H-STRUCTURES ON SP(2), SU(4) AND RELATED SPACES

CURTIS PAUL MURLEY
INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While
the most advanced technological means to photograph and reproduce this document
have been used, the quality is heavily dependent upon the quality of the original
submitted.

The following explanation of techniques is provided to help you understand
markings or patterns which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document
photographed is “Missing Page(s)”. If it was possible to obtain the missing
page(s) or section, they are spliced into the film along with adjacent pages.
This may have necessitated cutting thru an image and duplicating adjacent
pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it
is an indication that the photographer suspected that the copy may have
moved during exposure and thus cause a blurred image. You will find a
good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being
photographed the photographer followed a definite method in
“sectioning” the material. It is customary to begin photoing at the upper
left hand corner of a large sheet and to continue photoing from left to
right in equal sections with a small overlap. If necessary, sectioning is
continued again — beginning below the first row and continuing on until
complete.

4. The majority of users indicate that the textual content is of greatest value,
however, a somewhat higher quality reproduction could be made from
“photographs” if essential to the understanding of the dissertation. Silver
prints of “photographs” may be ordered at additional charge by writing
the Order Department, giving the catalog number, title, author and
specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as
received.

Xerox University Microfilms
300 North Zeeb Road
Ann Arbor, Michigan 48106
MURLEY, Curtis Paul, 1940-
H-STRUCTURES ON Sp(2), SU(4) AND
RELATED SPACES.
University of New Hampshire, Ph.D., 1974
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED.
H-STRUCTURES ON Sp(2), SU(4) AND RELATED SPACES

by

CURTIS P. MURLEY

M.S., University of Oregon, 1968

A THESIS

Submitted to the University of New Hampshire
In Partial Fulfillment of
The Requirements for the Degree of

Doctor of Philosophy
Graduate School
Department of Mathematics
November, 1974
This thesis has been examined and approved.

Arthur H. Copeland
Thesis director, Arthur H. Copeland, Jr. Prof. of Mathematics

Edward H. Batho
Edward H. Batho, Professor of Mathematics

M. Evans Munroe
M. Evans Munroe, Professor of Mathematics

Albert O. Shar
Albert O. Shar, Assistant Professor of Mathematics

Donovan H. Van Osdol
Donovan H. Van Osdol, Associate Prof. of Mathematics

November 25, 1974
Date
ACKNOWLEDGEMENTS

Many people made this thesis possible. I particularly wish to express my gratitude to Professor Arthur Copeland, my thesis advisor, whose help, encouragement, patience and hard work enabled me to complete it. I am also indebted to Professor Albert Shar, who provided some key ideas at the right time.

More generally, I want to thank those members of the Department of Mathematics of the University of New Hampshire who provided my graduate instruction and saw fit to recommend the financial support which made my studies possible. In this connection, I want to acknowledge the support provided by a National Defense Education Act Fellowship.

Finally, I am grateful to my wife, without whose encouragement and support none of this would have been possible.

Curtis P. Murley

Raymond, Maine

November, 1974
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>v</td>
</tr>
<tr>
<td>I. LOCALIZATION IN TOPOLOGY</td>
<td>1</td>
</tr>
<tr>
<td>II. H-STRUCTURES ON SIMPLY CONNECTED H-SPACES OF TYPE (3,7)</td>
<td>10</td>
</tr>
<tr>
<td>III. H-STRUCTURES ON SU(3)-BUNDLES OVER S^7</td>
<td>20</td>
</tr>
<tr>
<td>LIST OF REFERENCES</td>
<td>51</td>
</tr>
<tr>
<td>APPENDIX</td>
<td>54</td>
</tr>
</tbody>
</table>
ABSTRACT

H-STRUCTURES ON Sp(2), SU(4) AND RELATED SPACES

by

CURTIS P. MURLEY

The classical groups Sp(2) and SU(4) can be realized as the total spaces of fiber bundles over spheres. Associated in a natural way with these fiberings are a number of induced fiberings, some of whose total spaces also support H-structures. In this paper we consider the problem of determining the number of distinct H-structures supported by the classical group SU(4) and by the H-spaces associated with SU(4) and Sp(2).

The technique used to solve this problem involves the notion of localization of topological spaces. In Chapter I, we discuss localization and prove a general theorem which allows us to reduce the H-structure problem for a given space to a similar problem for its associated localized spaces.

The number of H-structures supported by Sp(2) is known. Using this number, the localization technique easily produces a complete solution for the total spaces of the associated fiberings. These computations are carried out in Chapter II.

For SU(4) the problem has not previously been solved. In Chapter III, using the general result from Chapter I, together...
with information about the structure of $Sp(2)$ and the generators of certain homotopy groups of spheres, we give a partial solution for $SU(4)$ and its related spaces.
CHAPTER I

LOCALIZATION IN TOPOLOGY

The main results of this paper are obtained using the technique of localizing topological spaces. In this chapter we will discuss localization and some of its properties. We will also prove an important theorem that is most useful in obtaining our main results.

We begin with a brief discussion of algebraic localization in the category $J/\mathbb{N}$ of nilpotent groups. This is necessary since topological localization is defined in terms of algebraic localization and, further, we will later have occasion to make some computations involving localizations of nilpotent groups.

In what follows $\mathbb{P}$ denotes the set of all primes and $P \subset \mathbb{P}$ is any subset. The symbol $\langle P \rangle$ is used to denote the multiplicative set generated by $P$ and $P'$ will denote $\mathbb{P} - P$.

**DEFINITION (1.1)** $G \in \mathcal{N}$ is said to be $P$-local if and only if the $n^{th}$ power map $e^n: G \rightarrow G$ is an isomorphism for all $n \in \langle P' \rangle$.

Let $\mathbb{Z}_P = \{ m/n \in \mathbb{Q} | n \in \langle P' \rangle \}$ where $\mathbb{Q}$ denotes the set of rational numbers. It is easily seen that $\mathbb{Z}_P$ is $P$-local and in the special cases $P = \mathbb{P}$ and $P = \emptyset$ we have $\mathbb{Z}_P = \mathbb{Z}$ and $\mathbb{Z}_\emptyset = \mathbb{Q}$.

**PROPOSITION (1.2)** If $A$ is an Abelian group the following are equivalent: 

(i) $A$ is $P$-local,
(ii) $A \cong A \otimes \mathbb{Z}_p$ and
(iii) $A$ is a $\mathbb{Z}_p$-module.

We denote by $\mathcal{N}_p$ the full subcategory of $\mathcal{N}$ consisting of $p$-local nilpotent groups and by $\iota_p: \mathcal{N}_p \rightarrow \mathcal{N}$ the inclusion functor.

**DEFINITION (1.3)** If $B \in \mathcal{N}_p$, then $\iota_p \text{Hom}_{\mathcal{N}}(G, \iota_p B)$ is said to $p$-localize $G$ if and only if given any $\iota_p G$ and $g \in \text{Hom}_{\mathcal{N}}(G, \iota_p H)$ there exists a unique $g' \in \text{Hom}_{\mathcal{N}_p}(B, H)$ such that $g = \iota_p(g') \cdot e$.

**DEFINITION (1.4)** $\iota_p \text{Hom}_{\mathcal{N}}(G, K)$ is said to be a $p$-isomorphism if and only if

(i) $g \in \ker \iota_p \Rightarrow \exists me \in \langle p \rangle$ such that $g^m = 1$ and

(ii) for all $k \in K$ there exists $n \in \langle p \rangle$ and $g \in G$ such that $\iota_p(g) = k^n$.

**THEOREM (1.5)** [Lazard (1954)]. There exists a functor $L_p: \mathcal{N} \rightarrow \mathcal{N}_p$. Further, for each $G \in \mathcal{N}$ there exists a morphism $\iota_p \text{Hom}_{\mathcal{N}_p}(G, \iota_p L_p(G))$ having the property that $\iota_p$ $p$-localizes $G$.

The functor $L_p$ of the above theorem is called the $p$-localization functor and the $p$-local nilpotent group $L_p(G)$, usually denoted by $G_p$, is called the $p$-localization of $G$. It is not difficult to show that for any $G \in \mathcal{N}$, the pair $L_p G$ and $\iota_p$ are uniquely determined up to isomorphism.

**THEOREM (1.6)** [Hilton (1973)]. If $H \in \mathcal{N}_p$ and $G \in \mathcal{N}$ then $\iota_p \text{Hom}_{\mathcal{N}}(G, \iota_p H)$ $p$-localizes $G$ if and only if $\iota_p$ is a $p$-isomorphism.

**COROLLARY (1.7)** $L_p: \mathcal{N} \rightarrow \mathcal{N}_p$ is an exact functor.

We now turn to a description of topological localization in the homotopy category of simply connected pointed CW complexes. We will denote this category by $\mathcal{G}$.

**DEFINITION (1.8)** $X \in \mathcal{G}$ is said to be $p$-local if and only if $\pi_*(X)$ is $p$-local.
We denote by $\mathcal{C}_p$ the full subcategory of $\mathcal{C}$ consisting of $P$-local spaces and by $i_p: \mathcal{C}_p \rightarrow \mathcal{C}$ the inclusion functor.

**DEFINITION (1.9)** If $Y \in \mathcal{C}_p$ then $f \in \mathcal{C}(X, i_p Y)$ is said to $P$-localize $X$ if and only if $f$ is universal with respect to maps from $X$ to $P$-local spaces, i.e.,

$$f^*: \text{Hom}(X, i_p Z) \rightarrow \text{Hom}(i_p Y, i_p Z)$$

is an isomorphism for all $Z \in \mathcal{C}_p$. If $f$ $P$-localizes $X$ then $Y$ is said to be a $P$-localization of $X$.

$P$-local spaces and $P$-localizations are characterized by the following.

**THEOREM (1.10) [Sullivan (1971)]** For $f \in \mathcal{C}(X, Y)$ the following are equivalent:

(i) $f$ $P$-localizes $X$,

(ii) $f_*$ $P$-localizes integral homology, i.e.,

$$f_*: \tilde{H}_*(X) \rightarrow \tilde{H}_*(Y)$$

$P$-localizes $\tilde{H}_*(X)$, and

(iii) $f_*$ $P$-localizes homotopy, i.e.,

$$f_*: \pi_*(X) \rightarrow \pi_*(Y)$$

$P$-localizes $\pi_*(X)$.

**COROLLARY (1.11)** For $X \in \mathcal{C}$ the following are equivalent:

(i) $X \in \mathcal{C}_p$,

(ii) $\tilde{H}_*(X) \in \mathcal{M}_p$, and

(iii) $\pi_*(X) \in \mathcal{M}_p$.

**COROLLARY (1.12)** If $f \in \mathcal{C}_p(X, Y)$ then the following are equivalent:

(i) $f$ is a homotopy equivalence,

(ii) $f_*: \tilde{H}_*(X) \rightarrow \tilde{H}_*(Y)$ is an isomorphism, and

(iii) $f_*: \pi_*(X) \rightarrow \pi_*(Y)$ is an isomorphism.

Note that Corollary (1.12) also holds in the more general category $\mathcal{C}$.
To see that $P$-localizations of spaces $X \in \mathcal{C}$ exist we follow Sullivan (1971) and outline a cellular construction. We begin by describing a $P$-local $n$-sphere.

Choose a cofinal sequence $\mathcal{I}$ from $\langle P \rangle$ and denote the elements of this sequence by $I_1, I_2, \ldots$. Choose maps $I_i : S^n \to S^n$ of degree $I_i$ and define the $P$-local $n$-sphere, denoted $S^n_P$, as

$$S^n_P = \lim_{\mathcal{I}} (S^n \xrightarrow{I_i} S^n).$$

A $P$-local CW complex is built inductively from a point or from a $P$-local 1-sphere by attaching cones over the $P$-local sphere using maps of the $P$-local sphere $S^n_P$ into the lower "local skeletons".

A point to notice about the above construction is that since there is no $P$-local 0-sphere, there is no $P$-local 1-cell.

THEOREM (1.13) [Sullivan (1971)]. If $X$ is a CW complex with one 0-cell and no 1-cells, then there is a $P$-local CW complex, denoted $L_P(X)$, and a cellular map $e_p : X \to L_P(X)$ such that

(i) $e_p$ induces a bijection between the cells of $X$ and the $P$-local cells of $L_P(X)$ and

(ii) $e_p : \pi_*(X) \to \pi_*(L_P(X))$ P-localizes $\pi_*(X)$.

COROLLARY (1.14) There exists a functor $L_p : \mathcal{C} \to \mathcal{C}_p$. Further, for each $X \in \mathcal{C}$ there is a canonical map $e_p : \text{Hom}_\mathcal{C}(X, L_P(X))$ having the property that $e_p$ $P$-localizes $X$.

As before, we will write $X_P$ for $L_P(X)$ and call $X_P$ the $P$-localization of $X$. Again, note that the universality condition means that $X_P$ is uniquely determined up to homotopy equivalence.

The following proposition, which, among other places, appears in Mimura-Nishida-Toda (1971), shows that localization behaves nicely with respect to some important concepts and constructions of algebraic topology.
PROPOSITION (1.15) In $\mathcal{C}$ $P$-localization preserves fibrations and cofibrations.

COROLLARY (1.16) If $X, Y \in \mathcal{C}$ then

(i) $(X \times Y)_P \simeq X_P \times Y_P$,

(ii) $(X \vee Y)_P \simeq X_P \vee Y_P$, and

(iii) $(X \wedge Y)_P \simeq X_P \wedge Y_P$

where $X \vee Y$ denotes the "wedge product", i.e., the one point union and $X \wedge Y$ denotes the "smash product", i.e., the quotient space $(X \times Y) / (X \vee Y)$.

THEOREM (1.17) [Mimura-Nishida-Toda (1971)]. For $X \in \mathcal{C}$ let $P_i$, $i \in I$, be a family of subsets of $P$ and set $\overline{P} = \bigcap P_i$ and $P = \bigcup P_i$. If we let $\prod_{i \in I} X_{P_i}$ denote the pull-back of the canonical maps

$$e_{P_i} : X_{P_i} \rightarrow X_{\overline{P}}$$

then $\prod_{i \in I} X_{P_i} \simeq X_P$.

When $P \subset \overline{P}$ is a singleton, $P = \{p\}$, we will denote $X_P$ by $X(p)$.

COROLLARY (1.18) [Mimura-Nishida-Toda (1971)]. $X \in \mathcal{C}$ is homotopy equivalent to $\prod_{p \in \mathbb{P}} X(p)$, the pull-back of $e(p) : X(p) \rightarrow \emptyset$ over $X \emptyset$ for all primes $p$.

Let $\mathcal{H}$ denote the subcategory of $\mathcal{C}$ of finite CW complexes, and, as is usual, for topological spaces $X$ and $Y$ let $[X, Y]$ denote the set of homotopy classes of maps from $X$ to $Y$.

THEOREM (1.19) [Hilton-Mislin-Roitberg (1973)]. Let $X, Y \in \mathcal{H}$ and let $P_i$, $\overline{P}$ and $P$ be as in Theorem (1.17). Then

$$[X, Y] \simeq \prod_{i \in I} [X, Y_{P_i}]$$

in the category of sets, where $\prod_{i \in I} [X, Y_{P_i}]$ is the pull-back of the maps $e_{P_i} : [X, Y_{P_i}] \rightarrow [X, Y_{\overline{P}}]$ over $[X, Y_{\overline{P}}]$. 


The following theorem provides a criteria for determining if a topological space is an $H$-space. This result appears in several slightly differing forms in the literature, see for example Sullivan (1971), Mimura-Nishida-Toda (1971), or Hilton-Mislin-Roitberg (1973). Some of these versions appear to have incorrect proof, Sullivan (1971) for example, although all results are reported to be true. We state and prove a variation of these results in a form which will be useful to us later.

