TRIANGULAR BLOCK MATRIX RINGS

PAUL LIVINGSTON ESTES

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by

PAUL L. ESTES

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ABSTRACT

TRIANGLE BLOCK MATRIX RINGS

by

PAUL L. ESTES

In this paper, we investigate full lower-triangular block matrix rings, hereafter called "full T-rings", over an arbitrary ring with unity. Although much has been written about full T-rings over a field or over a commutative ring, very little seems to be known about more general T-rings.

After developing some basic properties in Chapter One, we explore in Chapter Two the ideal structure of a full T-ring. We describe every ideal as a direct sum of "homogeneous" additive subgroups whose entries form ideals of the base ring which in turn satisfy some very nice containment relations. We then determine some special ideals such as the prime ideals, the prime and Jacobson radicals, and the annihilating ideals. A necessary and sufficient condition is given for a full T-ring to be decomposable into a direct sum of ideals.

Suppose a full T-ring R has F as base ring and diagonal blocks of sizes $n_1, n_2, \ldots, n_T$. One way of representing R is $F^{n_1, \ldots, n_T}$. A second representation, FB, is in terms of F and a set of matrix units $B = \{e_{ij} \mid (i,j) \in T\}$ where T is an appropriate index set. Chapter Three is an investiga-
tion of the uniqueness of these two representations. Examples are produced which show that these representations are highly non-unique. Theorems are then proved which give conditions for varying degrees of uniqueness.

The purpose of Chapter Four is to give a description of the right and left orders in a full T-ring. This is motivated by the Faith-Utumi Theorem (1965) and its subsequent generalization by Johnson. The latter gives an internal description of a given right order in a total matrix ring over a local ring. This result is extended here to T-rings. It is shown that any right order in a full T-ring over a local ring contains a special right order which can be expressed in terms of several right orders of a suitable base ring.
1.1 Overview - An Intuitive Summary

This thesis is an investigation of full triangular block matrix rings which, for the sake of brevity, we will call full T-rings. These are rings of matrices of the form

\[
\begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1r} \\
B_{21} & B_{22} & \cdots & B_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
B_{r1} & B_{r2} & \cdots & B_{rr}
\end{pmatrix}
\]

where each block $B_{ij}$ is an $n_i \times n_j$ matrix. If $r = 1$, then the system is simply the ring of $n_1 \times n_1$ matrices, or a total matrix ring. Thus, every theorem about full T-rings yields a corollary on total matrix rings.

The only assumption on the base ring (the ring from which the entries are taken) is that it contain a unit element. A search of the literature has indicated that while much is known about special types of T-rings, for example T-rings over a field, very little has been written about T-rings in general. Thus we develop basic properties in Chapter One.

Chapter Two is a detailed analysis of the ideal structure of a full T-ring. In the first section we obtain
an internal characterization of all two-sided ideals. It is shown that every ideal can be visualized as a triangular array

\[
\begin{pmatrix}
K_{11} & 0 \\
K_{21} & K_{22} \\
. & . & . \\
. & . & . \\
. & . & . \\
K_{r1} & K_{r2} & \cdots & K_{rr}
\end{pmatrix}
\]

of homogeneous blocks $K_{ij}$. The entries occurring in the $(i,j)^{th}$ block are from an ideal $K_{ij}$ of the base ring. Furthermore, the blocks become more densely populated as we move leftward or downward. More precisely, the ideals $K_{ij}$ satisfy the rather nice containment relations

\[
\begin{align*}
K_{11} 
\cap K_{21} & \supset K_{22} \\
. & \cap . \\
. & \cap . \\
. & \cap . \\
K_{r1} & \supset K_{r2} & \supset \ldots & \supset K_{rr}
\end{align*}
\]

In section two, we find all the prime ideals and maximal right ideals and then use them to calculate the prime and Jacobson radicals. Next, we look briefly at left and right ideals and find that their structure is not so easily pinned down as that of two-sided ideals. In section four, it is proved that a full T-ring is decomposable into
a direct sum of ideals if and only if the base ring is also. Finally, we discover all of the left (and right) annihilating ideals. Every left annihilating ideal has the form

\[
\begin{pmatrix}
K_1 & 0 \\
K_1 & K_2 \\
. & . & . \\
. & . & . \\
. & . & . \\
K_1 & K_2 & . & . & K_r
\end{pmatrix}
\]

where the $K_i$ are left annihilating ideals of the base ring and $K_1 \supset K_2 \supset \ldots \supset K_r$.

Kaplansky raised the question of the uniqueness of a total matrix ring. That is, if $F_n$ is isomorphic to $K_n$, is $F$ isomorphic to $K$? In Chapter Three, we consider the analogous questions of uniqueness for full T-rings. For example, suppose two T-rings are isomorphic. Are their base rings then isomorphic? Are the sizes of the blocks the same? Or at least related in some manner? We give examples which show that these questions can be generally answered in the negative. Theorems then follow which give conditions for varying degrees of uniqueness.

In Chapter Four, we investigate the right and left orders in a full T-ring. We find that each right order in a full T-ring over a local ring contains a right order of the form
where the $k_i$ are right orders of a suitable base ring. This generalizes the Faith-Utumi Theorem.

1.2 Basic Concepts and Notation

A T-ring will be defined as a certain subring of a total matrix ring $F_n$. Therefore, we first define the latter as done in [20] and in [13].

1.2.1 Definition. For any ring with unity 1, a finite subset \( \{e_{ij} \mid i, j = 1, \ldots, n \} \) of $R$ will be called a total set of matrix units in $R$ if and only if

\[
\begin{align*}
\text{(1) } & e_{ij}e_{km} = \delta_{jk} e_{im} \quad \forall i, j, k, m \quad (\delta_{jk} = \text{Kronecker delta}) \\
\text{and } & \sum_{i=1}^{n} e_{ii} = 1.
\end{align*}
\]

Corresponding to each set $A$ of total matrix units for $R$, there is a subring $F$ of $R$ called the centralizer of $A$ in $R$:

\[F = \text{Cent } A = \{r \in R \mid ar = ra \quad \forall a \in A\} .\]

If $R$ has a total set $A = \{e_{ij} \mid i, j = 1, \ldots, n \}$ of matrix units with $F = \text{Cent } A$, then $R$ is called a total matrix ring.
of rank \( n \). When this is the case, \( R \) is denoted by \( F_n \) or \( FA \) depending upon whichever aspect of \( R \) we are focusing attention at the time.

Now in order to define a \( T \)-ring, we introduce the following notation which will remain fixed throughout.

1.2.2 Notation. Let \( n_0 = 0 \) and \( (n_1, n_2, \ldots, n_r) \) be an ordered set of positive integers with \( n_1 + n_2 + \ldots + n_r = n \).

For the natural numbers, integers and rationals we will use \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{Q} \) respectively. For \( i = 1, 2, \ldots, r \) let

\[
P_i = \{ j \in \mathbb{N} \mid n_0 + \ldots + n_{i-1} < j \leq n_1 + \ldots + n_i \},
\]

\[
Q_i = \bigcup_{k=1}^{i} P_k,
\]

\[
P = \bigcup_{i=1}^{r} P_i = \{1, 2, \ldots, r\},
\]

and \( T = \bigcup_{i \geq j} P_i \times P_j \).

1.2.3 Definition. Any subring of

\[
R = \{ (a_{ij}) \in F_n \mid a_{ij} = 0 \ \forall \ (i, j) \in P \times P - T \}
\]

will be called a \textit{(lower) triangular block matrix ring}, or a \( T \)-ring for convenience. \( R \) itself will be called a \textit{full \( T \)-ring} and will be denoted by \( F_{n_1, \ldots, n_r} \) when we wish to specify the \textit{block numbers} \( n_1, \ldots, n_r \). The integers \( n \) and \( r \) will be called the \textit{rank} and \textit{degree} of \( F_{n_1, \ldots, n_r} \) respectively.

1.2.4 Example. \( F_{2,1} \) is a full \( T \)-ring of rank 3, degree 2 and consists of the set of all matrices
1.2.5 Remarks. Note that the case \( r = 1 \) brings us back to a total matrix ring \( F_n \). We also observe that any full \( T \)-ring \( R = F_{n_1}, \ldots, n_T \) has a unit element since \( F_n \) does and that the three rings \( R, F \) and \( F_n \) all share the same unit element.

That a \( T \)-ring is in fact a ring will be seen below. First, let us give a second way of describing a \( T \)-ring.

1.2.6 Definition. Let \( A = \{e_{ij} \mid i, j = 1, \ldots, n\} \) be a total set of matrix units of \( F_n \) with \( F = \text{Cent} \ A \) and

\[
B = \{e_{ij} \in A \mid (i,j) \in T\}.
\]

B will then be called a set of matrix units for \( F_{n_1}, \ldots, n_T \).

1.2.7 Lemma. Let 0 be the zero of \( F \). Then \( B_0 = \{0\} \cup B \) is a semigroup.

Proof: Let \( b, c \in B \).

Case 1: \( b = 0 \) or \( c = 0 \) or \( b = e_{ij}, c = e_{km} \) for some \( k \neq j \). Then \( bc = 0 \in B_0 \).

Case 2: \( b = e_{ij}, c = e_{jk} \). Then \( (i,j) \in P_r \times P_s, (j,k) \in P_s \times P_t \Rightarrow (i,k) \in P_r \times P_t \subseteq T \Rightarrow bc = e_{ik} \in B. //^1

//^1 will be used to signify the end of a proof. The conclusion of a proof of a claim or lemma within a larger proof will be denoted by //.
Since $B_0$ is a semigroup, we can form the semigroup ring $F(B_0) = \{ \sum_{b \in B_0} f_b b \mid f_b \in F \}$. Since $f = \sum_{i,j} f_{ij} e_{ij} \forall f \in F$, $F \subset F(B_0)$. Comparing the definitions of $B$ and of $R$ in 1.2.3, we can see that $R = F(B_0)$.

**1.2.8 Theorem.** Let $R = F_{n_1, \ldots, n_r}$ and $B$ be defined as above. Then

(a) $R$ is a ring,

(b) $R = F(B)$, the semigroup ring of $B$ over $F$,

and (c) $R$ is a free left $F$-module with basis $B$.

**Proof:** (a) We showed above that $R = F(B_0)$ which is a ring.

(b) $R = F(B_0) = \{ \sum_{b \in B_0} f_b b \mid f_b \in F \}$

$$= \{ \sum_{b \in B} f_b b \mid f_b \in F \} = F(B).$$

(c) Define $f (\sum f_{ij} b)$ as $\sum f_{ij} b$. Clearly $R$ is closed under this operation and so is a submodule of the module $p(F_n)$. We already know $B$ is a system of generators for $pR$. Let $0 = \sum_{b \in B} r_b b$. Since $0 \in p(F_n)$ which is free with basis $A \supset B$, $r_b = 0 \forall b \in B$. Thus $B$ is a basis for $pR$. //

**1.2.9 Lemma.** Let $R = F(B)$ as above. Then $F = \text{Cent } B$.

**Proof:** Since $B \subset A = \{ e_{ij} \mid i,j = 1, \ldots, n \}$, $F = \text{Cent } A \subset \text{Cent } B$. Suppose $t = \sum_{(i,j) \in T} t_{ij} e_{ij} \in \text{Cent } B$. Then $\forall e_{rs} \in B$, $(\sum_{(i,j) \in T} t_{ij} e_{ij}) e_{rs} = e_{rs}(\sum_{(i,j) \in T} t_{ij} e_{ij})$,
or \( \Sigma t_{ir}e_{is} = \Sigma t_{sj}e_{rj} \). Since \( B \) is a basis for \( R \), \( t_{ir} = 0 \) \( \forall i \neq r \) and \( t_{rr} = t_{ss} \forall e_{rs} \in B \). Since \( e_{11}, e_{21}, \ldots, e_{n1} \in B \), \( t_{rr} = t_{11} \) for \( r = 1, \ldots, n \). Thus \( t = \sum_{i=1}^{n} t_{11}e_{ii} = t_{11} \in F \). //

1.2.10 Remark. The lemma eliminates any difficulty which might arise in regarding \( R \) as a right \( F \)-module since \((Ef_b)b = \Sigma (f_b)_{bb} = \Sigma b(f_b)\).

1.2.11 Notation. Let us designate a full \( T \)-ring \( R \) by \( FB \) whenever we wish to specify that \( B \) is a set of matrix units for \( R \) and \( F = \text{Cent } B \).

1.2.12 Lemma. (a) Let \( R = FB \) and \( R \longrightarrow S \) be a ring isomorphism. Then \( S = F'B' \).
(b) Let \( R = FB \). Then \( R = (uFu^{-1})(uBu^{-1}) \) for any unit \( u \) of \( R \).

Proof: (a) is obvious and (b) follows from (a) since the transform by \( u \) is an automorphism. //

1.2.13 Notation. Whenever \( B = \{e_{ij} \mid (i,j) \in T \} \) is a set of matrix units for \( F_{n_1, \ldots, n_r} \), let \( e_i = \sum_{j \in P_i} e_{jj} \) (\( i = 1, \ldots, r \)).
1.2.14 Theorem. Let $R = F_{n_1}, \ldots, n_r$. Then

(a) each $e_i R e_j$ is a subring of $R$ with $e_i R e_j \neq 0$ iff $i \geq j$

(b) $e_i R e_j = \sum_{(k,m) \in P_i \times P_j} Fe_{km}$

(c) $e_1, e_2, \ldots, e_r$ are orthogonal idempotents with $\sum_{i=1}^{r} e_i = 1$

(d) each $e_i R e_i$ is isomorphic to $F_{n_i}$ and has $e_i$ as unit element

(e) $e_i$ is a left unit element for the ring (right ideal)

$$\sum_{j=1}^{r} e_i R e_j = e_i R$$

(f) each $e_i R e_j$ is a left $e_i R e_i$-module

(g) each $e_i R e_j$ is a free left $F$-module with basis $B \bigcap e_i R e_j$

$$= \{e_{km} \mid (k,m) \in P_i \times P_j\}$$

(h) $R$ is a direct sum of the submodules $e_i R e_j : R = \bigoplus_{i \geq j} e_i R e_j$

(i) $x \in R \Rightarrow x = \sum_{i \geq j} e_i x e_j$

(j) for $i \geq j \geq k$, $(e_i R e_j)(e_j R e_k) = e_i R e_k$ but $\forall i,j,k,m,$

$$e_i R e_j)(e_m R e_k) = 0 \text{ if } j \neq m.$$

Remark: (j) does not mean that every element of $e_i R e_k$ is of the form $(e_i x e_j)(e_j y e_k)$. Consistent with the usual meaning for the product of two rings, it means that every element of $e_i R e_k$ is a finite sum of elements of the form $(e_i x e_j)(e_j y e_k)$. The following example shows that not every element of $e_i R e_k$ is of the form $(e_i x e_j)(e_j y e_k)$.

