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DYNAMIC PROBABILISTIC SYSTEMS WITH CONTINUOUS PARAMETER MARKOV CHAINS AND SEMI-MARKOV PROCESSES

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by

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ABSTRACT

DYNAMIC PROBABILISTIC SYSTEMS WITH
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ABSTRACT

The modeling as well as analysis aspects of dynamics probabilistic systems whose state transition processes are continuous parameter Markov chains and semi-Markov processes are investigated. The first three chapters present the Introduction and some preliminary materials for the research conducted. The major contributions for Markovian systems are presented in Chapters 4 through 6, and that for semi-Markovian systems are presented in Chapters 7 and 8.

A new and simplified approach is employed to derive the systems of differential and integral equations, which respectively govern the dynamical behavior of the state probabilities of Markovian and semi-Markovian systems. The system of differential equations for Markovian systems are derived in Sections 4. 2 and 4. 3 of Chapter 4, whereas the system of integral equations for semi-Markovian systems are derived in Section 8. 2 of Chapter 8. A general procedure is developed for modeling of systems having complex configurations. Simple examples of modeling are presented in Sections 4. 5 and 7. 2 through 7. 4 to illustrate the general methodology.

Using a unified matrix approach, closed form general solutions are derived for many commonly used system effectiveness measures.

The matrix approach is shown to be very useful and appropriate for analysis of systems with finite state space. The solutions for Markovian systems are developed in Sections 5. 2 through 5. 11 of Chapter 5, and in Section 6. 2 of Chapter 6. For semi-Markovian systems the solutions are developed in Sections 8. 3 and 8. 4 of Chapter 8.

Existence of asymptotic solutions for the state probabilities of stationary ergodic Markovian and semi-Markovian systems are proved. Matrix expressions for the solutions are developed in Sections 5. 6 and 8. 3 Appendix C presents a proof for the existence of the limiting solutions.

Important properties pertaining to the state transition-rate matrix, M, of stationary ergodic Markovian systems are studied. The most significant findings are that M is always singular, and that submatrices of M resulted from deleting any number of rows and the corresponding columns of M are nonsingular. This is the main theorem proved pertaining to the properties of M. Appendix D contains the proof of this theorem.

Special contributions are made in Chapter 6 on the probability distribution and statistical moments of the system first passage time. In particular, for a Markovian system comprised of independent subsystems, it is shown in Section 6. 5 that the mean-up-time and down-time of the system are expressible in terms of those of the subsystems. Thus, computation for mean up-time and down-time of large scale systems can be greatly simplified by means of this approach.

Chapter 1

INTRODUCTION

1.1 HISTORICAL BACKGROUND

It was as early as the turn of this century and before the theory of stochastic processes were available that Erlang (1878-1929) [1]* pioneered the study of trunking problems for telephone exchanges. This problem was later studied by Palm in 1943 [2] after the theory of stochastic processes was developed. Based on these successful experiences, Khintchin in 1932 [3] and Palm in 1947 [4] mathematically formulated the groundwork on machine operation and maintenance problems. After World War II, the accelerated advances in modern sciences and engineering have rapidly increased the magnitudes and complexity of technological, economical, and social problems. Keeping in step with these increases have been the general rise of interest in modeling and analysis of these problems, which in turn had a profound stimulating influence on the development of modern probability theory and the study of stochastic processes.

*Numbers in brackets refer to references listed at the end of this thesis.

Many books on probability theory, stochastic processes, and their applications have been written in recent years. Among the important ones are [5-14]. There have been a large amount of research papers and monographs written on stochastic processes which are related to physical problems. In the area of waiting lines and queuing problems, some examples are [15-23]. Renewal theory was developed as the study of problems related to failure and replacement of components. The original and significant publications on renewal theory include [24-32]. The study of machine operation and maintenance problems is the central part of reliability theory. Motivated by the experiences during World War II with unreliable complex military systems, and by the unsuccessful events of satellite launching in the early part of the space program, reliability theory was developed as a result of the demand for more reliable systems. Some of the early papers in this field are [33-41]. Economical and social problems were also modeled and studied in the framework of dynamic probabilistic systems. Among the significant publications are [42-46].

As the complexity and scope of dynamic probabilistic systems continue to evolve, much research effort has been expanded in recent years in the development of new approaches for modeling and solutions to the problems. As a result, a large amount of papers and textbooks have been written in this field. References [47-71] are among the publications which relate closely to the materials covered in this thesis.

1.2 PROBLEM STATEMENT AND THESIS SUMMARY

In order that a physical system may be accurately analyzed, a representative math model of the system must be developed. The dynamical behavior of a physical system is generally influenced by many uncertain physical phenomena and fluctuations of natural forces. Therefore, representative models for physical systems, in general, are probabilistic. This thesis considers modeling and analysis of dynamic probabilistic systems whose state transition processes are continuous parameter Markov chains or semi-Markov processes. For the purpose of clarity and so that the development is physically motivated, machine operation and maintenance systems are considered as the underlying problems of the development. It should, however, be noted that the concept of modeling developed and the solutions derived are completely general and applicable to all Markovian and semi-Markovian systems. A unified matrix approach has been used in the problem formulations as well as in their solutions.

In Chapter 2, many commonly used system effectiveness measures for dynamic probabilistic systems are presented. The measures are defined in terms of the probabilistic and statistical properties of the systems. The definitions presented here parallel to those of Barlow et. al. [33, 49], Truelove [34], Bellman [39], Shooman [63] and others.

Chapter 3 begins with a brief review on some fundamental relationships for probabilistic system analysis. In particular, the interrelationships between the failure-time probability density function, the failure-rate function, and the reliability function are developed. New results of this chapter include the relationship between the reliability function and the mth order moment of the time-to-failure. In addition a sufficient condition for the existence of the moments is derived.

Chapter 4 considers the math modeling of Markovian systems. Modeling of some two unit redundant systems were considered by Barlow and Hunter [33], Dick [53], Garver [54], Srinivarsan [65], and Osaki [68, 49]. In this chapter the modeling methodology developed by the author in [55] is generalized. An example is provided to illustrate the general methodology. A new proof of this chapter is the derivation of the general vector differential equation which governs the state transition process of a Markovian system. The characterization for stationarity in terms of the characterization of the probability density function (p. d. f.) of the state transition process is discussed.

Chapter 5 considers the effectiveness analysis of stationary Markovian systems. In references [33, 40, 50, 53, 54, 65, 68], for example, effectiveness analyses were performed on some specific system configurations. The work here provides solutions for general system configurations. The results in this chapter are extensions of the work of the author in [55-60]. This chapter, together with Appendices B through E, represent the main contributions of this thesis on Markovian systems analysis. Matrix solutions for the system effectiveness measures defined in Chapter 2 are derived. The existence of the limiting solution for the vector differential equation of the system is proved, and the solution derived. Important properties pertaining to the state transition-rate matrix of the system are studied. Also treated in this chapter is the first passage time of the system from one subset of system states to another subset of system states. In addition, solutions for the p. d. f. and the general mth order moment of the first passage time are developed.

Chapter 6 discusses three different types of system up-time and down-time intervals for stationary ergodic Markovian systems in the steady state. They are: complete up-time (down-time) intervals, conditional and unconditional remaining up-time (down-time) intervals. Einhorn [77] developed mean up-time (MUT) and mean down-time (MDT) solutions for a class of systems which obey birth and death processes [7, 9, 10]. His solutions were later extended by Epstein [78] to contain systems obeying birth and death processes with general state transition rates. The work here further generalizes that of Epstein. In addition, the research investigates in depth the various up-time and down-time moments. In particular, MUT and MDT of a system comprised of independent subsystems are thoroughly studied. The materials in this chapter are mostly new and related to the work of the author in [62] and Buzacott [61].

Chapter 7 begins with an introduction to semi-Markov processes. Such a process can be viewed as a combination of Markov and renewal processes [26-30]. Pyke [51, 52] studied the properties of Markov renewal processes. Osaki [67], and Branson and Shah [71] employed semi-Markov processes to model systems with general repair-time distributions. The work in this chapter generalizes the modeling methodology for semi-Markovian systems. A simple three unit system is employed as a vehicle to illustrate the general methodology. The principle results are the solutions for the state transition probabilities of the imbedded Markov chain, and that for the conditional holding time distribution functions, of a general semi-Markovian system.

Chapter 8 considers analysis of ergodic semi-Markovian systems. The major contributions include derivation of the system of integral equations which govern the dynamical behavior of the state transition process, derivation of the limiting solutions for the integral equations, and the development of solution for the mean first passage time for the system to pass from one set of system states to another set.

Chapter 9 presents the conclusions and some suggested topics for further study.

Chapter 2

SYSTEM EFFECTIVE MEASURES FOR DYNAMIC PROBABILISTIC SYSTEMS

2.1 INTRODUCTION

This section defines many probabilistic and statistical measures of effectiveness for dynamic probabilistic systems. The measures presented here pertain to systems of machine operation and repair. It should be noted that effectiveness measures pertaining to other types of physical systems are generally the same or closely related to the ones presented here. The choice of appropriate measures for a given system depends on the function performed by the system and the condition under which the system is operated. Therefore, in general, a measure which is suitable for one type of system may not be suitable for another. Among the many measures which can be used for effectiveness evaluation of dynamic probabilistic systems, we will present the definitions of those which are considered to be more basic and important.

2.2 DEFINITIONS OF SOME IMPORTANT MEASURES

The following measures will be defined:

1. Pointwise availability
2. Reliability
3. Interval reliability
4. Interval availability

5. Steady state availability
6. Limiting interval reliability
7. Mean time-to-first-system-failure
8. Several mean times of interest in the steady state

The first four measures are normally used in relation to an initial and limited time period of the system operation. Except Number 7, the other measures deal with the steady state probabilistic and statistical properties of the system. Therefore these other measures are particularly suitable for repairable systems which are subject to long-term operation.

> Pointwise availability: For a given initial condition (IC), pointwise availability of a system at time $t > 0$ is defined as the probability that the system will be up at that time epoch t; this measure is denoted by $A(t|IC)$. (2.2-1)

The above measure concerns the condition of the system at the specific time instant in question only.

> Reliability: For a given initial condition, reliability of a system for an interval $[0, t]$ is defined as the conditional probability that the system will be up during the entire time interval $[0, t]$; this measure is denoted by $R(t|IC)$. (2.2-2)

The measure $R(t|IC)$ is applicable to both repairable as well as non-repairable systems. For non-repairable systems, $R(t|IC)$ would be equivalent to the measure $A(t|IC)$. This is because a non-repairable system is up at time t if and only if it is up diring the entire interval $[0, t]$.

Interval Reliability: For a given initial condition, interval reliability of a system for an interval $[t_1, t_2]$, where $0 < t_1 < t_2$, is defined as the conditional probability that the system is up during the entire interval $[t_1, t_2]$; this measure is denoted by $IR(t_1, t_2 | IC)$. (2.2-3)

For a non-repairable system $IR(t_1, t_2 | IC)$ would be the same as $R(t_2 | IC)$. For a repairable system, however, this measure disregards the up or down condition of the system prior to time t_1.

Interval availability: For a given initial condition, the interval availability of a system for an interval $[t_1, t_2]$, where $0 \leq t_1 < t_2$, is defined as the expected fraction of time within the interval that the system will be up; this measure is denoted by $IA(t_1, t_2 | IC)$. (2.2-4)

From this definition, we see the relationship between $IA(t_1, t_2 | IC)$ and $A(t | IC)$ defined in (2.2-1) as:

$$IA(t_1, t_2 | IC) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A(t | IC) \, dt \tag{2.2-5}$$

The above measures of system effectiveness are applicable to an initial and a finite period of system operation. For a system with stationary statistics, we say that the system is in its steady state after it has operated for a very long period of time. The following two system effectiveness measures pertain to the steady state condition of a system.

Steady State Availability: Steady State availability of an ergodic stationary system is defined as the probability that the system is up at a random time epoch in the steady state; this measure is denoted by A. (2.2-6)

In this definition, initial conditions of the system is not specified. This is because the initial conditions of an ergodic and stationary system has no effect on its steady state properties. The steady state availability is equivalent to the expected fraction of time in the long run that the system is up. Therefore, the steady state availability is also known as the "limiting interval availability" [39], or "limiting efficiency [33].

$$A = \lim_{t_2 \to \infty} IA(t_1, t_2) \tag{2.2-7}$$

Limiting interval reliability: Limiting interval reliability of an ergodic stationary system for a period of time T is defined as the interval reliability of the system for a period of time T in the steady state; this measure is denoted by LIR(T) (2.2-8)

Therefore

$$LIR(T) = \lim_{t_1 \to \infty} IR(t_1, t_1 + T | IC) \tag{2.2-9}$$

where the initial condition would be arbitrary. This quantity is also known as "strategic reliability" [34].

In the foregoing definitions, the measures of system effectiveness are expressed in terms of the probability measures of the system. Other systems effectiveness measures which are of basic importance and widely used are the statistical measures. Their definitions are presented below.

> Mean time-to-first-system-failure (Repair): For a given initial condition, the Mean time-to-first-system failure (repair) of a system is defined as the conditional expected first passage time of the system to a system down (up) condition; this measure is denoted by $MTTFSF_{IC}$ ($MTTFSR_{IC}$). (2.2-10)

It can be seen that the $MTTFSF_{IC}$ or $MTTFSR_{IC}$ can be considered as a generalized first passage time of a system from one set of system conditions to another set of system conditions. This point will become clear when the mathematical expressions for these measures are developed.

For ergodic stationary systems, there are other statistical measures which relate to the up and down time intervals of the system in the steady state. They are known as: system mean up-time (MUT), system mean down-time (MDT), unconditional mean remaining up-time (MRUT), unconditional mean remaining down-time (MRDT), conditional mean remaining up-time ($MRUT_u$), and conditional mean remaining down-time ($MRDT_d$). The definitions for these measures are:

> System MUT (MDT): MUT (MDT) of an ergodic stationary system is defined as the average length of time the system continuosuly stays up (down) from an instant it just comes up (goes down) in the steady state. (2.2-11)

System MRUT (MRDT): MRUT (MRDT) of an ergodic stationary system is defined as the average remaining up-time (down-time) of the system as the system is observed at a random time epoch in the steady state. (2.2-12)

System $MRUT_u$ ($MRDT_d$): $MRUT_u$ ($MRDT_d$) of an ergodic stationary system is defined as the average remaining up-time (down-time) of the system under the condition that as the system is observed at a random time epoch in the steady state it is found to be up (down). (2.2-13)

In definition (2.2-12) the system's condition (up or down) at the random observation time epoch is not given. Therefore, system MRUT (MRDT) takes into account the probability that the system is down (up) at the random time epoch.

Chapter 3

SOME MATHEMATICAL PRELIMINARIES

3.1 INTRODUCTION

This section introduces the failure-time density function and the failure-rate function of a system. The failure-rate function is also known as the hazard rate function or as the age-specific failure-rate function [26]. By "failure-time" of a system we mean the time to failure of the system, or the life span of the system before it fails. It will be shown that the failure-time density function and the failure-rate function are related to each other. As such specification of one is equivalent to specification of the other. Their relationship will be derived in Section 3.2. In Section 3.3, the relationship between the mean time-to-first-system-failure and the system reliability function will be developed.

3.2 FAILURE-TIME DENSITY AND FAILURE RATE FUNCTIONS

The time to failure of a system is a non-negative random variable. We assume that probability density function of this random variable is absolutely continuous over the interval $(0, \infty)$. For a given initial condition of the system, let this function be denoted by $f(t|IC)$. Then

$f(t|IC)\,dt$ = the probability that the failure-time of the system lies in the interval $(t, t+dt)$. (3.2-1)

Let $h(t|IC)$ denote the failure-rate function of the system. The function $h(t|IC)$ is defined as:

$h(t|IC)dt$ = the conditional probability the system will fail in the interval $(t, t + dt)$ given that it has not failed up to time t.

(3.2-2)

The right-hand side (RHS) of Eq. (3.2-2) is equal to $f(t|IC)dt/\int_t^\infty f(x|IC)dx$. Therefore we obtain the relationship for $h(t|IC)$ in terms of $f(t|IC)$.

$$h(t|IC) = \frac{f(t|IC)}{\int_t^\infty f(x|IC)dx} \qquad (3.2\text{-}3)$$

The inverse relationship for $f(t)$ in terms of $h(t|IC)$ can be obtained by expressing the integral $\int_t^\infty f(x|IC)dx$ in terms of $h(t|IC)$. Note that by definition (2.2-1), this integral is nothing but the reliability function. That is:

$$R(t|IC) = \int_t^\infty f(x|IC)dx. \qquad (3.2\text{-}4)$$

Differentiating the above with respect to t and multiplying by -1 we have:

$$f(t|IC) = -\frac{d}{dt} R(t|IC) \qquad (3.2\text{-}5)$$

Substituting Eqs. (3.2-4) and (3.2-5) into Eq. (3.2-3) gives:

$$h(t|IC) = -\frac{1}{R(t|IC)}\frac{d}{dt}R(t|IC)$$

$$= -\frac{d}{dt}\ln R(t|IC) \qquad (3.2\text{-}6)$$

To express $R(t|IC)$ in terms of $h(t)$, we first integrate Eq. (3.2-6) from 0 to t, and then simplify to obtain:

$$R(t|IC) = R(0|IC)\exp\{-\int_0^t h(x|IC)dx\} \qquad (3.2\text{-}7)$$

where $R(0|IC)$ is known from the given initial conditions of the system. Note that Eq. (3.2-4) expresses the reliability function in terms of the failure-time density function, and Eq. (3.2-7) expresses the same function in terms of the failure-rate function. By Eqs. (3-2-3), (3.2-4) and (3.2-7) we can express $f(t|IC)$ in terms of $h(t|IC)$.

$$f(t|IC) = R(0|IC)\,h(t)\exp\{-\int_0^t h(x|IC)dx\}\ . \qquad (3.2\text{-}8)$$

Eqs. (3.2-3) and (3.2-8) show that the functions $f(t|IC)$ and $h(t|IC)$ are espressible in terms of each other.

3.3 MEAN TIME-TO-FIRST-SYSTEM-FAILURE

In this section we shall derive the mean time-to-first-system-failure in terms of the reliability function. If $f(t|IC)$ represents the first passage time density of the system from a given initial condition to failure, then by definition (2.2-10) $MTTFSF_{IC}$ is given by:

$$MTTFSF_{IC} = \int_0^\infty t\, f(t|IC)\, dt \qquad (3.3\text{-}1)$$

Substitute $f(t|IC)$ from Eq. (3.2-5) into the above we obtain:

$$MTTFSF_{IC} = -\int_0^\infty t \frac{d}{dt} R(t|IC)\, dt. \qquad (3.3\text{-}2)$$

Integrating by parts gives:

$$MTTFSF_{IC} = [-t\, R(t|IC)]_0^\infty + \int_0^\infty R(t|IC)\, dt. \qquad (3.3\text{-}3)$$

Since $R(t|IC)$ is bounded between 0 and 1, the first term on the RHS is obviously 0 for $t = 0$. For a physical system, the failure-rate function is always greater than zero, i.e.,

$$h(t) \geq \epsilon > 0 \qquad \text{for all } t. \qquad (3.3\text{-}4)$$

Under this condition we shall show that the first term on the RHS of Eq. (3.3-3) goes to 0 as $t \to \infty$. Using Eq. (3.2-7) we have:

$$tR(t|IC) = t\,R(0|IC)\exp\{-\int_0^t h(x)\,dx\}$$

$$\leq t\,R(0|IC)\,e^{-t\epsilon}. \qquad (3.3\text{-}5)$$

By L'Hospital's rule,

$$\lim_{t\to\infty} \frac{t}{e^{t\epsilon}} = \lim_{t\to\infty} \frac{1}{\epsilon e^{t\epsilon}} = 0. \qquad (3.3\text{-}6)$$

Therefore,

$$\lim_{t\to\infty} t\,R(t|IC) \leq 0 \qquad (3.3\text{-}7)$$

But, $t\,R(t|IC)$ is a non-negative quantity. Hence,

$$\lim_{t\to\infty} t\,R(t|IC) = 0 \qquad (3.3\text{-}8)$$

This shows that under the condition given by Eq. (3.3-4), the first term on the RHS of Eq. (3.3-3) is zero. The second term can be shown to exist under the same condition.

$$\int_0^\infty R(t|IC)dt = \int_0^\infty R(0|IC)\exp\{-\int_0^t h(x)\,dx\}\,dt$$

$$\leq R(0|IC)\int_0^\infty e^{-\epsilon t}\,dt$$

$$= \frac{R(0|IC)}{\epsilon} < \infty \qquad (3.3\text{-}9)$$

We have proved the following theorem.

Theorem 3.1

If the failure rate function of a system obeys the condition of Eq. (3.3-4), the mean time-to-first-system-failure of the system exists and is equal to the area under the reliability function,

$$MTTFSF_{IC} = \int_0^\infty R(t|IC)\, dt. \quad (3.3\text{-}10)$$

Under the condition of Eq. (3.3-4), we will now show that not only the first moment (the mean) exists, but also the moments of all finite orders exist. The nth moment is:

$$\int_0^\infty t^n f(t|IC)\, dt = -\int_0^\infty t^n \frac{dR(t|IC)}{dt}\, dt. \quad (3.3\text{-}11)$$

Integrating by parts gives:

$$\int_0^\infty t^n f(t|IC)dt = -[t^n R(t|IC)]_0^\infty + \int_0^\infty n\, t^{n-1} R(t|IC)\, dt. \quad (3.3\text{-}12)$$

The first term on the RHS is zero for t = 0. To evaluate for the upper limit, by virtue of Eq. (3.3-4) we write

$$t^n R(t|IC) \leq t^n R(0|IC)\, e^{-\epsilon t}$$

$$= R(0|IC) \frac{t^n}{e^{\epsilon t}}. \quad (3.3\text{-}13)$$

Then applying L'Hospital's rule, the RHS of Eq. (3.3-13) tends to zero as $t \to \infty$. Therefore, the nth moment is

$$\int_0^\infty t^n f(t|IC)\,dt = n \int_0^\infty t^{n-1} R(t|IC)\,dt \tag{3.3-14}$$

Under the condition of Eq. (3.3-4), we have

$$n \int_0^\infty t^{n-1} R(t|IC) \leq \int_0^\infty n\, t^{n-1} R(0|IC) e^{-\epsilon t} dt$$

$$= R(0|IC) \frac{n!}{\epsilon^n} < \infty . \tag{3.3-15}$$

We have proved the following theorem.

Theorem 3.2

If the failure-rate function of a system obeys the condition of Eq. (3.3-4), for any positive integer n, the nth order moment of the time-to-first-system-failure exists and is related to the reliability function by Eq. (3.3-14).

Corollary 3.2

The variance of the time-to-failure of a system whose failure-rate-function is bound away from zero is

$$\mathrm{Var}(TTFSF|IC) = 2\int_{o}^{\infty} t\,R(t|IC)\,dt - \left(\int_{o}^{\infty} R(t|IC)dt\right)^2 \quad (3.3\text{-}16)$$

The proof for this corollary follows directly from Eq. (3.3-14). We will present an alternative proof by using Laplace transform. Let $f^*(s|IC)$ denote the Laplace transform of $f(t|IC)$,

$$f^*(s|IC) = \int_0^{\infty} f(t|IC)\,e^{-st}dt. \quad (3.3\text{-}17)$$

For all positive integer n,

$$\int_o^{\infty} t^n f(t|IC)dt = \left[(-1)^n \frac{d^n}{ds^n}\; f^*(s|IC)\right]_{s=0} \quad (3.3\text{-}18)$$

If we denote the Laplace transform of $R(t|IC)$ by $R^*(s|IC)$, then the transform of Eq. (3.2-5) is

$$f^*(s|IC) = -[sR^*(s|IC) - R(0|IC)] \quad (3.3\text{-}19)$$

Substituting Eq. (3.3-19) into the RHS of Eq. (3.3-18), and simplifying for the cases of n = 1 and 2 we have:

$$\int_o^\infty tf(t|IC)dt = [R^*(s|IC)]_{s=0} \quad (3.3\text{-}20)$$

and

$$\int_o^\infty t^2 f(t)dt = -2[\frac{d}{ds} R^*(s|IC)]_{s=0} \quad (3.3\text{-}21)$$

Therefore, the Laplace transform of the Var(TTFSF) is

$$2[(-1) \frac{d}{ds} R^*(s|IC)]_{s=0} - [R^*(s|IC)]^2_{s=0} \quad (3.3\text{-}22)$$

Taking the inverse transform, we obtain

$$\text{Var(TTFSF)} = 2\int_o^\infty t\, R(t|IC)dt - [\int_o^\infty R(t|IC)\, dt]^2 \quad (3.3\text{-}23)$$

This completes an alternative proof for Corollary 3.2.

Chapter 4

MODELING OF MARKOVIAN SYSTEMS

4.1 INTRODUCTION

Development of a math model for a physical system is an essential part of system analysis. Depending on the type of analysis to be performed on the system, the model needed could be quite different. For example, if one is interested in the accuracy or performance of the outputs for different inputs, the model needed would be an input-output transfer function of the system. In general the inputs and outputs could be either deterministic or stochastic. For dynamic probabilistic systems analysis, however, the model required is quite different from an input-output model. This is because the purpose of such analysis is not to analyze the input-output accuracy of the system, but rather to analyze the stochastic behavior of the state transition process of the system.

In Section 4.2 we will give an introduction of discrete and continuous parameter Markov chains, and classification of states in a Markov chain. Section 4.3 gives a definition for Markovian systems and the development of the general vector differential equation which governs the dynamical behavior of the probability state vector of a Markovian system. In Section 4.4 the general characterization of a stationary Markovian system is discussed. Section 4.5 gives a simple example to illustrate the consturction of a math model for a physical system.

4.2 MARKOV CHAINS AND STATE CLASSIFICATION

There are two kinds of Markov chains depending upon their time parameters: discrete parameter Markov chain, and continuous parameter Markov chain. We shall restrict our attention to Markov chains which have finite state space. The notion of discrete Markov chain having a finite state space may be introduced by generalizing the notion of a sequence of independent trials. Consider that a sequence of consecutive trials is performed, in each of which one of the n_o mutually exclusive and exhaustive events $E_1, E_2, \ldots, E_{n_o}$ may be realized. Let the outcome of the kth trial be denoted by x_k, which is a random variable. We say that the sequence of trials or the associated random variables x_k form a Markov chain of first order if the conditional probability of occurrence of the event E_i, $i = 1, 2, \ldots, n_o$, in the kth trial depends only on which event has occurred in the (k-1)th trial and is not affected by what events have occurred in the earlier trials. If E_{i_k} denotes the event occurs on the kth trial, then

$$\Pr\{x_k = E_{i_k} | x_o = E_{i_o}, x_1 = E_{i_1}, \ldots, x_{k-1} = E_{i_{k-1}}\}$$

$$= \Pr\{x_k = E_{i_k} | x_{k-1} = E_{i_{k-1}}\} \text{ for all positive integers } k \tag{4.2-1}$$

We say that $\{x_k; k \in Z^+\}$ is a discrete parameter Markov chain with state space $\{E_1, E_2, \ldots, E_{n_o}\}$. The probability mass density function $\{Pr(x_o = E_i); i = 1, 2, \ldots, n_o\}$, of the random variable x_o is called the initial probability distribution of the Markov chain. The conditional probabilities, $Pr\{x_k = E_j | x_{k-1} = E_i\}$ for all i, j = 1, 2, ..., n_o, are called the state transition probabilities. The Markov chain is said to be time homogeneous if the state transition probabilities are independent of the number of trials (the discrete time parameter k). We denote the transition probabilities of a time homogeneous Markov chain by $p_{i,j}$, which are defined as:

$$p_{i,j} = Pr\{x_k = E_j | x_{k-1} = E_i\}, \quad \text{for all } k \in Z^+ \tag{4.2-2}$$

where Z^+ denotes the set of non-negative integers.

The transition probabilities of a time homogeneous, first order, discrete Markov chain can be arranged in a matrix form as follows:

$$(p_{i,j}) = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} \cdots & p_{1,n_o} \\ p_{2,1} & p_{2,2} & p_{2,3} \cdots & p_{2,n_o} \\ \vdots & & & \\ p_{n_o,1} & p_{n_o,2} & p_{n_o,3} \cdots & p_{n_o,n_o} \end{bmatrix} \tag{4.2-3}$$

In this matrix, the (i, j) element is the state transition probability from state E_i to state E_j of the Markov chain in a single transition. Therefore, elements of $(p_{i,j})$ necessarily possess the following properties:

$$p_{i,j} \geq 0 \qquad \text{for all } i, j = 1, 2, \ldots, n_o \tag{4.2.4}$$

$$\sum_{j=1}^{n_o} p_{i,j} = 1 \qquad \text{for all } \quad i = 1, 2, \ldots, n_o \tag{4.2.5}$$

A square matrix having the above properties is known as a stochastic matrix.

A continuous parameter Markov chain is a generalization of the discrete parameter Markov chain in the time parameter. In this case, the role of the one-step transition probabilities is played by the state transition rates (also known as transition intensities [9]) which will be defined later when we develop the differential equations for the system state probabilities.

