases as \( W \) increases, and that when \( W = \frac{\pi^2EI}{\ell^2} \), the critical Euler buckling load for each leg, the stiffness goes to zero.

Next we will calculate the undamped natural frequency \( \omega_0 \) which the structure would have if it vibrated freely without the external resistance of the water mass. To do this, we will employ the Rayleigh method. We will assume that the deflection curve for the structure can be approximated by that shown in Fig. 1.1 and represented by Eq. (1.2), or:
\[
\xi = \xi(z,t) = A(1 - \cos \frac{\pi z}{\ell}) \sin \omega_0 t
\] (1.4)
where \( A \) is an arbitrary constant, the amplitude of vibration, and \( \omega_0 \) is the natural frequency. The total potential
energy for the structural system can be written as (13):

\[ V = 2EI \int_0^L \left( \frac{\partial^2 E}{\partial z^2} \right) dz - 2W \int_0^L \left( \frac{\partial E}{\partial z} \right)^2 dz \quad (1.5) \]

where the first term on the right side of Eq. (1.5) is the strain energy in bending for the three legs, and the second term is the change in potential energy of the platform weight, or the product of the platform weight and its decrease in vertical height due to the bending of the legs. The maximum value of \( V \) occurs at the most extreme deflected position of the structure where the horizontal velocity and, therefore, the kinetic energy of the structure is zero. In this position, \( \sin \omega_o \tau = 1 \), and:

\[ V_{\text{max}} = \left[ EIL(\pi/L)^4 - W(L(\pi/L))^2 \right] L^2 \quad (1.6) \]

The translational kinetic energy of the structure due to the motion in the \( x \)-direction is given by:

\[ T = 2 \int_0^L \left( \frac{\partial E}{\partial \tau} \right)^2 \rho dz + \frac{2W}{g} \left( \frac{\partial E}{\partial z} \right)_{z=L} \quad (1.7) \]

where \( \rho \) is the mass density per unit length of one of the legs. The first term on the right in Eq. (1.7) is the kinetic energy of the legs, and the other term is the kinetic energy of the platform. The maximum value of \( T \) occurs when the structure is exactly vertical, and \( \cos \omega_o \tau = 1 \).

Using Eqs. (1.4) and (1.7), the result is:

\[ T_{\text{max}} = \left[ 3\rho L + \frac{8W}{g} \right] \omega_o^2 L^2 \quad (1.8) \]

Since energy is conserved in this undamped system:
\[ V_{\text{max}} = T_{\text{max}} \] (1.9)

From Eqs. (1.6), (1.8), and (1.9), the undamped natural frequency of the structural system, unencumbered by water mass, is given by:

\[ \omega_0^2 = \frac{\pi^2 g}{8L} \left( \frac{\pi^2EI}{m_1g + W} \right) \] (1.10)

where \( m_1 = \rho L \), the mass of one leg, and \( W \) is one-third of the platform weight, acting on one leg.

Recent model tests for single, flexible cylinders vibrating in water have been performed in order to find the effects of water on the bending frequencies. Measurements indicate that \( \omega_n \) is always less than \( \omega_0 \) (12), or:

\[ \omega_n = c \omega_0 \] (1.11)

where \( c \) varies from 0.5 to 0.8 and is influenced by cylinder diameter, the end constraints, and the bending stiffness.

The undamped natural frequency of the system in water can now be written as:

\[ \omega_n^2 = \frac{\pi^2 g}{8L} \left( \frac{\pi^2EI}{0.375 m_1g + W} \right) = \frac{12EI}{mL^2} \left( 1 - \frac{W}{\pi^2EI} \right) \] (1.12)

Now, for the structure shown in Fig. 1.1, when \( W, \ m_1, \ EI, L \) and \( c \) are known, the parameters \( \omega_n, k \) and \( m \) for the equivalent single-degree-of-freedom system can be calculated from Eq. (1.12).

To complete the mathematical model of the structural system, we must consider the nature of the wave forces on
the structure and also the damping effect of the water. With regard to the time varying nature of the forces, a conservative assumption would be to take a composite measure of all of the wave forces acting on the legs and apply them as a single horizontal force, $F(t)$, acting on the platform as shown in Fig. 1.1. Assume that this force has a known time history. Such traces are either deduced from direct measurements on existing similar structures, or made from wave measurements. Moreover, as we shall see, only the power spectral densities of $F(t)$ need be known to obtain mean square results (see Appendix B). The effect of damping on the equivalent system can be studied by the inclusion of a damping force in the equation of motion which is represented by Fig. 1.1c and given by:

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0 e^{i\omega t} \tag{1.13}$$

where $F(t)$ = time varying force on mass $m$

$c\dot{x}$ = viscous damping force (assumed linear)

$kx$ = spring force (linear)

The displacement must now be separated into a homogeneous and a particular part:

$$x = x_{\text{homog.}} + x_{\text{part.}} \tag{1.14}$$

where the homogeneous part is the solution for the damped system found by setting the equation of motion equal to zero (see Appendix A) to obtain:

$$x_{\text{homog.}} = x_h = e^{-\omega_n t} \left[ A \sin \omega_n t + B \cos \omega_n t \right] \tag{1.15}$$
We must now also solve for the particular part by writing the equation of motion as in Eq. (1.13) and letting:

\[ x = x_{\text{part}} = x_e^{i\omega t}. \]

Substituting into Eq. (1.13) we have:

\[ F_0 e^{i\omega t} = m(-\omega^2 x_e^{i\omega t} + c(\omega x_e^{i\omega t}) + k(x_e^{i\omega t}) \]

or

\[ F_0 = x(-m\omega^2 + c\omega + k) \]

or

\[ x = \frac{F_0}{k - m\omega^2 + c\omega} \quad \text{(1.16)} \]

In order to reduce this to a dimensionless quantity, we divide both numerator and denominator by \( k \), to obtain:

\[ x = \frac{F_0/k}{1 - m\omega^2/k + c\omega/k} \]

or

\[ x = \frac{X_{\text{static}}}{1 - (\omega/\omega_n)^2 + 2\zeta(\omega/\omega_n)} \quad \text{(1.17)} \]

where \( X_{\text{static}} = F_0/k \) is the deformation of this system when \( F(t) = F_0 \) is applied as a static load (i.e., when \( \omega = 0 \) or \( e^{i\omega t} = 1 \)), and where \( \omega_n \) is the undamped natural frequency and \( \zeta \) is the damping factor (see Appendix A).

The ratio, therefore, of the vibrational deflection \( X \) to the static deflection \( X_{\text{static}} \), indicates the amount of increased or decreased deflection to be expected in the system when it is vibrating as compared to when it is statically loaded with the same amount of force, \( F_0 \). This ratio is called the magnification factor or frequency response function \( G(\omega) \) and is seen to be a function of both the
damping factor and the ratio $\omega/\omega_n$.

$$G(\omega) = \frac{X}{X_{st}} = \frac{1}{\left[1-(\omega/\omega_n)^2\right] + 2i\zeta(\omega/\omega_n)} \quad (1.18)$$

Note that when both the vibrating frequency, $\omega$, is equal to the undamped natural frequency of the system, $\omega_n$, and damping in the system is removed ($\zeta = 0$), the denominator in Eq. (1.18) goes to zero, and the frequency response function is infinite. Therefore, the amplitude of the vibration is also infinite and the system will fail. The purpose of supplying damping to a system, therefore, is primarily to avoid such "resonant" failures.

The frequency response function, Eq. (1.18), is schematically plotted in Figure 1.2, as a function of both $\zeta$ and $\omega/\omega_n$.

![Diagram](image)

The frequency response function, $G(\omega)$.

Figure 1.2
For damped systems, resonance will occur when \( \omega = \omega_{nd} \). However, \( \omega_n \) and \( \omega_{nd} \) are usually approximately equal since system damping, and therefore \( \zeta \), are usually quite small.

The absolute magnitude of the frequency response function can be found by multiplying both sides of Eq. (1.18) by its complex conjugate and taking the square root of both sides, to obtain:

\[
G(\omega) = \left| \frac{X}{X_{st}} \right| = \left\{ \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[ 2\zeta \left( \frac{\omega}{\omega_n} \right) \right]^2 \right\}^{1/2} \left\{ A^2 + B^2 \right\}^{1/2}
\]

where:

\[
A = \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right], \quad \text{and} \quad B = 2\zeta \left( \frac{\omega}{\omega_n} \right)
\]

This magnitude, in turn, could be viewed as a vector with components \( A \) and \( B \), at right angles to one another, and phase angle \( \phi \) (see Fig. 1.3). Here the angle \( \phi \) is the phase angle, or lag, between the applied harmonic force, \( F_0 \), and the vibrating response, \( X \), and can be expressed as:

\[
\tan \phi = \frac{B}{A} = \frac{2\zeta \left( \frac{\omega}{\omega_n} \right)}{1 - \left( \frac{\omega}{\omega_n} \right)^2}
\]

Therefore, the particular solution becomes:

\[
\begin{align*}
X_p &= X e^{i\omega t} = \left\{ X \right\} e^{i\omega t} e^{i\phi} \\
&= \left\{ X \right\} e^{i\omega t}
\end{align*}
\]

which is the equation for a "steady state" vibration, oscillating with constant amplitude, \( |X| \), at a frequency, \( \omega \), and phase angle \( \phi \).

We can now write the complete solution for the response:
Components of $G(\omega)$ viewed as a vector.

Figure I.3

\[ x = x_h + x_p = e^{-\zeta \omega_n t} \left[ A \sin \omega_d t + B \cos \omega_d t \right] + \]
\[ + |x| e^{i\phi} e^{i\omega t} \quad (1.22) \]

The homogeneous part of this solution represents the transient or dissipating part of the response while the particular part represents the steady state or continuing response of this forced vibration (see Fig. I.4). Since we are usually concerned with what the response will be at some future time ($t \gg 0$), we will be concerned primarily with the particular, or steady state solution, as the transient response will quickly die out. If the system under investigation is comprised of ceramic materials or concrete which fail rapidly however, both the transient and steady state responses must be added to determine the full expected
displacement.

Particular and homogeneous parts of the response.

Figure 1.4

B. Transmission of Vibration

Another general case of system vibrations occurs when vibration forces are transmitted through the system's supports by means of adjacent displacements or accelerations (see Fig. 1.5).

Figure 1.5
Assuming that the support displacement, \( x_b \), and that the system is free to vibrate, we can write the equation of motion for this case as follows:

\[
\begin{align*}
\text{Let } x_b &= X_b e^{i\omega t} \\
\text{and } x &= x_p = X e^{i\omega t}
\end{align*}
\]

Also, from Fig. 1.5b, using Newton's second law:

\[
F = ma
\]

\[
k(x_b - x) + c(\dot{x}_b - \dot{x}) = m\ddot{x}
\]

or

\[
m\ddot{x} + c\dot{x} + kx = kx_b + c\dot{x}_b
\]

Substituting for \( x_b \) and \( x \), we have:

\[
(-m\omega^2 + ci\omega + k)Xe^{i\omega t} = (k + ci\omega)X_e e^{i\omega t}
\]

or

\[
\frac{X}{X_b} = \frac{k + ci\omega}{k - m\omega^2 + ci\omega}
\]

or, dividing numerator and denominator by \( k \), we have:

\[
\frac{X}{X_b} = \frac{1 + 2\zeta i(\omega/\omega_n)}{1 - (\omega/\omega_n)^2 + 2\zeta^2(\omega/\omega_n)^2}
\]

Here we see that the system's response is completely related to the amplitude and frequency of the transmitted vibration and to the damping constant and natural frequency of the system itself. This ratio of system response to transmitted forcing function is, like the frequency response function, dependent on the two parameters, \( \zeta \) and \( \omega/\omega_n \). The magnitude of this ratio is found by taking the square root of Eq. (1.26), and is called the transmissibility of the
system, and is drawn schematically in Fig. 1.6.

\[
\text{Transmissibility} = \left| \frac{X}{X_b} \right| = \sqrt{\frac{1 + \left[2\zeta \left(\frac{\omega}{\omega_n}\right)\right]^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \left(\frac{\omega}{\omega_n}\right)\right]^2}}
\]

(1.27)

![Graph of transmissibility vs. \(\omega/\omega_n\).](image)

Transmissibility vs. \(\omega/\omega_n\).

Figure 1.6

Note, as expected, that for an undamped system \((\zeta = 0)\), when the frequency of the forcing function, \(\omega\), is equal to the natural frequency of the system, the system response resonates to infinite values. Note also that if the ratio of \(\omega/\omega_n\) equals \(\sqrt{2}\), the transmissibility of the system is only unity and, in this particular instance, increasing the system's damping ability would only decrease \(\omega_n\), decrease the ratio \(\omega/\omega_n\), and, therefore greatly increase the transmissibility of the system, causing the system to oscillate more violently. In this case and in
others like it, therefore, added damping can hurt rather than help the system.

C. Response of Linear Oscillators to Impulsive Forcing Functions (Green's Method)

In the previous discussions we have mainly considered steady state oscillations. For many types of physical problems, however, the transient effects are quite important. In this section we shall investigate the transient behavior of a linear oscillator which is subject to a driving force that acts discontinuously. Of course a "discontinuous" force is an idealization, since it always takes a finite time to apply a force. However, if the application time is small compared to the natural period of the oscillation, the result of the ideal case is a close approximation to the actual physical situation.

The differential equation which describes the motion of a damped oscillator is:

$$\ddot{x} + 2\beta \dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$

(1.28)

The general solution, as we have seen, is composed of the homogeneous and particular solutions:

$$x(t) = x_h(t) + x_p(t)$$

We can write the homogeneous or complementary solution as:

$$x_h(t) = e^{-\beta t}(A_1 \cos \omega_1 t + A_2 \sin \omega_1 t)$$

(1.29)

where

$$\omega_1 = \sqrt{\omega_n^2 - \beta^2}$$

(1.30)
The particular solution \( x_p(t) \) will, of course, depend on the nature of the forcing function \( F(t) \).

Two types of idealized discontinuous forcing functions are of considerable help and interest; these are the unit step function and the impulse function, shown in Figs. 1.7a and 1.7b, respectively.

The unit step function \( U \) is given by:

\[
U = \begin{cases} 
0, & t < t_o \\
b, & t > t_o 
\end{cases} \quad (1.31)
\]

where \( b \) is a constant with the dimensions of acceleration, and where the argument \( t_o \) indicates that the time of application of the force is \( t = t_o \).

The impulse function \( I \) may be considered to be a positive step function applied at \( t = t_o \), followed by a negative step function applied at some later time \( t_1 \). Thus,
Although we write the step and impulse functions as \( U(t_o) \) and \( I(t_o, t_1) \) for simplicity, these functions, of course, depend on the time \( t \) and are more properly written as \( U(t, t_o) \) and \( I(t, t_o, t_1) \).

Response to a Step Function

For this case the differential equation which describes the motion is:
\[
\ddot{x} + 2\beta \dot{x} + \omega_n^2 x = b, \quad t > t_o
\]  
(1.33)

We consider the initial conditions to be \( x(t_o) = 0 \) and \( \dot{x}(t_o) = 0 \). The particular solution, therefore, is just a constant, and examination of Eq. (1.33) shows that it must be \( x_p(t) = b/\omega_n^2 \). Thus, the general solution for \( t > t_o \) is:
\[
x(t) = e^{-\beta(t-t_o)} \left[ A_1 \cos \omega_1(t-t_o) + A_2 \sin \omega_1(t-t_o) \right] + \\
\quad + \frac{b}{\omega_n^2}
\]  
(1.34)

Application of the initial conditions yields:
\[
A_1 = -\frac{b}{\omega_n^2} ; \quad A_2 = \frac{b}{\omega_1 \omega_n^2}
\]  
(1.35)

and, of course, \( x(t) = 0 \) for \( t \leq t_o \).
If, for simplicity, we take $t_0 = 0$, the solution can be expressed as

$$x(t) = \frac{U(0)}{\omega_n^2} \left[ 1 - e^{-\beta t} \cos \omega_1 t - \frac{\beta}{\omega_1} e^{-\beta t} \sin \omega_1 t \right]$$

(1.36)

This response function is shown in Fig. 1.8 for the case $\beta = 0.2 \omega_n$. It is clear that the ultimate condition of the oscillator (i.e., the steady state condition) is simply a displacement by an amount $b/\omega_n^2$.

In the event that there is no damping, $\beta = 0$, and $\omega_1 = \omega_n$, then, for $t_0 = 0$, we have:

$$x(t) = \frac{U(0)}{\omega_n^2} [1 - \cos \omega_n t] ; \beta = 0 \quad (1.37)$$

so that the oscillation is sinusoidal with amplitude extremes $x = 0$ and $x = 2b/\omega_n^2$ (see Fig. 1.8).

![Figure 1.8](image-url)

Response to a step function for cases of low and zero damping.

Figure 1.8
Response to an Impulse Function

If we consider the impulse function as the difference between two step functions separated by a time $t_1 - t_0 = \tau$ then since the system is linear, the general solution for $t > t_1$ is given by the superposition of the solutions (Eq. 1.36) for the two step functions taken individually:

$$x(t) = \frac{b}{\omega_n^2} \left[ 1 - e^{-\beta(t-t_0)} \cos \omega_1(t-t_0) + \frac{b \beta}{\omega_n} e^{-\beta(t-t_0)} \sin \omega_1(t-t_0) \right]$$

or

$$x(t) = \frac{b e^{-\beta(t-t_0)}}{\omega_n^2} \left[ e^{\beta \tau} \cos \omega_1(t-t_0 - \tau) - \cos \omega_1(t-t_0) + \frac{\beta e^{\beta \tau}}{\omega_1} \sin \omega_1(t-t_0 - \tau) - \frac{\beta}{\omega_1} \sin \omega_1(t-t_0) \right] \text{ for } t > t_1$$

The total response (i.e., Eqs. 1.36 and 1.38) to an impulse function of duration $\tau = 5 (2\pi/\omega_1)$, which is applied at $t = t_0$ is shown in Fig. 1.9 for the case $\beta = 0.2 \omega_n$.

If the duration $\tau$ of the impulse function is allowed to approach zero, then the response function will become vanishingly small. But if we allow $b \to \infty$ as $\tau \to 0$ in such a way that the product $b \tau$ is constant, then the
response will be finite. This particular limiting case is of considerable importance since it approximates the application of a driving force which is a "spike" at \( t = t_0 \) \( (i.e., \tau \ll 2\pi/\omega_1) \). A "spike" of this type is usually termed a delta function and is written as \( \delta(t-t_0) \).

The delta function has the property that \( \delta(t) = 0 \) for \( t \neq 0 \) and \( \delta(0) = \infty \) for \( t = 0 \), but

\[
\int_{-\infty}^{\infty} \delta(t-t_0) \, dt = 1
\]

Therefore, this is not a proper function in the mathematical sense, but it can be defined as the limit of a well behaved and highly local function (such as a normal distribution or Gaussian function) as the width parameter approaches zero, and can be of significant benefit in solving problems of the type we are now discussing.

Now, if we expand Eq. (1.38) and allow \( \tau \to 0 \) but with \( b\tau = d \), we obtain:
\[ x(t) = \frac{d}{\omega_1} e^{-\beta(t-t_0)} \sin \omega_1(t-t_0), \quad t > t_0 \]  

(1.39)

It is evident that as \( t \) becomes large, the oscillator will return to its original position of equilibrium.

The fact that the response of a linear oscillator to an impulsive driving force can be represented in the simple manner of Eq. (1.39) leads to a powerful technique for dealing with general forcing functions, which was developed by George Green, a self-educated English mathematician (1793-1841). Green's method is based upon representing an arbitrary forcing function as a series of impulses, as shown schematically in Fig. 1.10.

Representation of a function as a series of impulses.

Figure 1.10
If the driving system is linear, the principle of superposition is valid and it is permissible to express the inhomogeneous part of the differential equation as the sum of individual forcing functions \( F_m(t)/m \), which in Green's method are impulse functions:

\[
\ddot{x} + 2\beta \dot{x} + \omega_n^2 x = \sum_{m=-\infty}^{\infty} \frac{F_m(t)}{m} = \sum_{m=-\infty}^{\infty} I_m(t)
\]  

where

\[
I_m(t) = I(t_m, t_{m+1}) = \begin{cases} 
    b_m(t_m), & t_m < t < t_{m+1} \\
    0, & \text{otherwise}
\end{cases}
\]  

(1.41)

The interval of time over which \( I_m \) acts is \( t_{m+1} - t_m = \tau \), and \( t < 2\pi/\omega_1 \). The solution for the \( m \)th impulse is, according to Eq. (1.39):

\[
x_m(t) = \frac{b_m(t_m)}{\omega_1} e^{-\beta(t-t_m)} \sin \omega_1(t-t_m) \quad (1.42)
\]

\( \tau > t_m + \tau \)

and the solution for all of the impulses up to and including the \( m \)th impulse is

\[
x(t) = \sum_{m=-\infty}^{M} \frac{b_m(t_m) \tau}{\omega_1} e^{-\beta(t-t_m)} \sin \omega_1(t-t_m) \quad (1.43)
\]

\( t_m < t < t_{m+1} \)
If we allow the time interval \( t \) to approach zero and write \( t_m \) as \( t' \), then the sum becomes an integral:

\[
x(t) = \int_{-\infty}^{+} \frac{h(t')}{\omega_1} e^{-\beta(t-t')} \sin \omega_1(t-t') \, dt' \quad (1.44)
\]

We now define

\[
G(t, t') = \begin{cases} 
\frac{1}{m\omega_1} e^{-\beta(t-t')} \sin \omega_1(t-t'), & t \geq t' \\
0, & t < t'
\end{cases} \quad (1.45)
\]

then since \( ma(t') = F(t') \) \( (1.46) \)

we have

\[
x(t) = \int_{-\infty}^{+} F(t') G(t, t') \, dt' = F(t) * G(t) \quad (1.47)
\]

which is known as the **convolution integral** expression.

The function \( G(t, t') \) is known as the **Green's function** for the linear oscillator equation \( (1.28) \). The solution expressed by Eq. \( (1.47) \) is valid only for an oscillator initially at rest in its equilibrium position since the solution which we used for a single impulse (Eq. \( 1.39 \)) was obtained for just such an initial condition. For other initial conditions, the general solution may be obtained in an analogous manner.

Green's method is generally useful for the solution of linear, inhomogeneous differential equations. The main advantage of the method lies in the fact that the Green's function \( G(t, t') \), which is the solution of the equation for an infinitesimal element of the inhomogeneous part, "already contains the initial conditions," so that the general
solution, expressed by the integral of \( F(t') \) \( G(t,t') \), automatically contains the initial conditions also.

**An Application of Green's Method.** Consider an exponentially decaying forcing function which begins at \( t = 0 \):

\[
F(t) = F_0 e^{-\gamma t}, \quad t > 0 \quad \text{(a)}
\]

The solution for \( x(t) \) according to Green's method is

\[
x(t) = \frac{F_0}{m \omega_1^2} \int_0^t e^{-\gamma t'} e^{-\beta(t-t')} \sin \omega_1(t-t') \, dt' \quad \text{(b)}
\]

Making a change of variable to \( z = \omega_1(t-t') \), we find

\[
x(t) = \frac{F_0}{m \omega_1^2} \int_0^{\omega_1 t} e^{-\gamma z} e^{-\beta t} e^{\left((\gamma-\beta)/\omega_1\right)z} \sin z \, dz
\]

\[
= \frac{F_0/m}{(\gamma-\beta)^2 + \omega_1^2} \left[ e^{-\gamma t} - e^{-\beta t} \left( \cos \omega_1 t - \frac{\gamma-\beta}{\omega_1} \sin \omega_1 t \right) \right]
\quad \text{(c)}
\]

This response function is illustrated in Fig. 1.11 for three different combinations of the damping parameters, \( \beta \) and \( \gamma \). When \( \gamma \) is large compared to \( \beta \), and if both are small compared to \( \omega_n \), then the response approaches that for a "spike."

When \( \gamma \) is small compared to \( \beta \), the response approaches the shape of the forcing function itself, i.e., an initial increase followed by an exponential decay. The lower curve in Fig. 1.11 shows a decaying amplitude on which is superimposed a residual oscillation. When \( \beta \) and \( \gamma \) are equal, Eq. (c) becomes
\[ x(t) = \frac{F_0}{m\omega_1} \cdot e^{-\beta t} (1 - \cos \omega_1 t) \quad ; \quad \beta = \gamma \]

Thus, the response is oscillatory with a "period" equal to \(2\pi/\omega_1\), but an exponentially decaying amplitude, as shown in the middle curve of Fig. 1.11.

Response to an exponentially decaying forcing function for three different combinations of the damping parameters, \(\beta\) and \(\gamma\).

Figure 1.11
D. Autocorrelation, Spectral Density

and Mean Square Response

Summarizing previous results, now, we have from Eqs. (1.18) and (1.45):

\[ G(\omega) = \frac{1}{1 - (\omega/\omega_n)^2 + 2\zeta(\omega/\omega_n)} \]

and

\[ g(t) = e^{-\zeta\omega_n t} \sin \left( (1-\zeta^2)^{\frac{1}{2}} \omega t \right) \]

where: \( \zeta\omega_n = \beta \) and \( \omega_n (1-\zeta^2)^{\frac{1}{2}} = (\omega_n^2 - \beta^2)^{\frac{1}{2}} \)

\( G(\omega) \) and \( g(t) \) form a transform pair for the Green's function and are related by the Fourier integral, as follows (see Appendix E):

\[ G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt; \quad g(t) = \int_{-\infty}^{\infty} \frac{G(\omega)}{k} e^{i\omega t} \frac{d\omega}{2\pi} \quad (1.48) \]

Similarly:

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt; \quad f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \frac{d\omega}{2\pi} \]

and

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt; \quad x(t) = \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \frac{d\omega}{2\pi} \quad (1.49) \]

From the convolution expression, Eq. (1.47), with \( t' \) replaced by \( \tau \) and letting \( t \) approach infinity, we can write:

\[ x(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau \]

or

\[ x(t) = g(t) * f(t) \leftrightarrow \frac{1}{k} G(\omega) F(\omega) = X(\omega) \quad (1.50) \]
where the double pointed arrow (\(\leftrightarrow\)) represents trans-
formation from frequency domain to time domain.

The autocorrelation function of the response \(x(t)\)
can now be written as (see Appendix B):

\[
R_{xx}(\tau) = \langle x(t) \cdot x(t+\tau) \rangle = E\{x(t) \cdot x(t+\tau)\} \quad (1.51)
\]

where:

\[
x(t) = \int_{-\infty}^{\infty} g(\tau_1) f(t-\tau_1) d\tau_1; \quad x(t+\tau) = \int_{-\infty}^{\infty} g(\tau_2) f(t+\tau-\tau_2) d\tau_2
\]

or

\[
R_{xx}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau_1) g(\tau_2) f(t-\tau_1) f(t+\tau-\tau_2) d\tau_1 d\tau_2
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau_1) g(\tau_2) R_{ff}(\tau+\tau_1-\tau_2) d\tau_1 d\tau_2 \quad (1.52)
\]

where \(R_{ff}(\tau+\tau_1-\tau_2) = \langle f(t-\tau_1) f(t+\tau-\tau_2) \rangle\)
is the autocorrelation function of the source.

The mean square spectral density of the response
(see Appendix B) is now written as:

\[
S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega \tau} d\tau \quad (1.53)
\]

and

\[
R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega \tau} \frac{d\omega}{2\pi} \quad (1.54)
\]

and

\[
R_{ff}(\tau+\tau_1-\tau_2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{ff}(\omega) e^{i\omega (\tau+\tau_1-\tau_2)} \quad (1.55)
\]

Eq. (1.53) now becomes:

\[
S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \tau} d\tau \int_{-\infty}^{\infty} g(\tau_1) d\tau_1 \int_{-\infty}^{\infty} g(\tau_2) d\tau_2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{ff}(\omega) e^{i\omega (\tau+\tau_1-\tau_2)}
\]

\[
= \int_{-\infty}^{\infty} e^{-i\omega \tau} d\tau \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{ff}(\omega) e^{i\omega \tau} \int_{-\infty}^{\infty} g(\tau_1) e^{i\omega \tau_1} d\tau_1 \int_{-\infty}^{\infty} g(\tau_2) e^{-i\omega \tau_2} d\tau_2
\]
\[
e^{-i\omega \tau} \left[ \int_{-\infty}^{\infty} S_{ff}(\omega) \frac{G(\omega)G(-\omega)}{k^2} e^{i\omega \tau} d\omega \right] d\tau
\]

or

\[S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \tau} \left[ \frac{1}{k^2} \int_{-\infty}^{\infty} S_{ff}(\omega) G(\omega) \left| e^{i\omega \tau} \frac{d\omega}{2\pi} \right| \right] d\tau \] (1.56)

Now, comparing Eq. (1.56) with Eqs. (1.53) and (1.54) we can write:

\[S_{xx}(\omega) = \frac{1}{k^2} \left| G(\omega) \right|^2 S_{ff}(\omega) \] (1.57)

and

\[R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega \tau} d\omega = \frac{1}{k^2} \int_{-\infty}^{\infty} \left| G(\omega) \right|^2 S_{ff}(\omega) e^{i\omega \tau} d\omega \] (1.58)

Eqs. (1.57) and (1.58), therefore, represent the "input-output relation" for autocorrelations and spectral densities. Note that when \( \tau = 0 \), we obtain \( R_{xx}(0) = \langle x^2(t) \rangle \), the mean square response.

Since \( |G(\omega)| \) has already been evaluated in Eq. (1.19), a knowledge of the power spectral density of the force is all that is needed to calculate the mean square value of the displacement. \( S_{ff}(\omega) \) can be determined from \( F(t) \) either by electronic measurements or by assumption from experience based on analysis of source ensembles (32).

Sample Narrow Band Filter
Center frequency = \( \omega_0 \)
Band width = \( \Delta \omega \)

Averaging Device
\[ z(t) = \frac{1}{T} \int_{t-T}^{t} y(\tau) d\tau \]

Measured quantity \( z(t) \)
Electronically we can determine $S_{ff}(\omega)$ from $F(t)$ as follows (see diagram on previous page): A sample function $F(t)$ is filtered, squared, and averaged over a relatively long interval $T$. The measured quantity $z(t)$ depends on the sample function $F(t)$ and also on the three parameters $\omega_0$, $\Delta \omega$ and $T$ of the measuring system. It can be shown (34) that when $\Delta \omega$ is small and $T$ is long, then $z$ is an approximation to $\Delta \omega S_1(\omega_0)$ where $S_1(\omega)$ is the individual spectral density of $F(t)$. Strictly speaking, the true spectral density $S_1(\omega_0)$ can only be obtained by a limiting process in which $T \to \infty$ and then $\Delta \omega \to 0$ (the order of the limiting processes here cannot be interchanged (34)). In practice when $T$ is long and $\Delta \omega$ is small, the reading $z(t)$ is a slowly varying quantity which fluctuates about the desired value $\Delta \omega S_1(\omega_0)$.

If a large number of different samples $F^{(j)}$ of a stationary random process, each of duration $T$, were available then it would be possible to process each sample as we have done above, and then to form an ensemble average of the $z^{(j)}$. It can be shown that this results in an approximation to $\Delta \omega S_{ff}(\omega_0)$ where the "process" spectral density is $S_{ff}(\omega)$. Again the true value of $S_{ff}(\omega_0)$ is only obtained by a double limiting process in which first the number of samples becomes infinite and then the width of the filter $\Delta \omega$ is made to approach zero.

When the random process can be taken as ergodic, it is only necessary to have a single sample function with
sufficiently long averaging time $T$ in order to obtain a good estimate of the process spectral density $S_{ff}(\omega)$. This is because, for an ergodic process, $S_{ff}(\omega_0)$ is identical to the individual densities $S_1(\omega_0)$ of the representative samples of the ergodic process.

In order to obtain the spectral density as a function of $\omega$ over a prescribed frequency range, it is necessary to repeat the operation just described for a large number of filter center frequencies $\omega_0$, thus producing the complete spectrum automatically for a given sample function $F(t)$.

To obtain $S_{ff}(\omega)$ from $F(t)$ based on an analysis of source ensembles we first define the temporal autocorrelation function $\Phi(\tau)$ of an individual function $F(t)$ (see Appendix B):

$$\Phi(\tau) = \int_{-\infty}^{\infty} S_1(\omega)e^{i\omega\tau}d\omega = \langle F(t)F(t+\tau) \rangle$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} F(t)F(t+\tau)d\tau$$

$$S_1(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\tau)e^{-i\omega\tau}d\tau$$

The interpretation of $S_1(\omega)$ as a temporal mean square spectral density follows from putting $\tau = 0$, and obtaining:

$$\Phi(0) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} F^2(t)dt = \int_{-\infty}^{\infty} S_1(\omega) d\omega$$
The total mean square is the sum over all frequencies of $S_1(\omega) d\omega$. Thus $S_1(\omega)$ plays the role of a density of distribution of the mean square along the frequency axis.

Individual spectral densities $S_j(\omega)$ can be obtained for individual sample functions $x(j)$ of a stationary random process. In general, these would differ from one another (and also from the spectral density $S_{ff}(\omega)$ of the random process). The ensemble average of the $S_j$'s would, however, be equal to $S_{ff}(\omega)$.

$$E \left[ S_1(\omega) \right] = S_{ff}(\omega)$$

For an ergodic process the differences among the individual $S_j(\omega)$ disappear, and we have:

$$S_j(\omega) = S_{ff}(\omega)$$

for a representative sample function $x(j)(t)$.

**E. Response to Ideal White Noise**

For ideal white noise, $S_{ff}(\omega) = S_o = \text{constant}$. In this case, the mean square displacement is, from Eq. (1.58), with $\tau = 0$:

$$\langle x^2(t) \rangle = R_{xx}(0) = \frac{2S_o}{k^2} \int_0^\infty \frac{d\omega}{\left[1-(\omega^2/\omega_n^2)\right]^2 + \left[2\zeta(\omega/\omega_n)\right]^2}$$

$$= \frac{\omega_n S_o}{4\zeta k^2}$$

(1.59)
Integrations such as these can be found in mathematical tables or can be performed by the method of residues. This result is physically reasonable, since one would expect small displacements for strong springs, or high k, and for high damping or large $\zeta$. It should be observed that the units are consistent if frequencies are given in radians/unit time.

The power spectral density for the displacement response from Eq. (1.57) is:

$$S_{xx}(\omega) = \frac{S_0}{k^2} \left| G(\omega) \right|^2 = \frac{S_0}{k^2} \frac{1}{\left[ 1 - (\omega^2/\omega_n^2) \right]^2 + [2\zeta(\omega/\omega_n)]^2}$$

(1.60)

which is sketched in Fig. 1.12 for the case of light damping ($\zeta = 0.05$). It is noted that $S_{xx}(\omega)$ resembles a typical narrow band spectrum, with the peak value at the natural frequency $\omega_n$. 

Response statistics for the single-degree-of-freedom model with white noise excitation.

Figure 1.12
An infinitely wide band excitation ($S_{1f} = \text{constant}$) is never realized in the physical world. Nevertheless, the conclusions based on this idealization often provide a good approximation to $x(t)$ for a lightly damped system.

**F. Response to Band-Limited White Noise**

Some deep water ocean structures are being planned which have frequencies $\omega_n$ within the range of wave frequencies. Suppose that the wave frequencies, and therefore the frequencies of $F(t)$, lie predominantly within a narrow band, $\omega_1 \leq \omega \leq \omega_2$, and that $\omega_n$ lies in the middle of this range. Suppose further that $S_{1f}(\omega)$ is idealized as a dotted rectangular function. The mean square displacement is found from Eq. (1.58) for $\omega_1 < |\omega| < \omega_2$ or

$$\begin{align*}
\frac{1}{\sigma_x^2(t)} &= \int_{-\omega_2}^{-\omega_1} \frac{S_0}{k^2} |G(\omega)|^2 \frac{d\omega}{2\pi} + \int_{\omega_1}^{\omega_2} \frac{S_0}{k^2} |G(\omega)|^2 \frac{d\omega}{2\pi} \\
&= \frac{\omega_n S_0}{4 \zeta k^2} \left[ I\left(\frac{\omega_1}{\omega_n}; \zeta\right) - I\left(\frac{\omega_2}{\omega_n}; \zeta\right) \right] 
\end{align*}$$

(1.61)

where the integral factor $I\left(\frac{\omega}{\omega_n}; \zeta\right)$ is

$$I\left(\frac{\omega}{\omega_n}; \zeta\right) = \frac{1}{\pi} \tan^{-1} \left[ \frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2} \right] +$$

$$\frac{\zeta}{2\pi \sqrt{1-\zeta^2}} \ln \left[ \frac{1+(\omega/\omega_n)^2 + 2\sqrt{1-\zeta^2} (\omega/\omega_n)}{1+(\omega/\omega_n)^2 - 2\sqrt{1-\zeta^2} (\omega/\omega_n)} \right]$$

(1.62)
A graph of Eq. (1.62) for two values of $\zeta$ is shown in Fig. 1.13. For ideal white noise

$$I(\infty, \zeta) - I(0, \zeta) = 1$$

and for limited-band excitation

$$I\left(\frac{\omega_2}{\omega_1}, \zeta\right) - I\left(\frac{\omega_1}{\omega_n}, \zeta\right) < 1$$

It is clear from Fig. 1.13, however, that the factor in brackets in Eq. (1.61) is only slightly less than one as long as $\omega_1$ and $\omega_2$ span $\omega_n$ and the damping is light.

Plot of the function given by Equation (1.62).

Figure 1.13

The power spectral density for the displacement is, from Eq. (1.57):
which is sketched in Fig. 1.14. This result should be compared with the result for $S_{xx}(\omega)$ of Fig. 1.12.

**Figure 1.14**

Power spectral densities for a narrow band input $S_{ff}(\omega)$, and the response $S_{xx}(\omega)$ of a single-degree-of-freedom system.

