INTERACTION OF SOUND AND TURBULENCE WITH PARTICULAR APPLICATION TO THE SCATTERING OF SOUND FROM A TURBULENT WAKE

MOSES ROBINSON SWIFT

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INTERACTION OF SOUND AND TURBULENCE WITH
PARTICULAR APPLICATION TO THE
SCATTERING OF SOUND FROM A TURBULENT WAKE

by

MOSES ROBINSON SWIFT
B.S., University of New Hampshire, 1971

A THESIS

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**NOMENCLATURE**

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<th>Definition</th>
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<td>$c_0$</td>
<td>Sound velocity</td>
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<td>$E$</td>
<td>Energy spectrum function</td>
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<td>$f$</td>
<td>Longitudinal velocity correlation coefficient; non-dimensionalized mean velocity</td>
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<td>$R_{ij}$</td>
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</table>
\( T_{ij} \)  
**Momentum flux tensor**

\( u_i \)  
**Component of turbulent fluid velocity**

\( \bar{u} \)  
**Mean fluid velocity**

\( \bar{u}_o \)  
**Velocity of the self-propelled body**

\( u_s \)  
**Velocity scale**

\( \bar{v}_f \)  
**Local flow velocity**

\( v_i \)  
**Component of the total fluid velocity**

\( v_i \)  
**Component of the fluid velocity due to the incident sound field**

\( \Delta V_i \)  
**Subvolumes of the wake**

\( V_t \)  
**Total volume of the turbulence**

\( \varepsilon \)  
**Rate of turbulence dissipation; small parameter such that \( \varepsilon c_o \) is the incident wave velocity amplitude**

\( \lambda \)  
**Second coefficient of viscosity**

\( \lambda_o \)  
**Dissipation length**

\( \mu \)  
**First coefficient of viscosity**

\( \nu \)  
**Kinematic viscosity**

\( \omega \)  
**Frequency**

\( \omega_o \)  
**Incident wave frequency**

\( \phi_{ij} \)  
**Fourier space transform of velocity correlation tensor**

\( \rho \)  
**Density**

\( \rho_o \)  
**Equilibrium density**

\( \beta \)  
**Density fluctuation**

\( \sigma_{ij} \)  
**Stress tensor**
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ABSTRACT

A study of the interaction of acoustic waves with turbulence is developed from the basic equations of fluid dynamics. Particular application is made to the scattering of sound, in the ocean, by turbulence found in wakes.

After an introduction and a review of previous work in the subject area, the basic theory of sound-turbulence interaction is developed taking into account the wave nature of the phenomena. The fundamental result of this section gives the power spectrum of the differential scattering cross-section of the scattered wave, in the far-field approximation, for the case of a plane acoustic wave incident upon a volume of homogeneous, isotropic turbulence, where the convective effects of the macro-eddies is taken into account. From this relation, it is shown that the size of the eddies responsible for most of the scattering are of the order of the wave-length of the incident sound wave when there is appreciable scattering. Because of this, a geometrical optics approach is invalidated under the conditions of the problem. The relation also indicates that the power spectrum of the scattered wave depends upon the turbulence energy spectrum function, E(k).

This is the contribution to the energy per unit mass of the turbulence fluctuations from that part of wave-number space between k and k+dk. Therefore, this quantity must be determined in order to explicitly evaluate the power spectrum of the scattered wave.

E(k) is found for turbulence of both high and low Reynolds number, and, with this information, the power spectrum of the scattered wave is evaluated for each case. In addition, the scattering of high frequency sound (whose
wave-length is much smaller than the size of any of the turbulent eddies) is treated using a somewhat different approach.

The results of the scattering of sound from a volume of homogeneous, isotropic turbulence is then extended to the inhomogeneous case in which the properties of the fluid flow vary with position. The conditions for the application of the analysis are that: (1) the statistical parameters of the flow vary very little over a distance larger than the size of the eddies responsible for most of the scattering, and that (2) the local isotropy assumption be valid for these eddies.

The general expression for the power spectrum of waves scattered from inhomogeneous turbulence is then applied to the problem of the scattering of an acoustic wave from the turbulent wake of a totally immersed, axisymmetric, self-propelled body. The determination of the spatial variation of fluid properties, needed in the power spectrum evaluation, is done using a self-preservation hypothesis. The results for the scattered wave power spectrum take into account the direction of the motion of the self-propelled body, as well as the position of the observer of the scattered wave.
I. INTRODUCTION

The existence of a turbulent wake behind vessels or any moving body operating in the ocean may, under certain circumstances, disturb the performance of acoustic systems on such vessels. This occurs when the turbulence of the wake scatters the sound waves, thus attenuating the original signal and creating a scattered wave. The purpose of this thesis will be to investigate the non-linear interaction of sound and turbulence responsible for this phenomenon and, in particular, develop a theory for sound scattering from the wake of a self-propelled body.

A. Review of Previous Investigations

The problem of scattering of sound by turbulence has been considered by several investigators, beginning with the fundamental work of Lighthill [1], Kraichnan [2], and Batchelor [3]. This early research was concerned with developing the basic formalism for this class of problems and was general in nature. For instance, the differential scattering cross-section, which characterizes the scattered waves, was expressed in terms of the spatial spectrum of turbulence correlations. The task remained, however, to supply the specific information on turbulent flows needed in order to explicitly evaluate solutions for particular scattering problems of practical interest. Subsequent research has been carried out with this purpose in mind [4-8].

Among the contributions related to the problem considered here was that of Meecham and Ford [4]. They considered the power spectrum of sound waves scattered from a volume of high Reynolds number, homogeneous, isotropic turbulence and showed how the convective effects of the large-scale eddies introduce Doppler shifts in the scattered waves. More recently, Celikkol, Swift, and
Yildiz [5] used a similar approach to treat sound scattering from low Reynolds number, homogeneous, isotropic turbulence. In this latter work, the turbulence spatial energy spectrum function, which is necessary in the scattered wave power spectrum evaluation, was derived for low Reynolds number turbulence. As a result, the dependence of the solution on $k_d = 2 k_0 \sin \theta/2$ (see Figure (1) for meaning of $k_d$) differed from earlier predictions and is more appropriate to the case considered.

B. Results Contained in Thesis

This work begins with a derivation of the basic hydrodynamic relations of sound-turbulence interaction that were first reported by Lighthill. Next, the power spectrum of the scattered waves is found for the case of homogeneous, isotropic turbulence, and this is evaluated in the high Reynolds number case to obtain the result due to Meecham and Ford, and in the low Reynolds number case to determine the solution given by Celikkol, Swift, and Yildiz. The derivation presented here, however, differs from that in the original papers. In particular, a "frozen-flow" assumption is used, in contrast to the published methods, to predict the convective effects of the larger turbulent eddies. The eventual analytical results are the same, and the derivation presented here serves to give additional physical insight into the dynamical processes involved.

For completeness, the solution to the problem of high-frequency sound scattering by turbulence, due largely to Lighthill, is also included.

The known theory of sound scattering from a volume of homogeneous, isotropic turbulence is then extended, with appropriate assumptions, to the case where the turbulence is inhomogeneous. This is then applied to the problem of sound scattering from the inhomogeneous turbulent wake of a totally immersed, axisymmetric, self-propelled body. The power spectrum of the scattered waves is
evaluated explicitly after an analysis of the turbulent fluid flow in the wake itself.

C. Analytical Approach

In each case, the analysis will treat the scattering of a plane, harmonic incident wave. This is done because the results for a plane incident wave can be applied to any incident wave which is approximately plane over distances of the order of those eddy sizes which bear most of the turbulent energy. In addition, the results of the scattering of a harmonic wave can be used to evaluate those of any arbitrary steady wave if its intensity spectrum is known. This is because a single-scattering theory is used so that the scattered wave from each Fourier component of an arbitrary incident wave is uncorrelated with any other, in which case, the intensities from each scattered wave may be added.

Results will be given in terms of the power spectrum of the differential scattering cross-section, which is the most general and convenient form of the solution. This is the mean flux of energy in the scattered wave per unit solid angle in the direction of an observer, per unit scattering volume, per angular frequency interval, and per incident intensity.

The analysis is based on the hydrodynamic equations of continuity and the Navier-Stokes equation. However, because of the complicated nature of turbulent processes, it is often necessary to make certain assumptions and approximations. These are normally based on physical arguments, dimensional analysis, or order of magnitude analysis. When used here, an attempt will be made to explain their physical significance. And it should be mentioned that all those employed here have been substantiated, either directly or indirectly, by experiment and have been incorporated into well-known classical theories of turbulence. Thus, the general analytical approach is mathematical in nature, and is based on fundamental principles, but relies also on somewhat intuitive, though well established, assumptions and approximations.
II. SOUND-TURBULENCE INTERACTION

A. Hydrodynamic Relations

The motion of a fluid medium is governed by the continuity equation

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \]  

and the equation of motion (in the case of no body forces)

\[ \rho \frac{dv_i}{dt} = \rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial}{\partial x_j} v_i = \frac{\partial}{\partial x_j} \sigma_{ji} . \]  

For a Newtonian, viscous fluid the stress tensor is

\[ \sigma_{ij} = -\rho \delta_{ij} + \lambda \frac{\partial v_i}{\partial x_j} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) . \]  

The last two relations may be combined to give the Navier-Stokes equation for a compressible, linearly viscous fluid:

\[ \rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial}{\partial x_j} v_i = -\frac{\partial p}{\partial x_i} + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j} v_j + \mu \frac{\partial^2}{\partial x_i \partial x_m} v_m . \]  

Equation (4) may be written in a slightly different form by use of equation (1)

\[ \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j) = -\frac{\partial p}{\partial x_i} + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j} v_j + \mu \frac{\partial^2}{\partial x_i \partial x_m} v_m . \]  

Since there are five unknowns \((p, \rho \text{ and components } v_i)\), but only four independent equations, an addition relation is needed involving the thermodynamic equation of state. If the fluid motion is assumed adiabatic and that the fluctuations in pressure and density are assumed small and almost reversible (therefore isentropic), one may write:

\[ \frac{\partial p}{\partial t} + c_o^2 \frac{\partial \rho}{\partial x_i} \]  

where \(c_o\) is a constant with the dimensions of velocity (the velocity of propagation of infinitesimal sound waves).
By taking the divergence of equation (5) and employing equations (1) and (6), the following relation may be found:

\[
\frac{\partial^2}{\partial t^2} \rho - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \rho = -\frac{1}{c_o^2} \frac{\partial^2}{\partial x_i \partial x_j} \rho \nu_i \nu_j + \frac{\lambda + 2\mu}{c_o^2} \frac{\partial^2}{\partial x_m^2} \frac{\partial \nu_i}{\partial x_i}
\]  

(7)

Employing the scaler Green's function for the wave equation, the solution to equation (7) in an infinite medium may be written as

\[
\rho(\vec{r},t) = \rho_h(\vec{r},t) + \frac{1}{4\pi c_o^2} \int dt' \int \, d^3\vec{r}' \frac{\delta(t-t' - |\vec{r}-\vec{r}'|)}{c_o} \left[ \frac{\partial^2 \rho \nu_i \nu_j}{\partial x_i \partial x_j'} \right] - (\lambda + 2\mu) \frac{\partial^2}{\partial x_m^2} \frac{\partial \nu_i}{\partial x_i'}
\]

(8)

where \( \rho_h(\vec{r},t) \) is a solution to the homogeneous part of (7); that is, the wave equation.

Consider a region of turbulence of low Mach number confined to some localized region through which a small-amplitude sound wave is sent. No physically realistic case will be omitted if the velocity field at infinity is considered zero. The density changes due to the sound wave, if there was no turbulence, is given by \( \rho_h(\vec{r},t) \), since that part satisfies the wave equation. The second term on the right-hand side of equation (8) would then represent the fluctuations in density due to the interaction of the turbulence and the incident sound field. If the Born approximation, or single-scattering approximation, is used, then the integral in equation (8) can be evaluated by taking the velocity in the integrand as the sum of the velocities of the incident sound field alone and the turbulent velocity field alone; in equation form this is, respectively,

\[ v_i = v_i + u_i \]  

(9)
Using equation (9) with equation (8), the fluctuations in density due to the interaction of sound and turbulence is

$$\beta(\vec{r}, t) = \frac{\rho_0}{4\pi c_o^2} \int dt' \int d^3\vec{r}' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c_o})}{|\vec{r}-\vec{r}'|} \rho_0 \frac{\partial^2}{\partial x_i' \partial x_j'} \left[ v_i(\vec{r}', t') + u_i(\vec{r}', t') \right]$$

$$\times \left[ v_j(\vec{r}', t') + u_j(\vec{r}', t') \right] + \frac{\partial^2}{\partial x_i' \partial x_j'} [\rho(\vec{r}', t') - \rho_0] \left[ v_i(\vec{r}', t') + u_i(\vec{r}', t') \right]$$

$$\times \left[ v_j(\vec{r}', t') + u_j(\vec{r}', t') \right] + (\lambda + 2\mu) \frac{\partial^2}{\partial x_m^2} \frac{\partial}{\partial x_i'} v_i(\vec{r}', t') \right]$$

(10)

since turbulence is rotational and, therefore, has no divergence. The second term within {} is of a higher order than the first term, since density fluctuations are small, and it may be neglected for low Mach number turbulence. The last term within {} represents the viscous loss in the incident sound field. This may be considered small for most fluids and is, at any rate, of no interest here since it is quite independent of the turbulence. Equation (10) is rewritten keeping only the important term concerning the non-linear interaction of sound and turbulence:

$$\beta(\vec{r}, t) = \frac{\rho_0}{4\pi c_o^2} \int dt' \int d^3\vec{r}' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c_o})}{|\vec{r}-\vec{r}'|} \frac{\partial^2}{\partial x_i' \partial x_j'} v_i(\vec{r}', t') v_j(\vec{r}', t')$$

(11)

where $v_i$ is given by equation (9). The momentum flux tensor $\rho_0 v_i v_j$ is usually denoted $T_{ij}$ and represents the rate at which momentum in the $x_i$ direction crosses unit surface area in the $x_j$ direction. Using this notation, one writes,

$$\beta(\vec{r}, t) = \frac{1}{4\pi c_o^2} \int dt' \int d^3\vec{r}' \left[ \frac{\partial^2}{\partial x_i' \partial x_j'} T_{ij}(\vec{r}', t') \right]^2 \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c_o})}{|\vec{r}-\vec{r}'|}$$
is zero at infinity, so the first volume integral on the right-hand side of relation (12) is zero by application of the divergence theorem. Also, if \( f \) is some arbitrary function of \(|\mathbf{r} - \mathbf{r}'|\), one may say that

\[
- \frac{\partial}{\partial x_i^i} f(|\mathbf{r} - \mathbf{r}'|) = - \frac{\partial}{\partial x_i} f(|\mathbf{r} - \mathbf{r}'|),
\]

so, equation (12) becomes

\[
\hat{\rho}(\mathbf{r},t) = \frac{-1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \int dt' \int d^3\mathbf{r}' \left[ \frac{\partial}{\partial x_j} T_{ij}(\mathbf{r}',t') \right] \frac{\delta(t-t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c_0})}{|\mathbf{r} - \mathbf{r}'|}
\]

(14)

After repeating this procedure again, the density fluctuation becomes

\[
\hat{\rho}(\mathbf{r},t) = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int dt' \int d^3\mathbf{r}' \left[ T_{ij}(\mathbf{r}',t') \right] \frac{\delta(t-t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c_0})}{|\mathbf{r} - \mathbf{r}'|}
\]

(15)

Now, the integration over \( t' \) may be completed using the properties of the Dirac delta function, leaving the expression:

\[
\hat{\rho}(\mathbf{r},t) = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\delta(t-t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c_0})}{c_0}
\]

(16)

This is the basic result first reported by Lighthill [9]. It should be noted that this expression is not restricted to the scattering problem for which it was just derived; its original purpose was to explain the aerodynamic generation of sound. In this case, there is no incident sound field, so
\( \rho_h(\mathbf{r},t) \) is zero, and the Born approximation is unnecessary because the velocities within the integrand of equation (8) are exactly the turbulence velocities alone, since \( V_i \) is zero. The last term in the integrand of equation (8), which was neglected because it represented only the damping of the incident wave when propagating by itself, is also negligible. The low Mach number assumption of small density changes may or may not be made according to the circumstances. Equation (16), in this case, shows how sound may be generated by the non-linear integration of turbulence velocities themselves.

Returning to the problem of the scattering of small amplitude sound waves from a volume of low Mach number turbulence, the momentum flux tensor, \( T_{ij} \), may be evaluated using the definition given previously and equation (9):

\[
T_{ij} = \rho_0 v_i v_j = \rho_0 V_i V_j + \rho_0 (V_i u_j + V_j u_i) + \rho_0 u_i u_j
\]

(17)

The first term on the right-hand side of equation (17) represents the self-modification of the incident wave due to its finite amplitude. Since the amplitude of this wave was considered small, the finite amplitude effect may be neglected. The last term on the right represents the self-noise of the turbulence as discussed in the previous paragraph. Since the sound generation by this term is independent of the interaction of the incident sound and turbulence, it will not be considered here. The middle term on the right-hand side of equation (17) is the one of interest in the scattering problem. Therefore, in subsequent expressions the momentum flux tensor in equation (16) will be taken as

\[
T_{ij} = \rho_0 (V_i u_j + V_j u_i)
\]

(18)

where \( u_i \) is the fluctuating velocity of the turbulence. (If the turbulence has a mean flow, then there will be a refracted wave as well as the scattered wave which is being treated here.)
When $T_{ij}$ is expressed in this way, the integrand in equation (16) is zero outside the volume $V_t$ containing the turbulence. The volume integration in equation (16) can then be restricted to this region and one may speak of a far field point outside $V_t$ in the radiation field of the scattered waves. The position vector $\vec{r}$ is such that

$$|\vec{r}-\vec{r}'| \sim r >> (v_t)^{1/3}$$  \hspace{1cm} (19)

if the point at $\vec{r}$ is a far field point. If the differentiation in equation (16) is carried out and the far field approximation is applied, one obtains

$$\hat{\beta}(\vec{r},t) = \frac{1}{4\pi c_o^6} \sum_{i,j} \frac{x_i x_j}{r^3} \frac{\partial^2}{\partial t^2} \int d^3\vec{r}' \ T_{ij}(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c_o})$$  \hspace{1cm} (20)

In this approximation, only terms falling away more rapidly than the inverse first power of the distance are excluded, so that this expression will still give an exact result for the total energy radiated and its directional distribution.

It is now supposed that a plane acoustic wave of angular frequency $\omega_o$ and propagation vector $\vec{k}_o$ is incident upon the turbulent region $V_t$ along the $x_1$ axis (see Figure (1) for problem geometry). The results of this analysis may be applied to other types of incident wave forms since, if the solution of a plane incident wave is worked out, then it may be applied to any incident wave which is approximately plane over distances of the order of those eddy-sizes which bear most of the turbulent energy. Also, if the results for a harmonic incident wave are determined, then those for an arbitrary steady incident wave can be deduced if its intensity spectrum is known.