A continuous parameter stochastic process $\{s(t);\ t \geq 0\}$ with discrete and finite state space $S = \{1, 2, \ldots, n_o\}$ is said to be a continuous parameter Markov chain if, for any set of k time points $t_1 < t_2 < t_3 \ldots < t_k$ in the index set of the process,

$$\Pr\{s(t_k) = i_k | s(t_{k-1}) = i_{k-1},\ s(t_{k-2}) = i_{k-2}, \ldots, s(t_1) = i_1\}$$

$$= \Pr\{s(t_k) = i_k | s(t_{k-1}) = i_{k-1}\} \quad \text{for all } i_r \in S \tag{4.2-6}$$

The principle property of this process is the Markov property which can be stated as: Given the state of the process at any time t, future changes of the process is not influenced by any past history of the process prior to time t. This property leads to the Chapman-Kolmogorov equation for the Markov process. Let $\tau < r < t$ be time points in the index set of the process. Further, let $s(\tau) = i$, and $s(t) = j$. The passage of the process from state $i \in S$ at time τ to state $j \in S$ at time t must occur via some state $k \in S$ at time r. This gives:

$$\Pr\{s(t) = j | s(\tau) = i\} = \sum_{k \in S} \Pr\{s(t) = j | s(r) = k,\ s(\tau) = i\} \cdot \Pr\{s(r) = k | s(\tau) = i\} \tag{4.2-7}$$

Applying the Markov property, we have:

$$\Pr\{s(t) = j | s(\tau) = i\} = \sum_{k \in S} \Pr\{s(t) = j | s(r) = k\} \Pr\{s(r) = k | s(\tau) = i\} \tag{4.2-8}$$

identically for all $\tau < r < t$. This is the Chapman-Kolmogorov equation. It is necessary that the state transition probabilities possess the following properties:

$$\Pr\{s(t) = j | s(\tau) = i\} \geq 0 \quad \text{for all } i, j \in S \tag{4.2-9}$$

and

$$\sum_{j \in S} \Pr\{s(t) = j \mid s(\tau) = i\} = 1 \quad \text{for all } i \in S \tag{4.2-10}$$

The states of a Markov chain may be classified according to their communicative properties. The following definitions are equivalent to those given in [14].

> An ergodic set of states: is a set in which every state can be reached from every other state, and which cannot be left once entered.
>
> A transient state: is a state such that if the process started from that state, the probability that the process ever returns to the state is less than 1.
>
> An absorbing state: is a state which once entered is never left.

A Markov chain is said to be ergodic if it is possible to go from every state to every other state. That is to say, an ergodic chain is one whose states form a single ergodic set. A Markov chain is said to be absorbing if there is at least one absorbing state, and such that an absorbing state can be reached from every state of the chain.

In view of the algebraic theory of order relations, it can be shown that the states of a Markov chain can be partitioned into eigenvalue classes by some equivalence relation, and that the minimal elements of an induced partial ordering of the eigenvalue classes are ergodic sets. This is shown in Appendix A.

4.3 MARKOVIAN SYSTEMS

We shall be interested in dynamical systems whose state transition processes are special continuous parameter Markov chains. The specialization being the following additional assumption that the chain must satisfy.

Assumption 1: To every pair of states $i, j \in S$ with $i \neq j$, there corresponds a continuous function $m_{i,j}(t) \geq 0$ such that as $\Delta t \to 0$

$$\frac{\Pr\{s(t+\Delta t) = j \mid s(t) = i\}}{\Delta t} \to m_{i,j}(t). \tag{4.3-1}$$

The function $m_{i,j}(t)$ is known as the state transition rate from state i to state j. A Markovian system is defined as a system whose state transition process is a continuous parameter Markov chain satisfying Assumption 1.

The probabilistic interpretation of Assumption 1 is: given that at time t the system is in state i, the probability that during $(t, t+\Delta t)$ the system changes to state j is $m_{i,j}(t)\,\Delta t + 0(\Delta t)$, and the probability of more than one change occurs is $0(\Delta t)$. The term $0(\Delta t)$ denotes a quantity which is of smaller order of magnitude than Δt. That is to say, as $\Delta t \to 0$, the ratio $\frac{0(\Delta t)}{\Delta t} \to 0$.

Before proceeding to derive the differential equations which govern the dynamical behavior of the state probabilities of the system, the following notations are introduced.

$$P_i(t) = \Pr\{s(t) = i\} \tag{4.3-2}$$

$$P_i(t|\cdot) = \Pr\{s(t) = i|\cdot\} \tag{4.3-3}$$

and

$$\underline{P}(t) = [P_1(t), P_2(t), \ldots, P_{n_o}(t)] \tag{4.3-4}$$

To derive the differential equations, we start with the Chapman-Kolmogorov equation. If $\tau < t$ and $\underline{P}(\tau)$ is given, then the Chapman-Kolmogorov equation corresponding to Eq. (4.2-8) is:

$$P_j(t+\Delta t|\underline{P}(\tau)) = \sum_{k \in S} P_j(t+\Delta t|s(t) = k)P_k(t|\underline{P}(\tau)) \tag{4.3-5}$$

We use $S - \{j\}$ to denote the complement of the set $\{j\}$ with respect to S, i.e., the set of elements that belong to S but not to $\{j\}$. Then Eq. (4.3-5) can be written as:

$$P_j(t+\Delta t|\underline{P}(\tau)) = P_j(t+\Delta t|s(t)=j)P_j(t|\underline{P}(\tau)) + \sum_{k \in S-\{j\}} P_j(t+\Delta t|s(t)=k)P_k(t|\underline{P}(\tau)) \tag{4.3-6}$$

By the necessary condition given by Eq. (4.2-10), $P_j(t+\Delta t|s(t) = j)$ can be replaced by:

$$P_j(t+\Delta t|s(t)=j) = 1 - \sum_{k \in S-\{j\}} P_k(t+\Delta t|s(t) = j). \tag{4.3-7}$$

Substituting Eq. (4.3-7) into Eq. (4.3-6), and after rearranging terms we have:

$$P_j(t+\Delta t|\underline{P}(\tau)) - P_j(t|\underline{P}(\tau)) = \sum_{k\in S-\{j\}} [P_j(t+\Delta t|s(t)=k)P_k(t|\underline{P}(\tau)) - P_k(t+\Delta t|s(t)=j)P_k(t|\underline{P}(\tau))] \tag{4.3-8}$$

First divide both sides of the above by Δt, and then let $\Delta t \to 0$, we obtain

$$\frac{\partial P_j(t|\underline{P}(\tau))}{\partial t} = \sum_{k\in S-\{j\}} [\lim_{\Delta t\to 0} \frac{P_j(t+\Delta t|s(t)=k)}{\Delta t} P_k(t|\underline{P}(\tau)) - \lim_{\Delta t\to 0} \frac{P_k(t+\Delta t|s(t)=j)P_j(t|\underline{P}(\tau))}{\Delta t}] \tag{4.3-9}$$

By Assumption 1, the above can be written as:

$$\frac{\partial P_j(t|\underline{P}(\tau))}{\partial t} = \sum_{k\in S-\{j\}} [m_{k,j}(t)\, P_k(t|\underline{P}(\tau)) - m_{j,k}(t)\, P_j(t|\underline{P}(\tau))] \tag{4.3-10}$$

For notational convenience, we define $m_{j,j}(t)$ as:

$$m_{j,j}(t) = -\sum_{k\in S-\{j\}} m_{j,k}(t) \tag{4.3-11}$$

Then Eq. (4.2-24) becomes

$$\frac{\partial}{\partial t} P_j(t|\underline{P}(\tau)) = \sum_{k \in S} m_{k,j}(t)\, P_k(t|\underline{P}(\tau)). \qquad (4.3\text{-}12)$$

This equation holds for all $j \in S$. Let $M(t)$ denote the matrix of $m_{i,j}(t)$ with the (i, j) element of M(t) being $m_{i,j}(t)$, that is

$$M(t) = (m_{i,j}(t)). \qquad (4.3\text{-}13)$$

Then the system of partial differential equations can be compactly written in the form of a vector partial differential equation:

$$\frac{\partial}{\partial t} \underline{P}(t|\underline{P}(\tau)) = \underline{P}(t|\underline{P}(\tau))\, M(t) \qquad (4.3\text{-}14)$$

In the above equation, the parameter τ of the vector $\underline{P}(\tau)$ is not a variable. In fact $\underline{P}(\tau)$ is a given probability state vector of the process at time τ. Therefore, Eq. (4.3-14) is simply a first order vector differential equation with t as the only variable.

$$\frac{d}{dt} \underline{P}(t|\underline{P}(\tau)) = \underline{P}(t|\underline{P}(\tau))\, M(t) \qquad (4.3\text{-}15)$$

When M(t) is a function of t, we say that the system is time varying. In the case when M(t) is a constant matrix,

$$M(t) = M \qquad (4.3\text{-}16)$$

we say that the system is time homogeneous or stationary.

We have proved the following theorem.

Theorem 4.1

The dynamical behavior of the state probabilities of a Markovian system is governed by a system of linear differential equations described by Eq. (4.3-15).

4.4 CHARACTERIZATION OF STATIONARITY IN MARKOVIAN SYSTEMS

In the preceeding section, we defined a Markovian system to be one whose state transition process obeys a continuous parameter Markov chain and the state transition rates of the system satisfy Assumption 1. We say that the system is stationary when the transition rates are independent of time. Let i, j be any two states of the system such that $i \neq j$. Given that the system is in state i, the waiting time of the transition process from state i to state j is a random variable. We denote this random variable by $\tau_{i,j}$ and the probability density function by $f_{i,j}(t)$. In this section we study the characterization for stationarity of a Markovian system in terms of the characterization for the density function $f_{i,j}(t)$ and vice versa.

From Eq. (4.3-1), the interpretation for the state transition rate function $m_{i,j}(t)$ is

$m_{i,j}(t)\,dt$ = the conditional probability that the system will be in state j at time t+dt, given that it is in state i at time t.

(4.4-1)

We observe the equivalence of the above interpretation to the definition for the failure-rate function defined in Eq. (3.2-2). Therefore, by Eq. (3.2-8) the probability density function $f_{i,j}(t)$ can be expressed in terms of $m_{i,j}(t)$ as follows:

$$f_{i,j}(t) = m_{i,j}(t) \exp\{-\int_0^t m_{i,j}(x)\,dx\} \qquad (4.4\text{-}2)$$

If the system is given to be stationary, Eq. (4.2-2) reduces to

$$f_{i,j}(t) = m_{i,j} e^{-m_{i,j}t} \qquad (4.4\text{-}3)$$

which is an exponential density function. The above shows that, if a Markovian system is given to be stationary, then the probability density functions for all $\tau_{i,j}$, where $i, j \in S$ and $i \neq j$, are exponentially distributed.

Now consider a Markovian system which is such that, for all $i, j \in S$ and $i \neq j$, $f_{i,j}(t)$ is an exponential distribution. That is,

$$f_{i,j}(t) = a_{i,j} e^{-a_{i,j} t} \qquad (4.4\text{-}4)$$

where

$$a_{i,j} \geq 0 \qquad (4.4\text{-}5)$$

Then by Eq. (3.2-3), the state transition rate from state i to state j is

$$m_{i,j}(t) = \frac{a_{i,j} e^{-a_{i,j} t}}{\int_t^{\infty} a_{i,j} e^{-a_{i,j} x} \, dx} \qquad (4.4\text{-}6)$$

Simplifying the RHS gives

$$m_{i,j}(t) = a_{i,j} \qquad (4.4\text{-}7)$$

This holds for all $i \neq j$. For $i = j$, by Eq. (4.3-11) we have

$$m_{i,i}(t) = - \sum_{k \in S - \{i\}} a_{i,k} \qquad (4.4\text{-}8)$$

Eqs. (4.4-7) and (4.4-8) show that all the transition rates are independent of time, which means that the system is stationary.

We have proved the following theorem.

<u>Theorem 4.2</u>

A Markovian system is stationary if and only if for all $i, j \in S$ and $i \neq j$ the probability density function for $\tau_{i,j}$ is exponentially distributed.

We give an interpretation of the implication of this theorem on physical systems of machine operation and repair. The states of such a system are defined by different combinations of the up and down conditions of the units comprising the system. A transition from one state to another occurs when the condition of a unit changes, physically this means when an operating unit fails or when a repair is completed on a failed unit. The implication of Theorem 4. 2 is that, a physical system is Markovian and stationary if and only if the time-to-failure and the time-to-repair for each unit in the system are exponentially distributed.

4. 5 MATH MODELING OF A TWO UNIT REDUNDANT SYSTEM

In this section we shall illustrate the methodology for construction of the transition-rate matrix of a physical system. A very simple two unit system will be used for this purpose. Figure 4-1 shows the block diagram of this system.

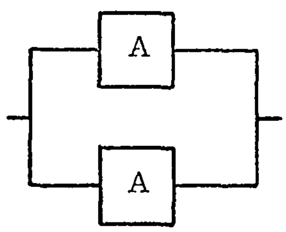

Figure 5-1 Block Diagram of the Sample System

It is assumed that the two A units are identical and functionally redundant.

The operation and repair policies of the system are as follows:

Operation policy:	When both units are operable (up), one unit is used on-line and the other unit is active off-line. If the on-line unit fails, the off-line unit is instantaneously switched on-line provided the off-line unit is up. Perfect switching is assumed.
Repair policy:	One repair crew is available to service the failures, and the policy for service is first-come, first-served.

The failure and repair rates of the units are:

λ_1 = failure rate of an on-line unit

λ_2 = failure rate of an active off-line unit

μ_1 = repair rate of a unit when failed from operating on-line

μ_2 = repair rate of a unit when failed from active off-line operation

For this system, five system states are possible. They are defined as follows:

Representation	System State Description
1	Both units are up, one is used on-line and the other is active off-line.
2	One unit is down from operating on-line, and the other is up on-line.
3	One unit is down from active off-line operation, and the other is up on-line.
4	One unit is down from operating on-line, and the other is down from active off-line operation.
5	Both units are down from operating on-line.

Note that the state that both units are down from active off-line operation does not exist. This is because the operation policy is such that whenever there is only one up unit, this unit will be operated on-line.

The transition rates from one state to another can be found by reasoning as follows:

First consider state 1 and state 2. Transition from state 1 to state 2 occurs when the on-line unit fails. Since, by hypothesis, the failure rate of the on-line unit is λ_1, we have:

$$m_{1,2} = \lambda_1 \qquad (4.5\text{-}1)$$

It can be seen that transition from state 2 to state 1 occurs when a repair is completed on the unit failed from operated on-line. Since the repair rate for such a unit is μ_1, we have

$$m_{2,1} = \mu_1 \tag{4.5-2}$$

By similar reasonings, the transition rates between state 1 and state 3 are found to be:

$$m_{1,3} = \lambda_2 \tag{4.5-3}$$

$$m_{3,1} = \mu_2 \tag{4.5-4}$$

Now consider state 1 and state 4. A one-step transition from state 1 to state 4 is not possible since such a transition would require two or more changes for the conditions of the units in an arbitrarily small interval. Therefore,

$$m_{1,4} = 0 = m_{4,1} \tag{4.5-5}$$

By using reasonings as above, transition rates between all other pairs of states can be determined. The transition-rate matrix for the system is:

$$
M = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & 0 & 0 & \lambda_1 \\ \mu_2 & 0 & -(\lambda_1+\mu_2) & \lambda_1 & 0 \\ 0 & \mu_2 & 0 & -\mu_2 & 0 \\ 0 & \mu_1 & 0 & 0 & -\mu_1 \end{bmatrix} \tag{4.5-6}
$$

Note that the diagonal terms in this matrix are determined by applying Eq. (4.3-11). Once M is found, the math model of the system is established. The two A units of the system are functionally redundant means that the system is up (i.e., operating satisfactorily) if at least one A unit is operating on-line. This in turn means that states 1, 2 and 3 are the up-states, and states 4 and 5 are the down-states of the system.

Suppose at the start of the system, the condition of both A units are known to be up. It is customary to count the initial starting time of a system as time 0. Then the initial probability state vector of the system is

$$
\underline{P}(0) = [1, 0, 0, 0, 0] \tag{4.5-7}
$$

Under this condition, $\underline{P}(t|\underline{P}(0))$ is the state probability vector of the system at t time units after start.

The preceding example, in spite of its simplicity, illustrates the general methodology and the essential steps in developing the math model of a physical system. In the next section, we will develop solutions for many measures of system effectiveness for the general Markovian system model.

Chapter 5

EFFECTIVENESS ANALYSIS OF STATIONARY MARKOVIAN SYSTEMS

5.1 INTRODUCTION

We recall the definitions of many system effectiveness measures stated in Section 2.2. The main effort of this section is to develop general solutions for these measures for stationary Markovian systems. First, two notations for subsets of the set of system states will be introduced.

S_u = the subset containing all up-states of the system. (5.1-1)

S_d = the subset containing all down-sets of the system. (5.1-2)

Since a system state is either an up-state or a down-state, the following relations are obvious.

$$S_u \cup S_d = S \quad (5.1\text{-}3)$$

and

$$S_u \cap S_d = \emptyset \quad (5.1\text{-}4)$$

We shall assume the order of S_u to be k_o. This implies the order of S_d to be $(n_o - k_o)$ since the order of S is n_o. Without loss of generality, we shall assume the elements in S_u and S_d are as follows:

$$S_u = \{1, 2, 3, \ldots, k_o\} \qquad (5.1\text{-}5)$$

and

$$S_d = \{k_o+1, k_o+2, k_o+3, \ldots, n_o\} \qquad (5.1\text{-}6)$$

We will use $\underline{P}_u(\cdot)$ and $\underline{P}_d(\cdot)$ to denote the subvectors for the system up-states and down-states respectively. That is,

$$\underset{(1\times k_o)}{\underline{P}_u(\cdot)} = [P_1(\cdot), P_2(\cdot), P_3(\cdot), \ldots, P_{k_o}(\cdot)] \qquad (5.1\text{-}7)$$

and

$$\underset{(1\times(n_o-k_o))}{\underline{P}_d(\cdot)} = [P_{k_o+1}(\cdot), P_{k_o+2}(\cdot), P_{k_o+3}(\cdot), \ldots, P_{n_o}(\cdot)] \qquad (5.1\text{-}8)$$

In terms of these subvectors, the system state vector can be written as:

$$\underset{(1\times n_o)}{\underline{P}(\cdot)} = [\underline{P}_u(\cdot)\ \underline{P}_d(\cdot)] \qquad (5.1\text{-}9)$$

A $(1\times n)$ vector of zeros will be denoted by:

$$\underline{0}_n = [0, 0, \ldots, 0] \qquad (5.1\text{-}10)$$

Two more vector notations will be introduced.

$\underline{u}(i, j)$ = a $(1\times(i+j))$ vector with i 0's followed by j 1's.

$$= [\underbrace{0, 0, \ldots, 0}_{i\ 0's}, \underbrace{1, 1, \ldots, 1}_{j\ 1's}] \qquad (5.1\text{-}11)$$

$\underline{v}(i, j)$ = a $(1 \times (i+j))$ vector with i 1's followed by j 0's.

$$= [\underbrace{1, 1, \ldots, 1}_{i \text{ 1's}}, \underbrace{0, 0, \ldots, 0}_{j \text{ 0's}}] \tag{5.1-12}$$

Using these notations, a $(1 \times k)$ unit vector may be expressed in two ways:

$$\underline{u}(0, k) = \underset{(1xk)}{[1, 1, \ldots, 1]} = \underline{v}(k, 0) \tag{5.1-13}$$

5.2 POINTWISE AVAILABILITY FUNCTION

Recall definition (2.2-1) of Section 2.2, the pointwise availability of a Markovian system at time t, for a given initial state vector, is the conditional probability that the system is in one of the up-states at that time. The past state history of the system is irrelevant. Suppose the initial probability state vector of the system is given to be $\underline{P}(0)$, then the pointwise availability at time t is denoted by $A(t|\underline{P}(0))$. Solution of the vector differential equation

$$\frac{d}{dt}\underline{P}(t|\underline{P}(0)) = \underline{P}(t|\underline{P}(0))M \tag{5.2-1}$$

gives the probability state vector at time t. In Appendix B, the general solution for the time varying vector differential equation is derived. By Appendix B, the solution for Eq. (5.2-1) is

$$\underline{P}(t|\underline{P}(0)) = \underline{P}(0)\,\Phi(t,0) \quad \text{for } 0 \le t < \infty \qquad (5.2\text{-}2)$$

where the transition or fundamental matrix $\Phi(t, 0)$ is given by the Peano-Baker series:

$$\Phi(t,0) = I + \int_0^t M d\sigma_1 + \int_0^t M \int_0^{\sigma_1} M d\sigma_2 d\sigma_1 + \ldots.$$

$$= I + Mt + \frac{M^2 t^2}{2!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{M^n t^n}{n!} \qquad (5.2\text{-}3)$$

In the above series, we have adopted the convention

$$\frac{M^0\, t^0}{0\,!} = I_{n_o} \qquad (5.2\text{-}4)$$

Due to the similarity of the series in the RHS of Eq. (5.2-3) to the exponential series, the series is usually denoted by e^{Mt} and known as the matrix exponential. That is,

$$e^{Mt} = \sum_{n=0}^{\infty} \frac{M^n t^n}{n!} \qquad (5.2\text{-}5)$$

Therefore, the solution for Eq. (5.2-1) can be written as:

$$\underline{P}(t|\underline{P}(0)) = \underline{P}(0)\, e^{Mt} \qquad (5.2\text{-}6)$$

The probability that system is up at time t is given by the sum of the probabilities that the system is in one of the up-states at that time. By Eq. (5.1-5), we have assumed that the up-states of the system are from 1 through k_o, therefore

$$A(t|\underline{P}(0)) = \underline{P}(0)\, e^{Mt}\, \underline{v}^T(k_o, n_o - k_o) \quad \text{for } 0 \leq t < \infty \qquad (5.2\text{-}7)$$

The following theorem has been proved.

<u>Theorem 5.1</u>

The states of a stationary Markovian system are such that the up-states are denoted by 1 through k_o, and the down-states are denoted by (k_o+1) through n_o. If M is the state transition-rate matrix and if the initial probability state vector of the system is $\underline{P}(0)$, then the pointwise availability function of the system for $0 \leq t < \infty$ is given by Eq. (5.2-7).

5.3 RELIABILITY FUNCTION

Definition (2.2-2) defines the reliability of a system for an interval [0, t]. For an initial state vector $\underline{P}(0)$, the reliability function is denoted by $R(t|\underline{P}(0))$. The essential difference between $R(t|\underline{P}(0))$ and $A(t|\underline{P}(0))$ is that the former represents the probability that the system is up for the entire interval [0, t], whereas the later represents the probability that the system is up at the time epoch t regardless of the system condition during [0, t]. In order to compute $R(t|\underline{P}(0))$ it is necessary to first compute, for each $i \epsilon S_u$, the probability that the system is in state i at time t and the system has not entered any state in S_d during (0, t). For this, we need to treat the states in S_d as absorbing states. Mathematically, we need to set to zero the state transition rates of the states in S_d. That is to say, the elements of $\bar{M}$ are to be modified such that

$$m_{i,j} = 0 \quad \text{for all } i \epsilon S_d \tag{5.3-1}$$

We shall denote the resultant matrix, after such modification, by B. Then the math model for computing the reliability function is

$$\dot{\underline{r}}(t|\underline{P}(0)) = \underline{r}(t|\underline{P}(0))\, B \quad \text{for } 0 \leq t < \infty \tag{5.3-2}$$

where $\underline{r}(t|\underline{P}(0))$ is obviously a $(1 \times n)$ vector. For each $i \epsilon S_u$, the interpretation for $r_i(t|\underline{P}(0))$ is:

$$r_i(t|\underline{P}(0)) = \Pr\{s(t) = i \text{ and } s(x) \epsilon S_u \text{ for all } 0 \leq x < t|\underline{P}(0)\} \tag{5.3-3}$$

However, for each $i \epsilon S_d$, the interpretation of $r_i(t|\underline{P}(0))$ is quite different:

$$r_i(t|\underline{P}(0)) = \text{the probability that the system started with } \underline{P}(0) \text{ is down via state i by time t} \tag{5.3-4}$$

The solution for Eq. (5.3-2) is

$$\underline{r}(t|\underline{P}(0)) = \underline{P}(0)\, e^{Bt} \qquad (5.3\text{-}5)$$

Therefore, the reliability function is:

$$R(t|\underline{P}(0)) = \underline{P}(0)e^{Bt}\ \underline{v}^T(k_o, n_o - k_o) \qquad (5.3\text{-}6)$$

$$\text{for } 0 \leq t < \infty$$

In the above equation, evaulation of the RHS involves the summation of the exponential series of $(n_o \times n_o)$ matrices. The smaller the value of n_o, the lesser is the computation required in the evaluation. We now attempt to develop an alternative formula which has significant computational advantage for evaluating $R(t|\underline{P}(0)$.

Recall the elements of S_d in Eq. (5.1-6), it can be seen that modification of M according to Eq. (5.3-1) means setting the elements of the last $(n_o - k_o)$ rows of M to 0. Therefore in the matrix B, elements of the last $(n_o - k_o)$ rows are 0. This implies that elements of the last $(n_o - k_o)$ rows of B^i are zero for all $i \geq 1$. This can be seen by first partitioning B into submatrices as follows:

$$B = \left[\begin{array}{c|c} \underset{(k_o x k_o)}{B_{1,1}} & \underset{(k_o x(n_o - k_o))}{B_{1,2}} \\ \hline \underset{((n_o - k_o) x k_o)}{\underline{\underline{0}}} & \underset{(n_o - k_o) x (n_o - k_o)}{\underline{\underline{0}}} \end{array}\right]. \qquad (5.3\text{-}7)$$

Then it is trivial to show that

$$B^i = \left[\begin{array}{c|c} B_{1,1}^i & B_{1,1}^{i-1} B_{1,2} \\ \hline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right] \qquad (5.3\text{-}8)$$

It follows that

$$\sum_{i=1}^{\infty} B^i = \left[\begin{array}{c|c} \sum_{i=1}^{\infty} B_{1,1}^i & \sum_{i=1}^{\infty} B_{1,1} B_{1,2} \\ \hline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right] \qquad (5.3\text{-}9)$$

In Appendix B it is shown that the type of series in the above equation is absolutely convergent. By Eq. (5.3-9), Eq. (5.3-6) can be rewritten as

$$R(t|\underline{P}(0)) = \underline{P}(0)\,[I_n + C(t)]\,\underline{v}^T(k_o, n_o - k_o) \qquad (5.3\text{-}10)$$

where

$$C(t) = \left[\begin{array}{c|c} \sum_{i=1}^{\infty} B_{1,1}^{i} \frac{t^i}{i!} & \sum_{i=1}^{\infty} B_{1,1}^{i-1} B_{1,2} \frac{t^i}{i!} \\ \hline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right] . \qquad (5.3\text{-}11)$$

Because of the $(n_o - k_o)$ 0's in $\underline{v}^T(k_o, n_o - k_o)$, Eq. (5.3-10) simplifies to

$$R(t|\underline{P}(0)) = \underline{P}_u(0)[I_{k_o} + \sum_{i=1}^{\infty} B_{1,1}^{i} \frac{t^i}{i!}] \underline{v}^T(k_o, 0) \qquad (5.3\text{-}12)$$

Therefore, an alternative expression for the reliability function is given by

$$R(t|\underline{P}(0)) = \underline{P}_u(0)\, e^{B_{1,1}t}\, \underline{v}^T(k_o, 0) \qquad (5.3\text{-}13)$$

$$\text{for } \tau \leq t < \infty$$

Notice that Eq. (5.3-6) is of the same form as Eq. (5.3-13). However, much computational advantage can be gained by using the latter due to the dimension of $B_{1,1}$ being smaller than that of B. The following theorem has been proved.

Theorem 5.2

For the same postulateas as in Theorem 5.1, the reliability function of the system for an interval $[0, t]$ is given by Eq. (5.3-13).

5.4 INTERVAL RELIABILITY FUNCTION

By definition (2.2-3), given an initial state vector $\underline{P}(0)$, the interval reliability of a Markovian system for an interval $[t_1, t_2]$ is the conditional probability that the system will be up during the interval $[t_1, t_2]$. This function is denoted by $IR(t_1, t_2 | \underline{P}(0))$. If we allow t_1 to equal t_2, the interval $[t_1, t_2]$ becomes a point at t_2. Then $IR(t_1, t_2 | \underline{P}(0))$ reduces to $A(t_2 | \underline{P}(0))$. That is

$$IR(t, t | \underline{P}(0)) = A(t | \underline{P}(0)) \tag{5.4-1}$$

On the other hand if $t_1 = 0$, we see that $A(t_1, t_2 | \underline{P}(0))$ reduces to $R(t_2 | \underline{P}(0))$. That is

$$IR(0, t | \underline{P}(0)) = R(t | \underline{P}(0)) \tag{5.4-2}$$

The above shows that the interval reliability function is more general than the pointwise availability and the reliability functions. To find the solution for $IR(t_1, t_2 | \underline{P}(0))$, for $0 < t_1 < t_2 < \infty$, we first find the probability state vector at time t_1. By Eq. (5.2-6)

$$\underline{P}(t_1 | \underline{P}(0)) = \underline{P}(0)\, e^{Mt_1} \tag{5.4-3}$$

From the definition of $IR(t_1, t_2 | \underline{P}(0))$ we see that

$$IR(t_1, t_2 | \underline{P}(0)) = R(t_2 - t_1 | \underline{P}(t_1 | \underline{P}(0)) \qquad (5.4\text{-}4)$$

Therefore, by Eq. (5.3-6) we have

$$IR(t_1, t_2 | \underline{P}(0)) = \underline{P}(t_1 | \underline{P}(0)) e^{B(t_2 - t_1)} \underline{v}^T(k_o, n_o - k_o) \qquad (5.4\text{-}5)$$

Substituting Eq. (5.4-3) into (5.4-5) yields

$$IR(t_1, t_2 | \underline{P}(0)) = \underline{P}(0) e^{Mt_1} e^{B(t_2 - t_1)} \underline{v}^T(k_o, n_o - k_o) \qquad (5.4\text{-}6)$$

Instead of using Eq. (5.3-6) we may use Eq. (5.3-13) for the reliability function. In which case the interval reliability function is given by

$$IR(t_1, t_2 | \underline{P}(0)) = \underline{P}_u(t_1 | \underline{P}(0)) \, e^{B_{1,1}(t_2 - t_1)} \underline{\check{v}}^T(k_o, 0) \qquad (5.4\text{-}7)$$

It should be noted that in Eq. (5.4-6) there are two matrix exponentials. Only under very special situations can these two matrix exponentials be combined into one. The special situation being that $t_1 = t_2 - t_1$ and $MB = BM$.