In our forthcoming investigation of the dynamics of ocean structures, we will be concentrating on three types of structural elements: cables, plates and beams. We are able to do this with very little loss of generality since the general equations governing these three elements can be easily extended to model a multitude of more complicated structures. Furthermore, we will be calculating the
statistical mean square response for each of these to both stationary and nonstationary types of random inputs. These calculations and subsequent results (in equation, tabular and graphical forms) will yield an extensive body of data upon which to base specific designs of ocean structures subject to any type of dynamic loading, experimentally recorded or assumed.
SECTION II

RESPONSE OF CABLES TO RANDOM FORCE FIELDS

A. Nonstationary Random Response of a Finite Cable

Evaluation of Nonstationary Response Function

The correlation function for the displacement, $u$, to a nonstationary random source, $f$, is given by the relation:

$$u(x_1,t_1)u(x_2,t_2) = \mathbb{E}\left[ \sum_{n=1}^{\infty} \frac{1}{p_n(x_1)p_n(x_2)} \left( \eta_0(t-t') \right) \beta_n(t_1-t') \beta_n(t_2-t') \right] \cdot$$

If the equation of motion of a cable is

$$\left( \frac{c^2}{L} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) u = -(f/m)(4\pi)$$

then the appropriate Green's function is given by:

$$g(x,t;x',t') = \sum_{n=1}^{\infty} \frac{p_n(x)p_n(x')}{p_n(x_1)p_n(x_2)} \beta_n(t_1-t') \beta_n(t_2-t') \cdot$$

where $p_n = (2/L)^{1/2} \sin k_n x$ are modes of vibrations, attenuation $\eta_0 = 2\omega_n \zeta$, $\omega_n = ck_n$, $c$ is the speed of sound in the cable $k_n = (n\pi/L)$, $\zeta$ is the damping, $\omega_n^d = \omega_n (1-\zeta^2)^{1/2}$ and $\eta(t-t')$ is the unit step function.

By using the normal mode technique, the Green's function can be separated into the spatial and temporal components by
\begin{align*}
g_n(x^*, t^*; x', t') &= p_n(x)p_n(x')g_n(t^* - t') f(x', t') = q(x')s(t'), \\
\text{and } s(t') &= e(t')a(t') \quad (11.4)
\end{align*}

We are to determine the mean square response, \( E[u^2(x, t)] \), when \( e(t') \) is both a unit step and a rectangular step, and \( a(t') \) has the correlation functions:

\[
R_a(\tau) = 2\pi k_0 \delta(\tau) \quad \text{and} \quad R_{\alpha}(\tau) = k_0 e^{-\beta |\tau|} \cos \Omega \tau
\]

(11.5)

where \( \tau = t^* - t_1 \). The first correlation is that for white noise, while the second is for correlated noise. Here we are assuming a one-dimensional forcing function, \( f(x', t') \), whose spatial part is purely random; therefore, \( \langle q(x')q(x'') \rangle = D\delta(x'' - x') \). Substituting now into Eq. (11.1) and operating on the delta function, we obtain:

\[
\begin{aligned}
&u(x, t_1)u(x, t_2) = \frac{D}{2}\int_0^L \int_0^L \int_0^L p_n(x)p_n(x')p_m(x)p_m(x')dx' \\
&\cdot g_n(t_1)g_m(t_2) \langle s(t_1)s(t_2) \rangle dt_1 dt_2 \quad (11.6)
\end{aligned}
\]

Now from the orthogonality condition, we can replace the integral over \( x' \) with a kronecker delta \( \delta_{mn} \), and letting \( m \) approach \( n \), we obtain:

\[
\begin{aligned}
&\langle u(x, t_1)u(x, t_2) \rangle = \frac{2D}{2} \int_0^L \int_0^L \int_0^L p_n(x)p_n(x')p_m(x)p_m(x')dx' \\
&\cdot g_n(t_1)g_n(t_2) \langle s(t_1)s(t_2) \rangle dt_1 dt_2 \quad (11.7)
\end{aligned}
\]

Now looking at the time integration separately, we have:
\[ \int g_n(t_1)s(t-t_1)dt_1 = s(t_1)*g_n(t_1) = \int d\omega G_n(\omega)S(\omega_1) \cdot e^{i\omega_1 t_1/2\pi} \] 

Eq. (11.7) therefore becomes:

\[ <u(x,t_1)u(x,t_2)> = \frac{2D}{m^2L^2} e^{2\sin^2 \frac{\pi n}{2L}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \cdot \cdot \cdot G_n^{*}(\omega_1) e^{-i\omega_1 t_1} G_n(\omega_2) e^{i\omega_2 t_2} <S^*(\omega_1)S(\omega_2)> \]

(11.9)

where \( G_n^{*}(\omega_1) \) and \( S^*(\omega_1) \) are complex conjugates of \( G_n(\omega_1) \) and \( S(\omega_1) \). A further investigation into the statistical time part of the response, represented by the term in pointed brackets in Eq. (11.9), is now necessary.

\[ <S^*(\omega_1)S(\omega_2)> = \frac{2D}{m^2L^2} \sum_{m} e^{2\sin^2 \frac{\pi m}{2L}} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \......
\( \omega_1 = \omega_2, S_\omega(\omega, \omega_1) \) and \( S_\omega(\omega_2 - \omega) \) are complex conjugates of one another. Eq. (11.12) now becomes:

\[
\langle u^2(x, t) \rangle = \frac{2D}{mL} \sum_{n} \sin^2 k_n \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |A(n, \omega)|^2 p_\alpha(\omega) \quad (11.14)
\]

where \( A(n, \omega) = \int_{-\infty}^{\infty} d\omega G^*(\omega) S_\omega(\omega - \omega_1)e^{-i\omega_1 t/2\pi} \).

We shall now proceed to calculate the mean square response by investigating Eq. (11.14) for various envelope functions and source correlations. We see from Eq. (11.14) that the nonstationary nature of the response can be investigated by examining the integral expression alone. We therefore let

\[
E \left\{ r_n^2(t) \right\} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |A(n, \omega)|^2 p_\alpha(\omega); \quad (11.15)
\]

and

\[
\langle u^2(x, t) \rangle = \frac{2D}{mL} \sum_{n} \sin^2 k_n \int_{-\infty}^{\infty} d\omega G^*(\omega) S_\omega(\omega - \omega_1)e^{-i\omega_1 t/2\pi}; \quad (11.16)
\]

**Unit Step Envelope Function**

When the envelope function \( e(t) \) is a unit step function defined by \( \eta(t) \), the integral representation of the unit step is defined by the following expression:

\[
\eta_+(t-t') = -i \int \frac{d\omega}{2\pi} S_\omega(\omega)e^{i\omega(t-t')} = i \int \frac{d\omega}{2\pi} e^{i\omega(t-t')} = -i \int \frac{d\omega}{2\pi} (p(1/\omega) + i\pi\delta(\omega)) e^{-i\omega(t-t')} \quad (11.17)
\]

Then the frequency shifted unit step envelope transformation function becomes

\[
S_\omega(\omega_2 - \omega) = -i(\pi\delta(\omega_2 - \omega) \pi + \frac{p}{\omega_2 - \omega}) = \int e^{-it(\omega_2 - \omega)} \eta_+(t) dt \quad (11.18)
\]
Substitution of Eq. (11.18) into Eq. (11.16) and the evaluation of the resultant integral gives

\[ |A(t,\omega)|^2 = |G(\omega)|^2 M(t,\omega) \]  

(11.19)

where

\[ M(t,\omega) = 1 + \Gamma_1(t) + \Gamma_2(t) \left[ \frac{B^2 - A^2 + \omega^2}{A^2} \right] - 2 \Gamma_3(t) \cos \omega t - 2 \Gamma_4(t) \] .

(11.20)

with

\[ \Gamma_1(t) = e^{-2Bt} (1 + B/A \sin 2At) \]

\[ \Gamma_2(t) = e^{-2Bt} \sin^2 At \]

\[ \Gamma_3(t) = e^{-Bt} (\cos At + (B/A) \sin At) \]

\[ \Gamma_4(t) = e^{-Bt} \sin At \]

\[ A = \omega_n(1-\xi^2)^{1/2}; \quad B = \frac{1}{2} = \omega_n \xi; \quad C = \omega_n \]

Hence, the mean square response via Eq. (11.15) becomes

\[ E[r^2(t)] = \int |G(\omega)|^2 M(t,\omega) P_\alpha(\omega) d\omega \]  

(11.22)

**White Noise Inputs.** If the input noise is assumed white, then the spectral density function \( P_\alpha(\omega) \) becomes a constant \( P_0 \). Thus the mean square response becomes

\[ E[r^2(t)] = P_0 \int_{-\infty}^{\infty} |G(\omega)|^2 M(t,\omega) d\omega = \frac{\pi P_0}{2BC} \]

\[ \cdot \left[ 1 - e^{-2Bt}(1 + B/A \sin 2At + \frac{2B^2}{A^2} \sin^2 At) \right] \]  

(11.23)
Correlated Input Excitation. In this case, the input excitation is assumed correlated and is indicated by
\[ R(\tau) = K_0 e^{-\beta|\tau|} \cos \Omega \tau. \]
The spectral density then has the form
\[ P_\alpha(\omega) = K_0 \beta (\beta^2 + \Omega^2 + \omega^2) / \pi (\omega^2 - \omega_3^2) (\omega^2 - \omega_4^2) \] (11.24)
where \( \omega_3 = \Omega + i\beta \) and \( \omega_4 = -\Omega + i\beta \). Upon substitution of the spectral density for correlated noise in Eq. (11.24) into expression (11.22), the mean square becomes
\[ E[f_n^2(t)] = K_0 \left[ F_1 L_1(t) + F_3 L_3(t) - F_3 L_2(t) - F_3 L_4(t) \right] \] (11.25)
where
\[ L_1(t) = \left[ 1 - \Gamma_1(t) \right] \frac{A}{2B} \]
\[ L_2(t) = -\Gamma_2(t) \]
\[ L_3(t) = \left[ 1 + \Gamma_1(t) + \frac{B^2 - A^2 + \Omega^2 - \beta^2}{A^2} \right] \Gamma_2(t) - \left[ \frac{1}{2} \right] \left[ \frac{1}{r_3(t) + 2 \Gamma_4(t) / A^2} \right] e^{-\beta t} \cos \Omega t - \left[ \frac{1}{2} \right] \left[ \frac{1}{r_3(t) + 2 \Gamma_4(t) / A^2} \right] e^{-\beta t} \sin \Omega t - \frac{2(\Omega/A) \Gamma_4(t) e^{-\beta t} \cos \Omega t}{A^2} \]
\[ L_4(t) = 2 \beta \Omega \Gamma_2(t) / A^2 - 2 \left[ \Gamma_3(t) + \beta \Gamma_4(t) / A \right] \cdot e^{-\beta t} \sin \Omega t + 2 A \Omega \Gamma_4(t) e^{-\beta t} \cos \Omega t / A \]
and
\[ F_1 = \text{Re} \left[ \frac{\Omega^2 + \beta^2 + \omega^2}{\omega (\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \right] \frac{\beta}{A^2}; \]
\[ F_3 = \text{Re} \left[ \frac{1}{(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \right]; \] (II.26)
\[ G_1 = \text{Imag} \left[ \frac{\Omega^2 + b^2 + \omega_1^2}{\omega_1 (\omega_1^2 - \omega_3^2) (\omega_3^2 - \omega_2^2)} \right] \beta/A^2; \]

\[ G_3 = \text{Imag} \left[ \frac{1}{(\omega_3^2 - \omega_1^2) (\omega_3^2 - \omega_2^2)} \right] \quad (11.27) \]

Rectangular Step Envelope Function

For a rectangular step envelope function of duration \( t' \), we have \( e(t) = \eta(t) - \eta(t-t') \). Upon substitution into Eq. (11.14), we obtain the rectangular step envelope transformation function defined as

\[ S_{e}(\omega_2 - \omega) = \left[ 1 - e^{-i(\omega_2 - \omega)t'} \right] \left[ \pi \delta(\omega_2 - \omega) + 1/i(\omega_2 - \omega) \right] \quad (11.28) \]

Substituting the last expression into Eq. (11.16), we obtain

\[
\begin{align*}
\left| \Delta_n(t, \omega) \right|^2 &= G_n(\omega) \left[ \left| (t, \omega) \eta(t) + (r_1(t) - M(t, \omega)) + (r_1(t-t') + \frac{B^2 - A^2 + \omega_2^2}{A^2} [r_2(t) + r_2(t-t')] - 2[r_3(t) r_3(t-t')] + (\omega^2/A^2) [r_4(t) r_4(t-t')] \right| \cos \omega t' + 2(\omega/A) \left[ r_3(t) r_4(t-t') - r_3(t-t') r_4(t) \right] \sin \omega t' \right] \eta(t-t') \right|^2, \quad (11.29)
\end{align*}
\]

Hence, from Eq. (11.15) the mean square response becomes

\[
\begin{align*}
E \left[ r_n^2(t) \right] &= \int_{-\infty}^{\infty} d\omega \left| G_n(\omega) \right|^2 \nu_n(\omega) M_n(t, \omega) \quad \text{for } 0 \leq t \leq t' \\
E \left[ r_n^2(t) \right] &= \int_{-\infty}^{\infty} d\omega \left| G_n(\omega) \right|^2 \nu_n(\omega) M_n(t, \omega) \quad \text{for } t > t' \quad (11.30)
\end{align*}
\]

where \( M_n(t, \omega) \) is given by Eq. (11.19) and
\[
M_r(t, \omega) = \Gamma_1(t) + \Gamma_1(t-t') + (B^2-A^2+\omega^2)/A^2 \left[ \Gamma_2(t) + \\
+ \Gamma_2(t-t') \right] - 2 \left[ \Gamma_3(t)\Gamma_3(t-t') + (\omega^2/A^2)\Gamma_4(t)\Gamma_4(t-t') \right] \cdot \\
\cdot \cos \omega t' + 2(\omega/A) \left[ \Gamma_3(t)\Gamma_4(t-t') - \Gamma_3(t-t')\Gamma_4(t) \right] \sin \omega t' \\
(11.31)
\]

**White Noise Input.** If the input excitation is assumed white, then

\[
E\left[ r_n^2(t) \right] = P_0 \int_{-\infty}^{\infty} d\omega |G(\omega)|^2 M_n(t, \omega) \quad \text{for } 0 \leq t \leq t' \\
E\left[ r_n^2(t) \right] = P_0 \int_{-\infty}^{\infty} d\omega |G(\omega)|^2 P_{\alpha}(\omega) M_n(t, \omega) \quad \text{for } t \geq t' \quad (11.32)
\]

The first integral is exactly Eq. (11.23), and the second integral is

\[
E\left[ r_n^2(t) \right] = (\pi P_0/2BC^2) \left\{ \Gamma_1(t) + \Gamma_1(t-t') + 2(B^2/A^2) \left[ \Gamma_2(t) - \\
- \Gamma_2(t-t') \right] - 2 \left[ \Gamma_3(t)\Gamma_3(t') + (\omega^2/A^2)\Gamma_4(t)\Gamma_4(t') \right] \cdot \\
\cdot \Gamma_3(t-t') + 2(C^2/A^2) \left[ (2\beta/A)\Gamma_4(t)\Gamma_4(t') - \Gamma_4(t)\Gamma_3(t') + \\
+ \Gamma_3(t)\Gamma_4(t') \right] \Gamma_4(t-t') \right\} \quad \text{for } t \geq t' \quad (11.33)
\]

**Correlated Input Excitation.** If the input excitation is assumed correlated, then \( P_{\alpha}(\omega) \) is given by Eq. (11.24).

Substituting Eq. (11.24) into Eq. (11.30) and evaluating the resultant integral will give

\[
E\left[ r_n^2(t) \right] = K_0 \left\{ F_1 L_1(t) + G_1 L_2(t) + F_3 L_3(t) - G_3 L_4(t) \right\} \quad \text{for } 0 \leq t \leq t' \\
E\left[ r_n^2(t) \right] = K_0 \left\{ F_1 L_{11}(t) - G_1 L_{22}(t) + F_3 L_{33}(t) - G_3 L_{44}(t) \right\} \quad \text{for } t > t'
\]
where

\[ L_{11}(t) = \frac{A}{2B} \left\{ \left[ \Gamma_1(t) + \Gamma_1(t-t') \right] - 2 \left[ \Gamma_3(t) + \frac{B}{A} \right] \right. \]

\[ \cdot \left[ (B/A) \Gamma_2(t) + (B/2A^2) \right] - 2 \left[ \left( \frac{B}{A} \right) \Gamma_2(t) + \left( \frac{B}{2A^2} \right) \right] \right. \]

\[ \cdot \left( \frac{B}{A} \right) \Gamma_2(t-t') \right\} \]

\[ L_{22}(t) = \frac{A}{B} \left\{ \left[ \left( \frac{B}{A} \right) \Gamma_2(t) + \frac{B}{A} \right] \right. \]

\[ \cdot \left[ (B/A) \Gamma_3(t) + \left( \frac{B}{2A^2} \right) \right] - 2 \left[ \left( \frac{B}{A} \right) \Gamma_3(t) + \left( \frac{B}{2A^2} \right) \right] \right. \]

\[ \cdot \left( \frac{B}{A} \right) \Gamma_3(t-t') \right\} \]

\[ L_{33}(t) = \Gamma_1(t) + \Gamma_1(t-t') + \left( \frac{B^2-A^2+\Omega^2-B^2}{A^2} \right) \left[ \Gamma_2(t) + \right. \]

\[ + \left( \frac{B}{A} \right) \Gamma_2(t-t') \right\] - 2 \left[ \left( \frac{B}{A} \right) \Gamma_3(t) + \left( \frac{B}{2A^2} \right) \right] \Gamma_3(t-t') \left. \right. \]

\[ \cdot \left( \frac{B}{A} \right) \Gamma_4(t) \right\} \Gamma_4(t-t') \right\} \right. \]

\[ L_{44}(t) = 2 \left[ \left( \frac{B}{A^2} \right) \left[ \Gamma_2(t) + \Gamma_2(t-t') \right] \right. \]

\[ \cdot \left[ \Gamma_3(t) + \Gamma_3(t-t') \right] \right. \]

\[ \cdot \left[ \Gamma_4(t) + \Gamma_4(t-t') \right] \right\} \right. \]

\[ \cdot \left[ \Gamma_4(t-t') \right] \right\} \right. \]

Here, \( B \) is the decay constant of the correlated noise and \( \Omega \) is the harmonic frequency of the correlated noise.
Lyon's Check. We should note that for white noise:

\[
P_0 = \lim_{\beta \to \infty} \beta P_\alpha(\omega) = \frac{K_0}{2\pi} \tag{11.34}
\]

and this expression is useful for checking the consistency of our work. We can now compare our mean square response to a white noise unit step input with that of Lyon (1) (where Lyon uses \(1/m^2 = (4\pi/D_\alpha)^2\)). Using Eqs. (11.34) and (11.23), the nonstationary response becomes:

\[
\langle u^2(x,t) \rangle = 16\pi^2D/P_\alpha^2 \sum_n p^2_n(x) \cdot \frac{K_0}{4BC} \left[1 - e^{-2Bt} \cdot \left(1 + \frac{B}{\lambda} \sin 2At + \frac{2B^2}{\lambda^2} \sin^2 At\right)\right] \tag{11.35a}
\]

To quote Lyon, "the coefficient of \(p^2_n(x)\) is the mean square amplitude or \(n^{th}\) mode which we shall call \(\langle u^2 \rangle_{nAV}\)." Now with \(K_0 = 1\), we obtain Lyon's stationary expression for the mean square amplitude by letting \(t\) go to \(\infty\) in Eq. (11.35), and find:

\[
\langle u^2 \rangle_{nAV} = 8\pi^2D/P_\alpha^2 \sum_n \omega^2_n \tag{11.35b}
\]

This is Lyon's Eq. (11), page 392.

For real \(\omega\), the Green's function \(g(x,\omega)\) is usually divided into two parts: a dissipative part and a reactive part. In our case, and more generally when the system is stationary, these are given, respectively, by the imaginary and real parts of \(g(x,\omega)\), defined as: \(g''(x,\omega)\) and \(g'(x,\omega)\), defining \(g(x,\omega) = g'(x,\omega) + g''(x,\omega)\); where:

\[
g''(x,\omega) = \frac{-1_{n}}{\omega^2_n - \omega^2} \left[\frac{\omega^2_n - \omega^2}{\omega^2_n - \omega^2} + 1_n^2\right] \tag{a}
\]

\[
g'(x,\omega) = \frac{(\omega^2_n - \omega^2)}{\omega^2_n - \omega^2} + 1_n^2 \tag{b}
\]
The Fourier transform of Eq. (a) is the imaginary odd function of time:
\[-ig''(x,t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} g''(x,\omega)e^{i\omega(t-t')} = -ie^{\frac{1}{2}|t-t'|}.\]

Likewise, the Fourier transform of Eq. (b) is the real even function of time:
\[g'(x,t-t') = \frac{1}{2}\left[g(x,t-t') + g(x,t-t')\right] = e^{-\frac{1}{2}|t-t'|}.\]

Since the response is causal, or equivalently, since \(g(x,\omega)\) which we have defined above is analytic in the upper or lower half-plane, the real and imaginary parts of \(g(x,\omega)\) are related by the Hilbert transform according to the relations:
\[\begin{align*}
g'(x,\omega) &= P\int_{-\infty}^{\infty} \frac{d\omega}{\pi} g''(x,\omega)/\omega - \omega; \\
g''(x,\omega) &= -P\int_{-\infty}^{\infty} \frac{d\omega}{\pi} g'(x,\omega)/\omega - \omega.
\end{align*}\]  

where \(P\) implies principle value integral, that is, an integral symmetrical about the singularity. Also:
\[g(x,\omega) = \int_{-\infty}^{\infty} d(t-t')e^{i\omega(t-t')}g(x,t-t') = \int_{0}^{\infty} d(t-t')e^{i\omega(t-t')}g(x,t-t')\]

where the last transformation reflects the causal nature of \(g(x,t-t')\). It is also convenient to define \(g(x,z)\) which is a function of the complex variable: \(z = \omega + i\epsilon\), for \(z\) in either the upper or lower half complex plane according
to the choice of the sign of the exponential. The function \( g(x,z) \) approaches \( g(x,\omega) \) as \( \varepsilon \) approaches 0, and is clearly analytical and bounded in the defined upper or lower half \( z \)-plane.

B. Stationary Response of a Finite Cable to a Correlated Moving Turbulent Force Field

The problem of predicting the motion of a finite cable excited along its length by a moving random pressure fluctuation is studied. A one-dimensional source correlation for this turbulent field deals with the expected loss of correlation (due to flow viscous loss and the introduction of new random components) by assuming a "modified" exponentially decaying correlation function.

A coincidence effect appears between the flow velocity of the forcing field and the velocity of the waves on the cable. The larger the damping of the cable, however, the more diminished is the magnitude of any coincidence effects which appear.

In 1956 Lyon (1) investigated the response of a finite ribbon to a moving, random pressure fluctuation. In this section we calculate the mean square response of various modes of the cable by assuming a source correlation function not simply as exponentially decaying (to reflect a loss of correlation) but as harmonically varying exponential decay (17). This assumption is seen as being a step closer than Lyon's results toward reflecting the random nature of the correlation loss. The results obtained
yield a more complete expression for the mean square response, which reduces to results previously obtained by Lyon when the assumed harmonicity of the source correlation is omitted.

The correlation function for the displacement, \( u \), to a random source, \( f \), is given by:

\[
\langle u(x,t)u(x',t') \rangle_{AV} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{L} \int_{0}^{L} dx_0 dx_0' G(x,t;x_0,t_0) \cdot g(x',t';x_0',t_0') \langle f(x_0,t_0)f(x_0',t_0') \rangle_{AV}
\]

where \( G(x,t;x_0,t_0) \) is the response at the point \( x,t \) due to an impulse source at the point \( x_0,t_0 \). If the equation of motion of the cable is:

\[
c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - \beta \frac{\partial u}{\partial t} = -f
\]

and the modes of vibration are:

\[
\rho_m(x) = (\frac{2}{L})^{\frac{1}{2}} \sin \left( \frac{m\pi}{L} x \right)
\]

then the impulse response is:

\[
G_m(x,t;x_0,t_0) = \sum_m \frac{1}{\omega_d} \rho_m(x) \rho_m(x_0) e^{-\frac{\beta m^2}{2}(t-t_0)} \sin \omega_d(t-t_0)
\]

where:

\[
\omega_d = \left( c^2 \frac{n^2 \pi^2}{L^2} - \frac{\beta^2}{4} \right)^{\frac{1}{2}} = \left( \frac{\omega^2}{n} - \frac{\beta^2}{4} \right)^{\frac{1}{2}}
\]

Note that the impulse response can be separated into real and imaginary parts according to the Hilbert transform (17)
as follows:

\[ G_m(x,t;x_o,t_o) = G'_m(x,t;x_o,t_o) + iG''_m(x,t;x_o,t_o); \]

where

\[ G'_m(x,t;x_o,t_o) = \sum_m \frac{1}{\omega_d} p_m(x)p_m(x')e^{-\frac{\beta}{2}t-t_o}\sin \omega_d(t-t_o) \]

\[(11.40)\]

and

\[ G''_m(x,t;x_o,t_o) = \sum_m \frac{1}{\omega_d} p_m(x)p_m(x')e^{-\frac{\beta}{2}t-t_o}\sin \omega_d(t+t_o) \]
\[(11.41)\]

Now making the following coordinate transformations, we let:

\[ x_o + x' = \rho \]
\[ t_o + t' = \mu \]
\[ x_o - x' = \sigma \]
\[ t_o - t' = \xi \]
\[(11.42)\]

From the Jacobian of this transformation, we obtain:

\[ dx_0dx'_o = \frac{1}{2} d\rho d\sigma; \quad dt_0dt'_o = \frac{1}{2} d\mu d\xi \]
\[(11.43)\]

These are used to find new limits for \( \mu \) and \( \xi \).

The mean square response, therefore, for the \( m^{th} \) mode is:

\[ \langle u^2_m \rangle = \frac{1}{2\omega_d^2} \int_0^{2t} d\mu e^{-\frac{\beta}{2}(2t-\mu)} \int_{-\infty}^{\infty} d\xi \sin \omega_d \left[ t-\frac{\beta}{2}(\mu+\xi) \right] 
\cdot \sin \omega_d \left[ t-\frac{\beta}{2}(\mu-\xi) \right] R_{mm}(\xi) \]

where

\[ R_{mm}(\xi) = \int_0^L dx_0 \int_0^{x_0} dx'_o p_m(x_o)p_m(x'_o) \left\langle f(x_o,t_o)f(x'_o,t'_o) \right\rangle \]
\[(11.44)\]
and:

\[
\langle f(x_o, t_o) f(x'_o, t'_o) \rangle = D S^2(\alpha) \delta(\sigma - v \xi) e^{-|\xi|/\Theta} \cos \Omega \xi \quad (11.45)
\]

Note that we have assumed a source correlation which models the expected loss of correlation as harmonically varying, exponentially decaying; \( \Theta \) is a mean lifetime for the statistical state. We have also transformed the space coordinate to a moving system. The dependence of the strength of the fluctuations on the average flow velocity is given by \( S(\alpha) \). Using this assumed correlation field, we now proceed to calculate the mean square response of various modes of the cable. Letting \( \gamma = 2t - \mu \), we rewrite Eq. (11.45) as:

\[
\langle u^2_m \rangle = \frac{D S^2(\alpha)}{2\omega_m^2} \int_0^L \int_0^L dx_o dx'_o \rho_m(x_o) \rho'_m(x'_o) \int_0^\infty d\eta \ e^{-\frac{B}{2} \eta} \cdot
\]

\[
\cdot \int_{-\gamma}^{\gamma} d\xi \ e^{-|\xi|/\Theta} \left\{ \cos \omega_m \xi - \cos \omega_m \eta \right\} \cos \Omega \xi \delta(\sigma - v \xi)
\]

(11.46)

Now integrating Eq. (11.46) with respect to \( \xi \), we operate on the delta function, letting \( \xi = \sigma/v \), and obtain:

\[
\langle u^2_m \rangle = \frac{D S^2(\alpha)}{2\omega_m^2} \int_0^L \int_0^L dx_o dx'_o \rho_m(x_o) \rho'_m(x'_o) e^{-\left(\frac{B}{2v} + \frac{1}{\Theta_v}\right)} \cdot
\]

\[
\cdot \left\{ \frac{\omega_m^2}{\omega_m^2} \left[ \cos(\omega_m + \Omega) \frac{\sigma}{v} + \cos(\omega_m - \Omega) \frac{\sigma}{v} \right] + \frac{\omega_m}{2\omega_m^2} \right\} \cdot
\]

\[
\cdot \left[ \sin(\omega_m + \Omega) \frac{|\sigma|}{v} + \sin(\omega_m - \Omega) \frac{|\sigma|}{v} \right] \quad (11.47)
\]
We again employ Eqs. (11.42) and (11.43) to transform the space integrations and find new limits for \( \rho \) and \( \sigma \). The integration over the space variables extends over the region shown in Fig. 11.1. Note the integrand is symmetric about the lines \( \rho = L \) and \( \sigma = 0 \); hence we may now integrate over the shaded triangle and multiply by 4, to obtain:

\[
\left< u^2_m \right> = \frac{DS^2(\Omega)^2}{2\omega_{dm}^2 \omega_m v L} \left\{ \int_0^L d\rho \int_0^\rho d\sigma \ e^{-\frac{\beta}{2v} + \frac{i}{\sigma v}} \right\} \left\{ -\cos k_m \rho \right. \\
\cdot \left[ \frac{2\omega_{dm}}{\beta} \left( \cos \omega_{dm} \frac{\sigma}{v} + \cos \omega_{dm} \frac{\sigma}{v} \right) + \left( \sin \omega_{dm} \frac{\sigma}{v} + \\
+ \sin \omega_{dm} \frac{\sigma}{v} \right) \right] + \frac{1}{2} \left[ \sin \left( k_m (1 + \frac{\delta_m'}{\alpha}) \right) \sigma - \sin \left( k_m \right) \sigma \\
\cdot \left( 1 - \frac{\delta_m'}{\alpha} \right) \sigma + \sin \left( k_m (1 + \frac{\delta_m''}{\alpha}) \right) \sigma - \sin \left( k_m (1 - \frac{\delta_m''}{\alpha}) \right) \sigma \right] + \\
+ \frac{\omega_{dm}}{\beta} \left[ \cos \left( k_m (1 + \frac{\delta_m'}{\alpha}) \right) \sigma + \cos \left( k_m (1 - \frac{\delta_m'}{\alpha}) \right) \sigma + \\
+ \cos \left( k_m (1 + \frac{\delta_m''}{\alpha}) \right) \sigma + \cos \left( k_m (1 - \frac{\delta_m''}{\alpha}) \right) \sigma \right] \right\} \tag{11.48}
\]

where: \( \omega_{dm}' = \omega_{dm} + \Omega; \omega_{dm}'' = \omega_{dm} - \Omega; \)

\( \delta_m = \frac{\omega_{dm}}{\omega_m}; \omega_m = k_m c; \alpha = v/c; \)

\( \delta_m' = \delta + \frac{\Omega}{\omega_m}; \delta_m'' = \delta - \frac{\Omega}{\omega_m} \)

Figure 11.1
We shall now allow the integral over $\sigma$, from $0 \rightarrow \rho$, to go to $\pm \infty$, since most of the contribution is near the line $\sigma = 0$. Therefore, we must have $c\Theta \ll L$ in order that our results be valid; we find:

$$\langle u_m^2 \rangle = \frac{D \Sigma^2(\alpha)}{2v\omega \alpha_m^2} \left\{ \frac{k_m}{2} \left( 1 + \frac{\delta_m'}{\alpha} \right) + \frac{\omega_m}{\beta} \left( \frac{\beta}{2v} + \frac{1}{\Theta v} \right) \right\} +$$

$$\frac{\omega_m}{\beta} \left( \frac{\beta}{2v} + \frac{1}{\Theta v} \right) + \frac{k_m}{2} \left( 1 - \frac{\delta_m'}{\alpha} \right) \right\}$$

$$\frac{\omega_m}{\beta} \left( \frac{\beta}{2v} + \frac{1}{\Theta v} \right) + \frac{k_m}{2} \left( 1 - \frac{\delta_m'}{\alpha} \right) \right\}$$

$$\frac{\omega_m}{\beta} \left( \frac{\beta}{2v} + \frac{1}{\Theta v} \right) + \frac{k_m}{2} \left( 1 - \frac{\delta_m'}{\alpha} \right) \right\}$$

By cross-multiplying the first and second terms with each other and likewise for the third and fourth terms, we could rewrite this as:

$$\langle u_m^2 \rangle = \frac{D \Sigma^2(\alpha)}{2v} \frac{2\beta}{\alpha_m^2} \frac{2\beta}{\alpha_m^2} \frac{4\beta}{\alpha_m^2} \frac{4\beta}{\alpha_m^2} \left\{ \frac{A}{B} + \frac{C}{D} \right\} \quad (11.49)$$

where:

$$A = \left[ \left( \frac{Q_m + m\pi \lambda}{2} \right)^2 + \left( \frac{m\pi \lambda Q_m}{2} \right)^2 \right] + \frac{\alpha^2}{Q_m^2} \left[ \frac{1}{2\omega_m} \left( 1 + \frac{\Omega \alpha^2}{2\omega_m} \right) \right]$$

$$+ \left( \frac{1}{2\omega_m} \right) \left\{ \left( \frac{1}{2\omega_m} \right)^2 + \left( \omega_m \Theta + \Omega \Theta \right)^2 \right\} \quad (11.51)$$
\[ C = \left( Q_m + m \pi \lambda \right)^2 + (m \pi \lambda Q_m)^2 \left\{ \alpha^2 + 1 + \frac{m \pi \lambda}{Q_m} \left( 1 - \frac{\alpha^2}{2 \omega_{dm}} \right) \right\} + \Omega (m \pi \lambda Q_m) \cdot \left\{ \frac{1}{\omega_{dm}^2} (Q_m + m \pi \lambda)(\Omega - 2\omega_{dm} \theta) + \frac{1}{2 \omega_{dm}} \left( (1 + m \pi \lambda)^2 + (\omega_{dm} \theta - \pi \lambda)^2 \right) \right\} \]

\[ (11.52) \]

\[ B = \left( Q_m + m \pi \lambda \right)^2 + 2 Q_m (m \pi \lambda)^2 (\alpha^2 + 1) (Q_m + m \pi \lambda) + (m \pi \lambda)^4 \left[ Q_m^2 (\alpha^2 - 1)^2 + \alpha^2 \right] + \frac{\Omega}{2 \omega_{dm}^2} \left( 2 \omega_{dm} + \Omega \right) \left\{ (m \pi \lambda)^2 (2 Q_m + m \pi \lambda)^2 - (m \pi \lambda)^4 \left[ 4 Q_m^2 (\alpha^2 - 1) + \right. \right. \]

\[ \left. \left. + 1 \right] + 2 \Omega (m \pi \lambda Q_m)^2 \left( 2 \omega_{dm} \theta^2 + \Omega \theta^2 \right) \right\} \]

\[ (11.53) \]

\[ D = \left( Q_m + m \pi \lambda \right)^2 + 2 Q_m (m \pi \lambda)^2 (\alpha^2 + 1) (Q_m + m \pi \lambda) + (m \pi \lambda)^4 \left[ Q_m^2 (\alpha^2 - 1)^2 + \alpha^2 \right] + \frac{\Omega}{2 \omega_{dm}^2} \left( \Omega - 2 \omega_{dm} \right) \left\{ (m \pi \lambda)^2 (2 Q_m + m \pi \lambda)^2 - (m \pi \lambda)^4 \left[ 4 Q_m^2 \cdot \right. \right. \]

\[ \left. \left. \cdot (\alpha^2 - 1) + 1 \right] + 2 \Omega (m \pi \lambda Q_m)^2 (\Omega \theta^2 - 2 \omega_{dm} \theta^2) \right\} \]

\[ (11.54) \]

where: \( Q_m \beta = k_m C, \lambda = l/l = C \theta / L, \) and \( \omega_m^2 = k_m^2 C^2 \)

Now looking at terms constant to both numerators (A and C) and to both denominators (B and D), recall that there is still a \( v \) left as yet unused in the denominator of the \( DS^2 (\alpha) \) term of Eq. (11.50). Bringing this \( v \) into the constant denominator term, we have:

\[
\frac{\text{NUM}}{\text{DEN}} = \frac{2 \theta}{C^3 \alpha^3 \theta^3} \cdot \frac{\alpha^4 C^4 \theta^4}{v \omega_m^2 \beta^2} = \frac{2 \theta}{\omega_m^2 \beta}
\]

Now multiplying both numerator and denominator by \( m \lambda Q_m \), we obtain:
Now substituting Eqs. (11.51) - (11.54) and the above expression for typical numerator-denominator factor into Eq. (11.50), we obtain:

\[ <u_m> = \frac{DS^2(v)m\lambda Q_m}{C^4} \left\{ \frac{A}{B} + \frac{C}{D} \right\} \]  

(11.55)

where we have defined a new constant, \( S(v) \), obtained by incorporating \( n/k_m^4 \) into \( S(\alpha) \).

Note that when we let \( \Omega \) go to zero, we obtain the same result as Lyon (1), where he used a \( (4\pi/\rho) \) factor in his forcing term (see Lyon's Eq. (21), p. 395).

Note also that Lyon's 1956 publication contains a typographical error on this same page 395, Eq. (21). The denominator of the bracketed term has a \( (m\pi\lambda)^4 \) term missing, and should read as does his thesis (15): 

\[ (Q_m + m\pi\lambda)^2 + 20_m(m\pi\lambda)^2(\alpha^2 + 1)(Q_m + m\pi\lambda) + (m\pi\lambda)^4 \left[ Q_m^2(\alpha^2 - 1) + \alpha^2 \right] \]

which agrees with our Eqs. (11.53) and (11.54) for \( \Omega = 0 \).

For completeness of results and for ease in cross-checking our results with those of Lyon (1), a listing of Lyon's results will be given below (with aforementioned correction to his Eq. (21) included).

A. The Finite String: Lyon finds the response of the various modes of a finite string subject to a purely random loading to be:

\[ <\phi_m^2>_{AV} = \frac{8\pi^2D/\beta}{\omega_m^2} \]
This is Lyon's Eq. (11), page 392, which corresponds, as mentioned above, to our Eq. (11.35b) with $k_0 = 1$, and the notation substitution of $\beta$ and $\omega_n$ substituted for $\beta$ and $\omega_m$.