**Theorem (1.20)** Let $X \in \mathcal{C}$. If $(X, m)$ is an $H$-space then $m$ induces a multiplication $m(p)$ on $X(p)$ for all $p \in \mathbb{P}$. Conversely, if $(X(p), n(p))$ is an $H$-space for each $p \in \mathbb{P}$ and, further if the multiplications induced on $X(p)$ by $n(p)$ and $n(q)$ are equal for all $p, q \in \mathbb{P}$, then $X$ is an $H$-space.

**Proof** Suppose $(X, m)$ is an $H$-space. Localizing the multiplication $m$ we get a map $m(p): (X \times X)(p) \rightarrow X(p)$ for each prime $p \in \mathbb{P}$. However, $(X \times X)(p)$ is homotopic to $X(p) \times X(p)$ and we have a map $m(p): X(p) \times X(p) \rightarrow X(p)$ which is easily seen to make $(X(p), m(p))$ an $H$-space.

Conversely, consider the following diagram where $p$ and $q$ are primes and the e-maps are the canonical localization maps.
By Theorem (1.17) the rectangle is a pull-back diagram. Let 
\[ e^\phi : X(p) \cup (q) \longrightarrow X^\phi \] 
Since \( \phi \subseteq \{p, q\} \)
\[ (e(p))^\phi \circ e(p) = e^\phi \text{ and } (e(q))^\phi \circ e(q) = e^\phi \]
Thus 
\[ (e(p))^\phi \circ [n(p)^\phi \circ (e(p) \times e(p))] = (n(p))^\phi \circ (e \times e) \]
and similarly 
\[ (e(q))^\phi \circ [n(q)^\phi \circ (e(q) \times e(q))] = (n(q))^\phi \circ (e \times e). \]
But by hypothesis \( (n(q))^\phi = (n(p))^\phi \) hence we have

\[ (e(p))^\phi \circ [n(p)^\phi \circ (e(p) \times e(p))] = (e(q))^\phi \circ [n(q)^\phi \circ (e(q) \times e(q))]. \]
Since \( X(p) \cup (q) \) is a pull-back, the above equality means there exists a unique map
\[ n(p) \cup (q) : X(p) \cup (q) \times X(p) \cup (q) \longrightarrow X(p) \cup (q) \]
which is easily seen to be a multiplication on \( X(p) \cup (q) \).

Inductively one can now define a multiplication, \( m \), on the infinite pull-back over all primes, which by Corollary (1.18) is homotopic to \( X \).

Q.E.D.

If \( G \in \mathcal{G} \) is a topological group then by a theorem of G. Whitehead, (1954), \([\ , \ G] \) defines a functor \([\ , \ G] : \mathcal{H} \longrightarrow \mathcal{M} \).

The following is a useful relationship between algebraic and topological localization.

**Proposition (1.21)** [Harrison-Scheerer (1972)]. For \( X \in \mathcal{H} \) and \( G \in \mathcal{G} \) where \( G \) is a topological group there is a natural isomorphism
\[ [X, G]_P \cong [X, G_P]. \]

**Corollary (1.22)** If \( X \in \mathcal{H} \) then \( \pi_*(X_P) \cong \pi_*(X) \otimes \mathbb{Z}_P \).
For $n \geq 2$, $\pi_n(X) \approx [S^{n-1}, \Omega X]$. But $\Omega X \simeq G(X)$ where $G(X)$ is a topological group, hence corollary is immediate from main proposition if one uses Proposition (1.2) together with the fact that $\pi_n(X)$ is Abelian for $n \geq 2$.

Q.E.D.

The next theorem is a restated localized version of a result due to Copeland (1972). Its proof is the same as that given by Copeland, since the restriction to the category of finitely generated CW complexes that is made by the author in the paper where the result appears is unnecessary.

**THEOREM (1.23) [Copeland (1972)].** Let $X \in \mathcal{E}$ be a finite product of spaces $X = X_1 \times X_2 \times \ldots \times X_n$ where $X_i \in \mathcal{E}$ and $(X_i)_P$ is an $H$-space for each $i = 1, 2, \ldots, n$ and some set of primes $P$. For all integers $u$ and $s$ with $1 \leq u, s \leq n$ let $\alpha = (i_1, \ldots, i_u, j_1, \ldots, j_s)$ be an $(u+s)$-tuple of integers with $1 \leq i_1 < i_2 < \ldots < i_u \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_s \leq n$. Set $# \alpha = u + s$, $A_n = \{\alpha \mid 2 \leq # \alpha \leq 2n\}$ and

$$(M_\alpha)_P = (X_{i_1})_P \wedge \ldots \wedge (X_{i_u})_P \wedge (X_{j_1})_P \wedge \ldots \wedge (X_{j_s})_P$$

then

$$\# [X_P \wedge X_P, X_P] = \prod_{t=1}^{n} \left( \prod_{\alpha \in A_n} # [M_\alpha)_P, (X_t)_P] \right).$$

Finally, we have the following theorem which will be most useful in obtaining our main results. Its proof requires a lemma.

**LEMMA (1.24)** If $X \in \mathcal{E}$ and $Y \in \mathcal{E}$ then $[X_P, Y_P] \simeq [X, Y_P]$ as sets.

**PROOF** That $e_p : X \to X_P$ induces a surjective function $e_p^* : [X_P, Y_P] \to [X, Y_P]$ follows immediately from the fact that $e_p P$-localizes $X$. To show that $e_p^*$ is injective one uses the fact that $e_p P$-localizes $X$.
together with the result that $(X \times I)_p \simeq X_p \times I$ which follows from Corollary (1.16).

Q.E.D.

**THEOREM (1.25)** (Arkowitz, Murley, Shar). Let $X \in \mathcal{C}$ be an H-space having the property that $[X \wedge X, X]_p$ is trivial. Let $X_1, X_2 \in \mathcal{C}$ be such that for some set $P \in \mathcal{P}$, $X_p \simeq (X_1)_p$ and $X_p \simeq (X_2)_p$. Then $(X_1)_p$ and $(X_2)_p$ are H-spaces and

$$
# [X \wedge X, X] = # [(X_1)_p \wedge (X_1)_p, (X_1)_p] = # [(X_2)_p \wedge (X_2)_p, (X_2)_p] .
$$

**PROOF** By Theorem (1.20) it is easy to see that $X_p$ and $X_p$ are H-spaces so it is immediate that $(X_1)_p$ and $(X_2)_p$ are H-spaces.

By Theorem (1.19) and the hypothesis that $[X \wedge X, X]$ is trivial we see that

$$
# [X \wedge X, X] = # [X \wedge X, X_p] = # [X \wedge X, X] .
$$

But by Lemma (1.24) and Corollary (1.16 (iii)) we have

$$
# [X \wedge X, X_p] = # [X_p \wedge X_p, X_p] = # [(X_1)_p \wedge (X_1)_p, (X_1)_p]
$$

and similarly

$$
# [X \wedge X, X_p] = # [X_p \wedge X_p, X_p] = # [(X_2)_p \wedge (X_2)_p, (X_2)_p] .
$$

Q.E.D.

We now proceed to the main results.
CHAPTER II

H-STRUCTURES ON SIMPLY CONNECTED H-SPACES OF TYPE (3,7)

The problem we consider in this chapter is that of determining, up to homotopy, the number of H-structures supported by H-spaces which are total spaces of principal $S^3$-bundles over $S^7$. Among these spaces are $S^3 \times S^7$, $Sp(2)$, and the famous Hilton-Roitberg H-space, Hilton Roitberg (1969), which was the first example not involving $S^7$ of a compact simply connected H-space not of the homotopy type of a Lie group. Later Hilton and Roitberg, Hilton-Roitberg (1970), showed that any simply connected H-space of type (3,7) has the homotopy type of the total space of a principal $S^3$-bundle over $S^7$. Thus, the problem we consider is that of determining the number of H-structures that a simply connected H-space of type (3,7) may support.

We are considering principal fibrations of the form

\[
\begin{array}{c}
\text{\(S^3\)} \\
\downarrow \\
\text{\(S^7\)}
\end{array}
\xrightarrow{\pi} X
\]

The classifying space for such fibrations is $B_{S^3}$ and so the number of homotopy classes of such fibrations is in one-to-one correspondence with $[S^7, B_{S^3}] \cong \pi_7(B_{S^3})$. Using the fact that $\Omega B_{S^3} \cong S^3$ we see that $\pi_7(B_{S^3}) \cong \pi_6(\Omega B_{S^3}) \cong \pi_6(S^3) \cong \mathbb{Z}/12$. Thus, there are twelve distinct homotopy classes of such fibrations.
The elements of \( \pi_7(B_{S^3}) \) corresponding to the distinct homotopy classes of fibrations are called the characteristic classes of the fibrations. For a fibration \( S^3 \rightarrow X \rightarrow S^7 \) we will denote its characteristic class by \( \kappa(X) \) and will consider \( \kappa(X) \) to be an element of \( \pi_6(S^3) \) under identification via the canonical isomorphism \( \pi_7(B_{S^3}) \cong \pi_6(S^3) \).

Let \( PB_{S^3} \) denote the space of based paths on \( B_{S^3} \) and \( p:PB_{S^3} \rightarrow B_{S^3} \) be the projection on the terminal point. With this notation \( \Omega_{BS^3} \rightarrow PB_{S^3} \xrightarrow{p} B_{S^3} \) is a fibration and any fibration of the form \( S^3 \rightarrow X \rightarrow S^7 \) is induced by \( \kappa(X) \) as follows:

\[
\begin{array}{ccc}
S^3 & \rightarrow & \Omega_{BS^3} \\
\downarrow & & \downarrow \\
X & \rightarrow & PB_{S^3} \\
\downarrow & & \downarrow \\
S^7 & \xrightarrow{\kappa(X)} & B_{S^3}
\end{array}
\]

The two unlabeled horizontal maps in the above diagram are induced by \( \kappa(X) \) in the usual fashion.

It is known that \( Sp(2) \) is the total space of a principal \( S^3 \)-bundle over \( S^7 \) and, further, that \( \kappa(Sp(2)) = \nu' + \alpha_1(3) \in \pi_6(S^3) \) where \( \nu' \) and \( \alpha_1(3) \) are the Toda, Toda (1962), generators of the direct summands \( \mathbb{Z}/4 \) and \( \mathbb{Z}/3 \) respectively of \( \pi_6(S^7) \cong \mathbb{Z}/12 \).

Let \( n:S^7 \rightarrow S^7 \) denote a map of degree \( n \) and \( X_n \) the total space of the principal fibration induced from \( S^3 \rightarrow Sp(2) \rightarrow S^7 \) by \( n \) as shown below.

\[
\begin{array}{ccc}
S^3 & \rightarrow & S^3 \\
\downarrow & & \downarrow \\
X & \xrightarrow{n} & Sp(2) \\
\downarrow & & \downarrow \\
S^7 & \xrightarrow{\nu' + \alpha_1(3)} & B_{S^3}
\end{array}
\]
It is now clear that with this notation \( \chi(X_n) = n(v' + \alpha(3)) \)
\( = nv' + n\alpha(3) \) and that \( X_n \), \( n = 0, 1, 2, \ldots, 11 \), is a complete list of
the total spaces of principal fibrations of the form \( S^3 \rightarrow X \rightarrow S^7 \).

Not all of the spaces \( X_n \) are of different homotopy type;
indeed, the following proposition shows that \( X_n \cong X_m \) if and only
if \( n \equiv \pm m \pmod{12} \).

**Proposition (2.1)** [Douglas, Hilton, Sigrist (1969)]. Let \( X_\alpha \)
denote the total space of an \( S^3 \)-bundle over \( S^n \) with \( \chi(X_\alpha) \)
\( = \alpha \in \pi_{n-1}(S^3) \). Then \( X_\alpha \cong X_\beta \) if and only if \( \alpha = \pm \beta \).

Thus, we find that there are only seven distinct homotopy
types of total spaces \( X_n \), namely those having the following repre-
sentatives: \( X_0, X_1, X_2, X_3, X_4, X_5 \) and \( X_6 \). Of these we are
interested only in the ones that are \( H \)-spaces. The question of
deciding which of these spaces are \( H \)-spaces has been answered,
although there is a minor problem involved.

From our notation it is clear that \( X_0 \cong S^3 \times S^7 \) and \( X_1 \cong Sp(2) \)
and, hence, are \( H \)-spaces. \( X_5 \) is the Hilton-Roitberg \( H \)-space and
Zabrodsky (1971) has shown that \( X_2 \) and \( X_6 \) are not \( H \)-spaces. The
problem lies with Stasheff's (1969) proof that both \( X_3 \) and \( X_4 \) are
\( H \)-spaces. It seems that his proof of this fact used a result
that has subsequently been shown to have had an incorrect proof.
However, the result Stasheff used is thought to be true, although
a correct proof has not yet appeared. To avoid this difficulty
we will argue that both \( X_3 \) and \( X_4 \) are \( H \)-spaces, using Theorem (1.20).

In order to use Theorem (1.20) to show that \( X_3 \) and \( X_4 \) are
\( H \)-spaces, we must verify that they satisfy the hypotheses of the
theorem, i.e., that \( (X_3)_p \) and \( (X_4)_p \) are \( H \)-spaces for each
prime \( p \) and, further, that the condition on the induced multipli-
cations in \((X, \emptyset)\) and \((X, \emptyset)\) is satisfied. That these facts are indeed true is a consequence of the following lemmas numbered \((2.3)\) and \((2.5)\). Both of these lemmas contain additional information about \(X\) which will be needed later.

In order to prove Lemma \((2.3)\) we need the next result, which is a localized version of Lemma \((2.3)\), Mimura-Toda (1964).

**Lemmas (2.2)**

Let \(\alpha \in \pi_m(X)\), \(m \geq 2\), be of finite order and \(X \in \Sigma\) be the total space of the fibration \(\Omega X \longrightarrow X \longrightarrow S^m\) induced by \(\alpha\) in the usual fashion. If \(n : S^m \longrightarrow S^m\) is a map of degree \(n\) then \((X_{n\alpha})(p) \cong (X_{\alpha})(p)\) for all primes \(p\) such that \((p,n) = 1\).

**Proof** The following diagram commutes where \(\bar{n}\) denotes the map induced on total spaces by \(n\).

Since localization preserves fibration we may localize the above diagram and consider the resulting homotopy exact sequences.

\[
\cdots \longrightarrow \pi_{i+1}(S^m_{(p)}) \longrightarrow \pi_i(\Omega_{X(p)}) \longrightarrow \pi_i(X_{n\alpha}(p)) \longrightarrow \pi_i(X_{n\alpha}(p)) \longrightarrow \pi_i(X_{\alpha}(p)) \longrightarrow \cdots
\]

\[
\cdots \longrightarrow \pi_{i+1}(S^m_{(p)}) \longrightarrow \pi_i(\Omega_{X(p)}) \longrightarrow \pi_i(X_{\alpha}(p)) \longrightarrow \cdots
\]
Since \((p,n) = 1\) and \(\pi_*(S^m_\infty) \approx \pi_*(S^m) \otimes \mathbb{Z}_p\) we see that \(n(p)^*\) is an isomorphism. We now conclude from the 5-lemma that \(n(p)^*\) is an isomorphism and thus, since all spaces are CW-complexes, that \((X_{\infty})(p) \simeq (X_{\infty})(p)^*\).

Q.E.D.