We have proved the following theorem.

Theorem 5.3

For the same postulates as in Theorem 5.1, the interval reliability function of the system for an interval $[t_1, t_2]$, $0 < t_1 < t_2 < \infty$, is given by Eq. (5.4-6) or (5.4-7).

5.5 INTERVAL AVAILABILITY FUNCTION

By definition (2.2-4), given an initial state vector $\underline{P}(0)$, the interval availability of a Markovian system for an interval $[t_1, t_2]$ is the expected faction of the time interval that the system is up. This function is denoted by $IA(t_1, t_2 | \underline{P}(0))$. By Eq. (2.2-5) $IA(t_1, t_2 | \underline{P}(0))$ is related to the pointwise availability function as follows:

$$IA(t_1, t_2 | \underline{P}(0)) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A(t | \underline{P}(0)) \, dt \qquad (5.5\text{-}1)$$

Substituting the expression for $A(t | \underline{P}(0))$ from Eq. (5.2-7) yields

$$IA(t_1, t_2 | \underline{P}(0)) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \underline{P}(0) \left(\sum_{i=0}^{\infty} \frac{M^i t^i}{i!} \right) \underline{v}^T(k_o, n_o - k_o) \, dt \qquad (5.5\text{-}2)$$

Each term of the matrix exponential series on the RHS is a continuous function of t. In Appendix B we have shown that such a series is absolutely and uniformly convergent. Therefore the order of summation and integration in Eq. (5.5-2) may be interchanged [72]. Thence

$$\int_{t_1}^{t_2} \sum_{i=0}^{\infty} \frac{M^i t^i}{i!} dt = \sum_{i=0}^{\infty} \left[\frac{M^i t^{i+1}}{(i+1)!} \right]_{t_1}^{t_2}$$

$$= \sum_{i=0}^{\infty} M^i \frac{t_2^{i+1} - t_1^{i+1}}{(i+1)!} \qquad (5.5\text{-}3)$$

Substituting the above into Eq. (5.5-2) we obtain

$$IA(t_1, t_2 | \underline{P}(0)) = \frac{1}{(t_2 - t_1)} \underline{P}(0) \sum_{i=0}^{\infty} M^i \frac{t_2^{i+1} - t_1^{i+1}}{(i+1)!} \underline{v}^T(k_o, n_o - k_o)$$

(5.5-4)

Theorem 5.4

For the same postulates as in Theorem 5.1, the interval availability function of the system for an interval $[t_1, t_2]$, $0 \leq t_1 < t_2 < \infty$, is given by Eq. (5.5-4).

5.6 STEADY STATE AVAILABILITY

The definition for steady state availability, A, of a system is given by (2.2-6). Since this measure of effectiveness is defined for the steady state condition, transient states of the system, if any, may be removed from consideration. Therefore, without loss of generality, we shall assume that the system is ergodic. By Eq. (2-2-7) the steady state availability can be considered as the limiting value of the interval availability, $IA(t_1, t_2 | \underline{P}(0))$, as $t_2 \to \infty$. It should be noted that Eq. (5.5-4) gives the solution of $IA(t_1, t_2 | \underline{P}(0))$ for finite t_2. Therefore, the steady state availability cannot be derived from Eq. (5.4-4) by taking the limit as $t_2 \to \infty$.

The following theorem, proved in Appendix C, shows the existance of the limiting solution.

Theorem 5.5

For an ergodic stationary Markovian system, the limiting solution of the system of differential equations governing the state probabilities exists, and this solution is independent of the initial condition of the system.

By this theorem, the system of equations

$$\frac{d}{dt} \underline{P}(t | \underline{P}(0)) = \underline{P}(t | \underline{P}(0)) M \qquad (5.6\text{-}1)$$

in the steady state, reduces to

$$\underline{0}_{n_o} = \underline{\pi} M \tag{5.6-2}$$

where $\underline{\pi}$ is the steady state probability vector of the system. This equation does not possess a unique solution since the rank of M is less than n_o. Otherwise the only solution for $\underline{\pi}$ would be $\underline{0}$, which is impossible since the system is ergodic and it is necessary that

$$\pi_i > 0 \quad \text{for all} \quad i \in S$$

and

$$\sum_{i=1}^{n_o} \pi_i = 1 \tag{5.6-3}$$

We now invoke the following important theorem which characterizes nonsingular submatrices of M. This is the main theorem on the properties of M. The proof for this theorem and its corollary are given in Appendix D.

Theorem 5.6

Let M be the transition rate matrix of an n_o-state stationary Markovian system which is such that n_1 of the states are transient states, and the remaining $(n_o - n_1)$ states form an ergodic set. If B is an $(m \times m)$ matrix resulted after deleting i, $1 \le i \le n_o - n_1$, rows and the corresponding i columns of M pertaining to i states of the ergodic set, then B is non-singular.

Corollary

If M is the transition-rate matrix of an n_o-state stationary Markovian system which is such that n_1 of the states are transient states, and the remaining $(n_o - n_1)$ states form an ergodic set, then M is singular.

By this theorem, a matrix resulted from striking out any ith row and the corresponding ith column of M of an ergodic system is non-singular. Hence it follows that if M is the transition rate matrix of an ergodic stationary Markovian system with n_o states, any $(n_o - 1)$ rows or any $(n_o - 1)$ column of M are linearly independent. That is to say the rank of M is $(n_o - 1)$.

Now consider replacing the last column of M by a column of 1's. Let the resultant matrix be denoted by W. That is,

$$W = \left[\begin{array}{ccc:c} & M' & & \begin{matrix}1\\1\\\vdots\end{matrix} \\ \hdashline m_{n_o,1} & m_{n_o,2} \cdots\cdots & m_{n_o,n_o-1} & 1 \end{array}\right] \qquad (5.6\text{-}4)$$

It follows that M' is non-singular. By Eq. (5.6-2), we have

$$\sum_{i=1}^{n_o} \pi_i \, m_{i,j} = 0 \quad \text{for all } j \in S \tag{5.6-5}$$

Since the system is ergodic, $\pi_i > 0$ for all $i \in S$. Therefore Eq. (5.6-5) can be written as:

$$\sum_{i=1}^{n_o - 1} \frac{\pi_i}{(-\pi_{n_o})} m_{i,j} = m_{n_o, j} \quad \text{for all } j \in S \tag{5.6-6}$$

Since M' is non-singular, there exists a unique set of coefficients

$$\underline{a} = [a_1, a_2, \ldots, a_{n_o - 1}] \tag{5.6-7}$$

such that

$$\sum_{i=1}^{n_o - 1} a_i m_{i,j} = m_{n_o, j} \quad \text{for } j = 1, 2, \ldots, n_o - 1 \tag{5.6-8}$$

By Eq. (5.6-6) the a_i's are given by:

$$a_i = - \frac{\pi_i}{\pi_{n_o}} \quad \text{for } i = 1, 2, \ldots, n_o - 1 \tag{5.6-9}$$

Therefore, all the a_i's are negative quantities. Hence

$$\sum_{i=1}^{n_o-1} a_i \neq 1 \tag{5.6-10}$$

This shows that no linear combination of the first (n_o-1) rows of W can yield the n_oth row of W. Therefore W is non-singular.

Replacing the n_oth scalar equation in Eq. (5.6-2) by Eq. (5.6-3) we have

$$\underline{u}(n_o-1, 1) = \underline{\pi}\, W \tag{5.6-11}$$

Since W is non-singular, we finally obtain the steady state availability of the system as:

$$A = \underline{u}(n_o-1, 1)\, W^{-1}\, \underline{v}^T(k_o, n_o-k_o) \tag{5.6-12}$$

The following theorem is now proved.

Theorem 5.7

The states of an ergodic stationary Markovian system is such that the up-states are denoted by 1 through k_o, and the down states are denoted by (k_o+1) through n_o. If W denotes the resultant matrix after replacing the last column of the system transition-rate matrix by a column of 1's, then the steady state availability of the system is given by Eq. (5.6-12).

It should be pointed out that the above theorem may be stated in a somewhat more generay way. If W_i denotes the resultant matrix after replacing the ith column of M by a column of 1's, and if $\underline{u}_i$ denotes a $(1 \times n_o)$ vector of zeros except the ith element being 1, then Eq. (5.6-12) generalizes to:

$$A = \underline{u}_i W_i^{-1} \underline{v}^T(k_o, n_o - k_o) \tag{5.6-13}$$

5.7 LIMITING INTERVAL RELIABILITY FUNCTION

By definition (2.2-9), the limiting interval reliability for a period τ, denoted by $LIR(\tau)$, is defined to be the limiting value of $IR(t, t=\tau | \underline{P}(\tau))$ as $t \to \infty$. The expressions derived for $IR(t_1, t_2 | \underline{P}(0))$ in Eqs. (5.4-6) and (5.4-7) are valid for finite t_1 and t_2 only. In Theorem 5.5, it has been established that for an ergodic stationary

Markovian system, the limiting value of $\underline{P}(t|\underline{P}(0))$ exists and this value is independent of the initial probability vector $\underline{P}(0)$. That is

$$\lim_{t \to \infty} \underline{P}(t|\underline{P}(0)) = \underline{\pi} \qquad (5.7\text{-}1)$$

We assume that the system is ergodic and stationary. Therefore, at any time instant in the steady state, the probability state vector of the system is $\underline{\pi}$. The limiting interval reliability for a period of τ is the reliability for a period of τ in the steady state. The probability state vector of the system at the beginning of the time period is $\underline{\pi}$. Therefore, by Theorem 5.2 we obtain the expression for $LIR(\tau)$.

$$LIR(\tau) = \underline{\pi}_u e^{B_{1,1}\tau} \underline{v}^T(k_o, 0) \qquad (5.7\text{-}2)$$

In the above, $\underline{\pi}_u$ is a subvector of $\underline{\pi}$ defined as follows:

$$\underset{(1 \times n_o)}{\underline{\pi}} = [\underset{(1 \times k_o)}{\underline{\pi}_u} \quad \underset{(1 \times (n_o - k_o))}{\underline{\pi}_d}] \qquad (5.7\text{-}3)$$

Observe that the expression for $LIR(\tau)$ is independent of any initial state vector $\underline{P}(0)$.

We have proved the following theorem.

Theorem 5.8

The limiting interval reliability of an ergodic stationary Markovian system exists and is independent of any initial condition of the system. The limiting interval reliability for a period of time $\tau < \infty$ is given by Eq. (5.7-2).

5.8 DISTRIBUTION OF TIME-TO-FIRST-SYSTEM-FAILURE

For a given initial probability state vector, the first passage time of the system to a system down-state is a random quantity. In this section we will derive the probability distribution of this random variable. The mean, variance and the general mth order moment of the variable will be treated in the sections to follow.

In Section 3.2 we derived some basic relationships between the failure-time density function, the reliability function, and the failure rate function of a system. Eq. (3.2-5) gives the relationship between the failure-time density function and the reliability function. Therefore, for an initial state vector $\underline{P}(0)$ of a Markovian system, the probability density function of the first passage time to system failure is

$$f(t|\underline{P}(0) = -\frac{d}{dt} R(t|\underline{P}(0)) \qquad (5.8\text{-}1)$$

But the reliability function of a stationary Markovian system has been found in Section 5.3. Substituting Eq. (5.3-13) into (5.8-1) we have:

$$f(t|\underline{P}(0) = -\frac{d}{dt}[\underline{P}_u(0)\, e^{B_{1,1}t}\, \underline{v}^T(k_o, 0)] \tag{5.8-2}$$

Since the series for the matrix exponential is uniformly and absolutely convergent, term by term differentiation of the series is valid. This gives

$$f(t|\underline{P}(0)) = -\underline{P}_u(0) \sum_{i=0}^{\infty} \frac{d}{dt}\left(\frac{B_{1,1}^i t^i}{i!}\right) \underline{v}^T(k_o, 0) \tag{5.8-3}$$

Simplifying we obtain

$$f(t|\underline{P}(0)) = -\underline{P}_u(0)\, B_{1,1} e^{B_{1,1}t}\, \underline{v}^T(k_o, 0) \tag{5.8-4}$$

The above expression indicates that if the system starts from state $i \in S_u$ at time 0, then the first passage time density function is given by the ith row-sum of the matrix $-B_{1,1}e^{B_{1,1}t}$. For the case where $B_{1,1}$ has distinct characteristic roots, the matrix exponential $e^{B_{1,1}t}$ can be simplified. If $\lambda_1, \lambda_2, \ldots, \lambda_{k_o}$ are the distinct characteristic roots of $B_{1,1}$, then there exists [73] a matrix G such that

$$B_{1,1} = G \Lambda G^{-1} \tag{5.8-5}$$

where Λ is a diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_{k_o}$ as its diagonal elements. Now rewrite the matrix exponential as:

$$e^{B_{1,1}t} = e^{G\Lambda G^{-1}t}$$

$$= \sum_{i=0}^{\infty} \frac{(G\Lambda G^{-1})^i t^i}{i!} \qquad (5.8\text{-}6)$$

Observe that the general terms of the above infinite series simplifies to:

$$\frac{(G\Lambda G^{-1})^i t^i}{i!} = \frac{G\Lambda^i G^{-1} t^i}{i!} \qquad (5.8\text{-}7)$$

Therefore Eq. (5.8-6) becomes

$$e^{B_{1,1}t} = G\, e^{\Lambda t}\, G^{-1} \qquad (5.8\text{-}8)$$

Since Λ is diagonal, the matrix exponential $e^{\Lambda t}$ can be written as:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ & & \ddots \\ 0 & & & e^{\lambda_{k_o} t} \end{bmatrix} \qquad (5.8\text{-}9)$$

This shows that the first passage time density function can be expressed in a closed form when $B_{1,1}$ has distinct characteristic roots. We recall that for a stationary Markovian system, the reliability function approaches zero as $t \to \infty$. That is

$$\lim_{t \to \infty} \underline{P}_u(0)\, e^{B_{1,1} t}\, \underline{v}^T(k_o, 0) \to 0 \qquad (5.8\text{-}10)$$

For the present case

$$\lim_{t \to \infty} \underline{P}_u(0)\, G\, e^{\Lambda t}\, G^{-1} \underline{v}^T(k_o, 0) \to 0 \qquad (5.8\text{-}11)$$

This means that the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_{k_o}$ have negative real parts. We have proved the following theorem.

Theorem 5.9

For a given initial state probability vector $\underline{P}(0)$ of a stationary Markovian system, the probability density function of the first passage time to system failure is given by Eq. (5.8-4). In the case when $B_{1,1}$ has distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_{k_o}$ the density function can be expressed in closed form as a linear function of $e^{\lambda_i t}$, for $i = 1, 2, \ldots, k_o$. The eigenvalues have negative real parts.

5.9 MEAN TIME-TO-FIRST-SYSTEM-FAILURE

By definition (2.2-10), for a given initial state probability vector $\underline{P}(0)$, the $MTTFSF_{\underline{P}(0)}$ of a Markovian system is the average first passage time of the system from the initial condition $\underline{P}(0)$ to a system down state. If $\underline{P}(0)$ is such that $P_i(0) = 0$ for all $i \in S_u$, then it is obvious that $MTTFSF_{\underline{P}(0)} = 0$ since the system is sure to start from one of the states in S_d. More generally speaking, $MTTFSF_{\underline{P}(0)} = 0$ only if and if $P_i(0) = 0$ for all $i \in S_u$. Several approaches may be used to derive the mean time-to-first-system-failure. One approach would be to apply Eq. (3.3-10) since the reliability function $R(t|\underline{P}(0))$ has already been found in Eq. (5.3-13). Using this approach we have:

$$MTTFSF_{\underline{P}(0)} = \lim_{\tau \to \infty} \int_0^\tau \underline{P}_u(0)\, e^{B_{1,1}t}\, \underline{v}^T(k_o, 0)\, dt$$

$$= \lim_{\tau \to \infty} \underline{P}_u(0) \int_0^\tau \sum_{i=0}^{\infty} \frac{B_{1,1}^i}{i!} t^i\, dt\, \underline{v}^T(k_o, 0) \qquad (5.9\text{-}1)$$

Since the infinite series is uniformly and absolutely convergent on the integrating interval $[0, \tau]$, the integration and summation in Eq. (5.9-1) may be interchanged. Therefore,

$$MTTFSF_{\underline{P}(0)} = \lim_{\tau \to \infty} \underline{P}_u(0) \sum_{i=0}^{\infty} \frac{B_{1,1}^i \tau^{i+1}}{(i+1)!}\, \underline{v}^T(k_o, 0) \qquad (5.9\text{-}2)$$

From Eq. (5.3-7) we see that $B_{1,1}$ is the resultant matrix after deleting from M all the rows and columns which correspond to the down-states of the system. By Theorem 5.6 such a matrix is non-singular. Therefore

Eq. (5.9-2) can be written as

$$\text{MTTFSF}_{\underline{P}(0)} = \lim_{\tau\to\infty} \underline{P}_u(0)\, B_{1,1}^{-1}[e^{B_{1,1}\tau} - I_{k_o}]\, \underline{v}^T(k_o, 0) \qquad (5.9\text{-}3)$$

Notice that the first term on the RHS of Eq. (5.9-3) is the reliability function of the system for the interval $[0, \tau]$. This function goes to 0 as τ becomes arbitrarily large.

$$\lim_{\tau\to\infty} \underline{P}_u(0)\, B_{1,1}^{-1}\, e^{B_{1,1}\tau}\, \underline{v}^T(k_o, 0)$$

$$= \lim_{\tau\to\infty} R(\tau \mid \underline{P}_u(0)\, B_{1,1}^{-1})$$

$$= 0 \qquad (5.9\text{-}4)$$

Hence, we obtain

$$\text{MTTFSF}_{\underline{P}(0)} = -\underline{P}_u(0)\, B_{1,1}^{-1}\, \underline{v}^T(k_o, 0) \qquad (5.9\text{-}5)$$

We will now show that the elements of the matrix $-B_{1,1}^{-1}$ have particular significance. Recall that

$$\int_0^\infty R(t \mid \underline{P}(0))\, dt = [\int_0^\infty r_1(t \mid \underline{P}(0))dt, \int_0^\infty r_2\, t \mid \underline{P}(0)dt, \ldots.$$

$$\int_0^\infty r_{k_o}(t \mid \underline{P}(0))dt]\; \underline{v}^T(k_o, 0)$$

$$(5.9\text{-}6)$$

Since Eqs. (5.9-5) and (5.9-6) hold for all k_o, it follows that

$$[\int_0^\infty r_1(t|\underline{P}(0))dt, \int_0^\infty r_2(t|\underline{P}(0))dt, \ldots, \int_0^\infty r_{k_o}(t|\underline{P}(0))dt] = -\underline{P}_u(0)B_{1,1}^{-1} \qquad (5.9\text{-}7)$$

Now consider the case when all the elements of $\underline{P}_u(0)$ is zero except the ith element being 1 (obviously $i \epsilon S_u$), then the RHS of Eq. (5.9-7) represents the ith row of $-B_{1,1}^{-1}$. Therefore, for all $i, j \epsilon S_u$

$$\int_0^\infty r_j(t|s(0) = i)dt = \text{the } (i, j) \text{ element of } -B_{1,1}^{-1} \qquad (5.9\text{-}8)$$

The LHS of the above equation represents the expected time the system will spend in state j before entering a system down-state the first time, given that the system starts from state i. We denote this quantity by $\psi_j(i)$:

$$\psi_j(i) = \int_0^\infty r_j(t|s(0) = i)\, dt \qquad (5.9\text{-}9)$$

Then, we have

$$-B_{1,1}^{-1} = (\psi_j(i)) = \begin{bmatrix} \psi_1(1) & \psi_2(1) & \psi_3(1) & \ldots \psi_{k_o}(1) \\ \psi_1(2) & \psi_2(2) & \psi_3(2) & \ldots \psi_{k_o}(2) \\ \vdots & & & \\ \psi_1(k_o) & \psi_2(k_o) & \psi_3(k_o) & \ldots \psi_{k_o}(k_o) \end{bmatrix} \qquad (5.9\text{-}10)$$

Since each $\psi_j(i)$ represents the expected time the system spends in state j before entering a system down-state the first time, given that the system starts from state i,

$$\psi_j(i) \geq 0 \quad \text{for all } i, j \in S_u \tag{5.9-11}$$

That is to say all elements of $B_{1,1}^{-1}$ are non-negative. It is not hard to see that if the up-states of the system form an ergodic set, then the inequality in Eq. (5.9-11) is strict.

We have proved the following theorem.

Theorem 5.10

In a stationary Markovian system, let $B_{1,1}$ be the resultant matrix after deleting from M all the rows and columns which correspond to the system down-states. The matrix $B_{1,1}$ is non-singular. The (i, j) element of $-B_{1,1}^{-1}$ represents the expected time the system spends in state j before entering a system down-state the first time given that the system starts from state i initially. If the initial state vector of the system is $\underline{P}(0)$, then the mean time-to-first-system-failure is given by Eq. (5.9-5).

5.10 VARIANCE OF TIME-TO-FIRST-SYSTEM-FAILURE

For a given initial state vector $\underline{P}(0)$, the variance of the first passage time to system failure, denoted by $\mathrm{var}(TTFSF|\underline{P}(0))$, can be found by employing Eq. (3.3-16).

$$\mathrm{Var}(\mathrm{TTFSF}|\underline{P}(0)) = 2\int_0^\infty t\, R(t|\underline{P}(0))\, dt - \left(\int_0^\infty R(t|\underline{P}(0))\, dt\right)^2 \tag{5.10-1}$$

The first term on the RHS represents the second moment of the first passage time. We will first evaluate the integral of this term. By Eq. (5.3-13),

$$\int_0^\infty t\, R(t|\underline{P}(0))\, dt$$

$$= \lim_{\tau\to\infty} \int_0^\tau t\, \underline{P}_u(0)\, e^{B_{1,1} t}\, \underline{v}^T(k_o, 0)\, dt$$

$$= \lim_{\tau\to\infty} \underline{P}_u(0) \int_0^\tau t \sum_{i=0}^\infty \frac{B_{1,1}^i\, t^i}{i!}\, dt\, \underline{v}^T(k_o, 0) \tag{5.10-2}$$

Integrating by parts we have

$$\int_0^\tau t \sum_{i=0}^\infty \frac{B_{1,1}^i\, t^i}{i!}\, dt$$

$$= \left[t \sum_{i=0}^\infty B_{1,1}^i \frac{t^{i+1}}{(i+1)!}\right]_0^\tau - \int_0^\tau \sum_{i=0}^\infty \frac{B_{1,1}^i\, t^{i+1}}{(i+1)!}\, dt \tag{5.10-3}$$

Simplifying gives

$$\int_0^{\tau} t \sum_{i=0}^{\infty} B_{1,1}^{i} \frac{t^{i}}{i!} dt$$

$$= \tau \sum_{i=0}^{\infty} \frac{B_{1,1}^{i} \tau^{i+1}}{(i+1)!} - \sum_{i=0}^{\infty} \frac{B_{1,1}^{i} \tau^{i+2}}{(i+2)!} \quad (5.10\text{-}4)$$

Since $B_{1,1}$ is non-singular this equation can be written as:

$$\int_0^{\tau} t \sum_{i=0}^{\infty} B_{1,1}^{i} \frac{t^{i}}{i!} dt$$

$$= \tau B_{1,1}^{-1}[e^{B_{1,1}\tau} - I_{k_o}] - B_{1,1}^{-2}[e^{B_{1,1}\tau} - I_{k_o} - B_{1,1}\tau]$$

$$= \tau B_{1,1}^{-1} e^{B_{1,1}\tau} - B_{1,1}^{-2} e^{B_{1,1}\tau} + B_{1,1}^{-2} \quad (5.10\text{-}5)$$

Substituting Eq. (5.10-5) into Eq. (5.10-2) gives

$$\int_0^{\infty} t R(t|\underline{P}(0)) dt$$

$$= \lim_{\tau\to\infty} \underline{P}_u(0)[\tau B_{1,1}^{-1} e^{B_{1,1}\tau} - B_{1,1}^{-2} e^{B_{1,1}\tau} + B_{1,1}^{-2}]\underline{v}^T(k_o, 0)$$

$$= \lim_{\tau\to\infty} \tau R(\tau|\underline{P}_u(0)B_{1,1}^{-1}) - \lim_{\tau\to\infty} R(\tau|\underline{P}_u(0)B_{1,1}^{-2}) + \underline{P}_u(0)B_{1,1}^{-2}\underline{v}^T(k_o, 0)$$

$$(5.10\text{-}6)$$

For a stationary Markovian system, the failure rate function is bounded from below by some constant $\epsilon > 0$. Therefore, by Eq. (3.3-8) the limiting value of the first two terms on the RHS of Eq. (5.10-6) is zero. Hence, the second moment of the first passage time is:

$$2\int_0^\infty t\,R(t|\underline{P}(0))dt = 2\,\underline{P}_u(0)\,B_{1,1}^{-2}\,\underline{v}^T(k_o, 0) \qquad (5.10\text{-}7)$$

Substituting Eqs. (5.9-5) and (5.10-7) in (5.10-1) we obtain:

$$\mathrm{Var}(\mathrm{TTFSF}|\underline{P}(0)) = 2\,\underline{P}_u(0)\,B_{1,1}^{-2}\underline{v}^T(k_o, 0) - [\underline{P}_u(0)\,B_{1,1}^{-1}\,\underline{v}^T(k_o, 0)]^2 \qquad (5.10\text{-}8)$$

We have proved the following theorem.

Theorem 5.11

Given that the initial probability state vector of a stationary Markovian system is $\underline{P}(0)$, the variance of the first passage time to system failure is given by Eq. (5.10-8).

5.11 THE mth ORDER MOMENT OF TIME-TO-FIRST-SYSTEM-FAILURE

In the preceeding two sections, we have derived the first and the second order moments of the first passage time to system failure. In this section we will derive an expression for the general case mth order moment.

Theorem 5.12

Given that the initial probability state vector of a stationary Markovian system is $\underline{P}(0)$, the mth order $(m \geq 1)$ moment of the time-to-first-system failure is given by

$$(-1)^m m! \ \underline{P}_u(0) B_{1,1}^{-m} \underline{v}^T(k_o, 0) \tag{5.11-1}$$

We will give an inductive proof. By Eqs. (5.9-5) and (5.10-7) we see that the theorem is true for the cases of m equals 1 and 2. Now, suppose it is true for $m = i > 2$. This means

$$\int_0^\infty t^i f(t|\underline{P}(0)) = (-1)^i i! \ \underline{P}_u(0) B_{1,1}^{-i} \underline{v}^T(k_o, 0) \tag{5.11-2}$$

By Eqs. (3.3-14), the above is equivalent to

$$i\int_0^\infty t^{i-1} R(t|\underline{P}(0)) \, dt = (-1)^i i! \ \underline{P}_u(0) B_{1,1}^{-i} \underline{v}^T(k_o, 0) \tag{5.11-3}$$

We will show that the theorem is true for $m = i+1$. Again by Eq. (3.3-14),

$$\int_0^{\infty} t^{i+1} f(t|\underline{P}(0))\, dt$$

$$= \lim_{\tau\to\infty} (i+1) \int_0^{\tau} t^i\, R(t|\underline{P}(0))\, dt$$

$$= \lim_{\tau\to\infty} (i+1)\, \underline{P}_u(0) \int_0^{\tau} t^i \sum_{j=0}^{\infty} \frac{B_{1,1}^j t^j}{j!}\, dt\, \underline{v}^T(k_o, 0) \qquad (5.11\text{-}4)$$

Carrying out the integration by parts we have:

$$\int_0^{\tau} t^i \sum_{j=0}^{\infty} \frac{B_{1,1}^j t^j}{j!}\, dt$$

$$= t^i \sum_{j=0}^{\infty} \frac{B_{1,1}^j t^{j+1}}{(j+1)!} - \int_0^{\tau} i\, t^{i-1} \sum_{j=0}^{\infty} \frac{B_{1,1}^j t^{j+1}}{(j+1)!}\, dt$$

$$= \tau^i B_{1,1}^{-1}[e^{B_{1,1}\tau} - I_{k_o}] - i B_{1,1}^{-1} \int_0^{\tau} t^{i-1}[e^{B_{1,1}t} - I_{k_o}]\, dt$$

$$= \tau^i B_{1,1}^{-1}[e^{B_{1,1}\tau} - I_{k_o}] - i B_{1,1}^{-1} \int_0^{\tau} t^{i-1} e^{B_{1,1}t}\, dt + B_{1,1}^{-1}\tau^i$$

$$= \tau^i B_{1,1}^{-1} e^{B_{1,1}\tau} - i B_{1,1}^{-1} \int_0^{\tau} t^{i-1} e^{B_{1,1}t}\, dt \qquad (5.11\text{-}5)$$

Substituting Eq. (5.11-5) into Eq. (5.11-4) we obtain:

$$\int_i^{\infty} t^{i+1} f(t|\underline{P}(0))\, dt$$

$$= (i+1)[\lim_{\tau\to\infty} \tau^i R(\tau|\underline{P}(0)\, B_{1,1}^{-1}) - \lim_{\tau\to\infty} i\, \underline{P}_u(0) \int_0^{\tau} t^{i-1} e^{B_{1,1}t}\, dt\, B_{1,1}^{-1} \underline{v}^T(k_o, 0)$$

$$(5.11\text{-}6)$$

For a stationary Markovian system, the first term on the RHS → 0 as $\tau \to \infty$. From Eq. (5.11-3) we have:

$$\lim_{\tau \to \infty} i \int_0^{\tau} t^{i-1} e^{B_{1,1} t} \, dt = (-1)^i \, i! \, B_{1,1}^{-1} \tag{5.11-7}$$

Hence we obtain

$$\int_0^{\infty} t^{i+1} f(t|\underline{P}(0)) \, dt = (-1)^{i+1} (i+1)! \, \underline{P}_u(0) \, B_{1,1}^{-(i+1)} \underline{v}^T(k_o, 0) \tag{5.11-8}$$

This completes the proof.

