ON NORMAL FORMATIONS AND RELATED CLASSES OF GROUPS

STEPHEN ALLAN BACON

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AND RELATED CLASSES OF GROUPS

by

STEPHEN A. BACON

M.S., State University of New York at Albany, 1965

A THESIS

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ABSTRACT

ON NORMAL FORMATIONS AND RELATED CLASSES OF GROUPS

by

STEPHEN A. BACON

This thesis is an investigation of the properties of several well-defined classes of groups and the relationships between them. The first of these is the normal formation. A formation is a nonempty class \( \mathcal{F} \) of groups which is closed with respect to epimorphic images and also has the property that for normal subgroups \( M \) and \( N \) of a group \( G \), \( G/M \in \mathcal{F} \) and \( G/N \in \mathcal{F} \) imply \( G/M \cap N \in \mathcal{F} \). A formation which satisfies the additional property that normal subgroups of members of \( \mathcal{F} \) are also in \( \mathcal{F} \) is called a normal formation.

Each formation \( \mathcal{F} \) gives rise to a normal formation \( \mathcal{F}' \) defined by \( \mathcal{F}' = \{ G \mid \text{for all } N \in G, N \in \mathcal{F} \} \). Each normal formation in turn gives rise to a polyclass, i.e., a class \( \mathcal{B} \) of groups which is closed with respect to normal subgroups and epimorphic images and has the additional property that for each \( N \trianglelefteq G \), if \( N \in \mathcal{B} \), \( G/N \in \mathcal{B} \), then \( G \in \mathcal{B} \). The polyclass \( \mathcal{B}^* \) associated with the normal formation \( \mathcal{F} \) is the class of all groups \( G \) which have a normal series of the form \( ... \).
\[ G = A_0 \supset A_1 \supset ... \supset A_n = E \]
with \( A_i / A_{i+1} \in \mathcal{F} \). This is the "smallest" polyclass which contains the normal formation \( \mathcal{F} \).

The third class of groups that is considered is called a Fitting formation. A Fitting formation \( \mathcal{D} \) is a normal formation which possesses the additional property that for each group \( G \) and for each pair of normal subgroups \( N, M \) of \( G \), \( N \in \mathcal{D}, M \in \mathcal{D} \) imply \( NM \in \mathcal{D} \). The classes of abelian, solvable and nilpotent groups are the obvious prototypes of these three general classes of groups. It is shown that many of the structural properties of the class of solvable groups are not dependent upon the fact that solvability is also a subgroup inherited property. Also, many of the relationships between the derived series and the Fitting series of a solvable group carry over to the corresponding series determined by Fitting formations and their associated polyclasses.

For a formation \( \mathcal{F} \) each finite group \( G \) has a characteristic subgroup \( \mathcal{F}(G) \) with the property that \( G/N \in \mathcal{F} \) if and only if \( \mathcal{F}(G) \subseteq N \). In this paper, the concept of formation is generalized to that of a strong formation so that infinite groups will also have this "commutator-type" subgroup. This generalization is made in such a way that the concepts of formation and strong formation are equivalent for finite groups. The notions of derived series and Fitting series for finite groups are generalized to \( \mathcal{S} \)-derived systems and \( \mathcal{F} \)-Fitting
systems with respect to a strong formation $\mathfrak{F}$. The properties of groups having either of these systems are examined fully. It is shown that the class of groups possessing an $\mathfrak{F}$-Fitting system is a polyclass; for $\mathfrak{F}$-derived systems, this is not the case. For classes of finite groups, a number of conditions which are equivalent to possessing a normal $\mathfrak{F}$-series are considered. It is shown that $\mathfrak{F} = \mathfrak{F}^*$ if and only if $\mathfrak{F}$ is a polyclass.

A class $\mathfrak{B}$ of groups is called saturated if and only if for each group $G$, $G/J(G) \in \mathfrak{B}$ implies $G \in \mathfrak{B}$. A number of conditions which imply saturation for the polyclass associated with a given normal formation are discussed. Among the conditions on $\mathfrak{F}$ that imply saturation for $\mathfrak{F}^*$ are the following: $\mathfrak{F}$ is subgroup inherited; $C \in \mathfrak{F}$ for each prime $p$; or $\mathfrak{F}$ is locally defined. Also, negative answers are provided to the following questions: Is every saturated formation subgroup inherited? Is every polyclass saturated?

Three examples are discussed in detail in order to illustrate the theory and clarify the relationships between the classes of groups which have been considered. The first of these, the class of solvable $K$-groups, i.e., the class of solvable groups that split over each normal subgroup, is shown to be a normal formation that is not subgroup inherited. A second example is a generalization of the class of $\pi$-separable groups, called $(\mathfrak{F}, \pi)$-separable groups. Results which generalize the theorems of P. Hall.
on the existence and conjugacy of Hall $\pi$-subgroups in $\pi$-separable groups are discussed. In addition, the class of groups having normal Hall $\pi$-subgroups which are members of a formation $\mathcal{F}$ is considered. It is shown that the class of $(\mathcal{F},\pi)$-separable groups is the polyclass associated with this class of groups.
INTRODUCTION

This thesis is an investigation of the properties of several well-defined classes of groups and the relationships between them. The first of these is the normal formation. A formation, as defined by Gaschütz [18], is a nonempty class $\mathcal{F}$ of groups which is closed with respect to epimorphic images and also has the property that for normal subgroups $M$ and $N$ of a group $G$, $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$ imply $G/M \cap N \in \mathcal{F}$. A formation which satisfies the additional property that normal subgroups of members of $\mathcal{F}$ are also in $\mathcal{F}$ is called a normal formation. Unlike the earlier work of Gaschütz and Carter [9] on classes of conjugate subgroups in finite solvable groups, this thesis proceeds in a different direction in that the emphasis is on the properties of normal formations and classes of groups which they generate. Also, applications of formation theory to both non-solvable and infinite groups are considered.