**Lemma (2.3)** Let \(p\) be a prime and \((p)'' = \mathbb{P} - \{p\}\), then

(a) \((X_3(3)) \simeq (S^3 \times S^7)(3)\) and \((X_3(3)) \simeq Sp(2)(3)\),

(b) \((X_4(2)) \simeq (S^3 \times S^7)(2)\) and \((X_4(2)) \simeq Sp(2)(2)\), and

(c) \((X_5(p)) \simeq (Sp(2))(p)\) for all \(p \in \mathbb{P}\).

**Proof** (a) Consider the following diagrams of fibrations and induced fibrations.

\[
\begin{array}{ccc}
S^3 & \xrightarrow{1} & S^3 \\
\downarrow & & \downarrow \\
S^3 \times S^7 & \xrightarrow{4} & X_3 \\
\downarrow & & \downarrow \\
S^7 & \xrightarrow{3} & Sp(2)
\end{array}
\]

Lemma (2.2) can now be applied to give (a).

(b) and (c) are proved similarly where in the proof of (c) we make use of the fact that \(X_5 \simeq X_7\).

Q.E.D.

In what follows we will use the notation \(\mu(X)\) to denote the number of distinct homotopy classes of multiplications that a given H-space \(X\) will support, and, as before \(\#\) will be used to denote set cardinality.

Before stating and proving Lemma (2.5) we record the following fundamental result which is needed not only in Lemma (2.5), but forms the basis for later computations.

**Theorem (2.4)** [Copeland (1959)]. If \(X \in \mathcal{E}\) is an H-space then
\( \mu(X) = \#([X \wedge X, X]). \)

**Lemma (2.5)** \( \mu((X_n)_\circ) = 1 \) for \( n = 0, 1, 3, 4 \) and 5.

**Proof** Since all spaces \( X_n, n = 0, 1, 3, 4, 5, \) are of type \((3,7)\) we know that \((X_n)_\circ \simeq K(\mathbb{Q}, 3) \times K(\mathbb{Q}, 7)\). Furthermore, we have set bijections,

\[
[(X_n)_\circ \wedge (X_n)_\circ, (X_n)_\circ] = [X_\wedge X_n, K(\mathbb{Q}, 3) \times K(\mathbb{Q}, 7)] = H^3(X_\wedge X_n; \mathbb{Q}) \oplus H^7(X_\wedge X_n; \mathbb{Q}).
\]

James-Whitehead (1954, p. 205), show that \( X_n \) has CW structure \( X_n \simeq S^3 \cup e^7 \cup e^{10} \) and, hence for dimensional reasons \( H^3(X_\wedge X_n; \mathbb{Q}) \oplus H^7(X_\wedge X_n; \mathbb{Q}) \) is trivial. Thus, by Theorem (2.4) there is only one homotopy class of multiplications on \((X_n)_\circ\).

Q.E.D.

From Lemma (2.3) we see that \((X_3)_p \simeq (Sp(2))_p\) for all primes \( p \neq 3 \) while \((X_3)_3 \simeq (S^3 \times S^7)_3\). Since both \( Sp(2) \) and \( S^3 \times S^7 \) are H-spaces, \((X_3)_p\) is an H-space for all primes \( p \) by Theorem (1.20). Similarly, one sees that \((X_4)_p\) is an H-space for all \( p \in \mathbb{P} \). Finally, Lemma (2.5) insures that \( X_3 \) and \( X_4 \) satisfy the condition on induced multiplications in \( X_\circ \). Thus, the hypotheses of Theorem (1.20) are satisfied and we may conclude that both \( X_3 \) and \( X_4 \) are H-spaces.

We thus know that there are five distinct homotopy types of simply connected H-spaces of type \((3,7)\), namely \( S^3 \times S^7 \), \( Sp(2) \), \( X_3 \), \( X_4 \) and \( X_5 \).

The basic result used in solving problems concerning the number of H-structures that a given H-space will support is Theorem (2.4) It turns out, however, that computing the order of the algebraic loop \([X \wedge X, X]\) is usually very difficult and comparatively little has been done in the way of specific computa-
tions, except for relatively simple spaces. However, using algebraic techniques, Arkowitz and Curjel proved the following general results about Lie groups.

**THEOREM (2.6) [Arkowitz-Curjel (1963)].** \( \mu(X) \) is infinite for

- \( X = SO(10), SO(14), SO(n) \) for \( n \geq 17 \),
- \( SU(n) \) for \( n \geq 6 \),
- \( Sp(n) \) for \( n \geq 8 \),

and the representatives of the exceptional groups \( E_6 \) and \( E_8 \). \( \mu(X) \) is finite for \( X \) any other classical group or representative of the other exceptional structures.

This means that in particular \( Sp(2) \) has a finite number of non-homotopic multiplications. Mimura subsequently computed \( \mu(Sp(2)) \) by finding \( \#[Sp(2) \wedge Sp(2), Sp(2)] \) via a direct assault on the cell structure of \( Sp(2) \wedge Sp(2) \).

**THEOREM (2.7) [Mimura (1969)].** \( \mu(Sp(2)) = 2^{20} \cdot 3 \cdot 5^5 \cdot 7 \).

The only other homotopy type of the five listed above for which the problem has been solved is \( S^3 \times S^7 \), which solution follows from Theorem (1.23). Using this result we carry out the computations in Proposition (2.8) below for two reasons, the first being that this result does not seem to appear in the literature and the second being that we will need some details of this computation later.

**PROPOSITION (2.8)** \( \mu(S^3 \times S^7) = 2^{38} \cdot 3 \cdot 5^5 \cdot 7 \).

**PROOF** Using Theorem (1.23) with \( P = \mathbb{P} \), the set of all primes, we have

\[
\#[(S^3 \times S^7) \wedge (S^3 \times S^7), S^3 \times S^7] = \prod_{j=3}^{7} (a_j b_j^2 c_j d_j^2 e_j^2 f_j)
\]

where

- \( a_j = \#[S^3 \wedge S^7, S^3, S^7] = \#[S^{20}, S^j] = (\pi_{20}(S^j)) \),
- \( b_j = \#[S^3 \wedge S^7, S^7, S^j] = \#[S^{17}, S^j] = (\pi_{17}(S^j)) \),
- \( c_j = \#[S^7 \wedge S^7, S^j] = \#[S^{14}, S^j] = (\pi_{14}(S^j)) \),
Using Toda's results, Toda (1962), on the homotopy groups of $S^3$ and $S^7$ we obtain

$$
a_j = 2^{4/3}, \quad a_7 = 2\cdot 3,
$$

$$
b_j = 2\cdot 3\cdot 5, \quad b_7 = 2^{4/3},
$$

$$
c_j = 2^{4/3}\cdot 7, \quad c_7 = 2^{3/3}\cdot 5,
$$

$$
d_j = 2^{3/3}, \quad d_7 = 2,
$$

$$
e_j = 3\cdot 5, \quad e_7 = 2^{3/3}, \text{ and}
$$

$$
f_j = 2^{2/3}, \quad f_7 = 1
$$

which gives the result.

Q.E.D.

We now come to the main result of this section, which together with the known results concerning $S^3 \times S^7$ and $Sp(2)$, Theorem (2.7) and Proposition (2.8), provides a complete solution to the problem of determining the number of $H$-structures supported by simply connected $H$-spaces of type $(3,7)$.

First, we prove an algebraic lemma.

**Lemma (2.9)** If $G$ is a finite nilpotent group of order $n$ then the order of $G_p$ for any set of primes $P$ is the product of the prime power factors of $n$ for those primes in $P$.

**Proof** We first note that a finite nilpotent group of order $n$ can be expressed as a direct sum of its Sylow $p$-subgroups, and that algebraic localization preserves sums, Corollary (1.7). The result now follows from the observation that for a $p$-group $H$,

$$
H(q) \cong \begin{cases} 
H & \text{if } q = p \\
1 & \text{otherwise.}
\end{cases}
$$
This last observation is an immediate consequence of Theorem (1.6) and the definition of p-group.

Q.E.D.

**THEOREM (2.10)** For the total spaces $X_n$, $n = 3, 4$ and $5$,

(a) $\mu(X_3) = 2^{20} \cdot 3^{15} \cdot 5^5 \cdot 7^5$,

(b) $\mu(X_4) = 2^{28} \cdot 3^5 \cdot 5^7$, and

(c) $\mu(X_5) = \mu(\text{Sp}(2)) = 2^{20} \cdot 3^5 \cdot 5^7$.

**PROOF** Lemmas (2.3) and (2.5) show that we can use Theorem (1.25) to obtain the following:

$$\mu(X_3) = \mu(\text{Sp}(2)(3)) \cdot \mu(\text{Sp}(2)(3)),$$

$$\mu(X_4) = \mu(\text{Sp}(2)(2)) \cdot \mu(\text{Sp}(2)(2)),$$

and

$$\mu(X_5) = \mu(\text{Sp}(2)_P) \cdot \mu(\text{Sp}(2)_P) \text{ where } P = \{2, 3, 5, 7\}.$$

The proof is now reduced to some simple calculations.

With regard to the calculations dealing with the various localizations of Sp(2), we note that Whitehead (1954) has shown that for a topological group $X$, the functor $\mu(X)$ takes values in the category of finitely generated nilpotent groups. Since Sp(2) is indeed a topological group, we have

$$\mu(\text{Sp}(2)_P) = \# \left[ \text{Sp}(2)_P \wedge \text{Sp}(2)_P, \text{Sp}(2) \right]$$

$$= \# \left[ \text{Sp}(2) \wedge \text{Sp}(2), \text{Sp}(2)_P \right]$$

$$= \# \left( \text{Sp}(2) \wedge \text{Sp}(2), \text{Sp}(2)_P \right)$$

where the above equalities follow from Theorem (2.4), Corollary (1.16), Lemma (1.24) and Proposition (1.21). Since $[\text{Sp}(2) \wedge \text{Sp}(2), \text{Sp}(2)]$ is a finite nilpotent group, we may apply Lemma (2.9) to obtain

$$\mu(\text{Sp}(2)(3)) = 2^{20} \cdot 5^5 \cdot 7,$$

$$\mu(\text{Sp}(2)(2)) = 3 \cdot 5^5 \cdot 7,$$
\( \mu(\text{Sp}(2)_p) = 2^{20} \cdot 3^5 \cdot 5^7 \), and
\( \mu(\text{Sp}(2)_{p^r}) = 1. \)

The result (c) now follows. To obtain (a) and (b) we must compute
\( \mu((S^3 \times S^7)_{(3)}) \) and \( \mu((S^3 \times S^7)_{(2)}) \).

To make these computations we use Theorem (1.23) along
with the observation that if \( X \) is a finite smash of spheres the
sum of whose dimensions is \( n \geq 2 \) and \( Ye \in \mathcal{G} \) then for any prime \( p \),
\[ \# [X(p), Y(p)] = \# [X, Y(p)] = \#(\pi_n(Y(p))) = \#(\pi_n(Y) \otimes \mathbb{Z}(p)). \]
With this it is an easy matter to see that
\( \mu((S^3 \times S^7)_{(3)}) = 3^{15} \)
and
\( ((S^3 \times S^7)_{(2)}) = 2^{38} \)
from the computations done in the proof of Proposition (2.8).
The results (a) and (b) now are obtained.

Q.E.D.
CHAPTER III

H-STRUCTURES ON SU(3)-BUNDLES OVER S^7

Curtis-Mislin (1970) have shown that all SU(3)-bundles over S^7 are H-spaces. These bundles, SU(3)→ Y → S^7, are classified by π_7(B SU(3)) = π_6(SU(3)) = ℤ/2 ⊕ ℤ/3. Let α+β ∈ π_7(B SU(3)) be a suitable generator where α has order two and β has order three. It is known that SU(4) is an SU(3)-bundle over S^7 and that for suitable choice of a generator, χ(SU(4)) = α+β. As in Chapter II, we define a total space Y_n, n = 0, 1, 2, 3, 4, or 5, to be the total space of the fibration induced from SU(3) → SU(4) → S^7 by a map n:S^7 → S^7 of degree n. Again, analogous to the results of Chapter II, we have six total spaces but only four distinct homotopy types, Y_0 = SU(3)x S^7, SU(4) = Y_1 ∼ Y_5, Y_2 ∼ Y_4 and Y_3.

In this chapter we consider the problem of determining the number of H-structures that each of the homotopy types Y_n, n = 0, 1, 2, 3 will support. Unlike Sp(2) and its associated spaces, nothing is known about this problem and, unfortunately, we will be able to give only a partial solution.

The first proposition of this chapter gives several equivalences that enable us to separate the solvable parts of the problem from the unsolvable.
PROPOSITION (3.1) Let $\mathcal{P}$ denote the set of all primes, then

(a) $(SU(3) \times S^7)_{\mathcal{P}-\{2\}} \cong (S^3 \times S^5 \times S^7)_{\mathcal{P}-\{2\}}$,

(b) $SU(4)(3) \cong (Sp(2) \times S^5)(3)$ and $SU(4)_{\mathcal{P}-\{3,2\}} \cong (S^3 \times S^5 \times S^7)_{\mathcal{P}-\{3,2\}}$,

(c) $(Y_2)(3) \cong (Sp(2) \times S^5)(3)$, $(Y_2)(2) \cong (SU(3) \times S^7)(2)$ and $(Y_2)_{\mathcal{P}-\{3,2\}} \cong (S^3 \times S^5 \times S^7)_{\mathcal{P}-\{3,2\}}$ and

(d) $(Y_3)(3) \cong (S^3 \times S^5 \times S^7)_{\mathcal{P}-\{2\}}$ and $(Y_3)(2) \cong SU(4)(2)$.

PROOF $SU(3)$ is the total space of an $S^3$-bundle over $S^5$ with $\chi(SU(3))$ having order two. A map of degree two $S^5 \to S^5$ then induces the product bundle, $S^3 \to S^3 \times S^5 \to S^5$, from $S^3 \to SU(3) \to S^5$ and (a) now follows from Lemma (2.2).

A map $S^7 \to S^7$ of degree six induces the product bundle $SU(3) \to SU(3) \times S^7 \to S^7$ from $SU(3) \to SU(4) \to S^7$ since $\chi(SU(4))$ has order six, thus the second part of (b) follows. To show that the first part is true, one uses the fact that $SU(4)$ is an $Sp(2)$-bundle over $S^5$ with $\chi(SU(4)) \in \pi_2(BSp(2)) \cong \pi_4(Sp(2)) \cong \mathbb{Z}/2$. As in the proof above, one now sees that $SU(4)(3) \cong (Sp(2) \times S^5)(3)$.

(c) and (d) follow in similar fashion using the fact that if one views $Y_2$ and $Y_3$ as $SU(3)$-bundles over $S^7$ then $\chi(Y_2)$ has order three and $\chi(Y_3)$ has order two.

Q.E.D.

Each of the spaces, $Y_n$, $n = 0, 1, 2, 3$, is of the type $(3, 5, 7)$ and, as in the case of the spaces $X_n$ of Chapter II, $[Y_n \wedge Y_n, (Y_n)_{\emptyset}]$ is trivial for dimensional reasons. Thus, we may use Theorem (1.25) which with the above proposition would give a complete solution to our problem, provided we could compute each of the following numbers:

(i) $\mu((SU(3) \times S^7)(2))$, 
(ii) $\mu(\text{SU}(4)_{(2)})$,
(iii) $\mu((\text{Sp}(2) \times S^5)_{(3)})$,
(iv) $\mu((S^3 \times S^5 \times S^7)_{P\{-2\}})$, and
(v) $\mu((S^3 \times S^5 \times S^7)_{P\{-3,2\}})$.