Appendix E presents an alternative derivation of Eq. (5.11-8) by using the Laplace transform approach.

Chapter 6

UP-TIME AND DOWN-TIME INTERVALS OF INTEREST FOR MARKOVIAN SYSTEMS IN THE STEADY STATE

6.1 INTRODUCTION

The development in this chapter concerns three different types of up-time and down-time intervals for ergodic stationary Markovian systems in the steady state. These intervals are named as below:

1. Complete up-time (down-time) interval
2. Unconditional remaining up-time (down-time) interval.
3. Conditional remaining up-time(down-time) interval.

Their definitions will next be given.

If one would plot the state of the system as a function of time, it could appear as shown in Figure 6-1.

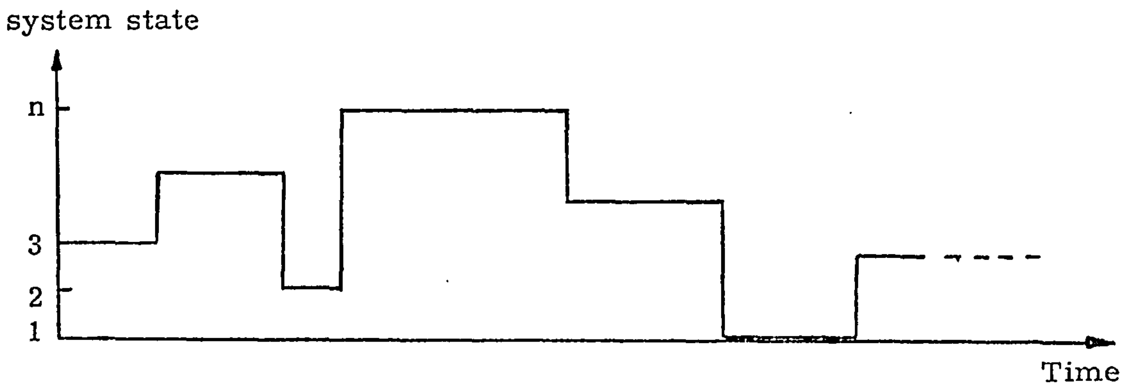

Figure 6-1 State of the System Versus Time Plot.

The system is said to be in an up-condition when it is in one of the states of S_u; and it is said to be in a down-condition when it is in one of the states of S_d. Therefore, from the plot of system state versus time, a plot of the system condition versus time may be made. A typical plot of system condition versus time is shown in Figure 6-2.

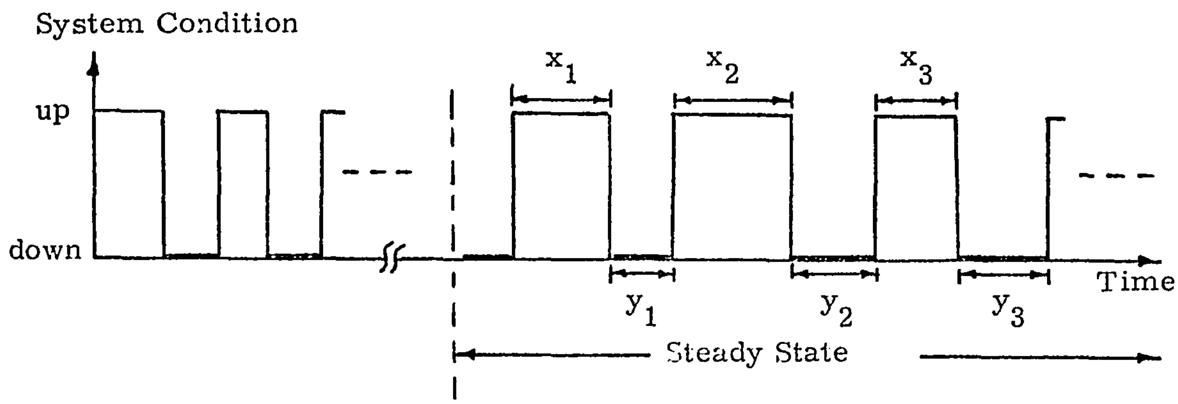

Figure 6-2 System Condition Versus Time Plot.

Each x_i shown in the figure denotes a complete up-time interval, which is a time interval beginning from the instant the system enters a state in S_u from a state in S_d up to the next time instant when it first enters a state in S_d. Each y_i denotes a complete down-time interval which is similarly defined. For simplicity, we shall use up-time and down-time intervals to mean the complete up-time and down-time intervals, respectively.

As the process of the system up and down condition continues in the steady state, if at a random time epoch an observation of the system is made, it is with zero probability that the observation time epoch will coincide with the beginning of an up-time or a down-time interval. At the observation time epoch, the system may be in either an up condition or a down condition. If no information is given regarding the condition of the system at the random instant it is observed, the remaining up-time or down-time interval is defined to be the unconditional remaining up-time or down-time interval.

Now, suppose as the system is observed at a random time epoch in the steady state, it is found to be in an up condition, but no information is given on the exact state of the system. The remaining up-time based on the condition that the system is up at the time epoch it is observed will be called the conditional remaining up-time. In a similar manner, the conditional remaining down-time is defined.

The following abbreviations will be used:

UT = complete up-time interval

DT = complete down-time interval

RUT = unconditional remaining up-time interval

RDT = unconditional remaining down-time interval

RUT_u = conditional remaining up-time interval

RDT_d = conditional remaining down-time interval

In the sections to follow, we will derive the probability density functions and the moments of UT, DT, RUT, RDT, RUT_u and RDT_d. In addition, a discussion will be presented on their interrelationships. The final section is devoted to the development of expressions for computing system mean up-time (MUT) and mean down-time (MDT) in terms of independent subsystems MUT's and MDT's.

6.2 COMPLETE UP-TIME AND DOWN-TIME INTERVALS

For an ergodic Markovian system, in general, there are more than one state in S_u that can be reached from some state or states in S_d by a single transition. This means that the system may not always begin an up-time interval from a fixed state in S_u. Similarly, all down-time intervals may not always begin from a fixed state in S_d. Therefore there are probability distributions governing the beginning states of the up-time and down-time intervals. The key step in computing the probability density functions and the moments for UT and DT lies in finding these probability distributions.

For each $j \in S$, and for an initial state vector $\underline{P}(0)$, the probability that the system will enter state j in the next dt time interval given that it is now in a down state is:

$$\Pr\{s(t+dt) = j \mid s(t) \in S_d, \underline{P}(0)\}$$

$$= \sum_{i \in S_d} \Pr\{s(t)=i,\ s(t+dt) = j \mid s(t) \in S_d, \underline{P}(0)\}$$

$$= \sum_{i \in S_d} \Pr\{s(t)=i \mid s(t) \in S_d, \underline{P}(0)\} \Pr\{s(t+dt) = j \mid s(t) = i\}$$

after employing Markov's property

$$= \sum_{i \in S_d} \frac{\Pr\{s(t)=i \mid \underline{P}(0)\}}{\Pr\{s(t) \in S_d \mid \underline{P}(0)\}} \Pr\{s(t+dt) = j \mid s(t) = i\}$$

$$= \frac{\sum_{i \in S_d} P_i(t \mid \underline{P}(0)\ m_{i,j} dt}{\sum_{i \in S_d} P_k(t \mid \underline{P}(0))} \qquad (6.2\text{-}1)$$

By Theorem 5.8, all $P_i(t \mid \underline{P}(0)) \to \pi_i$ as $t \to \infty$. Let $\nu(j \mid S_d)$ denotes the limiting value of Eq. (6.2-1) as $t \to \infty$. Therefore,

$$\nu(j \mid S_d) = \frac{\sum_{i \in S_d} \pi_i\, m_{i,j} dt}{\sum_{k \in S_d} \pi_k} \qquad (6.2\text{-}2)$$

This equation holds for all $j \in S$. To find the initial probability vector of an up-time interval, we need only those $\nu(j \mid S_d)$ such that $j \in S_u$. It can be seen that, in the steady state, for each $j \in S_u$

Pr{ system begins an up-time interval in state j}

$$= \frac{\nu(j|S_d)}{\sum_{j \epsilon S_u} \nu(j|S_d)} \quad (6.2\text{-}3)$$

Substuting Eq. (6.2-2) into (6.2-3) results in:

Pr{ system begins an up-time interval in state j}

$$= \frac{\sum_{i \epsilon S_d} \pi_i m_{i,j}}{\sum_{k \epsilon S_u} \sum_{i \epsilon S_d} \pi_i m_{i,k}} \quad \text{for all } j \epsilon S_u \quad (6.2\text{-}4)$$

Now, partition the system transition-rate matrix M into submatrices as follows:

$$M = \left[\begin{array}{c|c} \underset{(k_o x k_o)}{M_{1,1}} & \underset{(k_o x(n_o - k_o))}{M_{1,2}} \\ \hline \underset{((n_o - k_o)x k_o)}{M_{2,1}} & \underset{((n_o - k_o)x(n_o - k_o))}{M_{2,2}} \end{array}\right] \quad (6.2\text{-}5)$$

Notice that by comparing this partition of M with that of the matrix B in Eq. (5.3-7) shows the following equalities of submatrices:

$$M_{1,1} = B_{1,1} \quad (6.2\text{-}6)$$

and

$$M_{1,2} = B_{1,2} \tag{6.2-7}$$

Equation (6.2-4) can be written in vector form in terms of the submatrix $M_{2,1}$ thus

$$\underline{P}_u(0) = \frac{\underline{\pi}_d M_{2,1}}{\underline{\pi}_d M_{2,1} \underline{v}^T(k_o, 0)} \tag{6.2-8}$$

This is the initial probability vector as the system begins an up-time interval in the steady state.

Following a similar procedure we can readily show that the initial probability vector as the system begins a down-time interval in the steady state is:

$$\underline{P}_d(0) = \frac{\underline{\pi}_u M_{1,2}}{\underline{\pi}_u M_{1,2} \underline{v}^T(n_o - k_o, 0)} \tag{6.2-9}$$

Before proceeding, we will prove the following lemma.

<u>Lemma 6.1</u>

For an ergodic stationary Markovian system, the following expressions, which are functions of the steady state probabilities and the state transition rates, are equal.

$$-\sum_{j\epsilon S_u}\sum_{i\epsilon S_u}\pi_i m_{i,j} = \sum_{j\epsilon S_u}\sum_{i\epsilon S_d}\pi_i m_{i,j} = \sum_{j\epsilon S_d}\sum_{i\epsilon S_u}\pi_i m_{i,j} = -\sum_{j\epsilon S_d}\sum_{i\epsilon S_d}\pi_i m_{i,j} \tag{6.2-10}$$

Proof:

In terms of the submatrices $M_{i,j}$, Eq. (5.6-2) may be written as:

$$[\underline{\pi}_u \ \underline{\pi}_d] \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} = \underline{0}_{n_o} \tag{6.2-11}$$

On expanding we have

$$\underline{\pi}_d M_{2,1} = -\underline{\pi}_u M_{1,1} \tag{6.2-12}$$

and

$$\underline{\pi}_u M_{1,2} = -\underline{\pi}_d M_{2,2} \tag{6.2-13}$$

Recall the zero row sum property of the transition-rate matrix M. That is

$$\sum_{j=1}^{n} m_{i,j} = 0 \quad \text{for all } i \epsilon S \tag{6.2-14}$$

This means,

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \begin{bmatrix} \underline{v}^T(k_o, 0) \\ \underline{v}^T(n_o - k_o, 0) \end{bmatrix} = \underline{0}^T_{n_o} \qquad (6.2\text{-}15)$$

It follows that

$$M_{1,2}\underline{v}^T(n_o - k_o, 0) = -M_{1,1}\underline{v}^T(k_o, 0) \qquad (6.2\text{-}16)$$

and

$$M_{2,1}\underline{v}^T(k_o, 0) = -M_{2,2}\underline{v}^T(n_o - k_o, 0) \qquad (6.2\text{-}17)$$

By Eqs. (6.2-12), (6.2-13), (6.2-16) and (6.2-17) we obtain the following equalities:

$$-\underline{\pi}_u M_{1,1}\underline{v}^T(k_o, 0) = \underline{\pi}_d M_{2,1}\underline{v}^T(k_o, 0) = \underline{\pi}_u M_{1,2}\underline{v}^T(n_o - k_o, 0) = -\underline{\pi}_d M_{2,2}\underline{v}^T(n_o - k_o, 0) \qquad (6.2\text{-}18)$$

These are scalar quantities, and they can be written as:

$$-\sum_{j\epsilon S_u}\sum_{i\epsilon S_u}\pi_i m_{i,j} = \sum_{j\epsilon S_u}\sum_{i\epsilon S_d}\pi_i m_{i,j} = \sum_{j\epsilon S_d}\sum_{i\epsilon S_u}\pi_i m_{i,j} = -\sum_{j\epsilon S_d}\sum_{i\epsilon S_d}\pi_i m_{i,j}$$

This completes the proof.

For expediency, we will denote the value of the expressions in Eq. (6.2-10) by c_o. That is:

$$c_o = -\sum_{j\epsilon S_u} \sum_{i\epsilon S_u} \pi_i m_{i,j} = \sum_{j\epsilon S_u} \sum_{i\epsilon S_d} \pi_i m_{i,j} = \sum_{j\epsilon S_d} \sum_{i\epsilon S_u} \pi_i m_{i,j} = -\sum_{j\epsilon S_d} \sum_{i\epsilon S_d} \pi_i m_{i,j} \tag{6.2-19}$$

Since $\pi_i > 0$ for all $i \epsilon S$, and $m_{i,j} \geq 0$ for all $i \neq j$, we see that $c_o > 0$.

By Eq. (6.2-18), the denominators in Eqs. (6.2-8) and (6.2-9) are equal to c_o. Now applying Eqs. (6.2-12) and (6.2-13) to the numerators of Eqs. (6.2-8) and (6.2-9), respectively, we obtain:

$$\underline{P}_u(0) = -\frac{1}{c_o} \underline{\pi}_u M_{1,1} \tag{6.2-20}$$

and

$$\underline{P}_d(0) = -\frac{1}{c_o} \underline{\pi}_d M_{2,2} \tag{6.2-21}$$

We have proved the following theorem.

Theorem 6.1

In an ergodic stationary Markovian system, the state probability distributions of the system in the steady state at the beginning of the up-time and down-time intervals are given by Eqs. (6.2-20) and (6.2-21), respectively.

The probability density function of the up-time intervals will be denoted by $f_{UT}(t)$. Since $\underline{P}_u(0)$ has been found, this function can now be obtained by appealing to Theorem 5.9.

$$f_{UT}(t) = \frac{1}{c_o} \underline{\pi}_u M_{1,1}^2 e^{M_{1,1}t} \underline{v}^T(k_o, 0) \qquad (6.2\text{-}22)$$

Using a parallel approach, the probability density function of the down-time intervals can be derived. We denote this function by $g_{DT}(t)$.

$$g_{DT}(t) = \frac{1}{c_o} \underline{\pi}_d M_{2,2}^2 e^{M_{2,2}t} \underline{v}^T(n_o - k_o, 0) \qquad (6.2\text{-}23)$$

The mth order moments of UT and DT are obtained by applying Theorem 5.12. Thus, for $m \geq 1$,

$$UT^{(m)} = (-1)^{m+1} m! \frac{1}{c_o} \underline{\pi}_u M_{1,1}^{-(m-1)} \underline{v}^T(k_o, 0) \qquad (6.2\text{-}24)$$

and

$$DT^{(m)} = (-1)^{m+1} m! \frac{1}{c_o} \underline{\pi}_d M_{2,2}^{-(m-1)} \underline{v}^T(n_o - k_o, 0) \qquad (6.2\text{-}25)$$

It follows that the mean and variance of UT and DT are:

$$MUT = \frac{\underline{\pi}_u \underline{v}^T(k_o, 0)}{c_o} = \frac{A}{c_o} \qquad (6.2\text{-}26)$$

$$MDT = \frac{\underline{\pi}_d \underline{v}^T(n_o - k_o, 0)}{c_o} = \frac{1-A}{c_o} \qquad (6.2\text{-}27)$$

$$\mathrm{var(UT)} = -\frac{2}{c_o}\,\underline{\pi}_u\, M^{-1}_{1,1}\,\underline{v}^T(k_o, 0) - \left(\frac{A}{c_o}\right)^2 \qquad (6.2\text{-}28)$$

and

$$\mathrm{var(DT)} = -\frac{2}{c_o}\,\underline{\pi}_d\, M^{-1}_{2,2}\,\underline{v}^T(n_o-k_o, 0) - \left(\frac{1-A}{c_o}\right)^2 \qquad (6.2\text{-}29)$$

Equations (6.2-26) and (6.2-27) yield the following relationship between MUT, MDT and the system steady state availability:

$$\frac{\mathrm{MUT}}{\mathrm{MUT}+\mathrm{MDT}} = A \qquad (6.2\text{-}30)$$

6.3 CONDITIONAL AND UNCONDITIONAL REMAINING UP-TIME AND DOWN-TIME INTERVALS

Since the steady state probability vector $\underline{\pi}$ of an ergodic stationary Markovian system is independent of time, the probability that the system is in state i, for all $i \epsilon S$, at any random time epoch in the steady state is π_i. It follows from the definitions of RUT and RDT that the initial probability vectors for RUT and RDT are given respectively by

$$\underline{P}_u(0) = \underline{\pi}_u \qquad (6.3\text{-}1)$$

and

$$\underline{P}_d(0) = \underline{\pi}_d \qquad (6.3\text{-}2)$$

To find the probability density functions and the mth order moments for RUT and RDT we once again appeal to Theorems 5.9 and 5.12. Let $f_{RUT}(t)$ and $g_{RDT}(t)$ denote the probability density functions of RUT and RDT, respectively. Then,

$$f_{RUT}(t) = -\underline{\pi}_u \, M_{1,1} e^{m_{1,1} t} \, \underline{v}^T(k_o, 0) \quad (6.3\text{-}3)$$

$$g_{RDT}(t) = -\underline{\pi}_d \, M_{2,2} e^{M_{2,2} t} \, \underline{v}^T(n_o - k_o, 0) \quad (6.3\text{-}4)$$

$$RUT^{(m)} = (-1)^m \, m! \, \underline{\pi}_u \, M_{1,1}^{-m} \, \underline{v}^T(k_{o,} \; 0) \quad (6.3\text{-}5)$$

$$RDT^{(m)} = (-1)^m \, m! \, \underline{\pi}_d \, M_{2,2}^{-m} \, \underline{v}^T(n_o - k_o, 0) \quad (6.3\text{-}6)$$

For the case of RUT_u and RDT_d, the initial probability vectors in Eqs. (5.13-1) and (5.13-2) need to be normalized as follows:

$$\underline{P}_u(0) = \frac{\underline{\pi}_u}{\underline{\pi}_u \, \underline{v}^T(k_o, 0)} = \frac{\underline{\pi}_u}{A} \quad (6.3\text{-}7)$$

and

$$\underline{P}_d(0) = \frac{\underline{\pi}_d}{\underline{\pi}_d \, \underline{v}^T(n_o - k_o, 0)} = \frac{\underline{\pi}_d}{1 - A} \quad (6.3\text{-}8)$$

Once again, applying Theorems 5.9 and 5.12 we obtain:

$$f_{RUT_u}(t) = \frac{-1}{A} \underline{\pi}_u M_{1,1} e^{M_{1,1}t} \underline{v}^T(k_o, 0) \qquad (6.3\text{-}9)$$

$$g_{RDT_d}(t) = \frac{-1}{1-A} \underline{\pi}_d M_{2,2} e^{M_{2,2}t} \underline{v}^T(n_o - k_o, 0) \qquad (6.3\text{-}10)$$

$$RUT_u^{(m)} = \frac{(-1)^m m!}{A} \underline{\pi}_u M_{1,1}^{-m} \underline{v}^T(k_o, 0) \qquad (6.3\text{-}11)$$

$$RDT_d^{(m)} = \frac{(-1)^m m!}{1-A} \underline{\pi}_d M_{2,2}^{-m} \underline{v}^T(n_o - k_o, 0) \qquad (6.3\text{-}12)$$

By Eqs. (6.3-11) and (6.3-12) the mean and variances of RUT, RDT, RUT_u and RDT_d are as follows:

$$MRUT = -\underline{\pi}_u M_{1,1}^{-1} \underline{v}^T(k_o, 0) \qquad (6.3\text{-}13)$$

$$MRDT = -\underline{\pi}_d M_{2,2}^{-1} \underline{v}^T(n_o - k_o, 0) \qquad (6.3\text{-}14)$$

$$MRUT_u = \frac{-1}{A} \underline{\pi}_u M_{1,1}^{-1} \underline{v}^T(k_o, 0) \qquad (6.3\text{-}15)$$

$$MRDT_d = \frac{-1}{1-A} \underline{\pi}_d M_{2,2}^{-1} \underline{v}(n_o - k_o, 0) \qquad (6.3\text{-}16)$$

$$var(RUT) = 2\underline{\pi}_u M_{1,1}^{-2} \underline{v}^T(k_o, 0) - (\underline{\pi}_u M_{1,1}^{-1} \underline{v}^T(k_o, 0))^2 \qquad (6.3\text{-}17)$$

$$\text{var}(RDT) = 2\underline{\pi}_d M_{2,2}^{-2} \underline{v}^T(n_o - k_o, 0) - (\underline{\pi}_d M_{2,2}^{-1} \underline{v}^T(n_o - k_o, 0))^2 \quad (6.3\text{-}18)$$

$$\text{var}(RUT_u) = \frac{2}{A} \underline{\pi}_u M_{1,1}^{-2} \underline{v}^T(k_o, 0) - (\frac{1}{A} \underline{\pi}_u M_{1,1}^{-1} \underline{v}^T(k_o, 0))^2 \quad (6.3\text{-}19)$$

$$\text{var}(RDT_d) = \frac{2}{1-A} \underline{\pi}_d M_{2,2}^{-2} \underline{v}^T(n_o - k_o, 0) - (\frac{1}{1-A} \underline{\pi}_d M_{2,2}^{-1} \underline{v}^T(n_o - k_o, 0))^2 \quad (6.3\text{-}20)$$

6.4 RELATIONSHIPS BETWEEN THE PROBABILITY DENSITY FUNCTIONS AND THE MOMENTS OF THE VARIOUS UP-TIMES AND DOWN-TIMES

In Section 6.2, the probability density functions and the mth order moments of UT and DT are derived. They are given in Eqs. (6.2-22) - (6.2-25). In the preceeding section, the density functions and the moments for RUT, RDT, RUT_u, and RDT_d are developed. The expressions corresponding to RUT and RDT are given in Eqs. (6.3-3) - (6.3-6); and the expressions corresponding to RUT_u and RDT_d are given in Eqs. (6.3-9) - (6.3-12).

Comparison of the various density functions reveals no other relationship except the following obvious two:

$$f_{RUT}(t) = A\, f_{RUT_u}(t) \qquad (6.4\text{-}1)$$

and

$$f_{RDT}(t) = (1 - A)\, f_{RDT_d}(t) \qquad (6.4\text{-}2)$$

Comparison of Eqs. (6.2-24), (6.3-5) and (6.3-11) reveals the following interesting relationship between the moments.

$$UT^{(m+1)} = (\frac{m+1}{c_o})\, RUT^{(m)} = (m+1)(\frac{A}{c_o})\, RUT_u^{(m)} \qquad (6.4\text{-}3)$$

In view of Eq. (6.2-26), the above can be written as:

$$UT^{(m+1)} = (\frac{m+1}{A})(MUT)\, RUT^{(m)} = (m+1)(MUT)RUT_u^{(m)} \qquad (6.4\text{-}4)$$

This shows a relationship between the (m+1)th order moment of UT and the mth order moment of RUT or RUT_u. Similarly, from Eqs. (6.2-25), (6.3-6) and (6.3-12), and in light of Eq. (6.2-27) we obtain the relationship:

$$DT^{(m+1)} = (\frac{m+1}{1-A})(MDT)RDT^{(m)} = (m+1)(MDT)RDT_d^{(m)} \qquad (6.4\text{-}5)$$

It follows from the last two equations that

$$\mathrm{var(UT)} = \mathrm{MUT}[\frac{2}{A}\,\mathrm{MRUT} - \mathrm{MUT}] = \mathrm{MUT}[2\mathrm{MRUT}_u - \mathrm{MUT}] \quad (6.4\text{-}6)$$

and

$$\mathrm{var(DT)} = \mathrm{MDT}[\frac{2}{1-A}\,\mathrm{MRDT} - \mathrm{MDT}] = \mathrm{MDT}[2\,\mathrm{MRDT}_d - \mathrm{MDT}] \quad (6.4\text{-}7)$$

We now compare the first order moments of the various up-times and down-times. Recall that the system steady state availability A is bounded between 0 and 1, the following inequalities are obvious from Eqs. (6.4-3) and (6.4-4).

$$\mathrm{MRUT} < \mathrm{MRUT}_u \quad (6.4\text{-}8)$$

and

$$\mathrm{MRDT} < \mathrm{MRDT}_d \quad (6.4\text{-}9)$$

There exists no such fixed inequalities between MUT(MDT) and MRUT(MRDT) or MRUT_u(MRDT_d). This is illustrated by two simple examples in Appendix F.

6.5 RELATIONSHIP BETWEEN SYSTEM MUT (MDT) AND INDEPENDENT SUBSYSTEM MUT'S (MDT's)

In Section 6.2, expressions for system MUT and MDT, given by Eqs. (6.2-26) and (6.2-27), are derived for an ergodic stationary Markovian system in general. In the case when the system is comprised of independent subsystems, it would be computationally advantageous to compute the system MUT and MDT through computation of the subsystem MUT's and MDT's. This section is devoted to the development of the general relationship between system MUT (MDT) and independent subsystem MUT's (MDT's).

In general, a system is comprised of $n \geq 1$ independent subsystems. These subsystems could be interconnected in a complex configuration. Figure 6-3 shows a system comprised of 5 independent subsystems.

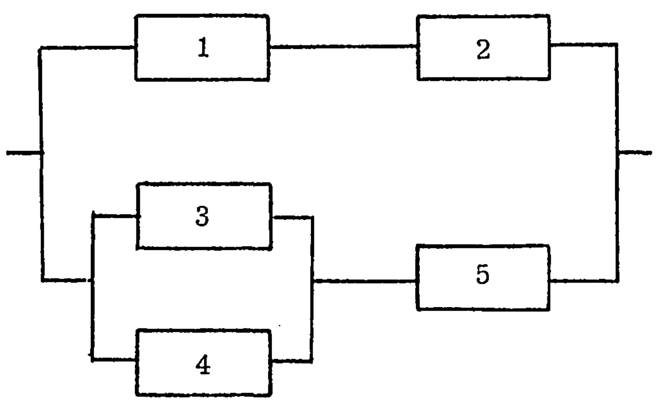

Figure 6-3 Configuration of a System Comprised of 5 Independent Subsystems.

It can be seen that the solution for any arbitrary system configuration can be achieved once the solutions for two independent subsystems in series as well as in parallel are developed. We shall first develop the solution for the series case.

Figure 6-4 shows the configuration of a system comprised of 2 independent subsystems in series.

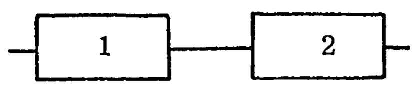

Figure 6-4 Two Independent Subsystems in Series

Let S_1 be the state space of subsystem 1,

S_2 be the state space of subsystem 2.

In addition, for i = 1, 2, let

S_{iu} = the set of up-states of subsystem i,

S_{id} = the set of the down-states of subsystem i.

It follows that

$$S_1 = S_{1u} \cup S_{1d} \tag{6.5-1}$$

and

$$S_2 = S_{2u} \cup S_{2d} \tag{6.5-2}$$

Since the 2 subsystems are independent and connected in series, the sets of the up-states and the down-states of the system are:

$$S_u = S_{1u} \times S_{2u} \tag{6.5-3}$$

$$S_d = (S_{1u} \times S_{2d}) U (S_{1d} \times S_{2u}) U (S_{1d} \times S_{2d}) \tag{6.5-4}$$

It can be seen that if n_i is the order of S_i, then the order of S is $n_1 n_2$. We note that the sets S_{iu} and S_{id}, $i = 1, 2$, are disjoint. From Eqs. (6.5-3) and (6.5-4) we see that each state of the system is defined as a combination of a state of subsystem 1 and a state of subsystem 2. Therefore an ordered pair of state notation may be used to represent a state in S. Our convention shall be as follows:

By $(a_1, a_2) \in S$, we mean that state (a_1, a_2) in S which represents the combination of states $a_1 \in S_1$ and $a_2 \in S_2$.

Recall the basic assumption made for a Markovian system in Section 4.3 that the probability of two or more changes occur in the system within an arbitrarily small interval is zero. It follows that the relationship between the state transition rates of the system and those of the subsystems are as given below.