Let $D$ be a domain and $R = D_{2,1,2}$. Consider the element $e_3 r e_1 \in e_3 R e_1$ where $r = e_{41} + e_{52}$. Suppose $e_3 r e_1 = (e_3 x e_2)(e_2 y e_1)$ for some $x, y \in R$. Then $e_3 r e_1 =$
\[ (e_{44} + e_{55})(\sum_{i,j} x_{ij} e_{ij} e_{33})(e_{33}, \sum_{i,j} y_{ij} e_{ij})(e_{11} + e_{22}) = \]

\[ (x_{43} e_{43} + x_{53} e_{53})(y_{31} e_{31} + y_{32} e_{32}) = \]

\[ x_{43} y_{31} e_{41} + x_{43} y_{32} e_{42} + x_{53} y_{31} e_{51} + x_{53} y_{32} e_{52} . \]

But \( e_3 r_1 = e_3 (e_{41} + e_{52}) e_1 = e_{41} + e_{52} \) so

\[ x_{43} y_{31} = 1 , x_{43} y_{32} = 0 , x_{53} y_{31} = 0 \text{ and } x_{53} y_{32} = 1 . \]

Since \( D \) is a domain, \( x_{43} = 0 \) or \( y_{32} = 0 \). However, neither of these is possible since \( x_{43} y_{31} = 1 \) and \( x_{53} y_{32} = 1 \).

Proof of the theorem: (a) through (h) are obvious.

(i) \( x = 1 \times 1 = \sum_{i,j} e_i x e_j = \sum_{i,j} e_i x e_j = \sum_{i \geq j} e_i x e_j + \sum_{i < j} e_i x e_j \)

\[ = \sum_{i \geq j} e_i x e_j + 0 . \]

(j) Since \( R e_j e_j R \subset R , (e_i R e_j)(e_j R e_k) \subset e_i R e_k \). Let\n
\[ x = e_1 x e_k = e_i x e_i e_i = \sum_{(p,q) \in P_i \times P_k} x_{pq} e_{pq} e_k . \]

Let \( s \in P_j \).

Then \( x = e_i \left( \sum_{(p,q) \in P_i \times P_k} x_{pq} e_{pq} e_{sq} e_k \right) \)

\[ = \sum_{e_i\left( x_{pq} e_{ps} e_{sq} e_k \right)} e_{j e_j (e_{sq}) e_k} e_i \in (e_i R e_j)(e_j R e_k) . \]

The last statement holds since \( e_{j e_m} = 0 \) whenever \( j \neq m \).

Remark: Clearly, we can state and prove analogues for (e) through (g) interchanging the words "left" and "right".
1.3 T-rings as Rings of Endomorphisms

Let \( R = \mathbb{F}_{n_1}, \ldots, n_r \subset \mathbb{F}_n \) and \( M = \mathbb{F}_{e_1} \oplus \ldots \oplus \mathbb{F}_{e_n} \) where \( A = \{e_{ij} \mid i,j = 1, \ldots, n\} \) is the given set of matrix units for \( \mathbb{F}_n \). That is, \( \mathbb{F}_n = FA \). We know that for \( a \in \mathbb{F}_n \), we have an endomorphism \( \bar{a} \in \text{End}_{\mathbb{F}_n}M \) which multiplies on the right by \( a \); also \( a \mapsto \bar{a} \) is an isomorphism between \( \mathbb{F}_n \) and \( \text{End}_{\mathbb{F}_n}M \). Henceforth, let us suppress the bar and regard each \( a \in \mathbb{F}_n \) as an endomorphism of \( \mathbb{F}_M \). For \( i = 1, \ldots, r \) let \( M_i = \oplus \Sigma_{j \in Q_i} \mathbb{F}_{e_{1j}} \). This gives us a chain of submodules of \( M \) which has a most important relationship to \( R \). The following theorem shows that \( R \) is precisely that subring of \( \mathbb{F}_n \) which holds invariant the chain \( 0 = M_o < M_1 < M_2 < \ldots < M_r = M \).

1.3.1 Theorem. Let \( R = \mathbb{F}_{n_1}, \ldots, n_r \) and \( M_i \) be defined as above. Then \( R = \{a \in \mathbb{F}_n \mid M_i a \subset M_i \ \text{for} \ i = 1, \ldots, r\} \).

Proof: Let \( a \in R \). Suppose \( e_{1j} \in M_i \). Is \( e_{1j}a \in M_i \)?

\[
a = \Sigma a_{st} e_{st} \in R \Rightarrow a_{st} = 0 \ \forall (s,t) \in P \times P - T = \bigcup_{k < m} P_k \times P_m
\]

and \( e_{1j} \in M_i \Rightarrow j \in Q_i = \bigcup_{k=1}^m P_k \cdot e_{1j}a = \Sigma e_{1js,t} a_{st} = \Sigma_{t \in X} e_{1jt} a_{jt} \).

Thus \( e_{1j}a \in M_i \). Since \( M_i \) is a left \( \mathbb{F} \)-module, \( \mathbb{F}_{e_{1j}}a \subset M_i \) and \( M_i a \subset M_i \).

Suppose \( a \in \mathbb{F}_n \) and \( M_i a \subset M_i \ \forall i \). Then \( \forall j \in Q_i \), \( e_{1j}a = \Sigma e_{1jt} \in M_i \), so \( \forall (j,t) \in Q_i \times (P - Q_i) \), \( a_{jt} = 0 \).
This is true for all $i$ so $(j,t) \in \bigcup_{i=1}^{r} (Q_i \times P - Q_i) \Rightarrow a_{jt} = 0$. But a routine set-theoretic argument shows that

$$\bigcup_{i=1}^{r} (Q_i \times (P - Q_i)) = P \times P - T.$$ 

Thus $(j,t) \in P \times P - T \Rightarrow a_{jt} = 0$ and hence $a \in R$. //

1.3.2 Remark: Let $R = F_{n_1},\ldots,n_r$, $M_i$ be as above and $N = \sum_{i>j} e_i R e_j$. Then by an argument similar to that in 1.3.1, $N = \{a \in F_n \mid M_i a \subseteq M_{i-1} \text{ for } i = 1,\ldots,r\}$.

1.3.3 Theorem: Let a module $K^M$ be a direct sum of isomorphic submodules: $K^M = N_1 \oplus \cdots \oplus N_n$, $K^M \xrightarrow{\phi_i} K N_i$, $\phi_i = \text{id}$. For $i = 1,\ldots,r$ let $M_i = \bigoplus_{j \in Q_i} N_j$. Then $R = \{a \in \text{End } K^M \mid M_i a \subseteq M_i \forall i\}$ is a T-ring. In particular, $R = F_{n_1},\ldots,n_r$ where $F \cong \text{End } K N_1$.

Proof: Define $e_{ij} \in \text{End } K^M$ by $e_{ij} \mid N_i = \phi^{-1}_i \phi_j$, $e_{ij} \mid N_m = 0$ if $m \neq i$. By theorem 2.1 in [20], $A = \{e_{ij} \mid i,j = 1,\ldots,n\}$ is a set of matrix units for $\text{End } K^M$ and hence $\text{End } K^M$ is a total matrix ring over $F = \text{Cent } A \cong \text{End } K N_1$. That is $\text{End } K^M = F_n$. Each $e_{ii}$ acts as the identity on $N_i$ and so $N_i F = N_i e_{ii} F = N_i F e_{ii} \subseteq N_i$.

Claim: $R = F_{n_1},\ldots,n_r$. Clearly $R$ is a subring of $\text{End } K^M$ and each $e_{ii} \in R$. Let $a = \sum_{pq} a_{pq} e_{pq} \in R$. Then $e_{ii} e_{ij} e_{jj} = a_{ij} e_{ij} \in R$ for $i,j = 1,\ldots,n$. Suppose
(i,j) ∈ P x P - T = \bigcup_{k<m} P_k x P_m. Then (i,j) ∈ P_k x P_m for some k < m. N_h a_{ij} e_{ij} = N_h e_{ij} a_{ij} = 0 ∀ h \neq i and N_i a_{ij} e_{ij} = N_i e_{ij} a_{ij} ⊆ N_j \cap M_k = 0. Thus M_{aij} e_{ij} = 0 so a_{ij} e_{ij} = 0. Hence a_{ij} = 0 ∀ (i,j) ∈ P x P - T and we have a ∈ F_{n_1},...,n_r.

Suppose e_{ij} ∈ F_{n_1},...,n_r. Then (i,j) ∈ P_k x P_m for some k ≥ m. Thus N_i e_{ij} = N_j ⊆ M_k. Since N_h e_{ij} = 0 ∈ M_g ∀ g, M_{ge_{ij}} ⊆ M_g ∀ g. Hence e_{ij} ∈ R. Since N_i F ⊆ N_i \forall i, F ⊆ R and we have F_{n_1},...,n_r = \sum_{i,j} e_{ij} R_j ⊆ R. //

The following is a special case of 1.3.3.

1.3.4 Corollary: Let \( kM \) be a free module with basis \{b_1,...,b_n\} where \( n_1 + ... + n_r = n, M_i = \sum_{j \in Q_i} k b_j \) for \( i = 1,...,r \) and \( R = \{a ∈ End_{kM} | M_i a ⊆ M_i \forall i\} \).

Then \( R = F_{n_1},...,n_r \) where \( F \cong K \).

Proof: For \( j = 1,...,n \) let \( N_j = Kb_j \). Then \( kM = N_1 ⊗ ... ⊗ N_n \) and \( kN_j \cong kN_1 ∀ j \). By 1.3.3, \( R = F_{n_1},...,n_r \) where \( F \cong End_{k(Kb_1)} \cong K \). //

1.3.5 Lemma. Let \( R = F_{n_1},...,n_r \) and \( N = \sum_{i > j} e_{ij} R_{ij} \).

Then \( N \) is nilpotent. In particular, \( N^r = 0, N^{r-1} \neq 0 \).
Proof: As before, let $M = \bigoplus_{j=1}^{n} F_{1j}$ and
\[ M_{i} = \sum_{j \in \mathbb{Q}_{i}} F_{ej}. \]
By 1.3.2, $N = \{ a \in F_{n} \mid M_{i} a \subseteq M_{i-1} \text{ for } i = 1, \ldots, r \}$. Let $x = x_{1}x_{2} \cdots x_{r} \in N^{r}$. Then
\[ Mx = (M_{r}x_{1})x_{2} \cdots x_{r} \subseteq (M_{r-1}x_{2})x_{3} \cdots x_{r} \subseteq \cdots \subseteq M_{r-(r-1)}x_{r} = M_{1}x_{r} = 0. \]
Hence $x = 0$ so $N^{r} = 0$.

Let $x_{1} = e_{(n_{1} + 1)1}$
\[ x_{2} = e_{(n_{1} + n_{2} + 1)(n_{1} + 1)} \]
\[ \vdots \]
\[ x_{r-2} = e_{(n_{1} + \ldots + n_{r-2} + 1)(n_{1} + \ldots + n_{r-3} + 1)} \]
\[ x_{r-1} = e_{(n_{1} + \ldots + n_{r-1} + 1)(n_{1} + \ldots + n_{r-2} + 1)} \]

Then $0 \neq x_{r-1}x_{r-2} \cdots x_{2}x_{1} = e_{(n_{1} + \ldots + n_{r-1} + 1)1} \in N^{r-1}$. //
CHAPTER II

ONE- AND TWO-SIDED IDEALS OF FULL T-RINGS

We have been using the notation $R = FB$ to indicate that $R$ is a full T-ring possessing $B$ as a set of matrix units with $F = \text{Cent } B$. Throughout Chapter Two, whenever $K$ and $A$ are subsets of a ring $R$, $KA$ will mean the set of all finite sums of elements of the form of $ka$ where $k \in K$ and $a \in A$. When $FB$ is intended to mean a full T-ring we will so indicate.

2.1 Internal Structure of Two-Sided Ideals

In this section, whenever $R = FB$ is a full T-ring with orthogonal idempotents $e_i$, we will denote $B \cap e_iRe_j$ by $B_{ij}$.

2.1.1 Lemma. Let $R = FB$ be a full T-ring and $K$ be an ideal of $F$. Then $S = KB$ is an ideal of $R$.

Proof: $S$ is an additive subgroup of $R$ since $K$ is an additive subgroup of $F$. Let $s \in S$ and $r \in R$. Then $sr = \sum_{b \in B} s_b b \sum_{c \in B} r_c c = \sum_{b,c \in B} (s_b r_c)(bc) \in KB = S$. Similarly, $rs \in S$.
In a total matrix ring over a ring with unity, every ideal is of this form. T-rings however, have a much more extensive ideal structure as we shall soon see. In particular, the following lemma shows that every "lower left rectangle" is an ideal.

2.1.2 Lemma. Let $R = F_{n_1, \ldots, n_r}$, $1 \leq q \leq p \leq r$, and $S_{pq} = \sum_{i \geq p} \sum_{j \leq q} e_i R e_j$. Then $S_{pq}$ is an ideal of $R$ and $(S_{pq})^2 = 0$ whenever $p \neq q$.

Proof: Let $S = S_{pq}$. It is clear that $S$ is an additive subgroup of $R$. By an argument similar to that in 1.3.1, $S = \{ s \in F_n \mid M_{p-1}s = 0 \text{ and } M_s \subseteq M_q \}$ where $M_i = \bigoplus_{j \in Q_i} e_j$ as before. Let $s \in S$ and $r \in R$. Then $M_{p-1}sr = 0r = 0$ and $Msr \subseteq M_\mathbb{Q}r \subseteq M_q$ so $sr \in S$. Also, $M_{p-1}rs \subseteq M_{p-1}s = 0$ and $Mrs \subseteq M_\mathbb{Q}s \subseteq M_q$ so $rs \in S$. Hence, $S$ is an ideal of $R$.