Of the above list (iv) and (v) are easily computed using Theorem (1.23). (iii) is computable but requires a knowledge of the cellular structure of $\text{Sp}(2)_{(3)}$ together with information about the generators of the 3-primary homotopy groups of certain spheres. (i) and (ii) are essentially non-computable by the techniques that yield (iii), (iv) and (v). The reason for this is that these methods would require a knowledge of the 2-primary unstable homotopy of certain spheres which is well beyond the range for which it has been computed. We will, however, be able to give a rough upper bound for (i) and (ii), modulo the cardinalities of some undetermined 2-primary homotopy groups of spheres.

We begin our computations with the easiest, namely (iv) and (v).

**PROPOSITION (3.2)**

(a) $\mu((S^3 \times S^5 \times S^7)_{P\{-2\}}) = 3^{105} \cdot 5^{30} \cdot 7^9 \cdot 11^5 \cdot 13$ and
(b) $\mu((S^3 \times S^5 \times S^7)_{P\{-3,2\}}) = 5^{30} \cdot 7^9 \cdot 11^5 \cdot 13$.

**PROOF** Using Theorem (1.23) with $P = P\{-2\}$ we have

$$
\# \left[ (S^3 \times S^5 \times S^7)_{P}, (S^3 \times S^5 \times S^7)_{P}, (S^3 \times S^5 \times S^7)_{P} \right] = \prod_{j=3,5,7} \left[ \prod_{k=1}^{5} a_{k_j} \prod_{h=1}^{7} b_{h_j} \right] \prod_{m=1}^{4} c_j \prod_{m=1}^{4} d_{m_j} \prod_{m=1}^{4} e_j \prod_{m=1}^{4} f_j
$$

where the values of the quantities $a_{k_j}, b_{h_j}, c_j, d_{m_j},$ and $e_j$ are given by the table below.
<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\varepsilon \cdot \varepsilon$</th>
<th>$\tau$</th>
<th>((d(\varepsilon S)\tau \mu )^# = \eta \eta_{\mu}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
<td>((d(\varepsilon S)\varepsilon \mu )^# = \eta_{\mu}^*)</td>
</tr>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\tau$</td>
<td>$\tau \cdot \varepsilon$</td>
<td>((d(\varepsilon S)\tau \varepsilon \mu )^# = \eta_{\varepsilon \mu}^*)</td>
</tr>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\varepsilon$</td>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>((d(\varepsilon S)\varepsilon \varepsilon \mu )^# = \eta_{\varepsilon \varepsilon \mu}^*)</td>
</tr>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\tau$</td>
<td>$\tau \cdot \varepsilon$</td>
<td>((d(\varepsilon S)\tau \varepsilon \mu )^# = \eta_{\varepsilon \mu}^*)</td>
</tr>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\tau$</td>
<td>$\tau \cdot \varepsilon$</td>
<td>((d(\varepsilon S)\tau \varepsilon \mu )^# = \eta_{\varepsilon \mu}^*)</td>
</tr>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\tau$</td>
<td>$\tau \cdot \varepsilon$</td>
<td>((d(\varepsilon S)\tau \varepsilon \mu )^# = \eta_{\varepsilon \mu}^*)</td>
</tr>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\tau$</td>
<td>$\tau \cdot \varepsilon$</td>
<td>((d(\varepsilon S)\tau \varepsilon \mu )^# = \eta_{\varepsilon \mu}^*)</td>
</tr>
<tr>
<td>$\varepsilon \cdot \varepsilon$</td>
<td>$\tau$</td>
<td>$\tau \cdot \varepsilon$</td>
<td>((d(\varepsilon S)\tau \varepsilon \mu )^# = \eta_{\varepsilon \mu}^*)</td>
</tr>
</tbody>
</table>

$\tau = \varepsilon$  
$\varepsilon = \varepsilon$  
$\varepsilon = \varepsilon$
The cardinalities of the homotopy groups above are obtained from Toda (1962) and Toda (1965). (a) is verified by adding appropriate exponents while (b) follows from (a) by setting $P = \mathbb{F}-\{3,2\}$.

Q.E.D.

In order to compute $\mu((Sp(2) \times S^5_3))$ we must first give some details about the cellular structure of $Sp(2)_3$, compute some homotopy groups, and develop some information regarding generators of the homotopy groups of certain spheres. We begin with a result about the cell structure of $Sp(2)_3$ and some of its related spaces.

Let $X^2$ be as in Chapter II, i.e., the total space of an $S^3$-bundle over $S^7$ having $\chi(X^2) \simeq \alpha_1(3)$, a generator of the 3-primary component of $\pi_6(S^3) \simeq \mathbb{Z}/12$. Using Lemma (2.2) it is easy to see that $(X^2)_3 \cong Sp(2)_3$.

**Lemma (3.3)**

(a) $X^2$ has cell structure $S^3 \cup \alpha_1(3) e^7 \cup e^{10}$,

(b) for $k \geq 2$, $E^k(X^2) \simeq (S^{k+3} \cup \alpha_1(k+3) e^{k+7}) \vee S^{k+10}$,

(c) $E^k(X^2 \wedge X^2) \simeq (S^{k+6} \cup \alpha_1(k+6) e^{k+10} \vee S^{k+10}) \cup E^k \beta$,

(d) $(X^2 \wedge X^2)/S^6 \simeq (S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}) \vee (S^{13} \cup \alpha_1(13) e^{17}) \vee (S^{13} \cup \alpha_1(13) e^{17})$.

**Proof** (a) is a result due to James-Whitehead (1954, p.205). (b)
follows immediately from Mimura (1969, Lemma 2.1(ii)) and (c) and 
(d) are obtained from Mimura (1969, Proposition 4.1). In the latter 
three cases the proofs are essentially the same as those given by 
Mimura, one merely ignores 2-primary generators. This is justified 
by the fact that $\text{Sp}(2)$ has cell structure $S^3 \cup_{\nu'} \alpha_1(3) \cup \alpha_1(3) \cup e^{10}$, 
which differs from that of $X_2$ only in that the attaching maps have 
2-torsion as well as 3-torsion since $\nu' \in \pi_6(S^3)$ has order four.

Q.E.D.

In later computations we will need to know the cardinalities 
of the homotopy groups $\pi_i((X_2)_3)$ for $8 \leq i \leq 30$. In order to compute 
these we first need some information about the homotopy of $S^3(3)$ 
and $S^7(3)$. In what follows the notation for the generators of the 
homotopy groups of spheres is that of Toda as found in Toda (1962), 
(1963) and (1966).

**LEMMA (3.4)**

<table>
<thead>
<tr>
<th>$i$</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_i(S^3(3)) \cong$</td>
<td>0</td>
<td>$\mathbb{Z}/3$</td>
<td>$\mathbb{Z}/3 \oplus \mathbb{Z}/3$</td>
<td>$\mathbb{Z}/3$</td>
<td>0</td>
</tr>
<tr>
<td>generators</td>
<td>$\gamma_2(3)$</td>
<td>$\alpha_6(3)$</td>
<td>$\tilde{u}_3(0, \beta_1 \beta_1)$</td>
<td>$\alpha_1(3) \beta_1(6) \beta_1(16)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i$</th>
<th>29</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_i(S^3(3)) \cong$</td>
<td>$\mathbb{Z}/3 \oplus \mathbb{Z}/3$</td>
<td>$\mathbb{Z}/3 \oplus \mathbb{Z}/3$</td>
</tr>
<tr>
<td>generators</td>
<td>$p_* \overline{Q}^2(\alpha_5)$</td>
<td>$\alpha_7(3)$</td>
</tr>
<tr>
<td></td>
<td>$p_* \overline{Q}^2(\beta_1 \beta_1)$</td>
<td>$p_* \overline{Q}^2(\beta_1 \beta_1)$</td>
</tr>
</tbody>
</table>

The following compositions also generate:

1. $\omega_1(3) \alpha_5(6) = \pm \gamma_2(3)$,
2. $\alpha_1(3) \alpha_5(6) = \pm p_* \overline{Q}^2(\alpha_5)$,
PROOF The entries in the table come from Toda (1966) since $\pi_3(S^3(3)) \cong \pi_4(S^3;3)$ by Corollary (1.22). The only facts requiring proof are that the compositions (1), (2), (3), and (4) also generate.

Using the exact sequence
\[ \cdots \to \pi_26(S^7;3) \xrightarrow{\Delta} \pi_24(S^5;3) \xrightarrow{G} \pi_25(S^3;3) \to \pi_25(S^7;3) \to \cdots \]
and information concerning $A$ and $G$ given in Toda (1962, Proposition 13.3) together with the facts that $\pi_24(S^5;3) \cong \mathbb{Z}/3$ has generator $\alpha_5(5)$, $\pi_26(S^7;3) \cong \mathbb{Z}/3$ has generator $\alpha_5(7) = E^2\alpha_5(5)$, and $\pi_25(S^3;3) \cong \mathbb{Z}/3$, we see that
\[ \Delta(\alpha_5(7)) = \Delta(E^2\alpha_5(5)) = 3\alpha_5(5) = 0 \]
and
\[ G(\alpha_5(5)) = \alpha_1(3)E\alpha_5(5) = \alpha_1(3)\alpha_5(6). \]
Thus, $G$ is an isomorphism and $\alpha_1(3)\alpha_5(6)$ generates $\pi_25(S^3;3)$.

This proves relation (1). To prove (2) and (3) one first notes the following concerning groups and generators: $\pi_30(S^7;3) \cong \mathbb{Z}/9 \oplus \mathbb{Z}/3$ has generators $\alpha_6'(7) = E^2\alpha_6'(5)$ of order 9 and $\alpha_1(7)\beta_1(10)\beta_1(20) = E^2(\alpha_1(5)\beta_1(8)\beta_1(18))$ of order 3, $\pi_30(S^5;3) \cong \mathbb{Z}/9 \oplus \mathbb{Z}/3$ has generators $\alpha_6'(5)$ of order 9 and $\alpha_1(5)\beta_1(8)\beta_1(18)$ of order 3.

By Toda (1962, Proposition 13.3),
\[ \Delta(\alpha_6'(7)) = \Delta(E^2\alpha_6'(5)) = 3\alpha_6'(5) \neq 0, \]
and
\[ \Delta(\alpha_1(7)\beta_1(10)\beta_1(20)) = \Delta(E^2(\alpha_1(5)\beta_1(8)\beta_1(18))) = 3\alpha_1(5)\beta_1(8)\beta_1(18) = 0. \]
Hence, $G(\alpha_1(5)\beta_1(8)\beta_1(18)) = \alpha_1(3)\alpha_1(6)\beta_1(9)\beta_1(19)$ generates and since $\alpha_6'(5) \notin \text{im} \Delta$, $G(\alpha_6'(5)) = \alpha_1(3)\alpha_6'(6)$ generates also.

That they map to the indicated generators is seen as follows:

By Toda (1965, (2.12)), $H = I*H^{(2)}$ and by Toda (1965 (6.3) and Lemma 6.1)
\[ H(p_*Q^2(\alpha_5)) = I*H^{(2)}(p_*Q^2(\alpha_5)) = IQ^1(\alpha_6') = I*1'(\alpha_6'(5)) = 0, \]
and
H(p \cdot Q^2(\beta_1 \beta_1)) = I \cdot H^2(p \cdot Q^2(\beta_1 \beta_1)) = I Q^1(\alpha_1 \beta_1 \beta_1) = I \cdot I^1(\alpha_1(5) \beta_1(8) \beta_1(18)) = 0

since I \cdot I^1 is two steps in exact sequence, Toda (1965, (2.5)). Hence each of the generators of \( \pi_{29}(S^3:3) \) is the image of an element of \( \pi_{28}(S^5:3) \) under G.

By Toda (1965, (1.3)),

\[
H^2(\pi_{29}(S^3:3) \rightarrow \pi_{26}(Q^1_2:3)
\]

is an isomorphism and by Toda (1966, (10.1)),

\[
H^2(\alpha_1(3) \alpha_1(6) \beta_1(9) \beta_1(19)) = H^2(\alpha_1(3) \alpha_1(3) \beta_1(6) \beta_1(16)).
\]

\[
H^2(\alpha_1(3)) \text{ generates } \pi_2(Q^1_2:3) \text{ which in Toda's notation, } (1965, (6.4)),
\]
is \( \iota_5 \) so using Toda (1965, (2.6), (6.3) and Lemma 6.1), we have

\[
H^2(\alpha_1(3) \alpha_1(6) \beta_1(9) \beta_1(19)) = (I \cdot I_5) \alpha_1(3) \beta_1(6) \beta_1(16)
\]

\[
= I' (\alpha_1(5) \beta_1(8) \beta_1(18))
\]

\[
= Q^1(\beta_1 \beta_1)
\]

\[
= H^2(p \cdot Q^2(\beta_1 \beta_1)).
\]

Thus \( \alpha_1(3) \alpha_1(6) \beta_1(9) \beta_1(19) = \pm p \cdot Q^2(\beta_1 \beta_1) \), and relations (2) and (3) are established.

To see that relation (4) holds, consider the following. By Toda (1966, (11.1)) we have an exact sequence

\[
\ldots \rightarrow \pi_{28}(S^3:3) \xrightarrow{p^2} \pi_{30}(S^3:3) \xrightarrow{H^2} \pi_{27}(Q^1_2:3) \xrightarrow{I} \pi_{27}(S^1:3) \rightarrow \ldots
\]

which means that \( H^2(\pi_{30}(S^3:3) \rightarrow \pi_{27}(Q^1_2:3) \) is an isomorphism.

By Toda (1965, Lemma 6.1 (i))

\[
H^2(p \cdot Q^2(\beta_1 \beta_1)) = I Q^1(\alpha_1 \beta_1 \beta_1)
\]

and from Toda (1966, (10.1) and (11.7)) we have

\[
H^2(\alpha_3(0, \beta_1 \beta_1) \alpha_1(27)) = H^2(\alpha_3(0, \beta_1 \beta_1) \alpha_1(24))
\]

\[
= I Q^1(\beta_1 \beta_1) \alpha_1(24).
\]

In the exact sequence Toda (1965, (2.5))

\[
\ldots \rightarrow \pi_{31}(S^7:3) \xrightarrow{A} \pi_{29}(S^5:3) \xrightarrow{I'} \pi_{27}(Q^1_2:3) \xrightarrow{I} \pi_{30}(S^7:3) \rightarrow \ldots
\]
\( \Delta \) is an isomorphism [Toda (1966, (vi) p.241)], and so \( \ker I \approx 0 \).

By definition [Toda, (1965, 6.3 (ii))],

\[
E^\infty I(\pi_{11}^1(\alpha_1^1 \beta_1^1 1)) = \pi_{11}^1 \alpha_1^1 \beta_1^1
\]

and by Toda (1965, (2.6)) and Toda (1962, (3.4))

\[
E^\infty I(\pi_{11}^1(\beta_1^1 \alpha_1^1)) = \pi_{11}^1 \beta_1^1 \alpha_1^1 = \pi_{11}^1 \alpha_1^1 \beta_1^1.
\]

But \( E^\infty \) is an injection in this case which means

\[
\pi_{11}^1(\alpha_1^1 \beta_1^1 1) - \pi_{11}^1(\beta_1^1 \alpha_1^1(24)) \in \ker I.
\]

Thus \( \pi_{11}^1(\alpha_1^1 \beta_1^1 1) = \pi_{11}^1(\beta_1^1 \alpha_1^1(24)) \) and since \( H^{(2)} \) is an injection, relation (4) now follows.

Q.E.D.