The incident acoustic wave may be described by specifying its velocity field:

$$V_i = \varepsilon c_o \cos \left[ k_o (x_1 - c_o t) \right] \delta_{i1}$$  \hspace{1cm} (21)
GEOMETRY OF THE PROBLEM

FIGURE 1.
Combining equations (18), (20), and (21), one writes

\[
\dot{\beta}(\vec{r}, t) = \frac{\rho_0 c}{4\pi c_0^3} \frac{x_1 x_i}{r^3} \frac{\partial^2}{\partial t^2} \int d^3\vec{r}' \{ \delta_{j1} \cos [k_0(x'_1 - c_o \tau')]u_j(\vec{r}', \tau')
\]

\[
+ u_i(\vec{r}', \tau') \delta_{j1} \cos [k_0(x'_1 - c_o \tau')] \}
\]

(22)

where \( \tau' = t - \frac{|\vec{r} - \vec{r}'|}{c_o} \).

The analysis will be restricted to situations in which the incident frequency is much larger than the natural frequencies of the turbulent fluid; that is:

\[
\omega_o >> \frac{<u'^2>_{c_o}}{L}
\]

(23)

where \( L \) is a linear dimension of the order of the size of the energy-containing eddies. It will be shown later that this does not omit any physically meaningful cases. By using this assumption, the time derivatives in equation (22) may be taken on the incident acoustic field alone, so that

\[
\dot{\beta}(\vec{r}, t) = -\frac{\rho_0 c k^2}{2\pi c_0} \frac{x_1 x_i}{r^3} \int d^3\vec{r}' \cos [k_0(x'_1 - c_o \tau')]u_i(\vec{r}', \tau')
\]

(24)

Now, the time autocorrelation of the density fluctuations at some far field point \( \vec{r} \), due to the scattering of sound by turbulence, may be formed:

\[
<\dot{\beta}(\vec{r}, t)\dot{\beta}(\vec{r}, t - \tau) = (-\frac{\rho_0 c k^2}{2\pi c_0})^2 \frac{x_1 x_i}{r^3} \frac{x_1 x_j}{r^3} \int d^3\vec{r}' \int d^3\vec{r}'' \cos [k_0(x'_1 - c_o \tau')] \times u_i(\vec{r}', \tau') \cos [k_0(x''_1 - c_o \tau'')]u_j(\vec{r}'', \tau'')>
\]

(25)

where \( \tau'' = t - \tau - \frac{|\vec{r} - \vec{r}''|}{c_o} \) and < > may be interpreted as a time average over an interval of time much larger than the period of the incident sound field, yet smaller than a length of time associated with overall changes in the turbulence.
structure. Since the acoustic and the turbulence field are statistically independent, the autocorrelation function may be written as

\[
\langle \hat{\rho}(\vec{r},t)\hat{\rho}(\vec{r},t-\tau) \rangle = \left(\frac{\rho_o E k^2}{2\pi c_o}\right)^2 \frac{x_1 x_2 x_3^2}{r^6} \int d^3\vec{r}' \int d^3\vec{r}'' \langle u_i(\vec{r}',\tau') u_j(\vec{r}'',\tau'') \rangle \\
\times \left\langle \cos[k_o(x'_1 - c_o \tau')\cos[k_o(x''_1 - c_o \tau'')] \right\rangle
\]

(26)

Using a trigonometric identity and by considering the geometry of the scattering points at \(\vec{r}'\) and \(\vec{r}''\) in relation to the far field observer at \(\vec{r}\), one obtains for the average of the cosine terms:

\[
\left\langle \cos[k_o(x'_1 - c_o \tau')\cos[k_o(x''_1 - c_o \tau'')] \right\rangle = \frac{1}{2} \left\langle \cos k_o(x'_1 + x''_1 - c_o \tau' + c_o \tau'') \right\rangle
\]

\[
+ \cos k_o(x'_1 - x''_1 - c_o \tau' + c_o \tau'')
\]

\[
= \frac{1}{2} \cos k_o(x'_1 - x''_1 - |\vec{r}'-\vec{r}| + |\vec{r}'-\vec{r}''| + \omega_o \tau)
\]

\[
= \frac{1}{2} \cos(k_o \cdot \vec{z} - \vec{k}_f \cdot \vec{z} + \omega_o \tau)
\]

(27)

where \(\vec{k}_f = k_o \frac{\vec{r}}{r}\) and \(\vec{z} = \vec{r}' - \vec{r}''\). Using relation (27) in equation (26), one obtains

\[
\langle \hat{\rho}(\vec{r},t)\hat{\rho}(\vec{r},t-\tau) \rangle = \left(\frac{\rho_o E k^2}{2\pi c_o}\right)^2 \frac{x_1 x_2 x_3^2}{r^6} \int d^3\vec{r}' \int d^3\vec{r}'' \langle u_i(\vec{r}',\tau') u_j(\vec{r}'',\tau'') \rangle \\
\times \left\{ \frac{1}{2} \Re e^{i\omega_0 \tau} e^{-i[(\vec{k}_f - \vec{k}_o) \cdot \vec{z}]} \right\}
\]

(28)

The velocity correlation tensor occurring in the integral is equal to

\[
\langle u_i(\vec{r}',t - \frac{|\vec{r}'-\vec{r}|}{c_o}) u_j(\vec{r}'',t - \tau - \frac{|\vec{r}'-\vec{r}''|}{c_o}) \rangle = \langle u_i(\vec{r}',t - \frac{|\vec{r}'-\vec{r}|}{c_o}) \rangle \\
\times \langle u_j(\vec{r}'',t - \frac{|\vec{r}'-\vec{r}|}{c_o} - \tau + \frac{\vec{r}/x \cdot \vec{z}}{c_o}) \rangle
\]

(29)
in the far field approximation. It is negligible if \( \vec{r}' \) and \( \vec{r}'' \) are points which have no eddy in common; that is, \( \varsigma \) must be smaller than the size of an eddy for the correlation to be non-zero. Therefore, the time delay represented by \( \frac{1}{c_o}(\vec{r}/\vec{r}'+\varsigma) \) is very small compared to a time significant in the turbulent fluctuations if the turbulent Mach number is low. Remembering that \( \langle \rangle \) may be interpreted as a time average, equation (28) can be written as

\[
\langle \hat{\beta}(\vec{r},t)\hat{\beta}(\vec{r},t-T) \rangle = \frac{\rho_o e_i k_o^2}{2\pi c_o} \frac{1}{i} \frac{1}{x^6} \int d^3\vec{r}'' \int d^3\vec{r}' \langle u_i(\vec{r}',t)u_j(\vec{r}'',t-T) \rangle \times \frac{1}{2} \Re \ e^{i \omega_0 T} e^{-i[(k_f-k_o) \cdot \varsigma]} \]

(30)

Within the inner integral, a new dummy variable will be defined by

\[
\vec{r}' = \varsigma + \vec{r}'' \quad , \quad d^3\vec{r}' = d^2\varsigma
\]

so that

\[
\langle \hat{\beta}(\vec{r},t)\hat{\beta}(\vec{r},t-T) \rangle = \frac{\rho_o e_i k_o^2}{2\pi c_o} \frac{1}{i} \frac{1}{x^6} \int d^3\varsigma \int d^3\varsigma' \langle u_i(\vec{r}''+\varsigma,t)u_j(\vec{r}'',t-T) \rangle \times \frac{1}{2} \Re \ e^{i \omega_0 T} e^{-i[(k_f-k_o) \cdot \varsigma]} \]

(32)

The velocity correlation tensor will be indicated with more compact notation by defining

\[
R_{ij}(\vec{r}'',\varsigma,T) = \langle u_i(\vec{r}''+\varsigma,t)u_j(\vec{r}'',t-T) \rangle
\]

(33)

The spectrum of the mean flux of energy in the scattered wave in the direction of an observer (indicated by the unit vector \( \vec{l} \)) is often expressed in terms of a quantity \( P(\omega,\vec{l}) \) called the "power spectrum of the differential
scattering cross-section." This is the power scattered per solid angle in the
direction of \( \hat{I} \), per angular frequency interval, per unit scattering volume
and per incident intensity. Using this definition and the well-known
theorem that the power spectrum is the Fourier transform of the autocorrelation,
one finds

\[
P(\omega, \xi) = \frac{\rho_0 c^3}{2 \pi I_i V_t} \int_{-\infty}^{\infty} e^{-i\omega \tau} \left\langle \hat{p}(\vec{r}, t) \hat{p}(\vec{r}, t-T) \right\rangle d\tau
\]

(34)

where \( I_i = \frac{1}{2} \rho_0 c^3 \varepsilon^2 \) (35)
is the intensity of the incident plane wave and \( V_t \) is the volume of the tur­
bulent region. Equations (32), (33), (34), and (35) may be combined to find the
following expression for the power spectrum \( P(\omega, \xi) \):

\[
P(\omega, \xi) = \frac{\rho_0}{4 \pi^2 V_t c^2} \frac{1}{\Xi} \frac{x_i x_j x_l}{x^n} \int_{-\infty}^{\infty} d\tau e^{-i\omega \tau} \int d^3 \vec{x} \int d^3 \vec{r} R_{ij}(\vec{x}, \vec{r}, \tau)
\]

\[
\times \frac{1}{2} \text{Re} e^{i k_d \cdot \vec{r}} e^{i \omega_0 \tau}
\]

(36)

where \( k_d = \vec{k}_f - \vec{k}_o \).

B. Specialization to Homogeneous, Isotropic Turbulence

Now consider the case when the turbulence is statistically homogeneous.
The term "statistically homogeneous" or "homogeneous", as it is used here, means
that the average properties which characterize the random fluid motion are in­
dependent of position in the fluid. When the turbulence is homogeneous,

\[
R_{ij}(\vec{x}, \xi, \tau) = \left\langle u_i(\vec{x}+\vec{z}, t) u_j(\vec{x}+\vec{z}, t-T) \right\rangle = R_{ij}(\xi, \tau)
\]

(37)

within the volume of turbulence and zero outside. The averaging may now be
considered either a time average or a spatial average over a large volume. In this case, equation (36) reduces to

$$P(\omega, \bar{k}) = \frac{k_0^j k_0^k k_0^l}{8 \pi^2 c_0^2} \int_{-\infty}^{\infty} d\tau e^{-i\omega \tau} \text{Re} \left[ \int d^3 \tau R_{ij}(\bar{\zeta}, \tau) e^{-i k_0^j \zeta} e^{i \omega \tau} d^3 \bar{\zeta} \right]$$

Next, the second order space-time velocity correlation $R_{ij}(\bar{\zeta}, \tau)$ will be considered. For this purpose, a frozen flow hypothesis will be employed. The term "frozen flow" indicates that if measurements of a transporting turbulent flow are taken at fixed points, the changes in the correlation due to the time delay $\tau$ are caused primarily by convection of the turbulence by the sum of the mean and macro-eddy velocities denoted by $\bar{V}_f$ [10]. This situation is one regularly observed in turbulent flows (at least homogeneous turbulent flows) and in the literature has often been called "frozen turbulence". In the drawing below, this concept is illustrated.

The turbulence structure is carried along at the local flow velocity $\bar{V}_f$, so the time-delayed correlation is essentially the same as one between two
fluid points separated by $\zeta + \bar{V}_f T$. Now, $\bar{V}_f$ is a random variable, so if the expected value of $\bar{V}_f$ is taken, time-delayed correlations between fixed points may be approximated by

$$R_{ij}(\zeta, T) = R_{ij}^F(\zeta, T) \sim \int p(\bar{V}_f) d^3\bar{V}_f R_{ij}^M(\zeta + \bar{V}_f T, T)$$  \hspace{1cm} (39)$$

where $R_{ij}^M$ is a velocity correlation taken with axes moving with local flow velocity. $p(\bar{V}_f)$ is the probability density of $\bar{V}_f$ and has been verified experimentally as Gaussian. In equation (38), the term in $\{\}$ is $(2\pi)^3$ the Fourier space transform of $R_{ij}^M(\zeta, T)$. It can be evaluated by using equation (39) as shown below:

$$\int d^3\zeta R_{ij}(\zeta, T) e^{-i\mathbf{k}_d \cdot \zeta} \sim \int p(\bar{V}_f) d^3\bar{V}_f \int d^3\zeta e^{-i\mathbf{k}_d \cdot \zeta} R_{ij}^M(\zeta + \bar{V}_f T, T)$$

$$= \int p(\bar{V}_f) d^3\bar{V}_f e^{i\mathbf{k}_d \cdot \bar{V}_f T} \int d^3\lambda e^{-i\mathbf{k}_d \cdot \lambda} R_{ij}^M(\lambda, T)$$

$$= [\int p(\bar{V}_f) d^3\bar{V}_f e^{i\mathbf{k}_d \cdot \bar{V}_f T}] (2\pi)^3 \phi_{ij}^M(k_d \cdot T)$$

$$= (2\pi)^3 < e^{i\mathbf{k}_d \cdot \bar{V}_f T} >_{\bar{V}_f} \phi_{ij}^M(k_d, T)$$  \hspace{1cm} (40)$$

where the new notation is obvious.

However, since the velocity correlation $R_{ij}^M(\zeta + \bar{V}_f T, T)$ is taken with respect to fluid particles and the moving fluid structure changes little during $T$, $R_{ij}^M(\zeta + \bar{V}_f T, T)$ is approximately $R_{ij}^M(\zeta + \bar{V}_f T, 0)$. Similarly, $\phi_{ij}^M(k_d, T) = \phi_{ij}^M(k_d, 0)$. Making these substitutions into equation (38), one obtains

$$P(\omega, \bar{\omega}) = \frac{k_s^4}{c_o^2} \frac{x \cdot x \cdot x^2}{r^4} \phi_{ij}(k_d) \int d\omega e^{-i\omega T} \Re e^{i\omega T} < e^{i\mathbf{k}_d \cdot \bar{V}_f T} >_{\bar{V}_f}$$  \hspace{1cm} (41)$$
It will now be assumed that the turbulence fluctuations are isotropic, though perhaps being carried along by a mean flow. By the phrase "isotropic turbulence", it is meant that all velocity correlations are independent of arbitrary rotations and reflections. In the isotropic case, the spectrum tensor $\phi_{ij}$ has the form:

$$\phi_{ij}(k) = \frac{E(k)}{4\pi k^4} (k^2 \delta_{ij} - k_i k_j)$$  \hspace{1cm} (42)

so that equation (41) may be written as:

$$p(\omega, \theta) = \frac{k_o^2}{c_o^2} \left\{ \frac{k_o^2}{4\pi k_d^2} \left[ \delta_{ij} - \frac{(k_d)_i (k_d)_j}{k_d^2} \right] \frac{x_i x_j}{r^2} \right\} \int_{-\infty}^{\infty} dt \, e^{-i\omega t} \left\langle e^{i(\omega_0 + k_d \cdot \overrightarrow{V_f} \tau)} \overrightarrow{V_f} \right\rangle$$  \hspace{1cm} (43)

The expression in $\{\}$ may be evaluated from consideration of the problem geometry in Figure (1):

$$\frac{k_o^2}{4\pi k_d^2} \left[ \delta_{ij} - \frac{(k_d)_i (k_d)_j}{k_d^2} \right] \frac{x_i x_j}{r^2} = \frac{x}{4\pi (2k_o \sin \theta/2)^2} \left[ \frac{r \cdot x}{r^2} - \left( \frac{k_d \cdot x}{k_d \cdot r} \right)^2 \right]$$

$$= \frac{1}{16\pi \sin^2 \theta/2} \cos^2 \theta \left[ 1 - \sin^2 \theta/2 \right]$$

$$= \frac{1}{16\pi} \cos^2 \theta \cot^2 \theta/2$$  \hspace{1cm} (44)

Combining the last two equations, the power spectrum becomes
It is seen that for sound scattering from homogeneous, isotropic turbulence, the power spectrum of the differential scattering cross-section depends on the spatial energy spectrum of the turbulence evaluated at $k_d$.

For the case when $\theta$ approaches zero, forward scattering, $k_d = 2k_o \sin \theta/2$ approaches zero, and the spatial energy spectrum evaluated at $k_d$ also becomes zero very quickly, there being no eddies of infinite extent. So there is no scattering within a cone where $\theta$ is close to zero. This being the case, one can say that $k_d$ is of the order of $k_o$ for values of $\theta$ for which there is scattering.

$$k_d \sim k_o$$  \hspace{1cm} (46)

This implies that, since the turbulence energy spectrum in equation (45) is evaluated at $k_d$, the eddies responsible for most of the scattering, i.e., the "scattering eddies", are of a size of the order of the wave-length of the incident sound field. If the frequency of the incident wave is such that its wave-length is not in the range of eddy sizes in the turbulence, there will be negligible scattering. At this point, the approximation that the incident wave frequency is much greater than the natural frequencies of the turbulence, expressed by equation (23), can be reviewed. It has been just shown that the physically meaningful cases occur when

$$\frac{1}{k_o} \sim L$$  \hspace{1cm} (47)

Since the analysis has been restricted to low Mach number,

$$\langle u^2 \rangle^{1/2} \ll c_o$$  \hspace{1cm} (48)
Thus, it can be seen from equations (47) and (48) that the incident wave frequencies for which there is appreciable scattering occur when

\[ \omega_o >> \frac{<u'^2>^{1/2}}{L}, \]  \hspace{1cm} (23)

which is just the condition stated by equation (23). Therefore, no physically meaningful situations have been omitted by restricting the analysis to high values of the incident wave frequency.

Furthermore, the conclusion that the "scattering eddies" are of size of the order of the incident field wave-length invalidates a geometrical optics approach. A necessary condition for a geometrical optics approach to apply is that the incident sound wave-length be shorter than significant lengths of change in the medium. This condition, from what has been shown, eliminates from consideration the very flow inhomogeneities which contribute most to the scattering. Thus, this approach is rejected.
III. EVALUATION OF POWER SPECTRUM FOR HOMOGENEOUS, ISOTROPIC TURBULENCE

A. High Reynolds Number Turbulence

It has been shown, equation (45), that the evaluation of the power spectrum of the differential scattering cross-section for homogeneous, isotropic turbulence essentially reduces to the calculation of the spatial energy spectrum of the turbulence. This will be specified in this section for high Reynolds number turbulence, and the power spectrum will be evaluated explicitly.

At this point, it will be necessary to review the "universal equilibrium theory" for high Reynolds number turbulence. The length dimension used in the Reynolds number definition, in this case, will be the characteristic length $L$ which is the approximate size of the "energy-containing" eddies. The characteristic velocity to be used will be $\left(\frac{1}{3} < u_i^2 >\right)^{\frac{1}{2}}$ which is also associated with the energy containing eddies. The Reynolds number being considered here is then

$$ R = \frac{\left(\frac{1}{3} < u_i^2 >\right)^{\frac{1}{2}} L}{v} \quad (49) $$

The picture of turbulent motion at large values of $R$ will now be described [10, 11]. The turbulence consists initially of a shear regime where the smaller wave-numbers of the eventual spectral distribution of energy are excited. These wave-numbers are of the order of magnitude of the reciprocal of the various linear dimensions of the turbulence source, and they receive energy directly from that mechanical system. The energy is quickly transferred by inertial forces to the higher wave-numbers. The excitement of the higher wave-numbers is accompanied, due to pressure forces, by a loss of directional preference. The range of wave-numbers containing most of the energy may be regarded as a definite group called the energy-containing eddies. A length $L$ has
already been associated with this group and can now be more specifically defined as the reciprocal of the wave-number at which the maximum of the energy spectrum function \( E(k) \) occurs. The spectrum decays monotonically from this maximum as \( k \to \infty \).