Suppose $x, y \in S$. Then $x = \sum_{i \geq j} e_i x e_j = \sum_{i \geq p} \sum_{j \leq q} e_i x e_j$ and likewise $y = \sum_{k \geq p} e_k y e_m$. Since $p > q$, each $e_i x e_j e_k y e_m = 0$ $k \geq p$ $m \leq q$ $(j \leq q < p \leq k)$. Thus $xy = 0$. //

Since sums of ideals are ideals, every "lower left staircase" $S_{p_1q_1} + S_{p_2q_2} + \ldots + S_{p_kq_k}$, where each $S_{p_iq_i}$ is of the form in lemma 2.1.2, is an ideal. In particular,
\[ N = S_{21} + S_{32} + S_{43} + \ldots + S_{r \cdot r-1} = \sum_{i \geq j} e_i R e_j \] is a very important ideal as we shall see later. We already saw in 1.3.5 that \( N \) is nilpotent.

Suppose we have a collection of ideals of \( F \):
\[ \{ K_{ij} \mid i, j = 1, \ldots, r ; i \geq j \} \] subject only to the following containment property:

\[ 2.1.3 \ K_{uv} \supseteq K_{ij} \] whenever \( u \geq i \) and \( v \leq j \).

If we form the direct sum of the additive groups \( K_{ij} B_{ij} \),
\[ X = \bigoplus_{i \geq j} K_{ij} B_{ij} , \]
then \( X \) is an ideal of \( R \). This follows from the observation that \( X \) is a sum of intersections of ideals:
\[ X = \sum_{p \geq q} (S_{p q} \cap K_{p q} B) . \]
Let us verify that \( X \) does equal \( \sum_{p \geq q} (S_{p q} \cap K_{p q} B) \). Each
\[ K_{ij} B_{ij} \subseteq S_{ij} K_{ij} B \] so
\[ X = \sum_{i \geq j} K_{ij} B_{ij} \subseteq \sum_{i \geq j} (S_{ij} \cap K_{ij} B) . \]
Also,
\[ S_{p q} \cap K_{p q} B = \sum_{i \geq p} K_{p q} B_{ij} \subseteq \sum_{i \geq p} K_{i j} B_{ij} = X \] by \( 2.1.3 \)
and so \( \sum_{p \geq q} (S_{p q} \cap K_{p q} B) \subseteq X \) since \( X \) is clearly closed under sums.

The following theorem shows that we have found all of the ideals of \( R \).
2.1.4 Ideal Decomposition Theorem. Let \( R = F B \) be a full T-ring of degree \( r \). Then \( X \) is an ideal of \( R \) if and only if there exists a set \( \{ K_{ij} \mid i,j = 1, \ldots, r \text{ and } i \geq j \} \) of ideals of \( F \) satisfying 2.1.3 such that \( X \) is a direct sum of the additive groups \( K_{ij}B_{ij} \): 

\[
X = \bigoplus_{i,j=1}^{r} K_{ij}B_{ij}.
\]

Proof: The "if" has been shown above. Now the converse: For \( i,j = 1, \ldots, r \) and \( i \geq j \) let \( K_{ij} = \{ k \in F \mid kB_{ij} \subseteq X \} \). Clearly, \( K_{ij} \) is an additive subgroup of \( F \). Let \( k \in K_{ij} \) and \( h \in F \). \( (kh)B_{ij} = (kB_{ij})h \subseteq Xh \subseteq X \) so \( kh \in K_{ij} \). Also, \( (hk)B_{ij} \subseteq hX \subseteq X \) so \( hk \in K_{ij} \). Thus \( K_{ij} \) is an ideal of \( F \).

Claim: \( X = \bigoplus_{i,j \geq j} K_{ij}B_{ij} \). Since \( k \in K_{ij} \) implies \( kB_{ij} \subseteq X \), \( X \supseteq \sum_{i \geq j} K_{ij}B_{ij} \). Let \( x = \sum_{(p,q) \in T} x_{pq}e_{pq} \in X \) and \( x_{pq}e_{pq} \in X \) and hence \( x_{pq} \in K_{ij} \). Therefore, \( x \in \sum_{i \geq j} K_{ij}B_{ij} \) and 

\[
X = \sum_{i \geq j} K_{ij}B_{ij}.
\]

The sum is direct since \( B \) is a basis for \( R \).

All that remains to be proved is property 2.1.3; but this follows from part (b) of
2.1.5 Lemma. (The Sliding Block Principle) Let $X$ be an ideal of a full T-ring $R = FB$ and $k \in F$.

(a) If $ke_{pq} \in X$, then $ke_{st} \in X$ whenever $s \geq p$ and $t \leq q$.

(b) If $kB_{ij} \subseteq X$, then $kB_{uv} \subseteq X$ whenever $u \geq i$ and $v \leq j$.

Proof: (a) Let $ke_{pq} \in X$ and $s \geq p$, $t \leq q$. Then $e_{sp}, e_{qt} \in R$ so $ke_{st} = e_{sp}(ke_{pq})e_{qt} \in X$.

(b) Let $kB_{ij} \subseteq X$ and $u \geq i$, $v \leq j$. Suppose $(s, t) \in P_u \times P_v$. For all $(p, q) \in P_i \times P_j$, $ke_{pq} \in X$ and $s \geq p$, $t \leq q$. Thus $ke_{st} \in X$ by (a). This concludes the proof of the lemma and of Theorem 2.1.4. //

Let us summarize what has been proved so far in this section. First, we have found two basic types of ideals of a full T-ring $R = FB$:

1. The F-derived ideals: those of the form $K B$ for some ideal $K$ of $F$.

2. The full "lower left rectangles": ideals of the form $S_{pq} = \sum_{i \geq p} e_i R e_{j q}$.

By taking sums and intersections of the two basic types, we get a multitude of new ideals. Theorem 2.1.4 and the remarks preceding it show that any given ideal is built up in this manner and is a direct sum of "homogeneous components" $K_{ij} B_{ij}$. Taking sums of ideals is our way of stacking blocks; lemma 2.1.5 shows that we cannot simply take blocks and stack them at random.
The containment property 2.1.3 shows (among other things) that \( e_r R e_1 \), besides being the lower left block of \( R \), is special in at least two ways. First, it is the unique minimal ideal of type 2 above. Secondly, if a member of \( F \) occurs as an entry of any member of an ideal \( X \), it occurs in \( K_{r1} \). Thus in any ideal, \( e_r R e_1 \) is the most densely populated block of \( R \).

As a special case of 2.1.4, we obtain the well-known characterization of the ideals in a total matrix ring over a ring with unity.

2.1.6 Corollary. Let \( R = F B \) be a total matrix ring over a ring \( F \) with unity. Then \( X \) is an ideal of \( R \) if and only if \( X = K B \) for some ideal \( K \) of \( F \).

Proof: This is the case \( r = 1 \). Therefore, the set \( \{ K_{ij} | i, j = 1, \ldots, r \} \) reduces to \( \{ K_{11} \} \).

2.2 Radicals

First, the notation: For any ring \( R \), let \( \text{rad} R \) and \( \text{Rad} R \) denote the prime and Jacobson radicals, respectively. Also, let us shorten \( B_{ii} = B \cap e_i R e_i \) to \( B_i \). Finally, recall that we use \( N \) for \( \sum_{i > j} e_i R e_j \).
2.2.1 Theorem. For any full T-ring $R = FB$,

(a) $\text{rad } R = \bigoplus \Sigma (\text{rad } F)B_i$

and (b) $\text{Rad } R = \bigoplus \Sigma (\text{Rad } F)B_i$.

To prove this theorem, let us discover what the prime ideals and the maximal right ideals look like:

2.2.2 Theorem. Let $R = FB$ be a full T-ring of degree $r$. Then $X$ is a

(a) prime ideal of $R$ iff there exist

(1) $j \in \{1, 2, \ldots, r\}$

and (2) a prime ideal $K$ of $F$

such that $X = \bigoplus \Sigma KB_j \bigoplus \Sigma FB_i$.

(b) maximal right ideal of $R$ iff there exist

(1) $j \in \{1, 2, \ldots, r\}$

and (2) a maximal right ideal $M$ of $e_jRe_j$

such that $X = \bigoplus \Sigma M \bigoplus \Sigma e_iRe_i$.

Proof: (a) Suppose there exist $j \in \{1, \ldots, r\}$ and a prime ideal $K$ of $F$ such that $X = \bigoplus \Sigma KB_j \bigoplus \Sigma FB_i$. By 2.1.4, $X$ is an ideal of $R$. Now let $Y$, $Z$ be ideals of $R$ satisfying $YZ \subseteq X$. By 2.1.4, there exist ideals $H_{ij}$, $K_{ij}$ of $F$ such that $Y = \Sigma H_{ij}B_{ij}$ and $Z = \Sigma K_{ij}B_{ij}$. Since $YZ \subseteq X$, we get $H_{jj}K_{jj} \subseteq K$ and thus $H_{jj} \subseteq K$ or $K_{jj} \subseteq K$. Hence $Y \subseteq X$ or
Now assume $X$ is a prime ideal of $R$. Since $N^r = 0 \subset X$, we obtain (by induction on $r$) $N \subset X$. Thus for some ideals $K_i$ of $F$, $X = N \oplus \Sigma K_i B_i$. Next, we show that at most one of the $K_i$'s is not equal to $F$. Suppose $K_j \neq F$ and $K_k \neq F$ ($j \neq k$). Let $Y = N \oplus FB_j$ and $Z = N \oplus FB_k$. Then $YZ \subset N \subset X$ but $Y \not\subset X$ and $Z \not\subset X$ in contradiction to the primeness of $X$. We now have, for some $j$ and some ideal $K = K_j$ of $F$, $X = N \oplus KB_j \oplus \Sigma FB_i$. All that remains to be shown is that $K$ is prime. If not, there exist ideals $H$ and $L$ of $F$ such that $HL \subset K$ but $H \not\subset K$, $L \not\subset K$; then for $Y = N \oplus HB_j$ and $Z = N \oplus LB_j$, we have $YZ \subset N + KB_j \subset X$ but $Y \not\subset X$, $Z \not\subset X$.

(b) First, we observe that whenever $S$ is any right ideal of $e_j R e_j$, then $Y = N \oplus S \oplus \Sigma e_i R e_i$ is a right ideal of $R$. ($YR \subset NR \oplus SR \oplus \Sigma e_i R e_i R$

$$\subset N \oplus (S \oplus N) \oplus (N \oplus \Sigma e_i R e_i)$$

$$\subset N \oplus S \oplus \Sigma e_i R e_i = Y.$$)

Clearly, if $M$ is a maximal right ideal of $e_j R e_j$, then $X = N \oplus M \oplus \Sigma e_i R e_i$ is a maximal right ideal of $R$.

Conversely, suppose $X$ is a maximal right ideal of $R$. Since $N$ is nilpotent, $N \subset \text{Rad } R \subset X$. For $i = 1, \ldots, r$ let $M_i = X \cap e_i R e_i$. Then each $M_i$ is a right ideal of $e_i R e_i$. If $M_j < e_j R e_j$ and $M_k < e_k R e_k$ for $k \neq j$, then $X < X + e_k R e_k < R$.
in contradiction to the maximality of $X$. ($X + e_kRe_k$ is a right ideal of $R$ since it is a sum of two right ideals: $X$ and $N + e_kRe_k$.) Thus, there exists a unique $j$ such that $M = M_j < e_jRe_j$. $X$ is now of the desired form: $X = N \oplus M \oplus \sum_{i \neq j} e_iRe_i$; and it is evident that since $X$ is maximal, so is $M$. //

Proof of Theorem 2.2.1: (a) Let $\mathcal{P}_R$ and $\mathcal{P}_F$ be the collections of prime ideals of $R$ and $F$ respectively. Then
\[
\text{rad } R = \bigcap \mathcal{P}_R = \bigcap \{N \oplus KB_j \oplus \sum_{i \neq j} FB_i \mid j \in \{1, \ldots, r\} \text{ and } K \text{ prime ideal of } F\}
\]
\[
= N \oplus \sum (\bigcap \mathcal{P}_F)B_i = N \oplus \sum (\text{rad } F)B_i .
\]

(b) Let $\mathcal{M}_R$ and $\mathcal{M}_i$ ($i = 1, \ldots, r$) denote the collections of maximal right ideals of $R$ and $e_iRe_i$ respectively. Then
\[
\text{Rad } R = \bigcap \mathcal{M}_R = \bigcap \{N \oplus M_j \oplus \sum_{i \neq j} e_iRe_i \mid j \in \{1, \ldots, r\}; M_j \text{ max right ideal of } e_jRe_j\}
\]
\[
= N \oplus \sum (\bigcap \mathcal{M}_i) = N \oplus \sum (\text{Rad } e_iRe_i)
\]
\[
= N \oplus \sum (\text{rad } F)B_i . //
\]
2.3 The Structure of One-Sided Ideals

We saw in 2.1 that every two-sided ideal has a decomposition into homogeneous blocks which satisfy a very nice containment property. Here we attempt a similar analysis for one-sided ideals. For the purpose of informal discussion, let us use the term horizontal (vertical) "segment" for the intersection of a row (column) with a block. We will see that a right (left) ideal can be visualized as a sum of homogeneous horizontal (vertical) segments with a containment property analogous to the one for two-sided ideals. However, a simple example shows that we cannot get the complete characterization that we did in Theorem 2.1.4.

If we line up horizontal segments starting from the left, we get the right analog of the lower left rectangles of 2.1:

2.3.1 Lemma. Let \( R = FB \) be a full T-ring, \( i \in P_k \) and \( q \leq k \). Define \( S_{iq} \) as \( \Sigma_{i=1}^n \Sigma_{j=0}^{q_1} F_{ij} \) Then \( S_{iq} \) is a right ideal of \( R \).