Each formation $\mathcal{F}$ gives rise to a normal formation $\mathcal{F}'$ defined by $\mathcal{F}' = \{G | \text{for all } N \leq G, N \in \mathcal{F}\}$. Although it is possible that $\mathcal{F}'$ might reduce to the identity formation, the existence of groups in $\mathcal{F}$ with common properties easily eliminates this possibility. Each normal formation in turn gives rise to a polyclass, i.e., a class $\mathcal{B}$ of groups which is closed with respect to normal subgroups and epimorphic images and has the additional property
that for each $N \triangleleft G$, if $N \in \mathfrak{B}$, $G/N \in \mathfrak{B}$, then $G \in \mathfrak{B}$. The polyclass $\mathfrak{F}$ associated with the normal formation $\mathfrak{F}$ is the class of all groups $G$ which have normal series of the form $G = A_0 \supset A_1 \supset \ldots \supset A_n = E$ with $A_i/A_{i+1} \in \mathfrak{F}$. It is the "smallest" polyclass which contains the normal formation $\mathfrak{F}$. The obvious prototype of this type of situation is the polyclass of solvable groups which is associated with the class of abelian groups.

For finite groups there is another interesting class of groups "between" the abelian groups and the solvable groups called the nilpotent groups. The nilpotent groups serve as a prototype for the classes of groups called Fitting formations. A Fitting formation $\mathfrak{D}$ is a normal formation which possesses the additional property that for each group $G$ and for each pair of normal subgroups $N, M$ of $G$, $N \in \mathfrak{D}$, $M \in \mathfrak{D}$ imply $NM \in \mathfrak{D}$. It is shown that many of the structural properties of the class of solvable groups are not dependent upon the fact that solvability is also a subgroup inherited property. Also, many of the relationships between the derived series and the Fitting series for a solvable group carry over to the corresponding series determined by Fitting formations and their associated polyclasses.

The classes of abelian, nilpotent, and solvable groups are each but one example of the more general classes of groups under consideration. As the theory evolves, a number of applications will be made to other classes of groups which satisfy the given properties. In general,
the theory of normal formations and their associated polyclasses provides a method for showing that classes of groups satisfying certain conditions have a number of common structural properties.

This thesis is divided into six chapters. The first chapter contains an exposition of the basic definitions and fundamental notions which will be needed in the succeeding chapters.

For a formation \( \mathfrak{F} \), each finite group \( G \) has a characteristic subgroup \( \mathfrak{F}(G) \) with the property that \( G/N \in \mathfrak{F} \) if and only if \( \mathfrak{F}(G) \subseteq N \). In Chapter II, the concept of formation is generalized to that of a strong formation so that each infinite group will also have this "commutator-type" subgroup. This generalization is made in such a way that the concepts of formation and strong formation are equivalent for finite groups. The first two sections of this chapter are concerned with generalizations of the derived series and the Fitting series of solvable groups with respect to a strong formation \( \mathfrak{F} \). Theorem 2.4 shows that any group which possesses an \( \mathfrak{F} \)-Fitting system is a polyclass. Example 2.1 shows that for groups having an \( \mathfrak{F} \)-derived system, this is not the case. In the third section of this chapter, classes of groups possessing normal \( \mathfrak{F} \)-series are considered. Theorem 2.5 gives the basic structural properties of the members of this class of groups.

Chapter III is concerned with classes of finite
groups. A number of equivalent conditions for a group to possess a normal $\mathcal{F}$-series are considered. Example 3.1 shows that there exist polyclasses which are "larger" than the class of solvable groups. Three equivalent conditions for a group to be a member of the polyclass $\mathfrak{P}$ associated with the normal formation $\mathcal{F}$ are provided in Theorem 3.2. This chapter also includes a number of theorems which generalize the relationships between the Fitting series and the derived series of a finite group. Theorem 3.7 gives a necessary and sufficient condition for a normal formation to be equal to its associated polyclass.

Saturated classes of groups are examined in the fourth chapter. A class $\mathcal{R}$ of groups is called saturated if and only if for each group $G$, $G/\mathcal{F}(G) \in \mathcal{R}$ implies that $G \in \mathcal{R}$. A number of conditions which imply saturation for the polyclass associated with a given normal formation are discussed. Also, negative answers are provided to the following questions: Is every saturated formation subgroup inherited? Is every polyclass saturated?

In Chapter V, three examples are considered which illustrate the material in the preceding sections and clarify the relationships between the classes of groups previously examined. The first of these, the class of solvable $K$-groups, i.e., the class of solvable groups that split over each normal subgroup, is shown to be a normal formation that is not subgroup inherited. A stronger
version of a theorem of C. Christensen concerning the complementation of the next to last term in the derived series of a solvable $K$-group is also included. A second example is a generalization of the class of $\mathcal{H}$-separable groups, called $(\mathcal{F}, \mathcal{M})$-separable groups. Results which generalize the theorems of P. Hall on the existence and conjugacy of Hall $\mathcal{H}$-subgroups in $\mathcal{H}$-separable groups are discussed. In addition, the class of groups having normal Hall $\mathcal{H}$-subgroups which are members of a formation $\mathcal{F}$ is considered. It is shown that the class of $(\mathcal{F}, \mathcal{M})$-separable groups is the polyclass associated with this class of groups.

The final chapter includes a summary of the relationships between the various classes of groups discussed in this thesis. Also, a number of unanswered questions which were motivated by this paper are discussed.

An attempt has been made to keep this thesis self-contained. Except for occasional references to well-known results, this aim has been achieved. The notation used in this thesis is that which is found in the standard references such as Huppert [23] and Scott [31].
CHAPTER I

FUNDAMENTAL CONCEPTS

This chapter is preliminary in nature. It contains definitions and fundamental notions that will be used in the succeeding chapters.