For all $(a_1, a_2) \epsilon S$ and $(b_1, b_2) \epsilon S$,

$$m_{(a_1, a_2), (b_1, b_2)} = \begin{cases} m_{a_1, b_1} + m_{a_2, b_2} & \text{if } a_1 = b_1 \text{ and } a_2 = b_2 \\ m_{a_2, b_2} & \text{if } a_1 = b_1 \text{ and } a_2 \neq b_2 \\ m_{a_1, b_2} & \text{if } a_1 \neq b_1 \text{ and } a_2 = b_2 \\ 0 & \text{if } a_1 \neq b_1 \text{ and } a_2 \neq b_2 \end{cases} \qquad (6.5\text{-}5)$$

By Eqs. (6.2-26), MUT for an ergodic stationary Markovian system in general is:

$$MUT = \frac{A}{c_o} \qquad (6.5\text{-}6)$$

For the series system under consideration

$$A = \sum_{(a_1, a_2) \epsilon S_{1u} x S_{2u}} \pi_{(a_1, a_2)}$$

$$= \sum_{(a_1, a_2) \epsilon S_{1u} x S_{2u}} \pi_{a_1} \pi_{a_2} \qquad \text{since the two subsystems are independent}$$

$$= \sum_{a_1 \epsilon S_{1u}} \pi_{a_1} \sum_{a_2 \epsilon S_{2u}} \pi_{a_2} \qquad \text{since } S_{1u} \text{ and } S_{2u} \text{ are disjoint}$$

(6.5-7)

By Lemma 6.1, c_o is given by any one of the four expressions in Eq. (6.2-19). Therefore, for the present system,

$$c_o = \sum_{(a_1, a_2)\epsilon S_u} \sum_{(b_1, b_2)\epsilon S_d} \pi_{(a_1, a_2)} m_{(a_1, a_2), (b_1, b_2)} \tag{6.5-8}$$

The sets S_u and S_d are given by Eqs. (6.5-3) and (6.5-4), respectively. Since the products $(S_{1u} \times S_{2u})$, $(S_{1u} \times S_{2d})$, $(S_{1d} \times S_{2u})$ and $(S_{1d} \times S_{2d})$ are disjoint, and since the 2 subsystems are independent, Eq. (6.5-8) can be written as:

$$\begin{aligned} c_o = & \sum_{(a_1, a_2)\epsilon (S_{1u} x S_{2u})} \sum_{(b_1, b_2)\epsilon (S_{1u} x S_{2d})} \pi_{a_1} \pi_{a_2} m_{(a_1, a_2), (b_1, b_2)} \\ & + \sum_{(a_1, a_2)\epsilon (S_{1u} x S_{2u})} \sum_{(b_1, b_2)\epsilon (S_{1d} x S_{2u})} \pi_{a_1} \pi_{a_2} m_{(a_1, a_2), (b_1, b_2)} \\ & + \sum_{(a_1, a_2)\epsilon (S_{1u} x S_{2u})} \sum_{(b_1, b_2)\epsilon (S_{1d} x S_{2d})} \pi_{a_1} \pi_{a_2} m_{(a_1, a_2), (b_1, b_2)} \end{aligned} \tag{6.5-9}$$

By Eq. (6.5-5) we see that the last term in the above equation is zero, and c_o can be expressed in terms of the transition rates of subsystems 1 and 2 thus:

$$c_o = \sum_{(a_1, a_2) \in (S_{1u} \times S_{2u})} \sum_{b_2 \in S_{2d}} \pi_{a_1} \pi_{a_2} m_{a_2, b_2}$$

$$+ \sum_{(a_1, a_2) \in (S_{1u} \times S_{2u})} \sum_{b_1 \in S_{1d}} \pi_{a_1} \pi_{a_2} m_{a_1, b_1} \quad (6.5\text{-}10)$$

Since the sets S_{1u} and S_{2u} are disjoint, we obtain:

$$c_o = \Big(\sum_{a_1 \in S_{1u}} \pi_{a_1} \Big) \cdot \Big(\sum_{a_2 \in S_{2u}} \sum_{b_2 \in S_{2d}} \pi_{a_2} m_{a_2, b_2} \Big)$$

$$+ \Big(\sum_{a_2 \in S_{2u}} \pi_{a_2} \Big) \Big(\sum_{a_1 \in S_{1u}} \sum_{b_1 \in S_{1d}} \pi_{a_1} m_{a_1, b_1} \Big)$$

(6.5-11)

Let A_i be the steady state availability of subsystem i. Therefore,

$$A_i = \sum_{a_i \in S_{iu}} \pi_{a_i} \quad (6.5\text{-}12)$$

In addition, let

$$c_i = \sum_{a_i \in S_{iu}} \sum_{b_i \in S_{id}} \pi_{a_i} m_{a_i, b_i} \quad (6.5\text{-}13)$$

Then Eq. (6.5-11)becomes

$$c_o = A_1 c_2 + A_2 c_1 \qquad (6.5\text{-}14)$$

and Eq. (6.5-7) becomes

$$A = A_1 A_2 \qquad (6.5\text{-}15)$$

Substituting Eqs. (6.5-14) and (6.5-15) into Eq. (6.5-6) results in:

$$\begin{aligned} MUT &= \frac{A_1 A_2}{A_1 c_2 + A_2 c_1} \\ &= \frac{\frac{A_1}{c_1} \frac{A_2}{c_2}}{\frac{A_1}{c_1} + \frac{A_2}{c_2}} \\ &= \frac{MUT_1 \; MUT_2}{MUT_1 + MUT_2} \end{aligned} \qquad (6.5\text{-}16)$$

where

$$MUT_i = \text{mean up-time of subsystem i.} \qquad (6.5\text{-}17)$$

Eq. (6.5-16) gives the expression for system MUT in terms of the subsystem MUT's for the case when the system is comprised of 2 independent subsystems in series.

To find system MDT, substitute Eqs. (6.5-14) and (6.5-15) into Eq. (6.2-27) we obtain:

$$\mathrm{MDT} = \frac{1 - A_1 A_2}{A_1 c_2 + A_2 c_1}$$

$$= \frac{A_1(1-A_2) + A_2(1-A_1) + (1-A_1)(1-A_2)}{A_1 c_2 + A_2 c_1} \qquad (6.5\text{-}18)$$

Dividing both the numerator and the denominator by $c_1 c_2$ the above equation becomes:

$$\mathrm{MDT} = \frac{\left(\frac{A_1}{c_1}\right)\left(\frac{1-A_2}{c_2}\right) + \left(\frac{A_2}{c_2}\right)\left(\frac{1-A_1}{c_1}\right) + \left(\frac{1-A_1}{c_1}\right)\left(\frac{1-A_2}{c_2}\right)}{\frac{A_1}{c_1} + \frac{A_2}{c_2}}$$

$$= \frac{\mathrm{MUT}_1 \mathrm{MDT}_2 + \mathrm{MUT}_2 \mathrm{MDT}_1 + \mathrm{MDT}_1 \mathrm{MDT}_2}{\mathrm{MUT}_1 + \mathrm{MUT}_2} \qquad (6.5\text{-}19)$$

where

MDT_i = mean down-time of subsystem i (6.5-20)

Eq. (6.5-19) gives the expression for system MDT in terms of the subsystem MUT's and MDT's for the case when the system is comprised of 2 independent subsystems in series.

The expressions for system MUT and MDT of Eqs. (6.5-16) and (6.5-19) can be generalized to the case when the system comprises $n \geq 2$ independent subsystems in series, as shown in Figure 6-5.

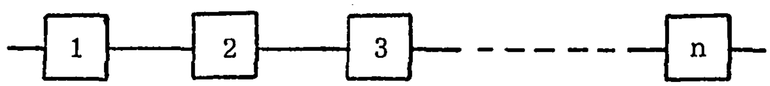

Figure 6-5 Configuration of a System Comprised of n Independent Subsystems in Series.

Define the following notations:

$MUT(i.j.---.k)_s$ = Mean up-time of subsystems $i, j, \ldots, k$ in series (6.5-21)

$MDT(i.j.---.k)_s$ = Mean down-time of subsystems $i, j, \ldots, k$ in series (6.5-22)

To develop the expression for the general case, first consider subsystems 1 and 2. The mean up-time of these two subsystems is given by Eq. (6.5-16) which can be written in another form as follows:

$$MUT(1 \cdot 2)_s = \frac{1}{\frac{1}{MUT_1} + \frac{1}{MUT_2}} \qquad (6.5\text{-}23)$$

Suppose it is true that

$$MUT(1.2.---.k)_s = \frac{1}{\sum_{i=1}^{k} \frac{1}{MUT_i}} \qquad (6.5\text{-}24)$$

Then for the case of (k+1) independent subsystems in series,

$$MUT(1.2.---.k+1)_s = \frac{1}{\frac{1}{MUT(1.2.---.k)_s} + \frac{1}{MUT_{k+1}}}$$

$$= \frac{1}{\sum_{i=1}^{k+1} \frac{1}{MUT_i}} \qquad (6.5\text{-}25)$$

Therefore, for the general case of n independent subsystems in series, we obtain:

$$MUT(1.2.---.n)_s = \frac{1}{\sum_{i=1}^{n} \frac{1}{MUT_i}} \qquad (6.5\text{-}26)$$

To find $MDT(1.2.---.n)_s$ we can make use of Eq. (6.2-30), which gives:

$$MDT(1.2.---.n)_s = MUT(1.2.---.n)_s\left[\frac{1-A}{A}\right] \qquad (6.5\text{-}27)$$

In the above equation A represents the availability of the n subsystems in series. Since the subsystems are independent,

$$A = A_1 A_2, \ldots, A_n = \prod_{i=1}^{n} A_i \qquad (6.5\text{-}28)$$

Substituting Eqs. (6.5-26) and (6.5-28) into Eq. (6.5-27),

$$MDT(1.2.---.n)_s = \frac{1}{\sum_{i=1}^{n} \frac{1}{MUT_i}} \quad \frac{1 - \prod_{i=1}^{n} A_i}{\prod_{i=1}^{n} A_i} \qquad (6.5\text{-}29)$$

For each A_i substitute

$$A_i = \frac{MUT_i}{MUT_i + MDT_i} \qquad (6.5\text{-}30)$$

Then Eq. (6.5-29) becomes

$$MDT(1.2.---.n)_s = \frac{1}{\sum_{i=1}^{n} \frac{1}{MUT_i}} \quad \frac{\prod_{i=1}^{n} (MUT_i + MDT_i) - \prod_{j=1}^{n} MUT_j}{\prod_{j=1}^{n} MUT_j}$$

(6.5-31)

We have proved the following theorem.

Theorem 6.2

For an ergodic stationary Markovian system which is comprised of n independent subsystems in series, the system MUT and MDT can be expressed in terms of the subsystem MUT's and MDT's. Eq. (6.4-26) gives the expression for system MUT and Eq. (6.5-31) gives the expression for system MDT.

To develop similar expressions for the case when the subsystems are connected in parallel, we start with a two subsystem configuration as shown in Figure 6-6.

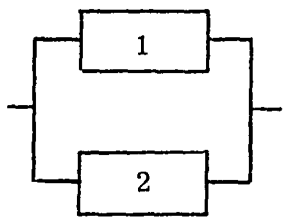

Figure 6-6 Configuration of a System Comprised of 2 Independent Subsystems in Parallel.

In this case, the sets of system up-states and down-states are as follows:

$$S_u = (S_{1u} \times S_{2u}) \cup (S_{1u} \times S_{2d}) \cup (S_{1d} \times S_{2u}) \qquad (6.5\text{-}32)$$

$$S_d = (S_{1d} \times S_{2d}) \qquad (6.5\text{-}33)$$

The steady state availability of the system is:

$$A = \sum_{(a_1, a_2) \in S_u} \pi_{(a_1, a_2)} \qquad (6.5\text{-}34)$$

Since the sets $(S_{1u} \times S_{2u})$, $(S_{1u} \times S_{2d})$ and $(S_{1d} \times S_{2u})$ are disjoint, and since the subsystems are independent,

$$A = \sum_{(a_1, a_2) \in (S_{1u} \times S_{2u})} \pi_{a_1} \pi_{a_2} + \sum_{(a_1, a_2) \in (S_{1u} \times S_{2d})} \pi_{a_1} \pi_{a_2} + \sum_{(a_1, a_2) \in (S_{1d} \times S_{2u})} \pi_{a_1} \pi_{a_2}$$

$$= \Big(\sum_{a_1 \in S_{1u}} \pi_{a_1} \Big) \Big(\sum_{a_2 \in S_{2u}} \pi_{a_2} \Big) + \Big(\sum_{a_1 \in S_{1u}} \pi_{a_1} \Big) \Big(\sum_{a_2 \in S_{2d}} \pi_{a_2} \Big)$$

$$+ \Big(\sum_{a_1 \in S_{1d}} \pi_{a_1} \Big) \Big(\sum_{a_2 \in S_{2u}} \pi_{a_2} \Big)$$

$$= A_1 A_2 + A_1(1 - A_2) + A_2(1 - A_1)$$

(6.5-35)

For this case, c_o is given by

$$c_o = \sum_{(a_1, a_2)\epsilon S_u} \sum_{(b_1, b_2)\epsilon S_d} \pi_{(a_1, a_2)} m_{(a_1, a_2), (b_1, b_2)} \qquad (6.5\text{-}36)$$

By Eqs. (6.5-5), (6.5-32) and (6.5-33), it follows that

$$c_o = \sum_{(a_1, a_2)\epsilon (S_{1u} x S_{2d})} \sum_{(b_1, b_2)\epsilon (S_{1d} x S_{2d})} \pi_{(a_1, a_2)} m_{(a_1, a_2), (b_1, b_2)}$$

$$+ \sum_{(a_1, a_2)\epsilon (S_{1d} x S_{2u})} \sum_{(b_1, b_2)\epsilon (S_{1d} x S_{2d})} \pi_{(a_1, a_2)} m_{(a_1, a_2), (b_1, b_2)}$$

$$= \left(\sum_{a_2\epsilon S_{2d}} \pi_{a_2} \right) \left(\sum_{b_1\epsilon S_{1d}} \sum_{a_1\epsilon S_{1u}} \pi_{a_1} m_{a_1, b_1} \right)$$

$$+ \left(\sum_{a_1\epsilon S_{1d}} \pi_{a_1} \right) \left(\sum_{b_2\epsilon S_{2d}} \sum_{a_2\epsilon S_{2u}} \pi_{a_2} m_{a_2, b_2} \right)$$

$$= (1 - A_2)c_1 + (1 - A_1)c_2 \qquad (6.5\text{-}37)$$

Therefore,

$$MUT = \frac{A_1 A_2 + A_1(1-A_2) + A_2(1-A_1)}{(1-A_2)c_1 + (1-A_1)c_2} \qquad (6.5\text{-}38)$$

Dividing both the numerator and the denominator by $c_1 c_2$ we obtain:

$$\text{MUT} = \frac{(\frac{A_1}{c_1})(\frac{A_2}{c_2}) + (\frac{A_1}{c_1})(\frac{1-A_2}{c_2}) + (\frac{A_2}{c_2})(\frac{1-A_1}{c_1})}{(\frac{1-A_2}{c_2}) + (\frac{1-A_2}{c_1})}$$

$$= \frac{\text{MUT}_1 \text{MUT}_2 + \text{MUT}_1 \text{MDT}_2 + \text{MUT}_2 \text{MDT}_1}{\text{MDT}_2 + \text{MDT}_1} \qquad (6.5\text{-}39)$$

Eq. (6.5-39) gives the expression for system MUT in terms of subsystem MUT's and MDT's for the case when the system is comprised of 2 independent subsystems in parallel.

To find MDT for this case, we recognize that

$$(1-A) = (1-A_1)(1-A_2) \qquad (6.5\text{-}40)$$

Therefore, by Eq. (6.2-27),

$$\text{MDT} = \frac{(1-A_1)(1-A_2)}{(1-A_2)c_1 + (1-A_1)c_2} \qquad (6.5\text{-}41)$$

Dividing both the numerator and the denominator by c_1 c_2, we obtain:

$$\text{MDT} = \frac{\left(\frac{1-A_1}{c_1}\right)\left(\frac{1-A_2}{c_2}\right)}{\left(\frac{1-A_2}{c_2}\right)+\left(\frac{1-A_1}{c_1}\right)}$$

$$= \frac{\text{MDT}_1 \ \text{MDT}_2}{\text{MDT}_1 + \text{MDT}_2} \qquad (6.5\text{-}42)$$

Eq. (6.5-42) gives the expression for system MDT in terms of subsystem MDT's for the case when the system is comprised of 2 independent subsystems in parallel.

To solve for the general case of n independent subsystems in parallel as shown in Figure 6-7, we first introduce the following notations.

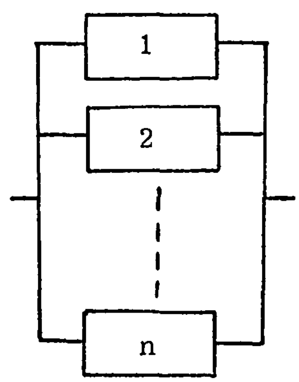

Figure 6-7 Configuration of a system comprised of n Independent Subsystems in Parallel.

$MUT(i.j.---.k)_p$ = MUT of subsystems i, j, . . . , k connected in parallel

(6.5-43)

$MDT(i.j.---.k)_p$ = MDT of subsystems i, j, . . . , k connected in parallel

(6.5-44)

First consider for the MDT of subsystems 1 and 2. By Eq. (6.5-42),

$$MDT(1\cdot 2)_p = \frac{1}{\frac{1}{MDT_1} + \frac{1}{MDT_2}} \tag{6.5-45}$$

It follows that

$$MDT(1\cdot 2\cdot 3)_p = \frac{1}{\frac{1}{MDT(1\cdot 2)_p} + \frac{1}{MDT_3}}$$

$$= \frac{1}{\sum_{i=1}^{3} \frac{1}{MDT_i}} \tag{6.5-46}$$

Suppose it is true for subsystems 1, 2, . . . , k that

$$MDT(1\cdot 2\cdot --- \cdot k)_p = \frac{1}{\sum_{i=1}^{k} \frac{1}{MDT_i}} \tag{6.5-47}$$

Then for subsystems 1, 2, . . . , k+1 we have

$$MDT(1\cdot 2\cdot ---\cdot k+1)_p = \frac{1}{\frac{1}{MDT(1\cdot 2\cdot ---k)_p} + \frac{1}{MDT_{k+1}}}$$

$$= \frac{1}{\sum_{i=1}^{k+1} \frac{1}{MDT_i}} \quad (6.5\text{-}48)$$

Therefore, for the general case of n independent subsystems in parallel, we have

$$MDT(1\cdot 2\cdot ---\cdot n)_p = \frac{1}{\sum_{i=1}^{n} \frac{1}{MDT_i}} \quad (6.5\text{-}49)$$

It should be noted that the above expression holds for the case of the system is considered to be up when at least one of the subsystem is up. By Eq. (6.2-30),

$$MUT(1\cdot 2\cdot ---\cdot n)_p = MDT(1\cdot 2\cdot ---\cdot n)_p \frac{A}{1-A} \quad (6.5\text{-}50)$$

For this case,

$$A = 1 - \prod_{i=1}^{n} (1 - A_i) \qquad (6.5\text{-}51)$$

First substitute Eq. (6.5-30) into Eq. (6.5-51) and then substitute the resultant equation into Eq. (6.5-50), we obtain after simplification

$$MUT(1\ 2\ \text{---}\ n)_p = \frac{1}{\sum_{i=1}^{n} \frac{1}{MDT_i}} \quad \frac{\prod_{i=1}^{n} (MUT_i + MDT_i) - \prod_{j=1}^{n} MDT_j}{\prod_{j=1}^{n} MDT_j}$$

(6.5-52)

We have proved the following theorem.

<u>Theorem 6.3</u>

For an ergodic stationary Markovian system which is comprised of n independent subsystems in parallel, the system is defined to be up if at least one of the subsystems is up, the system MUT and MDT can be expressed in terms of the subsystem MUT's and MDT's. Equation (6.5-49) gives the expression for system MDT, and Eq. (6.5-52) gives the expression for system MUT.

We now consider the case of 3 independent subsystems in parallel as shown in Figure 6-8.

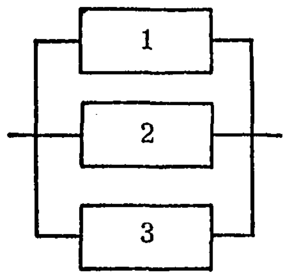

Figure 6-8 Configuration of 3 independent subsystems in Parallel.

Suppose this system is defined to be up when at least 2 of the 3 subsystems are up. It can be seen that the sets of up-states and down-states for this case are:

$$S_u = (S_{1u} \times S_{2u} \times S_{3u}) \cup (S_{1u} \times S_{2d} \times S_{3u}) \cup (S_{1d} \times S_{2u} \times S_{3u}) \cup (S_{1u} \times S_{2u} \times S_{3d}) \tag{6.5-53}$$

$$S_d = (S_{1d} \times S_{2d} \times S_{3u}) \cup (S_{1u} \times S_{2d} \times S_{3d}) \cup (S_{1d} \times S_{2u} \times S_{3d}) \cup (S_{1d} \times S_{2d} \times S_{3d}) \tag{6.5-54}$$

Since the subsystems are independent, by Eq. (6.5-53) the steady state availability of the system is

$$A = A_1A_2A_3 + A_1(1-A_2)A_3 + (1-A_1)A_2A_3 + A_1A_2(1-A_3)$$

(6.5-55)

We now compute the expression for c_o.

$$c_o = \sum_{(a_1, a_2, a_3)\epsilon S_u} \sum_{(b_1, b_2, b_3)\epsilon S_d} \pi_{(a_1, a_2, a_3)} m_{(a_1, a_2, a_3), (b_1, b_2, b_3)}$$

(6.5-56)

Invoke the basic assumption of Markovian system that the probability of two or more changes occur within an arbitrarily small interval is zero, we have

$$m_{(a_1, a_2, a_3)(b_1, b_2, b_3)} = \begin{cases} m_{a_1, b_1} + m_{a_2, b_2} + m_{a_3, b_3} & \text{if } a_i = b_i \text{ for } i = 1, 2, 3 \\ m_{a_1, b_1} & \text{if } a_2 = b_2 \text{ and } a_3 = b_3 \\ m_{a_2, b_2} & \text{if } a_1 = b_1 \text{ and } a_3 = b_3 \\ m_{a_3, b_3} & \text{if } a_1 = b_1 \text{ and } a_2 = b_2 \\ -0 & \text{otherwise} \end{cases}$$

(6.5-57)

Since the subsystems are independent,

$$\pi_{(a_1, a_2, a_3)} = \pi_{a_1} \pi_{a_2} \pi_{a_3} \tag{6.5-58}$$

Substituting Eqs. (6. 5-53) and (6. 5-58) into Eq. (6. 5-56), and after incorporating Eq. (6. 5-52) we obtain:

$$\begin{aligned}
c_o = & \sum_{(a_1, a_2, a_3) \epsilon (S_{1u} x S_{2d} x S_{3u})} \sum_{b_1 \epsilon S_{1d}} \pi_{a_1} \pi_{a_2} \pi_{a_3} m_{a_1, b_1} \\
& + \sum_{(a_1, a_2, a_3) \epsilon (S_{1u} x S_{2d} x S_{3u})} \sum_{b_3 \epsilon S_{3d}} \pi_{a_1} \pi_{a_2} \pi_{a_3} m_{a_3, b_3} \\
& + \sum_{(a_1, a_2, a_3) \epsilon (S_{1d} x S_{2u} x S_{3u})} \sum_{b_2 \epsilon S_{2d}} \pi_{a_1} \pi_{a_2} \pi_{a_3} m_{a_2, b_2} \\
& + \sum_{(a_1, a_2, a_3) \epsilon (S_{1d} x S_{2u} x S_{3u})} \sum_{b_3 \epsilon S_{3d}} \pi_{a_1} \pi_{a_2} \pi_{a_3} m_{a_3, b_3} \\
& + \sum_{(a_1, a_2, a_3) \epsilon (S_{1u} x S_{2u} x S_{3d})} \sum_{b_2 \epsilon S_{2d}} \pi_{a_1} \pi_{a_2} \pi_{a_3} m_{a_2, b_2} \\
& + \sum_{(a_1, a_2, a_3) \epsilon (S_{1u} x S_{2u} x S_{3d})} \sum_{b_1 \epsilon S_{1d}} \pi_{a'} \pi_{a_2} \pi_{a_3} m_{a_1, b_1}
\end{aligned} \tag{6.5-59}$$

Since the sets S_{iu} and S_{id}, for i = 1, 2, 3, are disjoint, we obtain

$$c_o = c_1(1-A_2)A_3 + A_1(1-A_2)c_3 + (1-A_1)c_2A_3 + (1-A_1)A_2c_3 + A_1c_2(1-A_3) + c_1A_2(1-A_3) \qquad (6.5\text{-}60)$$

Dividing Eq. (6.5-55) by Eq. (6.5-60) gives

$$MUT = \frac{A_1A_2A_3 + A_1(1-A_2)A_3 + (1-A_1)A_2A_3 + A_1A_2(1-A_3)}{\{c_1(1-A_2)A_3 + A_1(1-A_2)c_3 + (1-A_1)c_2A_3 + (1-A_1)A_2c_3 + A_1c_2(1-A_3) + C_1A_2(1-A_3)\}}$$

$$= \frac{MUT_1MUT_2MUT_3 + MUT_1MDT_2MUT_3 + MDT_1MUT_2MUT_3 + MUT_1MUT_2MDT_3}{MDT_2MUT_3 + MUT_1MDT_2 + MDT_1MUT_3 + MDT_1MUT_2 + MUT_1MDT_3 + MUT_2MDT_3} \qquad (6.5\text{-}61)$$

From Eq. (6.5-54) we see that

$$1-A = (1-A_1)(1-A_2)A_3 + A_1(1-A_2)(1-A_3) + (1-A_1)A_2(1-A_3) + (1-A_1)(1-A_2)(1-A_3) \qquad (6.5\text{-}62)$$

By Eqs. (6.5-60) and Eq. (6.5-62) we obtain:

$$MDT = \frac{1-A}{c_o}$$

$$= \frac{MDT_1 MDT_2 MUT_3 + MUT_1 MDT_2 MDT_3 + MDT_1 MUT_2 MUT_3 + MDT_1 MDT_2 MDT_3}{MDT_2 MUT_3 + MUT_1 MDT_2 + MDT_1 MUT_3 + MDT_1 MUT_2 + MUT_1 MDT_3 + MUT_2 MDT_3} \tag{6.5-63}$$

We have proved the following theorem.

Theorem 6.4

For an ergodic stationary Markovian system which is comprised of 3 independent subsystems in parallel, and the system is defined to be up if at least 2 of the 3 subsystems are up, then the MUT and MDT of the system, when expressed in terms of the subsystem MUT's and MDT's, are given by Eqs. (6.5-61) and (6.5-63), respectively.

In the special case where the 3 subsystems are identical, Eqs. (6.5-61) and (6.5-63) reduce to:

$$MUT = \frac{MUT_i(MUT_i + 3\ MDT_i)}{6\ MDT_i} \tag{6.5-64}$$

$$MDT = \frac{MDT_i(MDT_i + 3\ MUT_i)}{6\ MUT_i} \tag{6.5-65}$$

where i is either 1, 2 or 3.

The following corollary follows.

<u>Corollary 6.4</u>

If the 3 subsystems of the hypothesis of Theorem 6.4 are identical, then system MUT and MDT are given by Eqs. (6.5-64) and (6.5-65), respectively.

Chapter 7

MODELING OF SYSTEMS OBEYING SEMI-MARKOV PROCESSES

7.1 INTRODUCTION

In the last three chapters, we have devoted our effort to the modeling and analysis of stationary systems obeying continuous parameter Markov chains. By Theorem 4.2, the random time of the transition process between any two states of such a system necessarily possesses exponential probability distribution. For a physical system this means that the time-to-failure and time-to-repair of the units comprising the system necessarily possess exponential probability distributions. In this section we will consider systems which obey a more general semi-Markov model. A descriptive definition of a semi-Markov process is given below.

> A semi-Markov process is a stochastic process which moves from one state to another of a countable number of states with the successive states visited forming a Markov chain, and that the process stays in a given state for a random length of time, the distribution function of which is general and may depend on this state as well as on the next state to be visited.
>
> (7.1-1)

We will be mainly concerned with semi-Markov processes with finite state space. As before let the state space S be

$$S = \{1, 2, \ldots, n_o\} \tag{7.1-2}$$

The state transition probability matrix for the imbedded Markov chain will be denoted by

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n_o} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n_o} \\ \vdots & & & \vdots \\ p_{n_o,1} & p_{n_o,2} & \cdots & p_{n_o,n_o} \end{bmatrix} \tag{7.1-3}$$

Suppose $i, j \in S$ are such that the process can go from state i to state j by a single transition. Define the conditional holding time $w_{i,j}$ as follows:

$w_{i,j}$ = the holding time of the process in state i given that the process next visits state j. (7.1-4)

The probability distribution function of $w_{i,j}$ is denoted by $F_{i,j}(t)$. Therefore

$$F_{i,j}(t) = \Pr\{w_{i,j} \leq t\} \tag{7.1-5}$$

Note that $F_{i,j}(t)$ is the conditional holding time distribution function in state i, given that the process next visits state j. If w_i denotes the unconditional holding time in state i before the next transition, and if $H_i(t)$ denotes the distribution function for w_i, then it follows that

$$H_i(t) = Pr\{w_i \leq t\}$$

$$= \sum_{j \in S} p_{i,j} F_{i,j}(t) \qquad (7.1\text{-}6)$$

For simplicity of notation, define

$$Q_{i,j}(t) = p_{i,j} F_{i,j}(t) \qquad (7.1\text{-}7)$$

Then Eq. (7.1-6) becomes

$$H_i(t) = \sum_{j \in S} Q_{i,j}(t) \qquad (7.1\text{-}8)$$

The matrix $(Q_{i,j}(t))$ is known as the matrix of transition distributions [51].