Proof: It is evident that \( S_{iq} \) is an additive group and that \( S_{iq} = \{ a \in F_n \mid F_{ij}a = 0 \ \forall j \neq i \ and \ F_{il}a \subseteq M_q \} \) where \( M = F_{11} \oplus \ldots \oplus F_{1n} \) and \( M_q = \Sigma_{j=1}^q F_{ij} \) as usual. Let \( a \in S_{iq} \) and \( r \in R \). Then \( F_{ij}ar = 0r = 0 \ \forall j \neq i \) and \( F_{il}ar \subseteq M_q R \subseteq M_q \). //
We also have $F$-derived right ideals of $R$; right ideals of the form $KB$ where $K$ is a right ideal of $F$. (proof: same as in 2.1.1). Thus we have, as in the case of two-sided ideals, two basic types of right ideals from which we can obtain many more by taking sums and intersections.

Let us now construct the right analog for the ideal

$$\bigoplus_{i \geq j} K_i B_{ij}$$

of 2.1. Let

$$\{K_{iq} | i = 1, \ldots, n ; q \leq k \text{ where } i \in P_k\}$$

be a collection of right ideals of $F$ satisfying

$$2.3.2 \ K_{ip} \supseteq K_{iq} \text{ whenever } p \leq q.$$

Denote $B \cap e_i R e_q$ by $B_{iq}$. (Note: This is not the same as the previous $B_{iq}$.) Then $X = \bigoplus_{i, q} \Sigma K_{iq} B_{iq}$ is a right ideal of $R$. This follows immediately from the fact that $X = \Sigma (S_{iq} \cap K_{iq} B)$. 

To verify that $X$ does equal $\Sigma (S_{iq} \cap K_{iq} B)$, note that each $K_{iq} B_{iq} \subseteq S_{iq} \cap K_{iq} B$ so $X = \Sigma K_{iq} B_{iq} \subseteq \Sigma (S_{iq} \cap K_{iq} B)$; also $S_{iq} \cap K_{iq} B = \Sigma K_{iq} B_{ij} \subseteq \Sigma K_{ij} B_{ij} = X$ by 2.3.2 and so $\Sigma (S_{iq} \cap K_{iq} B) \subseteq X$ since $X$ is clearly closed under sums.

Not all right ideals of $R$ are of this form as is shown by the following example. Let $R = F_2$. Then $X =$
$(e_{11} + e_{21})R$ is a right ideal of $R$ not of the form
$\oplus \Sigma K_{iq}B_{iq}$ . However, theorem 2.3.3 below shows that if we
look at the entries occurring within each horizontal segment,
then every right ideal $X$ of $R$ does "look like" $\oplus \Sigma K_{iq}B_{iq}$
in an "entry-wise" sense. More precisely, $X$ is contained in
$\oplus \Sigma K_{iq}B_{iq}$ .

2.3.3 Theorem. Let $X$ be a right ideal of a full
T-ring $R = FB$ . Then there exists a set
\{ $K_{iq} \mid i = 1, \ldots, n$ ; $q \leq k$ where $i \in P_{k}$ \} of right ideals of
$F$ satisfying 2.3.2 such that each $e_{ii}X_{eq} = K_{iq}B_{iq}$ and
$X \subset \oplus \Sigma K_{iq}B_{iq}$ .

Proof: For $i = 1, \ldots, n$ and $q \leq k$ where $i \in P_{k}$ let
$K_{iq} = \{ k \in F \mid kB_{iq} \subset e_{ii}X \}$ .
If $h, k \in K_{iq}$ , then $(h-k)B_{iq} \subset hB_{iq} - kB_{iq} \subset e_{ii}X$ . If
$k \in K_{iq}$ and $f \in F$ , then $(kf)B_{iq} = (kB_{iq})f \subset e_{ii}Xf \subset e_{ii}X$ .
Hence, $K_{iq}$ is a right ideal of $F$ .

Claim: Each $e_{ii}X_{eq} = K_{iq}B_{iq}$ . Let $e_{ii}xe_{q} \in e_{ii}X_{eq}$ .
$e_{ii}xe_{q} = e_{ii}\Sigma x_{st}e_{st}e_{q} = \Sigma x_{it}e_{it}$ . Is each $x_{it} \in K_{iq}$ ?
Let $m \in P_{q}$ . Then $x_{it}e_{im} = (\Sigma x_{iz}e_{iz})e_{tm} \in e_{ii}X_{eq} \subset e_{ii}X$
so each $x_{it} \in K_{iq}$ and we have $e_{ii}X_{eq} \subset K_{iq}B_{iq}$ . Let
$ke_{im} \in K_{iq}B_{iq}$ . Then $ke_{im} \in e_{ii}X$ so $ke_{im} = ke_{im}q \in e_{ii}X_{eq}$ .
Since $e_{ii}Xe_q$ is an additive group, this completes the proof of the claim.

Now to show property 2.3.2: Suppose $p < q$. Let $k \in K_{iq}$ and $m \in \mathcal{P}_p$. For $t \in \mathcal{P}_q$, $ke_{im} = ke_{it}e tm \in e_{ii}XR \subseteq e_{ii}X$ so $k \in K_{ip}$.

The last statement of the theorem follows immediately from the claim. //

The single block case reduces to

2.3.4 Corollary. Let $X$ be a right ideal of a total matrix ring $R = F_n = FB$. Denote $B \bigcap e_{ii}R$ by $B_i$. Then there exists a set $\{K_i \mid i = 1, \ldots, n\}$ of right ideals of $F$ such that each $e_{ii}X = K_iB_i$ and $X \subseteq \bigoplus_i K_iB_i$.

Each of the above results for right ideals has the obvious analog for left ideals. In particular, the left analog for 2.3.3 is

2.3.5 Theorem. Let $X$ be a left ideal of a full T-ring $R = FB$. Denote $B \bigcap e_{pj}Re_{jj}$ by $B_{pj}$. Then there exists a set $\{K_{pj} \mid j = 1, \ldots, n ; p \geq k \text{ where } j \in P_k\}$ of left ideals of $F$ satisfying

$$K_{pj} \supseteq K_{qj} \text{ whenever } p \geq q$$

such that each $e_{pj}Xe_{jj} = K_{pj}B_{pj}$ and $X \subseteq \bigoplus_{p,j} K_{pj}B_{pj}$.

Proof: Similar to proof of 2.3.3. //
2.4 Decomposability

2.4.1 Definition. A ring $R$ is said to be decomposable iff $R$ can be expressed as a direct sum of nonzero ideals.

2.4.2 Theorem. Let $R = FB$ be a full T-ring. Then $R$ is decomposable iff $F$ is.

Proof: Suppose $F = H \oplus K$ where $H$ and $K$ are nonzero ideals of $F$. Then $R = FB = (H \oplus K)B = HB \oplus KB$ and we know by 2.1.1 that $HB$ and $KB$ are ideals of $R$.

Let $R = X \oplus Y$ for some $X$, $Y$ nonzero ideals of $R$. By 2.1.4, there exist ideals $H_{ij}$, $K_{ij}$ such that $X = \oplus \Sigma H_{ij}B_{ij}$ and $Y = \oplus \Sigma K_{ij}B_{ij}$. Then $E_{FB_{ij}} = R = X \oplus Y = E_{H_{ij}B_{ij}} \oplus E_{K_{ij}B_{ij}} = E_{(H_{ij}B_{ij} \oplus K_{ij}B_{ij})} = E_{(H_{ij} \oplus K_{ij})B_{ij}}$.

We conclude $F = H_{ij} \oplus K_{ij} \forall i,j$. //

The following corollary shows that we can say even more: that the decomposition of $R$ is an especially nice one.

2.4.3 Corollary. Let $R = FB$ be a full T-ring which is decomposable: $R = X \oplus Y$. Then there exist ideals $H$, $K$ of $F$ such that $X = HB$, $Y = KB$ and $F = H \oplus K$.

Proof: By 2.4.2, $F = H_{r1} \oplus K_{r1} = H_{ij} \oplus K_{ij} \forall i,j$. But $H_{r1} \supset H_{ij}$ and $K_{r1} \supset K_{ij} \forall i,j$. Thus $H_{r1} = H_{ij}$ and
\[ K_{r1} = K_{ij} \forall i, j \]. If we let \( H = H_{r1} \) and \( K = K_{r1} \), then
\[ X = \sum H B_{ij} = H B, \quad Y = \sum K B_{ij} = K B, \quad \text{and} \quad F = H \oplus K. \]

2.4.4 Corollary. Let \( R = FB \) be a full T-ring which is decomposable: \( R = H B \oplus KB \). Then \( B' = \{ l_{H} e_{ij} | e_{ij} \in B \} \) is a set of matrix units for \( HB \) and \( H = \text{Cent } B' \). Thus \( HB = HB' \) and \( HB' \) is a full T-ring over \( H \). The analogous results hold for \( B'' = \{ l_{K} e_{ij} | e_{ij} \in B \} \) and each \( e_{ij} = e'_{ij} + e''_{ij} \).

Proof: For \( e'_{ij} = l_{H} e_{ij}, \Sigma e'_{ii} = l_{H} F = l_{H} \) and \( e'_{ij} e'_{km} = l_{H} H e_{ij} e_{km} = \delta_{jk} e'_{im} \) so \( B' \) is a set of matrix units for \( HB \). By arguing as in 1.2.9, we get \( H = \text{Cent } B' \). Let \( e'_{ij} \in B' \) and \( e''_{ij} \in B'' \). Then \( e'_{ij} + e''_{ij} = l_{H} e_{ij} + l_{K} e_{ij} = (l_{H} + l_{K}) e_{ij} = l_{F} e_{ij} = e_{ij} \).

2.5 The Lattices of Annihilating Ideals

Let \( S \) be any ring and \( \mathcal{A}_{\lambda}(S) \) be the set of left-annihilating ideals of \( S \): \( L \in \mathcal{A}_{\lambda}(S) \) iff \( L \) is an ideal of \( S \) and \( L = \{ a \in S \mid aR = 0 \text{ for some } R \subseteq S \} \). If \( R \) is the set of all elements which are left-annihilated by \( L \), then \( R \) is an ideal of \( S \) and in fact \( R \) is then the right annihilator of \( L \). We will denote this by \( L = \lambda(R) \) and \( R = \pi(L) \). The notation \( \lambda(R) \) will be used only in this sense. Thus, \( L = \lambda(R) \).
implies $R$ is the set of all elements which are left-annihilated by $L$; hence $R$ is an ideal. The similar remarks hold for $\pi(L)$. It is well-known that $\mathcal{A}_L(S)$ is a complete lattice under inclusion with intersection as inf. However, $\mathcal{A}_L(S)$ is not a sublattice of the lattice of all ideals of $S$ since the sup of a subcollection $\mathcal{D}$ of $\mathcal{A}_L(S)$ 
\[(\inf \{L \in \mathcal{A}_L(S) \mid L \supseteq D \vee D \in \mathcal{D}\})\] is in general larger than the ideal-theoretic sum. Also, there is a dual isomorphism between $\mathcal{A}_L(S)$ and $\mathcal{A}_\pi(S)$, the set of right-annihilating ideals:

\[
\begin{align*}
\mathcal{A}_L(S) & \xrightarrow{\text{L}} \mathcal{A}_\pi(S) \\
\lambda(R) & \leftarrow \pi(L) \\
\end{align*}
\]

We now proceed to describe $\mathcal{A}_L(S)$ (and hence $\mathcal{A}_\pi(S)$) by the dual isomorphism above) where $S = F_{n_1, \ldots, n_t} = FB$.
First, we find two basic types of annihilators and then consider the most general kind. For $k = 1, \ldots, t$ let $L_k = S(e_1 + \ldots + e_k)$ and $R_k = (e_t + e_{t-1} + \ldots + e_{t-k+1})S$.

2.5.1 Lemma. If $S$ is a full T-ring of degree $t$, then the chain $0 = L_0 \subset L_1 \subset \ldots \subset L_t = S$ is contained in $\mathcal{A}_L(S)$. In particular, $L_k = \lambda(R_{t-k}) \forall k$.

Proof: $L_k$ is an ideal since it is a sum of $k$ of the "lower left rectangles" of 2.1.2, namely $L_k =$
$S_1 + S_2 + \ldots + S_{kk}$. Let $e = e_1 + \ldots + e_k$. Then $e$ is an idempotent and $R_{t-k} = (e_t + e_{t-1} + \ldots + e_{t-(t-k)+1})S = (1 - e)S$ so $L_k = Se = \ell((1 - e)S) = \ell(R_{t-k})$. //

2.5.2 Lemma. Let $S = FB$ be a full T-ring and $K = \ell(J) \in \mathcal{A}_\ell(F)$. Then $KB = \ell(JB) \in \mathcal{A}_\ell(S)$.

Proof: Since $K$ is an ideal of $F$, $KB$ is an ideal of $FB$. $(KB)(JB) \subseteq (KJ)B = 0$ so $KB \subseteq \ell(JB)$. Let $k = \Sigma k_b b \in \ell(JB)$. Then $kj = 0 \forall j \in J \subseteq JB$. Hence $k_b j = 0 \forall b \in B$ and $\forall j \in J$. Thus each $k_b \in \ell(J) = K$ and $k \in KB$. //

In the next two theorems, $B_i = B \cap Se_i$ and $D_i = B \cap e_i S$.

2.5.3 Theorem. Let $S = FB$ be a full T-ring of degree $t$. If $K_1, \ldots, K_t \in \mathcal{A}_\ell(F)$ satisfy $K_1 \supset \ldots \supset K_t$ and $L = \Sigma K_i B_i$, then $L \in \mathcal{A}_\ell(S)$. In particular, $L = \ell(\Sigma \pi(K_i) D_i)$.