DEFINITION 1.1. A nonempty class \( \mathcal{F} \) of groups is called a formation if and only if the following conditions are fulfilled:

1. Epimorphic images of members of \( \mathcal{F} \) are in \( \mathcal{F} \).
2. If \( M \) and \( N \) are normal subgroups of a group \( G \) and \( G/M \in \mathcal{F}, G/N \in \mathcal{F} \), then \( G/M \cap N \in \mathcal{F} \).

If the formation \( \mathcal{F} \) has the additional property that

3. Each normal subgroup of a member of \( \mathcal{F} \) is also in \( \mathcal{F} \),

then \( \mathcal{F} \) is called a normal formation.

From the definition of formation one immediately sees that finite direct products of members of \( \mathcal{F} \) are again in \( \mathcal{F} \). By simple induction it may be shown that for a normal formation \( \mathcal{F}, G \in \mathcal{F} \) implies that each subnormal subgroup of \( G \) is a member of \( \mathcal{F} \).

Since the class of groups consisting of only the unit subgroup is a formation, that is contained in every formation, every formation contains a normal formation. The following lemma provides a method for constructing a normal formation
from any given formation.

**LEMMA 1.1.** Let $\mathfrak{X}$ be a formation. If $\mathfrak{X}' = \{ G | \text{for each } N \leq G, N \in \mathfrak{X} \}$, then $\mathfrak{X}'$ is a normal formation.

Proof. Since $E \in \mathfrak{X}$, $\mathfrak{X}' \neq \emptyset$. Let $G \in \mathfrak{X}$ and consider an epimorphic image $G/N$ of $G$. If $K/N \leq G/N$, then $K \leq G$ and $K \in \mathfrak{X}$. But $\mathfrak{X}$ is a formation, thus $K/N \in \mathfrak{X}$ and $\mathfrak{X}'$ is closed with respect to epimorphic images.

Assume $G/M \in \mathfrak{X}'$, $G/N \in \mathfrak{X}'$. Let $K/M \cap N \in \mathfrak{X}'$. Then $K = G$. This implies that $K/M \leq G/M$, $KN/N \leq G/N$ and hence $K/M \in \mathfrak{X}$, $KN/N \in \mathfrak{X}$. By a basic isomorphism theorem, $K/M \cap K \cong K/M \in \mathfrak{X}$. Similarly $K/N \cap K \in \mathfrak{X}$. Thus since $\mathfrak{X}$ is a formation, $K/M \cap N \cong K/[(N \cap K)\cap (M \cap K)] \in \mathfrak{X}$. Therefore $\mathfrak{X}'$ is a formation.

Finally let $G \in \mathfrak{X}'$, $N \leq G$. If $K \leq N$, then $K \leq G$ and $K \in \mathfrak{X}$. Thus $\mathfrak{X}'$ is a normal formation.

If the definition of $\mathfrak{X}'$ as given in Lemma 1.1 is replaced by $\mathfrak{X}' = \{ G | \text{for each subgroup } N \text{ of } G, N \in \mathfrak{X} \}$, a similar argument would show that $\mathfrak{X}'$ is a subgroup inherited formation.

It is possible that with either definition $\mathfrak{X}'$ might reduce to the identity formation. The existence of groups in $\mathfrak{X}$ with common properties easily eliminates this possibility. For example, if for a prime $p$ the cyclic group of order $p$, denoted by $C_p$, is a member of $\mathfrak{X}$, then $\mathfrak{X}' \neq \{ E \}$. More
generally, if \( \mathcal{F} \) contains any simple group, then \( \mathcal{F} \neq \{ E \} \).

A recent result by Peter Neumann [30] shows that any formation of finite nilpotent groups, i.e., groups which can be expressed as a direct product of their Sylow subgroups, is subgroup inherited. This result, along with the easily proven result that the intersection of two formations is again a formation, indicates that any formation \( \mathcal{F} \) of finite groups which contains a nilpotent group has \( \mathcal{F} \neq \{ E \} \).

**Definition 1.2.** A nonempty class of groups \( \mathfrak{A} \) is called a **Fitting class** if and only if the following conditions are satisfied:

1. Each group isomorphic to a normal subgroup of a member of \( \mathfrak{A} \) is a member of \( \mathfrak{A} \).
2. For any two normal subgroups \( M \) and \( N \) of a group \( G \), if \( M \in \mathfrak{A}, N \in \mathfrak{A} \), then \( MN \in \mathfrak{A} \).

A class of groups which is both a formation and a Fitting class is called a **Fitting formation**.

**Definition 1.3.** A **polyclass** \( \mathfrak{B} \) is a nonempty class of groups with the following properties:

1. If \( G \in \mathfrak{B} \), then each normal subgroup and each epimorphic image of \( G \) is a member of \( \mathfrak{B} \).
2. If \( N \) is a normal subgroup of a group \( G \), and \( N \in \mathfrak{B}, G/N \in \mathfrak{B} \), then \( G \in \mathfrak{B} \).

The relationship between the above mentioned classes of groups is given in the following theorem due to Maier [29].
THEOREM 1.1. Every polyclass is a Fitting formation and hence a normal formation.

Proof. Let $\mathcal{B}$ be a polyclass. By definition, $\mathcal{B}$ is closed with respect to normal subgroups and factor groups. To show that $\mathcal{B}$ is a Fitting class, a stronger result than required will be proven. Let $N$ be a normal subgroup of $G$ and let $M$ be any subgroup of $G$ where $N \in \mathcal{B}$, $M \in \mathcal{B}$. By a basic isomorphism theorem, $MN/N \cong M/M \cap N$. But $M \in \mathcal{B}$, thus $M/M \cap N \in \mathcal{B}$. Therefore $MN/N$ is in $\mathcal{B}$. Since $\mathcal{B}$ is a polyclass, $MN/N, N \in \mathcal{B}$ imply that $MN \in \mathcal{B}$. Thus $\mathcal{B}$ is a Fitting class.