B. The Response of a Finite String to a Moving Turbulent Stream: Lyon finds the response of a finite string when acted upon by turbulence along its length to be:

$$<\Phi_m^2> = \frac{16\pi^2 D_g^2(v)m\lambda Q_m}{\rho_c^2 C} \cdot \left\{ \frac{(Q_m+m\pi\lambda)^2+(m\pi\lambda Q_m)^2(\alpha^2+1+m\pi\lambda/Q_m)}{(Q_m+m\pi\lambda)^2+2Q_m(m\pi\lambda)^2(\alpha^2+1)(Q_m+m\pi\lambda)+(m\pi\lambda)^4\left[\Phi_m^2(\alpha^2-1)+\alpha^2\right]} \right\}$$

This is the corrected version of Lyon's Eq. (21), page 395, which corresponds to our Eq. (11.55) with $\Omega \rightarrow 0$ and the notation substitution of $S(v)$ for $g(v)$. Note also that Lyon used a $(4\pi/\rho_c^2)$ factor in his forcing term which accounts for the added $(16\pi^2/\rho_c^2)$ factor in Lyon's Eq. (21).

Lyon also investigates two other cases of string response which we will also list here for added completeness:

C. The Piecewise Delayed Excitation: Here Lyon investigates a finite string extending from $0 \rightarrow L$ which is excited uniformly from $0 \rightarrow L/2$ by a purely random function of time $f(t)$. The signal is then delayed and fed to the second half of the string, $L/2 \rightarrow L$. The mean square response is then calculated for even and odd modes and found to be:

$$<\Phi_m^2>_{AV} = \frac{64LD_0}{\rho_c^2 \omega_1 \omega_2 \omega_m^2} e^{-\beta|\tau_o|/2} \left\{ 2\omega_1, m e^{\beta|\tau_o|/2} \left( \cos \omega_1, m \tau_o \right) - \beta\sin \omega_1, m |\tau_o| \right\}$$
These are Lyon's Eqs. (16) and (16a), pp. 393. Here $\omega_{1,m}$ is the damped natural frequency, $\omega_m$ is the undamped natural frequency, and Lyon again employs a $(4\pi/\rho L)$ term in his forcing function.

**D. Response of an Infinite String to Turbulent Flow:** Here Lyon investigates the response of an infinite string when excited by turbulent flow along its length. The response in this case must be the mean square velocity response since that is the energy density for the infinite string. The response is found to be:

$$
\langle u^2 \rangle = \frac{8\pi D\alpha^2 L^2}{C^2} \int_{-\infty}^{\infty} d\xi \psi(\xi)
$$

where:

$$
\psi(\xi) = \frac{\alpha^2}{(1-\alpha^2)} \left\{ \frac{4\pi^2}{\pi^2 \alpha^2} \left[ \frac{1+\xi\alpha}{\lambda\alpha^2} \right] + \frac{\xi^2}{\alpha^2 \pi^2 \alpha^2} \left[ \frac{1}{\lambda} + \frac{\pi}{10} \right] \right\}
$$

and

$$
k_{1,2}^2 = \frac{\alpha^2}{1-\alpha^2} \left\{ \frac{1}{\lambda} \left[ \frac{1}{\alpha^2} + 1 \right] \left[ \frac{1}{\lambda} + \frac{1}{L_o} \right] + \frac{1}{2L_o^2} \right\} +
$$

$$
+ \left\{ \frac{1}{\alpha^2} \left[ \frac{1}{\lambda} + \frac{1}{L_o} \right]^2 + \frac{1}{\lambda L_o^2} \left[ \frac{1}{\alpha^2} + 1 \right] \left[ \frac{1}{\lambda} + \frac{1}{L_o} \right] + \frac{1}{4L_o^2} \right\}^{1/2}
$$

These are Lyon's Eqs. (28) and (29), pp. 397. Eq. (29), however, is incorrect, due, most likely, to a typographical
error in the May 1956 issue of the Journal of the Acoustical Society of America. The errors were determined after carefully studying and checking Lyon's Ph.D. thesis (15) calculations. The corrected Eq. (29) should be:

$$\Psi(\xi) = \frac{\alpha^2}{(1-\alpha^2)} \left\{ 4\pi^2 \xi^2 \left[ 1 + \frac{\alpha^2}{\alpha^2} \right] + \frac{\xi^4}{\alpha^2} \frac{\alpha}{\Pi \lambda^2} \frac{\alpha}{\lambda} + \frac{\alpha}{10} \right\}$$

The assumption that the errors in Lyon's Eq. (29) are typographical is further substantiated by many similar errors of term omissions and misrepresentations in this 1956 article which disagree with Lyon's own Ph.D. thesis. In particular, his Eq. (21) on page 395, cited above, was also incorrectly written in the 1956 article.
SECTION III

RESPONSE OF PLATES TO RANDOM FORCE FIELDS

A. Temperature Dependent Response of a Viscoelastic Plate

To Nonstationary Random Excitation

We shall be investigating the temperature dependent mean square response of a viscoelastic plate under the influence of nonstationary random excitation. The nonstationary random excitation can be generated by multiplying a sample function from a stationary process by a deterministic envelope function. The excitation can therefore be written as \( s(t) = e(t)\alpha(t) \) where \( e(t) \) is a well-defined envelope function, and \( \alpha(t) \) is a Gaussian narrow band stationary statistical part of the excitation which has zero mean.

The mean square response will be developed in terms of the viscoelastic plate frequency response function (or Green's function in frequency domain) and the generalized spectral density of the input excitation. Both white noise and noise with an exponentially decaying harmonic correlation function are extended to include a rectangular unit step envelope function.

General Response Function Theory

The general plate displacement equation can be written as:

\[
\left( \frac{B}{M} v^4 + \alpha^2 + \dot{\alpha} \right) u = f/M \quad (III.1)
\]
where \( u(x, y, t) \) is the displacement of the neutral plane of the plate, 
\( B \) the bending stiffness, \( M \) the mass per unit area of the plate, \( \gamma \) is 
the attenuation coefficient. The bending stiffness is given by \( B = Eh^3/12(1-v^2) \) where \( E \) is Young's modulus, \( h \) the plate thickness, and 
\( v \) Poisson's ratio.

The mean square response is given by:

\[
\langle u(r_1, t)u(r_2, t_2) \rangle = \iint dr_1 dr_2 \iint g(r_1, t_1; r_2, t_2) \cdot \\
g(r_2, t_2; r_1, t_1) f(r_1, t_1) f(r_2, t_2) dt_1 dt_2 > (III.2)
\]

We consider here as an example a simply supported plate of dimensions 
\( L_x \) and \( L_y \). The deflection and bending moment are both zero at the plate 
edges, and the boundary conditions are: \( u = \frac{\partial^2 u}{\partial x^2} = 0 \) at \( x = 0, L_x \); 
\( u = \frac{\partial^2 u}{\partial y^2} = 0 \) at \( y = 0, L_y \). The normalized solutions of Eq. (III.1) 
which obey these boundary conditions are:

\[
P_{mn}(r) = \frac{2}{L_x L_y} \sin k_m x \sin k_n y; \quad k_m = m\pi/L_x; \quad k_n = n\pi/L_y \quad (III.3)
\]

The Green's function for the plate can now be written as:

\[
g_{mn}(r, t; r', t') = \sum_{m,n} \frac{P_{mn}(r) P_{mn}(r')}{\omega_{mn}^d} \exp [ - \gamma_{mn}(t-t')] \cdot \\
\sin [ \omega_{mn}^d (t-t')] \eta(t-t') \quad (III.4)
\]

where \( \omega_{mn}^d = \sqrt{\frac{1}{m \frac{E}{M} n^2 \gamma^2}} \) and \( \gamma_{mn} = \gamma_v + \gamma_t = \gamma \)

when we consider the temperature effects in the high frequency case.
For the ocean structures we have the following expressions for
\[ u_{mn}^2 = \frac{4}{\text{Imn}} \cdot \frac{B/M - \gamma^2_{mn}}{\gamma_{mn} = (\gamma_v)} + \frac{4}{\text{Imn}} \cdot \frac{B\eta/2M}{\omega_{mn}}, \]

where \( \eta \) is the hysteretic damping coefficient, \( \gamma_v \) is the viscous attenuation and \( \gamma_t \) is the temperature dependent attenuation coefficient. (For this model refer to Dyer's work).

We are assuming here that the displacement, \( u(r,t) \), can be separated into temporal and spatial parts:

\[ u_{mn}(r,t) = g_{mn}(r,t; r', t-t') * f(r', t') \quad (\text{III.5}) \]

and

\[ f(r', t') = q(x')q(y')s(t'); s(t') = e(t')a(t') \quad (\text{III.6}) \]

We are to determine the mean square response \( \langle u^2(r,t) \rangle \) when \( e(t') \) is both a unit step and a rectangular step, and \( \alpha(t') \) has the correlation functions:

\[ R_\alpha(\tau) = 2\pi K_o \delta(\tau); R_\alpha(\tau) = K_o \exp \left( \frac{-\beta|\tau|}{\Omega} \right) \cos \Omega \tau \quad (\text{III.7}) \]

where \( \tau = t_2 - t_1 \).

The first correlation is that for white noise, while the second is for correlated noise. Here we are assuming a two-dimensional forcing function, \( f(r', t') \), whose spatial part is purely random; therefore,

\[ \langle q(x')q(x') \rangle \langle q(y')q(y') \rangle = D \delta(x' - x') \delta(y' - y') \]

Substituting into Eq. (III.2) we obtain:
\[ u(r_1, t_1)u(r_2, t_2) = \frac{1}{M^2} \int \int \Sigma \Sigma \Sigma \Sigma dx_1^i dy_1^i dx_2^i dy_2^i \cdot \]
\[ \cdot P_m(x_1)P_n(y_1)P_m(x_1')P_n(y_1')P_p(x_2)P_q(y_2)P_p(x_2')P_q(y_2') \]
\[ \cdot \int \int g_{mn}(t_1-t_1')g(t_1-t_1')<q(x')q(x')><q(y')q(y')> \]
\[ \cdot <s(t')s(t')> dt' dt' \quad (III.8) \]

Operating now on both delta functions, we obtain:

\[ <u(r_1, t_1)u(r_2, t_2)> = \frac{D}{M^2} \int \int \Sigma \Sigma \Sigma \Sigma dx_1^i dy_1^i P_m(x_1)P_n(y_1) \cdot \]
\[ \cdot P_m(x_1)P_n(y_1)P_p(x_2)P_q(y_2)P_p(x_2')P_q(y_2') \int \int g_{mn}(t_1-t_1') \cdot \]
\[ \cdot g_{pq}(t_2-t_2')<s(t')s(t')> dt' dt' \quad (III.9) \]

Now, from orthogonality, we can replace the integral over \( x_1' \) with a Kronecker delta, \( \delta_{mp} \), and the integral over \( y_1' \) with another Kronecker delta, \( \delta_{nq} \), and operating on both Kronecker deltas now, we let \( p \) go to \( m \) and \( q \) go to \( n \), and obtain:

\[ <u(r_1, t_1)u(r_2, t_2)> = \frac{D}{M^2} \Sigma P_{mn}(r_1)P_{mn}(r_2) \int \int g_{mn}(t_1-t_1') \cdot \]
\[ \cdot g_{mn}(t_2-t_2')<s(t')s(t')> dt' dt' \quad (III.10) \]

Now, to obtain the mean square response, we let \( r_2 = r_1 = r \), and obtain:
\( \langle u(r, t_1)u(r, t_2) \rangle = D/M^2 \sum_{m,n} P_{mn}(r) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{mn}(t_1, t'_1) \times g_{mn}(t_2, t'_2) \times s(t_1) \times s(t_2) \times dt'_1 \times dt'_2 \)  

\[ (111.11) \]

Now looking at the time integrals separately, we have:

\[ \int_{-\infty}^{\infty} g_{mn}(t_1, t'_1) s(t'_1) dt'_1 = g_{mn}(t'_1) s(t'_1) \]

\[ (111.12) \]

or

\[ \langle u(r, t_1)u(r, t_2) \rangle = D/M^2 \sum_{m,n} P_{mn}(r) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} G_{mn}(\omega_1) S(\omega_1) e^{i\omega_1 t_1} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} G^*_{mn}(\omega_2) e^{i\omega_2 t_2} \langle S^*(\omega_1)S(\omega_2) \rangle \]

\[ (111.13) \]

where \( G_{mn}(\omega_1) \) and \( S^*(\omega_1) \) are complex conjugates of \( G_{mn}(\omega_1) \) and \( S(\omega_1) \). A further investigation into the statistical time part of the response, represented by the term in pointed brackets in Eq. (111.13), is now necessary.

\[ \langle S^*(\omega_1)S(\omega_2) \rangle = \int e^{i\omega_1 t_1} e^{-i\omega_2 t_2} e(t_1) e(t_2) \langle \alpha(t'_1)\alpha(t'_2) \rangle \]

\[ (111.14) \]

where:

\[ \langle \alpha(t'_1)\alpha(t'_2) \rangle = R_\alpha(\tau) = \int_{-\infty}^{\infty} P_\alpha(\omega) e^{i\omega(t'_2-t'_1)} \frac{d\omega}{2\pi} \]

\[ (111.15) \]

We can now find the mean square response by letting \( t_1 = t_2 = t \), and Eq. (111.13) becomes:
\[
\langle u^2(r, t) \rangle = D/M^2 \sum_{m,n} \left( \frac{2}{2\pi} \right) \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} G_{mn}^*(\omega_1) \cdot \\
G_{mn}(\omega_2) e^{-i\omega_1 t} e^{i\omega_2 t} \int dt_1 \int dt_2 e(t_1) e(t_2^*) e^{i\omega_1 t_1} e^{i\omega_2 t_2}. \\
e^{-i\omega_2 t_2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t_2^* - t_2)} P_\alpha(\omega)
\] (III.16)

Now define:

\[
S_e(\omega - \omega_1) = \int_{-\infty}^{\infty} \frac{dt_1}{2\pi} e^{-i(\omega - \omega_1)t_1^*} e(t_1) \\
S_e(\omega_2 - \omega) = \int_{-\infty}^{\infty} \frac{dt_2}{2\pi} e^{-i(\omega_2 - \omega)t_2^*} e(t_2)
\] (III.17)

Note that when \( \omega_1 = \omega_2 \), \( S_e(\omega - \omega_1) \) and \( S_e(\omega_2 - \omega) \) are complex conjugates of one another, and Eq. (III.16) becomes:

\[
\langle u^2(r, t) \rangle = D/M^2 \sum_{m,n} \left( \frac{2}{2\pi} \right) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |A(t, \omega)|^2 P_\alpha(\omega) \] (III.18)

where:

\[
|A(t, \omega)|^2 = A(m,n \omega_1) A^*(m,n \omega_2); \\
A(m,n \omega_1) = \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} G_{mn}^*(\omega_1) S_e(\omega - \omega_1) e^{-i\omega_1 t}; \\
A^*(m,n \omega_2) = \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} G_{mn}(\omega_2) S_e(\omega_2 - \omega) e^{i\omega_2 t}
\] (III.19)

We shall now proceed to calculate the mean square response, \( \langle u^2(r, t) \rangle \), by investigating Eq. (III.18) for various envelope functions and source correlations. We see from Eq. (III.18) that the nonstationary nature of the response can be investigated by examining the integral expression here which is purely time (or frequency) dependent.

We therefore let:
The mean square response to a unit step and rectangular step envelope function for both white noise inputs and correlated input excitations can now be found identically by referring to Eqs. (11.17) through (11.33), where we now have:

\[ A = \frac{ix_P}{\sqrt{2m}} \]  
\[ B = \frac{\gamma_{mn}}{2} \]

**Verification of Results**

For purposes of verification we can compare our expression for the cross product of two modes \((m,n)\) and \((p,q)\): our Eq. (11.18) with that of Dyer (2), Eq. (26). To do so, we will take the case of a unit step function with white noise input, \(\tau = 0\) and \(K_0 = 1\). Also we must make the appropriate substitutions in our notation to correspond with symbols used by Dyer, by letting: \(r \rightarrow r, r' \rightarrow r', t_1 \rightarrow t, t_2 \rightarrow t', t' \rightarrow t_0, t_0' \rightarrow t'_{0},\) \(u \rightarrow W, \omega_{mn} \rightarrow \omega_{mn}, \omega_{pq} \rightarrow \omega_{pq}, \gamma_{mn} \rightarrow a_{nn}, \gamma_{pq} \rightarrow a_{pq}.\) Also, our spatial part of the forcing function: \(D\delta(\xi_o)\delta(\xi_o)\) with \(\tau = 0.\) Making these substitutions now to our Eq. (11.18) and employing our Eq. (11.7) we obtain Dyer's Eq. (26).
\[ \langle W(r,t) W^*(r',t') \rangle_{mn,pq} = \frac{A r^2 \Phi_{mn}(r) \Phi_{pq}(r')}{\omega_{mn} \omega_{pq} M^2} \int_{-\infty}^{+} dt_0 \int_{-\infty}^{+} dt_0' \cdot \int_S dS_o \int_S dS'_o \Phi_{mn}(r_o) \Phi_{pq}(r'_o) \exp \left[ -a_{mn}(t-t_0) - a_{pq}(t'_o-t'_0) \right] \cdot \sin[\omega_{mn}(t-t_0)] \sin[\omega_{pq}(t'_o-t'_0)] \delta(\xi_o) \delta(\xi'_o) \]

where: \( P_m(x_1)P_n(y_1) \rightarrow \Phi_{mn}(r) \); \( P_m(x'_1)P_n(y'_1) \rightarrow \Phi_{mn}(r'_o) \)

\[ P_2(x_2)P_q(y_2) \rightarrow \Phi_{pq}(r') \]; \( P_2(x'_2)P_q(y'_2) \rightarrow \Phi_{pq}(r'_o) \)

and

\[ \int dx'_1 dy'_1 \rightarrow \int_S dS_o'; \quad \int dx'_2 dy'_2 \rightarrow \int_S dS'_o \]

Dyer then goes on to introduce new time variables along with corresponding limit changes on his integrals.

He then proceeds to give mean square response results for special cases (at coincidence, above, and below coincidence) by integrating his resulting equation in separate regions. However, in doing so, he makes certain limiting approximations for his \( \omega_{mn} \) and \( a_{mn} \) terms which render his final results non-exact. Later, in Section III B., we will compare our exact solution for the temperature dependent stationary response of a finite plate to a correlated moving force field to Dyer's approximate solution.

For real \( \omega \), the Green's function \( g(x,\omega) \) is usually divided into two parts: a dissipative part and a reactive part. In our case, and more generally when the system is stationary, these are given, respectively, by the imaginary and real parts of \( g(x,\omega) \), defined as: \( g''(x,\omega) \) and \( g'(x,\omega) \), defining \( g(x,\omega) = g'(x,\omega) + g''(x,\omega) \); where:
\[
g''(x, \omega) = -\frac{1}{\omega^2} \left[ \frac{\omega_n^2}{\omega_n^2 - \omega^2} \right]^2 + (\frac{1}{\omega n})^2; \quad (a)
\]
\[
g'(x, \omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2} \left[ \frac{\omega_n^2}{\omega_n^2 - \omega^2} \right]^2 + (\frac{1}{\omega n})^2 \quad (b)
\]

The Fourier transform of Eq. (a) is the imaginary odd function of time:

\[
-ig''(x, t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} g''(x, \omega) e^{i\omega(t-t')}
\]

\[
= -ie^{-(1/2) |t-t'| \sin \omega_n (1-\zeta^2)^{1/2}} \frac{\sin \omega_n (1-\zeta^2)^{1/2} |t-t'|}{2 \omega_n (1-\zeta^2)^{1/2}} \quad (c)
\]

Likewise, the Fourier transform of Eq. (b) is the real even function of time:

\[
g'(x, t-t') = \frac{1}{2} \left[ g(x, t-t') + g(x, t'-t) \right]
\]

\[
= \frac{e^{-(1/2) |t-t'| \sin \omega_n (1-\zeta^2)^{1/2} |t-t'|}}{2 \omega_n (1-\zeta^2)^{1/2}} \quad (d)
\]

Since the response is causal, or equivalently, since \(g(x, \omega)\) which we have defined above is analytic in the upper or lower half-plane, the real and imaginary parts of \(g(x, \omega)\) are related by the Hilbert transform according to the relations:

\[
g'(x, \omega) = P \int_{-\infty}^{\infty} \frac{d\omega}{\pi} g''(x, \omega) \delta(\omega' - \omega) \quad (e)
\]

\[
g''(x, \omega) = -P \int_{-\infty}^{\infty} \frac{d\omega}{\pi} g'(x, \omega) \delta(\omega' - \omega)
\]

where P implies principle value integral, that is, an integral symmetrical about the singularity. Also:
\[ g(x,\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(t-t')e^{-i\omega(t-t')} g(x,t-t') \]

where the last transformation reflects the causal nature of \( g(x,t-t') \). It is also convenient to define \( g(x,z) \) which is a function of the complex variable: \( z = \omega + i\epsilon \), for \( z \) in either the upper or lower half complex plane according to the choice of the sign of the exponential. The function \( g(x,z) \) approaches \( g(x,\omega) \) as \( \epsilon \) approaches 0, and is clearly analytical and bounded in the defined upper or lower half \( z \) - plane.

### B. Temperature Dependent Stationary Response of a Finite Plate to a Correlated Moving Force Field

The vibration of a plate exposed to the external forcing field \( f(r,t) \) is assumed to obey the classical thin plate equation (see Appendix D):

\[
\left( \frac{B}{M} \right) \nabla^4 + \frac{A}{t} + \frac{1}{\tau} \frac{\partial^2}{\partial t^2} u = \frac{f}{M} \tag{111.21}
\]

where \( u(r,t) \) is the displacement of the neutral plane of the plate, \( B \) is the plate bending-stiffness, \( M \) is the plate mass per unit area and \( \tau \) is the attenuation constant (consisting of two parts, \( \tau_T \) and \( \tau_V \), see Appendix F).

The bending-stiffness and attenuation constant are given, respectively, by:

\[
B = \frac{Eh^3}{12(1-\nu^2)} \; ; \; \; \tau_T = \frac{2\alpha h^2 E}{3C^2 h^2} \frac{(1+\nu)}{(1-\nu)} \tag{111.22}
\]
where $E$ is Young's Modulus, $h$ is the plate thickness, $\nu$ is Poisson's Ratio, $T$ is the temperature, $\alpha$ is the thermal conduction coefficient, $\alpha$ is the coefficient of thermal expansion and $C_p$ is the specific heat at constant pressure.

Boundary conditions (assumed time independent) and initial conditions are to be specified with Eq. (III.21) which, however, we shall leave to an application later.

We may associate with Eq. (III.21) the corresponding relation for the impulse response, $g(r,t; r_o, t_o)$, of the plate:

$$B/M \Delta^2 q + \left( \frac{\partial^2 g}{\partial t^2} \right) + \int \frac{\partial g}{\partial t} \delta(x-x_0)\delta(y-y_0)\delta(t-t_0)$$

(III.23)

The function $g$ is the response at $r$ and $t$ to a unit impulse at position $r_o$ occurring at time $t_o$. Then the integral expression for the displacement $u$ may be written as:

$$u(r,t) = \int_{-\infty}^{t} dt_o \int_{S} dS_o \; g(r,t; r_o, t_o) f(r_o, t_o)$$

(III.24)

where we have abbreviated $dS_o$ for $dx_o dy_o$. Eq. (III.24) is, in essence, a sum of all elemental responses of the plate for all times $t_o$ before time $t$.

We may obtain a formal representation of the impulse response in terms of the eigenfunctions of normal modes of oscillation of the plate. The normal mode $U_{mn}$ for a mode, designated by the two order numbers $m$ and $n$, is of the form:

$$U_{mn}(r,t) \sim \Phi_{mn}(r) \exp(-a_{mn} - i\omega_{mn} t)$$

(III.25)

where $a_{mn}$ is the modal damping and $\omega_{mn}$ is the damped resonance frequency. Both $a_{mn}$ and $\omega_{mn}$ are taken to be real and
positive.

The normal modes satisfy the equation:

$$B/M \, \nabla^4 \Phi_{mn} + \left( \frac{\alpha}{2} \frac{\Phi_{mn}}{\alpha t^2} \right) + \left( \frac{\Phi_{mn}}{\alpha t} \frac{\Phi_{mn}}{\alpha t} \right) = 0$$

(111.26)

Substitution of Eq. (111.25) into Eq. (111.26) gives the equation for the eigenfunctions, $\Phi_{mn}$:

$$B/M \, \nabla^4 \Phi_{mn} + \Phi_{mn}(-a_{mn} - i\omega_{mn})^2 + \Phi_{mn}(-a_{mn} - i\omega_{mn}) = 0$$

(111.26a)

or, taking only real terms:

$$\nabla^4 \Phi_{mn} + \left( \frac{M}{B}(a_{mn}^2 - \omega_{mn}^2) - \left( \frac{M}{B}a_{mn} \right) \right) \Phi_{mn} = 0$$

(111.26b)

or,

$$\nabla^4 \Phi_{mn} - \Gamma_{mn}^4 \Phi_{mn} = 0$$

(111.27)

where

$$\Gamma_{mn}^4 = \frac{M}{B}a_{mn} - \frac{M}{B}(a_{mn}^2 - \omega_{mn}^2) = \frac{M}{B} \left( \frac{\Phi_{mn}}{\alpha_{mn}^2 - (\omega_{mn}^2) \right}$$

and $\Gamma_{mn}$ are the eigenvalues, taken to be real. The eigenfunctions are orthogonal. Also, it is convenient to have the eigenfunctions normalized. Thus, they obey the integral:

$$\int_S \Phi_{mn} \Phi_{pq} \, dS = \delta_{mp} \delta_{nq}$$

(111.28)

where $\delta$ is the Kronecker delta. Solutions of Eq. (111.27) are to obey the same boundary conditions as those of Eq. (111.21). The eigenvalues, $\Gamma_{mn}$, are then determined by the particular boundary conditions of interest.

The modal damping and damped resonance frequency appearing in Eq. (111.25) are determined by the damping
coefficients and eigenvalues. From Eqs. (111.25), (111.26) and (111.27) we obtain two simultaneous equations in $a_{mn}$ and $\omega_{mn}$ as follows:

From the real terms of Eq. (111.26a) we obtained:

$$\nabla^4 \phi_{mn} + \left[ \frac{M}{B} \left( a_{mn}^2 - \omega_{mn}^2 \right) - \frac{1}{B} a_{mn} \right] \phi_{mn} = 0 \quad (111.26b)$$

From the imaginary terms of Eq. (111.26a) we obtain:

$$\left[ \frac{2M}{B} a_{mn} \omega_{mn} - \frac{1}{B} \omega_{mn} \right] \phi_{mn} = 0 \quad (111.26c)$$

or, from Eq. (111.26c), setting the bracketed term equal to zero, we obtain:

$$a_{mn} = \frac{1}{2} \quad (111.29)$$

Now, using Eq. (111.27), Eq. (111.26b) becomes:

$$\left[ \Gamma^4_{mn} + \frac{M}{B} \left( a_{mn}^2 - \omega_{mn}^2 \right) - \frac{1}{B} a_{mn} \right] \phi_{mn} = 0$$

where the term in brackets again must equal zero; therefore:

$$\omega_{mn}^2 = \frac{B}{M} \Gamma^4_{mn} - a_{mn}^2 \approx \frac{B}{M} \Gamma^4_{mn} \quad (111.30)$$

We now return to the representation for the impulse response. We introduce the Green's function $G$, which is the Fourier transform of $g$:

$$g(r,t; r_o, t_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(r,r_o; \omega) \exp \left[ -i\omega(t-t_o) \right] d\omega \quad (111.31)$$

Substituting into Eq. (111.23) we obtain:

$$\frac{B}{M} \nabla^4 G - \omega^2 G - i\omega \Gamma G = \delta(x-x_o)\delta(y-y_o)$$
The Green's function $G$ can be expanded in terms of the eigenfunctions $\Phi_{mn}$ as follows:

$$G(r, r_0; \omega) = \sum_{m, n} A_{mn}(r_0; \omega) \Phi_m(r) \Phi_n(r)$$  \hspace{1cm} (111.34)

We evaluate the coefficients $A_{mn}$ by placing Eq. (111.34) into Eq. (111.32), using Eq. (111.27), multiplying by $\Phi_{pq}$, integrating over $S$, and using Eq. (111.28), to obtain:

$$A_{mn}(r_0, \omega) = \frac{\Phi_{mn}(r_0)}{\frac{1}{4} \Gamma_{mn} - \Gamma}$$  \hspace{1cm} (111.35)

and

$$G(r, r_0; \omega) = \sum_{m, n} \frac{\Phi_m(r) \Phi_n(r_0)}{\frac{1}{4} \Gamma_{mn} - \Gamma}$$  \hspace{1cm} (111.36)

We then use Eq. (111.31) which is evaluated by the calculus of residues at the poles to obtain $g(r, r_0, t_0)$ as follows:

$$g(r, t; r_0, t_0) = \sum_{m, n} \frac{\Phi_m(r) \Phi_n(r_0)}{2\pi} \int_{-\infty}^{\infty} \frac{\exp\left[-i\omega(t-t_0)\right]}{\frac{1}{4} \Gamma_{mn} - \Gamma} d\omega$$  \hspace{1cm} (111.37)
\[ \begin{aligned} \frac{4}{\tau_{mn}} - \frac{4}{\tau} &= \frac{M}{B}(\omega_{mn}^2 + a_{mn}^2) - \frac{M}{B}(\omega^2 + i\omega) \\ &= \frac{M}{B}(\omega_{mn}^2 + a_{mn}^2 - \omega^2 - 2i\omega_{mn}) \end{aligned} \]

To find poles, we set the term in parentheses equal to zero, and taking the real part, we obtain:

\[ \omega^2 - \left[ \omega_{mn}^2 + a_{mn}^2 \right] = 0 \]

This is of the form \( ax^2 + bx + c = 0 \), where \( a = 1 \), \( b = 0 \), and \( c = -\left[ \omega_{mn}^2 + a_{mn}^2 \right] \), therefore:

\[ \omega_{1,2} = \pm \frac{1}{2} \left[ 4(\omega_{mn}^2 + a_{mn}^2) \right]^{1/2} = \pm \left[ (\omega_{mn} - ia_{mn})(\omega_{mn} + ia_{mn}) \right] \]

or, choosing to close the contour below, we have:

\[ \begin{aligned} \omega_1 &= \omega_{mn} - ia_{mn} \\ \omega_2 &= -\omega_{mn} - ia_{mn} \\ &= -(\omega_{mn} + ia_{mn}) \end{aligned} \]

Therefore, we have:

\[ \begin{aligned} \int_{-\infty}^{\infty} \frac{\exp \left[ -iz(t-t_0) \right]}{(z-\omega_1)(z-\omega_2)} \, dz &= -2\pi i \left[ \text{Res}(\omega_1) + \text{Res}(\omega_2) \right] \\ \text{or} \\ \int_{-\infty}^{\infty} \frac{\exp \left[ -iz(t-t_0) \right]}{(z-\omega_1)(z-\omega_2)} \, dz &= \frac{2\pi}{\omega_{mn}} \exp \left[ -a_{mn}(t-t_0) \right] \sin \omega_{mn}(t-t_0) \end{aligned} \]

(111.38)
Now, substituting Eq. (111.38) into Eq. (111.37), we obtain the Green's function for a plate as:

$$g(r,t;r_0,t_0) = \sum_{mn} \Phi_{mn}(r)\Phi_{mn}(r_0) \omega_{mn} \exp \left[ -a_{mn}(t-t_0) \right] \cdot \sin \left[ \omega_{mn}(t-t_0) \right] U(t-t_0)$$  \hspace{1cm} (111.39)

where $U(t-t_0)$ is the unit step function.

The correlation may be obtained from an ensemble average of the plate response (see Appendix B). With the use of Eq. (111.24), the correlation may be written as:

$$\langle u(r,t)u^*(r',t') \rangle = \int_{-\infty}^{\infty} dt_0 \int_{-\infty}^{\infty} dt'_0 \int_{S_0} dS_0 \int_{S'_0} dS'_0 \ g(r,t;r_0,t_0) \cdot \ \cdot \ g^*(r',t';r'_0,t'_0) \ <f(r_0,t_0)f(r'_0,t'_0)> \ \ \ \ (111.40)$$

Equations (111.39) and (111.40) form the general solution to the plate response problem. With the use of a particular form for the forcing function correlation, and with the adoption of plate boundary conditions for the determination of the eigenfunctions and eigenfrequencies, we may proceed to the solution of Eq. (111.40). In particular, we shall discuss the response of a finite plate to a force field, $f(r,t)$, whose correlation is described by:

$$\langle f(r,t)f^*(r',t') \rangle = A \sigma^2 \delta(\xi-v\tau) \epsilon(\xi) \exp(-|\xi|/\Theta) \cos \Omega \tau$$  \hspace{1cm} (111.41)

Response of a Simply Supported Plate

We consider here as an example a simply supported plate of dimensions $L_x$ and $L_y$. The deflection and bending
moment are both zero at the plate edges, and the boundary conditions are:

\[ u = \frac{\partial^2 u}{\partial x^2} = 0; \quad x = 0, L_x \]  
\[ u = \frac{\partial^2 u}{\partial y^2} = 0; \quad y = 0, L_y \]  

(III.42)

Normal solutions of Eq. (III.27) which obey these boundary conditions are:

\[ \Phi_{mn}(r) = \frac{2}{(L_x L_y)^{1/2}} \sin \frac{m\pi}{L_x} x \sin \frac{n\pi}{L_y} y \]  

(III.43)

where

\[ \Gamma_{mn}^2 = \left( \frac{m\pi}{L_x} \right)^2 + \left( \frac{n\pi}{L_y} \right)^2 \]  

(III.44)

With the use of Eqs. (III.41) and (III.39), we see that Eq. (III.40) for the plate displacement correlation involves the product of two doubly infinite sums, a typical term of which is the cross product of two modes \((m,n)\) and \((p,q)\):

\[ \langle u(r,t)u^*(r',t') \rangle_{mn,pq} = A_t \frac{2 \Phi_{mn}(r) \Phi_{pq}(r')}{\omega_{mn} \omega_{pq} M^2} \int_{-\infty}^{+\infty} dt_o \int_{-\infty}^{+\infty} dt'_o \]  

\[ \cdot \int_S d\Sigma_o \int_S d\Sigma'_o \Phi_{mn}(r_o) \Phi_{pq}(r'_o) \exp \left[ -a_{mn}(t-t_o) - a_{pq}(t'-t'_o) - \right] \]  

\[ \left[ \Gamma_{mn}^2 /\theta \right] \sin \omega_{mn}(t-t_o) \sin \omega_{pq}(t'-t'_o) \delta(\xi_o - \nu \tau_o) \delta(\xi'_o) \cos \Omega \tau_o \]  

(III.45)

where \(\xi_o = x_o - x'_o, \xi'_o = y_o - y'_o,\) and \(\tau_o = t_o - t'_o.\)
The $y_0$ integration is simple because of the delta function, $\delta(\xi_0) = \delta(y_0-y_0')$. Thus we have from:

$$\int_S dS_0 \int_S dS'_0 \Phi_{mn}(r_0) \Phi_{pq}(r'_0) \delta(y_0-y_0') = \delta_{nq}$$

The integration for $x$ is equally simple if we assume $v \Theta \ll L_x$. We have from:

$$\int_S dS_0 \int_S dS'_0 \Phi_{mn}(r_0) \Phi_{pq}(r'_0) \delta(z_0-v\tau_0) = \delta_{mp} \cos \alpha_m \tau_0$$

where $\alpha_m = \frac{m \pi v}{L_x}$.

Here $\alpha_m$ may be termed the convection frequency, and is interpreted as the frequency at which the turbulent field is convected past a length of plate equal to the m-model wave length. We see that by virtue of the Kronecker deltas, $\delta_{nq}$ and $\delta_{mp}$, the plate modes are statistically independent. Physically this is due to the largeness of the plate compared with the correlation size of the source. We can now rewrite Eq. (111.45) as follows:

$$\langle u(r,t) u^*(r',t') \rangle_{mn,pq} = A f \frac{\omega_{mn} \omega_{pq}}{\omega_{m} \omega_{p} M^2} \int_{-\infty}^{+} dt_0 \int_{-\infty}^{+} dt'_0 \cdot \exp \left[ -a_{mn}(t-t_0) - a_{pq}(t'-t'_0) - \left| t_0 \right| \right] \delta_{nq} \delta_{mp} \cos \alpha_m \tau_0 \cdot \cos \Omega \tau_0 \sin \omega_{mn}(t-t_0) \sin \omega_{pq}(t'-t'_0)$$

(111.46)

We now turn to the time integral of Eq. (111.46).
To facilitate integration, we introduce the variables:

\[ \gamma = (t' - t'_0) - (t - t_0) = \tau_0 - \tau \]

(III.47)

\[ \mu = (t' - t'_0) + (t - t_0) \]

From the Jacobian of this transformation, we obtain:

\[ \frac{d\tau_0}{d\gamma} \frac{dt'_0}{d\mu} = \frac{1}{2} dy \, d\mu \]

(III.48)

where the limits are: \(-\mu \leq \gamma \leq \mu; \quad 0 \leq \mu \leq \infty\)

Upon activating \(\delta_{mp}\) and \(\delta_{nq}\), Eq. (III.46) becomes:

\[ \left\langle u(r,t)u^*(r',t') \right\rangle = \frac{Af^2}{2} \frac{2}{M} \phi_{mn}(r)\phi_{mn}(r')I_{mn}(\tau) \]

(III.49)

where the time correlation integral, \(I_{mn}(\tau)\) is given by:

\[ I_{mn}(\tau) = \int_{-\infty}^{\infty} d\mu \int_{-\mu}^{\mu} d\gamma \, \exp \left[ -a_{mn} \mu - \frac{|\gamma + \tau|}{\theta} \right] \cos a_{m}(\gamma + \tau) \cos \Omega(\gamma + \tau) \left[ \cos \omega_{mn} \gamma - \cos \omega_{mn} \mu \right] \]

for \(\tau \geq 0 \)

(III.50)

The total plate displacement correlation is given by the sum over all \(m\) and \(n\) of terms like Eq. (III.49).