Proof: Let each $\pi(K_i) = J_i$ and $R = \Sigma J_i D_i$. Since the $K_i$ are ideals satisfying $K_1 \supset \ldots \supset K_t$, $L$ is an ideal of $S$. Now $LR$

$$= (\Sigma K_i B_i)(\Sigma J_i D_i) = \Sigma (K_i B_i)(J_i D_i)$$
\[= \sum_{i,h} (K_i J_h) (B_i D_h) \quad \text{(since } J_h \subseteq F = \text{Cent } B)\]

\[= \sum_{i} (K_i J_i) (B_i D_i) \quad \text{ (} i \neq h \Rightarrow \bigcap P_i \cap P_h = \emptyset \Rightarrow B_i D_h = 0)\]

\[= 0 \text{ since } J_i = \pi(K_i) \text{. Thus } L \subseteq \mathcal{L}(R) \text{.}\]

Let \( k = \sum_{j} k_{x,y} e_{xy} \in \mathcal{L}(R) \). Then for \( h = 1, \ldots, t \),

\[k_{x,y} e_{J_h} = 0 \quad \forall j \in J_h \text{. But } 0 = k_{x,y} e_{J_h} = \sum_{y \in P_h} k_{x,y} e_{xy} \quad \forall j \in J_h \Rightarrow k_{x,y} e_{J_h} = K_h \text{ whenever } y \in P_h \text{. Hence } k = \sum_{h=1}^{t} K_h B_h = L \text{.} \]

The next theorem shows that every left-annihilating ideal is of the form of \( L \) in 2.5.3.

2.5.4 Theorem. Let \( S = FB \) be a full T-ring of degree \( t \) and \( L \in \mathcal{A}_{\mathcal{L}}(S) \). Then there exist \( K_1, K_2, \ldots, K_t \in \mathcal{A}_{\mathcal{L}}(F) \) satisfying \( K_1 \supseteq K_2 \supseteq \cdots \supseteq K_t \) such that \( L = \sum_{j=1}^{t} K_j B_j \).

Proof: \( L = \mathcal{L}(R) \) for some \( R \in \mathcal{A}_{\mathcal{L}}(S) \). Since \( L \) is an ideal, there exist ideals \( K_{ij} \) of \( F \) satisfying \( K_{uv} \supseteq K_{ij} \) whenever \( u \geq i \) and \( v \leq j \) such that for \( B_{ij} = B \cap e_i R e_j \),

\[L = \sum_{i \geq j} K_{ij} B_{ij} \text{.}\]

Claim 1: For each \( j \), \( K_{ij} = K_{tj} \bigcup i \). Since \( S \) has degree \( t \), \( K_{ij} \subseteq K_{tj} \bigcup i \). Let \( k \in K_{tj} \) and \( e_{qw} \in B_{ij} \). We want to show that \( ke_{qw} R = 0 \). Let \( r = \sum_{x,y} e_{xy} \in R \). Then

\[ke_{qw} r = \sum_{y} r_{wy} e_{qw} e_{xy}.\]

For each \( y \), \( r_{wy} e_{wy} \in R \) and since \( w \in P_j \),
we have for \( z \in P_t \), \((ke_{zw})(r_{wy}e_{wy}) = kr_{wy}e_{zy} = 0 \). Thus \( kr_{wy} = 0 \ \forall y \) and we obtain \( ke_{qw}R = 0 \). Hence \( ke_{qw} \in L \) and therefore \( k \in K_{ij} \).

By claim 1, we can write \( K_{jj} = K_{j+1} = \ldots = K_{tj} = K_j \) and thus \( L = \sum_{j=1}^{t} K_j B_j \) where obviously \( K_1 \supset K_2 \supset \ldots \supset K_t \).

It remains to show that each \( K_i \) is a left annihilator in \( F \). By arguing as above, there exist ideals \( J_1 \subset J_2 \subset \ldots \subset J_t \) of \( F \) such that \( R = \sum_{i=1}^{t} J_i D_i \) where \( D_i = B \cap e_i S \).

**Claim 2:** \( K_i = \mathcal{L}(J_i) \). Let \( k \in K_i \). Then \( ke_i \in L \) so \( (ke_i)(je_i) = 0 \ \forall j \in J_i \). But \( 0 = (ke_i)(je_i) = (kj)e_i \Rightarrow kj = 0 \). Thus \( K_i \subset \mathcal{L}(J_i) \). Let \( k \in \mathcal{L}(J_i) \) and \( r = \sum_{x,y} r_{xy}e_{xy} \in R \). Then \( (ke_i)r = k \sum_{x,y} r_{xy}e_{xy} \in kJ_i D_i = 0 \) \( x, y \in P_i \) so \( ke_i \in \mathcal{L}(R) = L \). Hence \( k \in K_i \). //

Let \( L_k \) have the same meaning as in the preliminary remarks of this section.

**2.5.5 Corollary.** If \( S = FB \) is a full \( T \)-ring of degree \( t \) over a prime ring \( F \), then \( \mathcal{A}_2(S) = \{L_0, L_1, \ldots, L_t\} \).
Proof: By 2.5.1, $\mathcal{A}_\xi(S) \supseteq \{L_0, L_1, \ldots, L_t\}$. Suppose $L = \sum_{j=1}^{t} K_j B_j \in \mathcal{A}_\xi(S)$, $L \neq 0$. Since $S = L_t \supseteq L$ and the $L_i$ form a chain, there exists a smallest $k$ such that $L_k \supseteq L$. Because $F$ is a prime ring, $\mathcal{A}_\xi(F) = \{0, F\}$ so $K_j = F \ orall j < k$ and $K_j = 0 \ orall j > k$. Therefore, $L = \sum_{j=1}^{k} F B_j = L_k$. //

The converse of 2.5.5 is also true:

2.5.6 Proposition. If $S = FB$ is a full $T$-ring, then $\mathcal{A}_\xi(S) = \{L_0, \ldots, L_t\}$ only if $F$ is a prime ring.

Proof: $F$ not prime $\Rightarrow \exists K \in \mathcal{A}_\xi(F) - \{0, F\}$. Then $KB \in \mathcal{A}_\xi(S)$ but $KB$ is not one of the $L_i$. //

The following is needed in the next chapter.

2.5.7 Proposition. Let $S = FB$ be a full $T$-ring of degree $t > 1$ over a non-prime ring $F$. Then $\mathcal{A}_\xi(S)$ is not a chain.

Proof: Since $F$ is not prime, there exists $K \in \mathcal{A}_\xi(F) - \{0, F\}$ and therefore by 2.5.2, $KB \in \mathcal{A}_\xi(S) - \{0, S\}$. Let $e_1, \ldots, e_t$ have their usual meaning. We know by 2.5.1 that $Se_1 \in \mathcal{A}_\xi(S)$. Since $t > 1$, $KB \not\subseteq Se_1$; and since $K \neq F$, $KB \not\supsetneq Se_1$. Hence $\mathcal{A}_\xi(S)$ is not a chain. //
CHAPTER III

UNIQUENESS OF T-RINGS

Several questions arise which ask about the uniqueness of the representation of a full T-ring in the form 
\[ R = FB \] or 
\[ R = F_{n_1, \ldots, n_T} \]. For example, suppose \( FB \cong GC \).

Are \( F \) and \( G \) then isomorphic? Is there any nice relationship between \( B \) and \( C \)? If \( F_{n_1, \ldots, n_T} \cong G_{m_1, \ldots, m_S} \), are \( F \) and \( G \) isomorphic? Is \( r = s \)? If so, is \( n_i = m_i \) \( \forall i \)? Can we permute the block numbers without disturbing the ring? i.e. is 
\[ F_{n_1, \ldots, n_T} \] isomorphic to 
\[ F_{n_{p(1)}, \ldots, n_{p(r)}} \] for any permutation \( p \) of \( \{1, \ldots, r\} \)?

In general, the answer to each of these questions is no.

3.1 Examples of Non-Uniqueness

The total matrix ring case is enough to show the non-uniqueness of the representation \( R = FB \). Examples are given in [20].

Now let us look at some examples to show the non-uniqueness of the representation of a full T-ring in the form 
\[ F_{n_1, \ldots, n_T} \]. The first example shows that a total matrix ring can actually be isomorphic to a non-total full T-ring!
3.1.1 Example. Let $F$ be any ring with unity. Then

$$(F_2)_{1,1} \not\cong F_{2,2} \cong (F_1,1)_2.$$ 

Proof: Special case of 3.1.2 below.

$F_{2,2} \cong (F_1,1)_2$ shows that the degree of a full T-ring is not unique. It is a function only of the given representation $F_{n_1,\ldots,n_\tau}$. $(F_2)_{1,1} \cong F_{2,2}$ shows that the block numbers may not match up even when the degrees are the same. The case of $F$ a field shows the non-uniqueness of base rings. Example 3.3.1 below is another, less predictable illustration of non-uniqueness of degree and base rings. There, the degree of each representation is greater than one; hence example 3.1.1 and its generalization in 3.1.2 do not exemplify merely an isolated abnormality.

We now see that whenever the block numbers of a full T-ring are not relatively prime, then the ring can be "factored" in two different ways.

3.1.2 Theorem. For any ring $F$ with unity,

$$(F_m)_{n_1,\ldots,n_\tau} \cong F_{mn_1,\ldots,\tau mn} \cong (F_{n_1,\ldots,n_\tau})_m,$$

the isomorphisms being both ring isomorphisms and $F$-module isomorphisms.

Proof: (a) Let $A = \{e_{ij} \mid i,j = 1,\ldots,m\}$, $B = \{f_{pq} \mid (p,q) \in T\}$, and $D = \{g_{st} \mid (s,t) \in T'\}$ be the
given sets of matrix units for $F_m , (F_m)_{n_1}, \ldots, n_r$, and $F_{mn_1}, \ldots, mn_r$ respectively. A typical element of $(F_m)_{n_1}, \ldots, n_r$ is then of the form \[
\sum_{p,q} \left( \sum_{i,j} e_{ij} \right) f_{pq} = \sum_{p,q,i,j} e_{ij} f_{pq}.
\]
In fact, \(AB = \{e_{ij} f_{pq} \mid i,j = 1, \ldots, m; (p,q) \in \mathcal{T}\}\) is a basis for the free $F$-module $(F_m)_{n_1}, \ldots, n_r$. Thus we can define an $F$-homomorphism \[
\phi : (F_m)_{n_1}, \ldots, n_r \rightarrow F_{mn_1}, \ldots, mn_r
\]
by \[
\phi(e_{ij} f_{pq}) = g((p-1)m+i)((q-1)m+j).
\]
Suppose \(\phi(e_{ij} f_{pq}) = \phi(e_{hk} f_{st})\). i.e. \[
g((p-1)m+i)((q-1)m+j) = g((s-1)m+h)((t-1)m+k).
\]
Then \((p-1)m+i = (s-1)m+h\) and \((q-1)m+j = (t-1)m+k\) which yields \((p-s)m = h-i\) and \((q-t)m = k-j\). If \(p \neq s\), then \(m \mid (h-i)\); but \(h,i \in \{1, \ldots, m\} \Rightarrow h = i\) and \(p = s\) so we have a contradiction. Thus \(p = s\) and \(h = i\). Likewise \(q = t\) and \(k = j\); hence \(\phi\) is one-to-one. Since
\[
\text{Card } (AB) = m^2 \text{ Card } \mathcal{T} = m^2 \left( \sum_{i \geq j=1}^{r} n_in_j \right) = \sum_{i \geq j=1}^{r} (mn_i)(mn_j) = \text{Card } \mathcal{T}', \phi\text{ is a bijection between the two bases } AB \text{ and } D, \text{ hence it is an } F\text{-isomorphism.}
\]
Is \(\phi(e_{ij} f_{pq} \cdot e_{hk} f_{st}) = \phi(e_{ij} f_{pq}) \phi(e_{hk} f_{st})\)? If \(s = q\) and \(j = h\), then \[
\phi(e_{ij} f_{pq} \cdot e_{hk} f_{st}) = \phi(e_{ik} f_{pt}) = g((p-1)m+i)((t-1)m+k).
\]
If $s \neq q$ or if $j \neq h$, then an examination of cases as in the argument for $\Phi$ being one-to-one shows both sides equal to zero. Thus $\Phi$ is also a ring isomorphism.

(b) Let $B' = \{e_{ij} \mid (i,j) \in T\}$, $A' = \{f_{pq} \mid p, q = 1, \ldots, m\}$, and $D = \{g_{st} \mid (s,t) \in T'\}$ be the given sets of matrix units for $F_{n_1}, \ldots, n_r$, $(F_{n_1}, \ldots, n_r)_m$, and $F_{mn_1}, \ldots, mn_r$ respectively. Let

$$R = \sum_{x,y=0}^{m-1} \sum_{(i,j) \in T} F_{g(xn+i)(yn+j)}. \quad \phi' : (F_{n_1}, \ldots, n_r)_m \rightarrow R$$

defined by

$$\phi'((p,q,i,j \cdot x_{pqij} e_{ij} \cdot f_{pq})) = \sum_{p,q,i,j} x_{pqij} F_{g((p-1)n+i)((q-1)n+j)}$$

is a ring- and an $F$-isomorphism as in (a). We finish by showing that $R$ is ring- and $F$-isomorphic to $S = F_{mn_1}, \ldots, mn_r$. To accomplish this, we will use

3.1.3 Lemma. Suppose $FB$ is a full $T$-ring of rank $n$ with $C$, $D$ two subsemigroups of $B = \{e_{ij} \mid (i,j) \in T\}$. Let $R = FC$ and $S = FD$ be the corresponding subrings of $FB$. If there exists a permutation $p$ of $\{1, \ldots, n\}$ such that the correspondence $e_{ij} \rightarrow e_{p(i)p(j)}$ is a bijection between $C$ and $D$, then $R$ and $S$ are isomorphic as rings and as $F$-modules.
**Proof:** Since $C$ and $D$ are bases for $R$ and $S$ respectively,

$$
\Theta: \mathbb{F}^R \longrightarrow \mathbb{F}^S
$$

$$\Theta(e_{ij}) = e_{p(i)p(j)}$$

is an $\mathbb{F}$-isomorphism. But $\Theta(e_{ij}e_{km}) = \Theta(\delta_{jk}e_{im}) = \delta_{jk}e_{p(i)p(m)}$,

$$e_{p(i)p(j)}e_{p(k)p(m)} = \Theta(e_{ij})\Theta(e_{km}) \text{ so } \Theta \text{ is a ring isomorphism.} \ /$

**Remark:** The proof shows that the given correspondence between $C$ and $D$ is a semigroup isomorphism, even though we needed only to assume that it be a bijection.