**Lemma (3.5)**

<table>
<thead>
<tr>
<th>( i )</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_i(S^7(3)) \approx )</td>
<td>( \mathbb{Z}/3 )</td>
<td>( \mathbb{Z}/3 )</td>
<td>( \mathbb{Z}/3 )</td>
<td>( \mathbb{Z}/3 )</td>
<td></td>
</tr>
<tr>
<td>generators</td>
<td>( \alpha_5(7) )</td>
<td>( \beta_1(7)\beta_1(17) )</td>
<td>( u_2^{Q_1^4}(\beta_1^1) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( i )</th>
<th>29</th>
<th>30</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_i(S^7(3)) \approx )</td>
<td>( \mathbb{Z}/9 )</td>
<td>( \mathbb{Z}/9 )</td>
<td>( \mathbb{Z}/3 )</td>
</tr>
<tr>
<td>generators</td>
<td>( \alpha_6(7) )</td>
<td>( \beta_1(7)\beta_1(10)\beta_1(20) )</td>
<td>( p^{Q^4}(\beta_1^1) )</td>
</tr>
</tbody>
</table>

where \( u_3(1,\beta_1^1) \alpha_1(28) = \pi_{p^4}(\beta_1^1) \).

**Proof** As in previous lemma, table entries come from Toda (1965, 1966) while the relation \( u_3(1,\beta_1^1) \alpha_1(28) = \pi_{p^4}(\beta_1^1) \) is proven by Toda (1966, p.242).

Q.E.D.
As stated earlier, for the most part all we will need to know about the homotopy groups of \((X_2(3)^{^3})\) is their cardinalities; however, where possible in the next lemma we have specified the groups themselves.

**Lemma (3.6)**

<table>
<thead>
<tr>
<th>(i)</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi_i((X_2(3)^{^3})))</td>
<td>0</td>
<td>0</td>
<td>(\mathbb{Z}/3)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\mathbb{Z}/3)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(i)</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi_i((X_2(3)^{^3})))</td>
<td>0</td>
<td>(\mathbb{Z}/9)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\mathbb{Z}/3)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(i)</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
<th>29</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi'_i((X_2(3)^{^3})))</td>
<td>(\mathbb{Z}/3)</td>
<td>(\mathbb{Z}/3)</td>
<td>0</td>
<td>0</td>
<td>(\mathbb{Z}/9)</td>
<td>(\mathbb{Z}/9) or (\mathbb{Z}/3 \oplus \mathbb{Z}/3)</td>
</tr>
</tbody>
</table>

**Proof** Entries for \(8 \leq i \leq 23\) appear in Mimura-Toda (1964) since \((X_2(3)^{^3}) \cong \text{Sp}(2)(3)\) and \(\pi_1(\text{Sp}(2)(3)) \cong \pi_1(\text{Sp}(2);3)\). The entries for \(24 \leq i \leq 30\) are computed from the exact sequences

\[(3.6.1) \quad 0 \longrightarrow \text{coker}(\partial: \pi_{i+1}(S^{4n+3};3) \longrightarrow \pi_i(\text{Sp}(n);3)) \longrightarrow \pi_i(\text{Sp}(n+1);3) \longrightarrow \text{ker}(\partial: \pi_{i}(S^{4n+3};3) \longrightarrow \pi_{i-1}(\text{Sp}(n);3)) \longrightarrow 0\]

which are obtained from the fiberings \(\text{Sp}(n) \longrightarrow \text{Sp}(n+1) \longrightarrow S^{4n+3}\).

\(\partial\) is the boundary map and for the case \(n=1\) the following table gives the action of \(\partial: \pi_{i+1}(S^7;3) \longrightarrow \pi_i(S^3;3)\) on the various generators.
Mimura-Toda (1964a, Thm. 5.1) show in general that for
\[ \alpha \in \pi_1(S^{2n+2}) \]
(3.6.2) \[ \partial(\alpha E) = \partial \alpha E \]
Now \( \partial(\iota_\gamma) = \chi(X_2) = \alpha_1(3) \) which is a generator of \( \pi_6(S^3;3) \). Thus for \( \alpha \in \pi_1(S^7;3) \) which is a suspension element, that is \( \alpha = E\beta \) for some \( \beta \in \pi_{1-1}(S^6;3) \), we have
\[ \partial(E\beta) = \partial(\iota_\gamma E\beta) \]
\[ = \partial(\iota_\gamma \beta) \]
\[ = \alpha_1(3) \beta . \]

For \( i = 25, 26 \) and both 29's, \( \alpha \) in the above table is a suspension element and the images under \( \partial \) for these entries follow immediately from the above observation and Lemma (3.4). It remains to check the entries for \( i = 27 \) and 30.

Unfortunately neither \( u_3(1,\beta_1) \) nor \( p_*Q^4(\beta_1) \) are suspension elements so the technique used above does not apply. The entry for \( i = 27 \) will be needed to establish the entry for \( i = 30 \). We will establish the entry for \( i = 27 \) by showing that \( \pi_{27}((X_2)(3)) \approx \pi_{28}((X_2)(3)) \approx 0. \) That this gives
\[ \partial(u_3(1,\beta_1)) = \pm u_3(0,\beta_1\beta_1) \]
follows from the exact sequence

$$
... \rightarrow \pi_{28}(S^3;3) \rightarrow \pi_{28}((X_2)_3) \rightarrow \pi_{28}(S^7;3) \rightarrow \pi_{27}(S^3;3)
$$

by noting that $\pi_{28}(S^3;3) \approx \ker(d;\pi_{27}(S^7;3) \rightarrow \pi_{26}(S^3;3)) \rightarrow 0$ and $\pi_{28}(S^7;3) \approx \pi_{27}(S^3;3) \approx \mathbb{Z}/3$. Also note that this sequence implies that $\pi_{28}((X_2)_3) \approx \pi_{27}((X_2)_3)$.

Mimura-Toda (1964a) show that $\pi_{28}(Sp(6);3) \approx \pi_{27}(Sp(6);3) \approx 0$. Using exact sequence (3.6.1) with $n=2, 4$ and 5, $i = 27$ and 28 together with the facts that $\pi_{29}(S^{11};3) \approx \pi_{28}(S^{11};3) \approx \pi_{27}(S^{11};3)$

$\pi_{28}(S^{19};3) \approx \pi_{27}(S^{19};3) \approx \pi_{29}(S^{23};3) \approx \pi_{28}(S^{23};3) \approx \pi_{27}(S^{23};3)$

$\approx 0$ one can see that $\pi_{28}(Sp(5);3) \approx \pi_{27}(Sp(4);3) \approx 0$ and $\pi_{27}(Sp(3);3) \approx \pi_{27}((X_2)_3) \approx \pi_{28}(Sp(3);3) \approx \pi_{28}((X_2)_3)$.

Consider now the following pieces of exact sequences obtained from the fiberings $Sp(n) \rightarrow Sp(n+1) \rightarrow S^{4n+3}$ for $n = 3$ and 4.

(3.6.3) $\cdots \rightarrow \pi_{29}(S^{19};3) \rightarrow \pi_{28}(Sp(4);3) \rightarrow \pi_{28}(Sp(5);3) \rightarrow \cdots$

(3.6.4) $\cdots \rightarrow \pi_{29}(S^{15};3) \rightarrow \pi_{28}(Sp(3);3) \rightarrow \pi_{28}(Sp(4);3) \rightarrow \pi_{28}(Sp(5);3) \rightarrow \pi_{27}(Sp(3);3) \rightarrow \pi_{27}(Sp(4);3) \rightarrow \cdots$

Since $\pi_{28}(Sp(5);3) \approx 0$ and $\pi_{29}(S^{19};3) \approx \mathbb{Z}/3$ by (3.6.3) it must be that $\pi_{28}(Sp(4);3) \approx \mathbb{Z}/3$ or 0. If $\pi_{28}(Sp(4);3) \approx 0$ then since $\pi_{28}(S^{15};3) \approx \mathbb{Z}/3$ and $\pi_{29}(S^{15};3) \approx 0$ we see from (3.6.4) that $\pi_{28}(Sp(3);3) \approx 0$ and $\pi_{27}(Sp(3);3) \approx \mathbb{Z}/3$ which is a contradiction since this means that $\pi_{28}((X_2)_3) \not\approx \pi_{27}((X_2)_3)$. Thus $\pi_{28}(Sp(4);3) \approx \mathbb{Z}/3$. Consider now the homomorphism $d;\pi_{28}(S^{15};3)$

$\rightarrow \pi_{27}(Sp(3);3)$ in sequence (3.6.4). By Toda (1962) $\pi_{28}(S^{15};3) \approx \mathbb{Z}/3$ is generated by $\alpha_1(15)\beta_1(18)$ and thus $d(\alpha_1(15)\beta_1(18)) = d(\alpha_1(15))\beta_1(17)$ by (3.6.2). But $d(\alpha_1(15)) \not\in \pi_{17}(Sp(3);3)$ which is shown to be trivial by Mimura-Toda (1964a). Thus $d$ is trivial in (3.6.4) and we see
that \( \pi_{27}(\text{Sp}(3):3) \cong 0 \). Hence, we have shown that \( \pi_{28}((X_2)_{(3)}) \cong \pi_{27}((X_2)_{(3)}) \cong 0 \) and have also established the entry in boundary homomorphism table for \( i = 27 \).

To establish entry for \( i = 30 \) we first note that Mimura-Toda (1964, Thm. 2.5) show in general that \( \delta(\alpha \beta) = E^*(E \alpha \beta) \)
where \( E^* : \pi_{i+1}(S^4) \to \pi_i(S^3) \) has the property that \( E^*E = 1 \).

Using the relations given in Lemmas (3.5) and (3.4) we have

\[
\delta(p_*q^4(\beta_1)) = \delta(u_3(1, \beta_1)\alpha_1(28)) = E^*(E u_3(1, \beta_1)\alpha_1(28)) = E^*E(u_3(1, \beta_1)\alpha_1(27)) = \pm \beta_1(0, \beta_1)\alpha_1(27) = \pm p_*q^2(\beta_1)\beta_1.
\]

Using the table together with Lemmas (3.4) and (3.5) one can easily compute \( \text{coker} \delta \) and \( \text{ker} \delta \). This information, when used in exact sequence (3.6.1) with \( n = 1 \) establishes results \( \pi_i((X_2)_{(3)}) \) for \( i = 24, 25, 26, 29 \) and 30.

Q.E.D.

The next lemma completes the preliminary information needed to compute \( \mu((\text{Sp}(2) \times S^5)_{(3)}) \).

**Lemma (3.7)**

<table>
<thead>
<tr>
<th>( i )</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_i(S^5_{(3)}) \cong )</td>
<td>( \mathbb{Z}/3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{Z}/3 )</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{Z}/9 )</td>
<td>( \mathbb{Z}/9 )</td>
</tr>
<tr>
<td>generators</td>
<td>( \alpha_1(5) )</td>
<td>( \alpha_2(5) )</td>
<td>( \beta_1(5) )</td>
<td>( \alpha_1(5) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( i )</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_i(S^5_{(3)}) \cong )</td>
<td>0</td>
<td>( \mathbb{Z}/3 )</td>
<td>( \mathbb{Z}/3 )</td>
<td>( \mathbb{Z}/3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>generators</td>
<td>( \alpha_1(5) \beta_1(8) )</td>
<td>( p_*q^3(\alpha_1) )</td>
<td>( \alpha_1(5) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
where the following relations hold:

1. \( \alpha_2(5)\alpha_1(12) = \pm 3\beta_1(5) \),
2. \( \beta_1(5)\alpha_1(15) = \alpha_1(5)\beta_1(8) \),
3. \( \alpha'_3(5)\alpha_1(16) = \pm p_0^3(\alpha_1) \),
4. \( p_0^3(\alpha_2)\alpha_1(23) = 0 \),
5. \( \alpha_5(5)\alpha_1(24) = \pm 3\gamma_2(5) \), and
6. \( \alpha_4(5)\alpha_1(20) = 0 \).

**PROOF** The tabular entries can be obtained directly from Toda (1962) and Toda (1966). We will prove that the six relations hold below.

Mimura (1967) shows that \( \alpha_2(3)\alpha_1(10) = \pm \alpha_1(3)\alpha_2(6) \). Toda (1962, Lemma 13.8) shows that \( E^2(\alpha_2(3)\alpha_1(10)) = \pm E^2(\alpha_1(3)\alpha_2(6)) = \pm \alpha_1(5)\alpha_2(8) \), and (1) follows.

\( E^{\infty}: \pi_{18}(S^5) \longrightarrow (3\text{-primary component of stable 13-stem}) \) is an injection. The stable generators anti-commute, Toda (1962, (3.4)), hence

\[ E^{\infty}(\beta_1(5)\alpha_1(15)) = \beta_1\alpha_1 = \alpha_1\beta_1 = E^{\infty}(\alpha_1(5)\beta_1(8)) \] and (2) holds.

To show that relation (3) holds it suffices to show that \( \alpha'_3(5)\alpha_1(16) \neq 0 \) since the group involved is \( \mathbb{Z}/3 \).

Consider the homomorphisms

\( \beta_1(19)^*: \pi_{19}(S^5;3) \longrightarrow \pi_{29}(S^5;3) \).
and
\[ \alpha^*_{12}(26) : \pi_{26}(S^{53}) \to \pi_{29}(S^{53}). \]

\[ \beta^*_{1}(19) (\alpha^*_{12}(5) \alpha^*_{1}(16)) = \alpha^*_{1}(5) \alpha^*_{1}(16) / \beta^*_{1}(19) \]
\[ = \alpha^*_{1}(5) \beta^*_{1}(16) \alpha^*_{1}(26) \]
\[ = \alpha^*_{1}(26) (\alpha^*_{1}(5) \beta^*_{1}(16)). \]

By Toda (1966, p. 242) we see that \( \beta^*_{1}(26) \) generates \( \pi_{26}(S^{53}) \) and since \( \beta^*_{1}(26) \) generates \( \pi_{26}(S^{53}) \), we conclude that \( \alpha^*_{1}(26) \) is an isomorphism. Thus, to show that \( \alpha^*_{1}(5) \alpha^*_{1}(16) \neq 0 \) it suffices to show that \( \alpha^*_{1}(5) \beta^*_{1}(16) \neq 0 \).

By Toda (1966, (10.1) and Lemma 11.5)
\[ H^{(2)}(\alpha^*_{1}(5) \beta^*_{1}(16)) = H^{(2)}(\alpha^*_{1}(5) \beta^*_{1}(13)) \]
\[ = \overline{\alpha}^2(\alpha^*_{1} \beta^*_{1}(13) \]
where \( H^{(2)}(\alpha^*_{1} \beta^*_{1}(16)) \in \pi_{23}(Q^{3} : 3) \). By Toda (1965, (6.4)) \( \pi_{23}(Q^{3} : 3) \approx \mathbb{Z}/3 \) and has generator \( \overline{\alpha}^2(\alpha^*_{1} \beta^*_{1}) \).

By definition,
\[ E^\circ I(\overline{\alpha}^2(\alpha^*_{1} \beta^*_{1})) = \alpha^*_{1} \beta^*_{1} \]
and from Toda (1965, (2.6))
\[ E^\circ I(\overline{\alpha}^2(\alpha^*_{1} \beta^*_{1}(13))) = E^\circ \left[ (I(\overline{\alpha}^2(\alpha^*_{1})) \beta^*_{1}(16)) \right] \]
\[ = E^\circ (I \overline{\alpha}^2(\alpha^*_{1})) \alpha^*_{1} \beta^*_{1}(16) \]
\[ = \alpha^*_{1} \beta^*_{1}. \]

From the exact sequence Toda (1966, (11.2)) we see that
\[ I: \pi_{23}(Q^{3} : 3) \to \pi_{26}(S^{13} : 3) \]
is an isomorphism and since \( E^\circ \) is an injection we see that \( \overline{\alpha}^2(\alpha^*_{1} \beta^*_{1}(13)) = \overline{\alpha}^2(\alpha^*_{1} \beta^*_{1}) \).