Because of the absolute value sign in the integrand, we must integrate Eq. (III.50) in separate regions, depending upon whether \(\gamma\) is greater or less than \(-\tau\). Thus Eq. (III.50) becomes:
\[ I_{m n}(\tau) = \int_0^\infty d\mu \int_{-\mu}^\tau d\gamma \exp \left[ \frac{\gamma + \tau}{\Theta} \right] + \int_0^\infty d\mu \int_{-\mu}^\tau d\gamma \exp \left[ -\frac{\gamma - \tau}{\Theta} \right] + \int d\mu \int_{-\mu}^\tau d\gamma \exp \left[ -\frac{\gamma - \tau}{\Theta} \right] \exp \left[ -a_{mn} \mu \right] \cos \alpha_m (\gamma + \tau) \cos \Omega (\gamma + \tau) \cdot \left[ \cos \omega_m \gamma - \cos \omega_m \mu \right]; \quad \tau \geq 0 \quad (111.51) \]

The integral for \( \tau < 0 \) is identical to Eq. (111.51) except that \( \tau \) is replaced by \(-\tau \) throughout.

**Mean Response Below Coincidence**

In many applications, the damping is low \((a_{mn} \Theta \ll 1)\) and the mean convection speed is much smaller than the coincidence speed \((v \ll v_0)\). Consequently, we will investigate Eq. (111.51) for \( \alpha_m \ll \omega_m \) (or \( \alpha_m \approx 0 \)). Also we will expand \( \tau \) to include all values \(-\infty < \tau < \infty\), thus cancelling all sine terms which will occur when Eq. (111.51) is integrated for values of \( \tau \geq 0 \) only (i.e., those for \( \tau < 0 \) cancelling those for \( \tau \geq 0 \)), and we obtain:

\[
I_{m n}(\tau) = \frac{e^{-a_{mn} |\tau|} \cos \omega_{mn} \tau}{4a_{mn}} \left[ \frac{4}{1 + a_1^2 \Theta^2} + \frac{4 \Theta}{1 + b_1^2 \Theta^2} \right] + \frac{2 \Theta e^{-a_{mn} |\tau|} \cos \omega_{mn} \tau}{(1 + a_1^2 \Theta^2)(1 + b_1^2 \Theta^2)} \left[ \cos^2 \Omega_1 \tau - \sin^2 \Omega_1 \tau \right] + \left. e^{-|\tau| \Theta} \cos \Omega_1 \left[ \frac{(a_1^2 \Theta_1^2 - 1) \Theta_1^2}{2(1 + a_1^2 \Theta_1^2)} + \frac{(b_1^2 \Theta_1^2 - 1) \Theta_1^2}{2(1 + b_1^2 \Theta_1^2)} \right] + \frac{\Theta^2 (1 - a_1 \Theta^2)}{2(1 + a_1^2 \Theta^2)(1 + b_1^2 \Theta^2)} + \frac{\Theta^2 (1 - b_1 \Theta^2)}{2(1 + a_1^2 \Theta^2)(1 + b_1^2 \Theta^2)} \right] e^{-2|\tau| \Theta} e^{-a_{mn} \tau} \left[ \frac{(1 - a_1^2 \Theta_1^2) \Theta_1^2}{(1 + a_1^2 \Theta_1^2)} \right] \]

\[
+ \left. \left[ \cos (2 \Omega + \omega_m) \tau + \cos (\omega_m - 2 \Omega) \tau \right] + \frac{(1 - b_1^2 \Theta_1^2) \Theta_1^2}{(1 + b_1^2 \Theta_1^2)} \left[ \cos (2 \Omega + \omega_m) \tau \right. \right. \]

\[
+ \left. \cos (\omega_m - 2 \Omega) \tau \right] \right] + \frac{e^{-a_{mn} |\tau|} \cos \omega_{mn} \tau}{2} \left[ \frac{(1 + a_1^2 \Theta_1^2) \Theta_1^2}{(1 + a_1^2 \Theta_1^2)} + \frac{(1 - b_1^2 \Theta_1^2) \Theta_1^2}{(1 + b_1^2 \Theta_1^2)} \right] \]

\[(111.52)\]
where: \( a' = a + \Omega, a'' = a - \Omega, a = a' + \omega, b = a' - \omega mn, d = a'' + \omega mn, e = a' - \omega mn. \)

This Eq. (111.52) together with Eq. (111.49) thus gives the correlated response of a simply supported finite plate with low damping \( (\alpha_{mn} \theta \ll 1), \) below coincidence \( (\alpha \ll \omega_{mn}). \) Here we have let: \( \alpha_m \approx 0, \alpha'_m = \Omega, \alpha''_m = -\Omega, a = \Omega + \omega_{mn}, b = \Omega - \omega_{mn}, d = -\Omega + \omega_{mn} = -b, \) and \( e = -(\Omega + \omega_{mn}) = -a. \)

In order to verify this result we can compare our findings with those of Dyer (2). In order to do so, we must first adapt our force field correlation to the case of an exponentially decaying correlation by simply letting \( \Omega \rightarrow 0. \) Thus, we let: \( a = \omega_{mn}, b = -\omega_{mn} \) and \( a^2 = b^2, \) and obtain:

\[
I_{mn}(\tau) = e^{-a_{mn} |\tau|} \cos \omega_{mn} \tau \left[ \frac{2\theta}{a_{mn} (1 + \omega_{mn}^2 \theta^2)} + \frac{2\theta a_{mn}}{(a_{mn}^2 + \omega_{mn}^2)} + e^{-2|\tau|/\theta \left( 1 + \omega_{mn}^2 \theta^2 \right) \theta^2} \right] + e^{-a_{mn} |\tau|} (1 - \omega_{mn} \theta)^2 \frac{2}{(1 + \omega_{mn}^2 \theta^2)^2} + e^{-a_{mn} |\tau|} (1 - \omega_{mn} \theta)^2 \frac{2}{(1 + \omega_{mn}^2 \theta^2)^2}.
\]

We shall see below that Dyer (2) in his Eq. (41) has neglected the last four terms in Eq. (111.53) when rounding off in his estimate of the response below coincidence with low damping.

Summary of Dyer's Results (2)

Mean Square Response at Coincidence. Dyer first
considers the case where the mean convection speed, \( v \), of the random pressure field may be the same order as the free flexural phase velocity in the plate, \( C_b \), where:

\[
C_b = \omega^2 (B/M) 
\]  

(34)

The time correlation integral \( I_{mn}(\tau) \) is then specialized to \( \tau = 0 \) as well as \( \alpha_m = \omega_{mn} \) (i.e., \( v = v_0 \)). He finds:

\[
I_{mn}(0) \sim \frac{\Theta}{\alpha_m (\alpha_m \Theta + 1)} ; \quad \omega_{mn} \Theta > 1 
\]  

(37)

In the case of low damping (\( \alpha_m \Theta \ll 1 \)), Eq. (37) shows that the response increases linearly with increasing lifetime and decreasing damping. However, note that for high damping (\( \alpha_m \Theta > 1 \)) the response is proportional to \( \alpha_m^{-2} \), corresponding to the situation in which damping decreases the displacement at resonance without appreciably broadening the band width of the resonance.

If \( \alpha_m \) is close to but not quite equal to \( \omega_{mn} \), we obtain (for the case of low damping)

\[
\omega_{mn} \Theta < 1, \quad \alpha_m \Theta \ll 1 
\]  

(2)

\[
I_{mn}(0) \sim \frac{\Theta}{\alpha_m [1 + (\alpha_m \Theta - \omega_{mn} \Theta)^2]} 
\]  

(38)

Mean Square Response Below Coincidence. Dyer then investigates our Eq. (111.51) for \( \alpha_m \ll \omega_{mn} \) and specializes to the case \( \tau = 0 \), and obtains results for two special cases: low damping and high frequency, given respectively as:

\[
I_{mn}(0) \sim \frac{2\Theta}{\alpha_m (1 + \omega_{mn}^2 \Theta^2)} ; \quad \alpha_m \Theta \ll 1 
\]  

(41)
Dyer's Eq. (41) can be compared directly to our Eq. (III.53) with \( \tau = 0 \). We see again that Dyer has neglected the last four terms in our Eq. (III.53) in his estimate of the mean square response below coincidence with low damping. The second term in Eq. (III.53) could easily be neglected for the case of low damping. The last three however, can be significant in many applications and should be included in the results.

**Displacement Correlation Below Coincidence.** Dyer lastly considers \( l^m n = c o \) as in the previous section, but with \( \tau \) different from zero. For the case of low damping, he obtains:

\[
I_{mn}(\tau) \approx \frac{2\Theta}{\omega_{mn}^2} \frac{\exp \left[-a_{mn} |\tau|\right]}{a_{mn}(1 + \omega_{mn}^2)} \cos \omega_{mn} \tau; \quad a_{mn} \ll 1
\]

Eq. (III.49) then reduces to the result obtained by Eringen (8) for the case of zero lifetime.

In reviewing Dyer's results, therefore, we see that more complete expressions for all of his results, at and below coincidence, can be found from our Eqs. (III.52) and (III.53) with appropriate limits set on \( a_{mn} \), \( \omega_{mn} \) and \( \alpha_m \) for specific cases of interest.

**Comparison of our** \( I_{mn}(\tau=0) \)** with Dyer's** \( I_{mn}(0) \)** **Below Coincidence.** For \( \tau = 0 \), we could write Eq. (III.53) as:

\[
I_{mn}(0) \approx \frac{2}{\omega_{mn}^2} \left[ 1 + \frac{1}{a_{mn} \Theta} + \frac{1}{1 + \omega_{mn}^2} \right] \omega_{mn} \Theta > > 1
\]
\[
I_{mn}(0) = \frac{2\theta}{a_m (1 + \frac{\theta^2}{\omega_{mn}^2})} + \frac{2\theta a_{mn}}{\left(a_m^2 + \frac{\theta^2}{\omega_{mn}^2}\right)} + \frac{(2\omega_{mn}^2 \theta^2 + 1)\theta^2}{\left(1 + \frac{\theta^2}{\omega_{mn}^2}\right)^2}.
\]

Comparing now the first and second terms, we have:

\[
\frac{2\theta a_{mn}}{\left(a_m^2 + \frac{\theta^2}{\omega_{mn}^2}\right)} \cdot \frac{a_m (1 + \frac{\theta^2}{\omega_{mn}^2})}{2\theta} = \frac{a_m^2 (1 + \frac{\theta^2}{\omega_{mn}^2})}{a_m \left[1 + \left(\frac{\omega_{mn}^2}{a_m^2}\right)\right]} \approx
\]

\[
\approx \frac{\omega_{mn}^2 \theta^2}{\omega_{mn}^2 / a_m^2} = a_m^2 \theta^2
\]

where dimensionally, we have:

\[
a_m = \left[\frac{1}{\theta}\right]; \quad \theta = \left[\frac{1}{\omega_{mn}}\right]; \quad \text{and} \quad \omega_{mn} = \left[\frac{1}{\theta}\right]
\]

This term, \(a_m^2 \theta^2\), is thus a number showing second order differences from the second term of Eq. (111.53) with Dyer's results.

Now comparing the first and third terms, we have:

\[
\frac{(2\omega_{mn}^2 \theta^2 + 1)\theta^2}{\left(1 + \frac{\theta^2}{\omega_{mn}^2}\right)^2} \cdot \frac{a_m (1 + \frac{\theta^2}{\omega_{mn}^2})}{2\theta} \approx \frac{2\omega_{mn}^2 \theta^2 \cdot a_m \theta}{2 \left(1 + \frac{\theta^2}{\omega_{mn}^2}\right)} \approx
\]

\[
\approx \frac{2\omega_{mn}^2 \theta^2 \cdot a_m \theta}{2\omega_{mn}^2 \theta^2} = a_m \theta
\]

Therefore our expression for \(I_{mn}(0)\) can be written as:

\[
I_{mn}(0) \approx \frac{2\theta}{a_m (1 + \frac{\theta^2}{\omega_{mn}^2})} \left[1 + a_m \theta + (a_m \theta)^2\right]
\]
We can see from this that Dyer's results are good for his assumption of $\Theta = \omega_m \zeta \ll 1$ (i.e., low damping and short source lifetime), but become more inexact for systems with larger damping subjected to sources of relatively long lifetimes.

Assuming that the damping factor, $\zeta$, has values ranging from 0.1 to 1.0 (35), we see that even for low damping ($\zeta = 0.1$) and moderate vibration frequency ($\omega_m \approx 5$ cycles/second), a source lifetime, $\Theta$, of only 1 second would result in a 75 percent discrepancy in Dyer's estimate of the mean square response.
SECTION IV

RESPONSE OF BEAMS TO RANDOM FORCE FIELDS

A. Stationary Random Response of a Finite Beam

The cross beams of a deep ocean structure, assumed uniform in density and dimensions throughout, shown in Figure 1.1 will now be considered. The action of the currents and waves impinging normal to the beam causes forces which are purely random in sequence. The responses considered will be the mean square values of deflection, slope, bending moment, and shear force along the beam. With an estimate of the bending moments $M$, bending stresses $\sigma$ can be estimated to the same degree of certainty using the classical formula $\sigma = M c / I$. Here, $M$ represents the root mean square value, $\sqrt{M^2}$, where $c$ and $I$ are defined in Fig. IV.1 for the beam shown.

$I = \frac{2bc^3}{3}$

Figure IV.1

The Bernoulli-Euler beam model.
The structure shown in Fig. 1.1 will vibrate as a whole under the impinging forces. However, only the displacements of the cross beam relative to the flexible uprights, to which the ends are attached, give rise to cross beam stresses. For simplicity in calculating the mean square response, it is assumed that the uprights are rigid and that the cross beam is long, slender and pinned at its ends. When the end restraints and the motion of the uprights are neglected, one obtains a conservative estimate of the cross beam's maximum deflection and stresses. These results and the techniques used to find them can easily be extended to beams with different boundary conditions and deflections and stresses can be inferred from these results for beams of different geometry.

A small element of the beam in Fig. IV.1 is isolated as a free body upon which act shear forces $V$, bending moments $M$, and the external excitation force per unit length $f(x,t)$. The x-directed forces, set up by an axial stretching due to end restraints during bending, are assumed negligible. External damping is also included, just as it was for the single-degree-of-freedom system. Here, damping is assumed to have two forms: the usual Coulomb damping force which resists $y$-directional motion, and is linearly proportional to the vertical velocity $\dot{y}/\dot{t}$; and a second Coulomb damping force which resists beam rotation in the $x$-$y$ plane, and is proportional to the angular
velocity $\dot{\psi}/\dot{t}$. These damping coefficients, transverse and rotatory, are respectively $c_1$ and $c_2$.

The force and moment equilibrium equations, obtained from applying Newton's second law to the beam element, are combined. The result as shown in Appendix C, Eq. (C.8) is:

$$r^2a^2 \frac{d^2y}{dx^2} + \left( \beta_1 - \beta_2 \frac{a^2}{r^2} \right) \frac{d^2y}{dt^2} + \frac{d^2y}{dt^2} = \frac{f(x,t)}{\rho A} \quad (IV.1)$$

where

$$\beta_1 = c_1/\rho A \quad ; \quad \beta_2 = c_2/\rho I = c_2/\rho Ar^2 \quad ;$$

$$a^2 = \frac{E}{\rho} = \frac{EI}{\rho Ar^2}$$

and where $\rho$ is the mass density, $A$ is the cross-sectional area, $I$ is the cross-sectional moment of inertia, and $r$ is the cross-sectional radius of gyration of the beam. This is the Bernoulli-Euler beam in which transverse deformations due to shear force $V$, the effects of rotatory inertia, and all internal or material damping have been neglected. This is physically realistic here, considering the relatively low frequency range expected for $f(x,t)$ in the ocean environment.

Consider first the steady state responses to simple harmonic forces. If the load per unit length has the form

$$f(x,t) = e^{i\omega t} \sin \frac{n\pi}{L} x \quad (IV.2)$$

then a steady state solution to Eq. (IV.1) can be assumed in the form
\[ y(x,t) = G_y e^{i\omega t} \sin \frac{n\pi}{L} x \] (IV.3)

where the quantity \( G_y \), the complex frequency response for displacement, depends on the integer \( n \) and upon the excitation frequency \( \omega \). The term \( \sin \left( \frac{n\pi x}{L} \right) \) was chosen since, for all times, the end conditions of zero deflections and zero bending moments on the beam model are met. These end conditions are, respectively

\[ y(0,t) = y(L,t) = 0 \]

\[ \frac{\partial^2 y}{\partial x^2}(0,t) = \frac{\partial^2 y}{\partial x^2}(L,t) = 0 \]

For \( n = 1 \), the beam vibrates in the first mode shape; for \( n = 2 \), the second mode shape, and so on. In the single-degree-of-freedom example, only the first mode shape was considered. Now we are including all possible mode shapes \( (n = 1, 2, 3, ...). \) Although higher modes \( (n = 2, 3, 4, ...) \) tend to be unstable, they do have practical significance. (36)

When Eqs. (IV.2) and (IV.3) are substituted into Eq. (IV.1) the result is an algebraic equation which can be solved for \( G_y \), which is

\[ G_y = \frac{1}{\partial_A} \frac{1}{(a^2 r^2 \lambda_n^4 - \omega^2) + i(\beta_1 + \beta_2 r^2 \lambda_n^2)} \] (IV.4)

where

\[ \lambda = \frac{n\pi}{L} \ ; \quad a^2 = \frac{E}{\rho} \]
The complex frequency responses $G^\psi$, $G^M$, and $G^V$, for slope, bending moment, and shear force, respectively, are found by assuming solutions

$$\psi(x,t) = G^\psi e^{i\omega t} \cos \frac{n\pi}{L} x$$

$$M(x,t) = G^M e^{i\omega t} \sin \frac{n\pi}{L} x \quad (IV.5)$$

$$V(x,t) = G^V e^{i\omega t} \cos \frac{n\pi}{L} x$$

Since $\psi = \frac{\partial V}{\partial x}$ it follows from Eq. (IV.3) and the first of Eq. (IV.5) that

$$G^\psi = \lambda_n G^V \quad (IV.6)$$

The moment-curvature relation for small deflections is

$$M = M(x,t) = EI \frac{\partial^2 y}{\partial x^2} = EI \frac{\partial \psi}{\partial x}$$

It follows from Eq. (IV.3) and the second of Eq. (IV.5) that

$$G^M = (-EI\lambda^2_n) G^V \quad (IV.7)$$

From Newton's equations of motion for angular acceleration ($\Sigma T = I \ddot{\Theta}$), it follows that:

$$V = -\frac{\partial^2 M}{\partial x^2} + \rho A r^2 \left( \frac{\partial^3 y}{\partial x^2 \partial t^2} + \beta_2 \frac{\partial^2 \psi}{\partial \tau^2} \right)$$

Since $\psi = \frac{\partial y}{\partial x}$, the shear force becomes

$$V = -\frac{\partial^2 M}{\partial x^2} + \rho A r^2 \left( \frac{\partial^3 y}{\partial x^2 \partial t^2} + \beta_2 \frac{\partial^2 y}{\partial x \partial \tau} \right)$$

This, with Eq. (IV.3) and the last of (IV.5) gives

$$G^V = \rho A r^2 \lambda_n \left( \alpha^2 \lambda^2_n + i \omega \beta_2 \right) \cdot G^V \quad (IV.8)$$
Thus the responses $G_y$, $G_M$ and $G_Y$ for harmonic excitation can all be expressed as multiples of $G_Y$.

For a linear system of one degree of freedom, the impulse-response function was found to be in the form

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega$$

Note that in the present analysis, $G(\omega)$, which is equivalent to $G_y$, $G_y$, $G_M$ or $G_M$, is no longer dimensionless, and includes constants equivalent to $k$ of the previous system.

Suppose that there are now $n$ such linear systems whose corresponding complex frequency responses are $G(n,\omega)$ as first derived. The corresponding response $g(n,t)$ of the beam due to the $n^{\text{th}}$ impulse is of this same form. For instance, for the displacement this is given by

$$q_Y(n,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_Y e^{i\omega t} d\omega$$

The corresponding impulse responses for moment and shear are also given by Eq. (IV.9) where the subscript $y$ is replaced by $M$ and $V$, respectively.

Beam Responses to Stationary, Ergodic Forces

Assume that the stationary random load can be decomposed into $n$ loadings, each with a space variation of

$$\sin \left( \frac{n\pi x}{\lambda} \right),$$

or

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{\lambda} x$$

(IV.10)

where the coefficient of this Fourier sine series is

$$f_n(t) = \frac{2}{\lambda} \int_{0}^{\lambda} f(x,t) \sin \frac{n\pi}{\lambda} x \, dx$$

(IV.11)
Now, it was shown for a single-degree-of-freedom linear system (see Appendix C, Eq. (C.35)) that the response $x(t)$ was of the form

$$x(t) = \int_{-\infty}^{\infty} F(\theta) g(t-\theta) \, d\theta$$

where $g(t)$ was the response to a unit impulsive force $\delta(t)$. Again suppose that there are $n$ such systems, each with a mode shape $\sin \left( \frac{n\pi x}{l} \right)$. The response $y(x,t)$ for the beam can then be represented as the sum of $n$ individual impulse-responses $g_y(n,t)$.

$$y(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{l} x \int_{-\infty}^{\infty} f_n(\theta) g_y(n,t-\theta) \, d\theta \quad \text{(IV.12)}$$

Here, $g_y(n,t)$ is given by Eq. (IV.9) and $f_n(t)$ is given by Eq. (IV.11).

If one uses the same procedures as outlined for the single-degree-of-freedom system, the mean square responses for the beam can be calculated. The mean square displacement response at location $x$, or the autocorrelation function evaluated at time zero (see Appendix C, Eq. (C.51)), is:

$$\overline{y^2(x,t)} = \sum_{n=1}^{\infty} \sin^2 \frac{n\pi}{l} x \int_{-\infty}^{\infty} S_f(\omega) \left| G_y \right|^2 \, d\omega \quad \text{(IV.13)}$$

If one replaces the symbol $y$ in Eq. (IV.9) and Eq. (IV.12) by $\psi$, the mean square response of the slope is found to be

$$\overline{\psi^2(x,t)} = \sum_{n=1}^{\infty} \cos^2 \frac{n\pi}{l} x \int_{-\infty}^{\infty} S_f(\omega) \left| G_\psi \right|^2 \, d\omega \quad \text{(IV.14)}$$
In like manner, it follows that

\[ M^2(x,t) = \sum_{n=1}^{\infty} \sin^2 \frac{n\pi}{L} \times \int_{-\infty}^{\infty} S_f(\omega) |G_M|^2 d\omega \quad (IV.15) \]

and

\[ V^2(x,t) = \sum_{n=1}^{\infty} \cos^2 \frac{n\pi}{L} \times \int_{-\infty}^{\infty} S_f(\omega) |G_V|^2 d\omega \quad (IV.16) \]

Equations (IV.13) through (IV.16) represent time average responses. One can obtain space average responses by averaging each of these over the length of the beam.

For instance, for the displacement, this type of mean square average, represented simply by \( \bar{E}(y^2) \), is given by

\[ \bar{E}(y^2) = \frac{1}{L} \int_{0}^{L} y^2(x,t) \, dx \]

\[ = \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} S_f(\omega) |G_y|^2 \, d\omega \quad (IV.17) \]

Similar results can easily be obtained for \( \bar{E}(\psi^2) \), \( \bar{E}(M^2) \) and \( \bar{E}(V^2) \).

**Beam Responses to White Noise**

If the beam spectral density, \( S(\omega) = S_0 \), a constant value for all \( \omega \), the integrals of the form of Eq. (IV.17) can be evaluated. Such integrations are listed in Appendix C. Only one of the results will be repeated here. That is

\[ \int_{-\infty}^{\infty} |G_y|^2 \, d\omega = \frac{\pi}{(\rho A a r)^2 (\pi/L)^4} \left[ \frac{\pi}{\beta_1 n^4 + \beta_2 r^2 (\pi/L)^2 n^6} \right] \quad (IV.18) \]

The mean square response for displacement at location \( x \).
becomes, from Eq. (IV.13) and Eq. (IV.18)

\[
\hat{y}^2(x,t) = \frac{\pi S_0}{(\rho A ar)^2(\pi/\ell)^4} \sum_{n=1}^{\infty} \sin^2 \frac{n\pi x}{\ell} \left[ \beta_1 n^4 + \beta_2 r^2(\pi/\ell)^2 n^6 \right]
\]

(IV.19)

The mean square response averaged over \( x \) given by Eq. (IV.17) and Eq. (IV.18) is

\[
\overline{E}(\hat{y}^2) = \frac{\pi S_0}{2(\rho A ar)^2(\pi/\ell)^4} \sum_{n=1}^{\infty} \frac{1}{\left[ \beta_1 n^4 + \beta_2 r^2(\pi/\ell)^2 n^6 \right]}
\]

(IV.20)

With the result of Eq. (IV.18), we can now easily find the mean square responses for slope, moment, and shear. This follows because \( G_\psi, G_M, \) and \( G_\sigma \) are simple multiples of \( G_Y \), as shown in Eq. (IV.6), Eq. (IV.7) and Eq. (IV.8).

The responses to ideal white noise are now summarized:

1. Both \( \overline{E}(\hat{y}^2) \) and \( \overline{E}(\hat{\psi}^2) \) converge only if damping is present. At least one of the constants \( c_1 \) or \( c_2 \) must be non-zero.

2. Values of \( \overline{E}(M^2) \) diverge if rotatory damping is absent (\( c_2 = 0 \)). However \( \overline{E}(M^2) \) converges if rotatory damping is present, either alone (\( c_1 = 0 \)) or in combination with transverse damping (\( c_1 \neq 0 \)).

3. Values of \( \overline{E}(V^2) \) diverge for all combinations of damping.

The engineer is most interested in \( \overline{E}(M^2) \) because from this he can estimate the root mean square bending
stress, given by \( \sigma = c \sqrt{\frac{E(M^2)}{I}} \), where \( c \) is the beam dimension of Fig. IV.1. Without at least some rotatory damping, however, such a calculation is meaningless for ideal white noise.

The useful conclusions which a design engineer can draw from this problem is that mean square values of displacement and bending moment can be calculated from the analysis of the Bernoulli-Euler beam model when at least rotatory damping is included. Also, results based on the ideal white noise assumption where \( f(x,t) \) is stationary and ergodic may be quite useful in design, even if \( S_f(\omega) \) is narrow band. Finally, if \( f(x,t) \) is Gaussian, then the probability that an arbitrary value of displacement or moment can exceed its rms value can be calculated here, as it was for the single-degree-of-freedom system. Again, it would be a matter of judgment as to what the design limits should be. If design limits were on stress for instance, then careful account should be taken of the type of material to be used. Too many excursions of stress beyond a certain limit for a given material could cause fatigue failure of the beam.

**B. Stationary Response of a Finite Beam to a Correlated Moving Force Field**

Going back now to Eq. (IV.1), we can write:

\[
\left[ \alpha_t^2 + \left( \beta_1 - \frac{EI}{\rho A} \beta_2 \alpha_x^2 \right) \alpha_t + \frac{EI}{\rho A} \alpha_x^4 \right] y(x,t) = f(x,t)
\]

(IV.21)
We can now define

\[ y(x,t) = \sum_{n=1}^{\infty} y_n(t) \sin \frac{n\pi x}{L}; \quad (IV.22) \]

\[ f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{L}; \]

where

\[ \lambda_n = \frac{n\pi}{L} \]

Now, substituting Eqs. (IV.22) into Eq. (IV.21), we obtain:

\[
\left[ a_t^2 + \left( \beta_1 - \frac{EI}{\rho A} \beta_2 \lambda_n^2 \right) a_t + \frac{EI}{\rho A} \lambda_n^4 \right] y_n(t) = f_n(t) \quad (IV.23)
\]

If we now set:

\[ \beta_1 - \frac{EI}{\rho A} \beta_2 \lambda_n^2 = \gamma_n \quad \text{and} \quad \frac{EI}{\rho A} \lambda_n^4 = \omega_n^2 \]

we obtain the damped oscillator form for the beam equation:

\[
\left[ a_t^2 + \gamma_n a_t + \omega_n^2 \right] y_n(t) = f_n(t) \quad (IV.24)
\]

We can now obtain the impulse Green's function by writing Eq. (IV.24) as:

\[
\left[ a_t^2 + \gamma_n a_t + \omega_n^2 \right] g(t-t') = \delta(t-t') \quad (IV.25)
\]

Now, from Appendix E, we find the Green's function in time to be:

\[
g_n(t-t') = \eta(t-t') \frac{e^{-\gamma_n(t-t')}}{\omega_n} \sin \omega_n(t-t') \quad (IV.26)
\]

where

\[ \omega_{nd} = \left[ \omega_n - \left( \gamma_n/2 \right)^2 \right]^{\frac{1}{2}} = \omega_n \left( 1 - \zeta^2 \right)^{\frac{1}{2}} \]
The correlation function for the displacement $y$ to a random source, $f$, is given by:

$$
\langle y(x,t)y(x',t') \rangle_{AV.} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dt_0 dt'_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_0 dx'_0 .
$$

$$
\cdot \ G(x,t;x_0,t_0) \ G(x',t';x'_0,t'_0) \left\langle f(x_0,t_0)f(x'_0,t'_0) \right\rangle_{AV.}
$$

(IV.27)

where $g(x,t;x_0,t_0)$ is the response at the point $x,t$ due to an impulse source at the point $x_0,t_0$. With the equation of motion given by Eq. (IV.24), the modes of vibration are:

$$
\rho_n(x) = \left( \frac{2}{\lambda} \right)^{\frac{1}{2}} \sin \lambda_n x
$$

(IV.28)

The impulse response is then:

$$
G_n(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{1}{\omega_n} \rho_n(x) \rho_n(x') e^{-\frac{1}{2}(t-t')} \sin \omega_n(t-t') \eta(t-t')
$$

(IV.29)

Now making the following coordinate transformations, we let:

$$
\begin{align*}
&x + x'_0 = \rho ; \quad t_0 + t'_0 = \mu \\
&x - x'_0 = \sigma ; \quad t_0 - t'_0 = \xi
\end{align*}
$$

(IV.30)

From the Jacobian of this transformation, we obtain:

$$
\begin{align*}
&dx_0 dx'_0 = \frac{1}{2} d\rho d\sigma ; \quad dt_0 dt'_0 = \frac{1}{2} d\mu d\xi
\end{align*}
$$

These are used to find the new limits for $\mu$ and $\xi$. The mean square response, therefore, for the $n^{th}$ mode is:
\[ \langle y_n^2 \rangle = \frac{1}{2 \omega_n^2} \int_{-\infty}^{\infty} d\mu e^{-\frac{d_n}{2}(2t-\mu)} \int_{-(2t-\mu)}^{(2t-\mu)} \sin \omega_n \delta \left[ t-\frac{1}{2}(\mu+\xi) \right] R_{nn}(\xi) \]  

where

\[ R_{nn}(\xi) = \int \int d\xi' d\xi'' n(\xi') n(\xi'') \langle f(x_o, t_o) f(x_o', t_o') \rangle \]

and

\[ \langle f(x_o, t_o) f(x_o', t_o') \rangle = D \, S^2(\alpha) \delta(\sigma-\nu \xi) e^{-\xi/\Theta} \cos \Omega \xi \]  

Note that we have assumed here a source correlation which models the expected loss of correlation as harmonically varying, exponentially decaying. The mean lifetime for the stationary state is given by \( \Theta \). We have also transformed the space coordinates to a moving system. The dependence of the strength of the fluctuations on the flow velocity is given by \( S(\alpha) \). Using this assumed correlation field, we now proceed to calculate the mean square response of the various modes of the beam. Letting \( \gamma = 2t-\mu \), we proceed in exactly the same manner as we did for the finite cable in Section II. Referring then to Eqs. (II.46) through (II.55), we can simply write down the mean square response as:

\[ \langle y_n^2 \rangle = \frac{DS^2(\nu)n\lambda_0}{\omega_n^4 \lambda_n^4} \left\{ \begin{array}{l} A \\frac{1}{\lambda_n^4} + B \frac{1}{\lambda_n} \\ \frac{1}{\lambda_n^4} \end{array} \right\} \]  

where \( \lambda_n = \omega_n/\lambda \) = quality factor, \( \lambda = \) ratio of the correlation length for the moving source to the length of the
beam, \( \lambda_n = (n\pi/L) \), and \( A, B, C, \) and \( D \) are given by Eqs. (11.51) through (11.54).

C. Nonstationary Random Response of a Finite Beam

The correlation function for the displacement \( y \) to a nonstationary random source, \( f \), is given by:

\[
\langle y(x_1, t_1)y(x_2, t_2) \rangle = \int_0^L dx_1 \int_0^L dx_2 \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \ g(x_1, t_1; x_2, t_2) \cdot g(x_2, t_2; x_1, t_1) \langle f(x_1, t_1)f(x_2, t_2) \rangle \quad (IV.35)
\]

If the equation of motion of a beam is:

\[
(\partial_t^2 + n_n \partial_t + \omega_n^2)y = f/M \quad (IV.36)
\]

then the appropriate Green's function is given by:

\[
g(x, t; x', t') = \sum_{n=1}^{\infty} \frac{p_n(x)p_n(x')}{\omega_{nd}} e^{-\frac{n_n(t-t')}{2} \sin \omega_{nd}(t-t') \eta(t-t')} \quad (IV.37)
\]

where \( p_n(x) = (2/L)^{\frac{1}{2}} \sin \lambda_n x \) are modes of vibration.

The attenuation constant, \( n_n = 2\omega_n \zeta \), \( \lambda_n = (n\pi/L) \), \( \zeta \) is the damping (see Appendix A), \( \omega_{nd} = \omega_n(1-\zeta^2)^{\frac{1}{2}} \), \( M \) is the beam mass per unit length, and \( \eta(t-t') \) is the unit step function.

By using the normal mode technique, the Green's function can be separated into spatial and temporal components by:

\[
g(x, t; x', t') = \sum_n p_n(x)p_n(x') g_n(t-t') \quad (IV.38)
\]

also,

\[
f(x', t') = q(x')s(t'); \quad s(t') = e(t')\alpha(t') \quad (IV.39)
\]
We are to determine the mean square response

\[ E \left[ y(x,t) \right] \]

when \( e(t') \) is both a unit step and a rectangular step, and \( \alpha(t') \) has the correlation function:

\[ R_\alpha(\tau) = 2\pi k_o \delta(\tau) \quad \text{and} \quad R_\alpha(\tau) = k_o e^{\beta |\tau|} \cos \Omega \tau \]

where \( \tau = t_2 - t_1 \). The first correlation is that for white noise, while the second is for correlated noise.

Here we are assuming a one-dimensional forcing function, \( f(x',t') \), whose spatial part is purely random; therefore,

\[ \langle q(x') q(x'') \rangle = D \delta(x'' - x') \]. Substituting now into Eq. (IV.35) and operating on the delta function, we obtain:

\[ \langle y(x,t_1) y(x,t_2) \rangle = \int S_0(t') g_n(t_1) g_m(t_2) < s(t_1) s(t_2) > dt_1 dt_2 \]  

Now, from the orthogonality condition, we can replace the integral over \( x' \) with a Kronecker delta \( \delta_{mn} \), and letting \( m \) approach \( n \), we obtain:

\[ \langle y(x,t_1) y(x,t_2) \rangle = \int S_0(t') g_n(t_1) g_n(t_2) < s(t_1) s(t_2) > dt_1 dt_2 \] 

Now looking at the time integration separately, we have:

\[ \int g_n(t_1) s(t-t_1) dt_1 = s(t_1) \ast g_n(t_1) = \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \hat{g}_n(\omega) \cdot S(\omega_1) e^{i\omega_1 t_1} \]
Eq. (IV.42) therefore becomes:

\[
\langle y(x,t_1) y(x,t_2) \rangle = \frac{2D}{M\omega_1^2} \sum_{n} \sin \lambda_n x \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \cdot
\]

\[
\cdot G_n^*(\omega_1) e^{-i\omega_1 t_{1}} G_n(\omega_2) e^{i\omega_2 t_{2}} \langle S^*(\omega_1) S(\omega_2) \rangle \quad (IV.44)
\]

where \( G_n^*(\omega_1) \) and \( S^*(\omega_1) \) are complex conjugates of \( G_n(\omega_1) \) and \( S(\omega_1) \).

A further investigation into the statistical time part of the response represented by the term in pointed brackets in Eq. (IV.44) is now necessary.

\[
\langle S^*(\omega_1) S(\omega_2) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_1 t_1} e^{-i\omega_2 t_2} \langle \alpha(t') \rangle \langle \alpha(t') \rangle dt' \ dy' \quad (IV.45)
\]

Let:

\[
\langle \alpha(t') \rangle \langle \alpha(t') \rangle = R(\tau) = \int_{-\infty}^{\infty} P(\omega)e^{i\omega(t'-t')} d\omega \quad (IV.46)
\]

We can now find the mean square response by letting \( t_1 = t_2 = t \):

\[
\langle y^2(x,t) \rangle = \frac{2D}{M\omega_1^2} \sum_{n} \sin \lambda_n x \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \cdot
\]

\[
\cdot G_n(\omega_2) e^{-i\omega_1 t_{1}} G_n^*(\omega_1) e^{i\omega_2 t_{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau' \cdot \langle \alpha(t') \rangle \langle \alpha(t') \rangle e^{i\omega_1 t_{1}} \cdot
\]

\[
\cdot e^{-i\omega_2 t_{2}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t_2'-t_1')} P(\omega) \quad (IV.47)
\]
We now define:

\[
S_{e}(\omega - \omega_1) = \int_{-\infty}^{\infty} dt' e^{i(\omega - \omega_1)t'}
\]

(IV.48)

\[
S_{e}(\omega - \omega_2) = \int_{-\infty}^{\infty} dt' e^{i(\omega - \omega_2)t'}
\]

These are the delayed Fourier transforms. Note that when \(\omega_1 = \omega_2\), \(S_{e}(\omega - \omega_1)\) and \(S_{e}(\omega - \omega_2)\) are complex conjugates of one another. Eq. (IV.47) now becomes:

\[
< y^2(x, t) > = \frac{2D}{M^2} \sum_{n} \sin^2 \lambda_n x \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left| \Lambda_n(t, \omega) \right|^2 P_{\alpha}(\omega) \quad (IV.49)
\]

where:

\[
\Lambda_n(t, \omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_n(\omega, \omega_1) S_{e}(\omega - \omega_1) e^{-i\omega_1 t} \quad (IV.50)
\]

We shall now proceed to calculate the mean square response by investigating Eq. (IV.49) for various envelope functions and source correlations. We see from Eq. (IV.49) that the nonstationary nature of the response can be investigated by examining the integral expression alone. We, therefore, let:

\[
E \left[ r_n^2(t) \right] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left| \Lambda_n(t, \omega) \right|^2 P_{\alpha}(\omega) \quad (IV.51)
\]

and

\[
< y^2(x, t) >= \frac{2D}{M^2} \sum_{n} \sin^2 \lambda_n x \left\{ E \left[ r_n^2(t) \right] \right\} \quad (IV.52)
\]

The mean square response to a unit step and rectangular step envelope function for both white noise input and correlated input excitations can now be found identically by referring to Eqs. (11.17) through (11.33), where we now have \(A = \omega_n (1 - \zeta^2)^{\frac{1}{2}}\), \(B = \left( \omega_n / 2 \right)\).
SECTION V

GRAPHS AND DISCUSSION

Figures V.1-V.6: In this thesis, \( G(k, t-t') \) is the Green's function describing the non-equilibrium behavior of the viscoelastic member (beam, plate or cable). Specifically, it gives the response to a unit impulsive external force at time \( t' \). \( G(k, t-t') \) vanishes until an external force is applied. \( G(k, t-t') \) is therefore called the retarded response function, or Green's function. Recalling that the attenuation constant, \( \alpha \), has two parts (\( \alpha = \alpha_T + \alpha_V \), see Appendix F) where \( \alpha_T \) is the temperature dependent part and \( \alpha_V \) is the viscous part, the retarded response function is plotted as a function of temperature in Figure V.1.