To apply the lemma, note that $R$ and $S$ are subrings of $\mathbb{F}_{mn}$, $R = FC$, and $S = FD$ where

$$C = \{g(xn+i)(yn+j) \mid x,y = 0,\ldots,m-1; (i,j) \in T\}$$

and $D = \{g_{st} \mid (s,t) \in T'\}$.
We need a permutation $p$ of $\{1,\ldots,mn\}$ which induces a bijection between $C$ and $D$.

To obtain the desired $p$, we first partition the set $\{1,\ldots,mn\}$ in two different ways. We know that the block numbers $n_1,\ldots,n_r$ of $F_{n_1},\ldots,n_r$ determine a partition

$$P_1 \cup \ldots \cup P_r$$

of $\{1,\ldots,n\}$. Let $P'_1 \cup \ldots \cup P'_r$ be the corresponding partition of $\{1,\ldots,mn\}$ determined by the block numbers $mn_1,\ldots,mn_r$ of $S$; that is,

$$P'_1 = \{j \in \mathbb{N} \mid mn_0 + \ldots + mn_{i-1} < j \leq mn_1 + \ldots + mn_i\}.$$
Let $n + P_i = \{n + j \mid j \in P_i\}$ and $X_i = \bigcup_{k=0}^{m-1} (kn + P_i)$ . Then $X_1 \cup \ldots \cup X_r$ is a second partition of $\{1, \ldots, mn\}$.

Since $\text{Card}(X_i) = mn_i = \text{Card}(P'_i)$ , there exists a bijection $p_i : X_i \rightarrow P'_i$ (i = 1, \ldots, r) .

Let $p = \bigcup_{i=1}^{r} p_i$ . Since the $p_i$ have disjoint domains and disjoint ranges, $p$ is a bijection:

$$p : \bigcup_{i=1}^{r} X_i = \{1, \ldots, mn\} \rightarrow \bigcup_{i=1}^{r} P'_i = \{1, \ldots, mn\}.$$ 

Thus $p$ is a permutation of $\{1, \ldots, mn\}$.

It remains to show that $p$ induces a bijection $b$ between $C$ and $D$ . Let $g_{st} = g_{(xn+i)(yn+j)} \in C$ . Is $g_{p(s)p(t)} \in D$ ? $g_{st} \in C \Rightarrow (i,j) \in T = \bigcup_{h \geq k} P_h \times P_k \Rightarrow (i,j) \in P_h \times P_k$ for some $h \geq k$ $\Rightarrow (s,t) \in X_h \times X_k$ where $h \geq k \Rightarrow (p(s), p(t)) = (p_h(s), p_k(t)) \in P'_h \times P'_k \subseteq T' \Rightarrow g_{p(s)p(t)} \in D$ . Hence the induced function $b$ does at least map $C$ into $D$ . Since $p$ is 1-1 , so is $b$ , and since $\text{Card} C = m^2 \text{Card} T = m^2 \sum_{i \geq j=1}^{r} n_in_j = \sum_{i \geq j=1}^{r} (mn_i)(mn_j) = \text{Card} T' = \text{Card} D$ , $b$ is a bijection. //

3.1.4 Remark. At the start of the proof for 3.1.2 , we knew only that the product sets $AB$ and $B'A'$ were bases. Since they correspond to sets of matrix units under isomorphisms, we now know that they are also sets of matrix
units. In fact,

\[(FA)B \cong F(AB) \cong F(B'A') \cong (FB')A'\,.
\]

The overall picture of the relationships in 3.1.2 is

\[
\begin{array}{c}
(F_m)_{n_1,\ldots,n_r} \xrightarrow{\phi} S \xleftarrow{0} R \xleftarrow{\phi'} (F_{n_1,\ldots,n_r})_m \\
AB \xrightarrow{\phi} D \xleftarrow{b} C \xleftarrow{\phi'} B'A'
\end{array}
\]

3.2 Uniqueness Theorems

Now that we have seen how badly non-unique full T-rings can be in general, are there conditions we can impose which will guarantee uniqueness or at least some degree of uniqueness? The next few theorems begin answering that question. First, let us record what we mean by an IBN-ring.

3.2.1 Definition. A ring \(F\) with unity is said by P. M. Cohn in [6] to have an invariant basis number (or, to be an IBN-ring) iff for every free module \(FM\) with a finite basis, any two bases of \(FM\) have the same number of elements. If \(F\) is an IBN-ring and \(FM\) is a free module with a finite basis, then the number of elements in a basis of \(FM\) is called the rank of \(M\).

3.2.2 Lemma. Let \(R = FB\) be a full T-ring. Then \(R\) is an IBN-ring iff \(F\) is an IBN-ring.
Proof: If $R$ is an IBN-ring, so is $F$ by [7, Proposition 2.5]. Conversely, if $F$ is an IBN-ring, then $F_n$ (where $FB \subseteq F_n$) is an IBN-ring by [20, Theorem 3.5] and $R$ is hence an IBN-ring by [20, Theorem 3.3].

3.2.3 Definition. (a) An IBN-ring $F$ is called a \textbf{strong} IBN-ring iff for every free module $\oplus M$ of finite rank, whenever $M = \oplus M_i$ and $\oplus M_i \cong \oplus M_i \forall i$, where $n = \text{rank } M$, then each $\oplus M_i$ is free.

(b) An IBN-ring is called a \textbf{very strong} IBN-ring iff for every free module $\oplus M$ of finite rank, whenever $M = \oplus M_i$ and $\oplus M_i \cong \oplus M_i \forall i$, then each $\oplus M_i$ is free.

These two concepts are apparently due to Jategaonkar and Johnson respectively.

3.2.4 Theorem. Let $V = F_{n_1}, \ldots, n_v = FB$ and $W = G_{m_1}, \ldots, m_w = GC$ with $B = \{e_{ij} \mid (i,j) \in T\}$ and $C = \{f_{ij} \mid (i,j) \in T'\}$. If $V \cong W$ and either

(1) $F$ is prime and $w > 1$

or (2) $F$ and $G$ are both prime

then

(a) $v = w$ and each $e_iVe_i \cong f_iWf_i$

(b) if $F$ is a very strong IBN-ring, then there exists $q \in \mathbb{N}$ such that $G \cong F_q$ and $n_i = qm_i \forall i$,

(c) if both $F$ and $G$ are very strong IBN, then $F \cong G$, $n_i = m_i \forall i$ and there exists an isomorphism $\phi$
between $V$ and $W$ such that $\Phi[F] = G$ and $\Phi(e_{ij}) = f_{ij}$ for all $(i,j) \in T$.

Proof: Assume (1). By 2.5.5, $A_{\chi}(V)$ is a chain; hence $A_{\chi}(W)$ is also a chain. If $G$ were not prime, $A_{\chi}(W)$ would not be a chain by 2.5.7. Thus we conclude that $G$ must be prime. Hence (1) $\Rightarrow$ (2).

(a) Since $F$ and $G$ are prime, $A_{\chi}(V)$ and $A_{\chi}(W)$ consist precisely of two chains: $L_0 < \ldots < L_v$ and $L'_0 < \ldots < L'_w$ respectively. The given isomorphism $\Theta : V \rightarrow W$ induces a lattice isomorphism between $A_{\chi}(V)$ and $A_{\chi}(W)$. Therefore $v = w$ and $O[L_i] = L'_i \forall i$. Likewise, $A_\mu(V) = \{R_0, \ldots, R_v\}$, $A_\mu(W) = \{R'_0, \ldots, R'_v\}$ and $O[R_i] = R'_i \forall i$. Thus, $e_i V e_i \cong V/(L_{i-1} + R_{v-i}) \cong W/(L'_{i-1} + R'_{v-i}) \cong f_i W f_i$.

(b) Let $F$ be a very strong IBN-ring. $F_{n_i} \cong e_i V e_i \cong f_i W f_i \cong G_{m_i}$ so by [20, Theorem 4.2], there exist $q \in \mathbb{N}$ such that $G \cong F_q$ and $q m_i = n_i$. Now for $i = 2, \ldots, v$, $F_{n_i} \cong e_i V e_i \cong f_i W f_i \cong G_{m_i} \cong (F_q) m_i \cong F_{q m_i}$. Since $F$ is very strong IBN, $n_i = q m_i$.

(c) Now suppose $F$ and $G$ are both very strong IBN-rings. Then $q = 1$ so $n_i = m_i \forall i$ and $F \cong G$. Hence $F_n \cong G_n$. The last statement of the theorem now follows from

3.2.5 Lemma. Suppose $FB$ and $GC$ are two isomorphic full T-rings with the same ordered set of block numbers.
(We know there exist $A = \{e_{ij} \mid i, j = 1, \ldots, n\}$ and $D = \{f_{ij} \mid i, j = 1, \ldots, n\}$ such that $B \subseteq A$ and $C \subseteq D$.) If $FA \cong GD$ and $F$ is strong IBN, then there exists an isomorphism $\phi : FB \to GC$ such that $\phi[F] = G$ and $\phi[B] = C$. In particular, $\phi(e_{ij}) = f_{ij} \forall e_{ij} \in B$.

**Proof:** Let $FA \to GD$ be an isomorphism. Then $GD = F'A'$ by 1.2.9. By [20, Theorem 4.1], there exists a unit $u$ of $GD$ such that $D = uA'u^{-1}$ and $G = uF'u^{-1}$. In particular, $f_{ij} = ue_{ij}u^{-1} \forall i, j = 1, \ldots, n$. If $T_u$ denotes the transform by $u$, let $\phi = (T_u^o') | FB$. Then $\phi : FB \to GC$ and $\phi[F] = T_u[F'] = G$. Since the block numbers are the same, $B$ and $C$ have the same index set. Hence $\phi[B] = C$ and in particular, $\phi(e_{ij}) = f_{ij} \forall e_{ij} \in B$. The situation is summarized in the diagram

$$
\begin{array}{ccc}
FA & \xrightarrow{'} & F'A' \\
\cup & \phi = (T_u^o') | FB & \cup \\
FB & \xrightarrow{iso} & GC
\end{array}
$$

We now show that it is possible to do a little trading of the hypotheses of Theorem 3.2.4 and yet arrive at the same conclusions. We weaken the assumption of very strong IBN to strong IBN and add an assumption of equality between one pair of block numbers.
3.2.6 Theorem. Let $V = F_{n_1, \ldots, n_v}$ and $W = G_{m_1, \ldots, m_w}$ with $B = \{e_{ij} \mid (i,j) \in T\}$ and $C = \{f_{ij} \mid (i,j) \in T\}$. If $V \cong W$ and either

1. $F$ is prime and $w > 1$

or 2. $F$ and $G$ are both prime,

then

(a) $v = w$ and each $e_{ij} V e_{ij} \cong f_{ij} W f_{ij}$

(b) if $F$ is a strong IBN-ring and there exists $k$ such

that $m_k = n_k$, then $F \cong G$, $m_i = n_i$ for all $i$ and

there exists $\phi : V \xrightarrow{\text{iso}} W$ such that $\phi[F] = G$

and $\phi(e_{ij}) = f_{ij} \forall (i,j) \in T$.

Proof: As in 3.2.4, (1) $\Rightarrow$ (2) and (a) holds.

(b) Let $F$ be strong IBN and $m_k = n_k$. Since

$F_{m_k} \cong e_k V e_k \cong f_k W f_k \cong G_{m_k} = G_{n_k} \Rightarrow F \cong G$ by [20, Theorem 4.1].

For each $i$, we now have $F_{n_i} \cong e_{ij} V e_{ij} \cong f_{ij} W f_{ij} \cong G_{m_i} \cong F_{m_i}$.

On page 216 of [7], Cohn has proved

3.2.7 Lemma. The following are equivalent for any ring $F$ and all positive integers $r, s$:

(a) $F$ is an IBN-ring.

(b) If $M$ and $N$ are isomorphic free $F$-modules of rank $r$ and $s$ respectively, then $r = s$.

Therefore $m_i^2 = n_i^2$ and hence $m_i = n_i \forall i$.

The last assertion of the theorem now follows from Lemma 3.2.5. //
The case or a commutative base ring is a very important one which we shall now consider. In the next theorem, the fact that any commutative ring with unity is an IBN-ring [7, Theorem 2.6] proves to be very useful.

Let $Z(R)$ denote the center of a ring $R$.

3.2.8 Lemma. If $R = FB$ is a full T-ring, then $Z(R) = Z(F)$.

Proof: $Z(R) \subseteq \text{Cent } B = F$ so $Z(R) \subseteq Z(F)$. Also, since $F \subseteq R$, we have $Z(F) \subseteq Z(R)$. //

3.2.9 Theorem. Suppose $V = F_{n_1, \ldots, n_v} \cong W = G_{m_1, \ldots, m_w}$ where $F$ is commutative and $n_1 + \ldots + n_v = m_1 + \ldots + m_w$. Then

(a) $G \cong F$

(b) if $F$ is also semiprime and indecomposable, then $w = v$ and there exists a permutation $p$ of $\{1, \ldots, v\}$ such that $m_i = n_p(i) \forall i$

(c) if $F$ is prime, then $w = v$ and $m_i = n_i \forall i$.

Proof: (a) Kaplansky [23, page 160] has shown using results of Amitsur and Levitzki [1] that an $n$ by $n$ total matrix ring over a commutative ring satisfies the "standard" polynomial identity $P_{2n} = \sum_{\sigma \in S_{2n}} (-1)^{\sigma} x_\sigma(1) \cdots x_\sigma(2n)$.

Here, $S_k$ is the symmetric group on $k$ symbols and $(-1)^{\sigma}$ is
+1 or -1 according to whether \( \sigma \) is an even or odd permutation respectively. Since \( F_n \) satisfies \( P_{2n} \), so does \( V \) and hence also \( W \). Thus, \( \forall \ g, h \in G, \ P_{2n} \) is satisfied by the 2n elements \( gf_{nn}, hf_{nn}, f_n n-1, f_{n-1} n-1, f_{n-1} n-2, f_{n-2} n-2, \ldots, f_{22}, f_{21}, f_{11} \) of \( W \) (where the \( f_{ij} \) are matrix units of \( W \)). Plugging these 2n elements into the polynomial, we find that all terms but two vanish and we are left with \( ghf_{n1} - hgf_{n1} = 0 \). Consequently \( gh = hg \forall g, h \in G \) so \( G \) is commutative. By 3.2.8, \( Z(V) = F \) and \( Z(W) = G \) so if \( \Phi \) is the given isomorphism from \( V \) to \( W \), then \( \Phi[F] = G \).

