ABSOLUTELY UNIQUE FACTORIZATION IN A POLYNOMIAL DOMAIN IN NON-COMMUTING INDETERMINATES

JIANN-JER CHEN

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ABSOlutely unique factorization in a polynomial domain in noncommuting indeterminates.

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ABSOLUTELY UNIQUE FACTORIZATION IN A POLYNOMIAL DOMAIN
IN NONCOMMUTING INDETERMINATES

by

JIANN-JER CHEN

M.S., University of New Hampshire, 1968

A THESIS

Submitted to the University of New Hampshire
In Partial Fulfillment of
The Requirements for the Degree of

Doctor of Philosophy
Graduate School
Department of Mathematics
September, 1972
This thesis has been examined and approved.

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ACKNOWLEDGEMENTS

It is a great pleasure for me to express appreciation and gratitude to my thesis adviser, Professor Richard E. Johnson, for his guidance and encouragement in the development of this work.

I wish to thank the professors and my friends at the University of New Hampshire who took interest in my work. I am grateful to Miss Jean Gahan who carefully typed the manuscript. Especially I thank my mother, sister and brothers for their spiritual support.

This thesis is dedicated to Dr. Shing-Meng Lee of the Institute of Mathematics, National Tsing Hua University, Taiwan, my college mathematics professor, who has my deep respect. His dignity and strength both in and out of the classroom will always be a guide to those fortunate enough to know him.
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ABSTRACT

ABSOlutely UNIQUE FACTORIZATION IN A POLYNOMIAL
DOMAIN IN NONCOMMUTING INDETERMINATES

by

JIANN-JER CHEN

In this paper we obtain some results concerning absolutely
unique factorization in a polynomial domain \( R = \mathbb{F}[x_i | i \in \Gamma] \), where
\( \mathbb{F} \) is a (non-commutative) field and \( [x_i | i \in \Gamma] \) is a well-ordered
set of noncommuting indeterminates.

Whenever \( f \in R \) is represented as a product \( f = f_1 f_2 \cdots f_n \),
then \( f \) can be represented as a product

\[
(1) \quad f = (f_1 u_1)(u_1^{-1} f_2 u_2) \cdots (u_{n-1}^{-1} f_n)
\]

for any units \( u_1, u_2, \ldots, u_{n-1} \) of \( R \). We call a factorization \( f = \]
\( f_1 f_2 \cdots f_n \) of \( f \) as a product of elements of a stated type absolutely
unique if every other representation of \( f \) as a product of elements of
the stated type has the form (1) above.

In Chapter 3 it is shown that the prime factorization \( f = \]
\( f_1 f_2 \cdots f_n \) of \( f \) in \( R \) is absolutely unique iff there exist no \( p_1, p_2 \)
in \( R \) such that \( p_1 f_1 + f_1 p_2 = 1 \) for each \( i = 1, 2, \ldots, n-1 \). This
result was first proved by A. J. Bowtell and P. M. Cohn [3]. Moreover,
we show that if \( f = f_1 f_2 \cdots f_n \) is a prime factorization of \( f \) and for each \( i = 1, 2, \ldots, n-1 \) there exist no \( f_1, \ldots, f_{i+1} \) in \( R \) with \(\deg f_1 < \deg f_{i+1} \), \(\deg f_2 < \deg f_1 \) such that \( f_1 f_{i+1} = 1 \), then the prime factorization \( f = f_1 f_2 \cdots f_n \) is absolutely unique. In this chapter we also show several theorems about the prime factorizations of an element \( f \) in \( R \), if the prime factorization of \( f \) is not absolutely unique.

In Chapter 4 we define several types of factorizations of the elements in \( R \). We say \( f \in R \) is \( \Delta \)-partially homogeneous (\( \Delta \)-p.hom.) iff there exists a finite set \( \Delta \subset \Gamma \) such that \( f \) is homogeneous in \( \{x_i \mid i \in \Delta\} \). We call the degree of \( f \) in \( \{x_i \mid i \in \Delta\} \) \( \Delta \)-degree and denote it by \( \deg_\Delta f \). \( f \in R \) is called \( \Delta \)-nice iff it has a prime factorization \( f = \prod_{i=1}^n p_i \) such that \( \deg_\Delta p_i > 0 \) for all \( i \). We show that if \( f \) is \( \Delta \)-p.hom. in \( R \) and \( f = f_1 f_2 \cdots f_n \) is the factorization of \( f \) such that each \( f_i \) is \( \Delta \)-nice or of \( \Delta \)-degree = 0 and \( \deg_\Delta f_i = 0 \) implies \( \deg_\Delta f_{i+1} > 0 \); \( \deg_\Delta f_i > 0 \) implies \( \deg_\Delta f_{i+1} = 0 \), then this factorization of \( f \) is absolutely unique. If \( f \in R \) has a factorization \( f = f_1 f_2 \cdots f_n \) into primes, we say \( f \) has length \( n \). Let \( C_1 = \{p \in R \mid p \) is prime with zero constant term\} \), \( C_2 = \{q \in R \mid q \) is prime with nonzero constant term\} \), \( S_1 = \{f \in R \mid f = \prod_{i=1}^n p_i \) for some \( p_i \in C_1 \} \), \( S_2 = \{g \in R \mid g = q_1 q_2 \cdots q_m \) for some \( q_j \in C_2 \} \). Let \( f = f_1 f_2 \cdots f_n \). We call this factorization of \( f \) a \( (\max S_1, \min S_2) \) factorization provided: (1) If \( f_1 \in S_1 \), then \( f_1+1 \in S_2 \); if \( f_1 \in S_2 \), then \( f_1+1 \in S_1 \); (2) Each \( f_1 \in S_1 \) is of maximal length; (3) Each \( f_1 \in S_2 \) is of minimal length; (4) \( f_1 \in S_1 \), if \( f \) has an \( S_1 \)-factor (left factor); otherwise \( f_1 \in S_2 \). A \( (\max S_1, \max S_2) \) factorization, etc., is defined similarly. We show that if \( f = f_1 f_2 \cdots f_n \) is: (A) a \( (\max S_1, \min S_2) \) factorization, or (B) a
(max $S_1$, max $S_2$) factorization, or (C) a (max $S_2$, max $S_1$) factorization, or (D) a (max $S_2$, min $S_1$) factorization, then this factorization is absolutely unique.

In Chapter 5, we discuss the absolutely unique primary factorization of an element in $R$. An element $f \in R \setminus U(R)$ is called $p$-primary for some prime $p$ iff every prime factor of $f$ is similar to $p$, where $U(R)$ is the group of units of $R$. We call $f = f_1 f_2 \cdots f_n$ a primary factorization of $f$ iff each $f_i$ is $p_i$-primary for some prime $p_i$ and $p_i \neq p_j$ ($p_i$ is not similar to $p_j$) whenever $i \neq j$. We prove that a primary factorization $f = f_1 f_2 \cdots f_n$ of $f$ is absolutely unique iff there exist no $t_{i_1}, t_{i_2}$ in $R$ such that $t_{i_1} f_i f_{i+1} t_{i_2} = 1$ for each $i = 1, 2, \ldots, n-1$.

Throughout the paper examples are given following most of the theorems in order to illustrate the theory and indicate the necessity of some of the hypotheses.
An integral domain $R$ with unity is called a right weak Bezout domain by P. M. Cohn [4] iff $aR \cap bR$ and $aR + bR$ are principal right ideals of $R$ for all $a, b \in R$ such that $aR \cap bR \neq 0$. Since an integral domain is a right weak Bezout domain iff it is a left weak Bezout domain [4], there is no need to distinguish between the two. Some examples of weak Bezout domains other than principal right ideal domains are the polynomial domain $F[x_1|1 \in \Gamma]$ and the power series domain $F[[x_1|1 \in \Gamma]]$, where $F$ is a (noncommutative) field and $[x_1|1 \in \Gamma]$ is a well-ordered set of noncommuting indeterminates.

A noncommutative unique factorization domain was formally defined by P. M. Cohn [4], that is, an integral domain $R$ such that every nonunit $r$ of $R^* = R \setminus \{0\}$ has a factorization into primes: $r = p_1p_2 \cdots p_n$, and if $r = g_1g_2 \cdots g_m$ is another such factorization, then $n = m$ and there is a permutation $\sigma$ on $(1,2,\cdots,n)$ such that $p_1$ and $g_{\sigma(1)}$ are similar for $i = 1,2,\cdots,n$. As for the notion of similarity, two elements $a, b$ of an integral domain $R$ are right similar iff $R/\alpha R \cong R/\beta R$, as $R$-modules. Since in an integral domain $R/\alpha R \cong R/\beta R$ iff $R/\alpha R \cong R/\beta R$ [6], we simply call $a$ and $b$ satisfying the condition similar and denote it by $a \sim b$. It is known that a weak Bezout domain $R$ is a unique factorization domain iff every nonunit in $R^*$ has a prime factorization [4]. Evidently, the polynomial domain $F[x_1|1 \in \Gamma]$ and the power series domain $F[[x_1|1 \in \Gamma]]$ are unique factorization domains.

We call a factorization $a = a_1a_2 \cdots a_n$ of $a$ in an integral
domain $R$ as a product of elements of a certain type absolutely unique if every other representation of $a$ as a product of elements of the stated type has the form $a = (a_1 u_1)(u_1^{-1} a_2 u_2) \cdots (u_{n-1}^{-1} a_n)$, where $u_1, u_2, \ldots, u_{n-1}$ are units in $R$. It is known that every nonzero, nonunit element of the power series domain $F[[x_1 | i \epsilon \Gamma]]$ has absolutely unique factorization into primes [9]. But not every nonzero, nonunit element of the polynomial domain $F[x_1 | i \epsilon \Gamma]$ has absolutely unique factorization into primes. We know if a polynomial in $F[x_1 | i \epsilon \Gamma]$ is homogeneous, then it has absolutely unique factorization into primes. But, for example, the prime factorization $f = x_1(x_2x_1+1) = (x_1x_2+1)x_1$ in $Q[x_1, x_2]$ is not absolutely unique.

In the present paper we discuss how absolutely unique the factorization of an element in the polynomial domain $R = F[x_1 | i \epsilon \Gamma]$ is, and obtain some results concerning it. This paper is divided into five chapters. The first chapter is introduction.

Chapter 2 contains preliminary definitions and well-known results in a weak Bezout domain.

In Chapter 3 we start our work on the polynomial domain $R = F[x_1 | i \epsilon \Gamma]$. We mention the important properties in the polynomial domain $R$ and use the Euclidean algorithm to show $R$ is a weak Bezout domain. In Section 3 we apply the Euclidean algorithm to similar elements in $R$ and obtain some properties between two similar elements $f$ and $f'$ such as $\deg f = \deg f'$, $\lodeg f = \lodeg f'$, $\deg_{\Delta} f = \deg_{\Delta} f'$ and $\lodeg_{\Delta} f = \lodeg_{\Delta} f'$, where $\deg f$ is the degree of $f$, $\lodeg f$ is the low-degree of $f$ and $\deg_{\Delta} f$ ($\Delta$-degree) is the degree of $f$ in $\{x_1 | i \epsilon \Delta, \Delta$ is a finite subset in $\Gamma\}$. In Section 4 we define the absolutely unique factorization into primes of an element in $R$. By
mathematical induction we show if each pair of adjacent factors of the
prime factorization of an element \( f \) has absolutely unique prime
factorization, then \( f \) has absolutely unique prime factorization
(Theorem 3.19). We obtain necessary and sufficient conditions for a
prime factorization of an element to be absolutely unique (Theorem 3.22,
Theorem 3.24, and Theorem 3.30). The result was first proved by A. J.
Bowtell and P. M. Cohn [3]. We also apply the above-mentioned conditions
to obtain the different prime factorizations of an element \( f \) in \( R \),
if \( f \) has no absolutely unique factorization into primes (Theorem 3.36
and Theorem 3.37).

In Chapter 4 we discuss the absolutely unique factorization of
an element in \( R \) as a product of elements of certain types. We say
\( f \in R \) is \( \Delta \)-partially homogeneous (\( \Delta \)-p.hom.) iff there exists a finite
subset \( \Delta \subset \Gamma \) such that \( f \) is homogeneous in \( \{ x_1 \mid 1 \in \Delta \} \). \( f \in R \) is
called \( \Delta \)-nice iff \( f \) has a prime factorization \( f = p_1 p_2 \cdots p_n \) such
that \( \deg_{\Delta} p_i > 0 \) for all \( i \). We show if \( f \) is \( \Delta \)-p.hom. in \( R \) and
\( f = f_1 f_2 \cdots f_n \) is the factorization of \( f \) such that each \( f_i \) is \( \Delta \)-nice
or of \( \Delta \)-degree = 0 and \( \deg_{\Delta} f_1 = 0 \) implies \( \deg_{\Delta} f_{i+1} > 0; \deg_{\Delta} f_i > 0 \)
implies \( \deg_{\Delta} f_{i+1} = 0 \), then this factorization of \( f \) is absolutely
unique (Theorem 4.7). We also define several types of factorizations of
the elements in \( R \) and prove the factorizations of these types are
absolutely unique (Theorem 4.15, Theorem 4.18, and Theorem 4.20).

In Chapter 5 we discuss the absolutely unique primary factoriza-
tion of an element in \( R \). An element \( f \in R \setminus U(R) \) is called \( p \)-primary
for some prime \( p \) iff every prime factor of \( f \) is similar to \( p \),
where \( U(R) \) is the group of units of \( R \) [1]. We call \( f = f_1 f_2 \cdots f_n \n\) a primary factorization of \( f \) iff each \( f_i \) is \( p_i \)-primary for some
prime $p_i$ and $p_i \neq p_j$ whenever $i \neq j$ [1]. We know that not every element in $R$ has a primary factorization. It is known [1] that if $f \in R^* \setminus U(R)$ has two primary factorizations $f = f_1 f_2 \cdots f_m = g_1 g_2 \cdots g_n$, then $m = n$ and $f_i \sim g_{\sigma(i)}$ for some permutation $\sigma$ of $\{1, 2, \cdots, n\}$. In Section 2 we obtain necessary and sufficient conditions for a primary factorization of an element in $R$ to be absolutely unique (Theorem 5.12 and Theorem 5.13).
Throughout, all rings are integral domains with unity, denoted by 1.

\textbf{Definition 2.1.} An integral domain is called a right weak Bezout domain iff the sum and intersection of any two principal right ideals that have nonzero intersection are again principal.

A left weak Bezout domain is defined similarly. However, it can be shown (Cohn [4]) that the weak Bezout condition is left-right symmetric; that is, an integral domain is a right weak Bezout domain iff it is a left weak Bezout domain. Therefore, we simply call an integral domain satisfying the condition expressed in the definition a weak Bezout domain.

Moreover, we can weaken the condition that an integral domain be a weak Bezout domain by means of the following theorem which enables us to drop the condition that intersections be principal.

\textbf{Theorem 2.2.} Let \( R \) be an integral domain in which any two principal right ideals with nonzero intersection have a principal sum. Then \( R \) is a weak Bezout domain.

\textbf{Proof.} See Williams [10].

\textbf{Definition 2.3.} Two elements \( a, b \) of a ring \( R \) are said to be right similar iff \( R/aR \cong R/bR \), as right \( R \)-modules.
From the definition it is clear that right similarity is a reflexive, symmetric and transitive relation. The notion of left similarity may be defined in an analogous fashion; however, in an integral domain this is equivalent to right similarity, as follows from a more general result of Fitting [6]; that is, in an integral domain \( R/aR \simeq R/bR \) iff \( R/Ra \simeq R/Rb \). Therefore, we can omit the qualifier "right" and denote the similarity of \( a \) and \( b \) by \( a \sim b \). When \( R \) is commutative, the condition is not so important, because \( R/aR \simeq R/bR \) holds iff \( a \) and \( b \) are associated (\( b = au \) for some unit \( u \) in \( R \)).

In addition, similarity can be characterized as follows (see Jacobson[7], and Cohn [4]). Let \( R^* \) denote the semigroup of the nonzero elements in \( R \); that is, \( R^* = R \setminus \{0\} \).

\[(2.4) \text{ Let } a, a' \in R^*. a \text{ is similar to } a' \text{ iff there exists } b \text{ in } R^* \text{ such that } aR + bR = R \text{ and } aR \cap bR = ba'R.\]

In the case \( aR \cap bR = ba'R \) we have \( ba' = ab' \) for some \( b' \) in \( R \), then, by the definition, we have also \( b \sim b' \). It follows from the latter condition (2.4) that elements similar to units are units. It is easy to see that an element \( a \) of an integral domain \( R \) is similar to any of its associates \( a' \) (\( a' = uav \), where \( u, v \) are units in \( R \)).

In a weak Bezout domain the following theorem is a simple criterion for similarity.

**Theorem 2.5.** In a weak Bezout domain \( R \) the following statements
are equivalent:

1. $a, a' \in R^*$ and $a \sim a'$.

2. There exist $b, b' \in R^*$ such that $ab' = ba'$, where $aR + bR = R$ and $Ra' + Rb' = R$.

3. There exist $b, b' \in R^*$ such that $ab' = ba'$, where $aR \cap bR = ba'R$ and $Ra' \cap Rb' = Rba'$.

An element in $R^*$ is called a prime iff it is a nonunit which is not a product of two nonunits. Of interest are those elements in $R$ that can be expressed as a (finite) product of primes. If $R$ is a weak Bezout domain, it is well-known that any prime factorization of a nonunit in $R^*$ is unique up to order of factors (primes) and similarity as follows (see Cohn[4]).

**Theorem 2.6.** Let $R$ be a weak Bezout domain. If a nonunit $a \in R$ has two prime factorizations $a = p_1p_2\cdots p_n = q_1q_2\cdots q_m$, where $p_i, q_i$ are primes, then $n = m$ and $p_i \sim q_{\sigma(i)}$ for some permutation $\sigma$ of $\{1, 2, \ldots, n\}$.

**Remark 2.7.** By the above theorem we see if in the prime factorization $a = p_1p_2\cdots p_n$ a has exactly $k$ prime factors which are similar to one another, say, to a prime $p$, then any prime factorization of $a$ has exactly $k$ prime factors similar to $p$.

As for the prime factorizations of two similar elements $a$ and $a'$ in a weak Bezout domain, there are useful theorems as follows (Johnson-Beauregard [2]).
Theorem 2.8. Let $R$ be a weak Bezout domain and let $a, a' \in R$ with $a \sim a'$. If $a = a_1a_2$, then there exist $a_1', a_2' \in R$ such that $a_1 \sim a_1'$, $a_2 \sim a_2'$, and $a' = a_1'a_2'$.

Theorem 2.9. Let $R$ be a weak Bezout domain. If $a, a' \in R$ such that $a \sim a'$ and if $a = p_1p_2\cdots p_n$, $a' = q_1\cdots q_m$, where $p_i, q_i$ are primes, then $n = m$ and $p_i \sim q_{\sigma(i)}$ for some permutation $\sigma$ of $\{1, 2, \cdots, n\}$.

Corollary 2.10. In a weak Bezout domain any element similar to a prime is prime.
CHAPTER III

ABSOLUTELY UNIQUE PRIME FACTORIZATION IN A POLYNOMIAL DOMAIN

Throughout this thesis, fields are not assumed to be commutative.