One might also regard each degree of freedom of the motion (each wave-number component) as having its own Reynolds number. One definition might be

\[ \left[ k \, E(k) \right]^{1/k} \left( 1/k \right)/\nu, \]

which decreases in the higher wave-numbers above the maximum of \( E(k) \). Since the Reynolds number represents the ratio of inertial to viscous forces, the higher wave-number region is dominated by viscous forces, while in the lower wave-number range, inertial forces are most important. Thus, the physical picture of high Reynolds number turbulence (i.e., large \( R \)) may be extended by saying that the energy entering the turbulent flow at the low end of the spectrum and spread over the range of wave-numbers by inertial forces is finally dissipated in the sink provided by viscous damping at large wave-numbers.

Based on this physical picture, it is seen that there is a region of wave-numbers which is not excited directly by the external large-scale forces which generate the motion. The influence of the external conditions is then strongest for the small wave-numbers and increasingly less strong for the indirectly excited higher wave-numbers. It can now be argued that there is likely to exist a range of high wave-numbers whose Fourier coefficients are statistically steady and independent of the Fourier coefficients of the energy-containing range of wave-numbers. This high wave-number portion of the spectrum is called the equilibrium range. It is internally self-adjusting, because of the operation of inertia forces, and must depend only on the energy flux through this range and the rate of dissipation. Thus, the character of the turbulence in this range is specified by the energy supply, \( \varepsilon \), to the equilibrium range, defined by
and by the kinematic viscosity \( \nu \), since this determines the rate of dissipation.

These considerations have led to the hypothesis of universal equilibrium:

"The notion associated with the equilibrium range of wave-numbers is uniquely determined statistically by the parameters \( \varepsilon \) and \( \nu \)." \([10, 12]\)

From these two parameters, the length scale \( \eta \) and the velocity scale \( v \) may be defined from dimensional arguments,

\[
\eta = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4}, \quad v = (\nu \varepsilon)^{1/4}.
\]  \( (51) \)

When all lengths are non-dimensionalized by \( \eta \) and all velocities by \( v \), the resulting motion of the equilibrium range, expressed in this way, has a universal statistical form.

In particular, the energy spectrum function \( E(k, t) \) in the equilibrium range has the form

\[
E(k, t) = v^2 \eta E_e(\eta k)
\]  \( (52) \)

where \( E_e \) is a dimensionless universal function.

It should be noted that since the Reynolds number based on the length and velocity scales \( \eta \) and \( v \), respectively, is one, \( 1/\eta \) is of the order of those wave-numbers where dissipation takes place.

Now, consider the case of high Reynolds number (large R) turbulence in which the energy-containing range and the dissipation range of wave-numbers are so widely spaced that an inertial subrange of the equilibrium range exists such that

\[
\frac{1}{L} \ll k \ll \frac{1}{\eta}.
\]  \( (53) \)

This subrange will be statistically independent of both the energy-containing
eddies and the eddies responsible for dissipation. The motion of the inertial subrange is then specified uniquely by $\varepsilon$ only. In this case, the energy spectrum given by equation (52) may be further restricted. One can say that for the range of wave-numbers specified by relation (53), the function $E_e$ on the right-hand side of equation (52) must take a form such that $v^2\eta E_e$ is independent of $v$. From the defining relations (51), it can be seen that $E_e = K(\eta k)^{-5/3}$ has this form since the resulting energy spectrum,

$$E(k) = v^2\eta [K(\eta k)^{-5/3}] = K \varepsilon^{2/3} k^{-5/3},$$

(54)

(where $K$ is a non-dimensional constant) is then independent of $v$. This is frequently referred to as the Kolmogorov spectrum law.

Returning to the sound scattering problem, the results of this theory may be applied to the evaluation of $P(\omega, \Theta)$. In particular, if the turbulence is of large enough Reynolds number and if the frequency of the incident wave is high so that $k_d^2 = 2k_o \sin \theta/2$ is in the range of wave-numbers between that of the large-scale eddies and those of the very smallest eddies where viscous loss takes place; that is, if $k_d$ is within the inertial subrange, the Kolmogorov spectrum may be used to evaluate the spatial turbulence energy spectrum at $k_d$ in equation (45):

$$E(k_d) = K \varepsilon^{2/3} k_d^{-5/3},$$

(55)

where $A$ is a dimensionless constant and $\varepsilon$ is the rate of dissipation of the turbulence energy by viscous damping. The size of the eddies which this range represents is at least an order of magnitude less than that of the macro-eddies, i.e., the large energy containing eddies. This is consistent with the picture presented previously in the "frozen turbulence" discussion, where the mean velocity and velocities of the large-scale eddies sweep the small-scale structure along. In this case, the small-scale (high wave-number) eddies
within the inertial subrange are of the order of the incident sound wave-length and therefore, represent the scattering eddies. The convection of these eddies by the mean velocity and the macro-eddy velocities introduce Doppler effects in the scattered sound wave. The resulting frequency shift in the power spectrum will be shown in the subsequent analysis.

Substituting equation (49) into equation (45), one obtains

$$P(\omega, \Theta) = \frac{Kk^2 \epsilon^{2/3}}{16\pi c^2_o} \cos^2 \Theta \cot^2 \Theta / 2 (k_d)^{-5/3} \int d^3 \bar{V}_f p(\bar{V}_f) \frac{1}{k_d} \int_{-\infty}^{\infty} d(k_d \tau)$$

$$P(\omega, \Theta) = \frac{Kk^2 \epsilon^{2/3}}{8c^2_o} \cos^2 \Theta \cot^2 \Theta / 2 \left(\frac{1}{(k_d)^{5/3}}\right) \int d^3 \bar{V}_f p(\bar{V}_f) \delta\left(\frac{\omega - \omega_0}{k_d} + \frac{k_d}{k_d} \cdot \bar{V}_f\right) (56)$$

Performing the integration over components of $\bar{V}_f$ perpendicular to $k_d$; finally, upon performing the integration over components of $\bar{V}_f$ along the direction of $k_d$, one obtains

$$P(\omega, \Theta) = \frac{Kk^2 \epsilon^{2/3}}{8c^2_o} \cos^2 \Theta \cot^2 \Theta / 2 \left(\frac{1}{(2k_o \sin \Theta/2)^{8/3}}\right) P\left(\frac{\omega - \omega_0}{2k_o \sin \Theta/2}\right) (57)$$

As was previously mentioned, the probability density of the fluid velocity fluctuations has been experimentally determined to be Gaussian so that

$$P\left(\frac{\omega - \omega_0}{2k_o \sin \Theta/2}\right) = \frac{1}{(2\pi(2u^2))^{1/2}} \exp \left\{ - \frac{1}{2(u^2)} \left(\frac{\omega - \omega_0}{2k_o \sin \Theta/2} - V_o\right)^2 \right\} (58)$$

where $V_o$ is the mean velocity of the flow in the direction of $k_d$. The maximum
of the spectrum occurs when the exponent is one resulting in the expression:

$$\omega_{\text{max}} = \omega_0 + 2V_0 k_0 \sin \theta / 2.$$  (59)

It is seen that the maximum of the spectrum of the scattered waves is shifted from the incident wave frequency $\omega_0$ because of the mean velocity component in the direction of $k_d$.

The change in frequency $\Delta \omega$ from the maximum, $\omega_{\text{max}}$, which reduces the spectrum by a factor of $1/e$ is

$$\Delta \omega = (2 <u'^2>)^{1/2} (2k_0 \sin \theta / 2)$$  (60)

showing how the velocity fluctuations of the macro-eddies contribute to widening the spectrum of the scattered waves.

B. Low Reynolds Number Turbulence

In this section, sound scattering from low Reynolds number, isotropic, homogeneous turbulence will be considered. The procedure of the last section will again be used here. That is, the spatial energy spectrum will be calculated and then substituted into equation (45) in order to evaluate the power spectrum of the scattered waves. As before, the probability density of the fluid velocity fluctuations will be taken as Gaussian.

Since turbulence is usually associated by fluid dynamicists with high Reynolds numbers, the meaning of the term "low Reynolds number turbulence" should be explained. First of all, turbulence in wakes is generated by instabilities in the flow due to large inertial forces in an initial shear regime at the turbulence source at a high Reynolds number based on free-stream velocity, the kinematic viscosity, and some characteristic dimension of the source, i.e., $U_0 D/V$. However, in the wake itself, since the turbulent energy has immediately begun to decay through the action of viscosity, it is more appropriate to consider
a Reynolds number based on characteristic lengths and velocities associated with the turbulent eddies. This Reynolds number, determined by local conditions, decreases with distance from the source. In the latter portion of the wake, this Reynolds number is low, though the term "turbulence" can still be applied to the random fluid motion there. Batchelor has termed this part "the final period of decay". In cases where $U_0 D/v$ is just large enough for turbulence to be generated initially, the final region may constitute the significant portion of the wake, so that it is meaningful to discuss scattering of sound from low Reynolds number turbulence.

Since the Reynolds number may be considered the ratio of inertial forces to viscous forces and the Reynolds number in this case is low, the inertial term may be dropped from the Navier-Stokes equation leaving

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \quad (61)$$

Because the Mach number of the turbulent flow is restricted to small values, the density will be taken as a constant for the purpose of evaluating the turbulence energy spectrum. If the density is a constant, then the continuity equation reduces to

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (62)$$

Taking the divergence of equation (61) and keeping equation (62) in mind, it is seen that the pressure must satisfy the Laplace equation

$$\nabla^2 p = 0 \quad (63)$$

Now, the only solution to the Laplace equation which is finite throughout the whole space is a constant. Thus, the pressure must be independent of position, and from equation (61), the equation for $u_i$ becomes the diffusion equation,
\[ \frac{\partial u_i}{\partial t} = \nu \nabla^2 u_i \quad . \]  

Thus, each velocity component changes in time as does temperature in a solid, isotropic, homogeneous, heat-conducting medium. Next, equation (64) will be solved to determine \( u_i \) at some time \( t \) in terms of the initial distribution of \( u_i \) in space at \( t = 0 \). The first step will be to take the Laplace transform,

\[ \bar{u}_i(s) = \int_0^\infty u_i(t)e^{-st}dt \]

\[ u_i(t) = \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \bar{u}_i(s)e^{st}ds \quad , \]  

of equation (64), so that

\[ \nabla^2 \bar{u}_i(\vec{r},s) - \left( \frac{s}{\nu} \right) \bar{u}_i(\vec{r},s) = \frac{-1}{\nu} \bar{u}_i(\vec{r},t=0) \quad (66) \]

This is evaluated in the usual way (using a scalar Green's function) giving

\[ \bar{u}_i(\vec{r},s) = \int d^3\vec{r}' \left[ \frac{-e^{-s\sqrt{\nu}|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \right] \left[ \frac{-1}{\nu} \bar{u}_i(\vec{r}',t=0) \right] \quad (67) \]

Now, using equation (65) with equation (67), the velocity component \( u_i(\vec{r},t) \) may be written as

\[ u_i(\vec{r},t) = \int \frac{d^3\vec{r}'u_i(\vec{r}',0)}{4\pi\nu|\vec{r}-\vec{r}'|} \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} e^{-\sqrt{s/\nu}|\vec{r}-\vec{r}'|} e^{st}ds \quad \]  

The contour necessary for completing the right-hand integral is shown below:
Since there are no singularities within the contour, the total line integral is zero. In the limit as \( R \to \infty \) and \( r \to 0 \), the contributions along \( C_R \) and \( C_r \) approach zero leaving

\[
\int_{i\infty+\varepsilon}^{i\infty-\varepsilon} e^{s} |\frac{\mathbf{r}-\mathbf{r}'|}{\sqrt{\nu}}| \, e^{st} \, ds = 0 - \int_{0}^{\infty} e^{\frac{-|\mathbf{r}-\mathbf{r}'|}{\sqrt{\nu}}} \, e^{-\rho t} \, d\rho - \int_{0}^{\infty} e^{\frac{-|\mathbf{r}-\mathbf{r}'|}{\sqrt{\nu}}} \, e^{-\rho t} \, d\rho = 0 \quad (69)
\]

The two integrals on the right may now be easily completed to give the result that

\[
\int_{i\infty+\varepsilon}^{i\infty-\varepsilon} e^{s} |\frac{\mathbf{r}-\mathbf{r}'|}{\sqrt{\nu}}| \, e^{st} \, ds = \frac{i |\mathbf{r}-\mathbf{r}'|^2}{\nu t^{3/2}} e^{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4\nu t}} \quad (70)
\]

Combining equation (70) and equation (68), one has an expression for the fluid velocity component at some point \( \mathbf{r} \) and time \( t \) in terms of the initial velocity distribution at \( t = 0 \):

\[
u_i(\mathbf{r}, t) = \int d^3r' \frac{u_i(\mathbf{r}', t=0)}{(4\pi \nu t)^{3/2}} e^{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4\nu t}} \quad (71)
\]

This is the solution to the diffusion equation, equation (64), for an infinite medium in terms of an initial condition. Thus, the turbulence can be calculated if it is known at any instant.
However, only statistical characteristics are likely to be known at the initial instant, so that equation (71) is not useful in its present form. It can be used, though, to construct velocity correlations. In order to calculate these correlations, averaging must be done. Since the turbulence has been assumed to be statistically homogeneous, the averaging may be considered as a space average over a large volume of the turbulence:

\[<k(t)> = \frac{1}{V_t} \int \limits_{V_t} d^3 \bar{x} \ k(\bar{x},t)\]

(72)

A correlation which will be of interest is the double velocity correlation defined by

\[R(\bar{\xi},t) = <u(\bar{x},t)u(\bar{x}+\bar{\xi},t)>\]

(73)

that is, the average of the product of two parallel simultaneous velocity components at two points whose separation is represented by \(\bar{\xi}\). The vector \(\bar{\xi}\) has the projection \(\xi\) in the direction of the velocity components, and projections \(\eta\) and \(\zeta\) in two other orthogonal coordinate directions. Using the definitions given by equations (72) and (73), the double velocity correlation may be written as

\[R(\bar{\xi},t) = \frac{1}{V_t} \int \limits_{V_t} d^3 \bar{x} \ u(\bar{x},t)u(\bar{x}+\bar{\xi},t)\]

(74)

The integration may be carried out over all space since the velocity components are zero outside of the volume of turbulence. In this way, the shape of \(V_t\) does not affect the correlation. This is as one would expect if the volume of turbulence is large and the turbulence is statistically homogeneous.

Now, the result given by equation (71) is substituted into equation (74), giving,
\[
R(\xi, t) = \frac{1}{v_t} \int d^3\bar{x} \frac{1}{(4\pi vt)^{3/2}} \int d^3\bar{r}' u(\bar{r}', t=0) e^{-\frac{|x-x'|^2}{4vt}} \\
\times \frac{1}{(4\pi vt)^{3/2}} \int d^3\bar{r}'' u(\bar{r}'', t=0) e^{-\frac{|x+x'-x''|^2}{4vt}}
\]

\[
= \frac{1}{v_t} \frac{1}{(4\pi vt)^3} \int d^3\bar{x} \int d^3\bar{r}' \int d^3\bar{r}'' u(\bar{r}', t=0) u(\bar{r}'', t=0) \\
\times \int dx \int dy \int dz \exp \left\{ -\frac{1}{4vt} \left[ (x-x')^2 + (x+\xi-x'')^2 + (y-y')^2 + (y+\eta-y'')^2 \\
+ (z-z')^2 + (z+\zeta-z'')^2 \right] \right\}
\]

where \( \bar{x} \) has components \( x, y, z \) and \( \bar{\xi} \) has components \( \xi, \eta, \zeta \). Using the result that

\[
(x-x')^2 + (x+\xi-x'')^2 = 2 [(x-x')^2 + \xi-x'')^2 + \frac{(x'+\xi-x'')^2}{4}
\]

one may write

\[
\int dx \exp \left\{ -\frac{1}{4vt} [(x-x')^2 + (x+\xi-x'')^2] \right\}
\]

\[
= \exp \left\{ -\frac{1}{8vt} (x'+\xi-x'')^2 \right\} \int dx \exp \left\{ -\frac{1}{2vt} (x-x')^2 + \xi-x'' \right\}
\]

\[
= \sqrt{2\pi vt} \exp \left\{ -\frac{1}{8vt} (x'+\xi-x'')^2 \right\}.
\]

Therefore, relation (75) becomes

\[
R(\xi, t) = \frac{1}{v_t} \frac{1}{(4\pi vt)^3} \int d^3\bar{x} \int d^3\bar{r}'' u(\bar{r}', t=0) u(\bar{r}'', t=0)
\]
\[ x \cdot (2\pi vt)^{3/2} \exp \left\{ \frac{-1}{8vt} \left[ (x' + \xi - x'')^2 + (y' + \eta - y'')^2 + (z' + \zeta - z'')^2 \right] \right\} \] (78)

A new variable of integration may be defined for the right-hand integral with dummy variable \( \tilde{r}'' \); define \( \tilde{a} \) such that

\[ \tilde{r}'' = \tilde{a} + \tilde{r}', \quad x'' = x' + a, \quad y'' = y' + b, \quad z'' = z' + c. \] (79)

Then, equation (78) becomes

\[ R(\xi', t) = \frac{(2\pi vt)^{3/2}}{(4\pi vt)^3} \frac{1}{\sqrt{\text{v}_t}} \int d^3\xi' \int d^3\tilde{a} \ u(\xi', t=0)u(\tilde{r}'+\tilde{a}, t=0) \]

\[ \times \exp \left\{ \frac{-1}{8vt} \left[ (\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2 \right] \right\} \]

\[ = \frac{1}{(8\pi vt)^{3/2}} \int d\tilde{a} R(\tilde{a}, t=0) \exp \left\{ \frac{-1}{8vt} \left[ (\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2 \right] \right\} \]

\[ = \frac{1}{(8\pi vt)^{3/2}} e^{-|\tilde{\xi}|^2/8vt} \int da \int db \int dc \ R(a, b, c, 0) \exp \left\{ -\frac{2(2\xi a - 2b\eta - 2c\zeta + s^2)}{8vt} \right\} \] (80)

where \( |\tilde{\xi}|^2 = \xi^2 + \eta^2 + \zeta^2 \) and \( s^2 = a^2 + b^2 + c^2 \).

Now consider the asymptotic solution at large decay times; that is, times such that \( s_m^2/8vt \ll 1 \) where \( s_m \) is the largest value of \( s \) in which \( R(a, b, c, 0) \) still contributes to the integral. Then, the exponential term in the integral can be expanded in a Taylor series with only the lowest ordered terms being of significance:

\[ R(\xi', t) = e^{-|\tilde{\xi}|^2/8vt} \left\{ \int da \int db \int dc \ R(a, b, c, 0) \sum_{n=0}^{\infty} \frac{1}{n!} \frac{2(2\xi a - 2b\eta - 2c\zeta - s^2)}{8vt} \right\} \] (81)

Before proceeding, two relations must be established. The first is that
\[ \int db \int dc \, R(a,b,c,0) = \int d^3 x \, u(x,0) \int db \int dc \, u(x+a \hat{i} + b \hat{j} + c \hat{k},0) \]
\[ = 0 \quad (82) \]
since the net flow, due to the fluctuating velocity component \( u \), across a surface \( a = \text{constant} \) is zero. The second is that
\[ \int da \int db \int dc \, a^i b^j c^k \, R(a,b,c,0) = 0 \quad \text{when} \ i+j+k \text{ is an odd number.} \quad (83) \]
This follows because \( R(a,b,c,0) = R(-a,-b,-c,0) \).