Let $H$ and $K$ be normal subgroups of $G$ with $G/H \in \mathcal{B}, G/K \in \mathcal{B}$. Since $\mathcal{B}$ is closed with respect to normal subgroups, $G/H \in \mathcal{B}$ implies $HK/H \in \mathcal{B}$. Thus $K/H \cap K \subseteq HK/H \in \mathcal{B}$. Since $(G/H \cap K)/(K/H \cap K) \cong G/K \in \mathcal{B}$, it follows that $G/H \cap K \in \mathcal{B}$ by the extension property of $\mathcal{B}$. Therefore $\mathcal{B}$ is a formation and hence a Fitting formation.

The class of finite groups provides many common examples of the previously mentioned classes of groups. The class of solvable groups, i.e., the class of finite groups possessing a composition series whose factors are cyclic groups of prime order, and the class of $\Pi$-groups, i.e., the class of finite groups whose order contains only primes from a given set $\mathfrak{P}$ of primes, are well-known examples of polyclasses. The class of nilpotent groups is a Fitting formation but not a polyclass. The class
of supersolvable groups, i.e., the class of finite groups possessing an invariant series all of whose factors are cyclic, and the class of abelian groups are simply normal formations. This list of examples will be extended as the theory evolves.

One immediately notes that each of the above mentioned group properties is subgroup inherited and hence normal subgroup inherited. Next, a polyclass \( \mathcal{B} \), hence a normal formation, will be determined with the property that for each subgroup \( B \) of \( G \), \( G \in \mathcal{B} \) does not necessarily imply \( B \in \mathcal{B} \).

For a given set \( X \), denote by \( \Delta_X \) the set of all ordinal numbers which have cardinality at most the cardinality of the power set of \( X \). Let \( \mathcal{Z} \) denote the class of ordinal numbers.

The remaining definitions in this chapter follow the approach used in Kurosh [26] and Specht [32].

**DEFINITION 1.4.** A collection of subgroups \( \{ H_\alpha \mid \alpha \in \mathcal{Z} \} \) of a group \( G \) is called an ascending normal chain for \( G \) if and only if the following conditions are fulfilled:

1. \( H_0 = E \).
2. For each \( \alpha \in \mathcal{Z} \), if \( \alpha \) has an immediate predecessor \( \alpha' \in \mathcal{Z} \), then \( H_\alpha' \trianglelefteq H_\alpha \); otherwise let \( H_\alpha = \bigcup_{\delta < \alpha} H_\delta \).

If there exists a \( \tau \in \Delta_G \) such that \( H_\tau = G \) and \( H_{\alpha+1} \trianglelefteq H_\alpha \), \( 0 \leq \alpha \leq \tau \), then \( \{ H_\alpha \mid 0 \leq \alpha \leq \tau \} \) is called an ascending normal
system for $G$. An ascending normal system in which $H_\alpha \triangleleft G$, $\alpha \leq \omega \leq \gamma$, is called an ascending invariant system for $G$.

The subgroups $H_{\alpha-1}, H_\alpha$ are said to form a jump in $\{H_\alpha\}$ and the factor groups $H_\alpha/H_{\alpha-1}$ are called factors of $\{H_\alpha\}$.

**DEFINITION 1.5.** A collection of subgroups $\{K_\alpha | \alpha \in \Omega\}$ of a group $G$ is called a **descending normal chain** for $G$ if and only if:

1. $K_0 = G$.
2. For each $\alpha \in \Omega$, if $\alpha$ has an immediate predecessor $\alpha^{-1} \in \Omega$, then $K_\alpha \supseteq K_{\alpha^{-1}}$; otherwise let $K_\alpha = \bigcap_{\beta < \alpha} K_\beta$.

If there exists a $\tau \in \Delta_G$ such that $K_\tau = E$ and $K_\alpha \subseteq K_{\alpha^{-1}}$, $\alpha \leq \alpha \leq \tau$, then $\{K_\alpha | \alpha \leq \alpha \leq \tau\}$ is called a **descending normal system** for $G$. A descending normal system in which $K_\alpha \triangleleft G$, $\alpha \leq \alpha \leq \tau$, is called a **descending invariant system** for $G$.

**DEFINITION 1.6.** A finite normal system for a group $G$, $G = H_0 \supset H_1 \supset H_2 \supset \ldots \supset H_n = E$, is called a **normal series** of length $n$. An invariant series for $G$ is defined similarly.

**LEMMA 1.2 [32].** (a) An ascending (descending) normal system $\{H_\alpha | \alpha \leq \alpha \leq \gamma\}$ for a group $G$ induces an ascending (descending) normal system, possibly with repetitions, in each normal subgroup of $G$. Each factor of the induced normal system is isomorphic to a normal subgroup of a factor of $\{H_\alpha\}$. 

(b) An ascending normal system \( \{H_\alpha \mid 0 \leq \alpha \leq \gamma \} \) for a group \( G \) induces an ascending normal system on each epimorphic image of \( G \); the factors of the induced normal system are epimorphic images of the factors of \( \{H_\alpha \} \).

Proof. For (a) let \( N \triangleleft G \), \( \{H_\alpha \mid 0 \leq \alpha \leq \gamma \} \) an ascending normal system for \( G \). Define \( K_\alpha = H_\alpha \cap N \) for all \( \alpha \), \( 0 \leq \alpha \leq \gamma \). Thus \( K_0 = H_0 \cap N = E \) and \( K_\gamma = H_\gamma \cap N = N \). Also, \( K_\alpha = N \cap H_\alpha \leq N \cap H_{\alpha+1} = K_{\alpha+1} \), \( K_\alpha \leq K_{\alpha+1} \). If \( \mu \) is a limit ordinal, \( \mu \leq \gamma \), then \( \cup K_\alpha = \cup (N \cap H_\alpha) = N \cup \cup H_\alpha = N \cap H_\mu = K_\mu \).