In the literature, a semi-Markov process is denoted by $\{z(t); t \geq 0\}$. Let $N_i(t)$ denote the number of times the process enters state i in the half open interval (0, t]; and let $\underline{N}(t)$ denote the vector

$$\underline{N}(t) = [N_1(t), N_2(t), \ldots, N_{n_o}(t)] \qquad (7.1\text{-}9)$$

Then the stochastic process $\{\underline{N}(t); t \geq 0\}$ is known as the Markov renewal process. It can be seen that the z(t) process and the $\underline{N}(t)$ process are different aspects of the same underlying stochastic process. Therefore, studying of one is equivalent to studying of the other.

In the following sections we will consider modeling of systems with general distributions for the time-to-repair. It will be shown that such system models take the form of semi-Markov processes.

7.2 MODEL CONSTRUCTION FOR SYSTEMS WITH GENERAL REPAIR-TIME DISTRIBUTIONS

This section considers modeling of systems comprised of units with exponential distributions for time-to-failure, and general distributions for time-to-repair. The number of repair crews available for servicing the failed units is restricted to one so that the state space of the system is finite.

The first step in modeling is to define the various system states. Similar to the case of Markovian systems, the states for the more general type of system considered here are also defined according to the different combinations of the up and down conditions of the units comprising the system. There is, however, one important difference due to the general probability distributions for the time-to-repair. In the case of Markovian systems, given the present system state, the future behavior of the system is independent of the past. In the present type of system, the time-to-failure of the units would still be independent of the past. But the time-to-repair would not be so except at the so-called regeneration points. Starting from a regeneration point, the future behavior of the system is stochastically independent of the past. Therefore, in defining states for the more general system, we will need to identify the regeneration points in addition to the different combinations of the up and down conditions of the units. For the purpose of clarity, the 3-unit system shown in Figure 7-1 will be used as a vehicle to illustrate the general methodology.

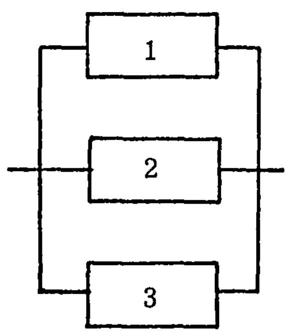

Figure 7-1 Configuration of a System Comprised of 3 Identical Units in Parallel.

The following operation and repair policies are assumed.

Operation Policy:

> All units will be turned on for as long as and as soon as they are operable.

Repair Policy:

> One repair crew services the failures on the first-come first-served basis.

At any given time instant, the units of the system may be in any one of the following four conditions.

(a) All 3 units are up

(b) One unit has failed and 2 units are up

(c) Two units have failed and 1 units is up

(d) All 3 units have failed.

If the system were Markovian, the above four system conditions would be the four states of the system. However, for the more general type of system under consideration, we need to identify the regeneration point or points within each condition. It can be seen that at the time instants when the system enters condition (a) or (b), its future behavior is stochastically independent of the past. Therefore, conditions (a) and (b) begin with regeneration points, and there are no other regeneration points within (a) or (b). Therefore, we define States 1 and 2 of the system as follows:

State 1: represents the state that at the time epoch the system enters this state, all the 3 units are up. (7.2-1)

State 2: represents the state that the time epoch the system enters this state, 1 unit has failed and repair is initiated on it. (7.2-2)

Now consider condition (c), this condition could be arrived from condition (b) due to failure of another unit before completion of repair on the failed unit. On the other hand, it could also be arrived at from a condition in which all the units have failed and repair on one of them has just been completed. Therefore, there are two regeneration points for condition (c). Hence, two system states are defined corresponding to this condition. Let states 3 and 4 be defined as follows:

State 3: represents the state that at the time epoch the system enters this state, 2 units have failed and one of the failed units has been in repair for some time. (7.2-3)

State 4: represents the state that at the time epoch the system enters this state, 2 units have failed and none of the failed units have received any repair. (7.2-4)

Note that according to the repair policy, during the time the system is in State 3, repair will be continued on the unit which already has been in repair for some time. As the system enters State 4, repair is initiated on the unit which has failed first.

Similar to condition (c), condition (d) also has two regeneration points. They correspond to the situations of different partial repairs completed on one of the failed units. Let State 5 and 6 be defined as follows:

State 5: represents the state that at the time epoch the system enters this state, all 3 units have failed and repair on one of the units has been started prior to the failures of the other two. (7.2-5)

State 6: represents the state that at the time epoch the system enters this state, all 3 units have failed and repair on one of the units has been started prior to the failure of only one of the other two units. (7.2-6)

Therefore, the state space for this system is

$$S = \{1, 2, \ldots, 6\} \quad (7.2\text{-}7)$$

From the state definitions, we obtain the state transition diagram of the system as shown in Figure 7-2.

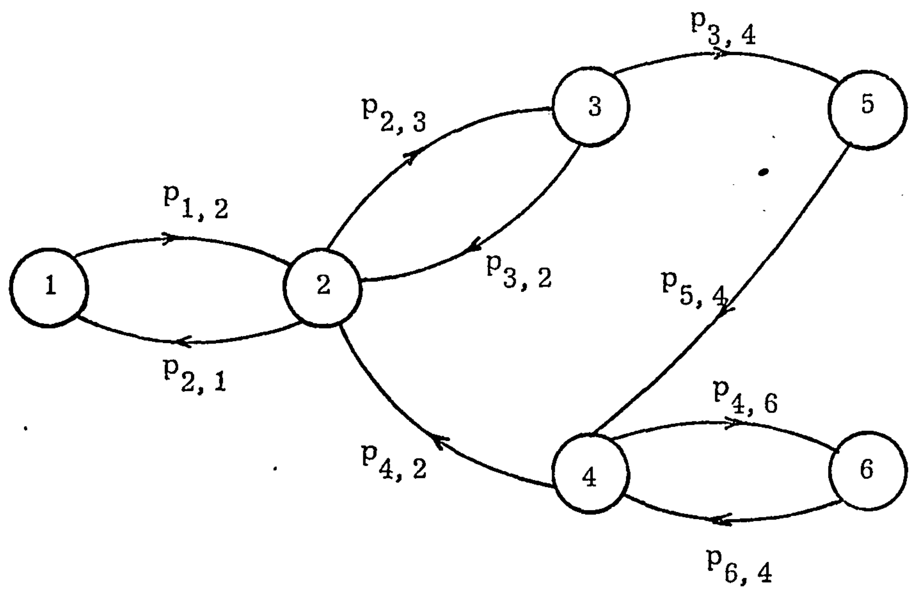

Figure 7-2 State Transition Diagram of the 3-Unit System.

From the above example, we see that the general methodology in defining states for systems with general repair-time distributions consists of two steps. The first is to enumerate the various possible system conditions corresponding to the combinations of the up and down conditions of the units. The second is to define system states based on the regeneration points within each system condition. Since each state is defined based on a regeneration point, the future state transition probabilities are independent of the past. Therefore, the state transition process of the system obeys a Markov chain. Due to the general repair-time distributions, the conditional holding time in each state has arbitrary distribution. Hence the dynamical model of the state transition process is semi-Markovian.

We have proved the following theorem.

Theorem 7.1

The dynamical behavior of a system comprised of units with exponential failure-time and general repair-time distributions can be modeled as a semi-Markov process.

We will call systems which can be modeled as semi-Markov processes as semi-Markovian systems. The next two sections consider the computation of the state transition probabilities, $p_{i,j}$, and the conditional holding time distribution functions, $F_{i,j}(t)$.

7.3 COMPUTATION OF STATE TRANSITION PROBABILITIES

This section considers the computation of state transition probabilities for the imbedded Markov chain of a semi-Markovian system. Since the failure-time and repair-time of the units in each state are independent random variables, computation of $p_{i,j}$ involves the computation of conditional probabilities of independent random variables. First consider the following general problem.

Let $x_1, x_2, \ldots, x_n$ be n independent random variables on the interval $[0, \infty]$ with probability density functions (p. d. f.) $f_{x_1}(t)$, $f_{x_2}(t), \ldots, f_{x_n}(t)$, respectively. Denote the distribution function for x_i by $F_{x_i}(t)$. That is

$$F_{x_i}(t) = \int_o^t f_{x_i}(\alpha)\, d\alpha \tag{7.3-1}$$

Let a random variable y be defined as

$$y = \min(x_1, x_2, \ldots, x_n) \tag{7.3-2}$$

Then, it follows that

$$\Pr\{y = x_i\} = \Pr\{x_i < x_j \text{ for all } j \neq i\}$$

$$= \int_0^\infty \Pr\{\alpha \leq x_i < \alpha + d\alpha \text{ and } x_j > \alpha \text{ for all } j \neq i\} \tag{7.3-3}$$

Since the random variables are independent, we obtain

$$\Pr\{y = x_i\} = \int_0^\infty f_{x_i}(\alpha) \prod_{\substack{j=1 \\ j \neq i}}^{n} [1 - F_{x_j}(\alpha)]\, d\alpha \tag{7.3-4}$$

This is the general equation for computing the state transition probabilities. Before illustrating the use of Eq. (7.3-4), an expression will be developed for the computation of remaining repair-time distributions. Now consider the following general problem.

Let a random variable η be defined as

$$\eta = \min(x_2, x_3, \ldots, x_n) \tag{7.3-5}$$

Next denote the difference between x_1 and η by ξ. That is,

$$\xi = x_1 - \eta \tag{7.3-6}$$

Since x_1 and η are random variables, ξ is a random variable. The conditional p. d. f. of ξ under the condition $x_1 > \eta$ will now be developed.

$$\Pr\{\xi \leq t | x_1 > \eta\}$$

$$= \iint_{(\alpha,\beta):\alpha-\beta\leq t} f_{x_1,\eta|x_1>\eta}(\alpha,\beta|\alpha>\beta)\,d\alpha\,d\beta$$

$$= \int_0^\infty \left[\int_0^{t+\beta} f_{x_1,\eta|x_1>\eta}(\alpha,\beta|\alpha>\beta)\,d\alpha\right] d\beta$$

or

$$\int_0^\infty \left[\int_{\alpha-t}^\infty f_{x_1,\eta|x_1>\eta}(\alpha,\beta|\alpha>\beta)\,d\beta\right] d\alpha \tag{7.3-7}$$

The conditional p. d. f. of ξ under the condition $x_1 > \eta$ is obtained by differentiating Eq. (7.3-8) with respect to t.

$$f_{\xi|x_1>\eta}(t) = \int_0^\infty f_{x_1,\eta|x_1>\eta}(t+\beta, \beta | t>0)\, d\beta$$

$$\text{or} \quad \int_0^\infty f_{x_1,\eta|x_1>\eta}(\alpha, \alpha-t | t>0)\, d\alpha$$

$$= \frac{\int_0^\infty f_{x_1,\eta}(t+\beta, \beta)\, d\beta}{\Pr\{x_1>\eta\}}$$

$$\text{or} \quad \frac{\int_0^\infty f_{x_1,\eta}(\alpha, \alpha-t)\, d\alpha}{\Pr\{x_1>\eta\}}$$

(7.3-8)

Since the random variables x_i's are defined only on the positive real line, $f_{x_1,\eta}(\alpha, \alpha-t)$ is zero for $\alpha < t$. Therefore, the lower limit of integration in the alternative expression on the RHS of Eq. (7.3-8) can be set to t. By the independent property of the random variables, Eq. (7.3-8) becomes

$$f_{\xi|x_1>\eta}(t) = \frac{\int_0^\infty f_{x_1}(t+\beta)\, f_\eta(\beta)\, d\beta}{\Pr\{x_1>\eta\}} = \frac{\int_t^\infty f_{x_1}(\alpha)\, f_\eta(\alpha-t)\, d\alpha}{\Pr\{x_1>\eta\}} \qquad (7.3\text{-}9)$$

We will find an expression for $\Pr\{x_1 > \eta\}$.

$$\Pr\{x_1 > \eta\} = \int_0^\infty \Pr\{\alpha \le x_1 < \alpha + d\alpha \text{ and } \eta \le \alpha\}$$

$$\text{or} \quad \int_0^\infty \Pr\{x_1 < \alpha \text{ and } \alpha \le \eta < \alpha + d\alpha\}$$

$$= \int_0^\infty f_{x_1}(\alpha)\, F_\eta(\alpha)\, d\alpha$$

$$\text{or} \quad \int_0^\infty [1 - F_{x_1}(\alpha)]\, f_\eta(\alpha)\, d\alpha \tag{7.3-10}$$

But,

$$F_\eta(\alpha) = \Pr\{\min(x_2, x_3, \ldots, x_n) \le \alpha\}$$

$$= 1 - \Pr\{\min(x_2, x_3, \ldots, x_n) > \alpha\}$$

$$= 1 - \prod_{i=2}^{n} \Pr(x_i > \alpha)$$

$$= 1 - \prod_{i=2}^{n} [1 - F_{x_i}(\alpha)] \tag{7.3-11}$$

Differentiating Eq. (7.3-11) with respect to α gives

$$f_\eta(\alpha) = \sum_{j=2}^{n} f_{x_j}(\alpha) \prod_{\substack{i=2 \\ i\neq j}}^{n} [1 - F_{x_i}(\alpha)] \tag{7.3-12}$$

Substitute Eqs. (7.3-11) and (7.3-12) into Eq. (7.3-10) we obtain

$$\Pr\{x_1>\eta\} = \int_0^\infty f_{x_1}(\alpha) \{1 - \prod_{i=2}^{n} [1 - F_{x_i}(\alpha)]\} d\alpha$$

$$\text{or } \int_0^\infty [1 - F_{x_1}(\alpha)] \{ \sum_{j=2}^{n} f_{x_j}(\alpha) \prod_{\substack{i=2 \\ i\neq j}}^{n} [1 - F_{x_i}(\alpha)]\} d\alpha \tag{7.3-13}$$

We have proved the following lemma.

<u>Lemma 7.1</u>

If $x_1, x_2, \ldots, x_n$ are independent random variables which are such that the probability density and distribution functions of x_i are denoted by $f_{x_i}(\cdot)$, respectively. If $\eta = \min(x_2, x_3, \ldots, x_n)$ and $\xi = x_1 - \eta$, then the conditional p.d.f. of ξ under the condition $x_1 > \eta$ is given by Eq. (7.3-9), in which the expressions for $f_\eta(\cdot)$ and $\Pr\{x_1>\eta\}$ are given by Eqs. (7.3-12) and (7.3-13), respectively.

Now consider the case when the p. d. f. of $x_2, x_3, \ldots, x_n$ are exponentially distributed as follows.

$$f_{x_i}(t) = \lambda_i e^{-\lambda_i t} \quad \text{for } i = 2, 3, \ldots, n \tag{7.3-14}$$

Then, by Eq. (7.3-12)

$$f_\eta(\alpha) = \sum_{j=2}^{n} \lambda_j e^{-\lambda_j \alpha} \prod_{\substack{i=2 \\ i \neq j}}^{n} e^{-\lambda_i \alpha}$$

$$= \lambda_s e^{-\lambda_s \alpha} \tag{7.3-15}$$

where

$$\lambda_s = \sum_{i=2}^{n} \lambda_i \tag{7.3-16}$$

By Eq. (7.3-13)

$$\Pr\{x_1 > \eta\} = \int_0^\infty f_{x_1}(\alpha) \{1 - \prod_{i=2}^{n} e^{-\lambda_i \alpha}\} d\alpha$$

$$= 1 - \int_0^\infty f_{x_1}(\alpha) e^{-\lambda_s \alpha} d\alpha \tag{7.3-17}$$

Substitute Eqs. (7.3-15) and (7.3-17) into Eq. (7.3-9) we obtain

$$f_{\xi | x_1 > \eta}(t) = \frac{\lambda_s \int_0^{\infty} f_{x_1}(t+\beta)\, e^{-\lambda_s \beta} d\beta}{1 - \int_0^{\infty} f_{x_1}(\alpha)\, e^{-\lambda_s \alpha} d\alpha} = \frac{\lambda_s e^{\lambda_s t} \int_t^{\infty} f_{x_1}(\beta) e^{-\lambda_s \beta} d\beta}{1 - \int_0^{\infty} f_{x_1}(\alpha) e^{-\lambda_s \alpha} d\alpha} \quad (7.3\text{-}18)$$

We have proved the following theorem.

Theorem 7.2

If $x_1, x_2, \ldots, x_n$ are independent random variables which are such that the p. d. f. of x_1 is $f_{x_1}(\cdot)$, and $\lambda_i e^{-\lambda_i t}$ is the p.d. f. of x_i for $i = 2, 3, \ldots, n$. If $\eta = \min(x_2, x_3, \ldots, x_n)$ and $\xi = x_1 - \eta$, then the conditional p. d. f. of ξ under the condition $x_1 > \eta$ is given by Eq. (7.3-18), in which λ_s is defined by Eq. (7.3-16).

To illustrate the use of Eq. (7.3-4) for computing the state transition probabilities of Figure 7-2, we assume that the failure-time p. d. f. of each unit in Figure 7-1 is

$$f(t) = \lambda e^{-\lambda t} \quad (7.3\text{-}19)$$

The repair-time p. d. f. is assumed to be the second Erlang distribution with parameter μ as follows:

$$g(t) = 4\mu^2 t\, e^{-2\mu t} \qquad (7.3\text{-}20)$$

By inspection of Figure 7-2, it is obvious that

$$\left.\begin{array}{l} p_{1,2} \\ p_{5,4} \\ p_{6,4} \end{array}\right\} = 1 \qquad (7.3\text{-}21)$$

Now consider State 2. If repair on the failed unit is completed before another unit fails, the system goes to State 1, otherwise the system goes to State 3. Therefore by Eq. (7.3-4),

$$p_{2,1} = \int_t^\infty 4\mu^2 t\, e^{-2\mu t}\, e^{-2\lambda t} dt$$

$$= \frac{\mu^2}{(\lambda+\mu)^2} \qquad (7.3\text{-}22)$$

and

$$p_{2,3} = 1 - p_{2,1} = \frac{\lambda(\lambda+2\mu)}{(\lambda+\mu)^2} \qquad (7.3\text{-}23)$$

Next consider State 3. By Theorem 7.2, the p.d.f. of the remaining repair-time on the unit under repair is

$$g_r(t) = \frac{2\lambda e^{2\lambda t} \int_t^\infty 4\mu^2 \alpha \, e^{-2(\lambda+\mu)\alpha} d\alpha}{1 - \int_0^\infty 4\mu^2 \alpha \, e^{-2(\lambda+\mu)\alpha} d\alpha}$$

$$= \frac{2\mu^2[2(\lambda+\mu)t+1]e^{-2\mu t}}{\lambda+2\mu} \qquad (7.3\text{-}24)$$

Therefore, by Eq. (7.3-4)

$$p_{3,2} = \int_0^\infty g_r(t) \, e^{-\lambda t} \, dt \qquad (7.3\text{-}25)$$

Substuting Eq. (7.3-24) into Eq. (7.3-25), we obtain after simplification

$$p_{3,2} = \frac{2\mu^2(3\lambda+4\mu)}{(\lambda+2\mu)^3} \qquad (7.3\text{-}26)$$

and

$$p_{3,4} = 1 - p_{3,2} = \frac{\lambda(\lambda^2+6\lambda\mu+6\mu^2)}{(\lambda+2\mu)^3} \qquad (7.3\text{-}27)$$

Next consider State 4. The p.d.f. for the repair-time in this state is g(t). It follows that

$$p_{4,2} = \int_0^\infty g(t) \, e^{-\lambda t} dt$$

$$= \frac{4\mu^2}{(2\mu+\lambda)^2} \qquad (7.3\text{-}28)$$

and

$$p_{4,6} = 1 - p_{4,2} = \frac{\lambda(\lambda + 4\mu)}{(2\mu + \lambda)^2} \qquad (7.3\text{-}29)$$

Thus all state transition probabilities are found by using Eq. (7.3-4) and Theorem 7.2.

7.4 CONDITIONAL HOLDING TIME DISTRIBUTION FUNCTIONS

This section considers the computation of conditional holding time distribution functions for semi-Markovian systems. We will start by treating the problem from a general point of view in the context of multiple independent random variables. Recall the random variables $x_1, x_2, \ldots, x_n$ and y defined in Section 7.3. Define an index set J as follows:

$$J = \{1, 2, \ldots, n\} \qquad (7.4\text{-}1)$$

Then,

$$\Pr\{y \leq t \mid x_i < x_j \text{ for all } j \in J - \{i\}\}$$

$$= \frac{\Pr\{x_i \leq t,\ x_i < x_j \text{ for all } j \in J - \{i\}\}}{\Pr\{x_i < x_j \text{ for all } j \in J - \{i\}\}}$$

$$= \frac{\int_0^t \Pr\{\alpha \le x_i < \alpha + d\alpha, x_j > \alpha \text{ for all } j \in J - \{i\}\}}{\int_0^\infty \Pr\{\alpha \le x_i < \alpha + d\alpha, x_j > \alpha \text{ for all } j \in J - \{i\}\}}$$

$$= \frac{\int_0^t f_{x_i}(\alpha) \prod_{j \in J - \{i\}} [1 - F_{x_j}(\alpha)]\, d\alpha}{\int_0^\infty f_{x_i}(\alpha) \prod_{j \in J - \{i\}} [1 - F_{x_j}(\alpha)]\, d\alpha} \tag{7.4-2}$$

This is the general equation for computing the conditional holding time distribution functions for semi-Markovian systems.

We have proved the following lemma.

Lemma 7.2

If $x_1, x_2, \ldots, x_n$ are independent random variables which are such that the probability density and distribution functions of x_i are denoted by $f_{x_i}(\cdot)$ and $F_{x_i}(\cdot)$, respectively. If $y = \min(x_1, x_2, \ldots, x_n)$, then the conditional distribution function of y under the condition that $x_i < x_j$ for all $j \in J - \{i\}$ is given by Eq. (7.4-2).

We now compute the $F_{i,j}(t)$ for the 3- unit system discussed in the preceeding section.

First consider State 1 in Figure 7-2. It is obvious that

$$F_{1,2}(t) = \int_0^t 3\lambda e^{-3\lambda\alpha} d\alpha$$

$$= 1 - e^{-3\lambda t} \qquad (7.4\text{-}3)$$

Next consider State 2. The system will go to State 1 if repair is completed before another unit fails. Therefore, applying Eq. (7.4-2),

$$F_{2,1}(t) = \frac{\int_0^t g(\alpha)\, e^{-2\lambda\alpha} d\alpha}{\int_0^\infty g(\alpha)\, e^{-2\lambda\alpha} d\alpha} \qquad (7.4\text{-}4)$$

Substituting Eq. (7.3-20) into the above, and after simplification we obtain

$$F_{2,1}(t) = 1 - [1 + 2(\lambda+\mu)t]e^{-2(\lambda+\mu)t} \qquad (7.4\text{-}5)$$

By similar reasoning,

$$F_{2,3}(t) = \frac{\int_0^t 2\lambda e^{-2\lambda\alpha}[1 - \int_0^\alpha g(\beta)\, d\beta] d\alpha}{\int_0^\infty 2\lambda e^{-2\lambda\alpha}[1 - \int_0^\alpha g(\beta)\, d\beta] d\alpha} \qquad (7.4\text{-}6)$$

Substituting Eq. (7.3-20) into the RHS of Eq. (7.4-6) we obtain after carrying out the integrations,

$$\text{Numerator} = 2\lambda\{\frac{\lambda+2\mu}{2(\lambda+\mu)^2} - e^{-2(\lambda+\mu)t}[\frac{\mu t}{\lambda+\mu} + \frac{\lambda+2\mu}{2(\lambda+\mu)^2}]\} \tag{7.4-7}$$

and

$$\text{Denominator} = \frac{\lambda(\lambda+2\mu)}{(\lambda+\mu)^2} \tag{7.4-8}$$

Therefore,

$$F_{2,3}(t) = 1 - [1+\frac{2\mu(\lambda+\mu)t}{(\lambda+2\mu)}]\, e^{-2(\lambda+\mu)t} \tag{7.4-9}$$

Following a similar approach, the remaining $F_{i,j}(t)$ are obtained.

$$F_{3,2}(t) = \frac{\int_0^t g_r(\alpha)\, e^{-\lambda\alpha}\, d\alpha}{\int_0^\infty g_r(\alpha)\, e^{-\lambda\alpha}\, d\alpha} \tag{7.4-10}$$

Substituting $g_r(\alpha)$ from Eq. (7.3-24) into the above, we obtain after simplification

$$F_{3,2}(t) = 1 - [1 + \frac{2(\lambda+\mu)(\lambda+2\mu)}{(3\lambda+4\mu)}\, t]\, e^{-(\lambda+2\mu)t} \tag{7.4-11}$$

Similarly,

$$F_{3,5}(t) = \frac{\int_0^t \lambda e^{-\lambda\alpha}[1 - \int_0^\alpha g_r(\beta)\, d\beta]\, d\alpha}{\int_0^\infty \lambda e^{-\lambda\alpha}[1 - \int_0^\alpha g_r(\beta)\, d\beta]\, d\alpha}$$

$$= 1 - [1 + \frac{2\mu(\lambda+\mu)(\lambda+2\mu)}{\lambda^2+6\lambda\mu+6\mu^2} t]\, e^{-(\lambda+2\mu)t} \qquad (7.4\text{-}12)$$

$$F_{4,2}(t) = \frac{\int_0^t g(\alpha)\, e^{-\lambda\alpha} d\alpha}{\int_0^\infty g(\alpha)\, e^{-\lambda\alpha} d\alpha}$$

$$= 1 - [1 + (\lambda+2\mu)t]e^{-(\lambda+2\mu)t} \qquad (7.4\text{-}13)$$

$$F_{4,6}(t) = \frac{\int_0^t \lambda e^{-\lambda\alpha}[1 - \int_0^\alpha g(\beta)\, d\beta]\, d\alpha}{\int_0^\infty \lambda e^{-\lambda\alpha}[1 - \int_0^\alpha g(\beta)\, d\beta]\, d\alpha}$$

$$= 1 - [1 + \frac{2\mu(\lambda+2\mu)}{\lambda+4\mu} t]\, e^{-(\lambda+2\mu)t} \qquad (7.4\text{-}14)$$

By inspection of Figure 7.2, we see that $F_{5,4}(t)$ and $F_{6,4}(t)$ are the remaining repair-time distribution functions due to uncompleted repairs in States 3 and 4, respectively. Therefore, by Theorem 7.2,

$$f_{5,4}(t) = \frac{\lambda e^{\lambda t} \int_t^{\infty} g_r(\alpha) e^{-\lambda\alpha} d\alpha}{1 - \int_0^{\infty} g_r(\alpha) e^{-\lambda\alpha} d\alpha}$$

$$= \frac{2\mu^2(3\lambda+4\lambda)+4\mu^2(\lambda+\mu)(\lambda+3\mu)t}{\lambda^2+6\lambda\mu+6\mu^2} e^{-2\mu t} \qquad (7.4\text{-}15)$$

Integrating from 0 to t gives

$$F_{5,4}(t) = \int_0^t f_{5,4}(\alpha)\, d\alpha$$

$$= 1 - [1 + \frac{2\mu(\lambda+\mu)(\lambda+2\mu)}{\lambda^2+6\lambda\mu+6\mu^2} t]e^{-2\mu t} \qquad (7.4\text{-}16)$$

Similarly,

$$f_{6,4}(t) = \frac{\lambda e^{\lambda t} \int_t^{\infty} g(\alpha)\, e^{-\lambda\alpha}\, d\alpha}{1 - \int_0^{\infty} g(\alpha)\, e^{-\lambda\alpha} d\alpha}$$

$$= \frac{4\mu^2}{\lambda+4\mu}[1 + (\lambda+2\mu)\, t]\, e^{-2\mu t} \qquad (7.4\text{-}17)$$

Integrating from 0 to t gives

$$F_{6,4}(t) = 1 - [1 + \frac{2\mu(\lambda+2\mu)}{\lambda+4\mu} t]\, e^{-2\mu t} \qquad (7.4\text{-}18)$$

Thus all the conditional holding time distribution functions are found.

Chapter 8

ANALYSIS OF SEMI-MARKOVIAN SYSTEMS

8.1 INTRODUCTION

This chapter considers the analysis of two important measures of effectiveness for semi-Markovian systems. One is the steady state availability which pertain to the limiting probabilities of the system, and the other is the mean first passage time of the system to a system down-state. In Section 8.2, the system of equations which governs the dynamical behavior of the state probabilities of a general semi-Markovian system is derived. Expressions for the steady state availability and the mean first passage time of the system to failure are developed in Sections 8.3 and 8.4, respectively.

8.2 SYSTEM OF EQUATIONS FOR THE STATE PROBABILITIES

In this section, the system of equations which govern the dynamical behavior of the state probabilities of a semi-Markovian system will be derived. The conditional probability that the system is in state j at time $t > 0$ given that it was in state i initially will be denoted by

$$P_{i,j}(t) = \Pr\{z(t) = j \mid z(0) = i\} \qquad (8.2\text{-}1)$$

First consider the case when $i = j$. The system, initially in state i, can be in the same state i at time t due to either one of the following events:

1. The system has never left state i during the entire interval [0, t].

2. The system left state i at least once during the interval (0, t), and returns back to state i by time t.

Since these two events are mutually exclusive, the sum of their probabilities gives the probability that the system is in state i at time t. Therefore,

$$P_{i,i}(t) = [1 - H_i(t)] + \sum_{k \in S} \int_0^t [p_{i,k} \, f_{i,k}(\tau) \, P_{k,i}(t-\tau) \, d\tau] \qquad (8.2\text{-}2)$$

The first bracketed term on the RHS is the probability that the system has never left state i during the interval [0, t]. The bracketed expression of the remaining term represents the probability of the sequence of events where the system left state i for state k at some time τ, $0 < \tau < t$, and then returns to state i in the remaining time interval $(\tau, t]$. This probability is summed over all possible states k and integrated over all time τ between 0 and t.