(b) Let \( F \) be semiprime and indecomposable. We will employ the following theorem of Jacobson (see [13, page 42]): If \( A \) is a ring such that \( A^2 = A \) and \( A = A_1 \oplus \ldots \oplus A_k \) where the \( A_i \) are indecomposable ideals, then this decomposition is unique.

Since \( F \) is semiprime, \( \text{rad } F = 0 \) and \( \text{rad } V = \sum_{i>j} e_i V e_j \).

Likewise, \( \text{rad } W = \sum_{i>j} f_i W f_j \). Thus, \( \bigoplus_{i=1}^{\nu} e_i V e_i \cong V/\text{rad } V \)

\[ \cong W/\text{rad } W \cong \bigoplus_{i=1}^{\nu} f_i W f_i \].

Since \( F \) is indecomposable, each \( e_i V e_i \) is indecomposable by 2.4.2. Likewise, each \( f_i W f_i \) is indecomposable. Of course, \( (\sum_{i=1}^{\nu} e_i V e_i)^2 = \sum_{i=1}^{\nu} e_i V e_i \) since this is true for any ring with unity. Therefore, by Jacobson's theorem, \( w = v \) and there
exists a permutation \( p \) of \( \{1, \ldots, v\} \) such that each
\[
G_{m_i} \cong f_i W_{f_i} \cong e_p(i) V_{e_p(i)} \cong F_{n_p(i)}.
\]
But \( F \cong G \) so \( F_{m_i} \cong G_{m_i} \cong F_{n_p(i)} \). Now since \( F \) is an IBN-ring, \( m_i = n_p(i) \)
by 3.2.7.

(c) Now suppose \( F \) is prime. Then \( G \) is also prime, so by 3.2.4, \( v = w \) and each \( e_i V_{e_i} \cong f_i W_{f_i} \). Again since \( F \cong G \) and \( F \) is an IBN-ring, \( m_i = n_i \) for all \( i \).

3.3 More Examples

We have yet to give an example to show the non-permutability of the block numbers. Toward that end, let \( F \) be a field and suppose \( F_{1,2} \cong F_{2,1} \). Since a field is prime and also a very strong IBN-ring, this contradicts theorem 3.2.4 (c). We have now answered negatively all of the questions in the first paragraph of Chapter Three.

Before going on to the next example, we observe that \( F_{1,2} \) and \( F_{2,1} \) are "similar" in at least one respect, and more generally, any full T-ring is similar in this same manner to the new full T-ring obtained by an arbitrary permutation of its block numbers.

3.3.1 Theorem. Let \( R = F_{n_1, \ldots, n_r} \), \( p \) be any permutation of \( \{1, \ldots, r\} \), and \( S = F_{n_{p(1)}, \ldots, n_{p(r)}} \). Then \( F_R \) and \( F_S \) are isomorphic as \( F \)-modules.
Proof: Let \( R = FB \) and \( S = FC \). Since \( B \) and \( C \) are \( F \)-bases of \( R \) and \( S \), it suffices to show that \( \text{Card } B = \text{Card } C \). Now \( \text{Card } B = \sum_{i \geq j=1}^{r} n_i n_j \) and \( \text{Card } C = \sum_{i \geq j=1}^{r} p(i) p(j) \) so let

\[ X = \{n_i n_j \mid i, j = 1, \ldots, r; i \geq j\} \]

and \( Y = \{p(i) p(j) \mid i, j = 1, \ldots, r; i \geq j\} \).

**Claim:** \( X = Y \). Let \( n_i n_j \in X \). Then \( i = p(k) \) and \( j = p(m) \) for some \( k, m \). If \( k \geq m \), then \( n_i n_j = n_{p(k)} n_{p(m)} \in Y \). If \( m > k \), then \( n_i n_j = n_{p(m)} n_{p(k)} \in Y \). Thus \( X \subseteq Y \). Similarly, \( X \supseteq Y \).

Hence \( \text{Card } B = \sum_{x \in X} x = \sum_{y \in Y} y = \text{Card } C \).

If \( F \) is commutative, we can say more about \( F_{1,2} \) and \( F_{2,1} \): They are anti-isomorphic. More generally, any full \( T \)-ring over a commutative ring is anti-isomorphic to the new full \( T \)-ring obtained by reversing its block numbers.

**3.3.2 Theorem.** If \( F \) is commutative, then \( F_{n_1, \ldots, n_r} \) is anti-isomorphic to \( F_{n_r, n_{r-1}, \ldots, n_1} \).

**Proof:** Let \( R = F_{n_1, \ldots, n_r} = FB \) with \( B = \{e_{ij} \mid (i,j) \in T\} \) and \( S = F_{n_r, \ldots, n_1} = FC \) with \( C = \{e_{ij} \mid (i,j) \in T\} \).
\{e_{ij} | (i,j) \in T^t\}. From the proof of 3.3.1 we know that Card B = Card C. Moreover, in this case, there is a natural one-to-one correspondence between B and C: the reflection about the lower-left-to-upper-right diagonal:

\[ e_{ij} \rightarrow e_{(n+1-j)(n+1-i)} \]

This function extends to an F-isomorphism

\[ \theta : R \rightarrow S \]

\[ \theta(\sum_{(i,j) \in T} x_{ij} e_{ij}) = \sum_{(i,j) \in T} x_{ij} e_{(n+1-j)(n+1-i)}; \]

and so all that remains is to show that \( \theta \) reverses products.

Let \( x = \sum_{(i,j) \in T} x_{ij} e_{ij} \) and \( y = \sum_{(k,m) \in T} y_{km} e_{km} \) be in \( R \). Since \( \theta(e_{ij} e_{km}) = \theta(\delta_{j,k} e_{im}) = \delta_{j,k} e_{(n+1-m)(n+1-i)} = e_{(n+1-m)(n+1-k)} e_{(n+1-j)(n+1-i)} = \theta(e_{km}) \theta(e_{ij}) \), we get

\[ \theta(xy) = \theta(\sum_{(i,j) \in T} x_{ij} e_{ij} \sum_{(k,m) \in T} y_{km} e_{km}) = \theta(\sum_{(i,j) \in T} x_{ij} y_{km} e_{ij} e_{km}) = \sum_{(i,j) \in T} x_{ij} y_{km} \theta(e_{ij}) \theta(e_{km}) = \sum_{(k,m) \in T} y_{km} \theta(e_{km}) \sum_{(i,j) \in T} x_{ij} \theta(e_{ij}) = \theta(y) \theta(x) \].

The following example shows that we cannot drop the assumption in 3.2.4 (1) or in 3.2.6 (1) that F is prime.
3.3.3 Example. There exist rings $F$, $G$ with unity, $F$ not prime, such that $F^{1,1,1} \cong G^{1,1}$.

Proof: Let $K$ be any ring with unity, $F = K^{1,1}$ and $G = K^{1,1,1}$.

Claim: $V = F^{1,1,1} = (K^{1,1})^{1,1,1}$ is isomorphic to $W = G^{1,1} = (K^{1,1,1})^{1,1}$. To show this, we follow the pattern of proof used in Theorem 3.1.2. Let $\{g_{st} \mid s, t = 1, \ldots, 6\}$ be the given set of matrix units for $K_6$ and

$$R = \sum_{x \geq y=0}^{2} \sum_{i \geq j=1}^{2} Kg(2x+i)(2y+j).$$

The function $\phi : V \rightarrow R$ defined by $\phi(e_{ij}^{pq}) = g(2(p-1)+i)(2(q-1)+j)$ is an isomorphism as in 3.1.2.

Likewise, for

$$S = \sum_{x \geq y=0}^{1} \sum_{i \geq j=1}^{3} Kg(3x+i)(3y+j),$$

$\phi' : W \rightarrow S$

$\phi'(e_{ij}^{pq}) = g(3(p-1)+i)(3(q-1)+j)$

is an isomorphism. Thus, we need only show $R \cong S$. $R = FC$ and $S = FD$ where

$$C = \{g(2x+i)(2y+j) \mid x \geq y = 0,1,2 ; i \geq j = 1,2\}$$

and

$$D = \{g(3x+i)(3y+j) \mid x \geq y = 0,1 ; i \geq j = 1,2,3\}.$$ 

Define a permutation $p$ of $\{1,2,3,4,5,6\}$ by
\[ p(1) = 1 \quad p(4) = 5 \]
\[ p(2) = 4 \quad p(5) = 3 \]
\[ p(3) = 2 \quad p(6) = 6 \]

By inspection of the diagram below, we see that \( p \) induces a bijection from \( C \) to \( D \).

We conclude \( R \cong S \) by 3.1.3 so \( V \cong R \cong S \cong W \).

3.3.4 Remarks. (a) Let us return for a moment to Theorems 3.2.4 and 3.2.6. Example 3.3.3 has served to show that the hypothesis of primeness for \( F \) in (1) of 3.2.4 and 3.2.6 cannot be eliminated. Now consider assumption (2) of the same theorem; this covers the case of \( w = 1 \). There we assumed that both \( F \) and \( G \) are prime. Can we drop the assumption of primeness for at least one of them? The answer is
no. For let $G = F_{1,1}$. Then $F_{2,2} \not\cong G_2$ (even when $F$ is a prime ring) by Theorem 3.1.2.

(b) To furnish the desired counterexamples above, we needed some non-prime rings. The non-prime rings we exhibited, namely $K_{1,1}$ and $F_{1,1}$, fail very badly to be prime. In fact, they are not even semiprime. (By 2.2.1, the prime radical of any full T-ring is non-zero.) This suggests the possibility of weakening the hypothesis of primeness in 3.2.4 or 3.2.6 to semiprimeness. Whether or not this can be done remains an open question.
CHAPTER IV
ORDERS IN T-RINGS

4.1 Preliminaries

4.1.1 Notation. For any ring Q with subsets R and S, let U(Q) denote the group of units of Q and RS⁻¹, R⁻¹S denote the sets \{rs⁻¹ | r ∈ R ; s ∈ S \cap U(Q)\}, \{r⁻¹s | r ∈ R \cap U(Q) ; s ∈ S\} respectively.

4.1.2 Definition. A subring R of a ring Q with unity will be called a right (left) order of Q when Q = RR⁻¹ (Q = R⁻¹R).

4.1.3 Definition. A ring F is called a local ring if the nonunits of F form an ideal.

The purpose of this chapter is to give a description of right (left) orders in full T-rings. In [9], Faith and Utumi prove a beautiful theorem which describes the right orders in a total matrix ring over a division ring. In [21], Johnson obtains a description of right orders in a total matrix ring over an arbitrary ring. In the local case, a given right order in the total matrix ring arises from a right order in a suitable base ring. In section 4.2, we generalize to obtain the analogous results for a right order in a full T-ring over a local ring.
4.1.4 Lemma. Suppose $R$ is a right order in $Q$ and $R \subseteq S \subseteq Q$ for some subring $S$ of $Q$. Then $S$ is also a right order of $Q$.

**Proof:** For all $q \in Q$, $q = xy^{-1} \in RR^{-1} \subseteq SS^{-1}$. //

The following result is well-known.

4.1.5 Lemma. Let $R$ be a right (left) order in $Q$ and $S$ some finite subset of $Q$. Then there exists $u \in R \cap U(Q)$ such that $Su \subseteq R(uS \subseteq R)$.

4.1.6 Theorem. Let $Q = FB$ be a full $T$-ring of rank $n$ and $B_j = B \cap Qe_{jj}$ for $j = 1, \ldots, n$. If $K_1, \ldots, K_n$ are right orders in $F$ and $R$ is any subring of $Q$ containing $\sum_{j=1}^{n} K_j B_j$, then $R$ is a right order in $Q$. In fact,

$$Q = (\sum_{j=1}^{n} K_j B_j)(\sum_{j=1}^{n} K_j e_{jj})^{-1}$$

**Proof:** Let $q = \sum_{ij} q_{ij} e_{ij} \in Q$. For each $j$, there exists (by 4.1.5) $c_j \in K_j \cap U(F)$ such that $q_{ij} c_j \in K_j$ for all $i$. If $c = \sum_{j=1}^{n} c_j e_{jj}$, then $qc = \sum(q_{ij} c_j)e_{ij} \in \sum K_j B_j$. Thus $q \in (\sum K_j B_j)(\sum K_j e_{jj})^{-1}$ and $R$ is a right order by 4.1.4. //
The first assertion of the next result is a corollary of 4.1.6; however, the last statement requires a different (yet similar) proof.

4.1.7 Corollary. Let $Q = FB$ be a full T-ring of degree $r$ and $B_j = B \cap Qe_j$ for $j = 1, \ldots, r$. If $K_1, \ldots, K_r$ are right orders in $F$, and $R$ is any subring of $Q$ containing $\sum_{j=1}^{r} K_j B_j$, then $R$ is a right order in $Q$. In fact,

$$Q = (\sum_{j=1}^{r} K_j B_j)(\sum_{j=1}^{r} K_j e_j)^{-1}.$$

Proof: Let $q = \sum_{j=1}^{r} \sum_{b \in B_j} q_{b} b \in Q$. For $j = 1, \ldots, r$ there exists $c_j \in K_j \cap U(F)$ such that $q_{b} c_j \in K_j \forall b \in B_j$.

Now let $c = \sum_{j=1}^{r} c_j e_j$. Then $qc = \left(\sum_{j=1}^{r} \sum_{b \in B_j} q_{b} b \right) \left(\sum_{i=1}^{r} c_i e_i\right) = \sum_{j=1}^{r} \sum_{b \in B_j} \sum_{i=1}^{r} (q_{b} c_j) b \in \sum_{j=1}^{r} K_j B_j$. Thus, $q \in \sum_{j=1}^{r} K_j B_j \left(\sum_{j=1}^{r} K_j e_j\right)^{-1}$. //

Remark. Theorem 4.2.1 below shows that every right order in a full T-ring over a local ring can be described as $R$ in 4.1.7.