From relation (82), it can be seen that the \( n = 0 \) term in the summation in equation (81) does not contribute. By using relation (83) for \( n = 1,2 \) in the expansion and keeping terms no smaller than of order \( s^2 / 8 v t \), one writes
\[ R(\xi,t) = \frac{e^{-|\xi|^2 / 8 v t}}{(8\pi vt)^{3/2}} \int da \int db \int dc \, R(a,b,c,0) \left[ -\frac{(a \xi + b \eta + c \zeta)^2}{2(4vt)^2} - \frac{s^2}{8vt} \right] \quad (84) \]
When the magnitude of \( |\xi| \) is set equal to zero in equation (84), \( R(\xi,t) \) reduces to \( < u^2(t) > \), and an expression for the decay of turbulence energy is obtained:
\[ < u^2(t) > = \frac{-\pi}{(8\pi vt)^{5/2}} \int dr_1 \int dr_2 \int dr_3 \, r_1^2 \, r_2^2 \, r_3^2 \, R(r_1,r_2,r_3,0) \quad (85) \]
When \( \eta = \zeta = 0 \) in equation (84), the vector separating the correlation points becomes parallel to the fluctuating velocity components and is of magnitude \( \xi \). In this case, the double velocity correlation degenerates to the longitudinal velocity correlation denoted by \( < u^2(t) > f(\xi,t) \). Equation (84), in this case,
becomes, with the aid of equation (82),

$$\langle u^2(t) \rangle = \chi(t) = \frac{-\pi e^{-\xi^2/8vt}}{(8\pi vt)^{5/2}} \int dr_1 \int dr_2 \int dr_3 \frac{r^2}{R(r_1, r_2, r_3, 0)}$$ (86)

Now, an expression for \( f(\xi, t) \) may be written from the last two equations:

$$f(\xi, t) = e^{-\xi^2/8vt}$$ (87)

At this point, the condition of statistical isotropy will be employed (only the homogeneity condition has been applied until now) to evaluate the integral in relation (85). For isotropic turbulence, the double velocity correlation can be expressed in terms of the velocity correlation tensor and, hence, in terms of the longitudinal velocity correlation coefficient \( f \):

$$R(\bar{r}, 0) = R_{11}(\bar{r}, 0)$$

$$= \langle u^2(0) \rangle \left[ f(r, 0) + \frac{r}{2} (1 - \frac{r^2}{r^2}) \frac{\partial f}{\partial r} (r, 0) \right]$$ (88)

The integral in equation (85) can then be integrated in spherical coordinates,

$$\langle u^2(t) \rangle = \frac{-\pi}{(8\pi vt)^{5/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} r \sin \theta \sin \phi \, r^2 \, \langle u^2(0) \rangle \left[ f(r, 0) + \frac{r}{2} \left( 1 - \cos^2 \theta \right) \frac{\partial f}{\partial r} (r, 0) \right]$$

$$= \frac{-2\pi^2}{(8\pi vt)^{5/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} r^4 \sin \theta \sin \phi \, <u^2(0)> \left[ f(r, 0) + \frac{r}{2} \sin^2 \theta \frac{\partial f}{\partial r} (r, 0) \right]$$
\[
\langle u^2(t) \rangle = \frac{48/3\pi^2}{(8\pi vt)^{5/2}} \langle u^2(0) \rangle \int_0^\infty dr \, r^4 \left[ f(r,0) + \frac{r}{3} \frac{\partial f}{\partial r}(r,0) \right]
\]

Denote the product \( \langle u^2(0) \rangle \int_0^\infty dr \, r^4 f(r,0) \) by \( L_0 \). \( L_0 \), as used here, is a constant (by definition) and is sometimes referred to as Loitsiansky's invariant. Using this notation, equation (89) becomes

\[
\langle u^2(t) \rangle = \frac{L_0}{48/2\pi (vt)^{5/2}}
\]

The results of the analysis of the double velocity correlation, equations (87) and (90), can now be used to find the energy spectrum of the turbulence. First, the trace of the velocity correlation tensor, \( R_{ii} \), will be expressed for homogeneous, isotropic turbulence in terms of \( \langle u^2(t) \rangle \) and \( f(r,t) \):

\[
R_{ii} = \langle u^2(t) \rangle \left[ 3 f(r,t) + \frac{r}{3} \frac{\partial}{\partial r} f(r,t) \right]
\]

\[
= \langle u^2(t) \rangle \, f(r,t) \left( 3 - \frac{r^2}{4vt} \right)
\]

Next (again from the theory of homogeneous, isotropic turbulence), one may write the energy spectrum function in terms of the spectrum tensor:

\[
E(k,t) := 2\pi k^2 \phi_{ii}(k,t) = 2\pi k^2 \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \, R_{ii}(r,t) e^{-i\mathbf{k} \cdot \mathbf{r}}
\]

Thus, in view of equations (87), (90), (91) and (92), the results of the double
velocity correlation analysis provide sufficient information to evaluate the turbulence energy spectrum.

By using spherical coordinates as shown below,

\[ \begin{array}{c}
\text{r}_3 \\
\text{r}_2 \\
\phi
\end{array} \]

equation (92) may be simplified:

\[
E(k,t) = \frac{k^2}{(2\pi)^2} \int_0^\infty \int_0^\pi \int_0^{2\pi} r \sin \theta drd\theta d\phi R_{ii}(r,t)e^{-ikr \cos \theta}
\]

\[
= \frac{k^2}{2\pi} \int_0^\infty dr \frac{r^2}{r} R_{ii}(r,t) \int_0^\pi \sin \theta dr d\phi e^{-ikr \cos \theta}
\]

\[
= \frac{k}{\pi} \int_0^\infty rdr \sin r R_{ii}(r,t)
\]

(93)

Now, equations (87), (90), and (91) are substituted, giving

\[
E(k,t) = \frac{k}{\pi} \left( \frac{L_0}{48\sqrt{2\pi} (vt)} \right)^{5/2} \int_0^\infty rdr \sin r e^{-r^2/4vt} (3 - r^2/4vt).
\]

(94)

Using the result that

\[
\int_0^\infty rdr \sin r e^{-r^2/\alpha^2} = \frac{\alpha^3 \sqrt{\pi k} e^{-k^2\alpha^2/4}}{4}.
\]

(95)

and

\[
\int_0^\infty r^3dr \sin r e^{-r^2/\alpha^2} = \frac{\alpha^4 e^{-k^2\alpha^2/4}}{4i} \left( 3/2i \alpha \sqrt{\pi} - \frac{ik^3 \alpha^3}{4\sqrt{\pi}} \right).
\]

(96)

the turbulence energy spectrum in equation (94) can be evaluated to give

\[
E(k,t) = \frac{L_0}{3\pi} k^4 e^{-2k^2vt}.
\]

(97)
The turbulence energy spectrum can also be written in terms of a "dissipation length parameter" denoted by \( \lambda_0 \) and defined by the expression

\[
\frac{d <u^2(t)>}{dt} = -(10\nu) \frac{<u^2(t)>}{\lambda_0^2} \quad (98)
\]

When the turbulence kinetic energy decay follows a power-law, that is, if \( <u^2(t)> \propto t^{-n} \), then

\[
\lambda_0^2 = \frac{10\nu t}{n} \quad (99)
\]

In the case of low Reynolds number, homogeneous, isotropic turbulence, which has just been considered, it is seen from equation (90) that there is a power low decay with \( n = \frac{5}{2} \). Thus, the dissipation length is

\[
\lambda_0 = 2\sqrt{\nu t} \quad (100)
\]

enabling the energy spectrum of the turbulence to be written in the form

\[
E(k) = E(k, \lambda_0) = \frac{L_0}{3\pi} k^4 e^{-k^2 \lambda_0^2/2} \quad (101)
\]

It should be mentioned that the analysis of low Reynolds number, homogeneous turbulence that has just been presented here was originally developed by Batchelor and Townsend [13]. They also verified the theoretical results by measuring the energy decay and longitudinal velocity correlation in the homogeneous, isotropic wake downstream of a fine-mesh grid.

Having completed the analysis of low Reynolds number, homogeneous, isotropic turbulence and found the turbulence energy spectrum, the power spectrum of sound waves scattered by this type of turbulence will be evaluated using equation (45):

\[
P(\omega, \theta) = \frac{L_0 k_0^2}{48\pi^2 c_0^2} \cos^2 \theta \cot^2 \theta/2 (k_d)^4 e^{-k_d^2 \lambda_0^2/2} \int_0^\infty dt \int d^3\vec{v}_f P(\vec{v}_f) \]

\[
\times e^{i(\omega_0 - \omega + k_d \cdot \vec{v}_f) \tau} \quad (102)
\]
Integrating over \( \tau \), one obtains

\[
P(\omega, \theta) = \frac{L_0 k^2}{24 \pi c_0^2} \cos^2 \theta \cot^2 \theta/2 \, k_d^3 e^{-k_d^2 \lambda^2/2} \int d^3 \vec{v}_f \, p(\vec{v}_f) \delta \left( \omega - \omega_0 - \frac{k_d}{k_d} \cdot \vec{v}_f \right) (103)
\]

Performing the integration over components of \( \vec{v}_f \) perpendicular to \( k_d \); finally, upon performing the integration over components of \( \vec{v}_f \) along the direction of \( k_d \), the integral is evaluated to give

\[
P(\omega, \theta) = \frac{L_0 k^2}{24 \pi c_0^2} \cos^2 \theta \cot^2 \theta/2 \, k_d^3 e^{-k_d^2 \lambda^2/2} \frac{\omega - \omega_0}{k_d} \frac{1}{p(\omega_0)} (104)
\]

where \( p \) is now the probability density of a velocity component parallel to \( k_d \).

As before, it will be taken as Gaussian, so that

\[
p(\omega_0) = \frac{1}{\sqrt{2\pi \langle u'^2 \rangle}} e^{-\frac{(\omega_0 - \omega_0')^2}{2 \langle u'^2 \rangle}} (105)
\]

where \( k_d = 2k_0\sin \theta/2 \) and \( V_0 \) is the mean velocity of the flow in the direction of \( k_d \). Combining the last two relations, one has

\[
P(\omega, \theta) = \frac{L_0 k^5}{3 \pi^2} \cos^2 \theta \cos^2 \theta/2 \sin \theta/2 e^{-2 \sin^2 \theta/2 k_0^2 \lambda_0^2}
\]

\[
\times \frac{1}{\sqrt{2\pi \langle u'^2 \rangle}} e^{-\frac{(\omega - \omega_0)^2}{2 \langle u'^2 \rangle}} \frac{1}{2k_0 \sin \theta/2} (106)
\]

Again, the maximum of the spectrum is

\[
\omega_{\max} = \omega_0 + 2V_0 k_0 \sin \theta/2 (107)
\]

and the change in frequency \( \Delta \omega \) from the maximum, \( \omega_{\max} \), which reduces the spectrum by a factor of \( 1/e \) is
\[ \Delta \omega = (2 \langle u' \rangle^2)^{1/2} (2k_0 \sin \theta / 2) \]  \hspace{1cm} (108)

If a comparison is made between the scattering of sound from low Reynolds number turbulence presented here and the results for high Reynolds number turbulence presented earlier, it is seen that the spectrum (\( \omega \) dependance) has a similar form. This is because the convective effects (frequency changing effects) of the macro-eddies and the mean velocity were treated in a similar way. However, the amount of energy received at a particular location of the observer (\( \theta \) dependance) differed greatly from the earlier case, since the turbulence energy function, \( E(k) \), was different.
IV. SCATTERING OF HIGH-FREQUENCY SOUND BY TURBULENCE

Another interesting case is the scattering of very high frequency sound. That is, the scattering of sound from a volume of turbulence where the wavelength of the incident field is very much smaller than the size of the energy containing eddies. Earlier, the power spectrum of the scattered waves was calculated when the incident field wavelength was small enough to be of the order of the size of the eddies in the inertial range of the turbulence. This size is also much smaller than that of the energy containing eddies, but the analysis was restricted to high Reynolds number, isotropic, homogeneous turbulence. In the calculation to follow, the total power of the scattered wave will be found, for the case of a high frequency incident wave, without restriction on the turbulence.

First, the intensity of the scattered energy, \( I_s \), will be defined as the average power of the scattered wave per unit area. According to this definition,

\[
I_s = \frac{c^3}{\rho_o} \langle \beta \beta \rangle
\]

Equation (30) with \( \tau = 0 \) may be used to evaluate the mean square of the density fluctuation so that

\[
I_s(\vec{r}) = \frac{\rho_o c^2}{\rho_o} \int d^3\vec{r}' \int d^3\vec{r}'' \left< u_i'(\vec{r}') u_i''(\vec{r}'') \right> \cos[(k_f - k_o) \cdot \vec{r}] \cos[(k_d - k_o) \cdot \vec{r}] \tag{110}
\]

In terms of the incident wave intensity,

\[
I_i = \frac{1}{2} \rho_o c^2 \tag{111}
\]

and defining

\[
F_{ij}(\vec{k}_d) = \frac{1}{(2\pi)^3} \int d^3\vec{r}'' \int d^3\vec{r} \cos(\vec{k}_d \cdot \vec{r}) \left< u_i(\vec{r}') u_j(\vec{r}'') \right> \tag{112}
\]
the intensity of the scattered wave becomes

\[ I_s = \frac{2\pi k^4}{c_o^2} \frac{x_1 x_1 x_1^2}{x_2} F_{ij}(\vec{k}_d) \]  

(113)

From the definition of \( \vec{k}_d \),

\[ (\vec{k}_d)_i = k_o \left( \frac{x_i}{r} - \delta_{il} \right), \]  

(114)

one can rearrange terms to find

\[ x_i = r \left( \frac{(k_d)_i}{k_o} + \delta_{il} \right) \]  

(115)

This last relation may be used in equation (113) so that

\[ I_s(\vec{r}) = \frac{2\pi k^4}{c_o^2} \frac{1}{r^2} \left[ \frac{(k_d)_i}{k_o} + 1 \right]^2 \left[ \frac{(k_d)_j}{k_o} + \delta_{ij} \right] F_{ij}(\vec{k}_d) \]  

(116)

The total power of the scattered wave is formed by integrating \( I_s(\vec{r}) \) over the surface of a large sphere with radius \( \vec{r} \) centered at the origin:

\[ P_t = \int I_s(\vec{r})d^2\vec{r} \]  

(117)

Using relation (115), the integration variable may be changed to \( \vec{k}_d \) so that

\[ P_t = \int I_s(k_d) \left[ \frac{r}{k_o} \right]^2 d^2\vec{k}_d \]  

\[ = \int d^2\vec{k}_d \frac{2\pi k^2}{c_o^2} \left[ \frac{(k_d)_i}{k_o} + 1 \right]^2 \left[ \frac{(k_d)_j}{k_o} + \delta_{ij} \right] F_{ij}(\vec{k}_d) \]  

(118)

where the area of integration is now, from equation (115), over a sphere in \( \vec{k}_d \)-space of radius \( k_o \) and with center at \( (-k_o', 0, 0) \).

Equation (112), which defines \( F_{ij} \), may be written as

\[ F_{ij}(\vec{k}) = R \frac{1}{(2\pi)^3} \int d^3\vec{r} e^{-i\vec{k}_d \cdot \vec{r}} \int d^3\vec{r}' \psi \psi_x u_i(\vec{r} + \vec{r}') u_j(\vec{r}') \]  

(119)

Expressed in this fashion, it can be seen that \( F_{ij}(\vec{k}_d) \) may be interpreted (from the Wiener theorem) as the real part of the spectrum of the volume.
integral
\[
\int d^3\bar{r}^\prime <u_i(\bar{r}^\prime)u_j(\bar{r}^\prime)> 
\] (120)
evaluated at \( \bar{k}_d \). With this interpretation in mind, it is obvious that \( F_{ij}(k_d) \) is small unless \( k_d \) is of the order of the reciprocal of a length scale associated with the turbulent eddies. If the incident sound wave frequency considered is high enough, then \( k_0 \) will be much larger than any value of \( k_d \) where \( F_{ij}(k_d) \) is significant. Physically, this means that an incident sound wave is being considered whose wave-length is much shorter than the size of the turbulent eddies. A picture showing the sphere of integration of equation (118) in \( \bar{k}_d \) space and the region in this space where \( F_{ij}(\bar{k}_d) \) is significant is given below:

The integration in equation (118) may be restricted to that part of the spherical surface where \( F_{ij}(\bar{k}_d) \) is significant. A good approximation to this part of the surface is the area of the plane \( (k_d)_1 = 0 \) where \( k_d \) is small. Using this approximation, equation (118) may be simplified to
\[
P_t = \int d(k_d)_2 \int d(k_d)_3 \frac{2\pi l_1 k^2}{c_0^2} F_{11}(0,(k_d)_2,(k_d)_3) 
\] (121)
The integral in equation (121) will be evaluated by returning to the definition of \( F_{ij} \) given by equation (119) and performing the following manipulations:
\[ F_{ij}(\vec{k}_d) = \int d^3\vec{r}'' \frac{1}{(2\pi)^3} \int d^3\vec{r} R e^{-i\vec{k}_d \cdot \vec{r}} R_{ij}(\vec{r}'', \vec{r}) \]

\[ \int d(k_d)_2^2 \int d(k_d)_3^3 F_{ij}(\vec{k}_d) = \int d^3\vec{r}'' \frac{1}{(2\pi)^3} \int d^3\vec{r} R_{ij}(\vec{r}'', \vec{r}) R e \int d(k_d)_2^2 \int d(k_d)_3^3 e^{-i\vec{k}_d \cdot \vec{r}} \]

\[ = \int d^3\vec{r}'' \frac{1}{2\pi} \int d^3\vec{r} R_{ij}(\vec{r}'', \vec{r}) \delta(\zeta_1) \delta(\zeta_2) R e^{-i(k_d)_1 \zeta_1} \]

\[ \int d(k_d)_2^2 \int d(k_d)_3^3 F_{11}(0, (k_d)_2^2, (k_d)_3^3) = \frac{1}{2\pi} \int d^3\vec{r}'' \int d\zeta_1 R_{ij}(\vec{r}'', \zeta_1, 0, 0) \quad (122) \]

Now (121) may be evaluated giving

\[ P_t = \frac{I_1 k_0^2}{c_0^2} \int d^3\vec{r}'' \int d\zeta_1 R_{ij}(\vec{r}'', \zeta_1, 0, 0) \quad (123) \]

In the case of homogeneous (but not necessarily isotropic) turbulence, equation (123) reduces to

\[ P_t = \frac{I_1 k_0^2}{c_0^2} \int \left[ \right. \int d\zeta_1 R_{ij}(\zeta_1) \quad (124) \]

Defining Taylor's "macro-scale of turbulence" by

\[ L_1 = \frac{1}{< u''^2 >} \int \left[ \right. d\zeta_1 R_{ij}(\zeta_1, 0, 0), \quad (125) \]

the power scattered (in all directions) by a volume of turbulence, denoted by \( P_{tv} \), becomes

\[ P_{tv} = \frac{2I_1 k_0^2 < u''^2 >}{c_0^2} L_1. \quad (126) \]
V. SOUND SCATTERING BY THE INHOMOGENEOUS WAKE OF A SELF-PROPELLED BODY

A. Extension of Scattering Theory to Inhomogeneous Turbulence

In the scattering problems that have been treated so far where the power spectrum of the scattered waves was evaluated explicitly, the turbulence was assumed to be completely homogeneous and isotropic throughout the turbulence volume. However, the turbulence found in wakes, jets and other cases of practical interest is both inhomogeneous and anisotropic. The flow and statistical characteristics change with position, so that it is inhomogeneous; and they vary with orientation, so that it is anisotropic. Thus the results of the analysis for homogeneous, isotropic turbulence cannot strictly be applied to these cases (though such solutions might be regarded as being a first approximation). To obtain an accurate solution for the more realistic inhomogeneous cases, therefore, the analysis for the homogeneous, isotropic case must be extended.