For the case when $i \neq j$, it is obvious that

$$p_{i,j}(t) = \sum_{k \in S} p_{i,k} \int_0^t f_{i,k}(\tau) \, P_{k,j}(t-\tau) \, d\tau \qquad (8.2\text{-}3)$$

For simplicity of notation, we denote the convolution integral in the preceeding equation as follows.

$$P_{k,j}(t) \otimes f_{i,k}(t) = \int_0^t P_{k,j}(t-\tau)\, f_{i,k}(\tau)\, d\tau \tag{8.2-4}$$

By a change of variable it is simple to show that

$$P_{k,j}(t) \otimes f_{i,k}(t) = f_{i,k}(t) \otimes P_{k,j}(t) \tag{8.2-5}$$

Equations (8.2-2) and (8.2-3) may be written in the combined form

$$P_{i,j}(t) = [1 - H_i(t)]\delta_{i,j} + \sum_{k \in S} p_{i,k}[f_{i,k}(t) \otimes P_{k,j}(t)] \text{ for all } i, j \in S \tag{8.2-6}$$

where $\delta_{i,j}$ is the Kronecker delta which is defined such that

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{8.2-7}$$

We have proved the following theorem.

Theorem 8.1

The dynamical behavior of the state probabilities of a semi-Markovian system is governed by a system of integral equations given by (8.2-6).

8.3 SYSTEM STEADY STATE AVAILABILITY

To determine the steady state availability A of an ergodic semi-Markovian, we need to solve for the limiting values of $P_{i,j}(t)$ from the system of integral equations (8.2-6). Let us denote the matrix with elements $P_{i,j}(t)$ by $\Phi(t)$.

$$\Phi(t) = (P_{i,j}(t)) \tag{8.3-1}$$

Since $i, j \epsilon S$, and S has order n_o, $\Phi(t)$ is an $(n_o \times n_o)$ matrix.

Let $R_i(t)$ denote complementary distribution functions of the unconditional holding time in state i. That is

$$R_i(t) = 1 - H_i(t) \tag{8.3-2}$$

Substituting Eq. (8.3-2) into Eq. (8.2-6) and then taking the Laplace transform, we have

$$P^*_{i,j}(s) = R^*_i(s) + \sum_{k \in S} p_{i,k} \, f^*_{i,k}(s) \, P^*_{k,j}(s) \quad \text{for all } i, j \in S \tag{8.3-3}$$

To write this system of equations in matrix form, define the "box" operation of two matrices to be element by element multiplication as follows:

$$\underset{(n_1 x n_2)}{B} \;\square\; \underset{(n_1 x n_2)}{C} = \underset{(n_1 x n_2)}{(b_{i,j} \, c_{i,j})} \tag{8.3-4}$$

Then, the system of equations in (8.3-3) can be written as

$$\Phi^*(s) = D^*_r(s) + [P \,\square\, \underline{\underline{f}}^*(s)]\Phi^*(s) \tag{8.3-5}$$

where $D^*_r(s)$ and $\underline{\underline{f}}^*(s)$ are defined by

$$D^*_r(s) = \begin{bmatrix} R^*_1(s) & & & 0 \\ & R^*_2(s) & & \\ & & \ddots & \\ 0 & & & R^*_{n_1}(s) \end{bmatrix} \tag{8.3-6}$$

and

$$\underline{\underline{f}}^*(s) = (f^*_{i,j}(s)) \tag{8.3-7}$$

From Eq. (8.3-5), we obtain

$$\Phi^*(s) = [I_{n_o} - P\Box \underline{\underline{f}}^*(s)]^{-1} D_r^*(s) \tag{8.3-8}$$

By final value theorem of Laplace transform,

$$\lim_{t\to\infty} \Phi(t) = \lim_{s\to 0} s\Phi^*(s) \tag{8.3-9}$$

We will denote this limiting matrix by Φ. Now from Eq. (8.3-5) we have

$$\Phi = \{\lim_{s\to 0} s[I_{n_o} - P\Box \underline{\underline{f}}^*(s)]^{-1}\}\{\lim_{s\to 0} D_r^*(s)\} \tag{8.3-10}$$

Note that for all $i \in S$

$$\lim_{s\to 0} R_i^*(s) = \int_0^\infty R_i(t)\, dt \tag{8.3-11}$$

By Theorem 3.1, the RHS of the above equation is the mean unconditional holding time in state i, we will denote this mean by $\bar{w}_i$. It then follows that

$$\lim_{s\to 0} D_r^*(s) = D_{\bar{w}} \tag{8.3-12}$$

where

$$D_{\overline{w}} = \begin{bmatrix} \overline{w}_1 & & & 0 \\ & \overline{w}_2 & & \\ & & \ddots & \\ 0 & & & \overline{w}_{n_o} \end{bmatrix} \qquad (8.3\text{-}13)$$

Now write Eq. (8.3-10) as

$$\Phi = (\lim_{s\to 0} T(s))\, D_{\overline{w}} \qquad (8.3\text{-}14)$$

where

$$T(s) = s[I_{n_o} - P\square \underline{\underline{f}}^*(s)]^{-1} \qquad (8.3\text{-}15)$$

This gives

$$T(s)[I_{n_o} - P\square \underline{\underline{f}}^*(s)] = s \qquad (8.3\text{-}16)$$

Note that for all $i, j \epsilon S$

$$\lim_{s\to 0} f^*_{i,j}(s) = \int_0^\infty f_{i,j}(t)\, dt = 1 \qquad (8.3\text{-}17)$$

Therefore, by taking the limit $s \to 0$ in Eq. (8.3-16) we obtain

$$T = TP \qquad (8.3\text{-}18)$$

in which

$$T = \lim_{s \to 0} T(s) \qquad (8.3\text{-}19)$$

Since the system is ergodic, the steady state probability vector $\underline{\pi}$ of the imbedded Markov chain is the unique solution of the following set of equations.

$$\underline{\pi} = \underline{\pi} P$$

and (8.3-20)

$$\sum_{i \in s} \pi_i = 1$$

Therefore, each row of T must be propoertional to $\underline{\pi}$. Let $\underline{t}_i$ denote the ith row of T. Then for all $i \in S$,

$$\underline{t}_i = k_i \underline{\pi} \qquad (8.3\text{-}21)$$

where k_i is some proportionality constant. Therefore,

$$T = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n_o} \end{bmatrix} \underline{\pi} \qquad (8.3\text{-}22)$$

Substituting Eq. (8.3-22) into Eq. (8.3-14) we have

$$(\phi_{i,j}) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n_o} \end{bmatrix} \underline{\pi}\, D_{\overline{w}} \tag{8.3-23}$$

To determine the constants k_i, we impose the necessary condition

$$\sum_{j\in S} \phi_{i,j} = 1 \quad \text{for all } i \in S \tag{8.3-24}$$

Therefore, from Eq. (8.3-23), for all $i \in S$

$$k_i = \frac{1}{\sum_{j\in S} \pi_j \overline{w}_j} \tag{8.3-25}$$

Note that k_i is not a function of i. It follows from Eq. (8.3-23) that

$$\phi_{i,j} = \frac{\pi_j \overline{w}_j}{\sum_{j\in S} \pi_j \overline{w}_j} \quad \text{for all } i \in S \tag{8.3-26}$$

Observe that the RHS is independent of i. We will therefore denote $\phi_{i,j}$ by ϕ_j. Hence the steady state availability of the system is

$$A = \frac{\sum_{j \in S_u} \pi_j \bar{w}_j}{\sum_{i \in S} \pi_i \bar{w}_i} \tag{8.3-27}$$

We have proved the following theorem.

Theorem 8.2

For an ergodic semi-Markovian system, the limiting state probabilities of the system are independent of the initial condition of the system, and depend only on the limiting probabilities of the imbedded Markov chain and the mean unconditional holding times of the states. For all $i \in S$, if π_i is the limiting state probability of the imbedded Markov chain for state i, and if $\bar{w}_i$ is the mean unconditional holding time in state i, then the steady state availability of the system is given by Eq. (8.3-27).

8.4 MEAN TIME-TO-FIRST-SYSTEM-FAILURE

Following our previous notations, we will use S_u and S_d to denote the sets of up-states and down-states, respectively, for a semi-Markovian system. Without loss of generality the elements of S_u and S_d are assumed to be the same as previously defined.

$$S_u = \{1, 2, \ldots, k_o\} \tag{8.4-1}$$

$$S_d = \{k_o+1, k_o+2, \ldots, n_o\} \tag{8.4-2}$$

Recall from Eq. (7.1-4) that $w_{i,j}$ is the conditional holding time of the process in state i given that the process next visits state j. We denote the expected value of $w_{i,j}$ by $\bar{w}_{i,j}$

$$\bar{w}_{i,j} = \int_0^\infty t\, f_{i,j}(t)\, dt \tag{8.4-3}$$

Let $\overline{W}$ denote the matrix of $\bar{w}_{i,j}$.

$$\overline{W} = (\bar{w}_{i,j}) \tag{8.4-4}$$

Now partition the matrices P, $\underline{f}(t)$, and $\overline{W}$ into submatrices as follows.

$$P = \left[\begin{array}{c|c} P_{1,1} & P_{1,2} \\ \hline P_{2,1} & P_{2,2} \end{array}\right] \tag{8.4-5}$$

$$\underline{\underline{f}}(t) = \left[\begin{array}{c|c} \underline{\underline{f}}_{1,1}(t) & \underline{\underline{f}}_{1,2}(t) \\ \hline \underline{\underline{f}}_{2,1}(t) & \underline{\underline{f}}_{2,2}(t) \end{array}\right] \tag{8.4-6}$$

$$\overline{W} = \left[\begin{array}{c|c} \overline{W}_{1,1} & \overline{W}_{1,2} \\ \hline \overline{W}_{2,1} & \overline{W}_{2,2} \end{array}\right] \qquad (8.4\text{-}7)$$

In the above partitions, the dimension of the (1, 1) submatrices is $k_o x k_o$; the dimension of the (1, 2) submatrices is $k_o x(n_o - k_o)$; the dimension of the (2, 1) submatrices is $(n_o - k_o)xk_o$; and the dimension of the (2, 2) submatrices is $(n_o - k_o)x(n_o - k_o)$.

Now let $g_{i,d}(t)$ denote the p. d. f. of the first passage time from state $i \epsilon S_u$ to a system down state in S_d. By using similar reasonings as in Section 8. 2, the system of integral equations governing $g_{i,d}(t)$ is

$$g_{i,d}(t) = \sum_{j \epsilon S_d} p_{i,j} f_{i,j}(t) + \sum_{k \epsilon S_u} p_{i,k} g_{k,d}(t) \otimes f_{i,k}(t) \quad \text{for all } i \epsilon S_u \qquad (8.4\text{-}8)$$

The first summation on the RHS accounts for the first passage times that the system enters a down-state in one transition after starting state i. Whereas the second summation accounts for the first passage times that the system enters a down-state after two or more transitions after starting from state i. Taking the Laplace transform of Eq. (8. 4-8) we have

$$g^*_{i,d}(s) = \sum_{j\epsilon S_d} p_{i,j} f^*_{i,j}(s) + \sum_{k\epsilon S_u} p_{i,k} f^*_{i,k}(s)\, g^*_{k,d}(s) \quad \text{for all } i \epsilon S_u \tag{8.4-9}$$

To write the above equation in vector form, define a $(k_o \times 1)$ vector

$$\underset{\sim}{g}^*(s) = \begin{bmatrix} g^*_{1,d}(s) \\ g^*_{2,d}(s) \\ \vdots \\ g^*_{k_o,d}(s) \end{bmatrix} \tag{8.4-10}$$

Then the set of equations in (8.4-9) can be written as

$$\underset{\sim}{g}^*(s) = [P_{1,2} \square \underline{\underline{f}}^*_{1,2}(s)]\, \underline{v}^T(n_o - k_o, 0) + [P_{1,1} \square \underline{\underline{f}}^*_{1,1}(s)] \underset{\sim}{g}^*(s) \tag{8.4-11}$$

The solution for $\underset{\sim}{g}^*(s)$ from Eq. (8.4-11) represents the Laplace transform of the vector of p.d.f.'s of the first passage times to a down-state in S_d when the system starts from the states in S_u.

Let $\tau_{i,d}$ denote the random first passage time when the system starts from state i. We will use $\bar{\tau}_{i,d}$ to denote the expected values of $\tau_{i,d}$. Define the vector $\underset{\sim}{\bar{\tau}}$ as

$$\bar{\underset{\sim}{\tau}} = \begin{bmatrix} \bar{\tau}_{1,d} \\ \bar{\tau}_{2,d} \\ \vdots \\ \bar{\tau}_{k_o,d} \end{bmatrix} \qquad (8.4\text{-}12)$$

It then follows that

$$\bar{\underset{\sim}{\tau}} = -\lim_{s \to 0} \frac{d}{ds} \underset{\sim}{g}^*(s) \qquad (8.4\text{-}13)$$

Now, rewrite Eq. (8.4-11) as follows:

$$[I_{k_o} - P_{1,1} \square \underline{\underline{f}}^*_{1,1}(s)] \underset{\sim}{g}^*(s) = [P_{1,2} \square \underline{\underline{f}}^*_{1,2}(s)] \underline{v}^T(n_o - k_o, 0) \qquad (8.4\text{-}14)$$

Differentiating the above with respect to s gives

$$[I_{k_o} - P_{1,1} \square \underline{\underline{f}}^*_{1,1}(s)] \frac{d}{ds} \underset{\sim}{g}^*(s) - [P_{1,1} \square \frac{d}{ds} \underline{\underline{f}}^*_{1,1}(s)] \underset{\sim}{g}^*(s)$$
$$= [P_{1,2} \square \frac{d}{ds} \underline{\underline{f}}^*_{1,2}(s)] \underline{v}^T(n_o - k_o, 0)$$
$$(8.4\text{-}15)$$

Taking the limit as $s \to 0$, and making use of Eq. (5.3-17) we obtain

$$-[I_{k_o} - P_{1,1}] \bar{\underset{\sim}{\tau}} + [P_{1,1} \square \overline{W}_{1,1}] \underline{v}^T(k_o, 0) = -[P_{1,2} \square \overline{W}_{1,2}] \underline{v}^T(n_o - k_o, 0)$$
$$(8.4\text{-}16)$$

Since $P_{1,1}$ is such that $P^k_{1,1} \to 0$ as $k \to \infty$, by the matrix inversion Lemma 5.1, $[I_{k_o} - P_{1,1}]$ is non-singular. Therefore,

$$\underset{\sim}{\bar{\tau}} = [I_{k_o} - P_{1,1}]^{-1}[(P_{1,1} \square \overline{W}_{1,1})\underline{v}^T(k_o, 0) + (P_{1,2} \square \overline{W}_{1,2})\underline{v}^T(n_o - k_o, 0)] \qquad (8.4\text{-}17)$$

Observe that the expression in the second bracket on the RHS is a column vector of dimension $(k_o \times 1)$, and the ith element of this vector is equal to $\sum_{j \in S} p_{i,j} \bar{w}_{i,j}$, which is the mean unconditional holding time in state i. If we let $\underset{\sim}{\omega}$ denote the vector

$$\underset{\sim}{\omega} = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_{k_o} \end{bmatrix} \qquad (8.4\text{-}18)$$

then, Eq. (8.4-17) takes the simple form

$$\underset{\sim}{\bar{\tau}} = [I_{k_o} - P_{1,1}]^{-1} \underset{\sim}{\omega} \qquad (8.4\text{-}19)$$

Since this equation gives all the mean first passage times from the initial states in S_u to a system down-state in S_d, it is the fundamental equation required for computing the mean first passage time for any given initial state vector $\underline{P}(0)$. Therefore it follows that

$$\text{MTTFSF}_{\underline{P}(0)} = \underline{P}_u(0)[I_{k_o} - P_{1,1}]^{-1} \underset{\sim}{\omega}. \tag{8.4-20}$$

We have proved the following theorem.

Theorem 8.3

In a semi-Markovian system which consists of transient states and one recurrent chain, and the set of states S_d is such that it does not contain any transient state, then the mean first passage time of the system from any state $i \in S_u$ to a state in S_d depends only on the submatrix $P_{1,1}$ and the mean unconditional holding times of the states in S_u. If the initial probability state vector of the system is $\underline{P}(0)$, then the MTTFSF of the system is given by Eq. (8.4-20).

Chapter 9

CONCLUSIONS AND RECOMMENDATIONS

9.1 SUMMARY OF RESULTS

The processes of Markovian and semi-Markovian systems treated in this thesis can be considered as generalizations of discrete parameter Markov chains as well as birth and death processes. There are many physical systems whose state transition processes obey the Markovian and semi-Markovian models. For example: single or multi-channel waiting line and trunking problems, machine operation and servicing problems, marketing problems, inventory and production problems, electrical power supply problems, etc. In this thesis we have considered both the modeling as well as analysis aspects of systems which are Markovian and semi-Markovian.

The system of differential equations which governs the dynamical behavior of the state probabilities of a general Markovian system is derived, and solutions for the equations are given. The characterization of stationarity of a Markovian system in terms of the characterization of the p. d. f. 's of the transition times among the states are discussed. The existance of and solutions for the limiting state probabilities of ergodic systems are established.

The development of math models for a physical system consists of several basic steps. The first is to identify and define all possible states. The second is to determine the transition rates or the transition probabilities among the states for the imbedded Markov chain. In addition, for semi-Markovian systems, all conditional holding time probability density or distribution functions of the states must be determined. To enhance clarity, simple machine operation and maintenance problems are used as vehicles to illustrate the general methodologies for modeling of Markovian as well as semi-Markovian systems.

Many commonly used measures of system effectiveness for dynamic probabilistic systems are defined. These measures can be grouped into three categories:

1. Time dependent probabilistic measures
2. Steady state probabilistic measures
3. Statistical measures

A unified matrix approach is used to develop the general expressions for the various effectiveness measures for stationary Markovian systems. The results developed are particularly suitable for analyzing large scale complex systems by employing a digital computer.

Of the many statistical measures treated, system MUT and MDT in the steady state are among the very important ones. Therefore special emphasis has been given in the treatment of system MUT and MDT. Expressions for system MUT and MDT in terms of independent subsystem MUT's and MDT's are derived. These results are especially valuable from the computational standpoint when solving for large scale systems.

It is shown that systems comprised of units with exponential p. d. f. 's for their time-to-failure, and general p. d. f. 's for their time-to-repair are semi-Markovian systems. The dynamical behavior of the state transition process of a semi-Markovian system is governed by a system of integral equations involving convolution integrals. Since the convolution of two functions is transformed to multiplication after Laplace or Laplace-Stieltjes transformation, such transformation is an effective tool for solving the system of integral equations. Expressions for system effective measures which pertain to the limiting state probabilities and the first passage time statistics of semi-Markovian systems are developed.

9.2 SUGGESTIONS FOR FURTHER STUDY

The treatment of Markovian systems in this thesis can be considered to be quite complete. Any other effectiveness measures which are not treated here may either be inferred from the results already developed, or they may be derived by using similar procedures employed in here. In the case of semi-Markovian systems, however, the treatment has been less complete. In particular, expressions for the time dependent probabilistic effectiveness measures have not been developed. This is because general analytical solutions for the system of integral equations in (8.2-6) are by no means easy. The following suggestions for further study pertaining to semi-Markovian systems are recommended.

1. Instead of solving for the most general solutions for the system of equations in (8.2-6), it is suggested that the system of equations be solved for a particular class of conditional holding time p.d.f.'s, for example the families of Erlang or Weibull p.d.f.'s. Once the time dependent solutions are obtained, expressions for the time dependent system effectiveness measures can be developed.

2. In Section 8.4, the general expression for the mean first passage time to system failure is developed. Since system MUT in the steady state can be considered as the mean first passage time corresponding to a particular initial probability state vector, it would not be

difficult to develop expressions for system MUT and MDT for semi-Markovian systems. The development of these expressions are suggested.

3. As a sequel to the preceeding development, it is suggested that expressions for system MUT and MDT in terms of independent subsystem MUT's and MDT's for semi-Markovian systems be developed.

4. In machine operation and maintenance problems, under certain operation and maintenance policies, the state space of the system could become countably infinite when there are more than one repair crew servicing the failures. It is suggested that modeling and analysis of multiple repair crew systems be studied.

APPENDIX A

AN ALGEBRAIC THEORETIC CLASSIFICATION OF THE STATES OF A MARKOV CHAIN

Consider the following binary relation defoned on the set of states S of a Markov chain. An element $i \in S$ is said to bear a relation T to $j \in S$, denoted by $i T j$, if it is possible for the chain to reach state j from state i. The following properties of T on the elements of S are obvious:

$$\text{Reflexive:} \quad iTi \quad \text{for all } i \in S \tag{A-1}$$

$$\text{Transitive:} \quad iTj \text{ and } jTk \text{ imply } iTk \text{ for all } i, j, k \in S \tag{A-2}$$

Therefore, the relation T is reflexive and transitive. Now define a binary relation R on S based on T. An element $i \in S$ is said to bear a relation R to an alement of $j \in S$ if iTj and jTi. In terms of communicative properties among states of a Markov chain, two states will have the relation R if it is possibly to reach from either state to the other state. It is not hard to see that R is reflexive, transitive as well as symmetric which is defined as follows:

$$\text{Symmetric:} \quad iRj \text{ implies } jRi \quad \text{for all } i, j \in S \tag{A-3}$$

Therefore, R is an equivalence relation on S. For any $i \in S$, let R(i) denote the set of all elements j in S which are equivalent to i, that is $j \epsilon R(i)$ if and only if $i R j$. $R(i)$ is known as an R-subset or an equivalence class of S. It can be shown [74] that the R subsets have the following properties:

1. For all $i, j \epsilon S$, $i R j$ if and only if $R(i) = R(j)$
2. Two R-subsets are either identical or have no elements in common, and the collection of all R-subsets is a partition of S.

In terms of these properties the states of a Markov chain can be grouped or classified according to the R-subsets. The classification will be such that any two states of the chain will belong to the same R-subset if and only if it is possible to reach from one state to the other.

Now, we will consider a partial order relation induced by T. Define a relation T^* on the set of equivalence classes as follows. For any two R-subsets u and v, uT^*v holds if every element of u bears the relation T to every element of v. It can be seen that T^* is reflexive, transitive and antisymmetric which is defined as follows:

Antisymmetric: If uT^*v and vT^*u, then $u = v$ (A-4)

Therefore, T^* is a partial order relation. It is known as a partial order induced by T. The minimal elements of the partial ordering of equivalent classes are classified as the ergodic sets of the Markov chain. By the minimal condition of partially ordered sets, there exists at least one ergodic set for every Markov chain.

APPENDIX B

SOLUTION OF THE VECTOR DIFFERENTIAL EQUATION OF A LINEAR HOMOGENEOUS SYSTEM

In this appendix we shall derive a solution for the vector differential equation:

$$\underset{1xn}{\dot{\underline{x}}(t)} = \underset{1xn}{\underline{x}(t)}\ \underset{nxn}{A(t)} \tag{B-1}$$

where the elements of A(t) are continuous functions of time on the interval $t_o \leq t \leq T$. Suppose $\underline{\phi}_1(t, t_o), \ldots, \underline{\phi}_n(t, t_o)$ are solutions of Eq. (3-1) for the following boundary conditions, respectively.

$$\begin{aligned} \underline{x}_1(t_o) &= [1, 0, 0, \ldots, 0] \\ x_2(t_o) &= [0, 1, 0, \ldots, 0] \\ &\vdots \\ x_n(t_o) &= [0, 0, 0, \ldots, 0, 1] \end{aligned} \tag{B-2}$$

Let $\Phi(t, t_o)$ denote the square matrix form by $\underline{\phi}_1(t, t_o), \underline{\phi}_2(t, t_o), \ldots, \underline{\phi}_n(t, t_o)$ as follows:

$$\Phi(t, t_o) = \begin{bmatrix} \underline{\phi}_1(t, t_o) \\ \underline{\phi}_2(t, t_o) \\ \vdots \\ \underline{\phi}_n(t, t_o) \end{bmatrix} \tag{B-3}$$

This is known as the transition [75] or fundamental [76] matrix of the linear system governed by Eq. (B-1). Since $\underline{\phi}_1(t)$, for all i = 1, 2, . . . n, are the solutions of Eq. (B-1) for the boundary conditions of Eq. (B-2), it follows that the solution for any initial vector $\underline{x}(t_o)$ would be

$$\underline{x}(t) = \underline{x}(t_o)\,\Phi(t, t_o). \tag{B-4}$$

The matrix $\Phi(t, t_o)$ is given by the solution of the matrix differential equation

$$\frac{d}{dt}\,\Phi(t, t_o) = \Phi(t, t_o)\,A(t) \tag{B-5}$$

for the boundary condition

$$\Phi(t_o, t_o) = I_n\,. \tag{B-6}$$

The following theorem establishes a series solution for $\Phi(t, t_o)$.

<u>Theorem B. 1</u>

If A(t) is a square (n x n) matrix whose elements are continuous functions of time on the interval $t_o \leq t \leq T$, then the series*

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*This is often known as the Peano-Baker series.

. . . . . .

$$I + \int_{t_o}^{t} A(\sigma_1)\, d\sigma_1 + \int_{t_o}^{t} A(\sigma_1) \int_{t_o}^{\sigma_1} A(\sigma_2)\, d\sigma_2 d\sigma_1 + \dots$$

is a solution of the matrix differential Eq. (B-5) for boundary condition Eq. (B-6) on the same interval.

Proof:

For simplicity of notation, let the given series be denoted by

$$B_o(t) + B_1(t) + B_2(t) + \dots \tag{B-7}$$

First we show that the series converges uniformly on the interval $t_o \leq t \leq T$. Recall that a series is said to converge uniformly if the sequence of its partial sum converges uniformly. Let $S_k(t)$ denote the general term for the partial sum of the given series:

$$S_k(t) = \sum_{i=0}^{k} B_i(t)$$

$$= I + \int_{t_o}^{t} A(\sigma_1) d\sigma_1 + \dots + \int_{t_o}^{t} A(\sigma_1) \int_{t_o}^{\sigma_1} A(\sigma_2) \dots \int_{t_o}^{\sigma_{n-1}} A(\sigma_k) d\sigma_k d\sigma_{k-1} \dots d\sigma_1 \tag{B-8}$$

To show that the sequence of matrices $\{S_k(t)\}$ converges uniformly, we have to show the sequence of each element of $S_k(t)$ converging uniformly. Let

$E_{i,j}[\cdot]$ denote the (i, j)th element of matrix $[\cdot]$; "a" denotes the maximum absolute value of the elements of A(t) on the interval $t_o \leq t \leq T$.

That is,

$$a = \max_{\substack{(i, j) \\ t_o \leq t \leq T}} |a_{ij}(t)| \qquad \text{(B-9)}$$

Note that "a" exists since the elements of A(t) are given to be continuous functions on the interval $t_o \leq t \leq T$. The absolute values of the elements in the terms of the partial sum Eq. (B-8) are bounded as follows:

$$|E_{ij}[B_o]| \leq 1$$

$$|E_{ij}[B_1]| \leq \int_{t_o}^{T} a \, d\sigma_1 = a(T-t_o)$$

$$|E_{ij}[B_2]| \leq \int_{t_o}^{T} a \int_{t_o}^{\sigma_1} a \, d\sigma_1 d\sigma_2 = \frac{a^2(T-t_o)^2}{2!}$$

$$\vdots$$

$$|E_{ij}[B_k]| \leq \int_{t_o}^{T} a \int_{t_o}^{\sigma_1} a \ldots \int_{t_o}^{\sigma_{n-1}} a \, d\sigma_k d\sigma_{k-1} \ldots d\sigma_1 = \frac{a^k(T-t_o)^k}{k!} \qquad \text{(B-10)}$$

The above is true for any (i, j)th element of the matrices B_m. We now invoke the Weierstress M-test theorem [72] which states that if there exists a convergent series $\sum_{n=0}^{\infty} M_i$, with M_i independent of t, such that $|u_i(t)| \leq M_i$ for all t on the closed interval $a \leq t \leq b$, then the series

$\sum_{n=0}^{\infty} u_i(t)$ is absolutely and uniformly convergent on that interval. We observe that the series

$$1 + a(T-t_o) + \frac{a^2(T-t_o)^2}{2!} + \ldots + \frac{a^i(T-t_o)^i}{i!} + \ldots \qquad \text{(B-11)}$$

is nothing but the Maclaurin's expansion of $e^{a(T-t_o)}$. Therefore, the series (B-11) converges, and each term of the series is independent of t. Hence, this establishes the uniform and absolute convergence of the given matrix series on the interval $t_o \leq t \leq T$. We now compute the term-by-term derivatives of the series:

$$\frac{d}{dt} B_o(t) = 0$$

$$\frac{d}{dt} B_1(t) = A(t)$$

$$\frac{d}{dt} B_2(t) = A(t) \int_{t_o}^{t} A(\sigma_2)\, d\sigma_2 = A(t)\, B_1(t)$$

In general, for $k \geq 2$

$$\frac{d}{dt} B_k(t) = A(t) \int_{t_o}^{t} A(\sigma_2) \int_{t_o}^{\sigma_3} A(\sigma_3) \ldots \int_{t_o}^{\sigma_{n-1}} A(\sigma_k) d\sigma_k d\sigma_{k-1} \ldots d\sigma_2$$

$$= A(t)\, B_{k-1}(t) \qquad \text{(B-12)}$$

Observe that the elements of the derivatives $\frac{d}{dt} B_k(t)$, for $k = 1, 2, 3, \ldots$, are continuous functions on the interval $t_o \leq t \leq T$. In addition $\sum_{k=0}^{\infty} \frac{d}{dt} B_k(t)$ is uniformly convergent on the interval $t_o \leq t \leq T$. This can be seen from

$$\sum_{k=0}^{\infty} \frac{d}{dt} B_k(t) = A(t) \sum_{k=0}^{\infty} B_k(t) \tag{B-13}$$

Therefore, $\sum_{k=0}^{\infty} \frac{d}{dt} B_k(t)$ is uniformly and absolutely convergent. Since the series $\sum_{k=0}^{\infty} B_k(t)$ and $\sum_{k=0}^{\infty} \frac{d}{dt} B_k(t)$ are uniformly convergent on the interval $t_o \leq t \leq T$, we have:

$$\begin{aligned} \frac{d}{dt} \sum_{k=0}^{\infty} B_k(t) &= \sum_{k=0}^{\infty} \frac{d}{dt} B_k(t) \\ &= A(t) \sum_{k=0}^{\infty} B_k(t) \end{aligned} \tag{B-14}$$

Observe that

$$\sum_{k=0}^{\infty} B_k(t_o) = I_n \tag{B-15}$$

By uniqueness theorem on the solution of differential equations, we conclude that $\sum_{k=0}^{\infty} B_k(t)$ is the solution of the matrix differential Eq. (B-5) for the boundary condition Eq. (B-6) on the interval $t_o \leq t \leq T$.