1. Basic Definitions and Concepts

Let $F$ be a field, $\Gamma$ be a well-ordered set and let $\{x_i | i \in \Gamma\}$ be a set of noncommuting indeterminates. We then form the polynomial ring $R = F[x_i | i \in \Gamma]$ in noncommuting indeterminates $x_i$ for all $i \in \Gamma$ over a field $F$. Let $M$ denote the free monoid on $\{x_i | i \in \Gamma\}$. For simplicity, we sometimes write the polynomial ring $R$ as $F[M]$ instead of $F[x_i | i \in \Gamma]$.

Every $\alpha \in M$ has a unique expression as a product $\alpha = x_{i_1}^{k_1}x_{i_2}^{k_2}\cdots x_{i_n}^{k_n}$, where the $k_j$ are non-negative integers and the $i_j \in \Gamma$. Such a product will be called a primitive monomial. By the degree of $\alpha$ we shall mean the integer $k_1 + k_2 + \cdots + k_n$ (which is $\geq 0$) and denote it by $\deg \alpha$. It is known that the free monoid $M$ can be ordered (that is, there exists a linear ordering $<$ in $M$ such that if $\alpha < \beta$, then $\alpha \gamma < \beta \gamma$ and $\gamma \alpha < \gamma \beta$ for all $\gamma \in M$) in such a way that if $\deg \alpha < \deg \beta$, then $\alpha < \beta$ (see Johnson [9]).

By a polynomial $f \in R = F[M]$ we mean a sum

$$f = \sum_{\alpha \in M} a_{\alpha} \alpha, \quad a_{\alpha} \in F,$$

where only a finite number of the $a_{\alpha}$ are nonzero. A polynomial $a_{\alpha}$,
where \( a \in F \), \( a \in M \), will be called a monomial (not necessary primitive).

Thus every element \( f = \sum_{a \in M} a_{a} \) of \( R = F[M] \) is either 0 or a sum of a finite number of nonzero monomials. If \( f = \sum_{a \in M} a_{a} \) is zero, we define its degree to be \(-\infty\); if \( f \) is nonzero, we define its degree to be the largest degree of any \( a \) for which \( a_{a} \neq 0 \) and denote it by \( \deg f \).

If all the nonzero monomials in this sum have the same degree, then \( f \) is said to be homogeneous. Similarly, we define the low-degree of \( f \) (\( \neq 0 \)) to be the least degree of any \( a \) for which \( a_{a} \neq 0 \) and denote it by \( \lodeg f \). Clearly, \( \lodeg f > 0 \) or \( \lodeg f = 0 \) means \( f \) has a zero or nonzero constant term. Let \( U(R) \) denote the group of units of \( R = F[M] \).

Evidently \( U(R) = F^{\times} = F \setminus \{0\} \).

For any two elements \( f = \sum a_{a} \), \( g = \sum b_{a} \) in \( R = F[M] \), we have in mind that \( f + g = \sum (a_{a} + b_{a})a \), \( fg = \sum c_{a}a \), where \( c_{a} = \sum_{\beta, \gamma M} a_{\beta}b_{\gamma} \).

We note that \( \deg fg = \deg f + \deg g \), \( \lodeg fg = \lodeg f + \lodeg g \).

Clearly, if \( f = gh \) in \( R^{\times} \setminus U(R) \), then \( f \) is homogeneous iff \( g \) and \( h \) are homogeneous. Since \( F \) is a field, it is easy to show that \( R = F[M] \) is an integral domain. Therefore, we call \( R = F[M] \) a polynomial domain.

2. The Euclidean Algorithm

In what follows \( R \) denotes the polynomial domain \( F[M] = F[x_{i} | i \in R] \).

It is well-known that the following propositions are true in the polynomial domain \( R \).
Theorem 3.1. If $f, g \in R$ such that $fR \cap gR \neq 0$ and $\text{lodeg } f \geq \text{lodeg } g$, then $f^{-1}gR = \{ t \in R | ft \in gR \} = hR$ for some $h \in R^*$ with $\text{lodeg } h = 0$.

Corollary 3.2. If $f, g \in R$ such that $fR \cap gR \neq 0$ and $\text{lodeg } f \geq \text{lodeg } g$, then $fR \cap gR = fhR$ for some $h \in R^*$ with $\text{lodeg } h = 0$.

Corollary 3.3. If $f, g \in R$ such that $f, g$ are homogeneous and $fR \cap gR \neq 0$, then either $fR \subseteq gR$ or $gR \subseteq fR$.

Theorem 3.4. If $f, g \in R$ such that $Rf \cap Rg \neq 0$ and $\text{lodeg } f \geq \text{lodeg } g$, then $Rf^{-1}g = \{ t \in R | tf \in Rg \} = Rh$ for some $h \in R^*$ with $\text{lodeg } h = 0$.

Corollary 3.5. If $f, g \in R$ such that $Rf \cap Rg \neq 0$ and $\text{lodeg } f \geq \text{lodeg } g$, then $Rf \cap Rg = Rhf$ for some $h \in R^*$ with $\text{lodeg } h = 0$.

Corollary 3.6. If $f, g \in R$ such that $f, g$ are homogeneous and $Rf \cap Rg \neq 0$, then either $Rf \subseteq Rg$ or $Rg \subseteq Rf$.

Theorem 3.7 (Division process in $R$). If $f, g \in R$ such that $fR \cap gR \neq 0$ and $\text{deg } f \geq \text{deg } g$, then $f = gq + r$ for some $q, r \in R$ with $\text{deg } r < \text{deg } g$. Moreover, $q, r$ are unique.

Theorem 3.8. If $f, g \in R$ such that $Rf \cap Rg \neq 0$ and $\text{deg } f \geq \text{deg } g$, then $f = qg + r$ for some $q, r \in R$ with $\text{deg } r < \text{deg } g$. Moreover, $q, r$ are unique.
We shall now develop the Euclidean algorithm in $R$, which will be used to show that $R$ is a weak Bezout domain.

3.9 Euclidean algorithm (Cohn [5]). As before, we assume $f, g \in R$ such that $fR \cap gR \neq 0$ and $\deg f > \deg g$. Since $fR \cap gR \neq 0$, we have

$$fg' = gr' \neq 0$$

for some $g', f' \in R$. By Theorem 3.7 it follows that

$$f = gq_1 + r_1, \ \deg r_1 < \deg g$$

for some $q_1, r_1 \in R$. Substituting from (2) into (1) we obtain

$$r_1g' = (f-gq_1)g' = g(f'-q_1g') \quad \text{if we put} \quad r_1' = f' - q_1g' \ , \text{we can write}$$

$$r_1g' = gr_1'.$$

We see by (3) and (2) that $\deg gr_1' = \deg g + \deg r_1' = \deg r_1 + \deg g' < \deg g + \deg g'$, and hence $\deg r_1' < \deg g'$. Therefore, we have $f' = q_1g' + r_1'$ with $\deg r_1' < \deg g'$, which is exactly what we obtained in Theorem 3.8. Thus there is a complete left-right symmetry. By (3) we find $r_1 = 0$ iff $r_1' = 0$. Otherwise, we can apply the same reasoning to (3) and thus obtain the chain of equations of the Euclidean algorithm. Moreover, we have two chains of equations,
one for left and one for right division:

(4) \[ f = gq_1 + r_1 \quad f' = q_1g' + r_1' \quad r_1g' = gr_1', \]
\[ g = r_1q_2 + r_2 \quad g' = q_2r_1' + r_2' \quad r_2r_1' = r_1r_2', \]
\[ r_1 = r_2q_3 + r_3 \quad r_1' = q_3r_2' + r_3' \quad r_3r_2' = r_2r_3', \]
\[ r_2 = r_3q_4 + r_4 \quad r_2' = q_4r_3' + r_4' \quad r_4r_3' = r_3r_4'. \]

We note that the remainders \( r_1, r_1' \) on the two sides are in general distinct, while the quotients \( q_1 \) are the same. The degrees of the remainders decrease strictly.

(5) \[ \deg g > \deg r_1 > \cdots , \deg g' > \deg r_1' > \cdots . \]

Hence, the remainders must vanish eventually. Let \( n \) be the least integer such that \( r_{n+1} = 0 \). Since \( r_{n+1}r_n = r_n r_{n+1} \), it follows that \( r_{n+1}' = 0 \).

If we have \( r_k' = 0 \) for some \( k \leq n \), then by symmetry \( r_k = 0 \), which contradicts the choice of \( n \). Hence, \( r_{n+1}' \) is the first vanishing remainder of the right division and it follows:

(4) \[ r_{n-3} = r_{n-2}q_{n-1} + r_{n-1} \quad r_{n-3}' = q_{n-1}r_{n-2}' + r_{n-1}' \quad r_{n-1}r_{n-2}' = r_{n-2}r_{n-1}', \]
\[ r_{n-2} = r_{n-1}q_n + r_n \quad r_{n-2}' = q_n r_{n-1}' + r_n' \quad r_n r_{n-1}' = r_{n-1} r_n', \]
\[ r_{n-1} = r_nq_{n+1} \quad r_{n-1}' = q_{n+1}r_n' \quad r_{n+1} = r_{n+1}' = 0 . \]

By (4) and (5) we see that

(6) \[ \deg q_i' > 0 \text{ for } i = 2,3,\ldots ,n+1 , \]
while \( \deg q_1 > 0 \) iff \( \deg g < \deg f \).

In view of (4) we see

\[
\begin{align*}
    r_1 &= f - gq_1 \in fR + gR, \\
    r_2 &= g - r_1q_2 \in r_1R + gR \subset fR + gR, \\
    &\vdots \\
    r_n &= r_{n-2} - r_{n-1}q_n \in r_{n-2}R + r_{n-1}R \subset fR + gR,
\end{align*}
\]

and then \( r_nR \subset fR + gR \).

Also, we see

\[
\begin{align*}
    r_{n-1} &= r_nq_{n+1} \in r_nR, \\
    r_{n-2} &= r_{n-1}q_n + r_n \in r_nR, \\
    &\vdots \\
    r_1 &= r_2q_3 + r_3 \in r_3R, \\
    g &= r_1q_2 + r_2 \in r_2R, \\
    f &= gq_1 + r_1 \in r_1R,
\end{align*}
\]

and then \( fR + gR \subset r_nR \).

It follows that

(7) \( fR + gR = r_nR \).
Similarly, we have

\[(8) \quad Rf' + Rg' = Rr_n' \cdot\]

Evidently, \(r_n\) is the greatest common left divisor (gcd) of \(f\) and \(g\); \(r_n'\) is the greatest common right divisor (gcrd) of \(f'\) and \(g'\).

Now by Corollary 3.2 and (7) it follows that

**Theorem 3.10.** \(R\) is a weak Bezout domain.

By the definition of the degree of a polynomial in \(R\), we see that every nonzero, nonunit element of \(R\) has a factorization into primes, which satisfies the property described in Theorem 2.6. If every nonzero, nonunit element of an integral domain has this property, P. M. Cohn calls the integral domain a unique factorization domain (see Cohn [4]).

3. Application of Euclidean Algorithm to Similar Elements

Let's consider nonzero, nonunit elements \(f\) and \(f'\) in \(R\). If \(f \sim f'\), by Theorem 2.5 there exist \(g, g'\) in \(R^*\) such that \(fg' = gf'\), where \(fR + gR = R\), \(Rf' + Rg' = R\). Moreover, if \(\deg g \geq \deg f\), then \(g\) is a nonzero, nonunit element. By the Euclidean algorithm we obtain

\[g = fq + r \quad g' = qf' + r' \quad fr' = rf'\]

for some \(q, r, r'\) in \(R\) with \(\deg r < \deg f\) and \(\deg r' < \deg f'\).

We note \(r \neq 0\). For if \(r = 0\), then \(g = fq\) and hence \(gR \subseteq fr \neq R\), which contradicts \(fR + gR = R\). Similarly, we assert \(r' \neq 0\).

Now consider \(fr' = rf'\). Since \(fR \cap rR \neq 0\), it follows that
$fR + rR = tR$ for some $t \in R$. Thus, $f = tf_1$, $r = tr_1$ for some $f_1, r_1 \in R$. Evidently, $g =fq + r = tf_1q + tr_1 = t(f_1q + r_1)$ and so $t$ is the common left divisor of $f$ and $g$. By $fR + gR = R$, where $f, g$ are nonzero, nonunit elements, it follows that $t$ is a unit. We conclude $fR + rR = R$. Similarly, we have $Rr' + Rr = R$. Conversely, if there exist $r, r'$ in $R^*$ with $\deg r < \deg f$ and $\deg r' < \deg f'$ such that $fr' = rf'$, where $fR + rR = R$, $Rf' + Rr = R$, then by Theorem 2.5 $f \sim f'$.

Therefore, we conclude that

**Theorem 3.11.** Two nonzero, nonunit elements $f, f'$ in $R$ are similar iff there exist $g, g'$ in $R^*$ with $\deg g < \deg f$, $\deg g' < \deg f'$ such that $fg' = gr'$, where $fR + gR = R$, $Rf' + Rg = R$.

**Corollary 3.12.** Let both $f$ and $f'$ be linear in $R$. Then $f \sim f'$ iff there exist $u, v$ in $F^*$ such that $fu = vf'$ ($f, f'$ are associated). Moreover, if one of the indeterminates in both $f$ and $f'$ has coefficient 1, then $u = v$; that is, in this case $f$ is similar to $f'$ iff $f$ and $f'$ are conjugate.

**Proof.** It is an immediate consequence of Theorem 3.11. Moreover, if one of the indeterminates in both $f$ and $f'$, say $x_j$, has coefficient 1, then in $fu = vf'$ we have $x_ju = vx_j$. Therefore $u = v$ and $f = uf'u^{-1}$.

**Corollary 3.13.** $x_1 - a$ is similar to $x_1 - b$ ($a, b \in F$) iff $a, b$ are conjugate in $F$. 
Proof. Obvious.

Using the hypothesis in Theorem 3.11 and by the Euclidean algorithm we see \( r_n, r'_n \in F^* \) in this case. It follows from (4) of 3.9 that

\[
\deg r_{n-1} = \deg(r_{n-1}r'_n) = \deg(r_nr'_{n-1}) = \deg r'_{n-1} \quad \text{since} \quad r_nr'_{n-1} = r_{n-1}r'_n,
\]

\[
\deg(r_{n-2}r'_n) = \deg(r_{n-2}r'_{n-1}) \quad \text{since} \quad r_{n-1}r'_{n-2} = r_{n-2}r'_{n-1},
\]

so \( \deg r'_{n-2} = \deg r_{n-2} \).

\[
\text{...............................................}
\]

and then \( \deg r_1 = \deg r'_1 \quad \text{(since} \quad r_2r'_1 = r_1r'_2) \),

\[
\deg g' = \deg g \quad \text{(since} \quad r_1g' = gr'_1).
\]

Hence, by \( fg' = gf' \) we conclude that \( \deg fg' = \deg gf' \) and \( \deg f = \deg f' \). Similarly, by the same reasoning, we have \( \operatorname{lodeg} f = \operatorname{lodeg} f' \).

Let \( \Delta \) be a finite subset of \( \Gamma \) and let \( \deg_{\Delta} f \) be the degree of \( f \) in \( \{ x_i \mid i \in \Delta \} \). If \( \Delta \) is a singleton, say \( \Delta = \{1\} \subset \Gamma \), for simplicity, we may write \( \deg_{1} f \) instead of \( \deg_{\Delta} f \). By the same reasoning as above, we have \( \deg_{\Delta} f = \deg_{\Delta} f' \) and \( \operatorname{lodeg}_{\Delta} f = \operatorname{lodeg}_{\Delta} f' \).

Furthermore, by the Euclidean algorithm we obtain

\[
f = g_1 \cdot r_1 + r_1
= (r_1q_2 + r_2)q_1 + r_1 = r_1(q_2q_1 + 1) + r_2q_1
\]
\[= (r_2 q_3 + r_3)(q_2 q_{1+1}) + r_2 q_1 = r_2(q_3 q_2 q_1 + q_3 q_1) + r_3(q_2 q_{1+1})\]
\[= (r_3 q_4 + r_4)(q_3 q_2 q_1 + q_3 q_1) + r_3(q_2 q_{1+1})\]
\[= r_3(q_4 q_3 q_2 q_1 + q_4 q_3 q_4 q_1 + q_2 q_{1+1}) + r_4(q_3 q_2 q_1 + q_3 q_1)\]

\[.............................\]
\[= r_n(q_{n+1} q_n q_{n-1} \cdots q_2 q_{n+1} q_n \cdots q_3 + q_{n+1} q_n \cdots q_4 q_{1+1} \cdots q_{n-2} \cdots q_1 \cdots ) \]
\[f' = q_1 g' + r_1\]
\[= q_1(q_2 r_1 + r_2) + r_1 = (q_1 q_2 + 1) r_1 + q_1 r_2\]
\[= (q_1 q_2^2 + 1)(q_3 r_2 + r_3) + q_1 r_2 = (q_1 q_2 q_3 + q_3 q_1) r_2 + (q_1 q_2 + 1) r_3\]
\[= (q_1 q_2 + 1)(q_3 q_4 + 1)(q_4 r_3 + r_4) + (q_1 q_2 + 1) r_3\]
\[= (q_1 q_2 + 1)(q_3 q_4 + 1)(q_4 q_1 + q_4 q_1^2 + 1) r_3 + (q_1 q_2 q_3 + q_3 q_1) r_4\]

\[.............................\]
\[= (q_1 q_2 \cdots q_{n+1} q_{n+1} q_{n+1} q_{n+1} q_1 q_2 q_3 q_4 \cdots q_{n-1} q_{n-2} \cdots q_1 \cdots ) r_n',\]

where \( \deg q_i > 0 \) for \( i = 1, 2, \ldots, n+1 \), since \( \deg g < \deg f \) and \( r_n \), \( r_n' \in F^* \). Therefore, we see from the above equations that the largest primitive monomial in \( f \) should be in \( q_{n+1} q_n \cdots q_2 q_1 \) and the largest primitive monomial in \( f' \) should be in \( q_1 q_2 \cdots q_n q_{n+1} \). Let \( a_1 \) be the largest primitive monomial in \( q_1 \) for \( i = 1, 2, \ldots, n+1 \), then the largest primitive monomial in \( f \) is \( a_{n+1} a_n \cdots a_1 \) and the largest primitive monomial in \( f' \) is \( a_1 a_2 \cdots a_n a_{n+1} \).

By what above-mentioned we conclude the following theorems.
Theorem 3.14. If two nonzero, nonunit elements $f$, $f'$ in $R$ are similar, then

1. $\deg f = \deg f'$ and $\lodeg f = \lodeg f'$.

2. If $f$ is homogeneous with degree $n$, so is $f'$.

3. $\deg_\Delta f = \deg_\Delta f'$ and $\lodeg_\Delta f = \lodeg_\Delta f'$ for every $\Delta \subset \Gamma$.
   In particular, $\deg_1 f = \deg_1 f'$ and $\lodeg_1 f = \lodeg_1 f'$ for every $i \in \Gamma$.

4. If $\Delta \subset \Gamma$ and $f$ is homogeneous in $\Lambda$ with degree $m$, so is $f'$.

Remarks 3.15. (I) By Theorem 3.14 (3) if $f \sim f'$ and $x_i$ is an indeterminate in $f$, then $x_i$ is also an indeterminate in $f'$.

(II) $ax_i + b$ is never similar to $a'x_j + b'$ if $i \neq j$, where $a, a' \in F^*$; $b, b' \in F$.

Theorem 3.16. If two nonzero, nonunit elements $f$, $f'$ in $R$ are similar, then the largest primitive monomials in $f$ and $f'$ can be expressed in the forms $a_1 a_2 \cdots a_n$ and $a_n a_{n-1} \cdots a_1$ respectively, where $a_i \in M$.