This implies that \( \alpha^*_{1}(5) \beta^*_{1}(16) \neq 0 \) and we have demonstrated (3).

By Toda (1966, (10.1)) and Toda (1965, Lemma 6.1 (iii)) we have
\[ H^{(2)}(\pi_3^*(\omega_3)\alpha_1(23)) = H^{(2)}(\pi_3^*(\omega_3)\alpha_1(20)) = \pm q^2(\omega_3^')\alpha_1(20). \]

From the exact sequence Toda (1966, (11.1)) and the fact that \( \pi_{24}(S^3;3) \) is trivial we see that \( H^{(2)}: \pi_26(S^5;3) \longrightarrow \pi_23(\omega_2^2;3) \) is an injection.

Similarly, from the exact sequence Toda (1966, (11.2)) and the fact that \( \pi_{25}(S^{11};3) \) is trivial we see that \( I: \pi_{23}(\omega_2^3;3) \longrightarrow \pi_{26}(S^{13};3) \) is an injection. But

\[ I(\omega_3^2(\omega_3^')\alpha_1(20)) = I(\omega_3^2(\omega_3^')\alpha_1(23)) = I(I'(\alpha_3(11))\alpha_1(23)) = 0 \]

by Toda (1965, (2.6)), definition of the generator \( \omega_3^2(\omega_3^') \), and the fact that \( I \circ I' \) is two steps in an exact sequence. Thus \( \pm \omega_3^2(\omega_3^')\alpha_1(20) = 0 \) and (4) follows.

\[ \alpha_5(5)\alpha_1(24) \in \pi_{27}(S^5;3) \text{ and } E: \pi_{27}(S^5;3) \longrightarrow \pi_{28}(S^6;3) \text{ is an injection. Using Toda (1962, Proposition 3.1) we have} \]

\[ E(\alpha_5(5)\alpha_1(24)) = \alpha_5(6)\alpha_1(25) = E^3\alpha_5(3)E^{22}\alpha_1(3) = -E^3\alpha_1(3)E^6\alpha_5(3) = E(-\alpha_1(5)\omega_5(8)). \]

Thus \( \alpha_5(5)\alpha_1(24) = -\alpha_1(5)\alpha_5(8) = -E^2(\alpha_1(3)\alpha_5(6)). \) By Lemma (3,4) relation (1),

\[ \alpha_1(3)\alpha_5(6) = t\nu_2(3) \]

and by Toda (1966, (11.8))

\[ E^2(\nu_2(3)) = 3\nu_2(5) \]
thus \( \alpha_5(5)\alpha_1(24) = 13\gamma_2(5) \) and (5) is established.

Finally, as above one can show that
\[
\alpha_1(5)\alpha_1(20) = -\alpha_1(5)\alpha_4(8).
\]
But \( \alpha_1(5)\alpha_4(8) = E^2(\alpha_1(3)\alpha_4(6)) \) with \( \alpha_1(3)\alpha_4(6) \in \pi_{21}(S^7;3) \). Toda (1966) shows that \( \pi_{21}(S^3;3) \cong \mathbb{Z}/3 \) and that it is generated by an element whose double suspension is trivial, hence \( \alpha_4(5)\alpha_1(20) = 0 \) and we have (6).

Q.E.D.

We are now ready to compute \( \mu((Sp(2)\times S^5)_{(3)}) \).

**Proposition (3.8)**

\[
\mu((Sp(2)\times S^5)_{(3)}) = 3^{46}
\]

**Proof** Since \((Sp(2)\times S^5)_{(3)} \cong Sp(2)_{(3)} \times S^5_{(3)} \cong (X_2^{(3)} \times S^5_{(3)})\) it suffices to compute \( \mu((X_2 \times S^5)_{(3)}) \). Using Theorem (1.23) we have
\[
\mu((X_2 \times S^5)_{(3)}) = \# \left[ (X_2 \times S^5)_{(3)} \wedge (X_2 \times S^5)_{(3)} , (X_2 \times S^5)_{(3)} \right]
\]
\[
= \left( \begin{array}{c|c}
6 & \hline 1 & a_1 \\
\hline 6 & \hline 1 & b_1 \\
\end{array} \right)
\]

where

\[
a_1 = \# [X_2 \wedge S^5 \wedge X_2 \wedge S^5 , (X_2^{(3)})] = \# [E^{10}(X_2 \wedge X_2), (X_2^{(3)})] = 3^2
\]
\[
a_2 = \# [X_2 \wedge S^5 \wedge X_2 \wedge S^5 , S^5_{(3)}] = \# [E^{10}(X_2 \wedge X_2), S^5_{(3)}] = 3^8
\]
\[
a_3 = \# [X_2 \wedge X_2 , (X_2^{(3)})] = 3^1
\]
\[
a_4 = \# [X_2 \wedge X_2 , S^5_{(3)}] = 3^1
\]
\[
a_5 = \# [S^5 \wedge S^5 , (X_2^{(3)})] = \# (\pi_{10}((X_2^{(3)}))) = 3^1
\]
\[
a_6 = \# [S^5 \wedge S^5 , S^5_{(3)}] = \# (\pi_{10}(S^5_{(3)})) = \# (\pi_{10}(S^5;3)) = 3^0
\]
\[
b_1 = \# [X_2 \wedge S^5 \wedge X_2 , (X_2^{(3)})] = \# [E^{5}(X_2 \wedge X_2), (X_2^{(3)})] = 3^7
\]
\[
b_2 = \# [X_2 \wedge S^5 \wedge X_2 , S^5_{(3)}] = \# [E^{5}(X_2 \wedge X_2), S^5_{(3)}] = 3^5
\]
\[
b_3 = \# [X_2 \wedge S^5 \wedge S^5 , (X_2^{(3)})] = \# [E^{10}(X_2), (X_2^{(3)})] = 3^0
\]
\[
b_4 = \# [X_2 \wedge S^5 \wedge S^5 , S^5_{(3)}] = \# [E^{10}(X_2), S^5_{(3)}] = 3^1
\]
\[ b_5 = \# [X_2^5, (X_2)_3] = \# [E^5(X_2), (X_2)_3] = 3^0 \]
\[ b_6 = \# [X_2^5, S^5(3)] = \# [E^5(X_2), S^5(3)] = 3^4. \]

The final result follows by adding exponents so it remains to verify the twelve values \( a_i \) and \( b_j \) for \( 1 \leq i, j \leq 6 \).

Since \( (X_2)_3 \cong \text{Sp}(2)_3 \), \( a_3 = 3^1 \) follows from Mimura's (1969) computation of \( \mu(\text{Sp}(2)) \). \( a_5 = 3^1 \) and \( a_6 = 3^0 \) comes from Lemmas (3.6) and (3.7).

By Lemma (3.3 (ii)) we have
\[ E^5(X_2) \cong S^8 \cup \alpha_1(8) \cup S^15 \]
and
\[ E^{10}(X_2) \cong S^{13} \cup \alpha_1(13) \cup S^{20}. \]

Thus
\[ [E^5(X_2), S^5(3)] \cong [S^8 \cup \alpha_1(8) \cup S^5(3)] \oplus \pi_{15}(S^5(3)), \]
\[ [E^5(X_2), (X_2)_3] \cong [S^8 \cup \alpha_1(8) \cup (X_2)_3] \oplus \pi_{15}((X_2)_3), \]
\[ [E^{10}(X_2), S^5(3)] \cong [S^{13} \cup \alpha_1(13) \cup S^5(3)] \oplus \pi_{20}(S^5(3)), \]
and
\[ [E^{10}(X_2), (X_2)_3] \cong [S^{13} \cup \alpha_1(13) \cup (X_2)_3] \oplus \pi_{20}((X_2)_3). \]

Letting \( Z \) represent \( S^5(3) \) or \( (X_2)_3 \), from the cofibrations
\[
\begin{array}{cccccccc}
\alpha_1(8) & 8 & 8 \cup \alpha_1(8) & 12 & 12 & 9 & \cdots \\
S^{11} & S^8 & S^8 \cup \alpha_1(8) & S^{12} & S^9 & \cdots \\
& & & & & \hline
\alpha_1(13) & 13 & 13 \cup \alpha_1(13) & 17 & 17 & 14 & \cdots \\
S^{16} & S^{13} & S^{13} \cup \alpha_1(13) & S^{17} & S^{14} & \cdots \\
\end{array}
\]
we obtain exact sequences
\[
\begin{array}{cccccccc}
\alpha_1(8)^* & \pi_{11}(Z) & \pi_8(Z) & [S^8 \cup \alpha_1(8) \cup 12, Z] & \pi_{12}(Z) & \pi_9(Z) & \cdots \\
\end{array}
\]
and
\[
\begin{array}{cccccccc}
\alpha_1(13)^* & \pi_{16}(Z) & \pi_{13}(Z) & [S^{13} \cup \alpha_1(13) \cup 17, Z] & \pi_{17}(Z) & \pi_{14}(Z) & \cdots \\
\end{array}
\]
Now \( \pi_1 (s^5 (3)) \approx \pi_9 (s^5 (3)) \approx 0 \) and \( \pi_8 (s^5 (3)) \approx \pi_{12} (s^5 (3)) \approx \mathbb{Z}/3 \)

means \( \# [s^8 \cup \alpha_1 (8) e^{12}, s^5 (3)] = 3^2 \) which together with \( \pi_{15} (s^5 (3)) \approx \mathbb{Z}/9 \)

implies \( b_6 = 3^4. \)

Similarly, \( \pi_8 ((x^2)_2 (3)) \approx \pi_{12} ((x^2)_2 (3)) \approx \pi_{15} ((x^2)_2 (3)) \approx 0 \)
gives \( b_5 = 3^0 \), and likewise \( \pi_{13} (Z) \approx \pi_{17} (Z) \approx \pi_{20} ((x^2)_2 (3)) \approx 0 \)
together with \( \pi_{20} (s^5 (3)) \approx \mathbb{Z}/3 \) gives \( b_4 = 3^1 \) and \( b_3 = 3^0. \)

To compute \( a_4 \) we use the cofibering

\[
S^6 \longrightarrow x_2 \wedge x_2 \longrightarrow (x_2 \wedge x_2) / S^6 \longrightarrow S^7 \longrightarrow E(x_2 \wedge x_2) \longrightarrow \ldots
\]

which gives an exact sequence

\[
\pi_6 (s^5 (3)) \longrightarrow [x_2 \wedge x_2, s^5 (3)] \longrightarrow [(x_2 \wedge x_2) / S^6, s^5 (3)] \longrightarrow \pi_7 (s^5 (3)) \longrightarrow \ldots
\]

which, since \( \pi_6 (s^5 (3)) \approx \pi_7 (s^5 (3)) \approx 0 \), means that

\[
a_4 = \# [(x_2 \wedge x_2) / S^6, s^5 (3)].
\]

By Lemma 3.3 (iv) we see that

\[
[(x_2 \wedge x_2) / S^6, s^5 (3)] \approx [(s^{10} \vee s^{10}) \cup C(s^{13} \vee s^{19}), s^5 (3)] \oplus [s^{13} \cup \alpha_1 (13) e^{17}, s^5 (3)] \oplus [s^{13} \cup \alpha_1 (13) e^{17}, s^5 (3)].
\]

But \( [s^{13} \cup \alpha_1 (13) e^{17}, s^5 (3)] \approx 0 \) so we have

\[
[(x_2 \wedge x_2) / S^6, s^5 (3)] \approx [(s^{10} \vee s^{10}) \cup C(s^{13} \vee s^{19}), s^5 (3)].
\]

As before, we use the appropriate cofibering to get an exact sequence

\[
\pi_{19} (s^5 (3)) \oplus \pi_{13} (s^5 (3)) \longrightarrow \pi_{10} (s^5 (3)) \oplus \pi_{10} (s^5 (3)) \longrightarrow [(s^{10} \vee s^{10}) \cup C(s^{13} \vee s^{19}), s^5 (3)] \longrightarrow \pi_{20} (s^5 (3)) \oplus \pi_{14} (s^5 (3)) \longrightarrow \pi_{11} (s^5 (3)) \oplus \pi_{11} (s^5 (3)) \longrightarrow \ldots
\]

Using the fact that \( \pi_{10} (s^5 (3)) \approx \pi_{11} (s^5 (3)) \approx \pi_{14} (s^5 (3)) \approx 0 \)

and \( \pi_{20} (s^5 (3)) \approx \mathbb{Z}/3 \) we see that
and so $a_4 = 3^{-1}$.

By Lemma (3.3 (ii)),

$$E^5(\Sigma_2) \simeq [(s^{11} \cup (\alpha_1(11) e^{15}) \cup \alpha_5 e^{19}) \cap (s^{13} \cup s^{19}), s^3(3)] \approx \mathbb{Z}/3$$

hence

$$b_1 = \#[(s^{11} \cup (\alpha_1(11) e^{15}) \cup \alpha_5 e^{19}, (x_2)_3)]$$

$$= \left( \#s^{18} \cup (\alpha_1(18) e^{22}, (x_2)_3) \right)^2 \cdot \#(\pi_{25}((x_2)_3))$$

and

$$b_2 = \#[(s^{11} \cup (\alpha_1(11) e^{15}) \cup \alpha_5 e^{19}, s^3(3))]$$

$$= \left( \#s^{18} \cup (\alpha_1(18) e^{22}, s^3(3)) \right)^2 \cdot \#(\pi_{25}(s^3(3)))$$

From Lemmas (3.6) and (3.7) we have

$$\#(\pi_{25}((x_2)_3)) = \#(\pi_{25}(s^3(3))) = 3^{-1}.$$
From the cofibering
\[ s^{18} \longrightarrow (s^{11} \cup \alpha_{1(11)}e^{15}v_{15}) \longrightarrow (s^{11} \cup \alpha_{1(11)}e^{15}v_{15}) \cup \beta e^{19} \longrightarrow \]
\[ s^{19} \longrightarrow (s^{12} \cup \alpha_{1(12)}e^{16}v_{16}) \longrightarrow \ldots \ldots \]

we get an exact sequence
\[ (3.8.1) \quad \pi_{18}(Z) \xrightarrow{(E^5)^*} [s^{11} \cup \alpha_{1(11)}e^{15}, \pi_{15}(Z)] \oplus \pi_{15}(Z) \]
\[ [s^{11} \cup \alpha_{1(11)}e^{15}v_{15}) \cup \beta e^{19}, \pi_{19}(Z)] \xrightarrow{(E^5)^*} \]
\[ [s^{12} \cup \alpha_{1(12)}e^{16}, \pi_{16}(Z)] \oplus \pi_{16}(Z) \]

The exact cofibration sequence
\[ (3.8.2) \quad \pi_{14}(Z) \xrightarrow{\alpha_{1(11)}^*} \pi_{11}(Z) \longrightarrow [s^{11} \cup \alpha_{1(11)}e^{15}, \pi_{15}(Z)] \xrightarrow{r^*} \]
\[ \pi_{15}(Z) \rightarrow \pi_{12}(Z) \longrightarrow \ldots \ldots \]

is needed to compute the group
\[ [s^{11} \cup \alpha_{1(11)}e^{15}, \pi_{15}(Z)]. \]

For \( Z = (X_2)^{(3)} \) we have \( \pi_{11}((X_2)^{(3)}) \approx \pi_{15}((X_2)^{(3)}) \approx \pi_{19}((X_2)^{(3)}) \approx 0 \) and we get \([s^{11} \cup \alpha_{1(11)}e^{15}, (X_2)^{(3)}] \oplus \pi_{15}((X_2)^{(3)}) \approx 0 \)
from (3.8.2) Thus exact sequence (3.8.1) gives
\[ [(s^{11} \cup \alpha_{1(11)}e^{15}v_{15}) \cup \beta e^{19}, (X_2)^{(3)}] \approx 0 \]
and we have
\[ b_1 = 3^7. \]

The computation for \( Z = S^5(3) \) is not as simple because of the existence of non-trivial homotopy groups. We begin by determining \([s^{11} \cup \alpha_{1(11)}e^{15}, S^5(3)]. \)

From Lemma (3.7) we have \( \pi_{11}(S^5(3)) \approx 0, \pi_{15}(S^5(3)) \approx \mathbb{Z}/9 \)
with generator \( \beta_1(5) \) and \( \pi_{12}(S^5(3)) \approx \mathbb{Z}/3 \) with generator \( \alpha_2(5) \). The
group in question is thus seen to be isomorphic to coker$\alpha_1(12)^*$ which is isomorphic to $\mathbb{Z}/3$ since

$$\alpha_1(12)^*(\alpha_2(5)) = \alpha_2(5)\alpha_1(12) = \pm 3\beta_1(5)$$

by Lemma (3.7 relation 1). Thus we see that

$$[S^{11} \cup \alpha_1(11)^{e^{15}}, S^5(3)] \approx \mathbb{Z}/3$$

and has generator

$$r^*(\beta_1(5))$$

where $r^*$ is induced by the collapsing map $r: S^{11} \cup \alpha_1(11)^{e^{15}} \rightarrow S^{15}$.