In order to do this, certain assumptions about the physical picture of statistically inhomogeneous turbulence, particularly wake turbulence, can be made. The first of these is that the variation of flow characteristics (such as mean velocity, turbulence intensity, etc.) change slowly with position and are not abrupt. For the types of inhomogeneous turbulence in which it might be important to understand how sound is scattered, for example, wake turbulence, this assumption is physically reasonable. The second assumption is based on the concept expressed in the discussion of universal equilibrium theory. Essentially it is the idea that only the largest eddies (lowest wave-numbers) are excited directly in the initial shear regime. All the higher wave-numbers receive their energy indirectly through the action of
inertial forces and, therefore, their random fluctuations tend to be statistically isotropic. It was also shown that the scattered waves depend mostly on the eddies whose size is approximately that of the wave-length of the incident sound or there is negligible scattering. So if the wave-number of the incident sound wave $k_0$ is not in the comparably short range of the very lowest wave-numbers, the turbulence fluctuations may be considered statistically isotropic for the purpose of scattering theory. Incidentally, it should be noted that if $k_0$ is in the range where anisotropic effects are concentrated, the assumption of statistical isotropy is not so inaccurate that the results are invalid. The anisotropic effects in fully developed turbulent wakes are not large so that an assumption of isotropy would be a good approximation. It can be said, however, that if $k_0$ is in the range of high wave-numbers of the turbulence fluctuations, the isotropy assumption is strictly correct.

With these two conditions in mind, sound scattering from an inhomogeneous wake may be treated by dividing the turbulence volume, $V_t$, into subvolumes, $\Delta V_i$, of a size much larger than that of the scattering eddies yet small enough so that the turbulence is essentially homogeneous and the mean velocity uniform within each subvolume. Using also the statistical isotropy assumption, it is seen that each subvolume $\Delta V_i$ within the inhomogeneous wake satisfied the conditions of complete homogeneity and isotropy. Therefore, the results of the analysis based on these restrictions, given by equation (45), applies to each $\Delta V_i$. It is not inconsistent with the single-scattering theory being used that each part of the turbulence sees the incident wave as if it was not interfered with by the rest of the turbulence. So the sound-turbulence interaction in each subvolume is quite independent of that in any other. Also, since the scattered waves from each $\Delta V_i$ are uncorrelated with that
from any other, their intensities may be added. Therefore, the power spectrum of the differential scattering cross-section of waves scattered by inhomogeneous turbulence, denoted by $P_t$ will be

$$P_t = \frac{1}{V_t} \left[ \frac{k^2}{8\pi c_o^2} \cos^2 \Theta \cot^2 \Theta / 2 \right] E_i \int_{-\infty}^{\infty} d\tau \int d^3 \bar{V}_f P_i(\bar{V}_f) e^{i(\omega - \omega_0 + k_d \cdot \bar{V}_f)\tau} \Delta V_i$$ \hspace{1cm} (127)

where the subscript $i$ means that the corresponding quantity is evaluated at the position of $\Delta V_i$. In this way, the variation of flow characteristics within the volume of inhomogeneous turbulence is taken into account.

Often, the changes with position of the energy spectrum function and the probability density of the fluid velocity fluctuations are known in the form of continuous functions of position with respect to some appropriate coordinate system. In this case, there will be little loss of accuracy, mathematically, if the summation is allowed to go over to a volume integral:

$$P_t = \frac{1}{V_t} \int d^3 \bar{\xi} \frac{k^2}{16\pi c_o^2} \cos^2 \Theta \cot^2 \Theta / 2 \int_{-\infty}^{\infty} d\tau \int d^3 \bar{V}_f P(\bar{V}_f, \bar{\xi}) e^{i(\omega - \omega_0 + k_d \cdot \bar{V}_f)\tau}$$

$$= \frac{1}{V_t} \int d^3 \bar{\xi} \frac{k^2}{8c_o^2 k_d} \cos^2 \Theta \cot^2 \Theta / 2 \int d^3 V_f P(\bar{V}_f, \bar{\xi}) \delta(\omega - \omega_0 - k_d \cdot \bar{V}_f)$$

$$= \frac{1}{V_t} \int d^3 \bar{\xi} \frac{k^2}{8c_o^2 k_d} \cos^2 \Theta \cot^2 \Theta / 2 \int d^3 V_f P(\bar{V}_f, \bar{\xi}) \frac{\omega - \omega_0 - k_d \cdot \bar{V}_f}{k_d}$$ \hspace{1cm} (128)

where $p$ is now the probability density, at position $\bar{\xi}$, as the component of velocity in the direction of $k_d$ evaluated at $\frac{\omega - \omega_0 - \bar{V}_o(\bar{\xi})}{k_d}$. Again, from the results of experiment, this will be taken to be Gaussian so that

$$P_t = \frac{k^2}{8c_o^2 k_d V_t} \cos^2 \Theta \cot^2 \Theta / 2 \left[ d^3 \bar{\xi} E(k_d, \bar{\xi}) \frac{1}{[2\pi < u^2(\bar{\xi})>]^2} \exp - \frac{[\frac{\omega - \omega_0 - \bar{V}_o(\bar{\xi})}{k_d}]^2}{2 < u^2(\bar{\xi})>} \right]$$ \hspace{1cm} (129)

where $\bar{V}_o(\bar{\xi})$ is the mean velocity in the direction of $k_d$ and $< u^2(\bar{\xi})>$ is the mean square of the velocity fluctuation at point $\bar{\xi}$. 
B. Application to the Wake of a Self-Propelled Body

A general expression has been derived, equation (129), for the power spectrum of the differential scattering cross section of acoustic waves scattered from inhomogeneous turbulence. This result will now be applied to the problem of the scattering of plane sound waves from the inhomogeneous turbulent wake of a totally immersed, axisymmetric, self-propelled body. The problem geometry is shown in Fig. 2. The self-propelled body is at the origin and is traveling along the $\xi_1$ axis as shown. Since the wake will have axisymmetric symmetry, the volume integration in equation (129), when applied to this case, is most easily done in cylindrical coordinates, $r$, $\phi$ and $z$, where $z$ is measured along the $\xi_1$ axis. Equation (129) then becomes

$$
P_t = \frac{\pi k^2 \cos^2 \theta \cot^2 \theta / 2}{4 c^2 V_o t} \int_0^{\infty} \int_0^{\infty} \frac{dz}{r^2} \frac{1}{2\pi <u^2(r,z)>} \exp \left\{ -\frac{\omega - \omega_0 - V_o(r,z)}{2 <u^2(r,z)>} \right\} \eta \right)
$$

In the case to be considered here, the wake will be of high Reynolds number. As discussed previously, the Kolmogoroff spectrum may be applied in this instance,

$$
E(k_d , r, z) = K[\epsilon(r,z)]^{2/3} k_d^{-5/3}
$$

so that

$$
P_t = \frac{\pi k^2 \cos^2 \theta \cot^2 \theta / 2}{4 / 2\pi \eta c^2 V_o t (k_d)} \frac{8}{3} \int_0^{\infty} \int_0^{\infty} \frac{dz}{r^2} \frac{1}{2 <u^2(r,z)>} \eta \exp \left\{ -\frac{\omega - \omega_0 - V_o(r,z)}{2 <u^2(r,z)>} \right\}
$$

In the above equation the term $V_o(r,z)$ is the component of the mean velocity in the direction of $\mathbf{k}_d$. In the wake, however, the mean velocity is very nearly parallel with the direction of the self-propelled body. One can then write:
GEOMETRY OF THE WAKE

FIGURE 2.
\[ v_0(r,z) = \frac{k_d}{k_d} \cdot \hat{e}_z U_z(r,z) \]

\[ = \frac{1}{2 \sin \theta/2} \left[ \frac{k_f}{k_o} - \frac{k_o}{k_o} \cdot \hat{e}_z \right] U_z(r,z) \]

\[ = \frac{1}{2 \sin \theta/2} [\cos \psi - \cos \phi] U_z(r,z) \]

\[ = D(\Theta, \psi, \phi) U_z(r,z) \quad (133) \]

where \( U_z(r,z) \) is the mean velocity of the fluid in the wake parallel to the wake axis.

Thus, it is seen that in order to evaluate explicitly the power spectrum of the scattered waves, it is necessary to analyze the fluid motion itself within the wake. In particular, the dissipation \( \varepsilon \), the turbulence intensity \( \langle u^2 \rangle \), and the axial mean velocity \( U_z \) must be known functions of position.

C. Determination of the Wake Flow Parameters

The analysis will begin by considering the continuity equation and the Navier-Stokes equation in the wake flow. The fluid velocity will be the sum of the mean velocity, \( \bar{U} \), and the fluctuating turbulent velocity, \( u \).

The pressure, also, will have a mean part \( \bar{P} \) and a fluctuating part \( P \). The coordinate system used here will be moving with the self-propelled body which is always at the origin and is heading in the negative z direction.

A sketch of the geometry is shown below:
The fluid flow, for the purpose of finding the needed wake characteristics, is assumed incompressible:

\[ \frac{\partial}{\partial x_j} (U_j + u_j) = 0 \]  

(134)

Taking an average of the above equation, one obtains

\[ \frac{\partial}{\partial x_j} U_j = 0 \]  

(135)

By comparing the last two relations, it is obvious that

\[ \frac{\partial u_i}{\partial x_j} = 0 \]  

(136)

The Navier-Stokes equation for incompressible flow is

\[ \frac{\partial}{\partial t} (U_i + u_i) + (U_j + u_j) \frac{\partial}{\partial x_j} (U_i + u_i) = - \frac{1}{\rho} \frac{\partial}{\partial x_i} (F + p) + \nu \frac{\partial^2}{\partial x_j^2} (U_i + u_i). \]  

(137)

If an average is taken and the mean flow is steady state, the above relation becomes

\[ U_i \frac{\partial}{\partial x_i} U_j = - \langle u_j \frac{\partial}{\partial x_j} u_i \rangle - \frac{1}{\rho} \frac{\partial}{\partial x_i} p + \nu \frac{\partial^2}{\partial x_j^2} U_i \]  

(138)

The two results given by equations (135) and (138) will now be written in cylindrical coordinates for the case where there is no mean velocity in the \(\Theta\)-direction or change of mean quantities in the \(\Theta\)-direction. For continuity, one writes

\[ \frac{1}{r} \frac{\partial}{\partial r} (rU_r) + \frac{\partial}{\partial z} U_z = 0. \]  

(139)

The equation of motion becomes

\[ U_r \frac{\partial}{\partial r} U_r + U_z \frac{\partial}{\partial z} U_r = - \frac{1}{r \partial r} \langle ru^2 \rangle - \frac{\partial}{\partial z} \langle u u \rangle + \frac{\partial^2}{r} \frac{\partial}{\partial r} U_r + \frac{1}{r} \frac{\partial}{\partial r} U_r + \frac{1}{r} \frac{\partial}{\partial r} U_z \]  

in the \(r\)-direction,
\[ 0 = \frac{\partial}{\partial r} \langle u_r u_\theta \rangle + \frac{\partial}{\partial z} \langle u_z u_\theta \rangle + \frac{2}{r} \langle u_\theta u_r \rangle \]  
(141)

in the \( \theta \)-direction, and

\[ \frac{u_z}{r} \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = - \frac{1}{r} \frac{\partial}{\partial r} r \langle u_r u_z \rangle - \frac{3}{\rho} \frac{\partial \rho}{\partial z} + \nu \left( \frac{\partial^2 u_z}{\partial r^2} \right) + \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial u_z}{\partial z} + \frac{3}{r^2} \frac{\partial^2 u_z}{\partial z^2} \]  
(142)

in the \( z \)-direction.

These equations will now be simplified by discarding small terms of negligible value. First of all, the Reynolds number has been assumed to be high, so terms representing the effects of molecular viscosity are neglected. The relative magnitude of the remaining terms will be determined by using the fact that the wake is long and narrow and the flow is nearly parallel to the \( z \)-axis. This implies that quantities change with position more rapidly in the \( r \)-direction than in the \( z \)-direction. This idea may be expressed mathematically by

\[ \frac{\partial}{\partial z} < \frac{\partial}{\partial r} \]  
(143)

Length and velocity scales will now be defined which will be used to indicate relative magnitude:

- \( U_o \) - velocity scale for uniform mean flow in the \( z \)-direction
- \( \bar{U}_z \) - velocity scale for velocity difference, \( U_z - U_o \), in the \( z \)-direction
- \( \bar{U}_r \) - velocity scale for velocity difference in the \( r \)-direction
- \( L_z \) - length scale in the \( z \)-direction
- \( L_r \) - length scale in the \( r \)-direction
- \( v^2 \) - velocity scale for the mean square turbulent fluctuation
- \( P_r \) - pressure scale for pressure changes in the \( r \)-direction  
(144)
The uniform mean flow, $U_0$, is much larger than velocity differences so that

$$U_0 \gg \bar{u}_z, \bar{u}_r$$  \hfill (145)

Equation (143) implies, in terms of the length scales,

$$\frac{1}{L_z} \ll \frac{1}{L_r}$$

$$\frac{L_z}{L_r} \gg 1.$$  \hfill (146)

From the continuity relation, equation (139), one can write

$$\frac{\bar{u}_z}{L_z} \approx \frac{U_r}{L_r}$$

$$\frac{\bar{u}_z}{U_r} \approx \frac{L_z}{L_r} >> 1.$$  \hfill (147)

And also, the turbulent velocity fluctuations will be at least of the order of differences in the mean axial velocity so that

$$\bar{v} \approx \bar{u}_z$$  \hfill (148)

Equations (140), (141), and (142), representing the equation of motion, will now be rewritten omitting the terms due to viscous stress. Below each equation, the order of magnitude of the terms will be indicated by the appropriate length, velocity, and pressure scales. In the line below this, a simplified version of the equation will be found by eliminating the smaller terms according to equations (145)-(148). Proceeding in this manner, one writes for the $\theta$-direction

$$\frac{\partial}{\partial \theta} \left( u_{\theta} \bar{u}_r \right) + \frac{\partial}{\partial z} \left( u_{\theta} \bar{u}_z \right) + \frac{2}{r} \left( u_{\theta} \bar{u}_r \right) = 0$$

$$\frac{v^2}{L_r} \quad \frac{v^2}{L_z} \quad \frac{v^2}{L_r}$$
\[
\frac{\partial}{\partial r} \langle u_\theta u_r \rangle + \frac{2}{r} \langle u_\theta u_r \rangle = 0
\]

\[
\langle u_\theta u_r \rangle \propto \frac{1}{r^2}
\]  

(149)

for the \( r \)-direction

\[
0 = -\frac{\partial}{\partial r} \langle u_r^2 \rangle - \frac{1}{\rho} \frac{\partial p}{\partial r}
\]

\[
\frac{1}{\rho} \int_0^R \frac{\partial p}{\partial r} \, dr + \int_0^R \frac{\partial}{\partial r} \langle u_r^2 \rangle \, dr = 0
\]

\[
p/\rho + \langle u_r^2 \rangle = p/\rho
\]

\[-1/\rho \frac{\partial p}{\partial z} = \frac{\alpha}{\beta} \langle u_z^2 \rangle \]

and for the \( z \)-direction (using the above result)

\[
U_z \frac{\partial u_z}{\partial r} + U_z \frac{\partial u_z}{\partial z} = \frac{3}{3} \langle u_z^2 \rangle - \frac{1}{\rho} \frac{\partial}{\partial r} \langle u_r u_z \rangle
\]

\[
\frac{\tilde{U} \tilde{U}_z}{L_z} + \frac{\tilde{U} \tilde{U}_z}{L_z} = 0 \quad \frac{v^2}{L_z}
\]

\[
U_z \frac{\partial u_z}{\partial z} + 1/r \frac{\partial}{\partial r} \langle u_r u_z \rangle = 0
\]  

(151)

Since the velocity difference, \( U_o - U_z \), is small to \( U_z \), there will be little additional error if equation (151) is further simplified as

\[
U_o \frac{\partial u_z}{\partial z} + 1/r \frac{\partial}{\partial r} \langle u_r u_z \rangle = 0
\]  

(152)
A self-preservation hypothesis will now be employed to find the self-preservation solution. This means, essentially, that quantities will be non-dimensionalized by scale velocities and scale lengths. The relations among these non-dimensionalized terms is then invariant along the wake axis, though the scale lengths and velocities themselves vary in the z-direction.

Proceeding in this manner, the following definitions are made:

\[ \lambda = \lambda(z) = \text{scale length}, \quad U_s = \left| U_{z_o} - U \right|_{\text{max}} = \text{scale velocity} \]

\[ \frac{U_{z_o} - U}{U_s} = f(r/\lambda) = f(\xi), \quad -\frac{<u_1 u_2 >}{U_s^2} = g(r/\lambda) = g(\xi) \quad (153) \]

Also, a turbulent eddy viscosity, \( \nu_T \), will be defined by the relation

\[ -<u_1 u_2 > = \nu_T \frac{\partial U_z}{\partial r} \quad (154) \]

so that

\[ \nu_T = \frac{\lambda U_s}{f'} g \quad (155) \]

and will be assumed to be not a function of \( r \).