Remark 3.17. If $f \sim f'$ and the largest monomial of $f$ has the indeterminates $\{x_i | i \in \Delta\}$, then so does the largest monomial of $f'$.

4. Absolutely Unique Prime Factorization

In the Section 2 of this chapter we have shown that $R = F[M]$ is a noncommutative unique factorization domain in the sense that every
nonzero, nonunit element $f$ of $R$ has a factorization into primes, and such that any two prime factorizations of $f$ have the same number of primes and these primes can be paired off into similar ones. For example, $f = x_1 + x_1x_2x_1$ in $\mathbb{Q}[x_1,x_2]$, where $\mathbb{Q}$ is the set of rational numbers, has two prime factorizations $f = x_1(1+x_2x_1) = (1+x_2x_1)x_1$, where $x_1 \sim x_1$ and $1 + x_2x_1 \sim 1 + x_1x_2$.

Here we are going to discuss a stronger prime factorization in $R$. Whenever $f \in R$ is represented as a product $f = p_1p_2 \cdots p_n$ of primes, then $f$ can also be represented as a product

$$f = (p_1u_1)(u_1^{-1}p_2u_2) \cdots (u_{n-1}^{-1}p_n)$$

into primes for any units $u_1, u_2, \cdots, u_{n-1}$ of $R$.

**Definition 3.18.** We call a factorization $f = p_1p_2 \cdots p_n$ of $f$ as a product of primes absolutely unique if every other representation of $f$ as a product of primes has the form (1) above for some choice of the units $u_1, u_2, \cdots, u_{n-1}$.

We say $f = p_1p_2 \cdots p_n$ and $f$ in the form (1) above have the same factorization into primes. For example, $g = x_1(x_2+x_3)$ in $\mathbb{Q}[x_1,x_2,x_3]$ has absolutely unique factorization into primes, while $f = x_1(1+x_2x_1) = (1+x_2x_1)x_1$ does not. We say that $f$ has two prime factorizations.

**Theorem 3.19.** If $f$ has factorization $f = p_1p_2 \cdots p_n$ into primes such that $f_i = p_ip_{i+1}$ has absolutely unique factorization for $i = 1, 2, \cdots, n-1$, then $f$ has absolutely unique factorization into primes.
Proof. We carry out mathematical induction on $n$. By hypothesis the theorem is true when $n = 2$.

By mathematical induction, we assume the theorem is true when $n = k$. Let $f = p_1p_2 \cdots p_k p_{k+1} = g_1 g_2 \cdots g_k g_{k+1}$ represent two prime factorizations of $f$. Assume that $p_1 p_{k+1}$ has absolutely unique factorization for each $i$. Note that $p_1 R \cap g_1 R \neq 0$, where $p_1, g_1$ are prime.

(A) If $p_1 R = g_1 R$, then $g_1 = p_1 u_1$ for some unit $u_1$ of $R$.

By $f = p_1 p_2 \cdots p_k p_{k+1} = g_1 g_2 \cdots g_k g_{k+1}$ it follows that $p_2 \cdots p_{k+1} = u_1 g_2 \cdots g_{k+1}$. By induction hypothesis the prime factorization $p_2 \cdots p_{k+1}$ is absolutely unique. Thus $u_1 g_2 = p_2 u_2$ (i.e., $g_2 = u_1^{-1} p_2 u_2$), $g_3 = u_2^{-1} p_3 u_3$, \ldots, $g_{k+1} = u_k^{-1} p_{k+1}$ for some units $u_2, u_3, \ldots, u_k$ of $R$.

Therefore the prime factorization $f = p_1 p_2 \cdots p_{k+1}$ is absolutely unique.

(B) If $p_1 R \neq g_1 R$, then, since $R$ is a weak Bezout domain and $p_1, g_1$ are prime, we have

$$p_1 R + g_1 R = R, \quad p_1 R \cap g_1 R = h R$$

for some $h \in R$, where $h = p_1 g_1 = g_1 p_1$ for some $g_1, p_1$ in $R$. By the Definition 2.4 of similarity we conclude that $p_1 \sim p_1$ and $g_1 \sim g_1$, and by Corollary 2.10 $p_1$ and $g_1$ are prime. Since $f = p_1 p_2 \cdots p_{k+1} = g_1 g_2 \cdots g_{k+1} \in p_1 R \cap g_1 R = h R$, it follows that $p_1 p_2 \cdots p_{k+1} = g_1 g_2 \cdots g_{k+1} = p_1 g_1 r$ for some $r \in R$. Thus $p_2 \cdots p_{k+1} = g_1 r$. By the induction hypothesis the prime factorization $p_2 \cdots p_{k+1}$ is absolutely unique. Thus $g_1 = p_2 u_2$ for some unit $u_2$ of $R$, and hence $p_1 p_2 u_2 = p_1 g_1 = g_1 p_1$.

Since the prime factorization $p_1 p_2$ is absolutely unique, it follows that $g_1 = p_1 u_1$ for some unit $u_1$ of $R$. Hence, $p_1 R = g_1 R$, which contradicts the assumption $p_1 R \neq g_1 R$. 

Therefore, only (A) will hold and so the prime factorization 
\( f = p_1 p_2 \cdots p_{k+1} \) is absolutely unique. The theorem is proved.

By Theorem 3.14 in Section 3 of this chapter we are able to 
figure out some sufficient conditions for a prime factorization of an 
element in \( R \) to be absolutely unique, which are practical in some sense.

**Theorem 3.20.** If \( f = f_1 f_2 \cdots f_n \) is a prime factorization of \( f \) 
in \( R \) such that for each \( i = 1, 2, \ldots, n-1 \) there exist some \( x_{1i} \) in \( f_i \), 
but \( x_{1i} \) not in \( f_{i+1} \) and some \( x_{12} \) in \( f_{i+1} \), but \( x_{12} \) not in \( f_i \), 
then the prime factorization \( f = f_1 f_2 \cdots f_n \) is absolutely unique.

**Proof.** In view of Theorem 3.19, it will suffice to prove that 
the theorem is true when \( n = 2 \). Let \( f = f_1 f_2 = g_1 g_2 \) represent two 
prime factorizations of \( f \) in \( R \). We know by hypothesis and (1) of 
Remarks 3.15 that \( f_1 \) is not similar to \( f_2 \). Since each \( g_i \) is similar 
to some \( f_j \), it follows that \( g_1 \) is not similar to \( g_2 \).

Assume \( f_1 \) is similar to \( g_2 \). It follows that \( f_2 \) is similar 
to \( g_1 \) and so \( f_1 \) is not similar to \( g_1 \); \( f_2 \) is not similar to \( g_2 \). 
Since by hypothesis \( x_{11} \) is in \( f_1 \), but it is not in \( f_2 \) and then by 
(1) of Remarks 3.15, it follows that \( x_{11} \) is in \( g_2 \) (since we assume 
f_1 \sim g_2), but \( x_{11} \) is not in \( g_1 \) (for if \( x_{11} \) is in \( g_1 \), then \( x_{11} \) is 
in \( f_2 \) which is similar to \( g_1 \)). Also, since \( x_{12} \) is in \( f_2 \), but it 
is not in \( f_1 \) and again by (1) of Remarks 3.15, it follows that \( x_{12} \) is in 
g_1 (since \( f_2 \sim g_1 \)), but \( x_{12} \) is not in \( g_2 \) (for if \( x_{12} \) is in \( g_2 \), 
then \( x_{12} \) is in \( f_1 \) which is similar to \( g_2 \)). Hence, the terms in 
f_1 f_2 involving \( x_{11}, x_{12} \) should have \( x_{11} \) preceding \( x_{12} \), while the
terms in \(g_1g_2\) involving \(x_1, x_2\) have \(x_2\) preceding \(x_1\). Now let
\(a_1\) be the largest term in \(f_1\), which has the indeterminate \(x_1\) and
\(\beta_1\) be the largest term in \(f_2\), which has the indeterminate \(x_2\); let
\(\gamma_1\) be the largest term in \(g_1\), which has the indeterminate \(x_1\) and
\(\delta_1\) be the largest term in \(g_2\), which has the indeterminate \(x_2\). Thus,
\(a_1\beta_1\) is a term of \(f_1f_2\), which is involving \(x_1, x_2\) and
\(\gamma_1\delta_1\) is a term of \(g_1g_2\), which is involving \(x_2, x_1\). It is impossible, since
\(f = f_1f_2 = g_1g_2\). We conclude that \(f_1\) is only similar to \(g_1\).

Note that \(f_1R \cap g_1R \neq 0\), where \(f_1, g_1\) are prime.

(A) If \(f_1R = g_1R\), then \(g_1 = f_1u_1\) for some unit \(u_1\) of \(R\).
By \(f_1^2 = g_1g_2 = f_1u_1g_2\) it follows that \(f_2 = u_1g_2\) and \(g_2 = u_1^{-1}f_2\).
Therefore, the prime factorization \(f = f_1f_2\) is absolutely unique.

(B) If \(f_1R \neq g_1R\), then, since \(R\) is a weak Bezout domain and
\(f_1, g_1\) are prime, we have

\[
f_1R + g_1R = R, \quad f_1R \cap g_1R = hR
\]

for some \(h \in R\), where \(h = f_1g_1' = g_1f_1'\) for some \(g_1', f_1' \in R\). By the
definition of similarity we conclude that \(f_1 \sim f_1'\) and \(g_1 \sim g_1'\), and
then by Corollary 2.10 \(f_1'\) and \(g_1'\) are prime. Now \(f = f_1f_2 =
g_1g_2 \in f_1R \cap g_1R = hR\), we obtain \(g_1g_2 = g_1f_1'r\) for some \(r \in R\). Thus
\(g_2 = f_1'r\). Because \(g_2\) and \(f_1'\) are prime, so \(r\) must be a unit of \(R\).
We now conclude that \(g_2 \sim f_1' \sim f_1\), which contradicts that \(f_1\) is only
similar to \(g_1\).

Therefore only (A) is true and so the theorem is proved.

Example 3.21. \(f = (x_1+1)(x_2x_3+1)x_1 \in Q[x_1, x_2, x_3]\), where \(Q\) is
the set of rational numbers, has absolutely unique prime factorization by
Theorem 3.20.

We shall now discuss necessary and sufficient conditions for a
prime factorization of an element \( f \) in \( R \) to be absolutely unique.
This result was first proved by A. J. Bowtell and P. M. Cohn [3].

**Theorem 3.22.** Let \( f = f_1 f_2 \cdots f_n \) be a prime factorization of
\( f \) in \( R \). If for each \( i = 1, 2, \ldots, n-1 \) there exist no \( p_1, p_2 \) in \( R \)
such that \( p_1 f_1 + f_1 p_2 = 1 \), then the prime factorization \( f = f_1 f_2 \cdots f_n \)
is absolutely unique.

**Proof.** By Theorem 3.19 it will suffice to prove that the theorem
is true when \( n = 2 \). Let \( f = f_1 f_2 \) be a prime factorization of \( f \)
satisfying the hypothesis (there exist no \( p_1 p_2 \) in \( R \) such that
\( p_1 f_1 + f_2 p_2 = 1 \)).

Assume \( f = f_1 f_2 = g_1 g_2 \) represents two prime factorizations of
\( f \). Note that \( f_1 R \cap g_1 R \neq 0 \), where \( f_1, g_1 \) are prime. If \( f_1 R \neq g_1 R \),
then, since \( R \) is a weak Bezout domain, it follows that

\[
f_1 R + g_1 R = R, f_1 R \cap g_1 R = hR
\]

for some \( h \in R \), where \( h = f_1 g_1 = g_1 f_1 \) for some nonzero, nonunit
elements \( g_1, f_1 \) in \( R \), since \( f_1 R + g_1 R = R \). Now \( f = f_1 f_2 = g_1 g_2 \in f_1 R \cap g_1 R = hR \), we obtain \( f = f_1 f_2 = g_1 g_2 = h t = g_1 f_1' t = g_1 f_1' t \) for
some \( t \in R \). Thus \( f_2 = g_1' t, g_2 = f_1' t \). Since \( f_2 \) and \( g_2 \) are prime
and \( g_1' f_1' \) are nonunit elements, \( t \) must be a unit of \( R \), and so
\[ g'_1 = f_2 t^{-1}, \quad f'_1 = g_2 t^{-1}. \]
Since \( f_1^* R + g_1^* R = R \), it follows that
\[
f_1 h_1 + g_1 h_2 = 1 \quad \text{for some } h_1, h_2 \in R^*
\]
and \( h = f_1^* g_1' = g_1 f_1' \). We see
\[
f_1 h_1 f_1 + g_1 h_2 f_1 = f_1
\]
and \( 0 \neq g_1 h_2 f_1 = f_1 (1-h_1 f_1) \in f_1 R \cap g_1 R = h_1 R \). Thus
\[
g_1 h_2 f_1 = f_1 (1-h_1 f_1) = h r_1 = f_1 g_1^* r_1 \quad \text{for some } r_1 \in R.
\]
Therefore \( 1 - h_1 f_1 = g_1^* r_1 \), and hence
\[
h_1 f_1 + g_1^* r_1 = 1.
\]
Thus
\[
h_1 f_1 + f_2 t^{-1} r_1 = 1 \quad \text{since } g_1' = f_2 t^{-1}
\]
If we put \( p_1 = h_1, p_2 = t^{-1} r_1 \), then we have \( p_1 f_1 + f_2 p_2 = 1 \),
which contradicts the hypothesis. We conclude that \( f_1^* R = g_1^* R \). Thus
\[
g_1 = f_1^* u_1 \quad \text{for some unit } u_1 \text{ of } R \quad \text{and so} \quad g_2 = u_1^{-1} f_2.
\]
Therefore, the prime factorization \( f = f_1^* f_2 \) is absolutely unique
and the theorem is proved.

**Remark 3.23.** In view of the proof of Theorem 3.22, we see that
if \( f = f_1^* f_2 = g_1^* g_2 \) represents two prime factorizations of \( f \), then
\( f_1 \sim g_2 \) and \( f_2 \sim g_1 \). By Theorem 3.14, we obtain \( \deg f_1 = \deg g_2 \)
and \( \deg f_2 = \deg g_1 \).

**Theorem 3.24 (the converse of Theorem 3.22).** If the prime factorization
\( f = f_1^* f_2 \cdots f_n \) of \( f \) in \( R \) is absolutely unique, then there
exist no \( p_1, p_2 \) in \( R \) such that \( p_1 f_1 + f_1 p_2 = 1 \) for each
\( i = 1, 2, \ldots, n-1 \).
Proof. Let's begin with \( n = 2 \) and let the prime factorization \( f = f_1 f_2 \) be absolutely unique. We are going to show that there exist no \( p_1, p_2 \) in \( R \) such that \( p_1 f_1 + f_2 p_2 = 1 \).

Assume that there exist \( p_1, p_2 \) in \( R \) such that
\[
p_1 f_1 + f_2 p_2 = 1.
\]
Then it follows that
\[
(1) \quad p_1 f_1 f_2 + f_2 p_2 f_2 = f_2, \quad 0 \neq p_1 f_1 f_2 = f_2 (1 - p_2 f_2).
\]
Therefore, we have
\[
p_1 R + f_2 R = R, \quad p_1 R \cap f_2 R \neq 0.
\]
Since \( R \) is a weak Bezout domain, it follows that
\[
p_1 R \cap f_2 R = h R \text{ for some } h \in R,
\]
where \( h = p_1 f_2' = f_2 p_1' \) for some \( f_2', p_1' \in R \). We note that \( p_1 \sim p_1' \), \( f_2 \sim f_2' \) and \( f_2' \) is a prime. We observe from (1) that \( 0 \neq p_1 f_1 f_2 = f_2 (1 - p_2 f_2') \in p_1 R \cap f_2 R = h R \). It follows that
\[
(2) \quad p_1 f_1 f_2 = f_2 (1 - p_2 f_2') = p_1 f_2' = f_2 p_1' \text{ for some } r \in R,
\]
\[
(3) \quad f_1 f_2 = f_2' \text{ and } 1 - p_2 f_2 = p_1'.
\]

(**)Claim. \( f_2' \) is another prime factorization of \( f \), which is different from the prime factorization \( f_1 f_2 \). Otherwise, we obtain from (3)
\[
(4) \quad f_2' = f_1 u_1, \quad r = u_1^{-1} f_2 \text{ for some unit } u_1 \text{ of } R.
\]
By (3) and (4) we see

\[ 1 = p_2 f_2 + p_1 r, \]

\[ = p_2 f_2 + p_1 u_1 f_2, \]

\[ = (p_2 + p_1 u_1) f_2. \]

This shows that \( f_2 \) is a unit of \( R \), which is a contradiction. But by the hypothesis, the prime factorization \( f = f_1 f_2 \) is absolutely unique. It comes a contradiction, since \( f_2 \) is another prime factorization of \( f \).

Therefore, if a prime factorization \( f = f_1 f_2 \) is absolutely unique, then there exist no \( p_1, p_2 \) in \( R \) such that \( p_1 f_1 + f_2 p_2 = l \).

Now let \( n > 2 \) and the prime factorization \( f = f_1 f_2 \cdots f_n \) be absolutely unique. Suppose there exist some \( p_1, p_{12} \) in \( R \) such that

\[ p_1 f_1 + f_{1+1} p_{12} = l \] for some \( i \in \{1, 2, \cdots, n-1\} \)

then, by the proof above for \( n = 2 \), the prime factorization \( f_1 f_{1+1} \) is not absolutely unique. This implies that the prime factorization \( f = f_1 \cdots f_{1+1} f_n \) is not absolutely unique, which contradicts the hypothesis. Therefore, if the prime factorization \( f = f_1 f_2 \cdots f_n \) is absolutely unique, then there exist no \( p_1, p_{12} \) in \( R \) such that

\[ p_1 f_1 + f_{1+1} p_{12} = l \] for each \( i = 1, 2, \cdots, n-1 \). The theorem is proved.

Corollary 3.25. Let \( f = f_1 f_2 \cdots f_n \) be a prime factorization of \( f \) in \( R \) and let \( a_i \) be the largest primitive monomial in \( f_i \) for each \( i = 1, 2, \cdots, n \). If for each pair of \( a_j, a_{j+1}, a_j, a_{j+1} \) can not be expressed as the forms \( u_j \delta_j, \epsilon_j u_j \) respectively, where \( u_j, \delta_j, \epsilon_j \in M \) (including 1) and \( \text{deg} u_j > 0 \), then the prime factorization \( f = f_1 f_2 \cdots f_n \).
is absolutely unique.

Proof. Consider the case \( n = 2 \). Let \( f = f_1 f_2 \) be a prime factorization satisfying the hypothesis.