Thus \( \{K_\alpha \mid 0 \leq \alpha \leq \gamma \} \) is an ascending normal system for \( G \); possibly with repetitions.

Consider an arbitrary factor group \( K_\alpha /K_{\alpha-1} \) of the ascending normal system for \( N \). By Zassenhaus' lemma,
\[
K_\alpha /K_{\alpha-1} = N \cap H_\alpha /N \cap H_{\alpha-1} \cong H_{\alpha-1}(N \cap H_\alpha) / H_{\alpha-1} \leq H_\alpha / H_{\alpha-1}.
\]
The proof for the descending normal system follows in a similar manner.

For (b) consider any factor group \( G/N \) of \( G \).
Define \( L_\alpha = H_\alpha N/N \), \( 0 \leq \alpha \leq \gamma \). Then \( L_0 = H_0 N/N = E \), \( L_\alpha = H_\alpha N/N \leq H_{\alpha+1} N/N = L_{\alpha+1} \), \( L_\alpha \leq L_{\alpha+1} \) and \( L_\gamma = H_\gamma N/N = G/N \). For each limit ordinal, \( \mu \leq \gamma \), \( \cup L_\alpha = \cup H_\alpha N/N = N \cup \cup H_\alpha N/N = N H_\mu N/N = L_\mu \).

Thus \( \{L_\alpha \mid 0 \leq \alpha \leq \gamma \} \) is an ascending normal system for \( G \).

Consider an arbitrary factor \( L_\alpha /L_{\alpha-1} \) of this induced normal system. Then, \( L_\alpha /L_{\alpha-1} \cong H_\alpha N/H_{\alpha-1} N \cong H_{\alpha-1} H_\alpha / H_{\alpha-1} N \cap H_\alpha = H_\alpha / H_{\alpha-1} (N \cap H_\alpha) \) by Zassenhaus' lemma. But \( H_\alpha / H_{\alpha-1} (N \cap H_\alpha) \cong (H_\alpha / H_{\alpha-1}) /[H_{\alpha-1} (N \cap H_\alpha) / H_{\alpha-1}] \). Thus \( L_\alpha /L_{\alpha-1} \) is an epimorphic image of \( H_\alpha / H_{\alpha-1} \).

Since the ordinals are a well ordered set, each set
of ordinals which indexes a collection of equal subgroups
in an ascending (descending) normal system has a first and
a last element. Thus replacing every such collection of
subgroups by a single subgroup introduces no new jumps in
the system. Therefore each ascending (descending) normal
system with repetitions can be transformed into a system
without repetitions.

DEFINITION 1.7. An ascending (descending) normal
system \( \{K_\alpha\} \) for a group \( G \) is called a proper refinement
of the ascending (descending) normal system \( \{H_\alpha\} \) if and
only if \( \{H_\alpha\} \subseteq \{K_\alpha\} \).

DEFINITION 1.8. An ascending (descending) normal
(invariant) system which fails to possess proper refine­
ments is called an ascending (descending) composition
(principal) system.

DEFINITION 1.9. A normal (invariant) series which
fails to possess proper refinements is called a composition
(principal) series.

The next theorem provides an example of a polyclass
which is not subgroup inherited.

THEOREM 1.2 [32]. The class of all groups which
possess an ascending composition system (series) is a
polyclass.

Proof. Let \( \mathcal{N} = \{H_\alpha | 0 \leq \alpha \leq \tau\} \) be an ascending
composition system for \( G \). Let \( N \triangleleft G \). For each \( \alpha, 0 \leq \alpha \leq \tau, \)
define $K_\alpha = N \cap H_\alpha$ and $L_\alpha = H_\alpha N/N$. By Lemma 1.2, \{K_\alpha\} and \{L_\alpha\} are ascending normal systems for $N$ and $G/N$, respectively. Each factor $K_\alpha/K_{\alpha-1}$ is isomorphic to a normal subgroup of $H_\alpha/H_{\alpha-1}$. Thus $K_\alpha/K_{\alpha-1} = E$ or $K_\alpha/K_{\alpha-1} \cong H_\alpha/H_{\alpha-1}$. Therefore \{K_\alpha\} is an ascending composition system for $N$.

Each factor $L_\alpha/L_{\alpha-1}$ is an epimorphic image of $H_\alpha/H_{\alpha-1}$. Thus again either $L_\alpha/L_{\alpha-1} = E$ or $L_\alpha/L_{\alpha-1} \cong H_\alpha/H_{\alpha-1}$ and \{L_\alpha\} is an ascending composition system for $G/N$.

Let $M \triangleleft G$ and assume that \{P_\alpha\} and \{R_\beta\} are ascending composition systems for $M$ and $G/M$, respectively. Then the \{P_\alpha\} and the inverse images of the members of \{R_\beta\} in $G$ form an ascending composition system for $G$. Thus the class of groups possessing an ascending composition system is a polyclass. The proof for composition series is analogous.

That the property of possessing an ascending composition series is not a subgroup inherited property is seen in the following well-known example.

**EXAMPLE 1.1.** Let $S_n$ denote the symmetric group on $n$ symbols and $A_n$ the alternating group on $n$ symbols for a given positive integer $n$. Consider $S = \bigcup_{n=1}^{\infty} S_n$. $S$ has a unique normal subgroup $A = \bigcup_{n=1}^{\infty} A_n$ of index 2 which is distinct from $S$ and $E$. The series $E \subseteq A \subseteq C \subseteq S$ is an ascending composition series for $S$. Let $H$ be the infinite abelian group generated by the transpositions
Each normal series $E = H_0 \subset H_1 \subset \ldots \subset H_n = H$ has a proper refinement and $H$ thus fails to have a composition series.
CHAPTER II
STRONG FORMATIONS

Descending $\mathcal{F}$-systems

Consider a formation $\mathcal{F}$ and let $\mathcal{F}(G)$ denote the intersection of all normal subgroups $N$ of a group $G$ for which $G/N \in \mathcal{F}$. For each finite group $G$, $\mathcal{F}(G)$ is characteristic in $G$ and $G/\mathcal{F}(G) \in \mathcal{F}$. In fact $\mathcal{F}(G)$ has the property that

\[(2') \text{ For each } N \trianglelefteq G, \text{ if and only if } \mathcal{F}(G) \subseteq N.\]

If all groups under consideration are finite, then upon replacing condition (2) in Definition 1.1 by $(2')$ one gets an equivalent definition of a formation. For infinite groups this is not the case since $G/\mathcal{F}(G)$ is not necessarily in $\mathcal{F}$. Consequently, modifications must be made.