Q. E. D.

APPENDIX C

LIMITING SOLUTION OF AN ERGODIC STATIONARY MARKOVIAN SYSTEM

In this Appendix we shall establish that for an ergodic Markovian system, the system of equations

$$\frac{d}{dt}\underline{P}(t|\underline{P}(0)) = \underline{P}(t)|\underline{P}(0))M \qquad \text{(C-1)}$$

possesses a limiting solution as $t \to \infty$. Furthermore, we shall show that this limiting solution is independent of the initial condition $\underline{P}(0)$.

Let S denote the set of system states of the system. For each $i \epsilon S$, let $\underline{\phi}_i(t, 0)$ be the solution of Eq. (C-1) when the given initial vector $\underline{P}(0)$ is such that the ith element of $\underline{P}(0)$ is 1 and all other elements are 0. In other words $\underline{\phi}_i(t, 0)$ is the probability state vector at time $t > 0$ given that the system is in state i at time 0. Denote the elements of $\underline{\phi}_i(t, 0)$ as follows:

$$\underline{\phi}_i(t, 0) = [\phi_{i,1}(t, 0), \phi_{i,2}(t, 0), \ldots, \phi_{i,n_o}(t, 0)] \qquad \text{(C-2)}$$

It follows that for all $i, j \epsilon S$

$$\phi_{i,j}(t, 0) \geq 0 \qquad \text{(C-3)}$$

and

$$\sum_{k \in S} \phi_{i,k}(t,0) = 1 \tag{C-4}$$

The system being ergodic, strict inequality holds in Eq. (C-3). That is

$$\phi_{i,j}(t,0) > 0 \quad \text{for all } i, j \in S \tag{C-5}$$

The system transition matrix $\Phi(t, 0)$ is

$$\Phi(t,0) = \begin{bmatrix} \phi_{1,1}(t,0) & \phi_{1,2}(t,0) & \cdots & \phi_{1,n_o}(t,0) \\ \phi_{2,1}(t,0) & \phi_{2,2}(t,0) & \cdots & \phi_{2,n_o}(t,0) \\ \phi_{n_o,1}(t,0) & \phi_{n_o,2}(t,0) & \cdots & \phi_{n_o,n_o}(t,0) \end{bmatrix} \tag{C-6}$$

It follows that the solution for Eq. (C-1) is

$$\underline{P}(t \mid \underline{P}(0)) = \underline{P}(0)\, \Phi(t,0) \tag{C-7}$$

But in Eq. (5.2-3) we have shown that

$$\Phi(t,0) = e^{Mt} \tag{C-8}$$

For all $\tau > 0$,

$$\Phi(t+\tau, 0) = e^{M(t+\tau)}$$

$$= e^{M\tau} e^{Mt}$$

$$= \Phi(\tau, 0)\, \Phi(t, 0) \qquad \text{(C-9)}$$

Now, for each $j \in S$, let

$$\alpha_j(t) = \max_{i \in S} \phi_{i,j}(t, 0) \qquad \text{(C-10)}$$

and

$$\beta_j(t) = \min_{i \in S} \phi_{i,j}(t, 0) \qquad \text{(C-11)}$$

This means $\alpha_j(t)$ and $\beta_j(t)$ are respectively the maximum and minimum elements of the jth column of $\Phi(t, 0)$. For all $i, j \in S$, the (i, j) element of $\Phi(t+\tau, 0)$ is

$$\phi_{i,j}(t+\tau, 0) = \sum_{k \in S} \phi_{i,k}(\tau, 0)\, \phi_{k,j}(t, 0)$$

$$\leq \sum_{k \in S} \phi_{i,k}(\tau, 0)\, \alpha_j(t) = \alpha_j(t) \qquad \text{(C-12)}$$

Similarly,

$$\phi_{i,j}(t+\tau, 0) \geq \sum_{k \in S} \phi_{i,k}(\tau, 0)\, \beta_j(t) = \beta_j(t) \qquad \text{(C-13)}$$

Since Eqs. (C-12) and (C-13) hold for all $i \in S$, it follows that

$$\alpha_j(t+\tau) \leq \alpha_j(t) \tag{C-14}$$

and

$$\beta_j(t+\tau) \geq \beta_j(t) \tag{C-15}$$

This establishes that, for all $j \in S$, $\alpha_j(t)$ is a monotone non-increasing function, and $\beta_j(t)$ is a monotone non-decreasing function. Since these functions are bounded between 0 and 1, their limits exist. Let

$$\lim_{t \to \infty} \alpha_j(t) = \alpha_j \tag{C-16}$$

and

$$\lim_{t \to \infty} \beta_j(t) = \beta_j \tag{C-17}$$

We will now prove that these limits are equal. Let $d_j(t)$ denote the difference of $\alpha_j(t)$ and $\beta_j(t)$. That is

$$d_j(t) = \alpha_j(t) - \beta_j(t) \tag{C-18}$$

We will show that $d_j(t) \to 0$ as $t \to \infty$. By Eqs. (C-14) and (C-15) it follows that

$$d_j(t) \geq d_j(t+\tau) \geq 0 \qquad \text{(C-19)}$$

This means $d_j(t)$ is a monotone non-increasing function.

Now let

$$c = \min_{i,j} \phi_{i,j}(\tau, 0) \qquad \text{(C-20)}$$

Recall that $\Phi(\tau, 0)$ is an $n_o x n_o$ matrix. It follows from Eqs. (C-4) and (C-5) that

$$0 < c < \frac{1}{n_o} \qquad \text{(C-21)}$$

By Eq. (C-9) we have for all positive integer n and all $\tau > 0$,

$$\phi_{i,j}((m+1)\tau, 0) = \sum_{k \epsilon S} \phi_{i,k}(\tau, 0)\, \phi_{k,j}(m\tau, 0) \qquad \text{(C-22)}$$

Let i be chosen such that

$$\phi_{i,j}((m+1)\tau, 0) = \alpha_j((m+1)\tau, 0) \qquad \text{(C-23)}$$

Now let q be chosen such that

$$\phi_{q,j}(m\tau, 0) = \beta_j(m\tau, 0) \qquad \text{(C-24)}$$

Then by Eq. (C-22)

$$\alpha_j((m+1)\tau, 0) = \phi_{i,q}(\tau, 0)\,\phi_{q,j}(m\tau, 0) + \sum_{k\epsilon S-\{q\}} \phi_{i,k}(\tau, 0)\,\phi_{k,j}(m\tau, 0)$$

$$\leq \phi_{i,q}(\tau, 0)\,\beta_j(m\tau, 0) + \alpha_j(m\tau, 0) \sum_{k\epsilon S-\{q\}} \phi_{i,k}(\tau, 0)$$

$$= \phi_{i,q}(\tau, 0)\beta_j(m\tau, 0) + \alpha_j(m\tau, 0)[1 - \phi_{i,q}(\tau, 0)]$$

$$= \alpha_j(m\tau, 0) - [\alpha_j(m\tau, 0) - \beta_j(m\tau, 0)]\,\phi_{i,q}(\tau, 0)$$

$$\leq \alpha_j(m\tau, 0) - [\alpha_j(m\tau, 0) - \beta_j(m\tau, 0)]c \tag{C-25}$$

Similarly, should i be chosen such that

$$\phi_{i,j}((m+1)\tau, 0) = \beta_j((m+1)\tau, 0) \tag{C-26}$$

and should q be chosen such that

$$\phi_{q,j}((m+1)\tau, 0) = \alpha_j((m+1)\tau, 0) \tag{C-27}$$

then by following a similar procedure as above would result in

$$\beta_j((m+1)\tau, 0) \geq \beta_j(m\tau, 0) + [\alpha_j(m\tau, 0) - \beta_j(m\tau, 0)]c \tag{C-28}$$

Subtracting Eq. (C-28) from Eq. (C-25) we obtain:

$$\alpha_j((m+1)\tau, 0) - \beta_j((m+1)\tau, 0) \leq (1-2c)[\alpha_j(m\tau, 0) - \beta_j(m\tau, 0)] \qquad \text{(C-29)}$$

The above equation implies that

$$d_j(m\tau, 0) \leq (1-2c)^m \, d_j(\tau) \qquad \text{(C-30)}$$

Since the order of S, n_o, is greater or equal to 2. Therefore, by (C-21)

$$0 < c < \frac{1}{2} \qquad \text{(C-31)}$$

Hence Eq. (C-30) shows that

$$\lim_{m \to \infty} d_j(m\tau, 0) = 0 \qquad \text{(C-32)}$$

We have established that

$$\alpha_j = \beta_j \quad \text{for all } j \in S \qquad \text{(C-33)}$$

Since $\alpha_j(t)$ and $\beta_j(t)$ are respectively the maximum and the minimum elements of the jth column of $\Phi(t, 0)$, Eq. (C-33) indicates that all elements of the same column of $\Phi(t, 0)$ converge to one limit as $t \to \infty$. We shall denote by π_j the limiting value of the elements of the jth column of $\Phi(t, 0)$. That is, for all $i \in S$

$$\pi_j = \lim_{t \to \infty} \phi_{i,j}(t, 0) \tag{C-34}$$

If Φ denotes the limiting value of $\Phi(t, 0)$ as $t \to \infty$, then

$$\Phi = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_{n_o} \\ \pi_1 & \pi_2 & \cdots & \pi_{n_o} \\ \vdots & & & \\ \pi_1 & \pi_2 & \cdots & \pi_{n_o} \end{bmatrix} \tag{C-35}$$

It follows that for any initial probability vector $\underline{P}(0)$

$$\lim_{t \to \infty} \underline{P}(t | (0)) = \underline{P}(0)\, \Phi$$

$$= [\pi_1, \pi_2, \ldots, \pi_{n_o}] \tag{C-36}$$

This completes the proof.

APPENDIX D

MAIN THEOREM FOR STATE TRANSITION-RATE MATRIX OF MARKOVIAN SYSTEMS

For convenience we restate Theorem 5.6 of Section 5.6 here.

Theorem 5.6

Let M be the transition rate matrix of an n_o-state stationary Markovian system which is such that n_1 of the states are transient states, and the remaining $(n_o - n_1)$ states forms an ergodic set. If B is an $(m \times m)$ matrix resulted after deleting i, $1 \leq i \leq n_o - n_1$, rows and the corresponding i columns of M pertaining to i states of the ergodic set, then B is non-singular.

The proof of this theorem requires the following matrix inversion lemma.

Lemma

If A is an $n \times n$ matrix such that A^k tends to $\underline{\underline{0}}$ (zero matrix) as k tends to infinity, then $(I_n - A)$ has an inverse, and

$$(I_n - A)^{-1} = I_n + A + A^2 + A^3 + \ldots = \sum_{k=0}^{\infty} A^k \qquad \text{(D-1)}$$

Proof of Lemma

Consider the identity

$$(I_n - A)(I_n + A + A^2 + \ldots + A^{k-1}) = I_n - A^k \qquad \text{(D-2)}$$

which can easily be verified by multiplying out the LHS. By hypothesis, the RHS tends to I_n as $k \to \infty$. Since the determinant of I_n is 1, for sufficiently large k the determinent of $I_n - A^k$ is non-zero. It follows that for sufficiently large k, the determinent of the LHS is non-zero. But the determinent of a product of two matrices is equal to the product of their determinents. Hence $(I_n - A)$ has a non-zero determinent. This is equivalent to saying that $(I_n - A)$ posseses an inverse. Multiplying both sides of Eq. (D-2) by $(I_n - A)^{-1}$ we have:

$$(I_n - A)^{-1}(I_n - A^k) = I_n + A + A^2 + A^3 + \ldots + A^{k-1} \qquad \text{(D-3)}$$

Now taking the limit as $k \to \infty$ we obtain Eq. (D-1).

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Proof of Theorem 5.6

Each state of the system being either transient or recurrent* all diagonal elements of M are negative. Therefore the diagonal elements of B are negative and B may be written as

$$B = (b_{1,j})$$

$$= \begin{bmatrix} b_{1,1} & & 0 \\ & b_{2,2} & \\ 0 & & \ddots \; b_{m,m} \end{bmatrix} [I_m - Q] \qquad \text{(D-4)}$$

where Q is the following (m x m) matrix.

*A recurrent state is defined to be a state of an ergodic set.

$$Q = \begin{bmatrix} 0 & \frac{b_{1,2}}{-b_{1,1}} & \frac{b_{1,3}}{-b_{1,1}} & --- & \frac{b_{1,m}}{-b_{1,1}} \\ \frac{b_{2,1}}{-b_{2,2}} & 0 & \frac{b_{2,3}}{-b_{2,2}} & --- & \frac{b_{2,m}}{-b_{2,2}} \\ \vdots & & & & \\ \frac{b_{m,1}}{-b_{m,m}} & \frac{b_{m,2}}{-b_{m,m}} & --------- & & 0 \end{bmatrix} \qquad \text{(D-5)}$$

Since all $b_{i,i} < 0$, the diagonal matrix of $b_{i,i}$ is non-singular. Then B will be non-singular if $I_m - Q$ is non-singular. If we can show that $Q^k \to 0$ as $k \to \infty$, then by the matrix inversion lemma, $[I_m - Q]$ is non-singular.

The following properties hold for the elements of Q:

(i) $q_{i,j} \geq 0$ (D-6)

(ii) $0 \leq \sum_{j=1}^{m} q_{i,j} \leq 1$ (D-7)

(iii) for at least one value of ℓ, $\sum_{j=1}^{m} q_{\ell,j} < 1$ (D-8)

Let ν_i be the sum of the ith row of Q, i.e.,

$$\nu_i = \sum_{j=1}^{m} q_{i,j} \qquad \text{(D-9)}$$

By properties (D-6) - (D-8), it follows that

$$\sum_{k=1}^{m} q_{i,k}\nu_k < 1 \quad \text{for all } i \tag{D-10}$$

Let $\theta = \max_i \{ \sum_{k=1}^{m} q_{i,k}\, \nu_k \}$ (D-11)

Therefore

$$\sum_{k=1}^{m} q_{i,k}\nu_k \leq \theta < 1 \quad \text{for all } i \tag{D-12}$$

Denote the (i, j) element of Q^k by $q_{i,j}(k)$, i. e.,

$$Q^k = (q_{i,j}(k)) \tag{D-13}$$

The (i, j) element of Q^2 is:

$$q_{i,j}(2) = \sum_{k=1}^{m} q_{i,k}q_{k,j} \leq \sum_{k=1}^{m} q_{i,k}\,\nu_k \leq \theta \tag{D-14}$$

The (i, j) element of Q^3 is:

$$q_{i,j}(3) = \sum_{k=1}^{m} q_{i,k}q_{k,j}(2) \leq \sum_{k=1}^{m} q_{i,k}\theta = \nu_i\theta \tag{D-15}$$

The (i, j) element of Q^4 is:

$$q_{i,j}(4) = \sum_{k=1}^{m} q_{i,k} q_{k,j}(3) \leq \sum_{k=1}^{m} q_{i,k} \nu_k \theta \leq \theta^2 \tag{D-16}$$

In general we have:

$$q_{i,j}(2k) \leq \theta^k \qquad \text{for } k \geq 1 \tag{D-17}$$

and

$$q_{i,j}(2k+1) \leq \nu_i \theta^k \qquad \text{for } k \geq 1 \tag{D-18}$$

Since $\theta < 1$ and $\nu_i \leq 1$, for all (i, j) elements, $q_{i,j}(k)$ is a non-increasing function of k, and $q_{i,j}(k) \to 0$ as $k \to \infty$. Therefore, by lemma 5.1 $[I_m - Q]$ is non-singular. Hence B is non-singular.

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The corollary states that the matrix M is singular. The following is a proof for the corollary.

Proof of Corollary

Let M' represent the matrix resulted from striking out the last row and the last column of M. Therefore M' is an $(n_o - 1) x (n_o - 1)$ matrix.

$$M' = \begin{bmatrix} m_{1,1} & m_{1,2} & \dots m_{1,n_o-1} \\ m_{2,1} & m_{2,2} & \dots m_{2,n_o-1} \\ \vdots & & \\ m_{n_o-1,1} & m_{n_o-1,2} & \dots m_{n_o-1,n_o-1} \end{bmatrix} \qquad \text{(D-19)}$$

By Theorem 5.6, M' is non-singular. Therefore, there exists a unique vector

$$\underline{a} = [a_1, a_2, \dots, a_{n_o-1}] \qquad \text{(D-20)}$$

such that

$$\underline{a}\, M' = [m_{n_o,1}, m_{n_o,2}, \dots, m_{n_o,n_o-1}] \qquad \text{(D-21)}$$

The preceeding equation represents the following set of scalar equations:

$$\sum_{i=1}^{n_o-1} a_i m_{i,j} = m_{n_o,j} \quad \text{for } j = 1, 2, \ldots, n_o-1 \tag{D-22}$$

Summing over j gives

$$\sum_{j=1}^{n_o-1} \sum_{i=1}^{n_o-1} a_i m_{i,j} = \sum_{j=1}^{n_o-1} m_{n_o,j} \tag{D-23}$$

Interchanging the order of summations on the LHS, and recognizing the fact that

$$\sum_{j=1}^{n_o-1} m_{i,j} = -m_{i,n_o} \quad \text{for all } i \in S \tag{D-24}$$

we obtain

$$\sum_{i=1}^{n-1} a_i m_{i,n_o} = m_{n_o,n_o} \tag{D-25}$$

By Eqs. (D-22) and (D-25), it follows that

$$a_1\underline{m}_1 + a_2\underline{m}_2 + \ldots + a_{n-1}\underline{m}_{n-1} = \underline{m}_{n_o} \tag{D-26}$$

where $\underline{m}_i$ denotes the ith row of M. Hence M is singular.

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APPENDIX E

AN ALTERNATIVE APPROACH OF DERIVING THE mth MOMENT OF THE TIME-TO-FIRST-SYSTEM-FAILURE

In this Appendix the expression for the general mth moment of the time-to-first-system-failure will be derived through Laplace transform approach. In contrast to the time domain approach used in deriving Eq. (5.11-8), the Laplace transform approach is more formal but reveals less insights to the problem.

First define the following vector notations:

$$\underline{r}_u(t|\underline{P}(0)) = [r_1(t|\underline{P}(0)),\ r_2(t|\underline{P}(0)), \ldots, r_{k_o}(t|\underline{P}(0))] \qquad \text{(E-1)}$$

$$\underline{r}_d(t|\underline{P}(0)) = [r_{k_o+1}(t|\underline{P}(0)), r_{k_o+2}(t|\underline{P}(0)), \ldots, r_{n_o}(t|\underline{P}(0))] \qquad \text{(E-2)}$$

The Laplace transform of the above vectors are:

$$\underline{r}_u^*(s|\underline{P}(0)) = \int_0^\infty \underline{r}_u(t|\underline{P}(0))e^{-st}dt \qquad \text{(E-3)}$$

$$\underline{r}_d^*(s|\underline{P}(0)) = \int_0^\infty \underline{r}_d(t|\underline{P}(0))e^{-st}dt \qquad \text{(E-4)}$$

Using the notations of Eqs. (E-1) and (E-2), Eq. (5.3-2) can be written as

$$[\dot{\underline{r}}_u(t|\underline{P}(0))\ \dot{\underline{r}}_d(t|\underline{P}(0))] = [\underline{r}_u(t|\underline{P}(0))\ \underline{r}_d(t|\underline{P}(0))] \begin{bmatrix} B_{1,1} & B_{1,2} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \qquad \text{(E-5)}$$

Taking the Laplace transform of Eq. (E-5), we have

$$s\underline{r}_u^*(s|\underline{P}(0)) - \underline{P}_u(0) = \underline{r}_u^*(s|\underline{P}(0))\, B_{1,1} \tag{E-6}$$

$$s\underline{r}_d^*(s|\underline{P}(0)) - \underline{P}_d(0) = \underline{r}_u^*(s|\underline{P}(0))\, B_{1,2} \tag{E-7}$$

Solving for $\underline{r}_u^*(s|\underline{P}(0))$ from Eq. (E-6) we obtain:

$$\underline{r}_u^*(s|\underline{P}(0)) = \underline{P}_u(0)\, [sI_k - B_{1,1}]^{-1} \tag{E-8}$$

Substituting Eq. (E-8) into Eq. (E-7) results in:

$$s\underline{r}_d^*(s|\underline{P}(0)) - \underline{P}_d(0) = \underline{P}_u(0)[sI_k - B_{1,1}]^{-1} B_{1,2} \tag{E-9}$$

The probability distribution function of the time-to-first-system-failure is:

$$\int_o^t f(t|\underline{P}(0))dt = \underline{r}_d(t|\underline{P}(0))\, \underline{v}^T(n_o - k_o, 0) \tag{E-10}$$

Differentiating gives

$$f(t|\underline{P}(0)) = \frac{d}{dt}\, \underline{r}_d(t|\underline{P}(0))\, \underline{v}^T(n_o - k_o, 0) \tag{E-11}$$

The Laplace transform of Eq. (E-11) is:

$$f^*(s|\underline{P}(0)) = [s\underline{r}_d^*(s|\underline{P}(0)) - \underline{P}_d(0)]\, \underline{v}^T(n_o-k_o, 0) \tag{E-12}$$

Substituting Eq. (E-9) into Eq. (E-12) we obtain:

$$f^*(s|\underline{P}(0)) = \underline{P}_u(0)[sI_{k_o} - B_{1,1}]^{-1} B_{1,2}\underline{v}^T(n_o-k_o, 0) \tag{E-13}$$

Since the elements of B are such that each row run is zero,

$$B_{1,2}\,\underline{v}^T(n_o-k_o, 0) = -B_{1,1}\underline{v}^T(k_o, 0) \tag{E-14}$$

By substituting Eq. (E-14) into Eq. (E-13), we obtain $f^*(s|\underline{P}(0))$ in terms of $B_{1,1}$.

$$f^*(s|\underline{P}(0)) = -\underline{P}_u(0)[sI_{k_o} - B_{1,1}]^{-1} B_{1,1}\underline{v}^T(k_o, 0) \tag{E-15}$$

This is the Laplace transform of the probability density function of the time-to-first-system-failure (TTFSF). The mth order moment, $TTFSF^{(m)}$, is related to the limiting value of the mth derivative of $f^*(s|\underline{P}(0))$ as follows:

$$TTFSF^{(m)} = \lim_{s \to 0} \{(-1)^m \frac{d^m}{ds^m} f^*(s|\underline{P}(0))\} \tag{E-16}$$

Substituting Eq. (E-15) into (E-16) and carrying out the derivative we obtain:

$$TTFSF^{(m)} = \lim_{s \to 0} - \underline{P}_u(0)\, m!\, [sI_{k_o} - B_{1,1}]^{-(m+1)} B_{1,1} \underline{v}^T(k_o, 0)$$

$$= (-1)^m\, m!\, \underline{P}_u(0)\, B_{1,1}^{-m}\, \underline{v}^T(k_o, 0) \qquad \text{(E-17)}$$

This completes the derivation.

APPENDIX F

ILLUSTRATIVE EXAMPLES OF RELATIVE MAGNITUDES BETWEEN MUT(MDT) AND MRUT (MRDT)

First consider System 1 with transition diagram as shown in Figure F-1.

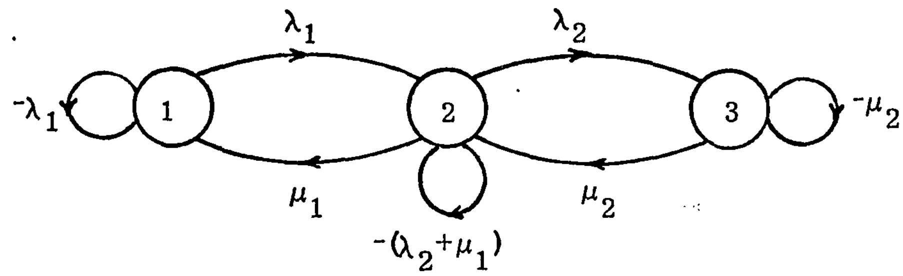

Figure F-1 Transition Diagram of System 1

For this system, let

$$S_u = \{1, 2\} \tag{F-1}$$

$$S_d = \{3\} \tag{F-2}$$

System 1 could rperesent, for example, a physical system with two identical units in parallel where at least one is needed for the system to be up. Either or both units are operated unless under repair, i. e., a unit is operated as long as and as soon as it is operable. Therefore, the system always starts from state 2 after it is restored from state 3, the only down state. It follows that

$$MUT = MTTFSF_{[0, 1, 0]} \tag{F-3}$$

Since state 3 is the only down state, by inspection of Figure F-1, we see that

$$MTTFSF_{[1, 0, 0]} > MTTFSF_{[0, 1, 0]} \tag{F-4}$$

The conditional mean remaining up-time for system 1 is:

$$MRUT_u = \frac{\pi_1}{\pi_1 + \pi_2} MTTFSF_{[1, 0, 0]} - \frac{\pi_2}{\pi_1 + \pi_2} MTTFSF_{[0, 1, 0]} \tag{F-4}$$

Subtracting Eq. (F-3) from Eq. (F-5) results in:

$$MRUT_u - MUT = \frac{\pi_1}{\pi_1 + \pi_2} [MTTFSF_{[1, 0, 0]} - MTTFSF_{[0, 1, 0]}] \tag{F-6}$$

By Eq. (F-4), the RHS of the above equation is greater than 0. Hence for system 1

$$MUT < MRUT_u \tag{F-7}$$

The mean down-times of the system can be found by inspection of Figure F-1.

$$\text{MDT} = \frac{1}{\mu_2} = \text{MRDT}_d \tag{F-8}$$

and

$$\text{MRDT} = \frac{\pi_3}{\mu_2} \tag{F-9}$$

Since $\pi_3 < 1$, the above two equations imply that for system 1

$$\text{MDT} > \text{MRDT} \tag{F-10}$$

By actually evaluating the expressions for MUT and MRUT of system 1, it can be shown that there exists no definite relationship between these two mean times for this system. Depending on the relative magnitudes of λ_i and μ_i $(i = 1, 2)$, either of these two mean times can be larger than the other.

Next consider system 2 with transition diagram as shown in Figure F-2.

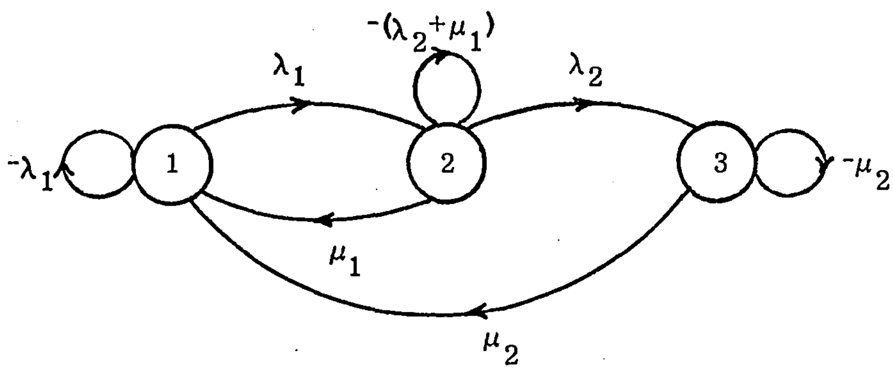

Figure F-2 Transition Diagram of System 2

Let the sets of up-states and down-states be the same as that specified in Eqs. (F-1) and (F-2). In terms of a physical system, system 2 could represent the same 2 units in parallel (as for system 1) except that the operational procedure is changed. For this case, when both units have failed, repairs must be completed on both units before operating the system again.

For system 2, the expression for $MRUT_u$ is the same as that given by Eq. (F-5), but the values for the π_i's $(i = 1, 2)$ for system 2 are different from those of system 1. By inspection of Figure F-2, we see that

$$MUT = MTTFSF_{[1,0,0]} \tag{F-11}$$

Therefore, for system 2

$$MRUT_u - MUT = \frac{-\pi_2}{\pi_1 + \pi_2} [MTTFSF_{[1,0,0]} - MTTFSF_{[0,1,0]}] \tag{F-12}$$

Since Eq. (F-4) remains true for system 2, the reversed inequality of Eq. (F-7) holds for system 2.

$$MUT > MRUT_u \tag{F-13}$$

These two simple examples show that there exists no fixed relationship between the magnitudes of MUT and MRUT or $MRUT_u$. Similarly, other simple examples may readily be constructed to demonstrate the nonexistance of fixed relationships between MDT and MRDT or $MRDT_d$.

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