Suppose the prime factorization \( f = f_1 f_2 \) is not absolutely unique. By Theorem 3.22 there exist \( p_1, p_2 \) in \( R^* \) such that

\[ p_1 f_1 + f_2 p_2 = 1. \]

If \( \deg p_1 > \deg f_2 \), then, applying the division process in \( R \) to
\[ 0 \neq p_1 f_1 f_2 = f_2 (1 - p_2 f_2), \]
we obtain

\[ p_1 = f_2 q + r_1 \text{ for some } q, r_1 \in R, \]

where \( \deg r_1 < \deg f_2 \) and \( r_1 \neq 0 \). For if \( r_1 = 0 \), it follows that \( p_1 = f_2 q \) and \( 1 = p_1 f_1 + f_2 p_2 = f_2 (q f_1 + p_2) \), which shows \( f_2 \) is a unit, a contradiction. Thus,

\[ r_1 f_1 = p_1 f_1 - f_2 q f_1, \]
\[ r_1 f_1 + f_2 p_2 = p_1 f_1 + f_2 p_2 - f_2 q f_1, \]
\[ r_1 f_1 + f_2 p_2 = 1 - f_2 q f_1 \text{ (since } p_1 f_1 + f_2 p_2 = 1) \]

and hence

\[ r_1 f_1 + f_2 (p_2 + q f_1) = 1. \]

Therefore, \( \deg r_1 f_1 = \deg f_2 (p_2 + q f_1) \) and \( \deg r_1 < \deg f_2 \), so that \( \deg (p_2 + q f_1) < \deg r_1 \). Let \( \beta_1, \delta_1 \) be the largest primitive monomials in \( r_1, p_2 + q f_1 \) respectively (in the case \( r_1 \in R^* \), its largest primitive monomial is 1). Thus we have

\[ \beta_1 a_1 = a_2 \delta_1 \]
with $\deg \beta_1 < \deg \alpha_2$, $\deg \delta_1 < \deg \alpha_1$. It follows that

$$a_2 = \beta_1 u_1 \text{ for some } u_1 \epsilon M \text{ with } \deg u_1 > 0,$$

$$a_1 = v_1 \delta_1 \text{ for some } v_1 \epsilon M \text{ with } \deg v_1 > 0$$

and then $\beta_1 v_1 \delta_1 = \beta_1 u_1 \delta_1$, which implies $v_1 = u_1$. Hence $a_1 = u_1 \delta_1$, $a_2 = \beta_1 u_1$ with $\deg u_1 > 0$. It is a contradiction to the hypothesis.

Therefore, $f = f_1 f_2$ is absolutely unique and then by Theorem 3.19 the corollary is proved.

**Corollary 3.26.** Let $f = f_1 f_2 \cdots f_n$ be a prime factorization of $f$ in $R$. If for each $i = 1, 2, \ldots, n-1$ $f_i$ and $f_{i+1}$ generate a proper ideal in $R$, then the prime factorization $f = f_1 f_2 \cdots f_n$ is absolutely unique. Moreover, the prime factorization $\overline{f} = f_1 f_{n-1} \cdots f_{n-2} f_{n-1}$ is also absolutely unique.

**Proof.** Since $f_i$ and $f_{i+1}$ generate a proper ideal in $R$, we see that $R f_i R + R f_{i+1} R < R$ for $i = 1, 2, \ldots, n-1$. This shows that there exist no $p_{1,1}, p_{1,2}$ in $R$ such that $p_{1,1} f_i + p_{1,2} f_{i+1} = 1$ for $i = 1, 2, \ldots, n-1$. By Theorem 3.22 the prime factorization $f = f_1 f_2 \cdots f_n$ is absolutely unique. Similarly, by the same reasoning we have that the prime factorization $\overline{f} = f_n f_{n-1} \cdots f_{2} f_1$ is absolutely unique.

**Remarks 3.27.** (1) The converse of Corollary 3.26 is not true. For instance, let $f = x_1(x_1 x_2 + 1)$ be a prime factorization of $f$ in $R = \mathbb{Q}[x_1, x_2]$. By Corollary 3.25 the prime factorization $f = x_1(x_1 x_2 + 1)$ is absolutely unique. But $x_1$ and $x_1 x_2 + 1$ generate $R$. 
since \( x_1(-x_2) + (x_1x_2+1) = 1 \).

(2) If the prime factorization \( f = f_1f_2 \) is absolutely unique, the prime factorization \( f = f_2f_1 \) is not necessarily absolutely unique. For example, the prime factorization \( f = x_1(x_1x_2+1) \) in \( \mathbb{Q}[x_1,x_2] \) as mentioned above is absolutely unique, but \( f = (x_1x_2+1)x_1 = x_1(x_2x_1+1) \) is not absolutely unique.

**Corollary 3.28.** If \( f = f_1f_2 \cdots f_n \) is a prime factorization of \( f \) in \( R \) such that for each \( i = 1, 2, \ldots, n \) \( f_i \) has zero constant term, then the prime factorization \( f = f_1f_2 \cdots f_n \) is absolutely unique.

**Proof.** Consider \( f_1 \) and \( f_{i+1} \) for \( i = 1, 2, \ldots, n \). Since both \( f_1 \) and \( f_{i+1} \) have zero constant term, so either \( p_1f_1 + f_{i+1}p_{i+1} = 0 \) or \( p_1f_1 + f_{i+1}p_{i+1} \in R \setminus F \) for any \( p_1, p_{i+1} \in R \). This shows that there exist no \( p_1, p_{i+1} \) in \( R \) such that \( p_1f_1 + f_{i+1}p_{i+1} = 1 \) for \( i = 1, 2, \ldots, n-1 \). By Theorem 3.22 the prime factorization \( f = f_1f_2 \cdots f_n \) is absolutely unique.

**Corollary 3.29.** If the polynomial \( f \) in \( R \) is homogeneous, then its prime factorization is absolutely unique.

**Proof.** It is easy to show that if \( f \) is homogeneous, then each prime factor of \( f \) is also homogeneous. Thus each prime factor of \( f \) has zero constant term, and then by Corollary 3.28 the prime factorization of \( f \) is absolutely unique.
In view of the proof in Theorem 3.24, we observe that if
\[ f = f_1 f_2 \] is a prime factorization such that \( p_1 f_1 + f_2 p_2 = 1 \) for some elements \( p_1, p_2 \) in \( R \), then \( p_1 R \cap f_2 R = hR \) for some \( h \in R \), where
\[
(1) \quad h = p_1 f'_2 = f_2 p'_1
\]
for some elements \( f'_2, p'_1 \in R \) with \( p_1 \sim p'_1, f_2 \sim f'_2 \).

Also, we have
\[
p_1 f_1 f_2 = f_2 (1 - p_2 f_2) = p_1 f_2' r = f_2 p_1' r \quad \text{for some} \quad r \in R ,
\]
\[
1 - p_2 f_2 = p'_1 r
\]
and \( f_1 f_2 = f_2' r \), where \( f_2' r \) is a prime factorization of \( f \), which is different from the prime factorization \( f_1 f_2 \) under the definition of absolutely unique prime factorization (see Claim (**) in the proof of Theorem 3.24).

We note that in the condition \( p_1 f_1 + f_2 p_2 = 1 \) when \( p_1 \) is fixed, \( p_2 \) is uniquely determined. Also, \( p_1 \) determines \( f_2' \), which is unique up to multiplying some unit element on the right of \( f_2' \) (since \( p_1 R \cap f_2 R = p_1 f_2' R \)) and then \( r \) is determined. Therefore we conclude \( p_1 \) uniquely determines a prime factorization \( f_2' r \) of \( f \), which is different from the prime factorization \( f_1 f_2 \) of \( f \).

If \( \deg p_1 > \deg f_2 \), by the same reasoning we did in the proof of Corollary 3.25, we obtain
\[
(2) \quad p_1 = f_2 q + r_1 \quad \text{for some} \quad q, r_1 \in R 
\]
and \( r_1 f_1 + f_2 (p_2 + q f_1) = 1 \), where \( 0 < \deg r_1 < \deg f_2 \) and \( 0 < \deg(p_2 + q f_1) < \deg f_1 \). We recall that \( p_1 \) uniquely determines a
prime factorization of $f$, which is different from the prime factorization $f_1f_2$, so does $r_1$.

**Claim.** $p_1$ and $r_1$ determine the same prime factorization of $f$.

Consider the case $r_1f_1 + f_2(p_2q_1) = 1$. This shows that

$$r_1R + f_2R = R, r_1R ∩ f_2R ≠ 0.$$  

Since $R$ is a weak Bezout domain, it follows that

$$r_1R ∩ f_2R = h'R$$  

for some $h' ∈ R$, where $h' = r_1f_2 = f_2r_1"$ for some $f_2, r_1" ∈ R$ and $f_2 ∨ f"_2, r_1 ∨ r_1"$.

We observe that $0 ≠ r_1f_1f_2 = f_2[1 - (p_2q_1)f_2] ∈ r_1R ∩ f_2R$. Thus,

$$r_1f_1f_2 = f_2[1 - (p_2q_1)f_2] = r_1f_2^"r' = f_2r_1'r'$$

for some $r' ∈ R$ and then

$$f_1f_2 = f_2^"r', 1 - (p_2q_1)f_2 = r_1'r'.$$

By the same reasoning as before, $r_1$ uniquely determines the prime factorization $f_2^"r'$ of $f$, which is different from $f_1f_2$.

We shall show that $f_2r'$ and $f_2^"r'$ are the same prime factorization of $f$ under the definition of absolutely unique prime factorization. Substituting from (2) into (1) we obtain

$$(f_2q^+r_1)f'_2 = f_2p_1'$$

and so $r_1f'_2 = f_2(p_1'-q_1f_2') ∈ r_1R ∩ f_2R$. By (3) we conclude that

$$r_1f'_2 = r_1f_2't$$  

for some $t ∈ R$.  

and then $f'_2 = f''_2 t$.

Since both $f'_2$ and $f''_2$ are prime, $t$ must be a unit of $R$.

By $f = f'_2 r = f''_2 r'$ and $f'_2 = f''_2 t$, we obtain $r = t^{-1} r'$. It says that $f'_2 r$ and $f''_2 r'$ are the same prime factorization of $f$.

Therefore we conclude

**Theorem 3.30.** If $f = f'_1 f'_2$ is a prime factorization of $f$ in $R$ such that $p'_1 f'_1 + f'_2 p'_2 = 1$ for some $p'_1, p'_2$ in $R$, then $p'_1$ uniquely determines a prime factorization $f'_1 r'$ of $f$, which is different from $f'_1 f'_2$. If $\deg p'_1 > \deg f'_2$, then we have $p'_1 = f'_2 q + r'_1$ for some $q, r'_1$ in $R$, $\deg r'_1 < \deg f'_2$, $r'_1 f'_1 + f'_2 (p'_2 q f'_1) = 1$ and $r'_1$ uniquely determines the same prime factorization of $f$ as $p'_1$ does.

Theorem 3.30 tells us that for the prime factorization $f = f'_1 f'_2$ with the condition $p'_1 f'_1 + f'_2 p'_2 = 1$ we need only consider $p'_1, p'_2$ in $R^*$ such that $\deg p'_1 < \deg f'_2$, $\deg p'_2 < \deg f'_1$. $p'_1$ might not be the only one satisfying the condition $p'_1 f'_1 + f'_2 p'_2 = 1$. We shall now discuss the prime factorizations of $f = f'_1 f'_2$, which are determined by different $p'_1$.

Recall that $p'_1 R \cap f'_2 R = hR$ for some $h \in R$, where $h = p'_1 f'_2 = f'_2 p'_1$ and $p'_1 \sim p'_1, f'_2 \sim f'_2$; $p'_1$ uniquely determines the prime factorization $f'_2 r'$ of $f$, which is different from $f'_1 f'_2$. If there exist $q'_1, q'_2$ in $R^*$, $q'_1 \neq p'_1, q'_2 \neq p'_2$ with $\deg q'_1 < \deg f'_2$, $\deg q'_2 < \deg f'_1$, satisfying $q'_1 f'_1 + f'_2 q'_2 = 1$, then by the same reasoning as before we have

$$q'_1 R + f'_2 R = R, \quad q'_1 R \cap f'_2 R = h''R.$$
for some $h'' \in R$, where $h'' = q_1 f_2'' = f_2 q_1''$ and $q_1 \sim q_1''$, $f_2 \sim f_2''$.

Then $q_1$ uniquely determines the prime factorization $f_2'' r'$ of $f$ for some $r' \in R$.

**Claim.** $f_2' r'$ and $f_2'' r'$ are two different prime factorizations of $f$, which are different from $f_1 f_2'$.

Assume that $f_2' r'$ and $f_2'' r'$ represent the same prime factorization of $f$, then $f_2'' = f_2' u$ for some unit $u$. By $q_1 f_1 + f_2 q_2 = 1$ we have $0 \neq f_2 q_2 q_1 = q_1 (1 - f_1 q_1) \in q_1 R \cap f_2 R$. Thus

$$f_2 q_2 q_1 = q_1 (1 - f_1 q_1) = q_1 f_2' t_1$$

for some $t_1 \in R$.

It follows that

$$1 - f_1 q_1 = f_2' t_1$$

and hence $f_2' t_1 + f_1 q_1 = 1$. Since we suppose $f_2'' = f_2' u$ for some unit $u$, we have

(1) $$f_2' u t_1 + f_1 q_1 = 1.$$  

Also, by $p_1 f_1 + f_2 p_2 = 1$ we have $0 \neq f_2 p_2 p_1 = p_1 (1 - f_1 p_1) \in p_1 R \cap f_2 R$. Thus

$$f_2 p_2 p_1 = p_1 (1 - f_1 p_1) = p_1 f_2' t_2$$

for some $t_2 \in R$.

It follows that $1 - f_1 p_1 = f_2' t_2$ and hence

(2) $$f_2' t_2 + f_1 p_1 = 1.$$  

By Remark 3.23 we see $\deg f_2' = \deg f_2$. It follows that $\deg p_1 < \deg f_2'$ and $\deg q_1 < \deg f_2'$. It is well-known that,

(3) for the condition $f_2' R + f_1 R = R(f_2', f_1)$ are nonzero, nonunit elements) there is only one $p_1 \in R$ with $\deg p_1 < \deg f_2'$ satisfying

$$f_2' t_2 + f_1 p_1 = 1$$

for some $t_2 \in R$.

Thus by (1), (2) and the hypothesis $p_1 \neq q_1$, we have a contradiction.
Consequently, $f_2^r$ and $f_2'^r$ represent two different prime factorizations of $f$, which are different from the prime factorization $f_1f_2$.

Therefore we conclude

**Theorem 3.31.** If $f = f_1f_2$ is a prime factorization of $f$ in $R$ such that $p_1f_1 + f_2p_2 = 1$ with $\deg p_1 < \deg f_2$, $\deg p_2 < \deg f_1$, then each $p_1$ determines a different prime factorization of $f$, which is different from the prime factorization $f_1f_2$.

In view of the proof in Theorem 3.22, we observe that if $f = f_1f_2 = g_1g_2$ represents two prime factorizations of $f$ in $R$, then we have

$$f_1R + g_1R = R,$$

$$f_1h_1 + g_1h_2 = 1 \text{ for some } h_1, h_2 \in R^*,$$

$$f_1R \cap g_1R = hR \text{ for some } h \in R,$$

where $h = f_1g_1' = g_1f_1'$, $f_1 \sim f_1'$, $g_1 \sim g_1'$ and also

$$p_1f_1 + f_2p_2 = 1,$$

where $p_1 = h_1$. We see that once $h_1$ in $f_1h_1 + g_1h_2 = 1$ (from $f_1R + g_1R = R$) is determined, $p_1$ is determined, since $p_1 = h_1$, and hence the condition $p_1f_1 + f_2p_2 = 1$ is determined. It is well-known that we can always find a unique $h_1 \in R^*$ with $\deg h_1 < \deg g_1$ such that $f_1h_1 + g_1h_2 = 1$ (see (3) in the proof of Theorem 3.31). By Remark 3.23 we have $\deg g_1 = \deg f_2$ and so $\deg p_1 = \deg h_1 < \deg f_2$. 
Therefore we conclude

**Theorem 3.32.** Let $f = f_1 f_2$ be a prime factorization of $f$, which has another prime factorization $g_1 g_2$, then there exists a unique $h_1$ in $R^*$ with $\deg h_1 < \deg g_1$ such that $f_1 h_1 + g_1 h_2 = 1$ and $h_1$ uniquely determines the condition $p_1 f_1 + f_2 p_2 = 1$ with $\deg p_1 < \deg f_2$ (automatically $\deg p_2 < \deg f_1$), where $p_1 = h_1$.

**Corollary 3.33.** If $f$ has a factorization $f = f_1 f_2 \cdots f_n$ into primes such that the $f_i$ are linear for all $i$. The prime factorization is absolutely unique if for there exist no $u_{i_1}, u_{i_2}$ in $F$ such that $u_{i_1} f_{i_1} + f_{i_1+1} u_{i_2} = 1$ for each $i = 1, 2, \ldots, n-1$. In particular, if $n = 2$ and there exist $u_1, u_2$ in $F^*$ such that $u_1 f_1 + f_2 u_2 = 1$, then $f = f_1 f_2$ has another prime factorization $(-f_1 u_1 + 1) f_1 u_2^{-1}$.

**Proof.** It is an immediate consequence of Theorem 3.32, Theorem 3.24 and Theorem 3.19. Moreover, if $n = 2$, then by $u_1 f_1 + f_2 u_2 = 1$ it follows that
\[
\begin{align*}
f_1 f_2 u_2 &= (1-f_1 u_1) f_1, \\
f_1 f_2 &= (1-f_1 u_1) f_1 u_2^{-1}.
\end{align*}
\]

If we apply what we did in the proof of Theorem 3.22 to the condition $u_1 f_1 + f_2 u_2 = 1$. Evidently, we have
\[
u_1 R + f_2 R = R
\]
and $u_1 R \cap f_2 R = R \cap f_2 R = f_2 R = u_1 (1-f_2 R) R = f_2 \cdot 1R$. Using the
notations in the proof of Theorem 3.22, it shows that

\[ f'_2 = u_1^{-1}f_2, \quad p'_1 = 1 \]

and so \( 1 - p_2 f_2 = 1 - u_2 f_2 = p'_1 r = 1 \cdot r = r. \)

Therefore \( f'_2 r = (u_1^{-1} f_2) (1 - u_2 f_2) \) is another prime factorization of \( f = f_1 f_2. \) We shall show that \((-f_1 u_1 + 1) f_1 u_2^{-1} \) and \( u_1^{-1} f_2 (1 - u_2 f_2)\) are the same prime factorization of \( f. \) For by \( u_1 f_1 + f_2 u_2 = 1 \) it follows that

\[ f_2 u_2 = 1 - u_1 f_1, \]

\[ u_1^{-1} f_2 u_2 u_1 = 1 - f_1 u_1. \]

We obtain \( u_1^{-1} f_2 = (1 - f_1 u_1)(u_2 u_1)^{-1} \), since \( u_1, u_2 \) are unit elements. Thus \( (1 - u_2 f_2) = (u_2 u_1) f_1 u_2^{-1}. \) Hence the corollary is proved.

**Remark 3.34.** By Corollary 3.33 it is not difficult to determine if a prime factorization \( f = f_1 f_2 \cdots f_n \) with the \( f_i \) being linear is absolutely unique or not. For example, if \( f = f_1 f_2 \) is a prime factorization of \( f \) such that the \( f_i \) are linear and \( f_1, f_2 \) do not have the same indeterminates, then the prime factorization \( f = f_1 f_2 \) is absolutely unique. For instance, \( f = (x_1 + x_2)(x_1 + 1) \in \mathbb{Q}[x_1, x_2] \) has absolutely unique prime factorization.