**DEFINITION 2.1.** A class $\mathcal{F}$ of groups is called a strong formation if and only if:

1. Epimorphic images of members of $\mathcal{F}$ are again in $\mathcal{F}$.

2. Each group $G$ possesses a unique characteristic subgroup $G_\mathcal{F}$ having the property that for each $N \trianglelefteq G$, $G/N \in \mathcal{F}$ if and only if $G_\mathcal{F} \subseteq N$.

If the strong formation $\mathcal{F}$ has the additional property that

3. Each normal subgroup of a member of $\mathcal{F}$ is
also in \( \mathcal{F} \).
then \( \mathcal{F} \) is called a **strong normal formation**.

An obvious example of a strong normal formation is the class \( \mathcal{A} \) of abelian groups. For each group \( G \), \( G_\alpha \) is the commutator subgroup.

Of course, each strong formation is a formation. If \( G/M \in \mathcal{F}, G/N \in \mathcal{F}, \mathcal{F} \) a strong formation, then \( G_\alpha \subseteq M \), \( G_\alpha \subseteq N \) imply that \( G_\alpha \subseteq M \cap N \). This is equivalent to \( G/M \cap N \in \mathcal{F} \). As with a normal formation, subnormal subgroups of members of a strong normal formation \( \mathcal{F} \) are again in \( \mathcal{F} \).

**DEFINITION 2.2.** Let \( \mathcal{A} \) be a formation (strong formation). A collection of subgroups \( \{ F_\alpha(G) \mid \alpha \in \mathcal{A} \} \) of a group \( G \) is called an **\( \mathcal{A} \)-derived chain** for \( G \) if and only if:

1. \( F_0(G) = G \).
2. For each \( \alpha \in \mathcal{A} \) having an immediate predecessor \( \beta \in \mathcal{A} \), \( F_\alpha(G) = \mathcal{A}(F_{\alpha-1}(G)) \); otherwise \( F_\alpha(G) = \bigcap_{\beta} F_\beta(G) \).

If there exists a \( \gamma \in \Delta_G \) such that \( F_\gamma(G) = E \) and \( F_\alpha(G) \subseteq F_{\alpha-1}(G), \alpha \leq \gamma \), then \( \{ F_\alpha(G) \mid 0 \leq \alpha \leq \gamma \} \) is called an **\( \mathcal{A} \)-derived system** for \( G \). An \( \mathcal{A} \)-derived system containing only a finite number of subgroups is called an **\( \mathcal{A} \)-derived series** and the least natural number \( k \) such that \( F_k(G) = E \) is called the **\( \mathcal{A} \)-derived length** of \( G \).

In order to simplify the above notation adopt the convention of letting \( F_\alpha(G) = F_\alpha \) whenever this will not
lead to confusion.

For an arbitrary formation $\mathcal{F}$ and any jump $F_{\alpha-1}$, $F_{\alpha}$ in the $\mathcal{F}$-derived chain of a group, $G$ the relation $F_{\alpha-1}/F_{\alpha} \in \mathcal{F}$ is not always valid. The next lemma shows that for a strong formation this property always holds.

**Lemma 2.1.** If $\mathcal{F}$ is a strong formation, then $\mathcal{F}(G) = G^\mathcal{F}$ for each group $G$.

**Proof.** Let $\mathcal{N} = \{N \mid N \triangleleft G$ and $G/N \in \mathcal{F}\}$. For each $N \in \mathcal{N}$, $G/N \in \mathcal{F}$ implies that $G^\mathcal{F} \leq N$ and hence that $G^\mathcal{F} \leq \mathcal{F}(G)$. But $G/G^\mathcal{F} \in \mathcal{F}$. Thus $G^\mathcal{F} \in \mathcal{N}$ and $\mathcal{F}(G) \leq G^\mathcal{F}$. Therefore $\mathcal{F}(G) = G^\mathcal{F}$.

**Lemma 2.2.** Let $\mathcal{F}$ be a strong formation, $G$ a group, and $N \triangleleft G$. Then $\mathcal{F}(G/N) = \mathcal{F}(G)N/N$.

**Proof.** Let $L/N = \mathcal{F}(G/N)$. Since $G/L \in \mathcal{F}$, $\mathcal{F}(G) \leq L$. Thus $\mathcal{F}(G)N \leq L$ and $\mathcal{F}(G)N/N$ is contained in $\mathcal{F}(G/N)$. But $G/\mathcal{F}(G) \in \mathcal{F}$ implies that $G/\mathcal{F}(G)N \in \mathcal{F}$. Therefore $\mathcal{F}(G/N) = \mathcal{F}(G)N/N$.

**Lemma 2.3.** Let $\mathcal{F}$ be a strong formation and $\{F_{\alpha} \mid \alpha \in \mathcal{X}\}$ the $\mathcal{F}$-derived chain for a group $G$. If $N \triangleleft G$ and $N \subseteq F_{\alpha}$, then $F_{\alpha}(G/N) = F_{\alpha}/N$. In particular for all $\alpha \leq \beta$, $F_{\alpha}(G/F_{\beta}) = F_{\alpha}/F_{\beta}$.