It is seen in Figure V.1 that an increase in temperature of the retarded responses corresponds to decaying oscillations. In order to observe the underdamped oscillations, the inequality \( \zeta < 1 \) has to be satisfied (see Appendix A). By using Hilbert and two-dimensional Fourier transforms, we can separate the real part, \( G'(k, t-t') \) (an even function of time), from the imaginary part, \( G''(k, t-t') \) (an odd function of time) of the retarded Green's function. These are illustrated in Figures V.2 and V.3. Similarly, for real \( \omega \) the Green's function \( G(k, \omega) \) is divided into two parts: a dissipative response (representing the imaginary
part of $G(k,\omega)$ which is an odd function of frequency) denoted by $G''(k,\omega)$, and a reactive response $G'(k,\omega)$ (representing the real part of $G(k,\omega)$ which is an even function of frequency). These are illustrated in Figures V.4 and V.5. Since $G(k,\omega)$ is complex in $\omega$, the absolute value is illustrated in Figure V.6.

In Figures V.7 and V.8, the ordinate axis represents the normalized rms response of the viscoelastic member (beam, plate or cable) given by:

$$
\omega_h(t) \cdot \left[ \frac{\zeta^2}{E} \right]^{1/2}
$$

and the abscissa axis represents the number of response cycles of the wave system given by Ct.

**Figure V.7:** This figure shows the normalized rms response to correlated noise modulated by a rectangular step function. Note that the middle curve has the smallest value of the harmonic part of the correlated noise, and also that for a constant $Q$ of the system, the normalized rms increases as $B/\beta$ increases.

**Figure V.8:** This figure shows the normalized rms response to white noise, modulated by a unit step envelope function, plotted for various values of $(A/Q)$, for specific values of the quality factor $Q$ and for specific values of $B/\beta$. This figure indicates that the damping value $B/C$ of the wave system affects the stationary value of the response as well as how quickly stationarity is achieved. The larger damping values corresponding to a lower $Q$ result in
lower stationary values and the mean square response obtains stationarity in a shorter duration.

Figures V.9 through V.18 are four-dimensional plots of the unit impulse response (Green's function) of the viscoelastic finite plate, plotted for two values of $\zeta$. The ordinate axis represents the response function (Eq. III.4) given by:

$$g_{mn}(r,t;r',t-t') = \sum_{mn} \frac{P_{mn}(r)P_{mn}(r')}{\omega_{mn} M} \exp \left[-\frac{1}{\omega_{mn} M} (t-t')\right] \cdot \sin \left[\omega_{mn} (t-t')\right] \eta(t-t')$$

where:

$$\omega_{mn} = \omega_{mn} (1-\zeta^2)^{\frac{1}{2}}; \quad 1_{mn} = 2\omega_{mn} \zeta;$$

$$\omega_{mn} = \frac{(Eh^3)^{\frac{1}{2}} k^2}{(12M)^{\frac{1}{2}} (1-\nu^2)^{\frac{1}{2}}}; \quad k^2 = k_m^2 + k_n^2; \quad \text{and}$$

$$P_{mn}(r) = \frac{2}{(L_xL_y)^{\frac{1}{2}}} \sin k_m x \sin k_n y; \quad k_m = \frac{m\pi}{L_x}; \quad k_n = \frac{n\pi}{L_y}$$

The abscissa axes represent distance $x$ across the plate and time. Figures V.9 through V.13 represent the unit impulse response for slices across the plate at $y = L_y/2$, $3L_y/8$, $L_y/4$, $L_y/8$, and $y=0$, respectively, plotted for $\zeta = 0.026$. Figures V.14 through V.18 similarly represent the unit impulse response for slices across the plate at the same $y$-values, plotted for $\zeta = 0.078$. All plots are for the response to a unit pulse applied at the midpoint of the plate. The larger damping value corresponds to a lower quality factor, $Q$, resulting in lower displacement response values and shorter vibration lifetimes.
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure_v1}
\caption{Green's function vs. time}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure_v2}
\caption{Real part of Green's function vs. time}
\end{figure}
Imaginary part of Green's function vs. time

![Figure V.3]

Imaginary part of Green's function vs. frequency

![Figure V.4]
Real part of Green's function vs. frequency

Figure V.5

Absolute value of Green's function vs. frequency

Figure V.6
Figure V.7

Normalized run response to the correlated noise modulated by a rectangular step function.

Figure V.8

Normalized run response to the C/QR, noise modulated by a unit step envelope function.
Unit impulse response of a finite, viscoelastic plate for a slice across the plate at $y = \frac{1}{2} L_y$ and $\zeta = 0.026$.

Figure V.10

Unit impulse response of a finite, viscoelastic plate for a slice across the plate at $y = L_y/2$ and $\zeta = 0.026$.

Figure V.9
Figure V.13

Unit impulse response of a finite, viscoelastic plate for a slice across the plate at $y = 0$ and $\zeta = 0.026$.

Figure V.14

Unit impulse response of a finite, viscoelastic plate for a slice across the plate at $y = L_y/2$ and $\zeta = 0.078$. 
Unit impulse response of a finite, viscoelastic plate for a slice across the plate at \( y = \frac{L_y}{4} \) and \( \zeta = 0.078 \).

Figure V.16

Unit impulse response of a finite, viscoelastic plate for a slice across the plate at \( y = \frac{3L_y}{8} \) and \( \zeta = 0.078 \).

Figure V.15
Figure V.17
Unit impulse response of a finite, viscoelastic plate for a slice across the plate at $y = L_y/2$ and $\zeta = 0.078$.

Figure V.18
Unit impulse response of a finite, viscoelastic plate for a slice across the plate at $y = 0$ and $\zeta = 0.078$. 
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This appendix is here attached for purposes of clarity and completeness. It contains much of the information found in the first two chapters of Thomson's Vibration Theory and Applications (31), and is presented here to show the derivation of many of the fundamental concepts and parameters of vibrating systems which are used extensively throughout this dissertation.

In general, mechanical systems may vibrate with as many as six possible degrees-of-freedom, i.e., displacements in the x-, y-, and z-directions and rotations about the x-, y-, and z-axes. Usually, however, only one, two or three of these modes are found occurring at one time in most engineering systems. The motion of the vibrating system is most easily represented by writing an equation of motion for each vibrational mode using Newton's second law.

Taking as an example the simple, undamped, single-degree-of-freedom system shown in Fig. A.1, neglecting the weight of the spring and choosing downward motion as positive, the equation of motion can be written as follows.

First, consider the system below its equilibrium position and traveling upward (Fig. A.1a). We can write:
Undamped single-degree-of-freedom system. 

(a) System below equilibrium position. 
(b) System above equilibrium position. 

Figure A.1

\[ F = ma \]
\[ -kx = m\ddot{x} \]

or \[ m\ddot{x} + kx = 0 \] (A.1)

Now, considering the system above its equilibrium position and traveling downward (Fig. A.1b), and noting that the spring force is now \[ F_{spring} = k(-x) \], we can write:

\[ F = ma \]
\[ k(-x) = m\ddot{x} \]

or \[ m\ddot{x} + kx = 0, \] as before (A.1)

Thus, as one might expect, the equation of motion for a particular system must be consistent, and can be written down, usually, by inspection.

If the motion is assumed to be harmonic in nature, we can let:

\[ x = A \sin \omega_n t + B \cos \omega_n t \]

and \[ \dot{x} = \omega_n A \cos \omega_n t - \omega_n B \sin \omega_n t \] (A.2)

and \[ \ddot{x} = -\omega_n^2 (A \sin \omega_n t + B \cos \omega_n t) = -\omega_n^2 x \]
Substituting into Eq. (A.1), therefore, we have:

\[-m\omega_n^2 (A \sin \omega_n t + B \cos \omega_n t) + k(A \sin \omega_n t + B \cos \omega_n t) = 0\]

or

\[\left[k - m\omega_n^2\right] (A \sin \omega_n t + B \cos \omega_n t) = 0 \quad (A.3)\]

Since the second term in parentheses cannot equal zero, the first term must equal zero, and we have:

\[k - m\omega_n^2 = 0\]

or

\[\omega_n = \sqrt{k/m} \quad (A.4)\]

where \(\omega_n\) = the undamped natural frequency of the motion in units of radians per unit time.

As a second example, consider the damped, single-degree-of-freedom system shown in Fig. A.2.

![Damped single-degree-of-freedom system](image)

Figure A.2

The equation of motion can now be quickly written down, as follows:

\[m\ddot{x} + c\dot{x} + kx = 0 \quad (A.5)\]
Now, choosing to express the displacement, velocity and acceleration as follows:

\[
\begin{align*}
x &= C e^{st} \\
\dot{x} &= sC e^{st} \\
\ddot{x} &= s^2C e^{st}
\end{align*}
\]  \hfill (A.6)

and substituting above, Eq. (A.5) becomes:

\[
(ms^2 + Cs + k) C e^{st} = 0 \quad \text{ (A.7)}
\]

If \( C = \text{vibration amplitude} = 0 \), the solution is trivial, therefore:

\[
ms^2 + Cs + k = 0
\]

Note: The displacement, \( x \), can be represented vectorially as a vector rotating at some frequency in the following way:

\[
x = \begin{cases} 
B \cos \omega_n t \\
A \sin \omega_n t
\end{cases}
\]

Therefore, \( x = C \sin (\omega_n t + \phi) \)

where \( C = \text{amplitude of vibration} \)

\[
= \sqrt{A^2 + B^2}
\]

and \( \phi = \text{phase angle} = \tan^{-1} \frac{B}{A} \)
and
\[ s = \frac{-C \pm \sqrt{C^2 - 4km}}{2m} \]
or
\[ s = -\frac{C}{2m} \pm \sqrt{\left(\frac{C}{2m}\right)^2 - \left(\frac{k}{m}\right)} \]  \hspace{1cm} (A.9)

or
\[ s_1 = -\frac{C}{2m} + \sqrt{\left(\frac{C}{2m}\right)^2 - \left(\frac{k}{m}\right)} \]
\[ s_2 = -\frac{C}{2m} - \sqrt{\left(\frac{C}{2m}\right)^2 - \left(\frac{k}{m}\right)} \]

and
\[ x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \]  \hspace{1cm} (A.10)

where the term under the square root sign can be either positive, negative, or zero. If this term is negative, the system is said to be "underdamped" (system continues to vibrate before motion ceases). If this term is positive, the system is said to be "overdamped" (motion ceases without vibrating), and if this term is zero, the system is said to be "critically damped" (motion rapidly ceases). (See Fig. A.3)
Let us first assume that the system is critically damped, \[ \sqrt{\frac{C^2}{(2m)^2} - \frac{k}{m}} = 0. \]

\[ \frac{C}{2m} \cdot \frac{C}{2m} = \frac{k}{m} \]

or

\[ C^2 = k \cdot \frac{4m^2}{m} = 4km \]

or

\[ C_{\text{crit.}} = 2 \sqrt{km} \quad (A.11) \]

Let us now define the standard damping factor \( \zeta \) as follows:

\[ \zeta = \frac{C}{C_{\text{crit.}}} = \frac{C}{2(km)^{\frac{1}{2}}} \quad (A.12) \]

where \( C \) is the damping constant for a particular system under investigation, and \( C_{\text{crit.}} \) is the damping constant required to make that same system critically damped. Note: \( \zeta \) is dimensionless.

Having thus defined the standard damping factor, let us look again at the square root term above in order to better understand its physical significance. To begin with, we have:

\[ \sqrt{(\frac{C}{2m})^2 - \frac{k}{m}} \]

Multiplying numerator and denominator of the first term by \( k \) and combining terms, we have:

\[ \sqrt{\frac{k}{m}} \sqrt{\frac{C^2}{4km} - 1} \]
Using Eqs. (A.4) and (A.12) this can be written as:

\[ \omega_n \int \zeta^2 - 1 = \omega'_n \]  

(A.13)

Since the term \( \int \zeta^2 - 1 \) is a constant for a particular system, it represents the effect of system damping on the natural frequency.

Looking now at Eq. (A.9), we see that \( \omega_n \) can now be written as:

\[ \omega_n = -\zeta \omega'_n \pm \omega_n \int \zeta^2 - 1 \]  

(A.14)

since

\[ \frac{C}{2m} = \frac{C}{2 \sqrt{\frac{k}{m}}} \cdot \frac{\sqrt{k}}{k} = \frac{C}{2 \sqrt{km}} \omega_n \]

Physically, therefore, the exponent "s" defines the shape of the exponentially decaying, time delayed response of a vibrating system, and Eq. (A.14) points out the dependence of the response shape on the damping constants of the system.

We can now re-establish the criteria for over-damping, underdamping, and critical damping in terms of \( \zeta \) as follows:

\[
\begin{align*}
\text{if } \int \left( \frac{-C}{2m} \right)^2 - \frac{k}{m} = \omega_n \int \zeta^2 - 1, & \text{ then:} \\
& \begin{cases} 
\text{overdamping occurs when } \int \zeta^2 - 1 = \text{positive, or } \zeta > 1 \\
\text{underdamping occurs when } \int \zeta^2 - 1 = \text{negative, or } \zeta < 1 \\
\text{and }
\end{cases} \\
& \begin{cases} 
\text{critical damping occurs when } \int \zeta^2 - 1 = 0, \text{ or } \zeta = 1 
\end{cases}
\end{align*}
\]  

(A.15)
The curves in Fig. A.3, together with Eq. (A.10), can now be reappraised in order to affirm the rapid exponential decay of both the overdamped and critically damped curves and the continued vibration of the underdamped curve.

If \( \zeta > 1 \), \( s_1 \) and \( s_2 \) are both negative; therefore Eq. (A.10) becomes:

\[
x = C_1 \frac{1}{e^{s_1 t}} + C_2 \frac{1}{e^{s_2 t}}
\]

which denotes rapid response delay due to the exponential term in the denominator.

Similarly, when \( \zeta = 1 \), \( s_1 = s_2 = -\zeta \omega_n \), and

\[
x = \left[ C_1 + C_2 \right] e^{-\zeta \omega_n t}
\]

which again denotes rapid decay.

But if \( \zeta < 1 \), and \( i = \sqrt{-1} \), we have:

\[
s_1 = -\zeta \omega_n + i \omega_n \sqrt{1-\zeta^2}
\]

\[
s_2 = -\zeta \omega_n - i \omega_n \sqrt{1-\zeta^2}
\]

\[
x = C_1 e^{-\zeta \omega_n t} + C_2 e^{-i \omega_n \sqrt{1-\zeta^2} t}
\]

Applying the trigonometric identity:

\[
e^{a+b} = e^a + e^b
\]

we can rewrite Eq. (A.18) as follows:

\[
x = e^{-\zeta \omega_n t} \left[ C_1 e^{i \omega_n \sqrt{1-\zeta^2} t} + C_2 e^{-i \omega_n \sqrt{1-\zeta^2} t} \right]
\]
or
\[
x = e^{-\zeta \omega_d t} \left[ A \sin \omega_d t + B \cos \omega_d t \right]
\]  \hspace{1cm} (A.20)

where we have defined:
\[
\omega_d = \omega_n \sqrt{1-\zeta^2}
\]  \hspace{1cm} (A.21)

as the damped natural frequency of the system.

Note Eq. (A.20), and therefore the underdamped system is characterized by both an exponentially decaying term and a periodic term which indicates that such a system is truly a vibrating system, since it continues to oscillate while slowly decaying.

**Logarithmic Decrement**

For the underdamped system Eq. (A.20) can be written in either of two ways as follows:
\[
x = e^{-\zeta \omega_d t} \left[ A \sin \omega_d t + B \cos \omega_d t \right]
\]  \hspace{1cm} (A.22)

or
\[
x = C e^{-\zeta \omega_d t} \sin (\omega_d t + \varphi)
\]  \hspace{1cm} (A.23)

where \( C = \sqrt{A^2 + B^2} \)

and \( \varphi = \tan^{-1} \frac{B}{A} \)

The constants \( A \) and \( B \) can be evaluated from the initial conditions, displacement \( x_0 \) and velocity \( \dot{x}_0 \), at time equals zero \( t_o \).

From Eq. \( (A.22) \), at \( t = 0 \), \( x = B \), and
\[ \dot{x}_0 = \left[ A \sin \omega_{nd} \hat{t} + B \cos \omega_{nd} \hat{t} \right] (-\zeta \omega_n e^{-\zeta \omega_n \hat{t}}) + \]
\[ + e^{-\zeta \omega_n \hat{t}} \left[ A \omega_{nd} \cos \omega_{nd} \hat{t} - B \omega_{nd} \sin \omega_{nd} \hat{t} \right] \]

or, at \( t = 0 \):

\[ \dot{x}_0 = -B \zeta \omega_n + A \omega_{nd} \]
\[ = -x_0 \zeta \omega_n + A \omega_{nd} \]

Therefore,

\[ A = \frac{\dot{x}_0 + x_0 \zeta \omega_n}{\omega_{nd}} \quad \text{(A.24)} \]

In Fig. A.4, a general plot of Eq. (A.23) represents the typical motion of an underdamped vibrating system.

**Figure A.4**

Typical underdamped system.

Often it is desirable to know what future amplitudes will develop, knowing only initial displacement, velocity or acceleration values. This can be determined by investigating the relationship between subsequent amplitudes in Fig. A.4, say \( x_1 \) and \( x_2 \), as follows:
\[ x_1 = C_1 e^{-\zeta \omega_n t_1} \left[ \sin (\omega_n t_1 + \phi) \right] \]
\[ x_2 = C_2 e^{-\zeta \omega_n (t_1 + T)} \left[ \sin (\omega_n (t_1 + T) + \phi) \right] \]
\[ \text{(A.25)} \]

Note: The bracketed terms are equal since they differ only by the amount \(2\pi\). We can now take the ratio of \(x_1\) to \(x_2\) to give us a relationship of amplitudes reflecting the decay in amplitude per period.

\[ \frac{x_1}{x_2} = \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n (t_1 + T)}} = \zeta \omega_n T \]
\[ \text{(A.26)} \]

Note: This is a constant for a particular system since \(\zeta\), \(\omega_n\), and \(T\) are all defined by the system constants \(k\), \(C\), and \(m\). Thus successive amplitudes (displacement, velocities or accelerations) are now known to decay by a constant factor per period. More precisely, this factor can be evaluated by taking the natural log of both sides of Eq. (A.26), giving us:

\[ \ln \frac{x_1}{x_2} = \zeta \omega_n T = \zeta \omega_n \frac{2\pi}{\omega_n} = \zeta \omega_n \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \]

or

\[ \ln \frac{x_1}{x_2} = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \]
\[ \text{(A.27)} \]

This factor is known as the logarithmic decrement of the system, and relates not only the value of \(x_1\) to \(x_2\), but also the values of any two consecutive amplitudes, i.e.:
In \( \frac{x_1}{x_2} = \text{same factor} = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}} = \ln \frac{x_2}{x_3} = \ln \frac{x_3}{x_4} = \ldots \)

or, more generally, the ratio of any two amplitudes, not necessarily consecutive, are related in the same way, as follows:

\[
\ln \frac{x_1}{x(1+n)} = \frac{2\pi \zeta n}{\sqrt{1-\zeta^2}}
\]

(A.28)

For example:

\[
\ln \frac{x_1}{x_3} = \frac{4\pi \zeta}{\sqrt{1-\zeta^2}} \quad \text{where } n = 2
\]

This relationship is quite useful, particularly in experimental work where system output data, such as that shown in Fig. A.4, has been obtained experimentally, and one wishes to find the damping constant, \( C \), of the system. (Usually \( k \) and \( m \) are easily obtained experimentally, but \( C \) is very difficult to find.) This can now be determined by reading \( x \)-values from the output data curve and using Eq. (A.28) to find the standard damping factor, \( \zeta \), and therefore, the system damping constant, \( C \). Also, in systems where damping is light, we can take \( x \)-values at larger intervals (say \( x_1, x_7, x_{13}, \text{etc.} \)) in order to facilitate the calculation of \( \zeta \).

In a practical sense, moreover, velocities or accelerations are usually measured experimentally, rather than displacements. The values \( \zeta \) and \( C \) can still be found from this data through the use of the log decrement
factor since the ratio of accelerations equals that of the velocities, which in turn equals that of the displacements. This can be shown from Eq. (A.6), as follows:

If \( x = C e^{st} \)

\[ \dot{x} = s \cdot C e^{st} = sx \]

\[ \ddot{x} = s^2 \cdot C e^{st} = s^2 x \]

Then

\[ \frac{\ddot{x}_1}{\ddot{x}_2} = \frac{s^2 x_1}{s^2 x_2} = \frac{x_1}{x_2} \]
This appendix is here attached for purposes of background information regarding many of the concepts of probability theory, distributions and averages employed throughout this dissertation. Much of the material contained herein is found in Crandall and Mark's Random Vibration in Mechanical Systems (32), Crandall's Random Vibration (33) and Caughey's "Nonstationary Random Input and Response" (11).

The use of the mathematical model of a random process to deal with complex vibrations such as in Fig. B.1(b) is a simplification of reality just as the use of a sinusoid to deal with simple vibrations such as in Fig. B.1(a) is also an idealization of reality.

A sinusoid is characterized by its amplitude and its frequency. For many purposes in vibration analysis, the phase is unimportant. A random vibration can often be adequately characterized by an average amplitude and by a decomposition in frequency. The average amplitude most commonly employed is the rms or root-mean-square value. The frequency decomposition is indicated by the mean square spectral density. Other statistical parameters can also be obtained to provide a more complex picture.
Vibration records: deflection or stress as a function of time. (a) Predominantly deterministic sinusoidal sample; (b) Predominantly random sample.

We shall, therefore, find it necessary to combine statistical description with description based on the concepts of harmonic analysis, suitably generalized to make it applicable to continuous records such as that of Fig. B.1(b). There are many occasions where the shape of a record is of more significance than its level distribution, and in such cases the methods of harmonic analysis provide
the better description. In particular, when we come to consider the response of a system to a randomly varying excitation, we shall see that a knowledge of the frequency content of the excitation makes it possible to make use of results obtained (experimentally or theoretically) for an excitation of discrete frequency. But those properties of a randomly varying quantity which we describe in terms of harmonic analysis can also be described in terms of statistics, and the availability of the two equivalent methods of describing the same properties is often a great convenience in analysis, where we can work with the most suitable method for a particular purpose.

Let us consider a randomly varying quantity, denoted by \( x(t) \), which may be any physical quantity such as force, pressure, stress, deflection, etc. If this is recorded over an interval \( \Delta t \) the result may be represented as in Fig. B.2.

![Figure B.2](image)

Figure B.2

Randomly varying quantity, \( x(t) \), vs. time.
Although it is possible to plot \( x(t) \) against \( t \) for any particular interval \( \Delta t \) during which \( x(t) \) has been measured, it is not possible to predict from this the precise value of \( x(t) \) at any future value of \( t \).

However, if we have reason to believe that the mechanisms by which \( x(t) \) is generated remain unchanged at all times, we can expect that the essential character of \( x(t) \) outside \( \Delta t \) remains unchanged also. Thus, for example, if the interval \( \Delta t \) is not too short \( x(t) \) will have the same mean value (represented by \( AB \)) outside \( \Delta t \) as it does within it, and the extreme values of \( x(t) \) will, except in rare instances, still lie between the limits represented by \( CD \) and \( EF \). This technique of description is capable of considerable refinement, and by means of statistical concepts it is, in fact, possible to describe the essential features of the whole signal \( x(t) \) with some precision based solely on its behavior in the interval \( \Delta t \). Before developing such a technique for a quantity \( x(t) \), it is helpful first to consider some rather more elementary statistical problems.

Statistical theory is based on the concept of probability. The measure of probability used is based on a scale such that the probability of the occurrence of an event which cannot possibly occur is taken to be zero; the probability of the occurrence of an event which is absolutely certain to occur is taken to be unity. Any other event clearly must have a probability between zero and unity, although it may not be easy to see at once how
any numerical value can be allotted to it.

In some cases an argument based on symmetry enables us to allot a precise measure of probability. Suppose that we spin a coin. We do not know beforehand whether to expect the results "heads" or "tails," but we can argue on grounds of symmetry that with a good coin the probability of the result "heads" is equal to that of the result "tails," or, in symbolic form, \( \Pr[H] = \Pr[T] \). As the probability of "either heads or tails," which will be equal to \( \Pr[H] + \Pr[T] \), must be unity, it follows that \( \Pr[H] = \Pr[T] = \frac{1}{2} \). Similarly we could argue that the probability of throwing any given number with a symmetrical six-sided die would be 1/6: all numbers from 1 to 6 are equally probable and their total probability must be unity.

Where no such argument as that of symmetry is available we have to base our measure of probability on the intuitively acceptable hypothesis that in any trial the probability of the occurrence of a particular event is equal to the relative frequency of its occurrence in a very large number of similar trials; i.e., if an event occurs in 500 out of a series of 1000 trials, we assume that the probability of its occurring in any one trial is approximately equal to \( \frac{1}{2} \). If we bear in mind the necessity for taking a sufficiently large sample before attempting to define probability, this empirical definition does prove satisfactory, and if there is any doubt as to the accuracy of a result, it may even be possible to make some statistical estimate of the uncertainty involved.
Let us now consider the six-sided die a little further. If the die is perfectly made we can say before any given throw that the result will be a number $N$, which is equally likely to turn out to be any number between 1 and 6. The probability that $N$ will be equal to 1 is equal to the probability that $N$ will be equal to 2, and so on up to 6, and the probability that $N$ will turn out to have any particular one of these values is thus $1/6$. We can write, therefore, $\Pr[N=n] = 1/6$, where $n$ can be given any specified value from 1 to 6. We can simplify this notation even further by writing $p(n) = \Pr[N=n]$ so that $p(n) = 1/6$, $(1 \leq n \leq 6)$. This can be plotted as a histogram against $n$, see Figure B.3(a). The distinction between $N$ and $n$ should be noted: $n$ represents any previously specified possible result, while $N$ is the previously unknown result of any trial. While $n$ can have any prescribed value, the value of $N$ cannot be known (until after the trial) and we can only assess the probability of its having a particular value. We note here the possibility of adding probabilities: the probability that $N$ is odd is equal to $\Pr[N=1 \text{ or } 3 \text{ or } 5]$ which is equal to $p(1) + p(3) + p(5) = \frac{1}{2}$.

It is convenient here to define a further probability and to introduce a new symbol: $P(n) = \Pr[N \leq n]$: that is, the probability that $N$ is not greater than any given number $n$. We see that, for example, $P(4) = p(1) + p(2) + p(3) + p(4) = 2/3$. Indeed, it is always the
case that \( P(n) = \sum_{1}^{n} p(r) \), and if it is plotted against \( n \), we obtain a uniform staircase as in Figure B.3(b).

The quantities \( p(n) \) and \( P(n) \) provide alternative means of describing the distribution of probability between the various possible values of \( n \). In what follows, the term probability distribution will be used to denote
the general properties of which the various defined quantities (e.g., \( p(n) \)) provide a quantitative description.

A further quantity which we use later is the expectation, \( E(N) \). This is the expected result in any given trial, assumed to be equal to the mean result of a very large number of trials. In the case of the symmetrical die we can assume that in many throws all the numbers from one to six will occur with equal frequency, so the expectation here is the mean of 1, 2, 3, 4, 5, 6, i.e.,

\[
E(N) = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3\frac{1}{2}
\]

(We could clearly also write this as \( \sum n \ p(n) \).)

With a loaded die the probability distribution will be modified somewhat. We could no longer infer the values of \( p(n) \) from symmetry, but an experiment consisting of a very large number of throws would indicate some such distribution as that shown in Fig. B.3(c). If we were now to plot \( P(n) \) the staircase would no longer be uniform but would be as shown in Fig. B.3(d), with \( P(n) = \sum \ p(r) \). The expectation would now be a weighted mean of the numbers 1 to 6, expressible only as \( E(N) = \sum_1^6 n \ p(n) \), some numbers occurring with greater frequency than others.

In both coin and die, there are certain discrete values which the random variable \( N \) can have: a coin has two, a die has six. The physical quantities in which we shall be interested, on the other hand, have a continuous range of possible values. We must, therefore, next consider
the probability distributions of quantities of this type.

Suppose for example that we have a very large number of pieces of string of different lengths, \( X \), and suppose that we wish to describe the distribution of lengths. We could attempt to do this by defining the probability that any piece taken at random would have a certain length \( x \), and do this for all possible lengths \( x \). This operation would, however, be unhelpful, for no string could be precisely, say, 12 inches long. Clearly \( \Pr[X=x] \) would be zero for any precise value of \( x \); the probability of the occurrence of any one out of an infinite number of possibilities can only be zero. So the use here of a quantity corresponding to \( p(n) \) would appear inapplicable.

But the probability, corresponding to \( P(n) \), that the length of the randomly chosen piece of string is less than a certain specified length is finite, and with a sufficient number of trials, it can be determined. We can therefore define

\[
P(x) = \Pr[X \leq x]
\]

without difficulty, and plot it with precision if a large enough number of trials can be made. This quantity is known as the distribution function, and it provides a precise quantitative description of the distribution of the lengths of all the pieces, or, because of our definition of probability, of the probability distribution associated with taking a single piece at random. If plotted
against $x$ it will, in general, give a continuous curve and, in the present case, will be somewhat like that shown in Fig. B.4. This may be seen to conform to certain obvious physical conditions: the probability that any string will have a negative length is zero; the probability of a very small positive length is small; the probability that a string will have any given value $x_1$ must be less than one; almost all strings will be found to have lengths less than some suitable large positive value of $x$. The plot of $P(x)$ will be seen to embody all the information we can expect to know about the values of the quantities in our sample, and so of the probabilities of a single trial. It is possible, however, to plot the information in other ways.
As probabilities can be added, they can also be subtracted. We can thus deduce from the distribution function of Fig. B.4 the probability of any randomly selected string having a length between two limits \( x_1 \) and \( x_2 \): this will, of course, be equal to the proportion of our very large number of pieces whose lengths are in this range. As

\[
Pr \left[ x_1 \leq X \leq x_2 \right] = Pr \left[ X \leq x_2 \right] - Pr \left[ X \leq x_1 \right] \\
= P(x_2) - P(x_1)
\]

this probability is given by the length AB in Fig. B.4.

This last result can be extended to give us a \( p(x) \) which does in fact correspond closely to our previous \( p(n) \). The probability that \( X \) lies between \( x \) and \( x + dx \), which we can call \( dP(x) \), can be obtained in the same way. Thus:

\[
dP(x) = Pr \left[ X \leq x + dx \right] - Pr \left[ X \leq x \right] \\
= P(x + dx) - P(x)
\]

Now this probability, if \( dx \) is small enough, will be proportional to \( dx \), and we can call it \( p(x)dx \). We then have \( dP(x) = p(x)dx \), which, when \( dP(x) \) and \( dx \) become infinitesimal, reduces to the differential relation

\[
p(x) = \frac{dP(x)}{dx}
\]  \hspace{1cm} (B.2)

The quantity \( p(x) \) is called the probability density and provides a second way of describing precisely the probability distribution of a random variable. When plotted it
will have the form of Fig. B.5. Here the higher portions of the curve correspond to the region in which most values in the sample are found, corresponding to the region of greatest slope of $P(x)$, and the lower parts of the curve indicate values of $x$ which occur only comparatively rarely.

![Figure B.5](image)

_Probability density vs. $x$."

Plots of both distribution function $P(x)$ and probability density $p(x)$ thus embody the identical facts concerning the probability distribution of a random variable, $X$, and either can be obtained if the other is known. We have defined $p(x)$ in terms of $P(x)$, and usually $P(x)$ is easier to obtain experimentally, but the reverse process can be carried out by writing

$$P(x) = \int_{-\infty}^{\infty} p(z)dz$$  \hspace{1cm} (B.3)

For pieces of string the lower limit of the integration would be zero, but in general, negative values of $x$ must be covered by the $p(x)$ and $P(x)$ curves.
The shape of the \( p(x) \) and \( P(x) \) curves gives a qualitative indication of the nature of a distribution even without detailed quantitative interpretation. For example, a variable whose values were closely clustered about a mean would obviously give a tall narrow \( p(x) \) curve or a \( P(x) \) curve steeply rising near the mean value.

A word often used when referring to a statistical distribution is "confidence." Thus we might say that the declared life of a given component is based on a 95% confidence level. By this we would mean that 95% of a large number of components tested had a life in excess of this declared value. This would be equivalent to saying that any sample taken would have a probability of 0.95 of exceeding the declared value, or a probability of 0.05 of failure within the declared value. A confidence level then, is simply a measure of probability and may be read off a curve giving the distribution function.

Ensembles and Higher Order Probability Distributions (32)

The central notion involved in the concept of a random process \( x(t) \) is that not just one time history is described but the whole family or ensemble of possible time histories which might have been the outcome of the same experiment are described.

Any single individual time history belonging to the ensemble is called a sample function. A random process can be portrayed schematically as in Fig. B.6. Each sample
Figure B.6

Schematic representation of a random process $x(t)$. Each $x^j(t)$ is a sample function of the ensemble.

function $x^j(t)$ is sketched as a function of time. The time interval involved is the same for each sample. It may be a finite interval, e.g., a certain 10-second interval during a missile flight, or it may be the infinite interval extending from $t = -\infty$ to $t = \infty$. There is a continuous infinity of different possible sample functions, of which only a few are shown. All of these represent possible outcomes of experiments which the experimenter considers to be performed under identical conditions. Because of variables beyond his control, the samples are
actually different. Some samples are more probable than others, and to describe the random process further, it is necessary to give probability information such as that discussed in the previous section.

The probability information associated with a random process can be given by first describing the distribution of values \( x(t_1) \) which occur in the ensemble for a fixed value of \( t = t_1 \). In some cases this distribution will be independent of \( t_1 \); in other cases a different distribution will be obtained for each \( t_1 \). Second, the joint distribution for pairs of values \( x(t_1) \) and \( x(t_2) \) which occur in the ensemble for a pair of fixed values of \( t_1 \) and \( t_2 \) is described. These joint distributions may be functions of both \( t_1 \) and \( t_2 \) or may, in some cases, only be functions of \( \tau = t_1 - t_2 \). Third, the joint distributions for triples of values \( x(t_1), x(t_2) \) and \( x(t_3) \) which occur in the ensemble for fixed values of \( t_1, t_2, \) and \( t_3 \) are described. In principle, the process is continued to include joint distributions of order \( n \) with \( n \to \infty \). In practice, much relevant information is obtained without going beyond distributions of second order. This is about the limit as far as experimental determination goes. In only a few theoretical cases (e.g., the Gaussian or Normal random process) it is then possible to indicate the general form of the higher-order joint distributions knowing the form of the first and second order distributions.

The first-order probability distribution for values
of \( x(t_1) \) at a fixed value \( t_1 \) can now be described by a graph exactly like that one shown in Fig. B.5 (previously used to describe the distribution along a particular sample function). Now graphs of this form can be used to show the probability density function \( p[ x(t_1) ] \) or simply \( p(x) \) (if there is no ambiguity concerning the value of \( t \)) where the sample is now taken across the ensemble. This function, like before, has the property that the fraction of ensemble members for which \( x(t_1) \) lies between \( x \) and \( x + dx \) is \( p(x)dx \).

Another way of looking at the term "probability" is that it is simply the fraction of favorable events out of all possible events. Probabilities are inherently non-negative, i.e., they can only be positive or zero. Probabilities of mutually exclusive events are additive. Thus the probability that a sample lies between \( a \) and \( b \) in Fig. B.5 is just the sum of the probabilities that a sample lies in each of the individual \( dx \) intervals which go to make up the integral

\[
\int_a^b p(x) \, dx
\]

The probability that the value of \( x \) lies between \(-\infty\) and \( \infty \) is unity; i.e., there is 100% certainty that \( x \) is somewhere in this interval (assuming \( x \) is a real number). This implies that the area under the curve of Fig. B.5 is unity or that

\[
\int_{-\infty}^{\infty} p(x) \, dx = 1 \quad (B.4)
\]
Note that the probability density $p(x)$ which can be interpreted as giving the "fraction of success" per unit of $x$ has the dimensions of $1/x$.

The second-order joint probability distribution for pairs of values $x(t_1)$ and $x(t_2)$ can be described by a surface such as in Fig. B.7 which shows the joint probability density $p(x_1, x_2)$ where the abbreviations $x_1$ and $x_2$ for the $x(t_1)$ and $x(t_2)$ have been introduced. The joint density has the property that the fraction of ensemble members for which $x(t_1)$ lies between $x_1$ and $x_1 + dx_1$ and $x(t_2)$ lies between $x_2$ and $x_2 + dx_2$ is $p(x_1, x_2) dx_1 dx_2$.

\[ p(x_1, x_2) \]

\[ \text{Figure B.7} \]

Second-order density function. The probability of a sample with $x_1$ between $x_1$ and $x_1 + dx_1$ and with $x_2$ between $x_2$ and $x_2 + dx_2$ is $p(x_1, x_2) dx_1 dx_2$.