Theorem 3.32 tells us that another prime factorization \( g_1 g_2 \) of \( f = f_1 f_2 \) uniquely determines the condition \( p_1 f_1 + f_2 p_2 = 1 \) with \( \deg p_1 < \deg f_2, \deg p_2 < \deg f_1. \) We remember that there exists a unique \( h_1 = (p_1) \) in \( \mathbb{R}^* \) with \( \deg h_1 < \deg g_1 \) such that

\[(1) \quad f_1 h_1 + g_1 h_2 = 1.\]
If \( f = f_1 f_2 = g_1 g_2 \) has another prime factorization \( t_1 t_2 \), then by Theorem 3.32 there exists a unique \( f_1' \) in \( R^* \) with \( \deg f_1' < \deg t_1 \) such that

\[
(2) \quad f_1' h_1' + t_1 h_2' = 1 \quad \text{for some } h_2' \in R
\]

and \( h_1' \) uniquely determines the condition

\[
p_1' f_1 + f_2 p_2' = 1 \quad \text{for some } p_1', p_2' \in R
\]

where \( \deg p_1' < \deg f_2 \) and \( p_1' = h_1' \).

Claim. \( h_1 \neq h_1' \).

Suppose \( h_1 = h_1' \). By (1) and (2) it follows that

\[
g_1 h_2 = t_1 h_2' .
\]

By the prime factorizations \( f = g_1 g_2 = t_1 t_2 \), it is easy to show that

\[
g_1 R \cap t_1 R = g_1 g_2 R = t_1 t_2 R .
\]

Now \( g_1 h_2 = t_1 h_2' \in g_1 R \cap t_1 R \), it follows that \( g_1 h_2 = t_1 h_2' = g_1 g_2 c \) for some \( c \in R \). Thus

\[
h_2 = g_2 c .
\]

Evidently, \( \deg h_2 \geq \deg g_2 = \deg f_1 \) (see Remark 3.23). By (1) we obtain \( \deg h_1 \geq \deg g_1 \), which contradicts that \( \deg h_1 < \deg g_1 \).

Consequently, we conclude \( h_1 \neq h_1' \), that is, \( p_1 \neq p_1' \). We have proved

Theorem 3.35. Let \( f = f_1 f_2 \) be a prime factorization of \( f \).
which is not absolutely unique. Then each prime factorization $g_1g_2$ of $f$, which is different from $f_1f_2$, determines a different $p_1$ such that $p_1f_1 + f_2p_2 = 1$ with $\deg p_1 < \deg f_2$.

We remember that if $f = f_1f_2$ is a prime factorization of $f$ and there exists $p_1$ in $R^*$ with $\deg p_1 < \deg f_2$ such that

$$p_1f_1 + f_2p_2 = 1$$

for some $p_2 \in R$, then $p_1$ uniquely determines another prime factorization $f_2'p_1$ of $f$, where $p_1R \cap f_2R = p_1f_2'R = f_2p_1R$. Conversely, we are going to show that the prime factorization $f_2'p_1$ uniquely determines the condition $p_1f_1 + f_2p_2 = 1$. By (1) we obtain $0 \neq f_2p_2p_1 = p_1(1 - f_1p_1) \in p_1R \cap f_2R$. Thus

$$p_1(1 - f_1p_1) = p_1f_2't$$

for some $t \in R$,

$$1 - f_1p_1 = f_2't$$

and hence $f_1p_1 + f_2't = 1$.

We note that $\deg p_1 < \deg f_2 = \deg f_2'$ (see Remark 3.23). Hence by Theorem 3.32 the prime factorization $f_2'p_1$ of $f$ uniquely determines the condition $p_1f_1 + f_2p_2 = 1$.

Now by Theorem 3.30, Theorem 3.31, Theorem 3.32, Theorem 3.35 and what we have discussed above, we have the following theorems.

**Theorem 3.36.** If $f = f_1f_2$ is a prime factorization of $f$ in $R$, then there is a bijection between the set \{ $p_1 \in R | p_1f_1 + f_2p_2 = 1$ \} for some $p_2 \in R$ with $\deg p_1 < \deg f_2$ and the set \{ $g_1R | g_1$ is prime, $g_1R \neq f_1R$, $f = g_1g_2$ for some $g_2 \in R$ \}, where $p_1$ corresponds to $g_1R$. 
such that \( p_1 \) determines the prime factorization \( f = g_1g_2 \); that is,
\[
\text{Card } \{ p_1 \in R \mid p_1f_1 + f_2p_2 = 1 \text{ for some } p_2 \in R \text{ with } \deg p_1 < \deg f_2 \} = \text{Card } \{ g_1R \mid g_1 \text{ is prime, } g_1R \neq f_1R, f = g_1g_2 \text{ for some } g_2 \in R \}.
\]

**Theorem 3.37.** If \( f = f_1f_2 \) is a prime factorization of \( f \) and
\[
P = \{ p_1 \in R \mid p_1f_1 + f_2p_2 = 1 \text{ for some } p_2 \in R \text{ with } \deg p_1 < \deg f_2 \}
\]
such that \( \text{Card } P = n \), then \( f \) has \( n + 1 \) different prime factorizations (including \( f_1f_2 \)). In particular, if \( \text{Card } P = 0 \), that is, \( P = \emptyset \), then the prime factorization \( f = f_1f_2 \) is absolutely unique.

**Example 3.38.** Consider \( f = x_1x_2 + x_1 = x_1(x_2x_1 + 1) \in \mathbb{Q}[x_1,x_2] \),
where \( f_1 = x_1 \), \( f_2 = x_2x_1 + 1 \). For the condition \( p_1f_1 + f_2p_2 = 1 \) with \( \deg p_1 < \deg f_2 \) we obtain
\[
p_1 = -x_2, \quad p_2 = 1
\]
and that is all we have for \( p_1 \) satisfying \( p_1f_1 + f_2p_2 = 1 \). Thus \( \text{Card } P = 1 \). By Theorem 3.37 we conclude that \( f \) has two different prime factorizations.

Applying what we have done in the proof of Theorem 3.24 to the example, we have \( p_1R \cap f_2R = (-x_2)R \cap (x_2x_1 + 1)R = (-x_2)(x_1x_2 + 1)R = (x_2x_1 + 1)(-x_2)R \), where \( f_2' = x_1x_2 + 1, p_1' = -x_2 \). Since \( 1 - p_2f_2 = p_1'^R \), we have \( 1 - 1 \cdot (x_2x_1 + 1) = -x_2' \) and so \( r = x_1 \). Therefore, \( f_2' = (x_1x_2 + 1)x_1 \) is the other prime factorization of \( f \).

**Example 3.39.** Let \( F \) be the field of real quaternions.
Consider \( f = x_1^2 + 1 = (x_1 + 1)(x_1 - 1) \in R = F[M] \). For the condition \( p_1(x_1 + 1) + (x_1 - 1)p_2 = 1 \) with \( \deg p_1 < \deg (x_1 - 1) = 1 \) for some \( p_2 \in R \),
it is easy to obtain

\[ p_1 = -\frac{1}{2}i + c j + d k , \]
\[ p_2 = -(-\frac{1}{2}i + c j + d k) , \]

where \( c, d \) are any real numbers. Therefore \( \text{Card } P = \infty \). By Theorem 3.37 we conclude that \( x_1^2 + 1 \) has infinitely many different prime factorizations.

In fact, we also note that

\[ x_1^2 + 1 = (x_1 + i)(x_1 - i) , \]
\[ = (x_1 + j)(x_1 - j) , \]
\[ = (x_1 + k)(x_1 - k) , \]
\[ \ldots \ldots \ldots \]
\[ = [x_1 + (l_1 i + l_2 j + l_3 k)][x_1 - (l_1 i + l_2 j + l_3 k)] , \]

where the \( l_1 \) are real numbers such that \( l_1^2 + l_2^2 + l_3^2 = 1 \) (we know that \( (l_1 i + l_2 j + l_3 k)^2 = -(l_1^2 + l_2^2 + l_3^2) = -1) \).

In view of Theorem 3.32 and Theorem 3.19, we are now able to restate Theorem 3.22 as follows

**Theorem 3.40.** Let \( f = f_1 f_2 \ldots f_n \) be a prime factorization of \( f \) in \( R \). If for each \( i = 1, 2, \ldots, n-1 \) there exist no \( p_{i1}, p_{i2} \) in \( R \) with \( \deg p_{i1} < \deg f_{i+1} \), \( \deg p_{i2} < \deg f_1 \) such that \( p_{i1} f_1 + f_{i+1} p_{i2} = 1 \), then the prime factorization \( f = f_1 f_2 \ldots f_n \) is absolutely unique.
Theorem 3.41. Let \( f = f_1f_2 \) be a prime factorization of \( f \) in \( R = F[M] \). If \( F \) has characteristic 0, then either

1. \( f = f_1^2 \) has absolutely unique prime factorization,
2. \( f = f_1f_2 \) has two prime factorizations (including \( f_1f_2 \)),
3. \( f = f_1f_2 \) has infinitely many prime factorizations.

Proof. It will suffice to prove that if \( f = f_1f_2 \) has more than two prime factorizations, then \( f = f_1f_2 \) has infinitely many prime factorizations.

By hypothesis \( F \) has characteristic 0, then \( F \) contains as a subfield an isomorphic image of the rational numbers. Since it is the smallest subfield of \( F \) containing 1, and has no automorphism except the identity, we identify it with \( Q \), the set of rational numbers. Evidently, \( Q \) is contained in the center of \( F \).

If \( f = f_1f_2 \) has more than two prime factorizations, then by Theorem 3.37 Card \( P \geq 2 \). Therefore there exist \( p_1^{(1)}, p_2^{(1)}, p_1^{(2)}, p_2^{(2)} \) in \( R^* \), where \( p_1^{(1)} \neq p_1^{(2)} \), such that

\[
p_1^{(1)}f_1 + f_2p_2^{(1)} = 1 \quad \text{with} \quad \deg p_1^{(1)} < \deg f_2,
\]
\[
p_1^{(2)}f_1 + f_2p_2^{(2)} = 1 \quad \text{with} \quad \deg p_1^{(2)} < \deg f_2.
\]

Now set \( p_1^{(n)} = \frac{1}{n} p_1^{(1)} + \frac{n-1}{n} p_1^{(2)} \), where \( n = 3, 4, \ldots \); \( i = 1, 2 \).

Clearly, we have

\[
p_1^{(n)}f_1 + f_2p_2^{(n)} = 1 \quad \text{with} \quad \deg p_1^{(n)} < \deg f_2
\]

for \( n = 1, 2, 3, 4, \ldots \). It is easy to show that \( p_1^{(n)} \neq p_1^{(m)} \) whenever \( n \neq m \). For if \( p_1^{(n)} = p_1^{(m)} \), then we have

\[
\frac{1}{n} p_1^{(1)} + \frac{n-1}{n} p_1^{(2)} = \]
\[ \frac{1}{m} p_1^{(1)} + \frac{m-1}{m} p_1^{(2)} \]. It follows that \( p_1^{(1)} = p_1^{(2)} \), which is a contradiction. Thus \( \text{Card } P = \infty \). By Theorem 3.37 \( f = f_1 f_2 \) has infinitely many prime factorizations. Hence, the theorem is proved.

**Theorem 3.42.** Let \( f = f_1 f_2 \) be a prime factorization of \( f \) in \( R \) such that \( f_1 \) is not similar to \( f_2 \), then either

1. \( f = f_1 f_2 \) has absolutely unique prime factorization,

or 2. \( f = f_1 f_2 \) has two prime factorizations (including \( f_1 f_2 \)).

**Proof.** It will suffice to show that if \( f = f_1 f_2 \) is not absolutely unique, then \( f = f_1 f_2 \) cannot have more than two prime factorizations.

Let \( f = f_1 f_2 = g_1 g_2 = h_1 h_2 \) be three prime factorizations of \( f \). Consider \( f_1 f_2 = g_1 g_2 \). By Remark 3.23 we conclude that

1. \( f_1 \sim g_2, \ f_2 \sim g_1 \).

Similarly, considering \( f_1 f_2 = h_1 h_2 \), we conclude that

2. \( f_1 \sim h_2, \ f_2 \sim h_1 \).

By the same reasoning, considering \( g_1 g_2 = h_1 h_2 \), we conclude that

3. \( g_1 \sim h_2, \ g_2 \sim h_1 \).

By (1), (2), and (3) it follows that

\[ f_1 \sim h_2 \sim g_1 \sim f_2, \]

which contradicts the hypothesis that \( f_1 \) is not similar to \( f_2 \).
Therefore we conclude that if the prime factorization \( f = f_1 f_2 \) is not absolutely unique, then \( f \) has only two prime factorizations, and the theorem is proved.

**Theorem 3.43.** Let \( f = f_1 f_2 f_3 \) be a prime factorization of \( f \) in \( R \). If \( f_1 f_2 \) has another prime factorization \( g_1 g_2 \) and \( f_2 f_3 \) has another prime factorization \( h_2 h_3 \); that is, \( f = f_1 f_2 f_3 = g_1 g_2 f_3 = f_1 h_2 h_3 \) represents three prime factorizations of \( f \), then \( g_2 f_3 \) is absolutely unique iff \( f_1 h_2 \) is absolutely unique.

**Proof.** Consider the prime factorizations \( f_1 f_2 = g_1 g_2 \). Let \( g_1 g_2 \) be determined by the condition \( p_1 f_1 + f_2 p_2 = 1 \) for some \( p_1, p_2 \in R^* \) with \( \deg p_1 < \deg f_2 \). By the proof in Theorem 3.24 we see \( p_1 R \cap f_2 R = h R \) for some \( h \in R \), where \( h = p_1 f_2 = f_2 p_1 \) for some \( f_2, p_1 \in R \). Moreover, we have

\[
(1) \quad 1 - p_2 f_2 = p_1 r \quad \text{for some} \quad r \in R,
\]

\[
f_1 f_2 = f_2^r,
\]

where \( f_2^r \) and \( g_1 g_2 \) are the same prime factorization, since we assumed that \( g_1 g_2 \) is determined by \( p_1 \). Similarly, consider the prime factorizations \( f_2 f_3 = h_2 h_3 \). Let \( h_2 h_3 \) be determined by the condition \( t_2 f_2 + f_3 t_3 = 1 \) for some \( t_2, t_3 \in R^* \) with \( \deg t_2 < \deg f_3 \). By the proof in Theorem 3.24 we see \( t_2 R \cap f_3 R = h' R \) for some \( h' \in R \), where \( h' = t_2 f_3 = f_3 t_2 \) for some \( f_3, t_2 \in R \). Moreover, we have

\[
1 - t_3 f_3 = t_2^r' \quad \text{for some} \quad r' \in R,
\]

\[
f_2 f_3 = f_3^{r'},
\]
where $f_3' r'$ and $h_2 h_3$ are the same prime factorization, since we assumed that $h_2 h_3$ is determined by $t_2$. For convenience sake, we write $f = f_1 f_2 f_3 = f_2' r' f_3' = f_1' f_3' r'$ instead of that in the hypothesis. Remember that $f_1 f_2 = f_2' r'$, $f_2 f_3 = f_3' r'$. Now we shall show that $rf_3$ is absolutely unique iff $f_1 f_3'$ is absolutely unique.

If $rf_3$ is not absolutely unique, then there exist $k_2, k_3$ in $R^*$ such that

$$k_2 r + f_3 k_3 = 1 \text{ with } \deg k_2 < \deg f_3.$$  

Thus $f_2 k_2 r + f_2 f_3 k_3 = f_2$, $f_2 k_2 r p_2 + f_3' r' k_3 p_2 = f_2 p_2$ (since $f_2 f_3 = f_3' r'$),

(2) $f_2 k_2 r p_2 + p_1 f_1 + f_3' r' k_3 p_2 = 1 \text{ (since } p_1 f_1 + f_2 p_2 = 1).$

By $p_1 f_1 + f_2 p_2 = 1$ we obtain $0 \neq f_2 p_2 p_1 = p_1 (1-f_1 p_1) \in p_1 R \cap f_2 R$. Thus $f_2 p_2 p_1 = f_2 p_1 w$ for some $w \in R$, and so $p_2 p_1 = p_1 w$. By (1) we have

$$p_1 r p_2 = p_2 (1-f_2 p_2),$$

$$= p_2 p_1 f_1,$$

$$= p_1 w f_1,$$

and so

(3) $r p_2 = w f_1$.

Substituting (3) into (2) we conclude $f_2 k_2 w f_1 + p_1 f_1 + f_3' r' k_3 p_2 = 1$, and so $(f_2 k_2 w p_1) f_1 + f_3' (r' k_3 p_2) = 1$. By Theorem 3.24 $f_1 f_3'$ is not absolutely unique. Conversely, if $f_1 f_3'$ is not absolutely unique, then there exist
In $R$ such that
\[ s_1f_1 + f_3's_2 = 1 \] with $\deg s_1 < \deg f'_3$.

Thus
\[ s_1f_1f_2 + f_3's_2f_2 = f_2, \]
\[ t_2s_1f_2'r + t_2f_3's_2f_2 = t_2f_2' \text{ (since } f_1f_2 = f_2'), \]
\[ t_2s_1f_2'r + t_2f_3's_2f_2 + f_3't_3 = 1 \text{ (since } t_2f_2' + f_3't_3 = 1), \]
and hence
\[ (t_2s_1f_2')r + f_3'(t_2s_2f_2't_3) = 1 \text{ (since } t_2f_3' = f_3't_2'). \] By Theorem 3.24 $rf_3$ is not absolutely unique.

Therefore, we conclude that $rf_3$ is absolutely unique iff $f_1f_3'$ is absolutely unique. The theorem is proved.

**Corollary 3.44.** Let $f = f_1f_2f_3 = g_1g_2g_3$ represent two prime factorizations of $f$ in $R$. If the prime factorization $f_1f_2$ is absolutely unique and $f_1R \neq g_1R$, $f_1R \cap g_1R = hR$ for some $h \in R$, where $h = f_1g_1' = g_1f_1'$ for some $g_1', f_1' \in R$, then $g_2 = f_1'u$, $g_3 = u^{-1}r$ for some unit $u$ and $r \in R$.

**Proof.** By the hypothesis we have $f = f_1f_2f_3 = g_1g_2g_3 = f_1g_1'r = g_1f_1'r$ for some $r \in R$. We note that $f_1f_2f_3$ and $f_1g_1'r$ represent two prime factorizations. For otherwise, we have $r = v f_3'$ for some unit $v$ in $R$ and so $f_1f_2f_3 = g_1f_1'r = g_1f_1'r f_3'$. It shows that $f_1f_2 = g_1f_1'v$ represents two prime factorizations, which contradicts the hypothesis that $f_1f_2$ is absolutely unique. Now $f = f_1g_1'r = f_1f_2f_3 = g_1f_1'r$ represents three prime factorizations of $f$. By Theorem 3.43 $f_1f_2$ is absolutely unique iff $f_1'r$ is absolutely unique. Then by the hypothesis,
we conclude that \( f \) is absolutely unique. Since \( g_1g_2g_3 = g_1f_1' \), hence \( g_2g_3 = f_1' \) and \( g_2 = f_1'u \), \( g_3 = u^{-1}r \) for some unit \( u \) of \( R \). The corollary is proved.

**Examples 3.45.** (1) Consider \( f = (x_1^2 + 1)x_1(x_2^2 + 1) \) in \( Q[x_1, x_2] \). We see that \( f = (x_1^2 + 1)x_1(x_2^2 + 1) = x_1(x_2 + 1)(x_2 - 1)x_1 \) represents three prime factorizations of \( f \), where both \( (x_1^2 + 1)(x_2^2 + 1) \) and \( (x_1^2 + 1)(x_1^2 - 1) \) are absolutely unique.