**Proof.** Proceed by transfinite induction. If $\alpha = 0$, then $F_0(G/N) = G/N = F_0/N$. Assume that the result holds for all $\alpha \leq \mu$. If $\mu$ is a limit ordinal, then $F_{\mu}(G/N) = \bigcap_{\alpha \in \mu} F_{\alpha}(G/N) = \bigcap_{\alpha \in \mu} (F_{\alpha}/N) = \bigcap_{\alpha \in \mu} F_{\alpha}/N = F_{\mu}/N$.

If $\mu$ has an immediate predecessor $\mu-1 \in \mathcal{X}$, then by
the induction hypothesis $F_\mu(G/N) = \mathcal{F}(F_{\mu-1}(G/N)) = \mathcal{F}(F_{\mu-1}/N)$.

Note that $(F_{\mu-1}/N)/(\mathcal{F}(F_{\mu-1})/N) \cong F_{\mu-1}/\mathcal{F}(F_{\mu-1}) \in \mathcal{F}$. Thus $\mathcal{F}(F_{\mu-1}/N) \subseteq \mathcal{F}(F_{\mu-1})/N$, i.e., $F_\mu(G/N) \subseteq F_\mu/N$. To show the opposite inclusion let $\mathcal{F}(F_{\mu-1}/N) = L/N$. Since $F_{\mu-1}/L \cong (F_{\mu-1}/N)/(L/N) \in \mathcal{F}$, $F_\mu = \mathcal{F}(F_{\mu-1}) \subseteq L$. Therefore

$$F_\mu/N = \mathcal{F}(F_{\mu-1})/N \subseteq L/N = \mathcal{F}(F_{\mu-1}/N) = F_\mu(G/N).$$

**DEFINITION 2.3.** Let $\mathcal{F}$ be a formation (strong formation). If $\{K_\alpha | \alpha \in \mathcal{K}\}$ is a descending normal chain in a group $G$ such that each factor $K_{\alpha-1}/K_\alpha \in \mathcal{F}$, then $\{K_\alpha\}$ is called a descending normal $\mathcal{F}$-chain for $G$.

Similar definitions can be made for descending normal systems and descending invariant systems.

**LEMMA 2.4.** Let $\mathcal{F}$ be a normal (strong normal) formation. If $\{K_\alpha | \alpha \in \mathcal{K}\}$ is a descending normal $\mathcal{F}$-chain for $G$, then $F_\alpha \subseteq K_\alpha$ for all $\alpha \in \mathcal{K}$, where $\{F_\alpha | \alpha \in \mathcal{K}\}$ is the $\mathcal{F}$-derived chain for $G$.

Proof. Proceed by transfinite induction. For $\alpha = 0$ the result trivially holds. Assume that the result holds for all $\alpha < \mu$. If $\mu$ is a limit ordinal, then $F_\mu = \bigcap_{\alpha \leq \mu} F_\alpha \subseteq \bigcap_{\alpha \leq \mu} K_\alpha = K_\mu$. If $\mu$ is not a limit ordinal, then $K_{\mu-1}$ exists and $K_{\mu-1}/K_\mu \in \mathcal{F}$. Consider $K_\mu F_{\mu-1} \subseteq K_{\mu-1}$. $K_\mu F_{\mu-1}/K_\mu \in \mathcal{F}$ since it is a normal subgroup of $K_{\mu-1}/K_\mu$. Therefore $F_{\mu-1}/K_\mu \cap F_{\mu-1} \subseteq K_\mu F_{\mu-1}/K_\mu \in \mathcal{F}$. Thus $F_\mu \subseteq K_\mu \cap F_{\mu-1}$ which implies $F_\mu \subseteq K_\mu$.

The proof for a strong normal formation is analogous.
THEOREM 2.1. Let $\mathcal{F}$ be a normal (strong normal) formation. A group $G$ possesses an $\mathcal{F}$-derived system if and only if it possesses a descending normal $\mathcal{F}$-system.

Proof. Let $G$ possess a descending normal $\mathcal{F}$-system $\{K_\alpha | 0 \leq \alpha \leq \gamma\}$ and let $\{F_\alpha | \alpha \in \mathcal{F}\}$ be the $\mathcal{F}$-derived chain for $G$. By the previous lemma $F_\gamma \subseteq K_\gamma = E$, and hence $G$ possesses an $\mathcal{F}$-derived system.

Conversely, let $G$ possess an $\mathcal{F}$-derived system $\{F_\alpha | 0 \leq \alpha \leq \gamma\}$. Let $\mathcal{N} = \{N | N \leq G$ and $G/N \in \mathcal{F}\}$. By hypothesis $\mathcal{F}(G) \neq E$ and hence $\mathcal{N} \neq \emptyset$. Let

$$G = N_0, N_1, \ldots, N_\alpha, \ldots, N_\rho, \rho \in \triangle G,$$

be a well-ordering of $\mathcal{N}$. Form the normal subgroups $K_\mu = N_\mu$ for all $\mu$, $0 \leq \mu \leq \rho + 1$.

After deletion of repetitions, $\{K_\alpha | 0 \leq \alpha \leq \rho + 1\}$ is a descending normal system for the jump $G = F_0$, $F_1 = \mathcal{F}(G)$. Since $K_0 = G$ and $F_1 = \mathcal{F}(G) = \cap N_\alpha = K_{\rho + 1}$, $\{K_\alpha | 0 \leq \alpha \leq \rho + 1\}$ forms a descending collection of subgroups between $G$ and $F_1$. If $\nu$ is a limit ordinal, then $K_\nu = \cap N_\alpha = \cap K_\alpha$. If $\nu$ is not a limit ordinal, then $\nu - 1$ exists and by the method of construction $K_\nu \subseteq K_{\nu - 1}$. Suppose that $\beta$ is not a limit ordinal, $0 < \beta \leq \rho + 1$. Then

$$K_\beta = \cap N_\alpha = \cap N_\alpha = (\cap N_\alpha) \cap N_{\beta - 1} = K_{\beta - 1} \cap N_{\beta - 1}.$$