Like first-order densities, second-order densities are inherently positive and probabilities of mutually exclusive events are additive. Thus the probability that a
random sample of the ensemble would have \( x_1 \) and \( x_2 \) lying in the following ranges

\[
a_1 < x_1 < b_1 \\
a_2 < x_2 < b_2
\]

is just the sum of the probabilities of occurrence in the individual elements of area \( dx_1 dx_2 \) which go to make up the integral

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} p(x_1, x_2) dx_1 dx_2
\]

The probability that \( x_1 \) is any real number and that \( x_2 \) is any real number is unity; i.e.,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1, x_2) dx_1 dx_2 = 1 \quad (B.5)
\]

which is analogous to Eq. (B.4).

Second-order joint probabilities imply a great deal more information than first-order probabilities. In particular, the first-order densities \( p(x_1) \) and \( p(x_2) \) are obtained from the joint density \( p(x_1, x_2) \) by integration.

\[
p(x_1) = \left\{ \int_{-\infty}^{\infty} p(x_1, x_2) dx_2 \right\} \quad (B.6)
\]

\[
p(x_2) = \left\{ \int_{-\infty}^{\infty} p(x_1, x_2) dx_1 \right\}
\]

The validity of the first of Eqs. (B.6) follows immediately on multiplying both sides by \( dx_1 \) and interpreting each side as the fraction of samples for which \( x_1 \) lies in the interval \( x_1 + dx_1 \).
This brief outline indicates how, in principle, it is possible to give complete probabilistic information about a random process. When for theoretical reasons such information is available it is a simple task to calculate statistical averages for the process. In dealing experimentally with a random process these statistical averages are much simpler to measure directly than are the probability distributions which underlie them.

Ensemble Averages, Mean and Autocorrelation (32)

Consider a fixed time \( t = t_1 \), and the ensemble of values \( x(t_1) \) or simply \( x \). Suppose that with each sample \( x \) we associate a value \( g(x) \) where \( g \) is a known function (say, \( x^2 - 2x \)). We wish to find the average of \( g(x) \) taken across the ensemble.

Consider first the experimental case where \( n \) sample values \( x^j(t) \), for \( j = 1, 2, \ldots, n \), are available. Under the assumption that those \( n \) samples are adequately representative of the process, the average of \( g \) across the ensemble would be simply the sum of the \( g \) values divided by the number of samples:

\[
\frac{1}{n} \sum_{j=1}^{n} q(x^j) \quad (B.7)
\]

An alternative interpretation of Eq. (B.7) is that it is a weighted sum of \( q \) values where each weighting factor gives the fraction of samples having that particular \( g \) value.
The second interpretation just given extends easily to the theoretical case in which there are infinitely many samples whose distribution is described by the first-order probability density \( p(x) \). The fraction of samples for which the \( x \) value lies between \( x \) and \( x + dx \) is \( p(x)dx \). The continuous analog of the discrete average Eq. (B.7) is thus

\[
E\left[ g(x) \right] = \int_{-\infty}^{\infty} g(x) p(x) \, dx
\]  

(B.8)

This ensemble average is called the mathematical expectation of \( g(x) \) and the operator \( E \) is used* to denote this kind of average.

When \( g(x) \) is simply \( x \) itself, the ensemble average, Eq. (B.8), becomes

\[
E\left[ x \right] = \int_{-\infty}^{\infty} x p(x) \, dx
\]  

(B.9)

which defines the mean of \( x \) or the expected value of \( x \) (recall the expectation \( E(N) = 3\frac{1}{2} \) for the discrete case of a six-sided uniform die).

When \( g(x) \) is the function \( x^2 \), Eq. (B.8) becomes

\[
E\left[ x^2 \right] = \int_{-\infty}^{\infty} x^2 p(x) \, dx
\]  

(B.10)

which defines the mean square value of \( x \). The square root of Eq. (B.10) is called the root mean square value or rms value.

*The notation for ensemble average is not uniform in literature. In addition to \( E\left[ g(x) \right] \) one finds \( \langle g(x) \rangle \) and \( \langle q(x) \rangle \).
An important statistical parameter is the variance of \( x \). It is obtained from Eq. (B.8) by setting \( g(x) = (x - \mathbb{E}[x])^2 \), i.e., the variance \( \sigma^2 \) is the ensemble average of the square of the deviation from the mean.

\[
\sigma^2 = \mathbb{E}[(x - \mathbb{E}[x])^2] = \int_{-\infty}^{\infty} (x - \mathbb{E}[x])^2 p(x) \, dx \tag{B.11}
\]

The square root of Eq. (B.11) is called the standard deviation \( \sigma \). An alternate expression for the variance can be obtained by evaluating the integral on the right of Eq. (B.11). We obtain

\[
\sigma^2 = \int_{-\infty}^{\infty} x^2 p(x) \, dx - 2 \mathbb{E}[x] \int_{-\infty}^{\infty} x p(x) \, dx + (\mathbb{E}[x])^2 \int_{-\infty}^{\infty} p(x) \, dx \\
= \mathbb{E}[x^2] - (\mathbb{E}[x])^2 \tag{B.12}
\]

on using Eqs. (B.10), (B.11) and (B.4). The result is Eq. (B.12) which relates the variance to the mean square and the mean. When the mean is zero, then the variance is identical with the mean square. When there is a mean value, this can always be found easily enough and subtracted from the total signal: in considering the statistics of a random process, therefore, it is not only simpler but also realistic to consider only signals having zero mean value.

Suppose that a signal \( x(t) \) has zero mean value; then it will give a record like that shown in Fig. B.8(a). This will have a distribution function plot extending on both sides of the origin, and a probability density plot with its centroid directly above the origin (Fig. B.8(b) and B.8(c)).
To describe completely the nature of the spread of $x(t)$ on either side of the zero level would require the
construction of one of these two plots. But the mean square value of \( x(t) \), i.e., \( E[x^2] \), does give an approximate measure of spread of a distribution—if this is small the \( p(x) \) curve has a narrow peak and \( P(x) \) is steep near \( x = 0 \).

**Autocorrelation and Covariance (32)**

Let \( t_1 \) and \( t_2 \) be two fixed values of \( t \) and use the abbreviations \( x_1 \) and \( x_2 \) to denote the ensemble of samples \( x(t_1) \) and \( x(t_2) \). Let \( f(x) \) and \( g(x) \) be known functions. We wish to obtain the ensemble average of \( f(x_1)g(x_2) \).

Consider first the experimental case where \( n \) sample functions \( x_j(t) \), \( j = 1, 2, \ldots, n \), are available. At the fixed times \( t_1 \) and \( t_2 \) these provide \( n \) pairs of values \( x_{1j} \) and \( x_{2j} \). Under the assumption that those samples are representative of the process, the average of \( f(x_1)g(x_2) \) would be simply

\[
\sum_{j=1}^{n} f(x_{1j})g(x_{2j}) \frac{1}{n} \quad (B.13)
\]

which can be interpreted as a weighted sum of \( f(x_1)g(x_2) \) values where each weighting factor gives the fraction of samples having that particular \( f(x_1)g(x_2) \) value. This interpretation permits easy generalization to the continuous case where the distribution of \( x_1 \) and \( x_2 \) is described by the second order probability density \( p(x_1, x_2) \). Since the fraction of samples for which \( x_1 \) lies between \( x_1 \) and \( x_1 + dx_1 \) and for which \( x_2 \) lies between \( x_2 \) and \( x_2 + dx_2 \) is \( p(x_1, x_2)dx_1 dx_2 \), the ensemble average or mathematical expectation of the product \( f(x_1)g(x_2) \) is
\[ E \left[ f(x_1)g(x_2) \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1)g(x_2)p(x_1,x_2)dx_1dx_2 \quad (B.14) \]

When \( f(x_1) = x_1 \) and \( q(x_2) = x_2 \) in Eq. (B.14), the resulting average \( E \left[ x_1x_2 \right] \) is called the **autocorrelation function**.

\[ E \left[ x(t_1)x(t_2) \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1x_2p(x_1,x_2)dx_1dx_2 \quad (B.15) \]

The prefix auto refers to the fact that \( x_1x_2 \) represents a product of values on the same sample at two instants. For fixed \( t_1 \) and \( t_2 \) this average is simply a constant; however, in subsequent applications \( t_1 \) and \( t_2 \) will be permitted to vary and the autocorrelation function will, in general, be a function of both \( t_1 \) and \( t_2 \). In an important special case, the autocorrelation function is a function only of \( \tau = t_1 - t_2 \).

A related average, the **covariance**, is obtained by averaging the product of the deviations from the means at two instants. Thus we set \( f(x_1) = x_1 - E\left[ x_1 \right] \) and \( g(x_2) = x_2 - E\left[ x_2 \right] \) in Eq. (B.14) to obtain

\[
E \left[ (x_1 - E\left[ x_1 \right])(x_2 - E\left[ x_2 \right]) \right] = \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - E\left[ x_1 \right])(x_2 - E\left[ x_2 \right])p(x_1,x_2)dx_1dx_2 \\
= E\left[ x_1x_2 \right] - E\left[ x_1 \right]E\left[ x_2 \right] \quad (B.16)
\]

as the covariance. Note that when \( x_1 \) and \( x_2 \) have zero mean, the covariance is identical to the autocorrelation. Some appreciation of the significance of the covariance can be had by considering the "normalized" form of the covariance which is obtained by dividing Eq. (B.16) by \( \sigma_1\sigma_2 \), the product
of the standard deviations of $x_1$ and $x_2$. It can be shown that the normalized covariance always lies between -1 and +1 for any possible distribution of $x_1$ and $x_2$. If on every sample $(x_1 - E[x_1])/\sigma_1$ is exactly equal to $(x_2 - E[x_2])/\sigma_2$ then the normalized covariance is unity, while if $(x_1 - E[x_1])/\sigma_1$ is equal to $-(x_2 - E[x_2])/\sigma_2$ on every sample, then the normalized covariance is -1. In these cases, we speak of 100% correlation, positive or negative. If, on the other hand, for any sub-ensemble with a certain range of $x_1$ values the corresponding values of $(x_2 - E[x_2])/\sigma_2$ are distributed both positively and negatively with zero mean the covariance will be zero and we say that $x_1$ and $x_2$ are uncorrelated.

When $t_1 = t_2$, the covariance, Eq. (B.16), becomes identical with the variance, Eq. (B.11), and the autocorrelation, Eq. (B.15), becomes identical with the mean square, Eq. (B.10).

**The Stationary and Ergodic Assumptions, Temporal Averages (32)**

A random process is an infinite ensemble of sample functions described by a set of probability distributions; e.g., by the first-order density $p[x(t_1)]$, the second-order density $p[x(t_1),x(t_2)]$, etc. In the foregoing sections, these distributions and certain average properties of these distributions have been examined for fixed instants $t_1, t_2, \ldots$. We turn now to the question of how these vary when $t_1, t_2, \ldots$, are assumed to vary.
At this point we make a major simplification. We assume that the random processes we are interested in belong to a special class known as stationary processes. This is somewhat analogous to the assumption of steady state forced vibration in ordinary vibration. Neither assumption is strictly true in practice but both provide useful engineering answers when they are suitably interpreted.

A random process is said to be stationary if its probability distributions are invariant under a shift of the time scale; i.e., the family of probability densities applicable now also applies ten minutes from now, or three weeks from now. In particular, the first-order probability density \( p(x) \) becomes a universal distribution, independent of time. This implies that all the averages based on \( p(x) \) (e.g., the mean \( E[x] \) and the variance \( \sigma^2 \)) are constants, independent of time. If the second-order probability density \( p(x_1, x_2) \) is to be invariant under a translation of the time scale, then it must be a function only of the lag between \( t_1 \) and \( t_2 \) and not a function of \( t_1 \) or \( t_2 \) individually. Setting \( t_2 - t_1 = \tau \), we can write the second-order density of a stationary process as \( p(t, t+\tau) \) and know that it is independent of \( t \). This implies that the autocorrelation function is also only a function of \( \tau \).

\[
E[x_1 x_2] = E[x(t)x(t+\tau)] = R(\tau) \quad (B.17)
\]

We will always use the notation \( R(\tau) \) to denote the autocorrelation function of a stationary random process. When more than one random process is involved, subscripts
will be used to identify each process (e.g., \( R_x(\tau) \) and 
\( R_y(\tau) \) if \( x \) and \( y \) are different stationary processes).

Note that \( R(0) \) reduces to the mean square \( E[ x^2 ] \).

In case \( x \) has zero mean, \( E[x] = 0 \), then the mean square 
is identical with the variance and \( R(0) = \sigma^2 \).

It is possible to partially verify the stationary assumption experimentally by obtaining a large family of 
sample functions and then calculating averages, such as 
the mean and autocorrelation, for many different times.
If the stationary hypothesis is warranted, there should be substantial agreement among the results at different times.

For a process to be strictly stationary, it can have no beginning and no end. Each sample must extend from 
\( t = -\infty \) to \( t = \infty \). Most real processes do, in fact, start and stop, and thus cannot be truly stationary. The nonstationary effects associated with starting and stopping are often neglected in practice if the period of stationary operation is long compared with the starting and stopping intervals. If changes in the statistical properties of a process occur slowly with time, it is sometimes possible to subdivide the process in time into several processes of shorter duration, each of which may be considered as reasonably stationary.

Temporal Averages

All of the averages discussed so far have been ensemble averages. To evaluate them it is necessary to have information about the probability distribution of the
samples or at least a large number of individual samples. Given a single sample $x_j$ of duration $T$, it is, however, possible to obtain averages by averaging with respect to time along the sample. Such an average is called a temporal average in contrast to the ensemble or statistical averages described previously.

Let $x_j = f(t)$ be a function of time defined from $t = -T/2$ to $t = T/2$. For our purposes it is well to think of $f(t)$ as representing a particular sample of a random process, although the following temporal averages apply to any function $f(t)$ and have nothing to do with random processes per se. The temporal mean of $f(t)$ is

$$< f > = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, dt \quad (B.18)$$

and the temporal mean square is

$$< f^2 > = \frac{1}{T} \int_{-T/2}^{T/2} f^2(t) \, dt \quad (B.19)$$

where we have adopted the notation $< f >$ for temporal mean. When $f(t)$ is defined for all time, the averages, Eqs. (B.18) and (B.19) are evaluated by considering the limits as $T \to \infty$. For such a function, a temporal autocorrelation function $\phi(\tau)$ can be defined as

$$\phi(\tau) = < f(t)f(t+\tau) > = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)f(t+\tau) \, dt \quad (B.20)$$

When $f(t)$ is only defined for a finite interval, a similar
expression can be utilized but care must be taken of end effects (e.g., the integral on the right of Eq. (B.20) can be used without the limit sign provided \( f(t) \) is defined from \( t = T/2 \) to \( t = T/2 + \tau \)). Such a finite average would probably only be used for values of \( \tau \) which are very small compared with \( T \). Note that \( \Phi(0) \) reduces to the temporal mean square.

Within the sub-class of **stationary** random processes there exists a further sub-class known as **ergodic** processes. An ergodic process is one for which ensemble averages are equal to the corresponding temporal averages taken along any representative sample function. Thus for an ergodic process \( x(t) \) with samples \( x^j \), we have

\[
E[x] = \langle x^j \rangle \\
R(\tau) = \Phi(\tau)
\]  

(B.21)

An ergodic process is necessarily stationary since \( \langle x^j \rangle \) is a constant while \( E[x] \) is generally a function of the time \( t = \tau_1 \) at which the ensemble average is performed, except in the case of a stationary process. A random process can, however, be stationary without being ergodic. Each sample of an ergodic process must be completely representative of the entire process.

It is possible to verify experimentally whether a particular process is or is not ergodic by processing a large number of samples, but this is a very time-consuming task. On the other hand, a great simplification results
if it can be assumed ahead of time that a particular process is ergodic. All statistical information can be obtained from a single sufficiently long sample. In situations where statistical estimates are desired, but only one sample of a stationary process is available, it is common practice to proceed on the "assumption" that the process is ergodic. These initial estimates can then be revised when further information becomes available.

Nonstationary Random Inputs and Responses (11)

It has been found useful in the theory of stochastic processes to divide such processes into two broad classes: stationary processes and evolutionary or nonstationary processes. Intuitively, a stationary process is one whose statistics do not change with time. A nonstationary process, on the other hand, is a process whose statistics evolve in time. Such processes are not in general ergodic, and therefore, their statistics cannot be obtained from the statistics of a single record but only from an ensemble of records. Since the characterization of an evolutionary process requires considerably more data than a stationary process, it is not surprising that most engineers are not as familiar with evolutionary processes as they are with stationary processes.

Many physical processes are inherently nonstationary in character. Examples in the area of ocean structures include the vibrations of submarine and other ships' hulls and superstructures and the vibration of structural elements
of offshore drilling structures subject to random loadings from ocean winds and waves. Examples in the area of rocket-boosted vehicles include the vibrational environment of a vehicle during takeoff, during transition between missile stage separation, and during periods of rapid deceleration. Other examples of evolutionary processes include earthquakes, motion of a confused sea, explosions, shocks, and other transient phenomena that can be described only statistically.

The purpose of this section is to provide the necessary theory to calculate the response of discrete, linear time-invariant dynamic systems to nonstationary inputs. The problems of acquiring and processing the necessary data to characterize the input process will not be discussed here; it will be assumed throughout that this information is already available.

One approach to the problem of developing mathematical models for physical phenomena evolving in a probabilistic manner is to characterize such processes in terms of the behavior of their mean and covariance.

The probability law for a process cannot in general be determined from a knowledge of the mean and covariance for the process unless the functional form of the probability law is known up to several unspecified parameters which are simply related to the mean and covariance of the process. If a stochastic variable has a normal or Gaussian distribution, then its mean and covariance serve to
characterize the variable completely. In the general case in which the probability law is unknown, the mean and covariance still serve to give a rough description of the probability law.

In the following analysis it will be assumed that the input process \( \{z(t)\} \) possesses a continuous mean value function \( m_z(t) \) and covariance kernel \( K_{zz}(t_1, t_2) \) defined for all \( t_1 \) and \( t_2 \), where

\[
m_z(t) = \mathbb{E}[z(t)] \tag{B.22}
\]

\[
K_{zz}(t_1, t_2) = \text{cov}[z(t_1), z(t_2)] = \mathbb{E}\left[\left\{z(t_1) - m_z(t_1)\right\}\left\{z(t_2) - m_z(t_2)\right\}\right] \tag{B.23}
\]

**Response of Single-Degree-of-Freedom System**

**To a Nonstationary Input (33)**

Consider the response of a single-degree-of-freedom linear system to a Gaussian stochastic input \( f(t) \), which is a member of a stochastic process \( \{f(t)\} \):

\[
m \dddot{x} + \beta \dot{x} + kx = f(t), \quad t > t_0
\]

\[
x(t_0) = a; \quad \dot{x}(t_0) = b \tag{B.24}
\]

Let

\[
\omega_n^2 = k/m; \quad \beta/m = 2\omega_n\zeta; \quad \frac{f(t)}{m} = z(t) \tag{B.25}
\]

then

\[
\dddot{x} + 2\omega_n\zeta \dot{x} + \omega_n^2 x = z(t), \quad t > t_0 \tag{B.26}
\]
Thus the response, in mean square is given by

\[
x(t) = a x_1(t-t_o) + b x_2(t-t_o) + \left\{ g(t-t) z(\tau) d\tau \right\}_{t_o}^t
\]

or

\[
x(t) = a x_1(t-t_o) + b x_2(t-t_o) + \left\{ g(\xi) z(t-\xi) d\xi \right\}_{0}^t
\]

where

\[
x_1(t) = e^{-\zeta \omega_n t} \left( \cos \omega_n t + \frac{\zeta \omega_n}{\omega_n} \sin \omega_n t \right)
\]

\[
x_2(t) = \frac{-\zeta \omega_n t}{\omega_n} \sin \omega_n t
\]

\[
g(t) = x_2(t)
\]

\[
\omega_n = \omega_n \sqrt{1-\zeta^2}, \quad 0 < \zeta < 1
\]

We shall consider only the oscillatory case \( \zeta < 1 \).

**Stochastic Average of \( x(t) \).** The mean \( m_x(t) \) of the output process is obtained by averaging across the ensemble

\[
m_x(t) = E \left[ x(t) \right] \quad (B.29)
\]

Thus,

\[
m_x(t) = a x_1(t-t_o) + b x_2(t-t_o) + \left\{ g(t-\tau) m_z(\tau) d\tau \right\}_{t_o}^t \quad (B.30)
\]

or

\[
m_x(t) = a x_1(t-t_o) + b x_2(t-t_o) + \left\{ g(\xi) m_z(t-\xi) d\xi \right\}_{0}^t \quad (B.31)
\]

We note in passing that for a system with infinite operating time \( t_o = -\infty \),
\[ m_x(t) = \int_0^\infty g(\xi) m_z(t-\xi) d\xi \quad (B.32) \]

**Covariance** \( K_{xx}(t_1, t_2) \) **of** \( x(t) \). The covariance kernel for the output process is easily computed as follows:

\[
K_{xx}(t_1, t_2) = E\left[ \left\{ \{ x(t_1) - m_x(t_1) \} \{ x(t_2) - m_x(t_2) \} \right\} \right] \quad (B.33)
\]

\[
= \int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 g(\tau_1 - \tau) g(\tau_2 - \tau) E\left[ \left\{ z(\tau_1) - m_z(\tau_1) \right\} \left\{ z(\tau_2) - m_z(\tau_2) \right\} \right].
\]

\[
\quad \left( B.34 \right)
\]

\[
= \int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 g(\tau_1 - \tau) g(\tau_2 - \tau) K_{zz}(\tau_1, \tau_2) \quad (B.35)
\]

or

\[
K_{xx}(t_1, t_2) = \int_{t_0}^{t_1-t_0} d\xi_1 \int_{t_0}^{t_2-t_0} d\xi_2 g(\xi_1) g(\xi_2) K_{zz}(t_1-\xi_1, t_2-\xi_2) \quad (B.36)
\]

**The Variance** \( \sigma^2_x(t) \) **of** \( x(t) \). The variance \( \sigma^2_x(t) \) of the output process is obtained by setting \( \tau_1 = t_2 = t \) in the expression for the covariance kernel:

\[
\sigma^2_x(t) = K_{xx}(t, t) = \int_{t_0}^{t-t_0} d\xi_1 \int_{t_0}^{t-t_0} d\xi_2 g(\xi_1) g(\xi_2) K_{zz}(t-\xi_1, t-\xi_2) \quad (B.37)
\]

In particular, for a system with infinite operating time,
\[ \sigma^2_x(t) = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 g(\xi_1) g(\xi_2) K_{zz}(t-\xi_1, t-\xi_2) \quad (B.38) \]

**Probability Density Functions \( p(x) \) for \( x(t) \).** If the input process to a linear system is normal or Gaussian, then the output process is also normal or Gaussian. Hence, the probability that \( x \) lies in the interval \( x \) to \( x + dx \) at time \( t \) is given by

\[
p(x)dx = \frac{1}{(2\pi)^{1/2} \sigma_x(t)} \exp \left\{ \frac{-(x-m_x(t))^2}{2\sigma_x^2(t)} \right\} dx
\]

where \( p(x) \) is the probability density function of \( x \) and \( \sigma_x(t) \) and \( m_x(t) \) are given by Eqs. (B.37) and (B.32), respectively.

**Probability of Large Deviations from the Mean.** If the stochastic variable \( x(t) \) is the stress in a structure, then it is not sufficient to know the average value of the stress. For safety we shall also be interested to know the probability of occurrence of stresses exceeding the specified working stress of the material of the structure.

If \( x(t) \) is Gaussian, then

\[
P\left[|x(t) - m_x(t)| > k\sigma_x(t)\right] = \int_{-\infty}^{k\sigma_x(t)} \frac{1}{(2\pi)^{1/2} \sigma_x} \exp \left\{ -\frac{E}{2\sigma_x^2(t)} \right\} d\xi + \int_{k\sigma_x(t)}^{\infty} \frac{1}{(2\pi)^{1/2} \sigma_x} \exp \left\{ -\frac{E}{2\sigma_x^2(t)} \right\} d\xi \]

\[
= (1 - \text{erf} \frac{k}{\sqrt{2}}) \quad (B.40)
\]
where erf is the error function.

Application of Fourier Transforms
To Nonstationary Processes

Fourier transforms play a central role in the analysis of stationary random variables relating the autocovariance to the power spectral density and vice versa.

Let \( k_x(t) \) be a member of a real-valued nonstationary random process \( \{k_x(t)\} \) defined for \(-\infty < t < \infty\). Define

\[
k_{x_T}(t) = \begin{cases} k_x(t), & |t| < T \\ 0, & \text{elsewhere} \end{cases}
\]  \hspace{1cm} (B.42)

For simplicity, we shall assume that \( E[k_x(t)] = 0 \). Let

\[
A_T(f, k_x) = \int_{-\infty}^{\infty} k_{x_T}(t)e^{-i2\pi ft} \, dt
\]  \hspace{1cm} (B.43)

With the use of Eqs. (B.42)

\[
A_T(f, k_x) = \frac{1}{T} \int_{-T}^{T} k_x(t)e^{-i2\pi ft} \, dt
\]  \hspace{1cm} (B.44)

Using the Fourier inversion theorem gives

\[
k_{x_T}(t) = \int_{-\infty}^{\infty} A_T(f, k_x)e^{i2\pi ft} \, df
\]  \hspace{1cm} (B.45)

Since \( k_{x_T}(t) \) is a real variable, it is equal to its complex conjugate \( k_{x_T}^*(t) \) and

\[
k_{x_T}(t) = \int_{-\infty}^{\infty} A_T^*(f, k_x)e^{-i2\pi ft} \, df
\]  \hspace{1cm} (B.46)
Hence,
\[
k_X(t_1, t_2) = \int_{-\infty}^{\infty} A_T^*(f, k_x) e^{-i2\pi f t_1} \int_{-\infty}^{\infty} A_T(f_2, k_x) e^{i2\pi f_2 t_2} df_1 df_2
\]
(B.47)

Thus,
\[
\text{Cov} \left[ k_X(t_1, k_x), k_X(t_2, k_x) \right] = \mathbb{E} \left[ k_X(t_1, k_x) k_X(t_2, k_x) \right] = K_{xx}(t_1, t_2, T)
\]
(B.48)

and
\[
K_{xx}(t_1, t_2, T) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E \left[ A_T^*(f, k_x) A_T(f_2, k_x) e^{-i2\pi f_1 t_1 - i2\pi f_2 t_2} \right] df_1 df_2
\]
(B.49)

The covariance of the process \( \{ k_X(t) \} \) is given by
\[
K_{xx}(t_1, t_2, T) = \lim_{T \to \infty} K_{xx}(t_1, t_2, T)
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xx}(f_1, f_2) e^{-i2\pi(f_1 t_1 - f_2 t_2)} df_1 df_2
\]
(B.50)

where
\[
S_{xx}(f_1, f_2) = \lim_{T \to \infty} E \left[ A_T^*(f_1, k_x) A_T(f_2, k_x) \right]
\]
(B.51)

The quantity \( S_{xx}(f_1, f_2) \) is called the generalized power spectral density function for the random process \( \{ k_X(t) \} \).

Applying the Fourier inversion theorem to Eq. (B.50) yields
\[
S_{xx}(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{xx}(t_1, t_2) e^{i2\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2
\]
(B.52)

Equations (B.50) and (B.52) may be rewritten in terms of the circular frequency \( \omega = 2\pi f \). Let
so that
\[ S_{xx}(\omega_1, \omega_2) = \frac{1}{4\pi^2} S_{xx}(f_1, f_2) \]  \hfill (B.54)

Substituting Eqs. (B.53) and (B.54) into Eqs. (B.50) and (B.52) yields

\[
\begin{align*}
\rho_{xx}(t_1, t_2) &= \iint_{-\infty}^{\infty} S_{xx}(\omega_1, \omega_2) e^{-i(\omega_1 t_1 - \omega_2 t_2)} d\omega_1 d\omega_2 \\
S_{xx}(\omega_1, \omega_2) &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \rho_{xx}(t_1, t_2) e^{i(\omega_1 t_1 - \omega_2 t_2)} dt_1 dt_2 \tag{B.55}
\end{align*}
\]

The pair of Eqs. (B.55) may be regarded as the generalization of the Wiener-Khinchine relations.

Response of Linear Systems to Nonstationary Inputs. (33) Let us now calculate the response \( k_y(t) \) of a linear system with infinite operating time to a nonstationary random input \( k_x(t) \), where \( k_x(t) \) is a member of a stochastic process \( \{k_x(t)\} \). Define

\[
k_{xT} = k_x(t), \quad |t| < T \hfill (B.56)
\]

\[ = 0, \text{ elsewhere} \]

The output \( k_y_T \) due to this input is determined by

\[
L \left( \begin{array}{c} k_y_T \\ \vdots \\ k_n_T \end{array} \right) = \sum_{i=0}^{n} a_i \frac{d}{dt} \left( \begin{array}{c} k_y_T \\ \vdots \\ k_n_T \end{array} \right) = k_x_T \hfill (B.57)
\]
so that
\[ k_y(t) = \int_{-\infty}^{\infty} g(\xi) x_T(t-\xi) d\xi \quad (B.58) \]

where \( g(\xi) \) is the impulse response of \( L \). Fourier transforming both sides of Eq. (B.58) gives
\[
B_T(f, k) = \int_{-\infty}^{\infty} k_y(t) e^{-i2\pi ft} dt
= G(f) A_T(f, kx) \quad (B.59)
\]

where
\[
G(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} dt \quad (B.60)
\]
\[
A_T(f, kx) = \int_{-\infty}^{\infty} x_T(t) e^{-i2\pi ft} dt \quad (B.61)
\]

If Eq. (B.59) is multiplied by its complex conjugate,
\[
B_T^*(f, k) B_T(f, k) = G_T^*(f, k) G_T(f, k) A_T^*(f, kx) A_T(f, kx) \quad (B.62)
\]
the generalized power spectrum of the random process \( \{k_y(t)\} \) is
\[
S_{yy}(f_1, f_2) = \lim_{T \to \infty} E \left[ B_T^*(f_1, k) B_T(f_2, k) \right] \quad (B.63)
\]

With the use of Eq. (B.62), Eq. (B.63) becomes
\[
S_{yy}(f_1, f_2) = G_T^*(f_1) G_T(f_2) \lim_{T \to \infty} E \left[ A_T^*(f_1, kx) A_T(f_2, kx) \right] \quad (B.64)
\]
so that
\[
S_{yy}(f_1, f_2) = G_T^*(f_1) G_T(f_2) S_{xx}(f_1, f_2) \quad (B.65)
\]

Equation (B.65) may be regarded as the generalization of
the familiar result, Eq. (B.66), for stationary processes:

\[ S_{yy}(f) = \left| G(f) \right|^2 S_{xx}(f) \]  

(B.66)

Using Eq. (B.50) we have

\[ K_{yy}(t_1, t_2) = \int_{-\infty}^{\infty} S_{yy}(f_1, f_2) e^{-i2\pi(f_1 t_1 - f_2 t_2)} df_1 df_2 \]  

(B.67)

In particular, since it was assumed that \( \{x\} \) had zero mean, we have

\[ K_{yy}(t, t) = E \left[ k^2_y(t) \right] = \int_{-\infty}^{\infty} S_{yy}(f_1, f_2) e^{-i2\pi t(1-f_2)} df_1 df_2 \]  

(B.68)

Using Eq. (B.65) gives

\[ E \left[ k^2_y(t) \right] = \int_{-\infty}^{\infty} G^*(f_1)G(f_2)S_{xx}(f_1, f_2) e^{-i2\pi t(1-f_2)} df_1 df_2 \]  

(B.69)

Examples

As an example of the application of the ideas developed, let us consider a very simple problem: that of calculating the response of a linear dynamic system with finite operating time to a stationary random process with a given power spectral density function \( \phi(\omega) \). Though the input is stationary, the output is nonstationary by virtue of the fact that the system has a finite operating time.

Consider Eq. (B.26) with \( t_0 = 0 \) and \( x(0) = \dot{x}(0) = 0 \); then

\[ \ddot{x} + 2\omega_n\dot{x} + \omega_n^2x = z(t), \quad t > 0 \]  

(B.70)

where \( z(t) \) is a member function of a stochastic process \( \{z(t)\} \) which is ergodic, stationary, with mean zero, and
has power spectral density function \( \varphi(\omega) \).

**Stochastic Average of** \( x(t) \). Using Eq. (B.32) and the fact that \( x(0) = \dot{x}(0) = m_z(t) = 0 \), we see that

\[
m_x(t) = \int_0^t g(t-\tau)m_z(\tau)d\tau = 0 \quad (B.71)
\]

**Variance of** \( x(t) \). Using Eq. (B.37) we see that

\[
\sigma_x^2(t) = K_{xx}(t) = \int_0^t dE_1 \int_0^t dE_2 g(E_1)g(E_2)K_{zz}(t-E_1, t-E_2) \quad (B.72)
\]

Since \( \{z(t)\} \) is stationary, \( K_{zz}(t-E_1, t-E_2) = R_{zz}(E_1-E_2) \); further, \( R_{zz}(E_1-E_2) \) is related to the power spectral density function \( \varphi(\omega) \) by the Wiener-Khinchine relation

\[
R_{zz}(E_1-E_2) = \int_0^\infty \varphi(\omega)\cos\omega(E_1-E_2)d\omega \quad (B.73)
\]

where it is assumed that \( \varphi(\omega) \) is such that

\[
\int_0^\infty \varphi(\omega) \, d\omega < \infty \quad (B.74)
\]

Inserting Eq. (B.73) into Eq. (B.72), we have

\[
\sigma_x^2(t) = \int_0^t dE_1 \int_0^t dE_2 g(E_1)g(E_2) \int_0^\infty \varphi(\omega)\cos\omega(E_1-E_2)d\omega \quad (B.75)
\]

Since the integrals involved in Eq. (B.75) are convergent, the order of integration may be reversed. With the use of Eq. (B.28), Eq. (B.75) may be written

\[
\sigma_x^2(t) = \int_0^\infty \frac{\varphi(\omega)}{\omega d\omega} \int_0^\infty \int_0^\infty e^{-\omega(E_1+E_2)} \sin\omega d\omega dE_1 \sin\omega d\omega dE_2 \cdot \cos\omega(E_1-E_2)dE_1 dE_2 d\omega \quad (B.76)
\]
The double integral on $t$ occurring in Eq. (B.76) may be evaluated after some tedious algebra, giving

$$
\sigma_x^2(t) = \int_0^\infty \frac{\phi(\omega) \, d\omega}{|z(\omega)|^2} \left\{ 1 + e^{-2\omega_n \zeta} \right\} \left\{ 1 + \frac{2\omega_n}{\omega_{dn}} \zeta \sin \omega_{dn} t \cos \omega_{dn} t - \right.
$$

$$
- e^{-\omega_n \zeta t} \left( 2\cos \omega_{dn} t + \frac{2\omega_n \zeta}{\omega_{dn}} \sin \omega_{dn} t \right) \cos \omega t - \left. - e^{-\omega_n \zeta t} \cdot \frac{2\omega_n}{\omega_{dn}} \sin \omega_{dn} t \sin \omega t + \frac{(\omega_n \zeta)^2 - \omega_{dn}^2 + \omega^2}{\omega_{dn}^2} \sin^2 \omega_{dn} t \right\}. \quad (B.77)
$$

where

$$
|z(\omega)|^2 = |\gamma(\omega)|^{-2} = (\omega^2 - \omega_n^2)^2 + (2\omega_n \zeta)^2 \quad (B.78)
$$

Equation (B.77) exhibits some interesting properties. They are:

1. As $t \to 0$,
   $$
   \sigma_x^2(t) \to 0, \quad \text{as would be expected.}
   $$

2. As $t \to \infty$,
   $$
   \sigma_x^2(t) \to \int_0^\infty \frac{\phi(\omega) \, d\omega}{|z(\omega)|^2}, \quad \text{the result which would be predicted from generalized harmonic analysis for the system with infinite operating time.}
   $$

3. If $\phi(\omega)$ is set equal to $(2\pi/\pi)$ and integral of Eq. (B.78) is evaluated by contour integration, then
\[ \sigma_x^2(t) = \frac{D}{2\zeta\omega_n^3} \left\{ 1 - e^{-2\omega_n\zeta t} \left[ \frac{\omega_n^2}{\omega_d} + \omega_n\omega_d\zeta \sin 2\omega_d t + \frac{(2\omega_n\zeta)^2}{2} \sin^2 \omega_d t \right] \right\} \]  

\[ \text{(B.79)} \]

**Approximate Evaluation of Eq. (B.77).** If \( \varphi(\omega) \) is given analytically, Eq. (B.77) can be evaluated by contour integration. If \( \varphi(\omega) \) is given numerically, then the integral may be evaluated numerically. However, if \( \varphi(\omega) \) is a smooth function of \( \omega \), having no sharp peaks, and if \( \zeta \) is small, then a very good approximation may be obtained in the following manner. If \( \zeta \) is small, the function \( \frac{1}{|z(\omega)|^2} \) is sharply peaked at \( \omega = \omega_n \); therefore, the main contribution to the integral comes from the region around \( \omega = \omega_n \). By analogy with Laplace's method of evaluating integrals, Eq. (B.77) may be approximated by

\[
\sigma_x^2(t) \approx \varphi(\omega_n) \left\{ \int_0^\infty \frac{1}{|z(\omega)|^2} \left\{ 1 + e^{-2\omega_n\zeta t} \left[ 1 + \frac{2\omega_n\zeta}{\omega_d} \sin \omega_d t \right] \right\} \cos \omega_d t - \right.
\]

\[
e^{-\omega_n\zeta t} \left\{ \frac{2\omega_n}{\omega_d} \sin \omega_d t \sin \omega t + \frac{(\omega_n\zeta)^2 - \omega_d^2 + \omega^2}{\omega_d^2} \right\} 
\]

\[
\sin^2 \omega_d t \right\} d\omega \right\} 
\]

\[ \text{(B.80)} \]
Evaluating the integral in Eq. (B.80) by contour integration gives

$$\sigma_x^2(t) \approx \frac{\pi \varphi(\omega_n)}{4\zeta \omega_n^2} \left\{ 1 - \frac{e^{-2\omega_n \xi \zeta^t}}{2\omega_n \zeta} \left[ 2 + \frac{(2\omega_n \xi)^2}{2} \sin^2 \frac{\omega_n}{\zeta} t + \omega_n \frac{\zeta \sin 2\omega_n t}{\zeta} \right] \right\}$$

(B.81)

Of special interest is the case where $\xi$ is zero. The results for this case can be obtained from Eq. (B.81) by a limiting process:

$$\sigma_x^2(t) \bigg|_{\xi=0} \approx \frac{\pi \varphi(\omega_n)}{4\omega_n^3} (2\omega_n t - \sin \omega_n t)$$

(B.82)

Plots of Eq. (B.81) are shown in Fig. B.9 for $\xi = 0, 0.025, 0.05,$ and $0.10$. It will be observed that for $\xi = 0.1$ the system reaches stationarity in roughly three cycles; on the other hand, for $\xi = 0.0$, the system never reaches stationarity.