(2) Consider \( g = (x_1^2 + 1)x_1(x_2^2 - 1) \) in \( F[x_1, x_2] \), where \( F \) is the field of real quaternions. We see that \( g = (x_1^2 + 1)x_1 \)

\[ (x_2^2 - 1) = x_1(x_2 + 1)(x_2 - 1) = (x_1^2 + 1)(x_1 - 1)x_1 \]

where both \( (x_2^2 + 1)(x_2 - 1) \) and \( (x_1^2 + 1)(x_1^2 - 1) \) have infinitely many prime factorizations.

**Lemma 3.46.** Let \( f = f_1f_2 \cdots f_n \) be the prime factorization of \( f \) in \( R \) such that \( f_i \neq f_j \) (not similar) whenever \( i \neq j \). If \( f = g_1g_2 \cdots g_n \) is any prime factorization of \( f \), then \( g_i \sim f_i \) for all \( i \) iff \( g_1g_2 \cdots g_n \) and \( f_1f_2 \cdots f_n \) are the same prime factorization of \( f \).

**Proof.** It is obvious that if \( g_1g_2 \cdots g_n \) and \( f_1f_2 \cdots f_n \) are the same prime factorization of \( f \), then \( g_i \sim f_i \) for all \( i \).

Conversely, if \( g_i \sim f_i \) for all \( i \), then \( g_i \neq f_j \) for \( j \in \{2, 3, \ldots, n\} \), since \( f_i \neq f_j \). We see that \( f_1R \cap g_1R \neq 0 \).

If \( f_1R \neq g_1R \), then \( f_1R + g_1R = R \) and \( f_1R \cap g_1R = hR \) for some \( h \in R \), where \( h = f_1g_1' = g_1f_1' \) for some \( g_1', f_1' \in R \) and \( f_1 \sim f_1', g_1 \sim g_1' \).

Since \( f = f_1f_2 \cdots f_n = g_1g_2 \cdots g_n \in f_1R \cap g_1R \), it follows that \( f_1f_2 \cdots f_n = f_1'g_1' \cdots g_n' \).
for some \( r \in R \) and so

\[
f_1 f_2^r f_3^r \cdots f_n^r = g_1^r.
\]

It implies that \( g_1^r = f_j^r \) for some \( j \in \{2,3,\ldots,n\} \) and hence \( g_1 = f_j \) for some \( j \in \{2,3,\ldots,n\} \), which is a contradiction. Therefore, it must be \( f_1 R = g_1 R \). It shows that \( g_1 = f_1 u_1 \) for some unit \( u_1 \) of \( R \) and so \( f_2 f_3^r \cdots f_n^r = u_1 g_2^r \cdots g_n^r \). By the same reasoning and continuing the process we have

\[
g_2 = u_1^{-1} f_2 u_2, \quad g_3 = u_2^{-1} f_3 u_3, \quad \ldots, \quad g_n = u_{n-1}^{-1} f_n
\]

for some units \( u_2, \ldots, u_{n-1} \) of \( R \). This proves that \( g_1 g_2^r \cdots g_n^r \) and \( f_1 f_2^r \cdots f_n^r \) are the same prime factorization of \( f \). The lemma is proved.

**Remark 3.47.** In view of Lemma 3.46, we note that using the hypothesis in that lemma, the prime factorization \( f = f_1 f_2^r \cdots f_n^r \) is absolutely unique iff any prime factorization \( f = g_1 g_2^r \cdots g_n^r \) of \( f \) implies \( g_1 = f_1 \) for all \( i \).

**Theorem 3.48.** If \( f \sim f' \) and the prime factorization \( f = f_1 f_2^r \cdots f_n^r \) is absolutely unique, where \( f_1 \not\sim f_j \) whenever \( i \neq j \), then \( f' \) has absolutely unique prime factorization. However, if the prime factorization of \( f \) is not absolutely unique, then \( f \) and \( f' \) have the same number of prime factorizations.

**Proof.** By Theorem 2.8 \( f' \) has a prime factorization \( f' = f_1' f_2' \cdots f_n' \), where \( f_1' \sim f_1 \) for all \( i \). Since \( f_1 \not\sim f_j \) whenever \( i \neq j \), \( f_1' \not\sim f_j' \) whenever \( i \neq j \). Assume that \( f' = f_1' f_2' \cdots f_n' \) is not absolutely unique and \( f' = g_1' g_2' \cdots g_n' \) is another prime factorization of \( f' \). By Lemma 3.46
there exists some $j \in \{1, 2, \ldots, n\}$ such that $g_j \neq f_j'$. Since
\[ f' = g_1' g_2' \cdots g_n' \sim f', \] again by Theorem 2.8 $f$ has a prime factorization
\[ f = g_1 g_2 \cdots g_n, \] where $g_i \sim g_i'$ for all $i$. Since $f_j' \neq g_j'$, it
follows that $g_j \neq f_j'$ and so $g_j \neq f_j$. Thus $f = f_1 f_2 \cdots f_n =
\prod_{i=1}^{n} g_i$ represents two prime factorizations of $f$, which contradicts
the hypothesis. Therefore $f'$ has absolutely unique prime factorization
\[ f' = f_1' f_2' \cdots f_n'. \]

However, suppose that $f = f_1 f_2 \cdots f_n$ is not absolutely unique.
Let $f = f_1 f_2 \cdots f_n$ correspond to $f' = f_1' f_2' \cdots f_n'$, where $f_i \sim f_i'$ for
all $i$. By Lemma 3.46 the mapping is well defined. If $f = g_1 g_2 \cdots g_n$
is another prime factorization of $f$, then $f$ corresponds to $f' =
\prod_{i=1}^{n} g_i'$, which is different from the prime factorization $f_1 f_2 \cdots f_n'$
by Lemma 3.46. Therefore, there exists an injection from the set of
different prime factorizations of $f$ into the set of different prime
factorizations of $f'$. By the same reasoning, we have an injection from
the set of different prime factorizations of $f'$ into the set of differ-
ent prime factorizations of $f$. Therefore, $f$ and $f'$ have the same
number of prime factorizations.
CHAPTER IV

ABSOLUTELY UNIQUE FACTORIZATION OF CERTAIN TYPES IN A POLYNOMIAL DOMAIN

As usual, $R$ denotes the polynomial domain $R = F[M]$.

**Definition 4.1.** $f \in R$ is called $\Delta$-partially homogeneous ($\Delta$-p.hom.) iff there exists a finite subset $\Delta \subseteq \Gamma$ such that $f$ is homogeneous in \{\(x_i \mid i \in \Delta\)\}.

We call the degree of $f$ in \{\(x_i \mid i \in \Delta\)\} $\Delta$-degree and denote it by $\deg_\Delta f$.

We observe that if $g$ and $h$ are $\Delta$-hom., then $f = gh$ is $\Delta$-p.hom.. Conversely, if $f = gh$ and $f$ is $\Delta$-p.hom., by comparing the product of the largest primitive monomial of $g$ in \{\(x_i \mid i \in \Delta\)\} and that of $h$ with the product of the least primitive monomial of $g$ in \{\(x_i \mid i \in \Delta\)\} and that of $h$, we see $g$ and $h$ are $\Delta$-p.hom.. Therefore, we conclude that if $f = gh$ in $R$, then $f$ is $\Delta$-p.hom. iff $g$ and $h$ are $\Delta$-p.hom..

**Definition 4.2.** $f \in R$ is called $\Delta$-nice iff it has a prime factorization $f = p_1 p_2 \cdots p_n$ such that $\deg_\Delta p_i > 0$ for all $i$.

We see that if $f$ is $\Delta$-p.hom., then $f$ has a factorization $f = f_1 f_2 \cdots f_n$, where each $f_i$ is $\Delta$-nice or of $\deg_\Delta f_i = 0$ and $\deg_\Delta f_i = 0$ implies $\deg_\Delta f_{i+1} > 0$; $\deg_\Delta f_i > 0$ implies $\deg_\Delta f_{i+1} = 0$. We note that if $f_i$ is $\Delta$-nice, then it is also $\Delta$-p.hom., since $f$ is $\Delta$-p.hom..

For example, $f = (x_1 + x_2)x_2(x_3 + 1)x_1 \in Q[x_1, x_2, x_3]$ is $\Delta$-p.hom. in \{\(x_1, x_2\)\}. 

with $\deg_\Delta f = 3$, where $f_1 = (x_1 + x_2)x_3(\Delta \text{-nice and } \Delta \text{-p.hom.})$, $f_2 = x_3 + 1$ ($\deg_\Delta f_2 = 0$), $f_3 = x_1(\Delta \text{-nice and } \Delta \text{-p.hom.})$.

**Lemma 4.3.** If $f$ is $\Delta \text{-p.hom.}$ and $f = gh$ is a prime factorization of $f$ such that $h$ is $\Delta \text{-p.hom.}$ with $\deg_\Delta h > 0$ and $\deg_\Delta g = 0$, then $f = gh$ is absolutely unique.

**Proof.** Let $f = gh = h'g'$ be two prime factorizations of $f$.

By Remark 3.23, $g \prec g'$, $h \prec h'$ and $\deg g = \deg g'$, $\deg h = \deg h'$.

(1) Suppose $\deg g > \deg h$. Let $a_g$ be the largest primitive monomial in $g$, $a_h$, $a_{h'}$, and $a_{g'}$, are defined similarly. By $gh = h'g'$ it necessarily has

$$a_g a_h = a_{h'} a_{g'}.$$

Since $\deg g > \deg h = \deg h'$, it follows that $a_g = a_{h'} \cdot \beta$ for some primitive monomial $\beta$. By Theorem 3.14 $\deg_\Delta a_{h'} = \deg_\Delta a_h > 0$, which implies that $\deg_\Delta a_g = \deg_\Delta (a_h \cdot \beta) > 0$ and $\deg_\Delta \beta > 0$. It contradicts the hypothesis. Hence $f = gh$ is absolutely unique in this case.

(2) Suppose $\deg g < \deg h$. Since $\deg g < \deg h = \deg h'$, by the Euclidean algorithm we obtain

$$h' = g p_1 + r_1, \quad h = p_1 g' + r_1', \quad r_1 g' = g r_1'$$

with $\deg p_1 > 0$, $\deg r_1 < \deg g$, $\deg r_1' < \deg g'$ and $r_1, r_1' \in \mathbb{R}^*$. For if $r_1 = 0$, then $h' = g p_1$. Since both $h'$ and $g$ are prime, $p_1$ must be a unit, and then $\deg h = \deg h' = \deg g$, which contradicts the assumption. Similarly, $r_1' \neq 0$. 
Claim. \( p_1 \) is \( \Delta\)-p.hom.

Let \( p_1 = a_k \beta_k + a_{k-1} \beta_{k-1} + \cdots + a_1 \beta_1 + a_0 \beta_0 \), where \( a_j \in F^* \), \( \beta_j \in M \) and \( \beta_k > \beta_{k-1} > \cdots > \beta_0 \). Since \( h \sim h' \), by Theorem 3.14 \( h' \) is \( \Delta\)-p.hom. By \( h' = gp_1 + r_1 \) and \( \deg \Delta g = 0 \), we note that \( \deg \Delta h' = \deg \Delta \beta_k \). Let \( \beta_1 \) be the largest primitive monomial in \( p_1 \) such that \( \deg \Delta \beta_1 \neq \deg \Delta h' \), then \( g \beta_1 \) is a primitive monomial in \( h' \). Thus \( \deg \Delta g \beta_1 = \deg \Delta \beta_1 \neq \deg \Delta h' \), which contradicts that \( h' \) is \( \Delta\)-p.hom. Therefore \( p_1 \) is \( \Delta\)-p.hom. with \( \deg \Delta p_1 = \deg \Delta h' \).

We see that \( r_1 = h' - gp_1 \) is also \( \Delta\)-p.hom. with \( \deg \Delta r_1 = \deg \Delta h' > 0 \). Since \( \deg r_1 < \deg g = \deg g' \), applying the same reasoning in (1) to \( r_1 g' = g r_1' \) we have a contradiction.

Therefore, \( f = gh \) is absolutely unique.

Lemma 4.4. If \( f \) is \( \Delta\)-p.hom. and \( f = hg \) is a prime factorization of \( f \) such that \( h \) is \( \Delta\)-p.hom. with \( \deg \Delta h > 0 \) and \( \deg \Delta g = 0 \), then \( f = hg \) is absolutely unique.

Proof. Similar to the proof of Lemma 4.3.

Lemma 4.5. Let \( f \) be \( \Delta\)-p.hom.. If \( f \) has a left prime factor with \( \Delta\)-degree > 0 (\( \Delta\)-degree = 0), then \( f \) does not have any left prime factor with \( \Delta\)-degree = 0 (\( \Delta\)-degree > 0).

Proof. Let \( f = c f_1 = d g_1 \) be two factorizations of \( f \) such that \( c, d \) are prime and \( \deg \Delta c > 0, \deg \Delta d = 0 \). Note that \( c \) is \( \Delta\)-p.hom. Since \( cR \cap dR \neq 0 \) and \( cR \nsubseteq dR \), \( dR \nsubseteq cR \), we have \( cR + dR = R \), \( cR \cap dR = hR \) for some \( h \in R \), where \( h = cd' = dc' \) for some \( d', c' \in R \).
We see that $c \sim c'$, $d \sim d'$, and $c', d'$ are prime. Thus $\deg_{\Delta} d' = \deg_{\Delta} d = 0$ and $\deg_{\Delta} c' = \deg_{\Delta} c > 0$. It shows that $cd' = dc'$ represents two prime factorizations, which is a contradiction by Lemma 4.4. The lemma is proved.

**Theorem 4.6.** Let $f$ be $\Lambda$-p.hom. in $R$. If $f = f_1 f_2 \cdots f_n$ is the factorization of $f$ such that each $f_i$ is $\Lambda$-nice or of $\Lambda$-degree = 0 and $\deg_{\Lambda} f_i = 0$ implies $\deg_{\Lambda} f_{i+1} > 0$; $\deg_{\Lambda} f_i > 0$ implies $\deg_{\Lambda} f_{i+1} = 0$, then this factorization of $f$ is absolutely unique.

**Proof.** Let $f = f_1 f_2 \cdots f_n = g_1 g_2 \cdots g_m$ be two factorizations of $f$ satisfying the hypothesis. By Lemma 4.5 either both $f_1$ and $g_1$ are $\Lambda$-nice or both $f_1$ and $g_1$ are of $\Lambda$-degree = 0. We see that if $f_1$ is $\Lambda$-nice, it is $\Lambda$-p.hom., since $f$ is $\Lambda$-p.hom.. Note that $f_1 R \cap g_1 R \neq 0$. Suppose $f_1 R \neq g_1 R$.

(1) Both $f_1$ and $g_1$ are $\Lambda$-nice. If $f_1 R < g_1 R$, then $f_1 = g_1 r$ for some nonunit $r \in R^*$. Since $f_1$ is $\Lambda$-nice, $r$ is $\Lambda$-nice. Now we have $rf_2 \cdots f_n = g_2 \cdots g_m$, which is $\Lambda$-p.hom., where $g_2$ is of $\deg_{\Lambda} g_2 = 0$. It is a contradiction by Lemma 4.5. Thus $f_1 R \neq g_1 R$. Similarly, $g_1 R \neq f_1 R$. Thus $f_1 R + g_1 R = rR$ for some $r \in R$. We note that $rR \neq f_1 R$, $rR \neq g_1 R$. It follows that $f_1 = rt_1$, $g_1 = rt_2$ for some nonunit elements $t_1, t_2 \in R^*$. Hence $t_1 R + t_2 R = R$. It is impossible, since $t_1, t_2$ are $\Lambda$-nice, which have zero constant term. Therefore, $f_1 R = g_1 R$ in this case.

(2) Both $f_1$ and $g_1$ are of $\Lambda$-degree = 0. By the same argument in (1) we conclude that $f_1 R \neq g_1 R$, $g_1 R \neq f_1 R$. Then $f_1 R \cap g_1 R =$
f_1 g_1' R = f_1 f_1' R for some nonunit elements g_1', f_1' \in R^*$. By Corollary 3.2, either \ldeg g_1' = 0 or \ldeg f_1' = 0. Say \ldeg g_1' = 0. We obtain $f_1 f_2 \cdots f_n = f_1 g_1's$ for some $s \in R$. Thus $f_2 f_3 \cdots f_n = g_1's$, which is \Delta-p.hom. Since a \Delta-nice factor of $f$ is \Delta-p.hom. and each factor of $g_1'$ is of low-degree $= 0$, it must have $\deg_{\Delta} g_1' = 0$. It is a contradiction by Lemma 4.5, since $f_2$ is \Delta-nice ($\deg_{\Delta} f_2 > 0$). Similarly, if \ldeg f_1' = 0, we have a contradiction. Therefore, $f_1 R = g_1 R$ in this case.

Hence $f_1 R = g_1 R$ and then $g_1 = f_1 u_1$ for some unit $u_1$ of $R$. It follows that $f_2 f_3 \cdots f_n = u_1 g_2 g_3 \cdots g_m$. By the same reasoning as in (1), (2) and continuing the process, we have $n = m$ and $g_2 = u_1^{-1} f_2 u_2$, $g_3 = u_2^{-1} f_3 u_3$, \ldots, $g_n = u_n^{-1} f_n$ for some units $u_2, u_3, \ldots, u_{n-1}$ of $R$. Therefore the theorem is proved.

**Example 4.7.** Consider $f = (x_3 + x_4 x_1) x_1 (x_2 x_1 + 1)x_3 \in Q[x_1, x_2, x_3, x_4]$, \Delta-p.hom. in $(x_3, x_4)$. The prime factorization is not absolutely unique, since $f = (x_3 + x_4 x_1) x_1 (x_2 x_1 + 1)x_3 = (x_3 + x_4 x_1)(x_1 x_2 + 1)x_1 x_3$. If we consider the factorization $f = (x_3 + x_4 x_1)[x_1 (x_2 x_1 + 1)]x_3$ of the stated type in Theorem 4.6, it is absolutely unique.

**Definition 4.8.** If $f \in R$ has a factorization $f = f_1 f_2 \cdots f_n$ into primes, we say that $f$ has length $n$.

By Theorem 2.6 a nonzero, nonunit element of $R$ has unique length.

Let $C_1 = \{p \in R | p$ is prime with zero constant term\}$,

$C_2 = \{q \in R | q$ is prime with nonzero constant term\}$,
Lemma 4.9. If \( f \in \mathbb{R}^* \) has no \( S_1 \) \( \ell \)-factor (left factor) and \( f_1 \) is an \( S_2 \) \( \ell \)-factor with minimal length, then the factorization \( f = f_1g \) is absolutely unique.

Proof. Let \( f = f_1g = f_2h \) be two factorizations of \( f \) satisfying the hypothesis. Since both \( f_1 \) and \( f_2 \) have the minimal length in the factorization, they have the same length. We note that \( g \) and \( h \) have \( s_1 \) \( \ell \)-factors. So we write the two factorizations of \( f \) in the following forms

\[
f = f_{11}f_{12}\cdots f_{1k}g_{11}g_{2},
\]

\[
= f_{21}f_{22}\cdots f_{2k}h_{11}h_{2},
\]

where the \( f_{1j} \) are \( C_2 \)-prime and \( g_{11}, h_{11} \) are \( C_1 \)-prime.