Since $\pi_{19}(S^5(3)) \approx \mathbb{Z}/3$ with generator $p^3(\alpha_1)$ and $\pi_{18}(S^5(3)) \approx \mathbb{Z}/3$ with generator $\alpha_1(5)\beta_1(8)$, to compute the group

$$[(S^{11} \cup \alpha_1(11)^{e^{15}}, S^5(3))]$$

from the exact sequence (3.8.1) we must determine $\ker(E^5\beta)^*$ and $\coker(E^6\beta)^*$.

By Lemma (3.3 (iii)), $E^6\beta \simeq \widetilde{\alpha}_1(15)\alpha_1(16)$ and so $(E^6\beta)^* \simeq (\widetilde{\alpha}_1(15))^* \oplus \alpha_1(16)^*$. But $\alpha_1(16)^*$ applied to the generator $\alpha_1'(5)$ of $\pi_{16}(S^5(3))$ is $\alpha_1'(5)\alpha_1(16)$ which generates $\pi_{19}(S^5(3))$ by relation (3) of Lemma (3.7). Thus $\coker(E^6\beta)^* \approx 0$.

Similarly, $E^5\beta \simeq \widetilde{\alpha}_1'(14)\alpha_1(15)$ and $(E^5\beta)^* \approx (\widetilde{\alpha}_1'(14))^* \oplus \alpha_1(15)^*$. Thus $\ker(E^5\beta)^*$ is determined by the images $(\widetilde{\alpha}_1'(14))^*(r^*(\beta_1(5)))$ and $\alpha_1(15)^*(\beta_1(5))$. Now

$$\alpha_1(15)^*(\beta_1(5)) = \beta_1(5)\alpha_1(15) = \pm \alpha_1(5)\beta_1(8)$$

by relation (2) of Lemma (3.7). Thus we see that $\ker\alpha_1(15)^* \approx \mathbb{Z}/3$.

For $\widetilde{\alpha}_1'(14)^*$ we get

$$\widetilde{\alpha}_1'(14)^*(r^*(\beta_1(5))) = r^*(\beta_1(5))\widetilde{\alpha}_1'(14) = \beta_1(5)r\widetilde{\alpha}_1'(14).$$

By Toda (1962, (1.18)) we have

$$r\widetilde{\alpha}_1'(14) = E\alpha_1(14) = \alpha_1(15)$$
which means

\[ \beta^*(r^*(\beta_1(5))) = \beta_1(5)\alpha_1(15) = \alpha_1(5)\beta_1(8) \]

by relation (2) Lemma (3.7). Thus \( \ker(\beta^*(\beta_1(5))) \approx 0 \) and we conclude

\[ \ker(\beta^*(\beta_1)) \approx \mathbb{Z}/9. \]

Thus, from exact sequence (3.11) we see that

\[ [(s^{11} \cup \alpha_1(11)e^{15} \cup s^{15} \cup \alpha_5^*e^{15}, s^5(3))] \approx \mathbb{Z}/3 \]

and we have shown that

\[ b_2 = 3^5. \]

As in the above computations for \( b_1 \) and \( b_2 \) it is similarly seen that

\[ a_1 = \#[(s^{16} \cup \alpha_1(16)e^{20} \cup s^{20} \cup \beta_1e^{24}, s^2(3)]. \]

\[ (\#[s^{23} \cup \alpha_1(23)e^{27}, s^2(3)]^2 \cdot \#(\pi_{30}(s^2(3))) \]

and

\[ a_2 = \#[(s^{16} \cup \alpha_1(16)e^{20} \cup s^{20} \cup \beta_1e^{24}, s^5(3). \]

\[ (\#[s^{23} \cup \alpha_1(23)e^{27}, s^5(3)]^2 \cdot \#(\pi_{30}(s^5(3))). \]

Beginning with the right-most cardinalities in the above products we have

\[ \#(\pi_{30}(s^2(3))) = 3^2 \]

and

\[ \#(\pi_{30}(s^5(3))) = 3^0 \]

from Lemmas (3.6) and (3.7).

The exact cofibration sequence

\[ \pi_{26}(z) \xrightarrow{\alpha_1(23)*} \pi_{23}(z) \xrightarrow{[s^{23} \cup \alpha_1(23)e^{27}, z]} \pi_{27}(z) \xrightarrow{\alpha_1(24)*} \pi_{24}(z) \]

together with the fact that \( \pi_{23}(s^2(3)) \approx \pi_{27}(s^2(3)) \approx 0 \) gives

\[ \#[s^{23} \cup \alpha_1(23)e^{27}, (x_2(3))] = 3^0. \]
For $Z = S^5(3)$ all relevant homotopy groups in the above exact sequence are non-trivial so we must determine $\ker \alpha_1(23)^*$ and $\text{coker} \alpha_1(24)^*$.

From Lemma (3.7) we have the following information about groups and generators:

\[ \pi_2(S^5(3)) \cong \mathbb{Z}/3, \text{ generator } \alpha_5(5) \]
\[ \pi_7(S^5(3)) \cong \mathbb{Z}/9, \text{ generator } \gamma_2(5) \text{ and } \]
\[ \pi_3(S^5(3)) \cong \mathbb{Z}/3, \text{ generator } \overline{p^3(\alpha_2)}. \]

By relations (4) and (5) of Lemma (3.7) we have
\[ \alpha_1(23)^*(p\cdot\overline{q^3(\alpha_2)}) = p\cdot\overline{q^3(\alpha_2)}\alpha_1(23) = 0 \]
and
\[ \alpha_1(24)^*(\gamma_2(5)) = \alpha_5(5)\alpha_1(24) = \pm 3\gamma_2(5) \]

and we thus conclude that
\[ \ker \alpha_1(23)^* \cong \text{coker} \alpha_1(24)^* \cong \mathbb{Z}/3 \]
which means
\[ \#[S^{23} \cup \alpha_1(23)^* e^{27}, S^5(3)] = 3^2. \]

The cofibration sequence
\[ (3.8.3) \]
\[ \pi_23(Z) \quad \xrightarrow{(E^{10})^*} \quad \pi_23(Z) \]
\[ [S^{16} \cup \alpha_1(16)^{e^{20}}, Z] \oplus \pi_20(Z) \]
\[ [(S^{16} \cup \alpha_1(16)^{e^{20}}, S^{20}) \cup E^{10}e^{24}, Z] \] \[ \pi_24(Z) \]
\[ [S^{17} \cup \alpha_1(17)^{e^{21}}, Z] \oplus \pi_21(Z) \]

is used to determine the remaining factors of $a_1$ and $a_2$. To use this sequence we must first compute
\[ [S^{16} \cup \alpha_1(16)^{e^{20}}, Z] \]
and
\[ [S^{17} \cup \alpha_1(17)^{e^{21}}, Z]. \]
These groups are obtained from the exact sequence

\[ (3.8.4) \]

\[ \begin{array}{ccccccc}
\pi_{19}(z) & \xrightarrow{\alpha_{1}(16)\ast} & \pi_{16}(z) & \xrightarrow{i\ast} & [S^{16} \cup \alpha_{1}(16) e^{20}, z] \\
\xrightarrow{r\ast} & \pi_{20}(z) & \xrightarrow{\alpha_{1}(17)\ast} & \pi_{17}(z) & \xrightarrow{} & [S^{17} \cup \alpha_{1}(17) e^{21}, z] \\
& \xrightarrow{} & \pi_{21}(z) & \xrightarrow{} & \ldots
\end{array} \]

where \( i\ast \) is induced from inclusion and \( r\ast \) by collapsing map.

For \( Z = (X_2)(3) \) computations are easy since \( \pi_{16}((X_2)(3)) \approx \pi_{20}((X_2)(3)) \approx 0 \approx \pi_{24}((X_2)(3)) \) which in \( (3.8.4) \) and \( (3.8.3) \) easily give

\[ \left[ (S^{16} \cup \alpha_{1}(16) e^{20} \cup S^{20}) \cup E_\beta e^{24}, (X_2)(3) \right] \approx 0, \]

that is

\[ \# \left[ (S^{16} \cup \alpha_{1}(16) e^{20} \cup S^{20}) \cup E_\beta e^{24}, (X_2)(3) \right] = 3^0. \]

At this point we may conclude that \( a_{1} = 3^2 \).

For \( Z = S^{5}(3) \) we have \( \pi_{21}(S^{5}(3)) \approx \pi_{17}(S^{5}(3)) \approx 0 \) and from \( (3.8.4) \) we see that

\[ \left[ S^{17} \cup \alpha_{1}(17) e^{21}, S^{5}(3) \right] \approx 0. \]

\( \pi_{20}(S^{5}(3)) \approx \mathbb{Z}/3 \) with generator \( \alpha_{4}(5) \), \( \pi_{17}(S^{5}(3)) \approx 0 \) and \( \ker\alpha_{1}(16) \ast \approx \mathbb{Z}/3 \) since \( \alpha_{1}(16)\ast \) applied to the generator \( \alpha_{3}'(5) \) of \( \pi_{16}(S^{5}(3)) \approx \mathbb{Z}/9 \) generates \( \pi_{19}(S^{5}(3)) \approx \mathbb{Z}/3. \) We also note at this point that since \( 3\alpha_{3}'(5) = \alpha_{3}(5) \) we may use \( \alpha_{3}(5) \) as a generator of \( \ker\alpha_{1}(16)\ast. \) In any event, we conclude from \( (3.8.4) \) that

\[ \left[ S^{16} \cup \alpha_{1}(16) e^{20}, S^{5}(3) \right] \approx \begin{cases} \mathbb{Z}/3 \oplus \mathbb{Z}/3 \\
\mathbb{Z}/9 \end{cases} \]

or

and that both \( r\ast(\alpha_{4}(5)) \) and \( \alpha_{3}(5) \) are non-zero where one or both
may be generators. The symbol $\alpha_3(5)$ represents an element having
the property that $i^*(\alpha_3(5)) = \alpha_3(5)$, i.e. $\alpha_3(5)$ is an extension of $\alpha_3(5)$.

Using exact sequence (3.8.3) we can determine the cardinality
of

$$[(S^{16} \cup \alpha_1(16)e^{20}vS^{20}) \cup E_{10}e^{24}, S^5(3)]$$

by noting that it is equal to the product

$$\#(\ker(E^{10} \beta)^*) \cdot \#(\pi_{24}(S^5(3))).$$

By Lemma (3.3 (iii)),

$$E_{10}^\beta = \alpha_1(19)v\alpha_1(20)$$

so

$$\ker(E^{10} \beta)^* \cong \ker(\alpha_1(19))^* \oplus \ker(\alpha_1(20))^*.$$  

By relation (6) of Lemma (3.7),

$$\alpha_1(20)^*(\alpha_4(5)) = \alpha_4(5)\alpha_1(20) = 0$$

so

$$\ker(\alpha_1(20))^* \cong \pi_{20}(S^5(3)) \cong \mathbb{Z}/3$$

i.e. $\#(\ker(\alpha_1(20))^*) = 3^1$.

We now show that $\#(\ker(\alpha_1(19))^*) = 3^2$ by demonstrating that
both $r^*(\alpha_4(5)) \alpha_1(19)$ and $\alpha_3(5) \alpha_1(19)$ are trivial.

We first consider the composition $\alpha_3(5)\alpha_1(19)$. By Toda
(1962, Proposition (1.7)) the set of all compositions \{\alpha_3(5)\alpha_1(19)\}
is equal the secondary composition

\{\alpha_3(5), \alpha_1(16), \alpha_1(19)\}

which is a coset of $\pi_{20}(S^5:3)\alpha_1(20) \oplus \alpha_3(5)\pi_{23}(S^{16}:3)$. However, $\pi_{20}(S^5:3)\alpha_1(20) \cong 0$ by relation (6) of Lemma (3.7) and $\alpha_3(5)\pi_{23}(S^{16}:3) \cong 0$ since $\alpha_3(5)\alpha_2(16) = E^2(\alpha_3(3)\alpha_2(14))$ and Toda
(1966) observes that the generator of $\pi_{23}(S^5:3)$ is not in the image
of $E^2$. Thus the coset $\{\alpha_3(5), \alpha_1(16), \alpha_1(19)\}$ contains the single
element $\alpha_3^*(5)\alpha_1^*(19)$. We may now use Toda (1962, Proposition (1.9)) to conclude that

$$r^*(\alpha_3^*(5)\alpha_1^*(19)) = \alpha_3^*(5)\alpha_1^*(16)$$

where as before $r^*$ is induced by collapsing map

$$r:S^{19}\cup_{\alpha_1^*(19)} S^{23} \longrightarrow S^{23}$$

and $\alpha_1^*(16)$ is an extension of $\alpha_1^*(16)$ to $S^{19}\cup_{\alpha_1^*(19)} S^{23}$.

Using the cofibration sequence

$$\pi_{22}(S^5(3)) \xrightarrow{\alpha_1^*(19)^*} \pi_{19}(S^5(3)) \xrightarrow{i^*} [S^{19}\cup_{\alpha_1^*(19)} S^{23}, S^5(3)]$$

$$\xrightarrow{r^*} \pi_{23}(S^5(3)) \xrightarrow{\alpha_1^*(20)^*} \pi_{20}(S^5(3)) \longrightarrow \ldots$$

and observing that $\alpha_1^*(20)^*$ is trivial we see that $r^*$ is an injection. Thus, to show that $\alpha_3^*(5)\alpha_1^*(19) = 0$ it suffices to show that $\alpha_3^*(5)\alpha_1^*(16) = 0$.

That this is indeed the case comes from the following observations:

1. $\alpha_3^*(5)\alpha_1^*(16) = (3\alpha_3^*(5))\alpha_1^*(16) = 3(\alpha_3^*(5)\alpha_1^*(16)) = 0$ by relation (3) of Lemma (3.7),

2. $\alpha_3^*(5)\alpha_1^*(16)$ may be chosen to be $\alpha_3^*(5)\alpha_1^*(16)$ since their restrictions to $S^{19}$ are equal, and finally

3. $\alpha_3^*(5)\alpha_1^*(16)$ is an extension of a trivial map and hence may be chosen to be trivial.

Finally, as to the triviality of $r^*(\alpha_4^*(5))\alpha_1^*(19)$, we note that

$$r^*(\alpha_4^*(5))\alpha_1^*(19) = \alpha_4^*(5) r \alpha_1^*(19) = \alpha_4^*(5) E\alpha_1^*(19) = \alpha_4^*(5)\alpha_1^*(20) = 0$$

by Toda (1962, (1.18)) and relation (6) Lemma (3.7).