![Figure B.9](image)

**Figure B.9**

Transient response of a dynamic system under random excitation.
APPENDIX C

RANDOM VIBRATION OF BEAMS (3)

This appendix is here attached for purposes of clarity and completeness regarding the derivation of the fundamental concepts and parameters of beam vibration used in Section IV of this dissertation. It contains much of the information found in Crandall and Yildiz's "Random Vibration of Beams" (3).

We consider a uniform beam of length $L$, simply supported, subjected to a transverse loading $f(x,t)$ per unit length. The geometrical relations for the Timoshenko beam and the forces for all models are shown in Figure C.1.

External viscous damping is assumed, transverse and rotatory, with coefficients $c_1$ and $c_2$, respectively. For uniformity of dimensions we introduce

$$
\beta_1 = \frac{c_1}{\rho A}, \quad \beta_2 = \frac{c_2}{\rho I} = \frac{c_2}{\rho Ar^2}
$$

(C.1)

where $\rho$ is the mass density, $A$ is the cross-sectional area, $I$ is the cross-sectional moment of inertia, and $r$ is the cross-sectional radius of gyration. Each of the new damping coefficients $\beta_1$ and $\beta_2$ has the dimension $T^{-1}$. The equilibrium equations obtained from Fig. C.1 are
(a) Simply supported beam; (b) Geometrical relationships; (c) Forces and moments.

\[
f + \frac{\partial V}{\partial x} = \rho A \left( \frac{\partial^2 y}{\partial t^2} + \beta_1 \frac{\partial y}{\partial t} \right) \quad (C.2)
\]

\[
V + \frac{\partial M}{\partial x} = \rho A r^2 \left( \frac{\partial^2 \psi}{\partial t^2} + \beta_2 \frac{\partial \psi}{\partial t} \right)
\]

We assume a Voigt type of viscoelasticity in the force-deformation relations.

\[
M = EI(1 + \varepsilon_1 \frac{\partial}{\partial t}) \frac{\partial \psi}{\partial x} \quad (C.3)
\]

\[
V = kGA(1 + \varepsilon_2 \frac{\partial}{\partial t})(\frac{\partial V}{\partial x} - \psi)
\]
where $EI$ is the bending modulus, $kGA$ is the Timoshenko shear modulus, $k$ is a numerical factor depending on the shape of the cross-section, and $\varepsilon_1$ and $\varepsilon_2$ are damping coefficients with dimension $\text{T}$. Combining Eqs. (C.2) and (C.3) we obtain the equations of motion for the Timoshenko beam

$$\frac{f}{\rho A} + \frac{a^2}{2}(1 + \varepsilon_2 \frac{\partial}{\partial t})(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x}) = \frac{\partial^2 y}{\partial t^2} + \beta_1 \frac{\partial y}{\partial t}$$

(C.4a)

$$\frac{\partial^2}{r^2} \left( 1 + \varepsilon_2 \frac{\partial}{\partial t} \right) \left( \frac{\partial y}{\partial x} - \psi \right) + \frac{\partial^2}{1} \left( 1 + \varepsilon_1 \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 \psi}{\partial x^2} \right) = \frac{\partial^2 \psi}{\partial t^2} + \beta_2 \frac{\partial \psi}{\partial t}$$

(C.4b)

where

$$\frac{EI}{\rho Ar^2} = \frac{E}{\rho} = a^2$$

$kGA = kG = a^2$

(C.5)

define the limiting wave velocities for the two modes of the Timoshenko beam.

A somewhat simpler model is obtained by eliminating the rotatory inertia in the Timoshenko model. This beam which still incorporates the shear deformation, we call, for simplicity, the "shear" beam. Its equations of motion are:

$$\frac{f}{\rho A} + \frac{a^2}{2}(1 + \varepsilon_2 \frac{\partial}{\partial t})(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x}) = \frac{\partial^2 y}{\partial t^2} + \beta_1 \frac{\partial y}{\partial t}$$

(C.6)

Rayleigh proposed a beam model which included the
rotatory-inertia effect but omitted the shear deformation.
Here we put \( \psi = \frac{\partial y}{\partial x} \) and omit the second of Equations (C.3).
The shear force \( V \) is defined by the Equations of Motion (C.2).
The equation of motion for the Rayleigh beam is obtained
by eliminating \( V \) and \( M \) and \( \psi \) from Eqs. (C.2) and (C.3) as
follows.

We begin by combining the first expression of Eq.
(C.3) with the second expression of Eq. (C.2), to obtain:

\[
V + EI(1 + \varepsilon_1 \frac{\partial}{\partial t} \frac{\partial^2 \psi}{\partial x^2}) = \rho A r^2 \left( \frac{\partial^2 \psi}{\partial t^2} + \beta_2 \frac{\partial \psi}{\partial t} \right)
\]

or, since \( \psi = \frac{\partial y}{\partial x} \), we have:

\[
V = \rho A r^2 \left( -\frac{\partial^4 y}{\partial x^2 \partial t^2} \right) + \beta_2 \frac{\partial^2 y}{\partial x^2 \partial t} - EI(1 + \varepsilon_1 \frac{\partial}{\partial t} \frac{\partial^3 y}{\partial x^3})
\]

Now substituting this expression into the first expres­sion of Eq. (C.2), we obtain:

\[
f + \rho A r^2 \left( -\frac{\partial^4 y}{\partial x^2 \partial t^2} \right) + \beta_2 \frac{\partial^2 y}{\partial x^2 \partial t} - EI(1 + \varepsilon_1 \frac{\partial}{\partial t} \frac{\partial^3 y}{\partial x^3}) = 0
\]

or, rearranging terms and using Eqs. (C.5), we obtain:

\[
r^2 a_1^2 (1 + \varepsilon_1 \frac{\partial}{\partial t} \frac{\partial^4 y}{\partial x^4}) + (\beta_1 - \beta_2 r^2 \frac{\partial^2}{\partial x^2}) \frac{\partial y}{\partial t} + (1 - r^2 \frac{\partial^2}{\partial x^2}) \frac{\partial^2 y}{\partial t^2} = f/\rho A
\]

Finally, the Bernoulli-Euler model is obtained by
omitting the rotatory-inertia effect from the Rayleigh
model. The equation of motion is
The four models of (C.4), (C.6), (C.7), and (C.8) will be studied for all combinations of transverse, rotary, and Voigt viscoelastic damping. For the lowest frequencies there is very little difference between the models. The fundamental natural frequencies are nearly the same. At this frequency the effect of small damping is essentially independent of whether the damping originates externally via the \( \beta \) or internally via the \( \epsilon \). At the other end of the frequency spectrum, however, there is a wide divergence between these models.

**Natural Frequencies and \( Q \) for Light Damping**

The free-motion equations for all four beams are satisfied (for the boundary conditions \( y = 0, M = 0 \) at \( x = 0, \) and \( x = L \)) by

\[
y = a_n e^{s_n t} \sin \left( \frac{n \pi x}{L} \right)
\]

\[
\psi = b_n e^{s_n t} \cos \left( \frac{n \pi x}{L} \right)
\]

Substitution of these leads to algebraic eigenvalue problems for \( a_n, b_n, \) and \( s_n \). For small damping the eigenvalues \( s_n \) have the form

\[
s_n = -\alpha_n + i\omega
\]
The ratio \( \omega_n / 2\alpha_n \) is called \( Q_n \) in analogy with the \( Q \) of a single-degree-of-freedom system (\( Q = 1 / 2 \zeta \), where \( \zeta \) is the damping ratio—see Appendix A). For small damping the frequency \( \omega_n \) in Eq. (C.10) can be approximated by neglecting the damping altogether. The decay rate \( \alpha_n \) is given approximately by considering the damping as a first-order perturbation of the undamped system.

For the Timoshenko beam the characteristic equation for \( s_n \) has the form

\[
 s_n^4 + A_3 s_n^3 + A_2 s_n^2 + A_1 s_n + A_0 = 0 \quad (C.11)
\]

where \( A_1 \) and \( A_3 \) are linear in the damping terms. \( A_0 \) has no damping terms while \( A_2 \) has zero-order terms and second-order damping terms. For each \( n \) there are two distinct pairs of roots (Eq. (C.10)). Neglecting damping we find, for large \( n \), the high and low mode frequencies

\[
 \omega_{n1} \approx \lambda_n^{a_1} \quad \omega_{n2} \approx \lambda_n^{a_2} \quad (C.12)
\]

where

\[
 \lambda_n = \frac{np}{L}
\]

The perturbation formulas for the corresponding decay rates are:

\[
 2\alpha_{n2} = \frac{A_4 - A_2 \omega_{n1}^2}{\omega_{n1}^2 - \omega_{n2}^2}; \quad 2\alpha_{n1} = \frac{A_3 \omega_{n1}^2 - A_1}{\omega_{n1}^2 - \omega_{n2}^2} \quad (C.13)
\]

Applying these formulas leads to the results shown in Table C.1 for the Timoshenko beam. Note that
<table>
<thead>
<tr>
<th>Beam Model</th>
<th>$\omega_n$</th>
<th>Damping Mechanism</th>
<th>$2\alpha_n$</th>
<th>$Q = \omega / 2\alpha_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli-Euler</td>
<td>$a_1r^2\lambda_n^2$</td>
<td>Viscoelastic</td>
<td>$\epsilon_1 a_1^2 r^2 \lambda_n^4$</td>
<td>$O(1/n^2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Rotatory</td>
<td>$\beta_2 r^2 \lambda_n^2$</td>
<td>$O(n^0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>$\beta_1$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>Rayleigh beam</td>
<td>$a_1\lambda_n$</td>
<td>Viscoelastic</td>
<td>$\epsilon_1 a_1^2 \lambda_n^2$</td>
<td>$O(1/n)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Rotatory</td>
<td>$\beta_2$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>$\beta_1/r^2 \lambda_n^2$</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>Shear beam</td>
<td>$a_2\lambda_n$</td>
<td>Viscoelastic</td>
<td>$\epsilon_2 a_2^2 \lambda_n^2$</td>
<td>$O(1/n)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Rotatory</td>
<td>$\beta_2$</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>$\beta_1$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Timoshenko beam</td>
<td>$a_1\lambda_n$</td>
<td>Viscoelastic</td>
<td>$\epsilon_1 a_1^2 \lambda_n^2$</td>
<td>$O(1/n)$</td>
</tr>
<tr>
<td>(high-mode)</td>
<td></td>
<td>Rotatory</td>
<td>$\beta_2$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>$\beta_1/r^2 \lambda_n^2 (a_2^2/a_1^2)$</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>Timoshenko beam</td>
<td>$a_2\lambda_n$</td>
<td>Viscoelastic</td>
<td>$\epsilon_2 a_2^2 \lambda_n^2$</td>
<td>$O(1/n)$</td>
</tr>
<tr>
<td>(low-mode)</td>
<td></td>
<td>Rotatory</td>
<td>$\beta_2$</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>$\beta_1$</td>
<td>$O(n)$</td>
</tr>
</tbody>
</table>

**Table C.1**

Asymptotic natural frequencies $\omega_n$ and decay rates $\alpha_n$ for high-frequency beam vibrations.

only the order of the factor in $Q_n$ is tabulated. For example, in the lower mode we find
\[ 2\alpha_{n} = \beta_{1} + \frac{\beta_{2}}{r^{2}} \frac{a_{n}^{4}}{\lambda_{n}^{2} (a_{1}^{2} - a_{2}^{2})^{2}} + \varepsilon \frac{\lambda_{n}^{2} a_{2}^{2}}{2\alpha_{n}^{2}} \quad (C.14) \]

where \( \lambda_{n} = n\pi/L \). For each type of damping separately the table gives the order of \( n \) in \( Q_{n} = \omega_{n}/2\alpha_{n} \). Thus for rotatory damping alone \((\beta_{1} = \varepsilon_{1} = \varepsilon_{2} = 0, \beta_{2} \neq 0)\)

\[ Q_{n} = \frac{\omega_{n}^{2}}{2\alpha_{n}^{2}} = \frac{\lambda_{n}^{2} a_{2}^{2}}{\beta_{2} r^{2}} \frac{a_{n}^{4}}{(a_{1}^{2} - a_{2}^{2})^{2}} = 0(n^{3}) \quad (C.15) \]

For the results in Table C.1 to be meaningful it is necessary for the damping to be small enough so that \( Q_{n} \) is in fact large compared with 1; e.g., greater than 10. This calls for delicate interpretation of the entries \( O(1/n) \) and \( O(1/n^{2}) \). In these cases the damping cannot be fixed and \( n \) increased without limit. These entries have meaning only for limited ranges of \( n \) in which \( Q_{n} \) remains large. When two or more forms of damping exist simultaneously the resultant is \( O(n^{k}) \) where \( k \) is the minimum exponent for the separate damping mechanisms.

Similar analysis for the other three beams leads to the other entries in Table C.1. We list here the perturbation formulas involved. For the shear beam the characteristic equation is

\[ A_{3}s_{n}^{3} + A_{2}s_{n}^{2} + A_{1}s_{n} + A_{0} = 0 \]

where as before \( A_{1} \) and \( A_{3} \) are linear in the damping terms while \( A_{2} \) has both zero-order and second-order damping terms.
Neglecting the damping we find, for large $n$

$$\omega_n \approx \lambda_n a_2 \quad (C.16)$$

The perturbation formula for the decay rate is

$$2\alpha_n = \frac{A_1 - A_3 \omega_n^2}{A_2} \quad (C.17)$$

where the second-order terms in $A_2$ are omitted. This model also has a single large negative real root corresponding to a highly damped nonoscillatory free motion with $s_n \approx -A_2 / A_3$.

For the Rayleigh beam and the Bernoulli-Euler beam the characteristic equation has the form

$$2s_n^2 + A_1 s_n + A_0 = 0 \quad (C.18)$$

where the damping appears only linearly in $A_1$. The exact roots are easy enough here; however, for small damping we have, for large $n$,

$$\omega_n \approx \lambda_n a_1 \ (\text{Rayleigh}), \quad \omega_n \approx a_1 r \lambda_n^2 \ (\text{Bernoulli-Euler}) \quad (C.19)$$

The perturbation formula for the decay rate is

$$2\alpha_n = A_1 / A_2 \quad (C.20)$$

It is interesting to note that the natural frequencies $\omega_n$ increase in linear proportion to $n$, for large $n$, for all models except the Bernoulli-Euler which continues to have an $n^2$ dependence. Note that the Rayleigh frequency is asymptotically the same as the high-mode
Timoshenko frequency and that the shear-beam frequency is asymptotically the same as the low-mode Timoshenko frequency. At high frequencies the Timoshenko beam is the superposition of the shear-beam mechanism and the Rayleigh-beam mechanism as far as the undamped systems go. The effect of Voigt viscoelastic damping is the same in the two modes of the Timoshenko beam as in the shear and Rayleigh beams separately. This is no longer precisely true for the effects of transverse and rotatory damping as indicated in Table C.1.

Transfer Functions

We consider first the stationary response to simple harmonic excitation. If the loading \( f(x,t) \) on the beam has the form

\[
f(x,t) = e^{i\omega t} \sin \frac{n\pi x}{L}
\]

then for all four beam models the stationary response has the form

\[
\begin{align*}
y(x,t) &= G_y(n,\omega) e^{i\omega t} \sin \frac{n\pi x}{L} \\
\psi(x,t) &= G_{\psi}(n,\omega) e^{i\omega t} \cos \frac{n\pi x}{L} \\
M(x,t) &= G_M(n,\omega) e^{i\omega t} \sin \frac{n\pi x}{L} \\
V(x,t) &= G_V(n,\omega) e^{i\omega t} \cos \frac{n\pi x}{L}
\end{align*}
\]

For each \( n \) the \( G(n,\omega) \) are functions of \( \omega \). They are called the transfer functions. Although established in terms of
simple harmonic motion they become the important building blocks for random excitation, by way of the Fourier transform.

When Eqs. (C.21) and (C.22) are inserted in the equations of motion of the beam models, the space and time variations cancel and we are left with simple algebraic equations for the transfer functions. For the Timoshenko and shear beams Equations (C.4) and (C.6), respectively, give both \( G_y \) and \( \psi \). For the Rayleigh and Bernoulli-Euler beams Equations (C.7) and (C.8), respectively, give only \( G_y \). In these latter two cases \( \psi \) is obtained from the relation \( \psi = \frac{dv}{dx} \). For the first two cases the bending moment and shear-force transfer functions \( G_M \) and \( G_Y \) are given by Eq. (C.3). In the latter two cases \( G_M \) is given by Eq. (C.3) but \( G_Y \) is obtained from the second of Eqs. (C.2). In this way we obtain the following sets of transfer functions.

**Timoshenko Beam**

\[
G_y = \left( \frac{1}{\rho A} \right) \left[ a_1^2 \lambda_n^2 + a_2^2 / r^2 + i \omega (\beta_2 + \epsilon_1 a_1^2 \lambda_n^2 + \epsilon_2 a_2^2 / r^2) - \omega^2 \right] \\
G_\psi = \left( \frac{\lambda_n}{\rho A} \right) \left[ a_2^2 / r^2 + i \omega \epsilon_2 a_2^2 / r^2 \right] \\
G_M = \left( -\lambda_n^2 a_1^2 a_2^2 / \Delta \right) \left[ 1 + i \omega (\epsilon_1 + \epsilon_2) - \omega^2 \epsilon_1 \epsilon_2 \right] \\
G_V = \left( a_2^2 \lambda_n / \Delta \right) \left[ a_1^2 \lambda_n^2 + i \omega (\beta_2 + (\epsilon_1 + \epsilon_2) a_1^2 \lambda_n^2) - \omega^2 (1 + \epsilon_2 B_2 + \epsilon_1 \epsilon_2 a_1^2 \lambda_n^2) - i \omega^3 \epsilon_2 \right] \\
\text{where} \quad \Delta = \omega^4 - i \omega^3 A_3 - \omega^2 A_2 + i \omega A_1 + A_0 \quad \text{(C.23)}
\]
and

\[ A_0 = a_1^2 a_2^2 \lambda_n \]

\[ A_1 = \beta_1 \left( a_1^2 \lambda_n + a_2^2 / r^2 \right) + \beta_2 a_2^2 \lambda_n + \left( \varepsilon_1 + \varepsilon_2 \right) a_1^2 a_2^2 \lambda_n \]

\[ A_2 = (a_1^2 + a_2^2) \lambda_n + a_2^2 / r^2 + \beta_1 \lambda_n + \varepsilon_1 \varepsilon_2 a_1^2 a_2^2 \lambda_n + \varepsilon_1 \varepsilon_2 a_2^2 / r^2 \]

\[ A_3 = \beta_1 + \beta_2 + \varepsilon_1 a_1^2 \lambda_n + \varepsilon_2 a_2^2 \lambda_n + 1 / r^2 \]  \hspace{1cm} (C.25)

Shear Beam

\[ G_y = \left( 1 / \rho A \Delta \right) \left[ a_1^2 \lambda_n + a_2^2 / r^2 + i \omega (\beta_2 + \varepsilon_1 a_1^2 \lambda_n + \varepsilon_2 a_2^2 / r^2) \right] \]

\[ G_{\psi} = \left( \lambda_n / \rho A \Delta \right) \left[ a_2^2 / r^2 + i \omega a_2^2 / r^2 \right] \]

\[ G_M = \left( -a_1^2 a_2^2 / \Delta \right) \left[ 1 + i \omega (\varepsilon_1 + \varepsilon_2) - \omega^2 \varepsilon_1 \varepsilon_2 \right] \]

\[ G_V = \left( a_2^2 \lambda_n / \Delta \right) \left[ a_1^2 \lambda_n + i \omega (\beta_2 + (\varepsilon_1 + \varepsilon_2) a_1^2 \lambda_n) - \right. \]

\[ \left. - \omega^2 (\varepsilon_2 \beta_2 + \varepsilon_1 \varepsilon_2 a_2^2 \lambda_n) \right] \]  \hspace{1cm} (C.26)

where in this case

\[ \Delta = -i \omega^3 A_3 - \omega^2 A_2 + i \omega A_1 + A_0 \]  \hspace{1cm} (C.27)

with

\[ A_0 = a_1^2 a_2^2 \lambda_n \]

\[ A_1 = \beta_1 \left( a_1^2 \lambda_n + a_2^2 / r^2 \right) + \beta_2 a_2^2 \lambda_n + \left( \varepsilon_1 + \varepsilon_2 \right) a_1^2 a_2^2 \lambda_n \]

\[ A_2 = a_1^2 \lambda_n + a_2^2 / r^2 + \beta_1 \lambda_n + \varepsilon_1 \varepsilon_2 a_1^2 a_2^2 \lambda_n + \varepsilon_1 \varepsilon_2 a_2^2 / r^2 \]

\[ A_3 = \beta_2 + \varepsilon_1 a_1^2 \lambda_n + \varepsilon_2 a_2^2 / r^2 \]  \hspace{1cm} (C.28)
Rayleigh Beam

\[ G_y = \frac{1}{\rho A \Delta} \]
\[ G_\psi = \frac{\lambda_n}{\rho A \Delta} \]
\[ G_M = (-a_1^2 \lambda_n^2 / \Delta) (1 + i \omega \varepsilon_1) \]
\[ G_V = \left( \frac{r^2 \lambda_n}{\Delta} \right) \left[ a_1^2 \lambda_n^2 + i \omega (\beta_2 + \varepsilon_1 a_1^2 \lambda_n^2) - \omega^2 \right] \]  \hspace{1cm} (C.29)

where \( \Delta = -\omega^2 A_2 + i \omega A_1 + A_o \)

with
\[ A_o = a_1^2 r^2 \lambda_n^4 \]
\[ A_1 = \beta_1 + \beta_2 r^2 \lambda_n^2 + \varepsilon_1 a_1^2 r^2 \lambda_n^4 \]
\[ A_2 = 1 + r^2 \lambda_n^2 \]  \hspace{1cm} (C.30)

Bernoulli-Euler Beam

\[ G_y = \frac{1}{\rho A \Delta} \]
\[ G_\psi = \frac{\lambda_n}{\rho A \Delta} \]
\[ G_M = (-a_1^2 r^2 \lambda_n^2 / \Delta) (1 + i \omega \varepsilon_1) \]
\[ G_V = \left( \frac{r^2 \lambda_n}{\Delta} \right) \left[ a_1^2 \lambda_n^2 + i \omega (\beta_2 + \varepsilon_1 a_1^2 \lambda_n^2) \right] \]  \hspace{1cm} (C.31)

where \( \Delta = -\omega^2 A_2 + i \omega A_1 + A_o \)

and with
\[ A_o = a_1^2 r^2 \lambda_n^4 \]
\[ A_1 = \beta_1 + \beta_2 r^2 \lambda_n^2 + \varepsilon_1 a_1^2 r^2 \lambda_n^4 \]
\[ A_2 = 1 \]  \hspace{1cm} (C.32)
In all of these formulas

\[ \lambda_n = \frac{n\pi}{L} \]  \hspace{1cm} (C.33)

is a kind of wave number, which provides the dependence on \( n \) in \( G(n,\omega) \).

Note that as \( \omega \to \infty \) all of the \( G \) approach zero except for \( G_V \) in the Rayleigh model which approaches the constant value

\[ \left[ \frac{\lambda_n}{(1 + 1/r^2\lambda_n^2)} \right]^{-1} \]

Random Vibration Analysis

In this section we obtain the formal response of our beam models to a random excitation \( f(x,t) \) which for simplicity is completely uncorrelated in \( x \) although it has an arbitrary correlation in time. Before beginning the analysis proper we recall some useful properties of transfer functions, Fourier transforms, autocorrelation functions, and spectral densities which will be employed later.

For a linear system with well-defined input and output, the frequency response or transfer function \( G(\omega) \) is the complex amplitude of the stationary sinusoidal output when the input is sinusoidal with frequency \( \omega \) and amplitude unity. Another important response is the unit-impulse response \( g(t) \) which is the output corresponding to a unit-impulse input at \( t = 0 \). It is to be noted that \( g(t) = 0 \) for \( t < 0 \). \( G(\omega) \) and \( g(t) \) are Fourier transforms of each other, i.e.,
\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} \, d\omega \]  
(C.34)

\[ G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} \, dt \]

Once the unit-impulse response \( g(t) \) is known it is possible to represent the response \( x(t) \) to an arbitrary excitation \( f(t) \) by the superposition integral or convolution,

\[ x(t) = \int_{-\infty}^{\infty} f(\theta) g(t-\theta) \, d\theta \]  
(C.35)

When \( f(t) \) is a stationary random process the statistical average or "average across the sample space" of the product of \( f(t) \) and \( f(t+\tau) \) is independent of \( t \) and is called the autocorrelation function of \( f \) denoted by \( R_f(\tau) \).

\[ R_f(\tau) = E \left[ f(t)f(t+\tau) \right] \]  
(C.36)

The average \( E \) in Eq. (C.36) is the expectation or statistical average. Note that the mean square value of \( f \) is just \( R_f(0) \). The following Fourier transform of \( R_f \) defines a spectral density \( S_f(\omega) \):

\[ S_f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_f(\tau) e^{-i\omega \tau} \, d\tau \]  
(C.37)

The inverse transform of Eq. (C.37) is

\[ R_f(\tau) = \int_{-\infty}^{\infty} S_f(\omega) e^{i\omega \tau} \, d\omega \]  
(C.38)

so that

\[ E \left[ f^2(t) \right] = R_f(0) = \int_{-\infty}^{\infty} S_f(\omega) \, d\omega \]  
(C.39)
Thus $S_f(\omega)$ is a spectral density giving the contribution to the mean square of $f(t)$ per unit of circular frequency where all frequencies from $-\infty$ to $\infty$ are required. This spectral density is $4\pi$ times smaller than the experimental unit which employs a frequency in cycles per unit time and considers only positive frequencies.

When the autocorrelation $R_f(\tau)$ of the input to a linear system is known, the autocorrelation function of the response $x(t)$ is

$$R_f(\tau) = \int_{-\infty}^{\infty} g(\Theta_1)g(\Theta_2)R_f(\tau+\Theta_2-\Theta_1)d\Theta_1d\Theta_2 \quad (C.40)$$

The Fourier transform of this gives the well-known result connecting the spectral densities of excitation and response

$$S_x(\omega) = \left|G(\omega)\right|^2 S_f(\omega) \quad (C.41)$$

We now return to our family of beams of length $L$ subjected to a stationary random transverse loading $f(x,t)$. Since the transfer functions discussed previously were based on loadings whose space variation was $\sin(n\pi x)/L$, we decompose the loading into these same space modes. We set

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x)/L \quad (C.42)$$

where

$$f_n(t) = \frac{2}{L} \int_{0}^{L} f(x,t) \sin(n\pi x)/L \, dx \quad (C.43)$$

We then consider that each value of $n$ defines a linear system whose frequency transfer functions $G(n,\omega)$ were tabulated previously. If desired the corresponding
impulse responses \( g(n,t) \) could be obtained from Eq. (C.34). The total response is the sum over \( n \) of all the individual responses. Thus for \( y(x,t) \), for example, we have, using Eq. (C.35):

\[
y(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \int_{-\infty}^{\infty} f_n(\theta) g_y(n,t-\theta) d\theta \quad (C.44)
\]

Introducing Eq. (C.43) yields

\[
y(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \int_{-\infty}^{\infty} g_y(n,t-\theta) d\theta \cdot \frac{2}{L} \int_{0}^{L} f_1(\xi,\theta) \sin \frac{n\pi \xi}{L} d\xi
\]

\[
(C.45)
\]

as a formal solution to the excitation-response problem for each sample excitation.

We next consider the statistical average over the sample space of the product \( y(x_1,t_1)y(x_2,t_2) \) which reduces to the mean square of \( y \) when \( x_2 = x_1 \) and \( t_2 = t_1 \). Using the operator \( \mathbb{E} \) to denote statistical average we have

\[
\mathbb{E} \left[ y(x_1,t_1)y(x_2,t_2) \right] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{n\pi x_1}{L} \sin \frac{m\pi x_2}{L} \cdot

\cdot \int_{-\infty}^{\infty} g_y(n,t_1-\theta) g_y(m,t_2-\theta) d\theta \cdot \frac{4}{L^2} \int_{0}^{L} \int_{0}^{L} f_1(\xi_1,\theta) f_2(\xi_2,\theta) \sin \frac{n\pi \xi_1}{L} \sin \frac{m\pi \xi_2}{L} d\xi_1 d\xi_2
\]

\[
(C.46)
\]

which relates the response statistical average to a similar statistical average of the excitation. Our further calculations are based on a particular choice of the form of this input average. We assume
\[
E \left[ f(\xi_1, \theta_1) f(\xi_2, \theta_2) \right] = \frac{L}{2} \delta(\xi_1 - \xi_2) R(\theta_1 - \theta_2) \quad (C.47)
\]
where \( \delta(x) \) is the Dirac delta function for the space correlation and \( R(\tau) \) is as yet an unspecified time correlation function. The process of simplifying Eq. (C.46) now begins. With Eq. (C.47) inserted, the last double integral in Eq. (C.46) becomes \( R(\theta_1 - \theta_2) \frac{L^2}{4} \) when \( m = n \) and is zero when \( m \neq n \). This reduces the double summation on \( m \) and \( n \) to a single summation on \( n \).

The remaining double integral is ostensibly a function of \( t_1 \) and \( t_2 \). With the assumption (C.47), however, it is actually only a function of \( \tau = t_1 - t_2 \). To see this we make the substitutions

\[
\theta_1 = t_1 - u_1 \quad \theta_2 = t_2 - u_2 \quad (C.48)
\]
to obtain

\[
E \left[ y(x_1, t + \tau) y(x_2, t) \right] = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x_1}{L} \right) \sin \left( \frac{n\pi x_2}{L} \right) \cdot \int_{-\infty}^{\infty} g_y(n, u_1) g_y(n, u_2) R(\tau + u_2 - u_1) du_1 du_2 \quad (C.49)
\]

This response is therefore a stationary process. By comparison with Eq. (C.40) we see that the double integral in Eq. (C.49) can be interpreted as an autocorrelation function for the \( n \)th-mode response. Since in the present case we have the \( G_y(n, \omega) \) and not the \( g_y(n, t) \) we make use of the Fourier transform to evaluate the double integral. Let the spectral density associated with the \( R(\tau) \) introduced in Eq. (C.47) be \( S(\omega) \). Then applying Eq. (C.38) to Eq. (C.41) yields
\[
E \left[ y(x_1, t + \tau) y(x_2, t) \right] = \sum_{n=1}^{\infty} \sin \frac{n\pi x_1}{L} \sin \frac{n\pi x_2}{L} \cdot \int_{-\infty}^{\infty} S(\omega) \left| G_y(n, \omega) \right|^2 e^{i\omega \tau} d\omega \quad (C.50)
\]

The mean square response at a location \( x \) is obtained by setting \( x_1 = x_2 = x \) and \( \tau = 0 \).

\[
E \left[ y^2(x, t) \right] = \sum_{n=1}^{\infty} \sin^2 \frac{n\pi x}{L} \int_{-\infty}^{\infty} S(\omega) \left| G_y(n, \omega) \right|^2 d\omega \quad (C.51)
\]

This result has been obtained for the deflection \( y \). Identical arguments lead to the following results for \( \psi \), \( M \) and \( V \):

\[
E \left[ \psi^2(x, t) \right] = \sum_{n=1}^{\infty} \cos^2 \frac{n\pi x}{L} \int_{-\infty}^{\infty} S(\omega) \left| G_\psi(n, \omega) \right|^2 d\omega
\]

\[
E \left[ M^2(x, t) \right] = \sum_{n=1}^{\infty} \sin^2 \frac{n\pi x}{L} \int_{-\infty}^{\infty} S(\omega) \left| G_M(n, \omega) \right|^2 d\omega \quad (C.52)
\]

\[
E \left[ V^2(x, t) \right] = \sum_{n=1}^{\infty} \cos^2 \frac{n\pi x}{L} \int_{-\infty}^{\infty} S(\omega) \left| G_V(n, \omega) \right|^2 d\omega
\]

These mean square results are still functions of the location of the response. A result which is independent of \( x \) is the space average of the mean squares. We use a bar to represent this average. Thus

\[
\bar{E}(y^2) = \frac{1}{L} \int_0^L E \left[ y^2(x, t) \right] dx
\]

\[
= \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} S(\omega) \left| G_y(n, \omega) \right|^2 d\omega \quad (C.53)
\]

with similar results for \( \bar{E}[\psi^2], \bar{E}[M^2], \bar{E}[V^2] \).
Spectral densities for (a) ideal white noise; (b) band-limited white noise.

We next consider the evaluation of the integrals appearing in Eqs. (C.52) and (C.53) and discuss the convergence of series of such integrals as in Eq. (C.53). We consider only two possibilities for $S(\omega)$ the spectrum of the correlation $R(\tau)$ introduced in Eq. (C.47). These two spectra are shown in Figure C.2. In case (a) the density is constant for all frequencies, while in case (b) it is constant out to a cut-off frequency, $\omega_c$.

Integration Formulas

In this section we give formulas for integrals of the form

$$I = \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega \quad (C.54)$$
where the denominators of \( G \) are polynomials in \( \omega \) of order 2, 3, or 4. These formulas are adapted from similar formulas given by James, Nichols, and Phillips (24).

For the following formulas to hold, it is necessary that \( G(\omega) \) be a quotient of polynomials; the order of the denominator should be at least one order higher than that of the numerator, and the roots of the denominator should lie in the upper half-plane; i.e., \( G(\omega) \) should be the transfer function of a linear passive system with positive damping and should vanish at least as fast as \( 1/\omega \) when \( \omega \to \infty \).

**Second-Order System**

\[
G(\omega) = \frac{i\omega B_1 + B_o}{-\omega^2 A_2 + i\omega A_1 + A_o}
\]

\[
\int_\infty^{-\infty} \left| G(\omega) \right|^2 \, d\omega = \pi \frac{B_o^2}{A_o} \frac{A_2 + B_1^2}{A_1 A_2}
\]

\[
(C.55)
\]

**Third-Order System**

\[
G(\omega) = \frac{-\omega^2 B_2 + i\omega B_1 + B_o}{-i\omega^3 A_3 - \omega^2 A_2 + i\omega A_1 + A_o}
\]

\[
\int_\infty^{-\infty} \left| G(\omega) \right|^2 \, d\omega = \pi \frac{B_o^2}{A_o} \frac{A_2 A_3 + A_1 B_2^2 + A_3 (B_1^2 - 2B_o B_2)}{A_1 A_2 A_3 - A_o A_2^2}
\]

\[
(C.56)
\]
Fourth-Order System

\[ G(\omega) = \frac{-i\omega^3 B_3 - \omega^2 B_2 + i\omega B_1 + B_0}{\omega^4 - i\omega^3 A_3 - \omega^2 A_2 + i\omega A_1 + A_0} \]

\[ \int_{-B}^{B} \left| G(\omega) \right|^2 d\omega = \frac{B^2}{A^2} \frac{(A_2 A_3 - A_1) + A_1 (B^2 - 2B_1 B_3) + A_3 (B^2 - 2B_0 B_2) + B^2 (A_1 A_2 - A_0 A_3)}{\pi} \frac{A_1 (A_2 A_3 - A_1) - A_0 A_3^2}{1 + 2 + 3 - A_1} \]

(C.57)

To discuss the convergence of the series Eq. (C.52) and Eq. (C.53) it is necessary to obtain order-of-magnitude results for the integrals therein. When the spectrum \( S(\omega) \) is that of Fig. C.2(a), these integrals reduce to those tabulated above.
APPENDIX D

DERIVATION OF THE PLATE EQUATION:

\[(B/M \quad \nabla^4 + \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t})u = f\]

Appendix D is here attached in order to present the derivation of the plate equation for a damped viscoelastic plate as well as the assumptions and implications of the small deflection theory of thin plates. It contains information which can be found in Brush and Almroth's *Buckling of Bars, Plates and Shells* (37), Saada's *Elasticity Theory and Applications* (38), as well as in numerous other texts.

We begin with the basic equations of elasticity:

- The Equilibrium Equations: \( \sigma_{ki,k} + F_i = 0 \) \hspace{1cm} (D.1)

- The Stress-Strain Relations: \( \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \) \hspace{1cm} (D.2)

- The Linear Strain-Displacement Relations:

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \] \hspace{1cm} (D.3)

- The Compatibility Equations:

\[ \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} \]

where \( \sigma_{ij} = \sigma_{ji} \), and \( \varepsilon_{ij} = \varepsilon_{ji} \) \((i,j = x,y,z)\), assuming the symmetry of the stress and strain tensors.

From assumption (2) below for in-plane displacements we have:
\[ u(x,y,z) = u_o(x,y) + za(x,y) \]

and

\[ v(x,y,z) = v_o(x,y) + z\beta(x,y) \]

where \( u_o \) and \( v_o \) are the in-plane middle surface \((z = 0)\) displacements, and \( \alpha \) and \( \beta \) are rotations as yet undefined.

Assumptions of Thin Plate Theory \((h \ll L_x, L_y)\)

If, when subjected to a load \( p = p(x,y) \), the deflection of the thin plate is small compared to its thickness, the following assumptions attributed to Kirchhoff may be made:

1. The middle plane remains constrained. This assumption will make it unnecessary to consider the equilibrium of the forces acting on the element of the plate in the \( x \)- and \( y \)-directions.