Assume \( Rg_2 \neq Rh_2 \). Evidently, \( g_2 \) and \( h_2 \) have the same length. So \( Rg_2 \neq Rh_2, Rh_2 \neq Rg_2 \). Then by \( Rg_2 \cap Rh_2 \neq \emptyset \) we have

\( Rg_2 \cap Rh_2 = Rh_2^{'}g_2 = Rh_2^{'}h_2 \) for some nonunit elements \( h_2^{'} \), \( g_2^{'} \in \mathbb{R} \), where \( h_2^{'}g_2 = g_2^{'}h_2^{'} \). By Corollary 3.5 either \( \text{lodeg } h_2^{'} = 0 \) or \( \text{lodeg } g_2^{'} = 0 \).

Say \( \text{lodeg } h_2^{'} = 0 \). We obtain \( h_2^{'} \in S_2 \). We see

\[
f = f_{11}f_{12}\cdots f_{1k}g_{11}g_{2} =
\]

\[
f_{21}f_{22}\cdots f_{2k}h_{11}h_{2} = rh_2^{'}g_2 = rg_2^{'}h_2^{'} \]

for some \( r \in \mathbb{R} \). Thus

\[
f_{11}f_{12}\cdots f_{1k}g_{11} = rh_2^{'},
\]

\[
f_{21}f_{22}\cdots f_{2k}h_{11} = rg_2^{'}.
\]
Since \( \text{ldeg} \ g_{11} > 0 \) and \( \text{ldeg} \ h_2' = 0 \), \( r \) must be a nonunit element.

Let \( r \) have the factorization \( r = r_{11}r_{12} \cdots r_{1m} \) into primes. We note that \( m \leq k \). Since \( g_{11} \) is \( C_1 \)-prime, by \( f_{11}f_{12} \cdots f_{1k}g_{11} = r_{11}r_{12} \cdots r_{1m}h_2' \) we conclude \( g_{11} \sim r_{1n} \) for some \( n \in \{1, 2, \ldots, m\} \). It implies that \( r_{1n} \) is \( C_1 \)-prime and the \( r_{1j} \) are \( C_2 \)-prime for \( j \neq n \), since each \( f_{1j} \) is \( C_2 \)-prime. Thus \( r_{11}r_{12} \cdots r_{1n-1} \in S_2 \) with \( r_{1n} \) being \( C_1 \)-prime and length \( (r_{11}r_{12} \cdots r_{1n-1}) = n - 1 \leq m - 1 \leq k - 1 < k \), which contradicts that \( k \) is the least length of the \( S_2 \) \( \ell \)-factor of \( f \). Similarly, if say \( \text{ldeg} \ g_2' = 0 \), we have a contradiction.

Therefore we conclude that \( Rg_2 = R_{n2} \). We obtain \( h_2 = u_2g_2 \) for some unit \( u_2 \) and

\[
f_{11}f_{12} \cdots f_{1k}g_{11} = f_{21}f_{22} \cdots f_{2k}h_1u_2.
\]

Therefore \( Rg_{11} \cap R_{n1}u_2 \neq 0 \). If \( Rg_{11} \neq R_{n1}u_2 \), then it has \( Rg_{11} + R_{n1}u_2 = R \), which is impossible, since both \( \text{ldeg} \ g_{11} > 0 \) and \( \text{ldeg}(h_{11}u_2) > 0 \). Hence \( Rg_{11} = R_{n1}u_2 \) and then \( h_{11}u_2 = u_1g_{11} \) for some unit \( u_1 \). It follows that \( f_2 = f_{21}f_{22} \cdots f_{2k} = f_{11}f_{12} \cdots f_{1k}u_1^{-1} = f_1u_1^{-1} \), \( h = h_{11}h_2 = u_1g_{11}u_2^{-1}g_2 = u_1g_{11}g_2 = u_1g_2 \). Therefore \( f_1g_2 \) is absolutely unique.

Lemma 4.10. If \( f = f_1g \) is a factorization of \( f \) such that \( f_1 \) is an \( S_1 \) \( \ell \)-factor with maximal length, then the factorization \( f = f_1g \) is absolutely unique.

Proof. Let \( f = f_1g = f_2h \) be two factorizations of \( f \) satisfying the hypothesis. Since both \( f_1 \) and \( f_2 \) have the maximal
length in the factorization, they have the same length. So \( f_1R \neq f_2R \) and \( f_2R \neq f_1R \).

Assume \( f_1R \neq f_2R \). We may assume without loss of generality that \( f_1 \) and \( f_2 \) are left relatively prime (since in \( R \), \( abR + acR = aR \) iff \( bR + cR = R \) for \( a, b, c \in R \)). Then \( f_1R + f_2R = R \), which is impossible, since both \( \text{lodeg} f_1 > 0 \) and \( \text{lodeg} f_2 > 0 \).

Consequently, we conclude \( f_1R = f_2R \) and then \( f_2 = f_1u \), \( h = u^{-1}g \) for some unit \( u \) of \( R \). Therefore \( f_1g \) is absolutely unique.

**Remark 4.11.** (1) In view of the proof in Lemma 4.10, we note that if \( f = f_1g = f_2h \) such that \( f_1 \) and \( f_2 \) are \( S_1 \) \( \ell \)-factors with the same length, then \( f_2 = f_1u \), \( h = u^{-1}g \) for some unit \( u \) of \( R \).

(2) If \( f = f_1g = f_2h \) such that \( f_1 \) and \( f_2 \) are \( S_2 \) \( \ell \)-factors with the same length, the conclusion in (1) does not follow. For example, consider \( f = (x_1+1)x_1(x_1^2+x_1+1) \in \mathbb{Q}[x_1, x_2] \). We note \( f = (x_1+1)x_1(x_1^2+x_1+1) = (x_1^2+x_1+1)x_1(x_1+1) \). But \((x_1^2+x_1+1) \neq (x_1+1)u\) for any unit \( u \) of \( R \). Moreover, \( f = (x_1+1)x_1(x_1^2+x_1+1) \) does not satisfy the hypothesis in Lemma 4.10, since \( f = x_1(x_1+1)(x_1^2+x_1+1) \).

**Lemma 4.12.** If \( f = f_1g \) is a factorization of \( f \) such that \( f_1 \) is an \( S_2 \) \( \ell \)-factor with maximal length, then the factorization \( f = f_1g \) is absolutely unique.

**Proof.** Let \( f = f_1g = f_2h \) be two factorizations of \( f \) satisfying the hypothesis. By the same reasoning as before we conclude that \( f_1R \neq f_2R \), \( f_2 \neq f_1R \).
Assume \( f_1 R \neq f_2 R \). Then by \( f_1 R \cap f_2 R \neq 0 \), we have

\[
f_1 R \cap f_2 R = f_1 f_2' R = f_2 f_1' R
\]

for some \( f_2', f_1' \in R \), where \( f_1 f_2' = f_2 f_1' \) and \( f_2', f_1' \) are nonzero, non-unit elements of \( R \). By Corollary 3.2 either \( \text{lodeg} f_2' = 0 \) or \( \text{lodeg} f_1' = 0 \). Say \( \text{lodeg} f_2' = 0 \). Therefore \( f_2' \in S_2 \). Now we see \( f = f_1 g = f_2 h = f_1 f_2' r \) for some \( r \in R \) and \( f_1 f_2' \in S_2 \) with length \( (f_1 f_2') > \text{length } f_1 \), which contradicts the hypothesis. Similarly, if \( \text{lodeg} f_1' = 0 \), we have a contradiction.

Therefore we conclude \( f_1 R = f_2 R \) and then \( f_2 = f_1 u \), \( h = u^{-1} g \) for some unit \( u \) of \( R \). Hence \( f_1 g \) is absolutely unique.

**Lemma 4.13.** If \( f \in R^* \) has no \( S_2 \) \( \ell \)-factor and \( f_1 \) is an \( S_1 \) \( \ell \)-factor with minimal length, then the factorization \( f = f_1 g \) is absolutely unique.

**Proof.** Let \( f = f_1 g = f_2 h \) be two factorizations of \( f \) satisfying the hypothesis. Since \( f_1 \) and \( f_2 \) are \( S_1 \) \( \ell \)-factors with the same length, by (1) of Remarks 4.11 we obtain \( f_2 = f_1 u \), \( h = u^{-1} g \) for some unit \( u \) of \( R \). Therefore, \( f_1 g \) is absolutely unique.

**Definition 4.14.** Let \( f = f_1 f_2 \cdots f_n \). We call this factorization of \( f \) a \((\text{max } S_1, \text{min } S_2)\) factorization provided:

1. If \( f_1 \in S_1 \), then \( f_{i+1} \in S_2 \); if \( f_1 \in S_2 \), then \( f_{i+1} \in S_1 \);
2. Each \( f_i \in S_1 \) is of maximal length;
3. Each \( f_i \in S_2 \) is of minimal length;
4. \( f_1 \in S_1 \) if \( f \) has an \( S_1 \) \( \ell \)-factor; otherwise, \( f_1 \in S_2 \).
A \((\max S_1, \max S_2)\) factorization, etc., is defined similarly.

**Theorem 4.15.** If \(f = f_1 f_2 \cdots f_n\) is a \((\max S_1, \min S_2)\) factorization of \(f\), then the factorization is absolutely unique.

**Proof.** The theorem is an immediate consequence of Lemma 4.10 and Lemma 4.9.

**Example 4.16.** (1) Consider \(f = (x_1 + x_2)(x_3 + 1)x_3(x_2 x_3 + x_2 + 1)\) \((x_1 x_2 + 1)x_1 \in \mathbb{Q}[x_1, x_2, x_3]\). If we write \(f\) in the \((\max S_1, \min S_2)\) factorization, we have \(f = [(x_1 + x_2) x_3][(x_3 + 1)(x_2 x_3 + x_2 + 1)](x_1)[x_2 x_1 + 1]\), where \(f_1 = (x_1 + x_2)x_3 \in S_1\), \(f_2 = (x_3 + 1)(x_2 x_3 + x_2 + 1) \in S_2\), \(f_3 = x_1 \in S_1\), \(f_4 = x_2 x_1 + 1 \in S_2\). Although the factorization \(f = [(x_1 + x_2) x_3][(x_3 + 1)(x_2 x_3 + x_2 + 1)](x_1)[x_2 x_1 + 1]\) is also a \((\max S_1, \min S_2)\) factorization, we note that the factorization is the same as that factorization above, since \((x_3 + 1)(x_2 x_3 + x_2 + 1) = (x_3 x_2 + x_2 + 1)(x_3 + 1)\).

(2) Consider \(g = (x_1 + 1)(x_4 x_1 + x_4 + 1)(x_3 + 1)(x_1 x_3 + x_1 + x_3) \in \mathbb{Q}[x_1, x_3, x_4]\). If we write \(g\) in the \((\max S_1, \min S_2)\) factorization, we have \(g = [x_1 x_4 + x_4 + 1][x_1 x_3 + x_1 + x_3][(x_1 + 1)(x_3 + 1)]\), where \(g_1 = x_1 x_4 + x_4 + 1 \in S_2\), \(g_2 = x_1 x_3 + x_1 + x_3 \in S_1\), \(g_3 = (x_1 + 1)(x_3 + 1) \in S_2\).

**Remark 4.17.** In (2) of Example 4.16 we see that \(g_1 = x_1 x_4 + x_4 + 1\) is an \(S_2\) \(\ell\)-factor with minimal length. By \(g = (x_1 + 1)(x_4 x_1 + x_4 + 1)(x_3 + 1)(x_1 x_3 + x_1 + x_3)\), \(x_1 + 1\) is an \(S_2\) \(\ell\)-factor, but \(x_1 + 1 \notin g_1\).

**Theorem 4.18.** If \(f = f_1 f_2 \cdots f_n\) is a \((\max S_1, \max S_2)\) factorization
or a \((\text{max } S_2, \text{max } S_1)\) factorization, then the factorization is absolutely unique.

**Proof.** The theorem is an immediate consequence of Lemma 4.10 and Lemma 4.12.

**Example 4.19.** Consider \(f = x_1(x_1^2+x_1+1)(x_1+1)x_1x_3(x_1x_3+x_3+1) \in Q[x_1,x_3]\). If we write \(f\) in the \((\text{max } S_1, \text{max } S_2)\) factorization, we have \(f = [x_1x_1][(x_1^2+x_1+1)(x_1+1)(x_1x_3+x_3+1)]x_3\), where \(f_1 = x_1x_1 \in S_1\), \(f_2 = (x_1^2+x_1+1)(x_1+1)(x_1x_3+x_3+1) \in S_2\), \(f_3 = x_3 \in S_1\). The factorization is absolutely unique. If we write \(f\) in the \((\text{max } S_2, \text{max } S_1)\) factorization, we have \(f = [(x_1^2+x_1+1)(x_1+1)]x_1x_3x_3[(x_1x_3+x_3+1)]\), where \(f_1 = (x_1^2+x_1+1)(x_1+1) \in S_2\), \(f_2 = x_1x_3 \in S_1\), \(f_3 = x_3 \in S_1\), \(f_4 = (x_2x_3+x_3+1) \in S_2\), \(f_5 = x_1 \in S_1\). Note that \(f\) does not have an \(S_2\) \(\ell\)-factor.

**Theorem 4.20.** If \(f = f_1f_2\cdots f_n\) is a \((\text{max } S_2, \text{min } S_1)\) factorization, then the factorization is absolutely unique.

**Proof.** The theorem is an immediate consequence of Lemma 4.12 and Lemma 4.13.

**Examples 4.21.** (1) Consider \(f = (x_1+x_2)x_3(x_2x_3+x_2+1)(x_3+1)\) \(x_1(x_2x_3+1) \in Q[x_1,x_2,x_3]\). If we write \(f\) in the \((\text{max } S_2, \text{min } S_1)\) factorization, we have \(f = [x_1+x_2][x_3+1][x_3][(x_2x_3+x_2+1)(x_1x_3+1)]x_1\), where \(f_1 = x_1 + x_2 \in S_1\), \(f_2 = x_3 + 1 \in S_2\), \(f_3 = x_3 \in S_1\), \(f_4 = (x_2x_3+x_2+1)(x_1x_3+1) \in S_2\), \(f_5 = x_1 \in S_1\). Note that \(f\) does not have an \(S_2\) \(\ell\)-factor.
(2) Consider \( g = (x_1+1)x_2(x_1x_2+x_2+1)(x_1x_4+x_4+1) \)
\((x_1x_3+x_1x_3)(x_1+1) \in \mathbb{Q}[x_1,x_2,x_3,x_4]\). If we write \( g \) in the \((\max S_2, \min S_1)\)
factorization, we have \( g = [(x_1+1)(x_2x_1+x_2+1)](x_2)(x_1x_4+x_4+1)(x_1+1) \)
\([x_3x_1+x_1x_3]\), where \( g_1 = (x_1+1)(x_2x_1+x_2+1) \in S_2 \), \( g_2 = x_2 \in S_1 \),
g_3 = (x_1x_4+x_4+1)(x_1+1) \in S_2 \), \( g_4 = x_3x_1 + x_1 + x_3 \in S_1 \). The factorization
is absolutely unique.
CHAPTER V

ABSOLUTELY UNIQUE PRIMARY FACTORIZATION IN A POLYNOMIAL DOMAIN

As before, \( R \) denotes the polynomial domain \( R = F[M] \) and \( U(R) \) denotes the group of units of \( R \).

1. Definitions and Consequences

**Definition 5.1** [1]. An element \( f \in R^* \setminus U(R) \) is called \( p \)-primary for some prime \( p \) iff every prime factor of \( f \) is similar to \( p \).

Evidently, if \( f \in R \) is prime, then \( f \) is \( f \)-primary. If \( f = p_1 p_2 \cdots p_n \) is a factorization of \( f \) into primes, then \( f \) is \( p \)-primary for some prime \( p \) iff \( p_i \sim p_j \) for all \( i, j \).

**Definition 5.2** [1]. We call \( f = f_1 f_2 \cdots f_n \) a primary factorization of \( f \) iff each \( f_i \) is \( p_i \)-primary for some prime \( p_i \) and \( p_i \not\sim p_j \) whenever \( i \neq j \).

By the definition we see that \( f \in R^* \) has a primary factorization iff \( f \) can be represented as a product of prime factors such that the similar prime factors are consecutive.

**Remarks 5.3.** (1) By Theorem 2.8 and Definition 5.1 we note that if \( f \in R \) is \( p \)-primary for some prime \( p \) and \( q \sim f \), then \( q \) is \( p \)-primary.

(2) By Theorem 2.8 and Definition 5.2 we note that if \( f = f_1 f_2 \cdots f_n \) is a primary factorization of \( f \) in \( R \), then \( f_i \not\sim f_j \).
whenever \( i \neq j \).

It is known that any primary factorization of a nonunit element in \( R^* \) is unique up to order of factors and similarity as follows (see Beauregard and Johnson [1]).

**Theorem 5.4.** If \( f \in R^* \setminus U(R) \) has two primary factorizations
\[
f = f_1 f_2 \cdots f_m = g_1 g_2 \cdots g_n,
\]
then \( m = n \) and \( f_1 \sim g_0(1) \) for some permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \).

2. **Absolutely Unique Primary Factorization**

If \( f \in R \) has a primary factorization \( f = f_1 f_2 \cdots f_n \), then \( f \) can be represented as a product
\[
(1) \quad f = (f_1 u_1)(u_1^{-1} f_2 u_2) \cdots (u_{n-1}^{-1} f_n)
\]
for any units \( u_1, u_2, \ldots, u_{n-1} \) of \( R \).

**Definition 5.5.** We call a primary factorization \( f = f_1 f_2 \cdots f_n \) of \( f \) **absolutely unique** if every other representation of \( f \) as a primary factorization has the form (1) above for some choice of the units \( u_1, u_2, \ldots, u_{n-1} \) of \( R \).

We shall now discuss necessary and sufficient conditions for a primary factorization of an element \( f \in R \) to be absolutely unique.
Lemma 5.6. If \( p \) is a prime factor of \( gh \), then \( p \) is similar to either a prime factor of \( g \) or a prime factor of \( h \).

Proof. Obvious.

It is well-known that the following lemma is true in \( R \).

Lemma 5.7. If \( fR \cap gR = fg'R = gf'R \) and \( fR + gR = hR \), where \( fg' = gf' \), then there exist \( f_o, g_o \in R \) such that \( f_oR \cap g_oR = f_o'g_R = g_o'f'R \) and \( f_oR + g_oR = hR \), where \( f_o'g' = g_o'f' \), \( f = hf_o \), \( g = hg_o \) and \( f_o \sim f', g_o \sim g' \).

Lemma 5.8. Let \( f = f_1f_2 \cdots f_n \) be a primary factorization of \( f \). If \( h \) is a factor of \( f \), say \( f = ghk \) for some \( g, k \in R \), such that \( h \sim f_i \) for some \( i \), where \( f_i \) is \( p_i \)-primary for some prime \( p_i \), then \( h \) is \( p_i \)-primary and no prime factor of \( g \) or \( k \) is similar to \( p_i \).

Proof. By (1) of Remarks 5.3 \( h \) is \( p_i \)-primary. If \( f_i \) has length \( k \), by the definition of primary factorization and Remark 2.7 \( f \) has exactly \( k \) prime factors which are similar to \( p_i \). By Theorem 2.8 \( h \) has length \( k \) and its \( k \) prime factors are similar to \( p_i \). Since \( h \) is a factor of \( f = ghk \), so no prime factor of \( g \) or \( k \) is similar to \( p_i \).

Lemma 5.9. If \( f \) has a primary factorization and \( f = h_1g \), where \( h_1 \) is \( q_1 \)-primary for some prime \( q_1 \) and no prime factor of \( g \) is similar to \( q_1 \), then \( g \) has a primary factorization; that is, \( h_1g \) has
a primary factorization with $h_1$ as the first primary factor.