Thus

$$\#\left[\left[S^{16}\cup_{\alpha_1^*(16)} S^{20}\right] \cup_{\varepsilon} S^{10} \cup_{\varepsilon} S^{24}, S^5(3)\right] = 4^4$$
and we conclude that
\[ a_2 = 3^8. \]
Q.E.D.

The next, and final lemma will be used to compute an upper bound for the numbers \( \mu(\text{SU}(4)_{(2)}) \) and \( \mu((\text{SU}(3) \times S^7)_{(2)}) \).

**Lemma (3.9)**

Let \( n_i, i = 1,2,\ldots , k \) be a sequence of positive integers greater than 1 and \( X \) be any space, then
\[
\#[s^{n_1} \cup \alpha_{n_1} e^{n_2} \cup \alpha_{n_2} e^{n_3} \cup \ldots \cup \alpha_{n_{k-1}} e^{n_k}, X] \leq \prod_{i=1}^{k} \#(\pi_{n_i}(X)).
\]

**Proof**

For \( k = 2 \) the inequality \( \#[s^{n_1} \cup \alpha_{n_1} e^{n_2}, X] \leq \#(\pi_{n_1}(X)) \cdot \#(\pi_{n_2}(X)) \)

follows from the cofibration sequence

\[
\pi_{n_2-1}(X) \xrightarrow{\alpha_{n_1}^*} \pi_{n_1}(X) \xrightarrow{[s^{n_1} \cup \alpha_{n_1} e^{n_2}, X]} \pi_{n_2}(X) \xrightarrow{(E\alpha_{n_1})^*} \pi_{n_1+1}(X) \xrightarrow{\cdots}
\]

since \( \#(\text{ker}\alpha_{n_1}^*) \leq \#(\pi_{n_1}(X)) \) and \( \#(\text{coker}\alpha_{n_1}^*) \leq \#(\pi_{n_2}(X)) \).

The result for any integer \( k \geq 2 \) now follows easily by induction.

Q.E.D.

For a given \( X \) and set of primes \( P \) we will use the symbol \( \mu_P(X) \) to denote the number of distinct homotopy classes of multiplications that the space \( X_P \) will support.

The main and final result of this chapter is the following theorem.
THEOREM (3.10) Let \( \mathcal{P} \) denote the set of all primes, then

(a) \( \mu_{\mathcal{P}-\{2\}}(SU(4)) = 3^4 \cdot 5^2 \cdot 7^9 \cdot 11^5 \cdot 13 \),

(b) \( \mu_{\mathcal{P}-\{2\}}(SU(3) \times S^7) = 3^{105} \cdot 5^3 \cdot 7^9 \cdot 11^5 \cdot 13 \),

(c) \( \mu_{\mathcal{P}-\{2\}}(Y_2) = 3^4 \cdot 5^2 \cdot 7^9 \cdot 11^5 \cdot 13 \),

(d) \( \mu_{\mathcal{P}-\{2\}}(Y_3) = 3^{105} \cdot 5^3 \cdot 7^9 \cdot 11^5 \cdot 13 \),

(e) \( \mu_{\{2\}}(SU(4)) = \mu_{\{2\}}(Y_3) \leq 2^{210} \cdot c_{27,3}^2 \cdot c_{30,3} \cdot c_{30,5} \cdot c_{30,7} \), and

(f) \( \mu_{\{2\}}(SU(3) \times S^7) = \mu_{\{2\}}(Y_2) \leq 2^{235} \cdot c_{27,3}^2 \cdot c_{30,3} \cdot c_{30,5} \cdot c_{30,7} \)

where \( c_{i,j} = \#(\pi_i(S^j;2)) \).

PROOF (a), (b), (c) and (d) follow immediately from Theorem (1.25) and Propositions (3.1), (3.2) and (3.8). Since \( Y_n \) has cellular structure

\[ S^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15} \]

it is easy to see that \( Y_n \wedge Y_n \) has cells in the numbers and dimensions given by the lattice below.

```
  18  20  22  23  25  27  30
  15  17  19  20  22  24  27
 13  15  17  18  20  22  25
 11  13  15  16  18  20  23
 10  12  14  15  17  19  22
  8  10  12  13  15  17  20
  6  08 10 11 13 15  18
```
We may use Lemma (3.9) to obtain the following inequality.

\[(3.10.1)\]
\[
\mu_{12}(SU(4)) \leq \#(\pi_6(SU(4):2)) \cdot \left[ \#(\pi_7(SU(4):2)) \right]^2 \left[ \#(\pi_{10}(SU(4):2)) \right]^3 \cdot \\
\left[ \#(\pi_{11}(SU(4):2)) \right]^2 \left[ \#(\pi_{12}(SU(4):2)) \right]^2 \left[ \#(\pi_{13}(SU(4):2)) \right]^4 \cdot \\
\left[ \#(\pi_{14}(SU(4):2)) \right] \cdot \left[ \#(\pi_{15}(SU(4):2)) \right] \cdot \left[ \#(\pi_{16}(SU(4):2)) \right] \cdot \\
\left[ \#(\pi_{17}(SU(4):2)) \right] \cdot \left[ \#(\pi_{18}(SU(4):2)) \right] \cdot \left[ \#(\pi_{19}(SU(4):2)) \right]^2 \cdot \\
\left[ \#(\pi_{20}(SU(4):2)) \right] \cdot \left[ \#(\pi_{22}(SU(4):2)) \right] \cdot \left[ \#(\pi_{23}(SU(4):2)) \right]^2 \cdot \\
\left[ \#(\pi_{24}(SU(4):2)) \right] \cdot \left[ \#(\pi_{25}(SU(4):2)) \right] \cdot \left[ \#(\pi_{27}(SU(4):2)) \right]^2 \cdot \\
\left[ \#(\pi_{30}(SU(4):2)) \right] .
\]

The cardinalities of all the groups involved, with the exception of the last four can be found in Mimura-Toda (1964). We may estimate the last four using the exact fibration sequence

\[
\ldots \rightarrow \pi_1(SU(3):2) \rightarrow \pi_1(SU(4):2) \rightarrow \pi_1(S^7:2) \rightarrow \ldots
\]

wherein we estimate \(\pi_i(SU(3):2)\) for \(i = 24, 25, 27\) and 30 using the fibration sequence

\[
\ldots \rightarrow \pi_1(S^3:2) \rightarrow \pi_1(SU(3):2) \rightarrow \pi_1(S^5:2) \rightarrow \ldots
\]

For \(SU(3)\) we get

\[
\#(\pi_{24}(SU(3):2)) \leq \#(\pi_{24}(S^3:2)) \cdot \#(\pi_{24}(S^5:2)) = 2 \cdot 2^4 = 2^5, \\
\#(\pi_{25}(SU(3):2)) \leq \#(\pi_{25}(S^3:2)) \cdot \#(\pi_{25}(S^5:2)) = 2 \cdot 2^3 = 2^4, \\
\#(\pi_{27}(SU(3):2)) \leq \#(\pi_{27}(S^3:2)) \cdot \#(\pi_{27}(S^5:2)) = 27 \cdot 2^3,
\]

and

\[
\#(\pi_{30}(SU(3):2)) \leq \#(\pi_{30}(S^3:2)) \cdot \#(\pi_{30}(S^5:2)) = 30 \cdot 2^3
\]

where the numerical values of the cardinalities involved are obtained from Toda (1962), Mimura-Toda (1963), and Mimura (1965).

[See Appendix I for tabulation of cardinalities]

Combining the above results for \(SU(3)\) with the fact that
\[ #(\pi_{24}(S^7;2)) = 2^4, \]
\[ #(\pi_{25}(S^7;2)) = 2^4, \]
\[ #(\pi_{27}(S^7;2)) = 2^3 \]
and we have
\[ #(\pi_{24}(SU(4);2)) \leq 2^9, \]
\[ #(\pi_{25}(SU(4);2)) \leq 2^8 \]
and
\[ #(\pi_{27}(SU(4);2)) \leq c_{27,3} \cdot 2^6. \]

Inequality (3.10.1) now becomes
\[
\sum_{\mu_{[2]}(SU(4))} 2^0 \cdot (2^3)^2 \cdot (2^4)^3 \cdot (2^2)^2 \cdot (2^5)^2 \cdot (2^4)^4 \cdot (2^6)^6 \cdot 2^7.
\]
\[
\cdot (2^6)^2 \cdot (2^3)^2 \cdot (2^2)^5 \cdot (2^8)^4 \cdot (2^7)^2 \cdot 2^9 \cdot (2^8)^2
\]
\[
c_{27,3} \cdot (2^6)^2 \cdot c_{30,3} \cdot c_{30,5} \cdot c_{30,7}
\]
\[
= 2^{210} \cdot c_{27,3} \cdot c_{30,3} \cdot c_{30,5} \cdot c_{31,7}
\]
and we have demonstrated (e). [See Appendix I for a tabulation of cardinalities.]

In a similar fashion one can check that (f) holds.

Q.E.D.
LIST OF REFERENCES


### APPENDIX I

<table>
<thead>
<tr>
<th></th>
<th>$#(\pi_1(S^3))$</th>
<th>$#(\pi_1(S^5))$</th>
<th>$#(\pi_1(S^7))$</th>
<th>$#(\pi_1(SU(3)))$</th>
<th>$#(\pi_1(SU(4)))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$2^2 \cdot 3^1$</td>
<td>$2^1$</td>
<td>$1$</td>
<td>$2^1 \cdot 3^1$</td>
<td>$1$</td>
</tr>
<tr>
<td>7</td>
<td>$2^1$</td>
<td>$2^1$</td>
<td>$\infty$</td>
<td>$1$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>8</td>
<td>$2^1$</td>
<td>$2^3 \cdot 3^1$</td>
<td>$2^1$</td>
<td>$2^2 \cdot 3^1$</td>
<td>$2^3 \cdot 3^1$</td>
</tr>
<tr>
<td>9</td>
<td>$3^1$</td>
<td>$2^1$</td>
<td>$2^1$</td>
<td>$3^1$</td>
<td>$2^1$</td>
</tr>
<tr>
<td>10</td>
<td>$3^1 \cdot 5^1$</td>
<td>$2^1$</td>
<td>$2^3 \cdot 3^1$</td>
<td>$2^1 \cdot 3^1 \cdot 5^1$</td>
<td>$2^4 \cdot 3^1 \cdot 5^1$</td>
</tr>
<tr>
<td>11</td>
<td>$2^1$</td>
<td>$2^1$</td>
<td>$1$</td>
<td>$2^1$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>12</td>
<td>$2^2$</td>
<td>$2^3 \cdot 3^1 \cdot 5^1$</td>
<td>$1$</td>
<td>$2^2 \cdot 3^1 \cdot 5^1$</td>
<td>$2^2 \cdot 3^1 \cdot 5^1$</td>
</tr>
<tr>
<td>13</td>
<td>$2^3 \cdot 3^1$</td>
<td>$2^1$</td>
<td>$2^1$</td>
<td>$2^1 \cdot 3^1$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>14</td>
<td>$2^4 \cdot 3^1 \cdot 7^1$</td>
<td>$2^3$</td>
<td>$2^3 \cdot 3^1 \cdot 5^1$</td>
<td>$2^3 \cdot 3^1 \cdot 7^1$</td>
<td>$2^5 \cdot 3^1 \cdot 5^1 \cdot 7^1$</td>
</tr>
<tr>
<td>15</td>
<td>$2^2$</td>
<td>$2^4 \cdot 3^2$</td>
<td>$2^3$</td>
<td>$2^2 \cdot 3^2$</td>
<td>$2^4 \cdot 3^2$</td>
</tr>
<tr>
<td>16</td>
<td>$2^1 \cdot 3^1$</td>
<td>$2^5 \cdot 3^2 \cdot 7^1$</td>
<td>$2^4$</td>
<td>$2^3 \cdot 3^1 \cdot 7^1$</td>
<td>$2^7 \cdot 3^2 \cdot 7^1$</td>
</tr>
<tr>
<td>17</td>
<td>$2^4 \cdot 3^1 \cdot 5^1$</td>
<td>$2^3$</td>
<td>$2^4 \cdot 3^1$</td>
<td>$2^2 \cdot 3^1 \cdot 5^1$</td>
<td>$2^6 \cdot 5^1$</td>
</tr>
<tr>
<td>18</td>
<td>$2^1 \cdot 3^1 \cdot 5^1$</td>
<td>$2^2 \cdot 3^1$</td>
<td>$2^4 \cdot 3^2 \cdot 7^1$</td>
<td>$2^2 \cdot 3^2 \cdot 5^1$</td>
<td>$2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1$</td>
</tr>
<tr>
<td>19</td>
<td>$2^2 \cdot 3^1$</td>
<td>$2^2 \cdot 3^1$</td>
<td>$1$</td>
<td>$2^3 \cdot 3^2$</td>
<td>$2^3 \cdot 3^1$</td>
</tr>
<tr>
<td>20</td>
<td>$2^4 \cdot 3^1$</td>
<td>$2^2 \cdot 3^1 \cdot 5^1$</td>
<td>$2^1 \cdot 3^1$</td>
<td>$2^3 \cdot 3^2 \cdot 5^1$</td>
<td>$2^3 \cdot 3^1 \cdot 5^1$</td>
</tr>
<tr>
<td>i</td>
<td>#(\pi_1(S^3))</td>
<td>#(\pi_1(S^5))</td>
<td>#(\pi_1(S^7))</td>
<td>#(\pi_1(SU(3)))</td>
<td>#(\pi_1(SU(4)))</td>
</tr>
<tr>
<td>----</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
<td>21</td>
<td>2^4 \cdot 3^1</td>
<td>2^2</td>
<td>2^5 \cdot 3^1</td>
<td>2^1 \cdot 3^1</td>
<td>2^5</td>
</tr>
<tr>
<td>22</td>
<td>2^3 \cdot 3^1 \cdot 11^1</td>
<td>2^4</td>
<td>2^6 \cdot 3^1 \cdot 11^1</td>
<td>2^2 \cdot 3^1 \cdot 11^1</td>
<td>2^8 \cdot 3^1 \cdot 5^1 \cdot 11^1</td>
</tr>
<tr>
<td>23</td>
<td>2^2</td>
<td>2^5 \cdot 3^1</td>
<td>2^4</td>
<td>2^3 \cdot 3^1</td>
<td>2^7 \cdot 3^1</td>
</tr>
<tr>
<td>24</td>
<td>2^-1</td>
<td>2^4 \cdot 3^1 \cdot 11^1</td>
<td>2^4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1</td>
<td>2^3 \cdot 3^1</td>
<td>2^4 \cdot 3^1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>c_{26,3} \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1</td>
<td>2^2 \cdot 3^1</td>
<td>2^4 \cdot 3^1 \cdot 11^1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>c_{27,3} \cdot 3^1 \cdot 5^1</td>
<td>2^3 \cdot 3^2 \cdot 5^1</td>
<td>2^3 \cdot 3^1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>c_{28,3}</td>
<td>c_{28,5} \cdot 3^3 \cdot 5^1 \cdot 7^1 \cdot 13^1</td>
<td>2^2 \cdot 3^1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>c_{29,3} \cdot 3^2</td>
<td>c_{29,5} \cdot 3^1 \cdot 5^1</td>
<td>2^6 \cdot 3^2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>c_{30,3} \cdot 3^2</td>
<td>c_{30,5}</td>
<td>c_{30,7} \cdot 3^3 \cdot 5^1 \cdot 7^1 \cdot 13^1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>