4.1.8 Corollary. Let $Q = FB$ be a full T-ring. If $K$ is a right order in $F$ and $R$ is any subring of $Q$ containing $KB$, then $R$ is a right order in $Q$. In fact,

$$Q = (KB)K^{-1}.$$
**Proof:** Let $q = \sum_{b \in B} q_b b \in Q$. There exists $c \in \mathcal{K} \cap \mathcal{U}(F)$ such that $q_b c \in \mathcal{K}$ for all $b \in B$. Thus $qc = \sum_{b \in B} (q_b c)b \in KB$ so $q \in (KB)K^{-1}$. //

**Remark.** Corollary 4.2.4 below shows that whenever $F$ is a local ring in which the intersection of any two right orders is a right order, then any right order in a full T-ring over $F$ can be described as $R$ in 4.1.8.

4.1.9 Lemma. Let $Q$ be a full T-ring and $q \in \mathcal{U}(Q)$. Then each $e_i q e_i \in \mathcal{U}(e_i Q e_i)$.

**Proof:** There exists $t$ in $Q$ such that $qt = tq = 1_Q$. Since $qt = 1_Q$, $\sum_{i \geq j} e_i q e_j \sum_{p \geq s} e_p t e_s = \sum_{i \geq j} e_i q e_j t e_s = 1_Q$ equals $e_i$. Each $e_i(q_e t)e_s e_i Q e_s$ so $(e_i q e_i)(e_i t e_i) = e_i$. Similarly, $tq = 1_Q$ implies $(e_i t e_i)(e_i q e_i) = e_i$. Hence $t e_i$ is the inverse in $e_i Q e_i$ of $e_i q e_i$. //

We are now in a position to state and prove the theorem which describes an arbitrary right order in a full T-ring.
4.2 Generalization of the Faith-Utumi Theorem

4.2.1 Theorem. Let \( R \) be a right order in a full \( T \)-ring \( Q \) of degree \( r \). Then

(i) there exist

1. a set \( B \) of matrix units for \( Q \),
2. \( u \in R \cap U(Q) \),

and (3) subrings \( K_1, \ldots, K_r \) of \( F = \text{Cent} B \) such that

(a) for \( B_j = B \cap Qe_j \), \( \sum_{j=1}^{r} K_j B_j \subseteq R \)

and (b) for \( N = \sum_{i>j} e_i Re_j \), \( uRu \subseteq N + \sum_{j=1}^{r} K_j B_j \);

(ii) if \( R \) is also a left order in \( Q \), then for any set \( B \) of matrix units for \( Q \), there exist the unit \( u \) and the subrings \( K_j \) as above such that (a) and (b) hold;

(iii) if \( F \) is a local ring, then each \( K_j \) is a right order in \( F \) and furthermore, \( Q = (\sum_{j=1}^{r} K_j B_j)(\sum_{j=1}^{r} K_j e_j)^{-1} \).

Proof: (i) There exists a set \( A = \{ f_{ij} \mid (i,j) \in T \} \) of matrix units for \( Q \) and \( E = \text{Cent} A \); so \( Q = EA \). By 4.1.5, there exists \( c \in R \cap U(Q) \) such that \( Ac \subseteq R \). Let \( e_{ij} = c^{-1}f_{ij}c \) for all \( (i,j) \in T \), \( B = \{ e_{ij} \mid (i,j) \in T \} \) and \( F = \text{Cent} B \). Then \( F = c^{-1}Ec \) and \( Q = FB \). Again by 4.1.5, there exists \( d \in R \cap U(Q) \) such that \( Bd \subseteq R \). If we now let \( u = dc \), then \( u \in R \cap U(Q) \), \( uB \subseteq R \), and \( Bu \subseteq R \).
For \( j = 1, \ldots, r \) let \( K_j = \{ x \in F : x B_j \subseteq R \} \). For all \( x, y \in K_j \), \((x-y)B_j \subseteq R\) and since \( B_j^2 = B_j \cup \{0\} \), \(xyB_j \subseteq xyB_j^2 = xB_jyB_j \subseteq R\). Thus, \( K_j \) is a subring of \( F \). By definition of \( K_j \), we immediately have (a): \( \sum_{j=1}^{r} K_j B_j \subseteq R \).

Now we show (b): \( uRu \subseteq N + \sum_{j=1}^{r} K_j B_j \). Let \( r \in R \) and \( uru = \sum_{i,j} r_{ij} e_{ij} \), \( r_{ij} \in F \). We know \( uru \in N + \sum_{j=1}^{r} FB_j \) so it remains to show that \( r_{km} \in K_j \) whenever \( k, m \in P_j \). Let \( k, m \in P_j \) and \( e_{st} \in B_j \). Then \( e_{sk}, e_{mt} \in B \) so \( r_{km} e_{st} = e_{sk} urue_{mt} \in R \). Thus \( r_{km} B_j \subseteq R \) and \( r_{km} \in K_j \).

(ii) Suppose \( R \) is a left as well as a right order in \( Q \) and that \( B \) is an arbitrary set of matrix units for \( Q \) with \( F = \text{Cent} B \). By 4.1.5, there exist \( c, d \in R \cap U(Q) \) such that \( cB \subseteq R \) and \( Bd \subseteq R \). For \( u = dc \), we have again \( uB \subseteq R \) and \( Bu \subseteq R \). Repeating the arguments in (i), we obtain the subrings \( K_1, \ldots, K_r \) of \( F \) satisfying (a) and (b).

(iii) Let \( F \) be a local ring. We want to show \( K_j K_j^{-1} = F \), so let \( f \in F \). Let \( m = m_j = n_1 + \ldots + n_{j-1} + 1 \). Since \( Q = RR^{-1} = (uRu)(uRu)^{-1} \), there exist \( a, c \in R \) such that \( f e_{mm} = (uau)(ucu)^{-1} \). By (i) (b), \( uau, ucu \in N + \sum_{j=1}^{r} K_j B_j \). Also, \( uau = \sum_{st} a_{st} e_{st} \) and \( ucu = \sum_{st} c_{st} e_{st} \) for some \( a_{st}, c_{st} \in F \) where \( a_{st}, c_{st} \in K_j \) whenever \( s, t \in P_j \).
Since $ucu$ is a unit in $Q$, $e_j(ucu)e_j$ is a unit in $e_jQe_j$ by 4.1.9. Thus, there exist $f_{xy} \in F$ such that

$$\left( \sum_{s,t \in P_j} c_{st} e_{st} \right) \left( \sum_{x,y \in P_j} f_{xy} e_{xy} \right) = e_j = \sum_{i \in P_j} le_{ii}.$$ Hence, for all $s \in P_j$, $\sum_{t \in P_j} c_{st} f_{ts} = 1$. But $F$ being local implies that for each $s \in P_j$, there exists $t \in P_j$ such that $c_{st}$ is a unit of $F$. In particular, $c_{mt}$ is a unit of $F$ for some $t \in P_j$. Since $f_{mm} ucu = uau$, we get $f_{cm} = a_{mt}$ and so

$$f = a_{mt} c_{mt}^{-1} \in K.K^{-1}.$$ All that remains to be proved is that $Q = (\sum_{j} K_{j}B_{j})(\sum_{j} K_{j}e_{j})^{-1}$; however, this now follows immediately from 4.1.7. //

The case $r = 1$ gives the following

4.2.2 Corollary (Johnson's Theorem). Let $R$ be a right order in a total matrix ring $Q$. Then there exist

(1) a set $B$ of matrix units for $Q$,
(2) $u \in R \cap U(Q)$,
and (3) a subring $K$ of $F = \text{Cent } B$

such that

$$uRu \subset KB \subset R.$$ If $F$ is a local ring, then $K$ is a right order in $F$ and $Q = (KB)K^{-1}$. 
If \( F \) is a (not necessarily commutative) field, then 4.2.2 is simply the now classical Faith-Utumi Theorem.

4.2.3 Example. It would be nice if we could eliminate the \( N \) in (i) (b) of 4.2.1. That is, we would like to be able to say \( uRu \subseteq \bigoplus_{j} j^j \subseteq R \) as in Johnson's Theorem. However, the following example shows that this is impossible.

Let \( Q = \mathbb{Q}_{1,1} \) with \( B = \{e_{11}, e_{21}, e_{22}\} \) as set of matrix units; that is \( Q = \mathbb{Q}B \). Now let \( R = e_{11} \mathbb{Z} + e_{21} \mathbb{Q} + e_{22} \mathbb{Z} \).

A convenient way to visualize \( R \) and \( Q \) is

\[
R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{pmatrix}, \quad Q = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}.
\]

Claim: \( R \) is a right and left order in \( Q \). Clearly, \( R \) is a subring of \( Q \). Let \( q = \sum q_{ij} e_{ij} \in Q \). Each \( q_{ij} = m_{ij} n_{ij}^{-1} \) for some \( m_{ij}, n_{ij} \in \mathbb{Z} \) with \( n_{ij} \neq 0 \). Therefore,

\[
q = m_{11}^{-1}n_{11}^{-1}e_{11} + m_{21}^{-1}n_{21}^{-1}e_{21} + m_{22}^{-1}n_{22}^{-1}e_{22}
= (m_{11}n_{11}^{-1}e_{11} + m_{21}n_{21}^{-1}n_{11}n_{22}^{-1}e_{21} + m_{22}n_{22}^{-1}e_{22})(n_{11}^{-1}n_{22}^{-1})^{-1}e_{RR}^{-1}
= (n_{11}n_{22})^{-1}(m_{11}n_{22}^{-1}e_{11} + m_{21}n_{11}^{-1}n_{22}^{-1}e_{21} + m_{22}n_{11}^{-1}e_{22})e_{R}^{-1}R.
\]

Hence there exist \( u \in R \cap U(Q) \) and subrings \( K_1 \) and \( K_2 \) of \( \mathbb{Q} \) such that \( K_1B_1 + K_2B_2 \subseteq R \) and \( uRu \subseteq \mathbb{N} + K_1B_1 + K_2B_2 \).
Claim: \( uRu \subseteq K_1B_1 + K_2B_2 \). \( u \in R \cap U(Q) \) implies
\[ u = ve_{11} + xy^{-1}e_{21} + we_{22} \text{ for some } v, x, y, w \in \mathbb{Z} \text{ with } v, y, w \neq 0. \]
Now let \( r = (|vw| + 1)^{-1}e_{21} \). Then \( uru = vw( |vw| + 1)^{-1}e_{21} \notin K_1B_1 + K_2B_2 \). This is true because \( K_1B_1 + K_2B_2 \subseteq R \) forces \( K_1 \subseteq \mathbb{Z} \) yet \( vw( |vw| + 1)^{-1} \notin \mathbb{Z} \).

If \( N \) were deleted in 4.2.1 (i) (b), then this example shows that (ii) would not hold. Since we have proved that (ii) does hold, that would be a contradiction. //

4.2.4 Corollary. Let \( R \) be a right order in a full T-ring \( Q \). Then

(i) there exist

(1) a set \( B \) of matrix units for \( Q \),
and (2) a subring \( K \) of \( F = \text{Cent } B \)

such that \( KB \subseteq R \);

(ii) if \( F \) is a local ring in which the intersection of any two right orders is a right order, then \( K \) is a right order in \( F \).

Proof: (i) By 4.2.1, there exist a set \( B \) of matrix units for \( Q \) and subrings \( K_j \) of \( F = \text{Cent } B \) such that
\( \Sigma K_jB_j \subseteq R \). Let \( K = \bigcap_j K_j \). Then \( K \) is a subring of \( F \) and
\( KB = \Sigma KB_j \subseteq \Sigma K_jB_j \subseteq R \).

(ii) Let \( F \) be a local ring with the property that the intersection of any two right orders is a right order. Then each \( K_j \) is a right order in \( F \) and hence so is \( K \). //
The following example shows that the property mentioned above is not vacuous; that is, there do exist local rings in which right orders are finite-intersection-closed.

4.2.5 Example. Right orders in the rationals are closed under finite intersections.

Proof: Let $R$ and $S$ be right orders in $\mathbb{Q}$. Any non-zero additive subgroup of $\mathbb{Q}$ contains a nonzero ideal of $\mathbb{Z}$. Hence there exist nonzero $m$, $n \in \mathbb{Z}$ such that $R \supset m \mathbb{Z}$ and $S \supset n \mathbb{Z}$. Evidently $mn\mathbb{Z} \subset R \cap S$ and $mn\mathbb{Z}$ is a right order in $\mathbb{Q}$. Thus, $R \cap S$ is a right order in $\mathbb{Q}$ by 4.1.4. //

The quotient field of a right Ore domain which is not left Ore is an example of a local ring with two right orders whose intersection is not a right order. For let $K$ be a right but not left Ore domain with quotient field $F$. There exist nonzero $x, y \in K$ such that $Kx \cap Ky = 0$. Then $Kx$ and $Ky$ are right orders in $F$ with intersection not a right order.

Each of the results of this chapter which has been stated for right orders has a left analog. For example, the left analog of Theorem 4.2.1 would read
4.2.6 **Theorem.** Let $L$ be a left order in a full T-ring $Q$ of degree $r$. Then

(i) there exist

1. a set $B$ of matrix units for $Q$,
2. $u \in L \cap \mathbb{U}(Q)$,

and (3) subrings $K_1, \ldots, K_r$ of $F = \text{Cent } B$

such that

(a) for $B_i = B \cap e_i Q$, $\sum_{i=1}^{r} K_i B_i \subseteq L$

and (b) for $N = \sum_{i>j} e_i \mathbb{R} e_j$, $uL u \subseteq N + \sum_{i=1}^{r} K_i B_i$;

(ii) if $L$ is also a right order in $Q$, then for any set $B$
of matrix units for $Q$, there exist the unit $u$ and the
subrings $K_i$ as above such that (a) and (b) hold;

(iii) if $F$ is a local ring, then each $K_i$ is a left order in $F$ and furthermore, $Q = (\sum_{i=1}^{r} K_i e_i)^{-1} (\sum_{i=1}^{r} K_i B_i)$. 
BIBLIOGRAPHY