**Proof.** By hypothesis, let $f = f_1 f_2 \cdots f_n$ be a primary factorization of $f$, where each $f_i$ is $p_i$-primary for some prime $p_i$. Of course $p_i \neq p_j$ whenever $i \neq j$ by definition. We shall prove the lemma by mathematical induction on $n$.

When $n = 2$, we have $f = f_1 f_2 = h_1 g$. Clearly, $g$ is a primary factor of $f$ and so the lemma is true in this case.

Assume that the lemma is true when $n = k$. Consider $f = f_1 f_2 \cdots f_k f_{k+1} = h_1 g$. Note that $f_1 R \cap h_1 R \neq 0$.

(1) If $f_1 R = h_1 R$, then $f_1 = h_1 u$ for some unit $u$ of $R$.

Thus $uf_2 f_3 \cdots f_k f_{k+1} = g$ and so the lemma is true.

(2) Assume $f_1 R \neq h_1 R$.

(A) Suppose $f_1 R < h_1 R$, then $f_1 = h_1 t_1$ for some nonunit element $t_1$ of $R$. It shows that $f_1, h_1$ and $t_1$ are $p_1$-primary and $p_1 \sim q_1$.

By $t_1 f_2 f_3 \cdots f_k f_{k+1} = g$, we see that a prime factor of $t_1$ is also a prime factor of $g$. Thus $g$ has a prime factor similar to $q_1$, which contradicts the hypothesis. Therefore (A) is not true.

(B) Suppose $h_1 R < f_1 R$, then $h_1 = f_1 t_2$ for some nonunit element $t_2$ of $R$. It shows that $f_1$ and $t_2$ are $q_1$-primary and $q_1 \sim p_1$.

By $f_2 f_3 \cdots f_k f_{k+1} = t_2 g$, we see that a prime factor of $t_2$ is similar to a prime factor of some $f_j$ for $j \neq 1$ by Lemma 5.6. It follows that $p_j \sim q_1 \sim p_1$, where $j \neq 1$. It is a contradiction. Therefore (B) is not true.

(C) If $f_1 R \cap h_1 R \neq f_1 R$, then we have $f_1 R \cap h_1 R = f_1 h_1 R = h_1 f_1 R$ and $f_1 R + h_1 R = s R$ for some $h_1, f_1, s \in R$, where $f_1 h_1 = h_1 f_1$. We note that $f_1, h_1$ are nonunit elements and $s R \neq f_1 R, h_1 R$. 
Suppose $s$ is a nonunit element. By Lemma 5.7 there exist $r_1, r_2 \in R$ such that

$$r_1 R \cap r_2 R = r_1 h_1' R = r_2 f_1' R,$$

where $r_1 h_1' = r_2 f_1'$, $f_1 = s r_1$, $h_1 = s r_2$ and $r_1 \sim f_1'$, $r_2 \sim h_1'$. We note that $r_1$ and $r_2$ are nonunits. We observe that $f_1$, $h_1$ and $r_1$ are $q_1$-primary, since $f_1$ and $h_1$ have the same factor $s$. Since $f = f_1 f_2 \cdots f_{k+1} = h_1 g \in f_1 R \cap h_1 R$, we have $f_1 f_2 \cdots f_{k+1} = h_1 g = f_1 h_1' t_3 = h_1 f_1' t_3$ for some $t_3 \in R$. Thus $g = f_1' t_3$. Since $f_1' \sim r_1$, by (1) of Remarks 5.3 $f_1'$ is $q_1$-primary. It follows that $g$ has a prime factor similar to $q_1$, which contradicts the hypothesis.

If $s$ is a unit, then we have $f_1 R + h_1 R = R$ and $f_1 R \cap h_1 R = f_1 h_1' R = h_1 f_1' R$. Thus $f_1 \sim f_1'$, $h_1 \sim h_1'$ and $f_1 f_2 \cdots f_{k+1} = h_1 g = f_1 h_1' t_3 = h_1 f_1' t_3$ for some $t_3 \in R$. It follows that

$$f_2 \cdots f_{k+1} = h_1' t_3,$$  
$$g = f_1' t_3.$$

Since $h_1' \sim h_1$, $h_1'$ is $q_1$-primary. By $g = f_1' t_3$, we see that no prime factor of $t_3$ is similar to $q_1$. Therefore, $f_2 \cdots f_{k+1} = h_1' t_3$ satisfies the hypothesis in the case $n = k$. By induction hypothesis, $t_3$ has a primary factorization. We note that a prime factor of $t_3$ is not similar to $p_1$. Since $f_1' \sim f_1$, $f_1'$ is $p_1$-primary. Therefore, $g = f_1' t_3$ has a primary factorization. The lemma is true when $n = k + 1$.

Hence, the lemma is proved.

**Lemma 5.10.** If $f$ has a primary factorization $f = f_1 f_2 \cdots f_n$ and $f = h_1 g$, where $h_1$ is similar to $f_1$ for some $i$, then $g$ has a primary factorization; that is, $h_1 g$ has a primary factorization with
h_1 as the first primary factor.

Proof. By Lemma 5.8 and Lemma 5.9.

Theorem 5.11. If \( f = f_1 f_2 \cdots f_n \) is a primary factorization of \( f \) in \( R \) such that \( f_1 f_{i+1} \) has absolutely unique primary factorization for \( i = 1, 2, \ldots, n-1 \), then \( f \) has absolutely unique primary factorization.

Proof. We carry out mathematical induction on \( n \). By hypothesis the theorem is true when \( n = 2 \).

Assume that it is true when \( n = k \). Let \( f = f_1 f_2 \cdots f_{k+1} = g_1 g_2 \cdots g_{k+1} \) be two primary factorizations of \( f \), where each \( f_i \) is \( p_i \)-primary; each \( g_i \) is \( q_i \)-primary. Assume that \( f_1 f_{i+1} \) has absolutely unique primary factorization for \( i = 1, 2, \ldots, k \).

Suppose \( f_1 R \neq g_1 R \).

(1) If \( f_1 R < g_1 R \), then \( f_1 = g_1 t_1 \) for some nonunit element \( t_1 \) of \( R \). It shows that \( f_1, g_1 \) and \( t_1 \) are \( p_1 \)-primary and \( p_1 \sim q_1 \). We obtain \( t_1 f_2 f_3 \cdots f_{k+1} = g_2 \cdots g_{k+1} \). By Lemma 5.6 a prime factor of \( t_1 \) is similar to a prime factor of \( g_j \) for some \( j \neq 1 \). It follows that \( p_1 \sim q_j \) for some \( j \neq 1 \) and so \( q_1 \sim q_j \) for some \( j \neq 1 \). It is a contradiction. Therefore (1) is not true.

(2) If \( g_1 R < f_1 R \), then by the same reasoning in (1) we have a contradiction. Therefore (2) is not true.

(3) If \( f_1 R \cap g_1 R \neq f_1 R \), \( g_1 R \), then by the same argument in (T) of the proof of Lemma 5.9 we obtain \( f_1 R + g_1 R = R \), \( f_1 R \cap g_1 R = f_1 g_1' R = g_1 f_1' R \), where \( f_1 g_1' = g_1 f_1' \) and \( f_1 \sim f_1' \), \( g_1 \sim g_1' \). Thus \( f_1 f_2 \cdots f_{k+1} = \)}
\[ g_1 g_2 \cdots g_{k+1} = f_1 g_1^r = g_1 f_1^r \quad \text{for some} \quad r \in R \quad \text{and then} \]

(A) \[ f_2 f_3 \cdots f_{k+1} = g_1^r. \]

Since \( g_1' \sim g_1 \), by Lemma 5.8 \( g_1' \) is \( q_1 \)-primary and no prime factor of \( f_1 \) or \( r \) is similar to \( q_1 \). Then applying Lemma 5.9 to (A) above we see that \( g_1^r \) has a primary factorization with \( g_1' \) as the first primary factor. By induction hypothesis \( f_2 f_3 \cdots f_{k+1} \) has absolutely unique primary factorization, hence we have \( g_1' = f_2 u_2 \) for some unit \( u_2 \) of \( R \).

Thus \( f_1 f_2 u_2 = f_1 g_1' = g_1 f_1' \). Since \( f_1' \sim f_1 \), by Lemma 5.10 and Theorem 5.4 \( g_1 f_1' \) is a primary factorization. By hypothesis \( f_1 f_2 u_2 \) has absolutely unique primary factorization, we conclude \( g_1 = f_1 u_1 \) for some unit \( u_1 \) of \( R \). Therefore \( g_1 R = f_1 R \), which contradicts the assumption \( g_1 R \neq f_1 R \).

Consequently, we have \( f_1 R = g_1 R \). Thus \( g_1 = f_1 u_1 \) for some unit \( u_1 \) and so \( f_2 f_3 \cdots f_{k+1} = u_1 g_2 g_3 \cdots g_{k+1} \). By induction hypothesis the primary factorization \( f_2 f_3 \cdots f_{k+1} \) is absolutely unique, so is the primary factorization \( f = f_1 f_2 \cdots f_{k+1} \). The theorem is true when \( n = k + 1 \). Therefore the theorem is proved.

Theorem 5.12. Let \( f = f_1 f_2 \cdots f_n \) be a primary factorization of \( f \) in \( R \), where each \( f_1 \) is \( p_1 \)-primary for some prime \( p_1 \). If for each \( i = 1, 2, \ldots, n-1 \) there exist no \( t_1, t_2 \) in \( R \) such that \( t_1 f_1 + f_{i+1} t_2 = 1 \), then the primary factorization \( f = f_1 f_2 \cdots f_n \) is absolutely unique.

Proof. By Theorem 5.11 it will suffice to prove that the theorem...
is true when \( n = 2 \). Let \( f = f_1f_2 \) be a primary factorization of \( f \) satisfying the hypothesis (there exist no \( t_1, t_2 \) in \( R \) such that \( t_1f_1 + f_2t_2 = 1 \)).

Suppose \( f \) has two primary factorizations \( f = f_1f_2 = g_1g_2 \), where each \( f_1 \) is \( p_1 \)-primary; each \( g_1 \) is \( q_1 \)-primary. We have \( f_1R \cap g_1R \neq 0 \) and \( f_1R \neq g_1R \). Using the same reasoning in (1) and (2) of the proof of Theorem 5.11 we conclude \( f_1R \not\subset g_1R, g_1R \not\subset f_1R \). Then by the same argument in (T) of the proof of Lemma 5.9 we finally obtain

\[
(1) \quad f_1R + g_1R = R, \quad f_1R \cap g_1R = hR
\]

for some \( h \in R \), where \( h = f_1g_1 = g_1f_1 \) for some \( g_1, f_1 \in R \) and \( f_1 \sim f_1', g_1 \sim g_1' \). Thus \( f_1f_2 = g_1g_2 = f_1'g_1r = g_1f_1'r \) for some \( r \in R \) and so \( f_2 = g_1r, g_2 = f_1'r \). If \( r \) is a nonunit element, then, since \( f_2 \) and \( g_2 \) have the same factor \( r \), \( f_2, g_2 \) and \( f_1' \) are \( p_2 \)-primary. Since \( f_1 \sim f_1', f_1' \) is \( p_1 \)-primary. It follows that \( p_1 \sim p_2 \), which is a contradiction. Therefore \( r \) is a unit. Thus

\[
(2) \quad f_2r^{-1} = g_1', \quad g_2r^{-1} = f_1'.
\]

From (1) we have \( f_1s_1 + g_1s_2 = 1 \) for some \( s_1, s_2 \in R^\times \). It follows that \( 0 \neq g_1s_2f_1 = f_1(1-s_1f_1) \in f_1R \cap g_1R \). Thus \( g_1s_2f_1 = f_1(1-s_1f_1) = f_1g_1'r_1 = g_1f_1'r_1 \) for some \( r_1 \in R \), and then

\[
l - s_1f_1 = g_1'r_1, \quad s_1f_1 \sim g_1'r_1 = 1, \quad s_1f_1 + f_2r^{-1}r_1 = 1 \quad (by \ (2)).
\]

If we put \( t_1 = s_1, t_2 = r^{-1}r_1 \), then we have \( t_1f_1 + f_2t_2 = 1 \). We
conclude that if there exist no $t_1, t_2$ in $R$ such that $t_1 f_1 + f_2 t_2 = 1$, then the primary factorization $f = f_1 f_2$ is absolutely unique. Therefore the theorem is proved.

Theorem 5.13 (the converse of Theorem 5.12). If the primary factorization $f = f_1 f_2 \cdots f_n$ is absolutely unique, then there exist no $t_1, t_2$ in $R$ such that $t_1 f_1 + f_2 t_2 = 1$ for each $i = 1, 2, \cdots, n-1$.

Proof. Let's begin with $n = 2$ and let the primary factorization $f = f_1 f_2$ be absolutely unique. We shall show that there exist no $t_1, t_2$ in $R$ such that $t_1 f_1 + f_2 t_2 = 1$.

Assume that there exist $t_1, t_2$ in $R$ such that $t_1 f_1 + f_2 t_2 = 1$. Then $0 \neq t_1 f_1 f_2 = f_2(1-t_2 f_2)$. Therefore we obtain

$t_1 R + f_2 R = R$, \hspace{0.5cm} $t_1 R \cap f_2 R = h R$

for some $h \in R$, where $h = t_1 f_1 = f_2 t_1'$ for some $f_2', t_1' \in R$ and $t_1 \sim t_1'$, $f_2 \sim f_2'$. Thus

(1) \hspace{0.5cm} $t_1 f_1 f_2 = f_2(1-t_2 f_2) = t_1 f_2 r = f_2 t_1^r$ for some $r \in R$

and then $f_1 f_2 = f_2^r$.

By Lemma 5.10 and Theorem 5.4, $f_2^r$ is a primary factorization.

Since the primary factorization $f_1 f_2$ is absolutely unique, it has $f_2' = f_1 u$ and $r = u^{-1} f_2$ for some unit $u$ of $R$. By (1) it has

$1 - t_2 f_2 = t_1^r$,
$t_2 f_2 + t_1 u^{-1} f_2 = 1$,
$(t_2 + t_1 u^{-1}) f_2 = 1$. 
It shows that \( f_2 \) is a unit, which is a contradiction. Therefore if the primary factorization \( f = f_1f_2 \) is absolutely unique, then there exist no \( t_1, t_2 \) in \( R \) such that \( t_1f_1 + f_2t_2 = 1 \).

We now assume \( n > 2 \) and the primary factorization \( f = f_1f_2 \cdots f_n \) is absolutely unique. Suppose that there exist some \( t_1, t_2 \) in \( R \) such that \( t_1f_1 + f_1f_2 = 1 \) for some \( i \). By the above proof for \( n = 2 \), the primary factorization \( f_1f_{i+1} \) is not absolutely unique. It implies that the primary factorization \( f = f_1f_2 \cdots f_n \) is not absolutely unique, which contradicts the hypothesis. Therefore the theorem is proved.

**Corollary 5.14.** Let \( f = f_1f_2 \cdots f_n \) be a primary factorization of \( f \). If for each \( i = 1, 2, \ldots, n-1 \) \( f_i \) and \( f_{i+1} \) generate a proper ideal in \( R \), then the primary factorization \( f = f_1f_2 \cdots f_n \) is absolutely unique. Moreover, the primary factorization \( \mathcal{F} = f_nf_{n-1} \cdots f_2f_1 \) is also absolutely unique.

**Proof.** Since \( f_1 \) and \( f_{i+1} \) generate a proper ideal in \( R \), we see that \( Rf_iR + Rf_{i+1}R < R \) for \( i = 1, 2, \ldots, n-1 \). This shows that there exist no \( t_1, t_2 \) in \( R \) such that \( t_1f_1 + f_{i+1}t_{i+1} = 1 \) for \( i = 1, 2, \ldots, n-1 \). By Theorem 5.12 the primary factorization \( f = f_1f_2 \cdots f_n \) is absolutely unique. Similarly, by the same reasoning the primary factorization \( \mathcal{F} = f_nf_{n-1} \cdots f_2f_1 \) is also absolutely unique.

**Corollary 5.15.** If \( f = f_1f_2 \cdots f_n \) is a primary factorization of \( f \) such that for each \( i = 1, 2, \ldots, n \) \( f_i \) has zero constant term, then the primary factorization \( f = f_1f_2 \cdots f_n \) is absolutely unique.
Proof. We see that for each $i = 1, 2, \ldots, n$ either $t_i f_i + f_{i+1} t_{i+2} = 0$ or $t_i f_i + f_{i+1} t_{i+2} \in R \setminus F$ for any $t_i, t_{i+2} \in R$. By Theorem 5.12 the corollary is proved.

Corollary 5.16. If the polynomial $f \in R$ is homogeneous and it has a primary factorization, then the primary factorization is absolutely unique.

Proof. It is an immediate consequence of Corollary 5.15.

Examples 5.17. (1) Not every element in $R$ has a primary factorization. For example, consider $f = x_1(x_1 + x_2)x_1$ in $\mathbb{Q}[x_1, x_2]$. Although the prime factorization is absolutely unique, $f$ does not have a primary factorization, since $x_1$ is $x_1$-primary, $x_1 + x_2$ is $(x_1 + x_2)$-primary and $x_1$ is $x_1$-primary, where $x_1 \sim x_1$.

(2) Consider $g = (x_1^2 + 1)(x_2^2 + 1)$ in $R = F[M]$, where $F$ is the field of real quaternions. Since $(x_1^2 + 1)j = j(x_1 - 1)$, where $j$ is a unit, it follows that $x_1 + 1 \sim x_1 - 1$ (see Corollary 3.13) and so by Definition 5.1 $x_1^2 + 1 = (x_1 + 1)(x_1 - 1)$ is $(x_1 + 1)$-primary. Similarly, $x_2^2 + 1$ is $(x_2^2 + 1)$-primary. We see that $x_1 + 1 \not\equiv x_2 + 1$ and $g = (x_1^2 + 1)(x_2^2 + 1)$ is a primary factorization of $g$. Obviously, the primary factorization $g = (x_1^2 + 1)(x_2^2 + 1)$ is absolutely unique. But $g$ does not have absolutely unique prime factorization, since $g = (x_1 + 1)(x_1 - 1)(x_2 + 1)(x_2 - 1) = (x_1 + j)(x_1 - j)(x_2 + j)(x_2 - j) = \cdots$.

(3) Consider $h = (x_1^2 + 1)x_1^2$ in $R = F[M]$, where $F$ is the field of real quaternions. We see that $x_1^2 + 1$ is $(x_1 + 1)$-primary and $x_1^2$ is $x_1$-primary. Thus $(x_1^2 + 1)x_1^2$ is a primary factorization of $h$. 
The primary factorization is not absolutely unique, since \( h = x_1^2(x_1^2+1) \) is another primary factorization of \( h \).

**Remark 5.18.** If \( f \) has absolutely unique prime factorization and it has a primary factorization, then the primary factorization of \( f \) is absolutely unique.

Recall that in Theorem 3.22 we have discussed necessary and sufficient conditions for a prime factorization of an element in \( R \) to be absolutely unique. Now we see that it has the same necessary and sufficient conditions for a primary factorization of an element in \( R \) to be absolutely unique. By the same method and reasoning as we did from Theorem 3.22 in Section 4 of Chapter 3 we can show that the theorems from Theorem 3.22 in Section 4 of Chapter 3 are also true when we consider the primary factorization (if it exists) of an element in \( R \).
BIBLIOGRAPHY