Using this last definition in equation (152), one can write

\[ \frac{\partial}{\partial z} U_{z_o} (U_{z_o} - U) = \frac{1}{r} \frac{\partial}{\partial r} r \nu_T \frac{\partial}{\partial r} (U_{z_o} - U) \quad (156) \]

Multiplying the above relation by \( r^n \) and integrating, with respect to \( r \), from 0 to \( \infty \) one obtains:

\[ \int_0^\infty dr \left\{ \frac{\partial}{\partial z} r^n U_{z_o} (U_{z_o} - U) \right\} = \nu_T \int_0^\infty r^{n-1} \frac{\partial}{\partial r} \left( r^n U_{z_o} - U \right) \]

\[ \frac{\partial}{\partial z} \int_0^\infty r^n U_{z_o} (U_{z_o} - U) dr = -\nu_T (n-1) \int_0^\infty r^{n-1} \frac{\partial}{\partial r} (U_{z_o} - U) dr \]

\[ = \frac{\nu_T}{2\pi \rho} (n-1)^2 \int_0^\infty r^{n-3} \rho (U_{z_o} - U) 2\pi r dr \quad (157) \]
Now if \( n = 3 \) and the conservation of mass of the fluid flow is considered, equation (157) becomes

\[
\frac{d}{dz} \int_{0}^{\infty} r^3 U_o (U-U_o)dr = 0 \quad (158)
\]

This means that the integral, which can be expressed in terms of the definitions (153), is a constant:

\[
U_o \xi^3 \int_{0}^{\infty} \xi^3 f(\xi) d\xi = \text{constant} \quad (159)
\]

Equation (152), the simplified equation of motion, will now be written in terms of the self-preservation solutions, defined by the relations (153), to obtain

\[
-\frac{U_o}{U_s^2} \frac{dU_s}{dz} + \frac{U_o}{U_s} \frac{dl}{dz} \xi f' = g' + g/\xi \quad (160)
\]

This equation must hold for any value of \( z \). In order for this to be true, the following two relations must be satisfied:

\[
\frac{1}{U_s^2} \frac{dU_s}{dz} = \text{constant}, \quad \frac{1}{U_s} \frac{dl}{dz} = \text{constant} \quad (161)
\]

These will be satisfied if

\[
\xi = z^n \quad \text{and} \quad U_s = z^{n-1} \quad (162)
\]

The value of \( n \) is obtained by substitution into equation (159) and using the fact that the exponent of \( z \) must be zero:

\[
(n-1) + 4n = 0, \quad n = 1/5 \quad (163)
\]

Using this value, the length and velocity scales become [14]

\[
U_s = A z^{1/5}, \quad \xi = B z^{1/5} \quad (164)
\]
The above result along with equation (155) are then substituted into equation (160) to obtain
\[
\frac{U_B}{5A} \left( \frac{2U}{V_T} \right) (4f + \xi f') + f'' + f'/\xi = 0 \tag{165}
\]

The \(z\)-dependence of \(v_T\) is determined since the term in \([\ ]\) cannot be a function of \(z\) for equation (165) to hold at arbitrary values of \(z\). The coefficient in the length scale, \(B\), can be defined by requiring the term in \([\ ]\) to be unity. Thus, the equation which \(f(\xi)\) must satisfy becomes
\[
(4f + \xi f') + f'' + f'/\xi = 0 \tag{166}
\]
The solution to this equation is
\[
f = (1/2\xi^2 - 1)e^{-\xi^2/2} \tag{167}
\]
so that the mean velocity in the \(z\)-direction becomes
\[
U_z = U_o + U_s [1 - \frac{1}{2} \left( r/l \right)^2] e^{-\xi^2/2} \tag{168}
\]
If the reference frame is fixed with respect to the undisturbed fluid (keeping the same orientation, however, and considering the instant of time that the self-propelled body is at the origin), the component of mean velocity in the \(z\)-direction is
\[
U_z = U_s [1 - \frac{1}{2} \left( r/l \right)^2] e^{-\xi^2/2} \tag{169}
\]
Besides the mean velocity, in view of equation (132), the spatial variation of the turbulence intensity, \(<u^2(r,z)\>\), must also be known in order to evaluate explicitly the total power spectrum of the scattered waves. The dissipation \(\varepsilon\), which is needed in addition, will be found with this information along with the length and velocity scales already developed.

The first step in the analysis is to write the turbulence energy equation. The rate of work done on a volume of fluid is the scalar product of the traction and the velocity at the surface integrated over the surface,
\[ \int_{\partial S} \mathbf{T} \cdot d\mathbf{s} = \int_{S} \mathbf{u} \cdot \mathbf{\sigma}_{ij} n_j \, dS = \int_{V} \frac{\partial}{\partial x_j} \mathbf{\sigma}_{ji} u_i \, dv, \]  

(170)

so that the rate of work done on a fluid element is

\[ \frac{\partial}{\partial x_j} \mathbf{\sigma}_{ji} u_i = u_i \frac{\partial}{\partial x_j} \mathbf{\sigma}_{ji} + \mathbf{\sigma}_{ji} \frac{\partial}{\partial x_j} u_i \]  

(171)

The first term on the right-hand side of the above equation is related to the increase in kinetic energy. Using the equation of motion, one can write

\[ \rho/2 \frac{d}{dt} (u_i u_i) = u_i \frac{\partial}{\partial x_j} \mathbf{\sigma}_{ji}. \]  

(172)

Equation (171) then becomes

\[ \frac{\partial}{\partial x_j} \mathbf{\sigma}_{ji} u_i = \rho/2 \frac{d}{dt} (u_i u_i) + \mathbf{\sigma}_{ji} \frac{\partial}{\partial x_j} u_i. \]  

(173)

The fluid will be considered incompressible,

\[ \frac{\partial u_i}{\partial x_i} = 0, \]  

(174)

and viscous, so the stress tensor can be written as

\[ \mathbf{\sigma}_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(175)

Combining the last three relations, one writes

\[ \frac{\partial}{\partial x_j} [-p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)] u_i = \rho/2 \frac{d}{dt} u_i u_i + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} \]  

(176)

If there is a mean velocity, as there will be in this case, the turbulent fluid velocity will be the sum of the mean and fluctuating parts so that

\[ \frac{\partial}{\partial x_j} [-\bar{p} + p] \delta_{ij} + \mu \left[ \frac{\partial}{\partial x_j} (u_i + u_{i}^*) + \frac{\partial}{\partial x_i} (u_j + u_{j}^*) \right] (u_i + u_{i}^*) = \]  

\[ = \rho/2 \frac{d}{dt} (u_i + u_{i}^*) (u_i + u_{i}^*) + \mu \left[ \frac{\partial}{\partial x_j} (u_i + u_{i}^*) + \frac{\partial}{\partial x_i} (u_j + u_{j}^*) \right] \frac{\partial}{\partial x_j} (u_i + u_{i}^*) \]  

(177)
where $U_1$ is the mean velocity and $u_1$ now denotes fluctuating part which has zero mean. After taking an average and rearranging terms, equation (177) becomes

$$
\frac{1}{2} \frac{\partial}{\partial t} U_{ij} U_{ij} + \frac{1}{2} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u_1 \left( \frac{p}{\rho} + \frac{1}{2} U_{ij} U_{ij} \right) + \frac{\partial}{\partial x_i} U_{ij} \left( \frac{1}{\rho} \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)
$$

\[ \begin{align*}
- \nu(\frac{\partial}{\partial x_j} U_i + \frac{\partial}{\partial x_i} U_j) \frac{\partial}{\partial x_j} U_j - \frac{\partial}{\partial x_i} u_1 \left( \frac{p}{\rho} + \frac{1}{2} U_{ij} U_{ij} \right) \\
- \frac{\partial}{\partial x_i} U_j u_1 u_j - \frac{1}{2} \frac{\partial}{\partial x_j} U_i U_{ij} + \nu \frac{\partial}{\partial x_i} U_j \left( \frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j \right) \\
- \nu \left( \frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j \right) \frac{3}{\partial x_i} U_j \end{align*} \]

(178)

The above equation can be simplified by use of the Navier-Stokes equation, expression (137), and the incompressibility conditions, equations (135) and (136). Averaging equation (137), one obtains

$$
\frac{\partial}{\partial t} U_i + \frac{\partial}{\partial x_j} U_{ij} = - \frac{1}{\rho} \frac{\partial}{\partial x_i} p + \nu \frac{\partial^2}{\partial x_j} U_i - \frac{\partial}{\partial x_j} u_1 u_i 
$$

(179)

After multiplying through by $U_i$ and rearranging the terms, equation (179) becomes

$$
\frac{1}{2} \frac{\partial}{\partial t} U_{ij} U_{ij} = - \frac{\partial}{\partial x_i} u_1 \left( \frac{p}{\rho} + \frac{1}{2} U_{ij} U_{ij} \right) + U_i \nu \frac{\partial}{\partial x_j} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - u_1 \frac{\partial}{\partial x_j} u_1 u_j 
$$

(180)

This is subtracted from equation (178) to obtain

$$
\frac{1}{2} \frac{\partial}{\partial t} <u_1 u_j> + \frac{1}{2} U_i \frac{\partial}{\partial x_i} <u_1 u_j> = - \frac{\partial}{\partial x_i} u_1 \left( \frac{p}{\rho} + \frac{1}{2} u_j u_j \right) - u_1 \frac{\partial}{\partial x_j} u_1 u_j \\
+ \nu \frac{\partial}{\partial x_i} <u_j \frac{\partial u_i}{\partial x_j}> + \frac{1}{2} \nu \frac{\partial^2}{\partial x_i} <u_j u_j> \\
- \nu \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} - \nu \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} 
$$

(181)
Equation (181) is the turbulence energy equation that had been mentioned as being necessary in determining the turbulence intensity. However, since the wake is axially symmetric, it will be more useful if expressed in the cylindrical coordinate system used in finding the mean velocity. Therefore, equation (181) is written in cylindrical coordinates for the case of steady, axially symmetric (with no spiral component to the mean velocity), incompressible, turbulent flow:

\[
\frac{1}{2} \left[ \frac{\partial \langle q^2 \rangle}{\partial x} + u_z \frac{\partial \langle q^2 \rangle}{\partial z} \right] = -\frac{1}{\rho} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \langle u_t \rangle \right) + \frac{\partial \langle u_t \rangle}{\partial z} \right] - \frac{1}{2} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \langle u_t q^2 \rangle + \frac{\partial \langle u_t q^2 \rangle}{\partial z} \right) \right] \\

- \left[ \langle u_x u_x \rangle \frac{\partial u_x}{\partial x} + \langle u_x u_z \rangle \left( \frac{\partial u_z}{\partial x} + \frac{\partial u_z}{\partial z} \right) + \langle u_x u_\theta \rangle \frac{u_x}{r} + \langle u_z u_z \rangle \frac{\partial u_z}{\partial z} \right] \\

+ v/r \left\{ \frac{\partial}{\partial z} \left[ \frac{\partial \langle q^2 \rangle}{\partial z} \right] + \frac{\partial \langle u_z \rangle}{\partial z} \right\} + \frac{\partial}{\partial r} \left( \frac{\partial \langle q^2 \rangle}{\partial r} \right) + \frac{\partial \langle u_x \rangle}{\partial r} \left( \frac{\partial \langle u_x \rangle}{\partial r} \right) \\

+ \frac{1}{r} \frac{\partial}{\partial r} \left( \langle u_x^2 \rangle \right) - \frac{\langle u_x^2 \rangle}{r} \} - \varepsilon/\rho
\]

where

\[ q^2 = u_x^2 + u_\theta^2 + u_z^2 \]

and

\[
\varepsilon = \mu \left\{ 2 \langle (-\frac{r}{\partial})^2 \rangle + \langle \left[ \frac{\partial u_x}{\partial r} - \frac{u_\theta}{r} \right]^2 \rangle + \langle \left[ \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \right]^2 \rangle + \langle \left[ \frac{\partial u_x}{\partial \theta} + \frac{u_\theta}{r} \right]^2 \rangle + \langle \left[ \frac{\partial u_z}{\partial \theta} + \frac{u_\theta}{r} \right]^2 \rangle \right\} + \frac{2}{r^2} \langle (\frac{\partial u_\theta}{\partial \theta} + u_\theta)^2 \rangle + \langle \left[ \frac{\partial u_x}{\partial z} + \frac{u_\theta}{r} \right]^2 \rangle + 2 \langle \left[ \frac{\partial u_z}{\partial z} \right]^2 \rangle \\
(182)
\]

In order to be useful, the turbulence energy equation above must be simplified somewhat. The dissipation term, \( \varepsilon \), will be dealt with first. It comes from the last two terms of equation (181),

\[
\varepsilon = \mu \left< \frac{\partial u_j}{\partial x_i} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \right) \right>
\]
and represents the dissipation of mechanical energy into heat by viscosity. Since dissipation is done by the smaller eddies whose structure is isotropic (as was discussed earlier), an assumption of local isotropy can be applied.

Then, the expanded form of equation (183),

\[
\varepsilon = \mu \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) + \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) + \left( \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) + \left( \frac{\partial u_3}{\partial x_3} + \frac{\partial u_1}{\partial x_1} \right)
\]

can be modified using

\[
\frac{\partial u_1}{\partial x_1}^2 = \frac{\partial u_2}{\partial x_2}^2 = \frac{\partial u_3}{\partial x_3}^2
\]

\[
\frac{\partial u_1}{\partial x_2}^2 = \frac{\partial u_2}{\partial x_2}^2 = \ldots
\]

\[
\left( \frac{\partial u_1}{\partial x_2} \right)^2 = \left( \frac{\partial u_2}{\partial x_2} \right)^2 = \ldots
\]

so that it is simplified to the form

\[
\varepsilon = 6\mu \left[ \frac{\partial u_1}{\partial x_1}^2 + \frac{\partial u_2}{\partial x_2}^2 + \frac{\partial u_3}{\partial x_3} \right] .
\] (184)

This expression may be reduced further using some of the results of isotropic turbulence found in the Appendix. In particular, one can write

\[
\frac{\partial u_1}{\partial x_2}^2 = \lim_{\bar{x}_2 \to x_2} \frac{\partial^2}{\partial x_2^2} \left[ u_1(x_1) u_1(x_2) \right] = - \frac{\partial^2}{\partial x^2} \left[ U_{11} \right] \left| x = 0 \right.
\] (185)
The relationship between $g$, the lateral correlation, and $f$, the longitudinal correlation, is

$$ g = f + r/2 \frac{\partial f}{\partial r} $$  \hspace{1cm} (186)$$

so that

$$ \frac{\partial^2 g}{\partial r^2} \bigg|_{r=0} = 2 \frac{\partial^2 f}{\partial r^2} \bigg|_{r=0} $$  \hspace{1cm} (187)$$

Similarly, $\langle \frac{\partial u_1}{\partial x_1} \rangle^2$ is related to $f$ by

$$ \langle \frac{\partial u_1}{\partial x_1} \rangle^2 = \lim_{x_2 - x_1 \to r} \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} u_1(x_1) u_1(x_2) $$

$$ = \lim_{x_2 - x_1 \to r} \frac{\partial^2}{\partial x_1^2} u_1(x_1) u_1(x_2) - \frac{\partial^2}{\partial x_1 \partial x_2} u_1(x_1) u_1(x_2) $$

$$ = - \langle u^2 \rangle \frac{\partial^2}{\partial x_1 \partial x_2} f \bigg|_{r=0} $$  \hspace{1cm} (188)$$

By comparing equations (185), (187) and (188), one sees that

$$ \langle \frac{\partial u_1}{\partial x_2} \rangle^2 = 2 \langle \frac{\partial u_1}{\partial x_1} \rangle^2 $$  \hspace{1cm} (189)$$

Next, the term $\langle \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \rangle$ will be related to $\langle \frac{\partial u_1}{\partial x_1} \rangle^2$. First, one writes

$$ \langle \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \rangle = \lim_{x_2 - x_1 \to r} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} u_1(x_1) u_2(x_2) $$

$$ = \lim_{x_2 - x_1 \to r} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} u_1(x_1) u_2(x_2) $$

$$ = \langle \frac{\partial u_1}{\partial x_1} \rangle^2 $$  \hspace{1cm} (190)$$
Continuity is then used to find

$$< \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} > = 0 \tag{191} $$

which is rearranged using isotropy to obtain

$$< \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} > = - \frac{1}{2} < \frac{\partial u_1}{\partial x_1} > \tag{192} $$

Therefore, in view of equations (190) and (192), the following result can be written:

$$< \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} > = - \frac{1}{2} < \frac{\partial u_1}{\partial x_1} > \tag{193} $$

Combining equations (189) and (193) with (184), the dissipation becomes simply

$$\varepsilon = 6 \mu \left[ < \frac{\partial u_1}{\partial x_1} >^2 + 2 \cdot < \frac{\partial u_1}{\partial x_1} > \cdot < \frac{\partial u_1}{\partial x_1} > - \frac{1}{2} < \frac{\partial u_1}{\partial x_1} >^2 \right]$$

$$\varepsilon = 15 \mu < \frac{\partial u_z}{\partial z} >^2 \tag{194} $$

Next, the term $\nu / \sigma \{ \} \sigma$ in equation (182) will be considered. It corresponds to the two terms

$$\nu \frac{\partial}{\partial x_i} < \frac{\partial u_j}{\partial x_j} > + \frac{1}{2} \nu \frac{\partial^2}{\partial x_i^2} < u_j u_j >$$

in equation (181) when written in the rectangular system. The above terms can further be identified since they may be written in the form

$$\frac{1}{\rho} \frac{\partial}{\partial x_i} \left( u_j \mu \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_i} \right) = \frac{1}{\rho} \frac{\partial}{\partial x_i} \left( u_j \sigma_{ij} \right)$$

$$\text{viscous} \tag{195} $$
where \((\sigma_{viscous})_{ij}\) is the viscous stress tensor due to the turbulent velocity fluctuations. Therefore, in view of equation (171), the term in question is the average work done per unit mass by the turbulent viscous forces.

Having identified the term in this way, it is remembered that the wake flow is of high Reynolds number. The high Reynolds number condition says that the viscous forces are small. Thus, the term being considered is negligible and will be dropped.