2. The normal strain \( \varepsilon_{zz} \) in the \( z \)-direction is small enough to be neglected, and the normal stress \( \sigma_{zz} \) is small compared to \( \sigma_{xx} \) and \( \sigma_{yy} \) so that it can be neglected in the stress-strain relations. Therefore,

\[
\begin{align*}
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0; \\
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_{xx} - \nu \sigma_{yy}); \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_{yy} - \nu \sigma_{xx}) \\
\end{align*}
\]

3. The normals to the middle plane before bending remain normal to this plane after bending. This means that the out-of-plane shear strains are small enough
to be neglected. Therefore,
\[ \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \approx 0; \quad \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \approx 0 \]

(D.6)

The only shearing strain is:
\[ \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

(D.7)

Equations (D.5) through (D.7) express the fact that all the strains and, consequently, all the stresses, can be written in terms of the deflection w of the middle plane. For example, Eq. (D.5) expresses the fact that w is only a function of x and y, i.e.,
\[ w = w(x,y) \]

(D.8)

Integrating Eqs. (D.6), we get
\[ u = -z \frac{\partial w}{\partial x} + f_1(x,y) \]

(D.9)

Now \( f_1 \) and \( f_2 \) are two functions which represent displacements in the middle plane and, according to our first assumption above, these displacements are negligible. Thus:
\[ u = -z \frac{\partial w}{\partial x} \quad \text{and} \quad v = -z \frac{\partial w}{\partial y} \]

(D.11)

Substituting Eqs. (D.11) into Eq. (D.5) and (D.6), we get:
\[ \varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \]

(D.12)
\[ \varepsilon_{yy} = -z \frac{\partial^2 w}{\partial y^2} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) \]

(D.13)
\[ \varepsilon_{xy} = -z \frac{2}{\alpha_x \alpha_y} = \frac{1}{2G} \sigma_{xy} \]  

(S.14)

Solving for the stresses, the following equations are obtained:

\[ \sigma_{xx} = \frac{E}{1-\nu^2} (\varepsilon_{xx} + \nu \varepsilon_{yy}) = \frac{-E}{1-\nu^2} z \left( \frac{2}{\alpha_x^2} + \nu \frac{2}{\alpha_y^2} \right) \]  

(D.15)

\[ \sigma_{yy} = \frac{E}{1-\nu^2} (\varepsilon_{yy} + \nu \varepsilon_{xx}) = \frac{-E}{1-\nu^2} z \left( \frac{2}{\alpha_y^2} + \nu \frac{2}{\alpha_x^2} \right) \]  

(D.16)

\[ \sigma_{xy} = 2G \varepsilon_{xy} = -\frac{E(1-\nu)}{1-\nu^2} z \frac{2}{\alpha_x \alpha_y} \]  

(D.17)

Thus, at a given point, \( \sigma_{xx}, \sigma_{yy}, \) and \( \sigma_{xy} \) vary linearly with \( z \).

Looking back at the previous assumptions, we see that the stresses can be grouped into three classes: the stresses parallel to the middle plane of the plate \( \sigma_{xx}, \sigma_{yy}, \) and \( \sigma_{xy}; \) the transverse normal stress \( \sigma_{zz}, \) and the transverse shear stresses \( \sigma_{xz} \) and \( \sigma_{yz}. \) \( \sigma_{zz} \) is of the order of magnitude of \( p, \) which rarely reaches values higher than 50 psi. It usually varies between 1 and 10 psi. This is negligible compared to the hundreds of psi reached by \( \sigma_{xx} \) and \( \sigma_{yy}. \) The total transverse load of the plate is of the order of \( pL^2. \) For equilibrium, this load must be balanced by transverse shear forces of the order of \( \sigma_{xz} Lh \) or \( \sigma_{yz} Lh. \) Therefore, \( \sigma_{xz} \) and \( \sigma_{yz} \) are of the order of \( p(L/h). \) If we consider the bending of a strip of the plate of unit width, the bending moment is of the order of \( pL^2 \) and the resisting
moment is of the order of \( \sigma_{xx} h^2 \) or \( \sigma_{yy} h^2 \). Therefore, \( \sigma_{xx} \), \( \sigma_{yy} \) (and, it is assumed, \( \sigma_{xy} \) also), are of the order of \( p(L/h)^2 \). Thus, since \( L/h \) is relatively large for thin plates, then \( \sigma_{xx}, \sigma_{yy} \) and \( \sigma_{xy} \) are greater than \( \sigma_{xz} \) and \( \sigma_{yz} \), and much greater than \( \sigma_{zz} \). Since \( \sigma_{xz}, \sigma_{yz} \) and \( \sigma_{zz} \) are relatively small, our neglecting of their effects on the displacement \( w \) is quite justified.

**Derivation of the Equilibrium Equations for a Plate**

The figure below shows the positive directions of stress quantities to be defined when the plate is subjected to lateral and in-plane loads.

![Plate element](image)

**Figure D.1**

Plate element.
where
\[ M_x = \int_{-h/2}^{h/2} \sigma_{xz} \, dz \quad \text{and} \quad M_y = \int_{-h/2}^{h/2} \sigma_{yz} \, dz \quad (D.18), (D.19) \]

\[ M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, dz \quad \text{and} \quad M_{yx} = \int_{-h/2}^{h/2} \sigma_{yx} \, dz \quad (D.20), (D.21) \]

Similarly the transverse shear elements are defined as:
\[ Q_x = \int_{-h/2}^{h/2} \sigma_{xz} \, dz; \quad Q_y = \int_{-h/2}^{h/2} \sigma_{yz} \, dz \quad (D.22), (D.23) \]

Finally, the in-plane stress resultants are defined as:
\[ N_x = \int_{-h/2}^{h/2} \sigma_{x} \, dz; \quad N_y = \int_{-h/2}^{h/2} \sigma_{y} \, dz \quad (D.24), (D.25) \]

\[ N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, dz; \quad N_{yx} = \int_{-h/2}^{h/2} \sigma_{yx} \, dz \quad (D.26), (D.27) \]

The procedure for obtaining the governing equations for plates from the equations of elasticity is to perform certain integrations on them. To proceed, we define: \( \tau_{1x} = \sigma_{xz} (+h/2); \tau_{2x} = \sigma_{xz} (-h/2) \). Recall \( \sigma_{ij} = \sigma_{ji} \); multiply Eq. (D.1) by \( (z \, dz) \) and integrate from \(-h/2\) to \(+h/2\); we have:

\[ \int_{-h/2}^{h/2} \left( z \frac{\partial \sigma_{x}}{\partial x} + z \frac{\partial \sigma_{xy}}{\partial y} + z \frac{\partial \sigma_{xz}}{\partial z} \right) \, dz = 0 \]

Upon integration, we obtain:
\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{h}{2}(\tau_{1x} + \tau_{2x}) - Q_x = 0 \quad (D.28)
\]

Likewise from Eq. (D.1), we can obtain:

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{h}{2}(\tau_{1y} + \tau_{2y}) - Q_y = 0 \quad (D.29)
\]

where

\[\tau_{1y} = \sigma_{yz}(+h/2); \quad \tau_{2y} = \sigma_{yz}(-h/2)\]

Looking again at Eq. (D.1), we can write:

\[
\frac{\partial}{\partial x} \sigma_{xz} + \frac{\partial}{\partial y} \sigma_{yz} + \frac{\partial}{\partial z} \sigma_{zz} = 0
\]

Multiplying this expression by \(dz\), and integrating from \((-h/2)\) to \((+h/2)\), we obtain:

\[
\frac{\partial}{\partial x} Q_x + \frac{\partial}{\partial y} Q_y + P_1(x,y) - P_2(x,y) = 0 \quad (D.30)
\]

where

\[P_1(x,y) = \sigma_z(+h/2); \quad P_2(x,y) = \sigma_z(-h/2)\]

Similarly from Eq. (D.1) we can write:

\[
\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{xy} + \frac{\partial}{\partial z} \sigma_{zx} = 0; \quad \frac{\partial}{\partial x} \sigma_{yx} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \sigma_{zy} = 0
\]

Multiplying each of these by \(dz\) and integrating from \((-h/2)\) to \((+h/2)\) in the \(x\)- and \(y\)-directions, respectively, we obtain:

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\tau_{1x} - \tau_{2x}) = 0; \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\tau_{1y} - \tau_{2y}) = 0
\]

\((D.31), (D.32)\)
Derivation of Plate Moment - Curvature Relations
and Integrated Stress Resultant Displacement Relations

We have from Eqs. (D.2) and (D.3):

\[
\varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu(\sigma_y + \sigma_z) \right] \quad \text{and} \quad \varepsilon_{yz} = \frac{1}{E} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)
\]

Combining these two expressions, and remembering that \( \sigma_z = 0 \) in the interior of the plate, we obtain:

\[
\frac{\partial u}{\partial x} = \frac{1}{E} \left[ \sigma_x - \nu \sigma_y \right]; \quad \frac{\partial v}{\partial y} = \frac{1}{E} \left[ \sigma_y - \nu \sigma_x \right] \quad \text{(D.33)},
\]

\[
\frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2G} \sigma_{xy}; \quad \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2G} \sigma_{yz} \quad \text{(D.35)},
\]

\[
\frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = \frac{1}{2G} \sigma_{xz} \quad \text{(D.37)}
\]

Now, using Eqs. (D.8) and (D.11), noting \( \varepsilon_{xz} = \varepsilon_{yz} = 0 \), we obtain from Eq. (D.3):

\[
\frac{\partial u}{\partial z} = - \frac{\partial w}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial z} = - \frac{\partial w}{\partial y}
\]

Hence, from Eq. (D.4b), we find:

\[
\alpha = - \frac{\partial w}{\partial x}; \quad \beta = - \frac{\partial w}{\partial y} \quad \text{(D.38)},
\]

Now using Eqs. (D.8), (D.11), (D.33), (D.34), and (D.35), multiplying by \((z \, dz)\) and integrating from \((-h/2)\) to \((+h/2)\), we obtain:

\[
\int_{-h/2}^{h/2} z^2 \frac{\partial a}{\partial x} \, dz = \int_{-h/2}^{h/2} \frac{1}{E} \left[ \sigma_x - \nu \sigma_y \right] z \, dz \quad \text{(D.40)}
\]
\[ \int_{-h/2}^{h/2} \frac{\partial v_o}{\partial y} \, dz + \int_{-h/2}^{h/2} z^2 \frac{\partial^2 \sigma_x}{\partial y^2} \, dz = \int_{-h/2}^{h/2} \frac{1}{E} \left[ \sigma_y - \nu \sigma_x \right] z \, dz \] \tag{D.41}

\[ \int_{-h/2}^{h/2} \left( \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \right) z \, dz + \int_{-h/2}^{h/2} \left( z^2 \frac{\partial^2 \sigma_y}{\partial y^2} + z^2 \frac{\partial^2 \sigma_x}{\partial x^2} \right) \, dz = \int_{-h/2}^{h/2} \frac{1}{G} \sigma_y z \, dz \] \tag{D.42}

Integrating Eqs. (D.40), (D.41), and (D.42), using Eqs. (D.38) and (D.39) yields:

\[ \frac{h^3}{12} \frac{\partial^2 \sigma_x}{\partial y^2} = \frac{1}{E} \left[ M_x - \nu M_y \right] = \frac{h^3}{12} \frac{\partial^2 w}{\partial x^2} \] \tag{D.43}

\[ \frac{h^3}{12} \frac{\partial^2 \sigma_y}{\partial x^2} = \frac{1}{E} \left[ M_y - \nu M_x \right] = \frac{h^3}{12} \frac{\partial^2 w}{\partial y^2} \] \tag{D.44}

\[ \frac{h^3}{12} \left( \frac{\partial \sigma_x}{\partial y} + \frac{\partial \sigma_y}{\partial x} \right) = \frac{1}{G} M_{xy} = -\frac{h^3}{6} \frac{\partial^2 w}{\partial x \partial y} \] \tag{D.45}

\[ M_{xy} = -(1-\nu)B \frac{\partial^2 w}{\partial x \partial y} \] \tag{D.46}

where:

\[ G = \frac{E}{2(1+\nu)} \quad \text{and} \quad B = \frac{E h^3}{12(1-\nu^2)} \]

Solving Eqs. (D.43) and (D.44) for \( M_x \) and \( M_y \) results in:

\[ M_x = -B \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] ; \quad M_y = -B \left[ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right] \] \tag{D.47}

Eqs. (D.46), (D.47), and (D.48) are known as the Moment Curvature Relations, and \( B \) is the flexural stiffness of the plate.
Likewise, substituting Eqs. (D.46), (D.47), and (D.48) into Eqs. (D.28) and (D.29) yields:

\[
Q_x = -B \frac{\partial}{\partial x} (V^2) + \frac{h}{2} (\tau_{1x} + \tau_{2x}); \quad Q_y = -B \frac{\partial}{\partial y} (V^2) + \frac{h}{2} (\tau_{1y} + \tau_{2y})
\]

(D.49)

\[
\n
\frac{\partial}{\partial x} + \frac{\partial}{\partial y}
\]

Also, using Eqs. (D.8) and (D.11), substituting them into Eqs. (D.33), (D.34), and (D.35), multiplying through by \( dz \) and integrating from \((-h/2)\) to \((+h/2)\) yields:

\[
N_x = K \left[ \frac{\partial u_o}{\partial x} + \nu \frac{\partial v_o}{\partial y} \right]; \quad N_y = K \left[ \frac{\partial v_o}{\partial y} + \nu \frac{\partial u_o}{\partial x} \right] \quad \text{(D.51), (D.52)}
\]

\[
N_{xy} = N_{yx} = G h \left[ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \right]; \quad \text{where } K = \frac{E h}{(1-\nu^2)} \quad \text{(D.53)}
\]

Eqs. (D.51) - (D.53) describe the in-plane forces and deformation behavior. \( K \) is the extensional stiffness.

**Derivation of the Governing Equations for a Plate**

The equations governing the lateral deflections, bending and shearing action of a plate are:

Eqs. (D.28), (D.29), (D.30), (D.47), (D.48), and (D.46)

The equations governing the in-plane stress resultants and in-plane midsurface displacements are:

Eqs. (D.31), (D.32), (D.51), (D.52), and (D.53)

First note that the plate can tell only the difference between normal tractions on the upper and lower surfaces. Hence we define:
\[ P_1(x, y) - P_2(x, y) \equiv P(x, y) \quad (D.54) \]

Substituting Eqs. (D.28) and (D.29) in Eq. (D.30) results in the following for the case of no shear stresses on the upper and lower surfaces of the plate:

\[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x, y) = 0 \]

Substituting Eqs. (D.47), (D.48), and (D.46) into this, we get this result:

\[ -B \left[ \frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] - 2(1-\nu)B \frac{\partial^4 w}{\partial x^2 \partial y^2} - B \left[ \frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] + p(x, y) = 0 \]

or

\[ B \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = p(x, y) \quad (D.55) \]

or

\[ B \nabla^4 w = p(x, y) \]

where

\[ \nabla^2 ( ) = \frac{\partial^2 ( )}{\partial x^2} + \frac{\partial^2 ( )}{\partial y^2} \quad \text{and} \quad \nabla^4 ( ) = \nabla^2 ( \nabla^2 ( ) ) \]

Eq. (D.55) can now be used to discuss some other similar equations. With \( B = \frac{Eh^3}{12(1-\nu^2)} \), Eq. (D.55) is:

\[ \frac{Eh^3}{12(1-\nu^2)} \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = p(x, y) \quad (D.55b) \]

Letting \( \frac{\partial ( )}{\partial y} = 0; \nu = 0 \) and multiplying both sides of Eq. (D.55) by \( b \), we obtain:
This is the governing differential equation for a beam, where \( b \) = beam width, \( I = \left( b h^3 \right)/12 \) for a rectangular beam, and \( q(x) \) = load per unit length of beam.

For a vibrating elastic plate, an inertial load per unit platform area is added as an equivalent force per unit area, resulting in:

\[
B \nabla^4 w = p(x,y,t) - \rho h \frac{\partial^2 w}{\partial t^2}
\]

where \( \rho \) = mass density; therefore \( \rho h = M \) = plate mass.

For a vibrating viscoelastic plate, we add a damping term, \( M \frac{\partial w}{\partial t} \). Now letting \( w \rightarrow u \), and \( p \rightarrow f \), we obtain:

\[
\left( B \nabla^4 + M \frac{\partial^2}{\partial t^2} + M \phi \frac{\partial}{\partial t} \right) u = f
\]

or

\[
\left( B/M \nabla^4 + \frac{\partial^2}{\partial t^2} + \phi \frac{\partial}{\partial t} \right) u = f
\]
APPENDIX E

IMPULSE GREEN'S FUNCTION (17)

The impulse Green's function can be found from

the general damped viscoelastic medium equation with

impulsive forcing function:

\[
\left[ \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_n^2 \right] q(t-t') = \delta(t-t') \tag{E.1}
\]

where, for our plate:

\[
\omega_n^2 = \frac{B}{M} k^4 = \frac{E h^3 k^4}{12(1-\nu^2)}
\]

and

\[
\gamma = 2\omega_n \zeta.
\]

Letting:

\[
g(t-t') = \left\{ \begin{array}{c}
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} i\omega(t-t') G(\omega) \\
\end{array} \right. 
\]

and

\[
G(\omega) = \left\{ \begin{array}{c}
\int_{-\infty}^{\infty} d(t-t') e^{-i\omega(t-t')} g(t-t') \\
\end{array} \right. \tag{E.2}
\]

and

\[
\delta(t-t') = \left\{ \begin{array}{c}
\int_{-\infty}^{\infty} i\omega(t-t') \frac{d\omega}{2\pi} \\
\end{array} \right. \tag{E.3}
\]

We have:

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t-t')} \left[ -\omega + i\gamma \right] \omega + \omega_n^2 G(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t-t')} \tag{E.4}
\]
or
\[ G(\omega) = \frac{1}{-\omega^2 + i\omega + \omega_n^2} = G'(\omega) + iG''(\omega) \quad (E.5) \]

Therefore,
\[ g(t-t') = \int_{-\infty}^{\infty} \frac{i\omega(t-t')}{2\pi} e^{i\omega(t-t')} G'(\omega) + i \int_{-\infty}^{\infty} \frac{i\omega(t-t')}{2\pi} e^{i\omega(t-t')} G''(\omega) \quad (E.6) \]

where, from Eq. (E.5), and upon multiplying both numerator and denominator by \((-\omega + \omega_n^2) - i\omega\), we have:
\[ G'(\omega) = \frac{-\omega^2 + \omega_n^2}{(-\omega + \omega_n^2) + \frac{\omega_n^2}{\omega}} \]

and
\[ G''(\omega) = \frac{-\omega^2 + \omega_n^2}{(-\omega + \omega_n^2) + \frac{\omega_n^2}{\omega}} \]

We now have:
\[ g'(t-t') = \int_{-\infty}^{\infty} \frac{i\omega(t-t')}{2\pi} e^{i\omega(t-t')} \cdot \frac{-\omega^2 + \omega_n^2}{(-\omega + \omega_n^2) + \frac{\omega_n^2}{\omega}} \quad (E.7) \]

\[ g''(t-t') = \int_{-\infty}^{\infty} \frac{i\omega(t-t')}{2\pi} e^{i\omega(t-t')} \cdot \frac{-\omega^2 + \omega_n^2}{(-\omega + \omega_n^2) + \frac{\omega_n^2}{\omega}} \]

where:
\[ q(t-t') = g'(t-t') + i \alpha''(t-t') \quad (E.8) \]

To evaluate \(g'(t-t')\) we must perform a contour integration; consider:
\[ \int_{C} \frac{iz(t-t')}{{-z^2 + \omega_n^2}} \cdot \frac{dz}{{-z^2 + \omega_n^2 + \omega_n^2 z^2}} \quad (\text{See figure on next page}) \]
where

\[ z = \text{Re} \theta \quad \text{and} \quad \frac{dz}{d\theta} = i \text{Re} \theta \]

We have poles at: \( z^2 - 2\omega_n z^2 + \omega_n^4 + \eta^2 z^2 = 0 \)

or

\[
\begin{align*}
\left(2\omega_n - \eta^2\right) + \left[(4\omega_n - 4\omega_n^2) + \eta^4 - 4\omega_n^4\right]^{-\frac{1}{2}}
\end{align*}
\]

\[
= \omega_n - 2\left(\frac{\eta^2}{2}\right) + 2\left(\frac{\eta^2}{2}\right) \left[\left(\frac{\eta^2}{2}\right)^2 - \omega_n^2\right]^{-\frac{1}{2}}
\]

or

\[ z = \pm \left[a + ib\right] \quad \text{where:} \quad a = \left[\frac{2}{\omega_n} - \left(\frac{\eta^2}{2}\right)^{-\frac{1}{2}}\right]
\]

and

\[ b = \frac{\eta}{2} \]

Find:

\[ \text{Res} (a+ib) = \frac{\left[-(a+ib)^2 + \omega_n^2\right] e^{i(t-t') (a+ib)}}{8ab(ia-b)} \]

\[ \text{Res} (-a+ib) = \frac{\left[-(-a+ib)^2 + \omega_n^2\right] e^{i(t-t') (-a+ib)}}{8ab(ia+b)} \]

\[ \text{Res} (-a-ib) = \frac{\left[-(-a-ib)^2 + \omega_n^2\right] e^{i(t-t') (-a-ib)}}{8ab(-ia+b)} \]

\[ \text{Res} (a-ib) = \frac{\left[-(a-ib)^2 + \omega_n^2\right] e^{i(t-t') (a-ib)}}{8ab(-ia-b)} \]
For \((t-t') > 0\) use \(C_1\), and for \((t-t') < 0\) use \(C_2\), where:

\[
\begin{align*}
&\text{for } (t-t') > 0,
&\oint_{C_1} = 2\pi i \left[ \text{Res} \ (a+ib) + \text{Res} \ (-a+ib) \right] \\
&\quad + \frac{e^{-b(t-t')}}{8ab} \left[ \frac{-\left((a+ib)^2 + \omega_n^2\right)(-ia-b)}{a^2 + b^2} \ e^{ia(t-t')} + \frac{-\left(-a+ib\right)^2 + \omega_n^2}{a^2 + b^2} \ e^{-ia(t-t')} \right] \\
&\quad = 2\pi i \frac{e^{-b(t-t')}}{8ab} \left[ -e^{ia(t-t')} - e^{-ia(t-t')} \right]
&\oint_{C_1} = \frac{\pi}{2ab} e^{-b(t-t')} \sin a(t-t'); \ \text{for } (t-t') > 0
\end{align*}
\]

Similarly, for \((t-t') < 0\),

\[
\begin{align*}
&\oint_{C_2} = 2\pi i \frac{e^{b(t-t')}}{8ab} \left[ e^{-ia(t-t')} - e^{ia(t-t')} \right] \\
&\oint_{C_2} = \frac{\pi}{2ab} e^{b(t-t')} \sin a(t-t')
\end{align*}
\]

Therefore, for all \((t-t')\):
\[
g'(t-t') = \frac{1}{2\pi} \left[ \frac{\pi d}{2ab} e^{-b|t-t'|} \sin a|t-t'| \right]
\]

\[
= \frac{1}{2} \left[ \frac{2}{\omega_n - \left( \frac{d}{2} \right)^2} \right]^{1/2} \frac{-|t-t'| d}{2} \sin |t-t'| \left[ \frac{2}{\omega_n - \left( \frac{d}{2} \right)^2} \right]^{1/2}
\]

(E.9)

Similarly, to evaluate \( g''(t-t') \), we perform a contour integration; consider:

\[
\oint_C \frac{z e^{+iz(t-t')}}{(-z^2+\omega_n^2)^2 + \frac{d^2}{4} z^2} \, dz
\]

with poles at:

\[
z = \pm \left\{ \left[ \frac{2}{\omega_n + \left( \frac{d}{2} \right)^2} \right]^{1/2} + i \left( \frac{d}{2} \right) = \pm (a \pm ib) \right. \]

Here,

\[
\text{Res} (a+ib) = \frac{-i}{8ab} = i(t-t')(a+ib)
\]

\[
\text{Res} (-a+ib) = \frac{i}{8ab} = i(t-t')(-a+ib)
\]

\[
\text{Res} (-a-ib) = \frac{-i}{8ab} = i(t-t')(-a-ib)
\]

\[
\text{Res} (a-ib) = \frac{i}{8ab} = i(t-t')(a-ib)
\]

for \((t-t') > 0\), use \( C_1 \), and for \((t-t') < 0\), use \( C_2 \), where:

\[
\oint_{C_1} = 2\pi i \left[ \text{Res} (a+ib) + \text{Res} (-a+ib) \right]
\]

or

\[
\oint_{C_1} = 2\pi i \frac{(-i)}{8ab} e^{-b(t-t')} \left[ e^{ia(t-t')} - e^{-ia(t-t')} \right]
\]
\[
\left\{ \begin{array}{l}
\frac{\pi}{2ab} e^{-b(t-t')} \sin a(t-t') \; \text{for } (t-t') > 0
\\
\frac{\pi}{2ab} e^{b(t-t')} \sin a(t-t') \; \text{for } (t-t') < 0
\end{array} \right.
\]

Similarly, for \((t-t') < 0\):

\[
\left\{ \begin{array}{l}
2\pi i \cdot \frac{1}{8ab} e^{b(t-t')} \left[ -e^{a(t-t')} + e^{-a(t-t')} \right]
\\
\frac{\pi}{2ab} e^{b(t-t')} \sin a(t-t')
\end{array} \right.
\]

Therefore, for all \((t-t')\):

\[
g''(t-t') = \frac{1}{2\pi} \left[ \frac{\pi}{2ab} e^{-b|t-t'|} \sin a(t-t') \right]
\]

\[
= -\frac{1}{2}\left[ \omega_0^2 - \left( \frac{\omega}{2} \right)^2 \right]^{\frac{1}{2}} e^{-\frac{1}{2} \frac{d}{2} \sin (t-t')} \left[ \omega_0^2 - \left( \frac{\omega}{2} \right)^2 \right]^{\frac{1}{2}}
\]

(E.10)

Combining Eqs. (E.9) and (E.10), we have:

\[
g(t-t') = g'(t-t') + ig''(t-t')
\]

\[
= \left\{ \begin{array}{l}
\frac{e^{-(t-t')d}}{2\left[ \omega_0^2 - \left( \frac{\omega}{2} \right)^2 \right]^{\frac{1}{2}}} \sin (t-t') \left[ \omega_0^2 - \left( \frac{\omega}{2} \right)^2 \right]^{\frac{1}{2}} \; \text{for } (t-t') > 0
\\
0; \; \text{for } (t-t') < 0
\end{array} \right.
\]

(E.11)

or, noting:

\[
\zeta = \frac{\frac{d}{2\omega_0}}{2}\; \text{we have:}
\]
Therefore, we can now write:

\[ g(t-t') = \eta(t-t') \frac{e^{-\frac{t-t'}{2}} \sin \left(\frac{t-t'}{2}\right)}{\omega_n (1-\zeta)^{1/2}} \]

where:

\[ \eta(t-t') = \begin{cases} 
1; & t > t' \\
0; & t < t' 
\end{cases} \]

(a) Odd part.

(b) Even part.

(c) Sum of odd and even parts.

Figure E.1 (18)

Green's function in time domain.
Figure E.2 (18)
Green's function in frequency domain.
Figure E.3 (18)

Absolute value of Green's function in frequency domain.
APPENDIX F

DETERMINATION OF THE TEMPERATURE DEPENDENT ATTENUATION CONSTANT (\(\gamma_T\)) FOR THE TRANSVERSE VIBRATION OF PLATES

Figure F.1
Bent plate.

We will see that \(\gamma\), the total attenuation constant for transverse plate vibrations, has two parts, i.e.,

\[
\gamma = \gamma_T + \gamma_V
\]

where \(\gamma_V\) is the well-known viscous part of the attenuation constant, and \(\gamma_T\) is the temperature dependent part.

To begin with, we indicate that \(1/K_{AD} = 1/K - T\alpha^2/C_p\) results if we use the Maxwell Relation within the appropriate Jacobian transformation:

\[
-\frac{1}{K_{AD}} = \frac{\partial V}{\partial P} = \frac{\partial (V,S)}{\partial (T,P)} = \frac{\partial (T,P)}{\partial (S,P)} = \frac{\partial V}{\partial P} \frac{\partial V}{\partial T} + \frac{T}{C_p} \left(\frac{\partial V}{\partial T}\right)^2
\]
where:

\[(\frac{\partial V}{\partial P})_T = -\frac{1}{K} \quad \text{and} \quad (\frac{\partial V}{\partial T})_P = \alpha\]

Starting with Eqs. (4.6), (4.7) and (6.8) of Eringen (7) (pages 1178-79 and 1180), after some algebraic modifications, we obtain the following expression for the dissipated energy per unit volume and per unit mass:

\[
diss. = \frac{-K(\nabla T)^2}{T} - 2\eta (u_{ik} - \frac{1}{3}) \frac{\partial u_{ik}}{\partial T} - \frac{1}{2} \frac{\partial \sigma_{ll}}{\partial t}
\]

where \(T\) is the temperature, \(\kappa\) is the coefficient of thermal conductivity, \(K\) is the plate bulk modulus, \(\alpha\) is the coefficient of thermal expansion, \(\eta\) and \(\zeta\) are the two coefficients of viscosity, \(V\) is the volume, \(P\) is the pressure, \(S\) is the entropy, \(C_p\) is the specific heat at constant pressure, \(\delta_{ik}\) is the Kronecker delta, \(\nabla\) is the del operator, \(u\) is the transverse displacement of a point on the neutral surface of the plate (see Fig. F.1), \(u_{ik}\) is the strain tensor \(= \frac{1}{2}(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i})\), and \(u_{ll} = (\frac{\partial u_l}{\partial x_l})\) is the sum of the diagonal elements of the strain tensor, and the dots (') above \(u_{ik}\) and \(u_{ll}\) represent differentiation with respect to time.

In order to extract the desired information prescribing the temperature dependent part of the attenuation constant, \(\gamma(T)\), it is necessary to understand the manner in which sound is dissipated in isotropic solids. The thermal conduction part of the energy dissipated in solids is:
\[ \epsilon_T = -(\kappa / T)(\nabla T)^2. \] On account of viscosity, an amount of energy \(2\varepsilon_v\) is dissipated per unit time and volume, so that the total viscosity part of \(\varepsilon_{\text{diss}}\) is \(-2\varepsilon_v\). Using the expression of Eringen (7) (4.18),

\[ \varepsilon_v = \eta (u_{ik} - (1/3)\delta_{ik} u_{ll})^2 + \frac{1}{2} \varepsilon_{\text{visc}}. \]

Adding these terms, we obtain Eq. (F.2). Of these two parts, however, the thermal conduction part is "anomalously large" for plates while the viscous part "is of the usual order of magnitude" and therefore "only the former need be calculated." (23)

To calculate the temperature dependent part of the attenuation constant, we use the fact that sound oscillations are adiabatic in the first approximation (23). Using the expression \(S(T) = S_0(T_0) + K\alpha u_{ll}\), for the entropy, we can write the adiabatic condition \(S_0(T_0) = S_0(T) + K\alpha u_{ll}\), where \(T_0\) is the temperature in the undeformed state. Expanding the difference \(S_0(T) - S_0(T_0)\) in powers of \((T-T_0)\), we have as far as the first order terms \(S_0(T) - S_0(T_0) = (T-T_0)(\alpha S_0/\alpha T_0) = C_V(T-T_0)/T_0\), where \(C_V\) is the specific heat at constant volume.

The derivative of the entropy is taken for \(u_{ll} = 0\); i.e., at constant volume. Thus, \((T-T_0) = -T\alpha u_{ll}/C_V\). Using also the relationships:

\[ K = K_{\text{iso.}} = C K_T / C_p \quad \text{and} \quad K_{\text{iso.}} / C_p = (C_T - 4/3 C_V^2), \]

we can write this result as:

\[ T - T_0 = -\frac{T\alpha \alpha(C_T^2 - (4C_V^2/3))}{C_p} u_{ll} \quad \text{(F.3)} \]
where $C_L$ and $C_T$ are the longitudinal and transverse velocities of sound and can be expressed in terms of elastic constants as:

$$C_L^2 = \frac{E(1-v)}{\rho(1+v)(1-2v)}; \quad C_T^2 = \frac{E}{2\rho(1+v)} \quad (F.4)$$

where $E$ is Young's modulus and $v$ is Poisson's ratio.

Let us now consider the dissipation of transverse sound waves. The sound dissipation coefficient (or attenuation constant) is defined (23) as the ratio of the absolute mean energy dissipation to twice the mean energy flux in the wave. This quantity gives the manner of variation of the wave amplitude with time. The amplitude decreases as $\exp[-\alpha(t-t')]$. Thus we find the following expression for the temporal attenuation constant:

$$\alpha = \frac{\rho c}{d}$$

From Eqs. (F.3) and (F.4), we have:

$$\nabla T = \nabla \left( T - T_o \right) = -\frac{\rho c^2}{3\rho} (3C_L^2 - 4C_T^2) \nabla u_{ii}$$

$$= -\frac{\rho c^2}{3\rho (1-2v)} \nabla u_{ii} \quad (F.6)$$

and

$$u_{ii} = u_{xx} + u_{yy} + u_{zz}$$

$$= -z \frac{\partial^2 x^2}{\partial x^2} - z \frac{\partial^2 y^2}{\partial y^2} + \frac{\nu}{1-v} \left( z \frac{\partial^2 x^2}{\partial x^2} + z \frac{\partial^2 y^2}{\partial y^2} \right)$$
\[ = \left( \frac{\nu}{1-\nu} - 1 \right) z \frac{a^2 c}{ax^2} \]

\[ = -\left( \frac{1-2\nu}{1-\nu} \right) z \frac{a^2 c}{ax^2} \quad (F.7) \]

and

\[ \nabla u_{ij} = -\left( \frac{1-2\nu}{1-\nu} \right) \nabla \left[ z \frac{a^2 c}{ax^2} \right] \quad (F.8) \]

\[ \nabla \left[ z \frac{a^2 c}{ax^2} \right] = \frac{az}{ax} \frac{a^2 c}{ax^2} + z \frac{a^2 c}{ax^2} + \frac{az}{ay} \frac{a^2 c}{ayx} + z \frac{a^3 c}{ax^2 ay} + \frac{az}{ax} \frac{a^2 c}{ax^2} + \frac{az}{ax} \frac{a^2 c}{ax^2 az} \]

\[ = z \frac{a^3 c}{ax^3} + \frac{a^2 c}{ax^2} = \left[ z \frac{a^2 c}{ax} + 1 \right] \frac{a^2 c}{ax^2} \]

\[ = \left[ -u_x + 1 \right] \frac{a^2 c}{ax^2} \quad (F.9) \]

where

\[ u_x = -z \frac{a^2 c}{ax} \quad (F.10) \]

Moreover, we can regard the plate as being of infinitesimal thickness, i.e., as being a geometrical surface, since we are interested only in the form which it takes under the action of the applied forces, and not in the distribution of the deformation inside it. The quantity \( c \) is then the displacement of points on the plate, regarded as a surface when it is bent. (23)

Therefore, we can let \( \frac{a^2 c}{ax^2} \) approach \( u_{xx} \), and we obtain:

\[ \nabla T = -\frac{TaF}{3c_p(1-\nu)} \left[ -u_x + 1 \right] u_{xx} \quad (F.11) \]
or

\[(\nabla T)^2 = \frac{T'^2 \alpha^2 E^2}{9C_p^2(1-\nu)^2} \left[ u_{xx}^2 - 2u_{xx} + 1 \right] u_{xx}^2 \quad (F.12)\]

where

\[u_x = u_{ox} \cos (kx - \omega t)\]
\[u_y = u_{oy} \cos (kx - \omega t)\]
\[u_{xx} = -u_{ox} k \sin (kx - \omega t)\]

Now, substituting for \(u_x\) and \(u_{xx}\) from Eq. (F.13) into Eq. (F.12), we obtain for \((E_{\text{diss}})_T\):

\[
(E_{\text{diss}})_T = -\frac{kT \alpha^2 E^2}{9C_p^2(1-\nu)^2} \int \int \left[ u_{ox}^2 \cos^2 (kx-\omega t) - 2u_{ox} \cos(kx-\omega t) + \right. \\
+ 1 \left. \right] u_{ox} \cos^2 (kx-\omega t) dV = \\
= -\frac{kT \alpha^2 E^2}{9C_p^2(1-\nu)^2} u_{ox}^2 k^2 \frac{V_o}{2} \quad (F.14)
\]

where the integrals of \(\sin^2\), \(\cos^2\), and \(\sin^2 \cos\) go to zero, and the integral of \(\sin^2 dV\) is \(V_o/2\).

Also, the mean total energy, \(\bar{E}\), is equal to twice the total free energy of the plate, \(F_{pl}\), given on page 45 by Eq. (11.6) of Landau and Lifschitz (23):

\[
\bar{E} = 2 \left\{ \frac{E h^3}{24(1-\nu^2)} \int \int \left[ \left( \frac{a^2 \xi}{a_x^2} + \frac{2a^2 \xi}{a_y^2} \right)^2 2(1-\nu) \left\{ \left( \frac{a \xi}{a_x a_y} \right)^2 - \\
- \frac{a^2 \xi}{a_x^2} \frac{a^2 \xi}{a_y^2} \right\} \right] dx \, dy \right\} \quad (F.15)
\]
or
\[ \bar{E} = \frac{Eh^3}{12(1-\nu^2)} \iint \left( \frac{\partial^2 \sigma}{\partial x^2} \right) \, dx \, dy \]  \hspace{1cm} (F.16)

Now, for reasons stated previously, we can let \( \frac{\partial^2 \sigma}{\partial x^2} \) approach \( u_{xx} \), and \( \bar{E} \) becomes:

\[ \bar{E} = \frac{Eh^3}{12(1-\nu^2)} u_{ox} k^2 A_o \frac{A_o}{2} \]  \hspace{1cm} (F.17)

where the integral of \( \sin^2 \) \( dx \, dy \) equals \( A_o/2 \).

Now,
\[ \eta_T = \left| \frac{E_{diss}}{1/2\bar{E}} \right| \]

or:
\[ \eta_T = \frac{E^2}{(1-\nu)^2} \cdot \frac{6(1-\nu)}{Eh^3} \cdot \frac{\kappa T a}{9c^2} \cdot \frac{V_o/2}{A_o/2} \cdot \frac{u_{ox} k^2}{u_{ox} k^2} \]

or
\[ \eta_T = \frac{2\kappa T a E (1+\nu)}{3c^2 h^2 (1-\nu)} \]  \hspace{1cm} (F.18)

where \( V_o/A_o = h \).