With this result and that expressed by equation (194), the turbulence energy equation, equation (182), becomes

\[
\frac{1}{2} \left[ U_x \frac{\partial}{\partial x} < q^2 > + U_z \frac{\partial}{\partial z} < q^2 > \right] = -\frac{1}{\rho} \left[ \frac{1}{r} \frac{\partial}{\partial r} r < u_r p > + \frac{\partial}{\partial z} < u_z p > \right]
\]

\[
- \frac{1}{2} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r < u_r^2 > + \frac{\partial}{\partial z} < u_z^2 >) \right] - \left[ < u_r u_z > \frac{\partial}{\partial r} u_r + < u_r u_z > \cdot \frac{\partial}{\partial z} u_z \right] - 15 \nu \left( \frac{\partial^2 u_z}{\partial z^2} \right)^2.
\]

The above expression will be simplified further using the technique that was previously applied to the equation of motion for the wake. The characteristic length and velocity scales, that have already been defined, will be used again here with the addition of the three scales defined below:

\[
\frac{\Delta v}{L_F} - \text{scale for the derivative of fluctuating quantities} \quad (\frac{\Delta v}{L_F} > \frac{\partial}{\partial z} \frac{\partial}{\partial z})
\]

\[
(\cdot)_r - \text{scale for the difference of} \quad < u_r, z > \text{or} \quad < u_r, q^2 > \text{in the} \quad r\text{-direction}
\]

\[
(\cdot)_z - \text{scale for the difference of} \quad < u_r, z > \text{or} \quad < u_r, q^2 > \text{in the} \quad z\text{-direction}
\]
Since quantities of this type change more rapidly (in the wake) in the r-direction than in the z-direction, the relationship between the last two new characteristic scales is

\[
\frac{\tau_r}{\tau_z} \gg 1
\]  

(198)

Equation (196) will now be rewritten below with the magnitudes of the various terms indicated underneath:

\[
-\rho \left[ <u_z^2> \frac{\partial}{\partial z} U_z + <u_r^2> \frac{\partial U_r}{\partial r} + <u_z^2> \frac{\partial U_z}{\partial r} + <u_r u_z> \frac{\partial U_r}{\partial z} + <u_z u_r> \frac{\partial U_z}{\partial z} \right] \\
\rho \frac{\tilde{u}}{L_z} \frac{\tilde{u}}{L_r} \rho \frac{\tilde{u}}{L_r} \rho \frac{\tilde{u}}{L_z} \rho \frac{\tilde{u}}{L_z} \rho \frac{\tilde{u}}{L_z}
\]

\[
1 \quad 1 \quad 1 \quad \frac{L_z}{L_r} \gg 1 \quad \frac{\tilde{u}}{\tau_z} \gg 1
\]

\[
-\left[ \frac{\partial}{\partial z} <u_z p> + \frac{1}{r} r <u_r p> \right] = \frac{\rho}{2} \left[ U_z \frac{\partial}{\partial z} <q^2> + U_r \frac{\partial}{\partial r} <q^2> \right]
\]

\[
\frac{\tau_z}{L_z} \frac{\tau_r}{L_r} \rho \frac{\tilde{u} v^2}{L_z} \rho \frac{\tilde{u} v^2}{L_r} \rho \frac{\tilde{u} v^2}{L_z} \rho \frac{\tilde{u} v^2}{L_r}
\]

\[
1 \quad \frac{\tau_z}{L_z} \sim 1 \quad \frac{\tau_r}{L_r} \quad \frac{L_z}{L_r} \gg 1 \quad \frac{\tilde{u}}{\tau_z} \gg 1 \quad 1
\]

\[
+ \frac{\rho}{2} \left[ \frac{\partial}{\partial z} <u_z q^2> + \frac{1}{r} r <u_r q^2> \right] + 15 \mu \left[ <\frac{\partial u_z}{\partial z}>^2 \right]
\]

\[
\rho \frac{\tau_z}{L_z} \rho \frac{\tau_r}{L_r} \rho \frac{\tilde{u} v^2}{L_z} \rho \frac{\tilde{u} v^2}{L_r} \rho \frac{\tilde{u} v^2}{L_z} \rho \frac{\tilde{u} v^2}{L_r}
\]

\[
\frac{\tau_z}{u \tau_z} \sim 1 \quad \frac{\tau_r}{v^2 \tau_z} \quad \frac{L_z}{L_r} \gg 1 \quad \rho \frac{v^2 \tilde{u} v^2}{L_z} \quad \rho \frac{v^2 \tilde{u} v^2}{L_r} \quad \frac{(\Delta v)^2 L_z}{\nu^2 U_z \nu^2 L_z^2} \gg 1
\]  

(199)
The equation containing only the largest terms is then

\[-\rho \langle u \frac{\partial u}{\partial r} \rangle \frac{\partial u}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} (r \langle u \rangle) = \frac{\rho}{2} \frac{\partial}{\partial z} \langle q^2 \rangle + \frac{\rho}{2} \frac{1}{r} \frac{\partial}{\partial r} (r \langle u^2 \rangle) + 15\mu \langle \frac{\partial u}{\partial z} \rangle^2 \]  

(200)

If the analysis is restricted to the downstream portion of the wake beyond the initial shear regime, the mean velocity distribution is nearly uniform over the cross-section and one can say that

\[ U_z \sim U_0 \quad \text{and} \quad \frac{\partial}{\partial r} U_z < 1 \]  

(201)

The local isotropy assumption will be applied giving the relation

\[ \langle u_z^2 \rangle = \frac{1}{3} \langle q^2 \rangle \]  

(202)

The term representing dissipation can be written in terms of the dissipation length parameter:

\[ 15\mu \langle \frac{\partial u}{\partial z} \rangle^2 = 15\mu \frac{\langle u_z^2 \rangle}{\lambda_o^2} \]  

(203)

where \( \lambda_o \) is assumed to be constant over the wake cross-section.

Another simplifying approximation is that

\[ \langle u_r (\frac{1}{2} q^2 + P/\rho) \rangle = -\epsilon_e \frac{\partial}{\partial r} (\frac{1}{2} \langle q^2 \rangle) \]  

(204)

since the left-hand side represents the diffusion of energy in the \( r \)-direction due to turbulence, while the right-hand side is written by analogy to Fourier's law in the diffusion of heat.

With the simplifications and assumptions (201) - (204), equation (200) becomes
The above equation will now be integrated over the cross-section of the turbulent wake:

\[
\int_0^R 2\pi r dr \{ U_o \frac{\partial}{\partial z} <u^2>_z - \frac{1}{r} \frac{\partial}{\partial r} (r \varepsilon \frac{\partial}{\partial r} <u^2>_r) + 10V \frac{<u^2>_z}{\lambda_o^2} = 0 \}
\]

where \( R_o \) extends outside the turbulent region. At this point, the self-preservation solution for the turbulence intensity will be introduced,

\[
\frac{<u^2>_z}{U_s^2} = h(\xi), \quad (207)
\]

so that equation (206) can be written as

\[
U_o \frac{\partial}{\partial z} \xi^2 U_s^2 \int_0^{R_o/\lambda} h \xi d\xi + \frac{10V}{\lambda_o^2} \xi^2 U_s^2 \int_0^{R_o/\lambda} h \xi d\xi = 0 \quad (208)
\]

with the result that

\[
\frac{10V}{\lambda_o^2} = \frac{-U_o}{\xi^2 U_s^2} \frac{\partial}{\partial z} \frac{\partial}{\partial z} \xi^2 U_s^2 \xi^2 U_s^2 d\xi = \frac{-U_o}{U_s^2} \frac{d}{dz} U_s^2 - \frac{U_o}{\xi^2} \frac{d}{dz} \xi^2 \quad (209)
\]

If the relations (207) and (209) are substituted into equation (205), one obtains:
\[
\frac{\partial}{\partial z} \left( \frac{1}{u} \frac{\partial}{\partial r} \right) [\sigma_e u^2 \frac{1}{r} \frac{\partial h}{\partial \zeta}] - \left[ \frac{1}{u^2} \frac{\partial}{\partial z} u^2 + \frac{1}{\lambda^2} \frac{\partial}{\partial z} \right] u^2 h = 0
\]

\[
\frac{dh}{d\zeta} + \frac{2u \lambda}{\varepsilon_e} \frac{d\lambda}{dz} \xi h = 0 \tag{210}
\]

In order for the self-preservation solution to exist, the term in parentheses must not be a function of \( z \). It is set equal to a constant,

\[
\frac{2u \lambda}{\varepsilon_e} \frac{d\lambda}{dz} = \frac{2}{w} \tag{211}
\]

so that equation (210) becomes

\[
\frac{dh}{h} = - \frac{2}{w} \xi d\xi \tag{212}
\]

Integrating the above expression, one finds

\[
h = C e^{-\xi^2/w} \tag{213}
\]

In terms of the length and velocity scales, the turbulence intensity becomes

\[
\langle u^2 \rangle = C u^2 e^{-\frac{r^2}{w \lambda^2}} \tag{214}
\]

Besides the turbulence intensity, the rate of energy dissipation must also be known in order to evaluate explicitly equation (132), the power spectrum of the scattered waves. Using equations (194) and (203), the dissipation, \( \varepsilon' \), may be expressed in terms of the dissipation length, \( \lambda_o \):

\[
\varepsilon = 15\mu \frac{\langle u^2 \rangle}{\lambda_o^2} \tag{215}
\]

The dissipation length, \( \lambda_o \), is evaluated with the use of equations (164) and (209) to give
Thus, all the information needed in equation (132) to evaluate the power spectrum of the scattered waves has been obtained from the preceding wake flow analysis.

It should be mentioned that the basis for the wake turbulence analysis was the self-preservation assumption. Naudascher, in this connection, has reported the results of experiments where the turbulent wake flow behind a totally immersed, axisymmetric, self-propelled body was studied [15]. These experiments verified the assumption of self-preserved flows and, therefore, serve to substantiate the preceding theoretical development.

D. Explicit Evaluation of the Power Spectrum

The results of the analysis of the turbulent wake of a self-propelled body will now be substituted into the expression for the power spectrum of the scattered waves, equation (132).

First, however, the limits to the volume integration must be restricted to the region of the wake turbulence since the relations to be substituted apply only in that part of the fluid. The beginning of the self-preserved portion of the wake (beyond the initial shear regime) along the z-axis is denoted by $z_0$. The distance along the wake axis to where the turbulence has decayed enough to be considered negligible is $L_0$. The width of the wake will be proportional to the length scale $l$, so the upper limit of the radial integration will be denoted by $Wl$.

Using these limits and substituting equations (133), (169), (214), (215), and (216), equation (132) becomes

$$
\lambda_0 = 5 \sqrt{\frac{\nu}{3U_0}} z^\frac{1}{2}
$$

(216)
\[
Pt = \frac{\pi KK^2 \cos^2 \theta \cot^2 \theta}{2} \int dz \int r dr \left( \frac{15 \mu \text{Cu} e^{-1/\gamma (r/\lambda)^2}}{z} \right)^{1/2} \frac{1}{(\text{Cu} e^{-1/\gamma (r/\lambda)^2})^{1/2}} \\
\times \exp \left\{ -\left( \frac{-\omega_0}{k_d} - DU_s (1-k^2 \gamma) e^{-k^2/\gamma} \right)^2 \right\} \frac{2 \text{Cu} e^{-\beta^2/\gamma}}{2} \\
\]

\[
= \frac{9U_o \rho}{4\sqrt{2\pi} \text{Cu} v (k_d)^{8/3}} \int dz \int r dr \left( \frac{U_1^{1/3}}{z^{2/3}} \right)^2 \int_0^W \beta d\beta e^{-\beta^2/\gamma} \\
\times \exp \left\{ -\left( \frac{-\omega_0}{k_d} - DU_s (1-k^2 \gamma) e^{-k^2/\gamma} \right)^2 \right\} \frac{2 \text{Cu} e^{-\beta^2/\gamma}}{2} \\
\]

\[
= \frac{g^2/3 \text{U}_o^{2/3} \rho^{2/3} \text{K}_o \cos^2 \theta \cot^2 \theta}{2} \int dz \int r dr \left( \frac{1}{z^{2/3}} \right)^2 \int_0^W \text{d}X \int_0^{W^2} \text{d}Y e^{-\gamma/w} \\
\times \exp \left\{ -\left( \frac{-\omega_0}{k_d} - DAz^{-4/5} (1-k \gamma) e^{-k \gamma} \right)^2 \right\} \frac{2 \text{C} \text{A}^2 \text{z}^{-8/5} e^{-\gamma/w}}{2} \\
\]

\[
= \frac{g^2/3 \text{U}_o^{2/3} \rho^{2/3} \text{K}_o \cos^2 \theta \cot^2 \theta}{2} \int dz \int r dr \left( \frac{1}{z^{2/3}} \right)^2 \int_0^W \text{d}X \int_0^{W^2} \text{d}Y e^{-\gamma/w} \\
\times \exp \left\{ -\left( \frac{-\omega_0}{k_d} - DAz^{-4/5} (1-k \gamma) e^{-k \gamma} \right)^2 \right\} \frac{2 \text{C} \text{A}^2 \text{z}^{-8/5} e^{-\gamma/w}}{2} \\
\]

\[
(217) 
\]
Now, the exponential term is expanded in a Taylor series about the origin:

\[
f(\gamma, \Lambda) = \exp \left\{ \frac{\Lambda - DA(1-\frac{k}{c}) e^{-k\gamma^2}}{2 CA^2 e^{-\gamma/w}} \right\}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!} \frac{1}{n!} \frac{\partial^m}{\partial \Lambda^m} \frac{\partial^n}{\partial \gamma^n} f(\gamma, \Lambda) \left|_{\Lambda=0}^{\Lambda=0} \right. \frac{\gamma^n}{(\omega - \omega_0)^m} \frac{\lambda^m \gamma^n}{k_d}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\lambda^m}{m!} \frac{\gamma^n}{n!} A_{mn} \frac{(\omega - \omega_0)^m}{k_d}
\]

where \( A_{mn} = \frac{\partial^m}{\partial \Lambda^m} \frac{\partial^n}{\partial \gamma^n} f(\gamma, \Lambda) \bigg|_{\Lambda=0}^{\Lambda=0} \frac{\omega - \omega_0}{k_d} \).

Substituting the expansion for the exponential in equation (217) and integrating term by term, one obtains

\[
P_t = \frac{g^2/3 \nu/2}{3!} \frac{\rho^2}{3} k_0^2 \cos^2 \theta \cot^2 \theta/2 C^{1/6} A^{1/3} B^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \frac{(\omega - \omega_0)^m}{k_d}
\]

\[
= \frac{(\nu/5)^{m+7/12}}{(z_0/5)^{m+7/12}} \frac{(\omega - \omega_0)^m}{k_d} \frac{(6w)^{n+1} \gamma_{ic}(n+1, \frac{w^2}{6w})}{k_d}
\]

where \( \gamma_{ic} \) is the incomplete gamma function. The series representation for the incomplete gamma function is given by

\[
Y_{ie}(1+n, x) = n! \left[ 1 - e^{-x} \left( \sum_{s=0}^{n} \frac{x^s}{s!} \right) \right]
\]

so that the power spectrum of the scattered waves becomes
\[

P_t = \frac{g^2/3 \delta^1/3 \rho^2/3 \theta \cos^2 \theta \cot^2 \theta / 2}{32\sqrt{2\pi} \kappa \eta t (k_d)^{1/3}} \frac{C^{1/6} A^{1/3} B^2}{\sum_{m=0}^{\infty} \frac{B_m}{m! k_d^m}} \times \left( \frac{1}{Z_o^{m+7/12}} - (Z_o^{m+7/12}) \right) \frac{(\omega - \omega_o)^m}{m+7/12}
\]

where

\[

B_m = \sum_{n=0}^{\infty} A_{mn} (6\omega)^{n+1} \left[ 1 - \frac{\Gamma^s}{s!} \right] e^{-\frac{W^2}{6\omega}}
\]

(221)

In discussing the above solution, it should be noted that the main contribution of the exponential term to the integral (217) comes when \( \gamma \) and \( \lambda \) are both small. For this reason, the series will converge quickly, and the lower ordered terms will be sufficient for most applications. (The first few values of \( A_{mn} \) are listed in the Appendix.) In the final result above, the spectrum is given by an expansion about \( \omega_o \), which is very near the maximum of the spectrum.

\( B_m \) or \( A_{mn} \), which appear in the expansion, each depends upon the angle the incident wave makes with the wave axis so that the solution depends on this parameter, as well as the position of the observer. In this way, the solution contains the dependence of the power spectrum on the direction of the self-propelled body. An analysis which assumes complete homogeneity and isotropy of the wake, of course, does not include this result. In addition, the relationship between the scattered waves and the wake hydrodynamics are identified through the parameters describing the mean flow and the wake dimensions.
VI. DISCUSSION OF RESULTS

A. Summary and Conclusions

After an introduction to the thesis problem is given in Chapter I, the basic theory of sound-turbulence interaction is developed in Chapter II. The fundamental result of this chapter is equation (45). This gives the power spectrum of the differential scattering cross-section of the scattered wave, in the far-field approximation, for the case of a plane acoustic wave impinging upon a volume of homogeneous, isotropic turbulence, where the convective effects of the macro-eddies are taken into account. It is shown, from this equation, that the size of the eddies responsible for most of the scattering, the "scattering eddies," is of the order of the wavelength of the incident sound field. Because of this, one can also conclude that the techniques of geometrical optics do not apply so that the approach used, which considers the wave nature of the sound field, is the only reasonable one. Equation (45) also shows that the power spectrum of the scattered waves depends upon the turbulence energy spectrum function evaluated at \( k_d \), \( E(k_d) \). This quantity must therefore be determined before the power spectrum can be explicitly evaluated.

In Chapter III, \( E(k_d) \) is found for both the high and low Reynolds number cases, and the power spectrum is evaluated for both types of turbulence. The results show that the convective effects of the mean flow and the larger turbulent eddies tends to widen and shift the spectrum of the scattered wave from the incident wave frequency \( \omega_o \).

In the next chapter, Chapter IV, the scattering of high frequency incident sound is investigated from a slightly different point of view. The scattered wave intensity is defined (not its power spectrum), and the analysis starts at
a point in Chapter II, equation (30), before the isotropy assumption is employed. The total power scattered, per volume of turbulence, is then calculated, with the high frequency assumption being the only restriction.

The results of Chapter V include an extension of the basic solutions in Chapter II to the case of sound scattering from inhomogeneous turbulence. The only restrictions are that the statistical parameters of the turbulence vary only slightly over distances of the order of the size of the scattering eddies and that the wavelength of the incident wave be smaller than the largest eddies. The generalization to inhomogeneous flows, equation (129), can normally be applied to many turbulence scattered sound problems of engineering interest such as wakes and jets. If it is necessary, the anisotropic effects of the larger eddies (this is where such effects would appear first) can even be taken into account when the angular dependence of the mean square of the velocity fluctuation in the direction of $k_d$ is known.

The general expression for the power spectrum of waves scattered from inhomogeneous turbulence was then applied to the problem of sound scattering from the wake of a self-propelled body. The terms which needed to be determined in order to evaluate the scattered wave power spectrum were found by analyzing the wake itself with the use of the hypothesis of self-preservation. The power spectrum is then evaluated, and the result is given by equation (221).

In examining this result and the definition of the $A_{mn}$ terms, it is seen that the solution depends not only on the position of the observer (as in the case of homogeneous, isotropic turbulence) but also on the angle the incident wave propagation vector makes with the wake axis. The solution also identifies the relationship between the wake hydrodynamics and the scattered wave.
B. Recommendations

There are several areas where further development of the results contained here is warranted. Among them are:

1. The implementation of an experimental program to study the scattering of sound waves from wakes. The configuration described by Naudascher could be used to simulate the wake of a self-propelled body.

2. The application of the general expression for the power spectrum of sound scattered from inhomogeneous turbulence, equation (129), to other inhomogeneous turbulence flows. This could be evaluated, with interesting results, for many types of wakes and jets in addition to the application given here.

3) The evaluation of the power spectrum of the scattered wave, for both the homogeneous and inhomogeneous cases, using other forms of the energy spectrum function which would be appropriate to different circumstances than those which have been previously presented. This will involve extensive numerical calculation since the turbulence energy spectrum function will usually not be given in closed form.


A. Some Relations from the Theory of Homogeneous, Isotropic Turbulence

Several relations, which have been used previously in the report, were taken without proof from the theory of homogeneous, isotropic turbulence. In this section these relations will be derived from such fundamental notions as continuity, the Navier-Stokes equation, and elementary results from statistical theory. The analysis will be restricted to obtaining the expressions that were necessary in the sound scattering problem. For a more comprehensive account of homogeneous, isotropic turbulence theory, the reader is referred to Batchelor's book [10] or the notes given by W. Mosberg and M. Yildiz [16].

In this theory the fluid is taken to be incompressible so that

$$\nabla \cdot \bar{u} = 0 \quad (A1)$$

where $\bar{u} = \bar{u}(\bar{x},t)$ is the fluctuating velocity. In this case, there will be no mean velocity. Or, equivalently, the motion is referred to axes traveling with the mean velocity of the fluid. The density $\rho$ and kinematic viscosity $\nu$ are assumed to be constants. This is appropriate for turbulence of low Mach number. Equation (A1) may also be applied when there is a weak compressional velocity field present (as in the sound scattering problem) since the turbulence velocity field $\bar{u}(\bar{x},t)$ is largely rotational. Also, the longitudinal velocity field $\bar{V}(\bar{x},t)$, due to the presence of the sound wave, has been restricted in the analysis to be of small amplitude. For this reason, the compressional field does not appreciably affect the rotational turbulence field; while the turbulence field, as has been shown, has a definite effect on the sound field. It should be noted that normally the velocities associated with the movement of fluid particles when a
Compressional wave passes are much smaller than their velocities when agitated by turbulence [3]. So no physically important cases have been eliminated by the small amplitude sound wave restriction.

With this in mind, it is recognized that the turbulence velocity field itself must obey the Navier-Stokes equation for an incompressible fluid.

\[ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} \]  

(A2)

This may be derived from the equation of motion for \( \vec{u}(\vec{x}, t) \),

\[ \frac{\partial}{\partial t} u_i + u_j \frac{\partial}{\partial x_j} u_i = \frac{1}{\rho} \frac{\partial}{\partial x_j} \sigma_{ij} \]  

(A3)

by substituting the stress-tensor for a Newtonian viscous fluid,

\[ \sigma_{ij} = -p \delta_{ij} + \lambda \delta_{ij} \frac{\partial u_m}{\partial x_m} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(A4)

and using the relation (A1).

The two equations (A1) and (A2) are then sufficient to determine the two unknowns \( \vec{u} \) and \( p \). There are no boundaries; the fluid extends to infinity. But the initial conditions will be known only in statistical terms. Therefore, the fluid velocity and pressure can be found, at best, only as random functions of position obeying certain probability laws. Thus, the analysis of turbulence is done in terms of conveniently defined statistical averages and correlations.

The most useful correlation in turbulence theory is the velocity correlation tensor defined by

\[ R_{ij}(\vec{r}) = \langle u_i(\vec{x}, t) u_j(\vec{x} + \vec{r}, t) \rangle \]  

(A5)

where \( \langle \rangle \) indicates a probabilistic mean is to be taken:

\[ R_{ij}(\vec{r}) = \int d^3u_1 \int d^3u_2 \ [\vec{u}_1 \cdot \hat{\vec{r}}] [u_2 \cdot \hat{\vec{r}}] p_u(\vec{u}_1, \vec{u}_2) \]  

(A6)
where $\vec{u}_1 = \vec{u}(\vec{x},t)$, $\vec{u}_2 = \vec{u}(\vec{x}+\vec{r},t)$, $e_i$ is a unit vector in the $i$th direction, and $p_{u}(\vec{u}_1,\vec{u}_2)$ is the joint probability density of $\vec{u}_1,\vec{u}_2$. The property of spatial homogeneity, in which $\vec{u}(\vec{x},t)$ is a stationary function of $\vec{x}$, says that the velocity correlation tensor is not a function of $\vec{x}$. This implies that the averaging may be made, equivalently, over a large region of space. In other words, $R_{ij}(\vec{r})$ may be defined in terms of a space average:

$$R_{ij}(\vec{r}) = \frac{1}{V_t} \int d^3\vec{x} \ u_i(\vec{x},t)u_j(\vec{x}+\vec{r},t)$$

For homogeneous turbulence, the relative configuration (given by $\vec{r}$) of the two points where the velocities are measured determines the spatial dependence of the velocity correlation. It is independent of the location of the two points. Therefore, the homogeneity condition with the geometry shown below,

establishes the equivalence of two correlations

$$\langle u_i(\vec{x})u_j(\vec{x}+\vec{r}) \rangle = \langle u_j(\vec{x})u_i(\vec{x}-\vec{r}) \rangle.$$  \hspace{1cm} (A8)

This last relation along with the definition of $R_{ij}(\vec{r})$ given by equation (A5) shows that

$$R_{ij}(\vec{r}) = R_{ji}(-\vec{r})$$  \hspace{1cm} (A9)
The Fourier space transform of $R_{ij}(\vec{r})$ is called the "energy spectrum tensor" or "spectrum tensor" and is defined by

$$\phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3} \int d^3 \vec{r} \ R_{ij}(\vec{r}) e^{-i\vec{k} \cdot \vec{r}}$$  \hspace{1cm} (A10)

The inverse transform expresses $R_{ij}(\vec{r})$ in terms of $\phi_{ij}(\vec{k})$:

$$R_{ij}(\vec{r}) = \int d^3 \vec{k} \ \phi_{ij}(\vec{k}) e^{i\vec{k} \cdot \vec{r}}$$ \hspace{1cm} (A11)

When $\vec{r} = 0$, this last expression reduces to

$$R_{ij}(0) = \langle u_i(\vec{x})u_j(\vec{x}) \rangle = \int d^3 \vec{k} \ \phi_{ij}(\vec{k})$$ \hspace{1cm} (A12)

showing that $\phi_{ij}(\vec{k})$ represents a density, in wavenumber space, of contributions to $\langle u_i(\vec{x})u_j(\vec{x}) \rangle$. Another property of $\phi_{ij}(\vec{k})$ is that, from equations (A9) and A(10),

$$\phi_{ij}(\vec{k}) = \phi_{ji}(-\vec{k}) = \phi_{ji}^{*}(\vec{k}).$$ \hspace{1cm} (A13)

That is, $\phi_{ij}(\vec{k})$ is a tensor with Hermitian symmetry.

Additional properties of $\phi_{ij}(\vec{k})$ and $R_{ij}(\vec{r})$ may be found from conditions imposed by the continuity equation, equation (A1). In particular, since the continuity equation says that the divergence of $\vec{u}(\vec{x},t)$ is zero, the following relations are valid:

$$0 = \frac{\partial u_i}{\partial x_j}(\vec{r})$$

$$= u_i(0) \frac{\partial u_i}{\partial x_j}(\vec{r})$$ \hspace{1cm} (A14)
Because the homogeneity condition says that the average is independent of location (so long as the relative position is maintained), one may write

\[ 0 = \frac{\partial}{\partial x_j} \langle u_i(x) u_j(x+r) \rangle = \frac{\partial}{\partial x_j} R_{ij}(\tilde{r}) = \frac{\partial}{\partial \tilde{r}_i} R_{ij}(\tilde{r}) \]  

(A16)
in view of equations (A5) and (A9).

From the last result and equation (A11), the corresponding relation for \( \phi_{ij}(\tilde{k}) \) is

\[ 0 = k_i \phi_{ij}(\tilde{k}) = k_j \phi_{ij}(\tilde{k}) \]  

(A17)

In the equations derived so far concerning the velocity correlation tensor and its Fourier space-transform, only the condition of spatial homogeneity was applied. If symmetry conditions due to isotropy of the turbulence are imposed, additional independent requirements can be found for the form of these tensors.

To establish the implications of statistical isotropy, the mean value of the product of two velocity components in the directions given by unit vectors \( \vec{a} \) and \( \vec{b} \) at two points separated by a vector \( \vec{r} \), \( Q(\vec{r}, \vec{a}, \vec{b}) \), will be defined by

\[ Q(\vec{r}, \vec{a}, \vec{b}) = \langle a_i u_i(x) b_j u_j(x+r) \rangle \]

\[ = a_i b_j R_{ij}(\vec{r}) \]  

(A18)

Isotropy imposes the condition that \( Q(\vec{r}, \vec{a}, \vec{b}) \) must be independent of arbitrary rigid rotations or reflections of the configuration formed by the vectors \( \vec{r} \), \( \vec{a} \) and \( \vec{b} \).

Geometrically, this means that the two configurations given below
will give identical values of $Q(\vec{r}, \vec{a}, \vec{b})$ for any value of $\theta$. This implies that $Q$ is a function only of the length of $\vec{r}$ and the angles (or scalar products) between pairs of the vectors $\vec{r}, \vec{a}$, and $\vec{b}$, that is

$$Q(r, a, b) = Q(\vec{r}, \vec{a}, \vec{b})$$ \hspace{1cm} (A19)

Since equation (A18) shows that $Q$ must be linear in $a_i$ and $b_i$, the most general form for $Q(\vec{r}, \vec{a}, \vec{b})$ is

$$Q(\vec{r}, \vec{a}, \vec{b}) = A(r_i r_j) a_i b_j + B(r_i r_j) a_i b_j$$

$$= a_i b_j [A(r^2) r_i r_j + B(r^2) \delta_{ij}]$$ \hspace{1cm} (A20)

In view of equation (A18), the form of the velocity correlation tensor for the case of homogeneous and isotropic turbulence can be written as

$$R_{ij}(\vec{r}) = A(r) r_i r_j + B(r) \delta_{ij}$$ \hspace{1cm} (A21)

For convenience in interpretation, $A(r)$ and $B(r)$ will be found in terms of the physically more meaningful longitudinal velocity correlation, $f(r)$, and the lateral velocity correlation $g(r)$. These new terms are defined by the expressions

$$f(r) = \frac{<u_p(x) u_p(x+r)>}{<u^2_p>} \quad \text{and} \quad g(r) = \frac{<u_n(x) u_n(x+r)>}{<u^2_n>}$$ \hspace{1cm} (A22)

where $u_p$ is the component of velocity parallel to $\vec{r}$ and $u_n$ is the component normal to $\vec{r}$. The geometry is shown below:
Consider the velocity correlation tensor when \( i = j = 1 \) and \( \vec{r} \) is along the \( x_1 \) axis. In this case, \( u_1 = u_j = u_1 = u_p \) and \( \vec{r} = \vec{r}_1 \), so that

\[
R_{11} = Ar^2 + B = \langle u_p(x)u_p(x+\vec{r}) \rangle = \langle u^2 \rangle f
\tag{A23}
\]

When \( \vec{r} \) is parallel to the \( x \) axis and \( i = j = 2 \), \( u_i = u_j = u_2 = u_n \) and \( r_2 = 0 \) giving for this case

\[
R_{22} = B = \langle u_n(x)u_n(x+\vec{r}) \rangle = \langle u^2 \rangle g
\tag{A24}
\]

Using the relations (A23) and (A24), the velocity correlation tensor given by equation (A21) may be expressed in terms of \( f \) and \( g \). \( R_{ij} \) becomes

\[
R_{ij}(\vec{r}) = \langle u^2 \rangle \left[ \frac{(f-g)}{r} r_i r_j + g \delta_{ij} \right]
\tag{A25}
\]

The continuity expression may now be applied to find a relation between \( f \) and \( g \). Applying continuity in the form given by equation (A16) to equation (A25), one finds

\[
0 = \frac{\partial}{\partial r_j} \left[ \frac{f(r)-g(r)}{r^2} r_i r_j + g(r) \delta_{ij} \right]
\tag{A26}
\]

Completing the differentiation and solving for \( g(r) \) in terms of \( f(r) \), one obtains

\[
g = f + \frac{r}{2} \frac{\partial f}{\partial r}
\tag{A27}
\]

so that \( R_{ij}(\vec{r}) \) can be written as

\[
R_{ij}(\vec{r}) = \langle u^2 \rangle \left[ (\frac{-1}{2r} \frac{\partial f}{\partial r}) r_i r_j + (f + \frac{r}{2} \frac{\partial f}{\partial r}) \delta_{ij} \right]
\tag{A28}
\]

Thus, only one scalar function is needed to specify the velocity correlation tensor when the turbulence is isotropic as well as homogeneous.
When the \( i \) and \( j \) indices are contracted, equation (A28) becomes

\[
R_{ij}(\vec{r}) = \langle u^2 \rangle [3f + x \frac{\partial f}{\partial x}] \tag{A29}
\]

The form of \( \phi_{ij}(k) \), the Fourier space-transform of \( R_{ij}(\vec{r}) \), will now be considered. It should be mentioned, first, that the form of \( R_{ij}(\vec{r}) \) given by equation (A21) is general, and any isotropic, second order tensor which depends on a single vector argument will have similar form. Since \( \phi_{ij}(\vec{k}) \) is also a second order, isotropic tensor with a single argument, \( \vec{k} \), it may be written analogously as

\[
\phi_{ij}(\vec{k}) = C(k)k_ik_j + D(k)\delta_{ij} \tag{A30}
\]

The requirement of continuity, as expressed in equation (A17), is applied to the above relation to give

\[
k_j\phi_{ij} = Ck_ik^2 + Dk_i = 0 \tag{A31}
\]

so that

\[
D = -k^2C. \tag{A32}
\]

Eliminating \( D \) from equation (A30), one writes

\[
\phi_{ij} = A(k_ik_j - k^2\delta_{ij}) \tag{A33}
\]

Now define the energy spectrum function \( E(k) \) by the relation

\[
E(k) = -4\pi k^6A(k) \tag{A34}
\]

Equation (A33) then becomes

\[
\phi_{ij}(\vec{k}) = \frac{E(k)}{4\pi k^6} (k^2\delta_{ij} - k_ik_j) \tag{A35}
\]

The physical meaning of \( E(k) \) may be obtained by first contracting the above equation,
Using this in the contracted form of equation (All) with \( r = 0 \) and dividing by \( \frac{1}{2} \),

\[
\frac{1}{2} R_{ii}(0) = \frac{1}{2} \langle u^2 \rangle \int d^3k \frac{1}{2} \phi_{ii}(k) = \int d^3k \frac{E(k)}{4\pi k^2} = \int dk E(k),
\]

the meaning of \( E(k) \) is made clear. It is the contribution to the energy per unit mass from that part of wavenumber space between spheres of radii \( k \) and \( k+dk \).

### B. Dynamics of Decay

Next, some relations having to do with the general dynamics of decay will be found. The purpose here will be to relate the energy spectrum function to the decay of the turbulence energy and make a physical interpretation of the results.

Let \( u' \) and \( p' \) denote the velocity and pressure at the point \( x' = x + r \) and write the Navier-Stokes equation at the points \( x \) and \( x' \).

\[
\frac{\partial u_i}{\partial t} = - \frac{\partial u_i u_k}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2}{\partial x_m \partial x_m} u_i
\]

\[
\frac{\partial u'_j}{\partial t} = - \frac{\partial u'_j u'_k}{\partial x'_k} - \frac{1}{\rho} \frac{\partial p'}{\partial x'_j} + \nu \frac{\partial^2}{\partial x'_m \partial x'_m} u'_j
\]

After the first of the above equations is multiplied by \( u^j \) and the second by \( u^j \), the resulting two equations are added and an average taken to give

\[
\langle \frac{\partial}{\partial t} u_i u^j \rangle = - \langle u_i \frac{\partial u^j}{\partial x'_k} \rangle + \langle u^j \frac{\partial u_i}{\partial x'_k} \rangle - \frac{1}{\rho} \langle \frac{\partial p}{\partial x_i} \rangle + \frac{\partial}{\partial x'_m} \frac{\partial^2}{\partial x'_m \partial x'_m} \langle u_i \rangle\]

\[
+ \nu \left( \langle u_i \frac{\partial^2 u^j}{\partial x'_m \partial x'_m} \rangle + \langle u^j \frac{\partial^2}{\partial x'_m \partial x'_m} u_i \rangle \right)
\]

(A39)
Since the operations of averaging and differentiation permute, $u_i$ and $p$ are independent of $\overline{r}'$, and $\partial / \partial x'_i$ and $\partial / \partial x_i$ can be replaced by $\partial / \partial x_i$ and $- \partial / \partial x'_i$, respectively, when they are taken on a two-point mean value; one may write

$$\frac{\partial}{\partial t} \langle u_i u'_j \rangle = \frac{\partial}{\partial x_k} \langle u_i u'_k u'_j \rangle - \langle u_i u'_k u'_j \rangle + \frac{1}{\rho} \frac{\partial}{\partial x_i} \langle p u'_j \rangle - \frac{\partial}{\partial x'_j} \langle p' u'_i \rangle$$

$$+ 2 v \frac{\partial^2}{\partial x_m \partial x_m} R_{ij}(\overline{r}, t)$$

(A40)

Using the expression for the velocity correlation tensor given by equation (A31), the above equation can be written as

$$\frac{\partial}{\partial t} \langle u_i u'_j \rangle = \frac{\partial}{\partial x_k} \langle u_i u'_k \rangle - \langle u_i u'_k \rangle + \frac{1}{\rho} \frac{\partial}{\partial x_i} \langle p u'_j \rangle - \frac{\partial}{\partial x'_j} \langle p' u'_i \rangle$$

$$- 2v \int d^3k k^2 \phi_{ij}(k) e^{i\overline{k} \cdot \overline{r}}$$

(A41)

If the indices $i$ and $j$ are contracted and $\overline{r}$ is zero so that $u_i = u'_i$ and $p = p'$, then equation (A41) becomes

$$\frac{\partial}{\partial t} \langle u_i u'_i \rangle = -2v \int d^3k k^2 \phi_{ii}(k)$$

(A42)

By dividing by two and substituting $\phi_{ii}$ from equation (A36), the decay of the turbulence energy per unit mass is found in terms of the energy spectrum function:

$$\frac{\partial}{\partial t} \frac{1}{2} \langle u_i u'_i \rangle = -2v \int d^3k \frac{E(k)}{4} = -2v \int dk k^2 E(k)$$

$$\frac{\partial}{\partial t} \int dk E(k, t) = -2v \int dk k^2 E(k, t)$$

(A43)
The left-hand side of the above equation represents the rate of change of the turbulence energy due to the dissipation which is represented by the right-hand side. Since \( E(k) \) in the right-hand side integral is weighted by a factor \( k^2 \), it is seen that it is the higher wavenumbers which are responsible for viscous dissipation. Typical examples of \( E(k) \) and \( k^2 E(k) \) are drawn below:

Often, as in the case shown, the range of wavenumbers containing the turbulence energy is quite distinct from the range of high wavenumbers responsible for the dissipation. This supports the physical picture where the energy in the large, "energy-containing eddies" (small wavenumbers) excites, through the action of inertial forces, the smaller eddies (high wavenumbers) so that the energy flows from the large eddies to the sink provided by viscous damping in the dissipation range. The idea of inertial forces spreading the energy from lower to higher wavenumbers is an important concept in turbulence, particularly in the discussion of the theory of universal equilibrium.

C. Values of \( A_{mn} \)

The first few values of \( A_{mn} \) are listed below:

\[
A_{\infty} = \exp \left( -\frac{\left( \cos \psi - \cos \phi \right)^2}{8C \sin^2 \phi / 2} \right)
\]

\[
A_{10} = \frac{\left( \cos \psi - \cos \phi \right)}{2C \sin \theta / 2} \exp \left( -\frac{\left( \cos \psi - \cos \phi \right)^2}{8C \sin^2 \theta / 2} \right)
\]
\[ c - \frac{\sin \theta \sqrt{z}}{z (\cosh - \cos \theta)} - \frac{\cos \theta \sqrt{z}}{z (\cosh - \cos \theta)} \left[ z \frac{\frac{\sqrt{z}}{1 - M}}{\frac{\sqrt{z}}{1 - M}} - \frac{\frac{\sqrt{z}}{1 - M}}{\frac{\sqrt{z}}{1 - M}} \right] + \frac{\csc \theta \sqrt{z}}{z (\cosh - \cos \theta)} \]