Consider each of the factors $K_{\beta - 1}/K_\beta$. Thus

$$K_{\beta - 1}/K_\beta = K_{\beta - 1}/K_{\beta - 1} \cap N_{\beta - 1} \subseteq K_{\beta - 1} N_{\beta - 1}/N_{\beta - 1} \subseteq G/N_{\beta - 1} \in \mathcal{F}.$$

In a similar manner it can be shown that for each jump $F_{\alpha - 1}, F_\alpha$ in the $\mathcal{F}$-derived system $\{F_\alpha | 0 \leq \alpha \leq \gamma\}$, there exists a descending normal $\mathcal{F}$-system. The insertion of
the descending normal $\mathcal{F}$-systems in the $\mathcal{F}$-derived system refines the $\mathcal{F}$-derived system. This refinement is a descending normal $\mathcal{F}$-system for $G$.

One notes that if $\mathcal{F}$ is a strong formation, then the converse is trivially valid. Hence the theorem also holds for strong formations.

**THEOREM 2.2.** If $\mathcal{F}$ is a strong normal formation, then the following are equivalent for each group $G$:

(a) $G$ possesses a descending invariant $\mathcal{F}$-system.
(b) $G$ possesses a descending normal $\mathcal{F}$-system.
(c) $G$ possesses an $\mathcal{F}$-derived system.

Proof. (a) implies (b) by the definitions of descending invariant and descending normal $\mathcal{F}$-systems.

(b) implies (c) by Theorem 2.1.

Since $\mathcal{F}$ is a strong formation each factor of the $\mathcal{F}$-derived system is a member of $\mathcal{F}$ and thus (c) implies (a).

For each strong normal formation $\mathcal{F}$, let $\mathcal{F}_*$ denote the class of all groups possessing a descending normal $\mathcal{F}$-system.

**LEMMA 2.5.** Let $\mathcal{F}$ be a strong normal formation.

(1) Each normal subgroup of a member of $\mathcal{F}_*$ is also in $\mathcal{F}_*$.

(2) If $N \triangleleft G$ and $N \in \mathcal{F}_*$, $G/N \in \mathcal{F}_*$, then $G \in \mathcal{F}_*$.

Proof. For (a) let $G \in \mathcal{F}_*$, $N \triangleleft G$, and let $\{H_x | x \in \mathcal{T}\}$ be a descending normal $\mathcal{F}$-system for $G$. By
Lemma 1.2(a) \( \mathcal{N} = \{ N \cap H_\alpha \} \), after deletion of repetitions, is a descending normal system for \( N \) with each factor isomorphic to a normal subgroup of the corresponding factor of \( \{ H_\alpha \} \). Since by hypothesis each factor \( H_\alpha / H_\alpha \in \mathfrak{X} \) and \( \mathfrak{X} \) is a strong normal formation, each factor of \( \mathcal{N} \) is also in \( \mathfrak{X} \). Thus \( \mathcal{N} \) is a descending normal \( \mathfrak{X} \)-system for \( N \) and \( N \in \mathfrak{X}^* \).

For (b) let \( \{ K_\alpha \mid 0 \leq \alpha \leq \gamma \} \) be a descending normal \( \mathfrak{X} \)-system for \( K \triangleleft G \) and \( \{ L_\alpha \mid 0 \leq \alpha \leq \mu \} \) a descending normal \( \mathfrak{Y} \)-system for \( G/K \). The inverse images of the members of \( \{ L_\alpha \} \) in \( G \) and \( \{ K_\alpha \} \) form a descending normal \( \mathfrak{X} \)-system for \( G \).

The next example shows that \( \mathfrak{X}^* \) is not necessarily a formation.

**EXAMPLE 2.1.** Let \( \mathfrak{F} \) be the class of abelian groups and \( \mathfrak{F}(G) \) the commutator subgroup of a group \( G \). It has been shown by Kurosh [26, p.38] that for the first infinite ordinal number \( \omega \), the \( \omega \)-th derived group of a free group is \( E \). Thus every free group is in \( \mathfrak{F}_* \). If \( G \in \mathfrak{F}_* \) implies that \( G/N \in \mathfrak{F}_* \) for each \( N \triangleleft G \), then every group is contained in \( \mathfrak{F}_* \) because every group can be represented as a factor group of a free group. Since there exist groups which aren't in \( \mathfrak{F}_* \), \( \mathfrak{F}_* \) is not closed with respect to epimorphic images and hence \( \mathfrak{F}_* \) is not a formation.
**Ascending \( \mathcal{F} \)-systems**

**DEFINITION 2.4.** Let \( \mathcal{F} \) be a formation (strong formation). If \( \{ H_\alpha \mid \alpha \in \tau \} \) is an ascending normal chain in a group \( G \) such that each factor \( H_\alpha /H_{\alpha-1} \in \mathcal{F}, \) then \( \{ H_\alpha \} \) is called an **ascending normal \( \mathcal{F} \)-chain** for \( G \).

Ascending normal \( \mathcal{F} \)-systems and ascending invariant \( \mathcal{F} \)-systems are defined similarly.

**THEOREM 2.3.** Let \( \mathcal{F} \) be a normal (strong normal) formation. The class of all groups possessing an ascending normal \( \mathcal{F} \)-system is a polyclass.

Proof. Let \( G \) possess an ascending normal \( \mathcal{F} \)-system \( \{ H_\alpha \mid 0 \leq \alpha \leq \tau \} \). According to Lemma 1.2, for each \( N \trianglelefteq G, \) \( \mathcal{N} = \{ N \cap H_\alpha \} \) and \( \mathcal{M} = \{ NH_\alpha /N \} \) form ascending normal systems for \( N \) and \( G/N, \) respectively. Each factor \( N \cap H_\alpha /N \cap H_{\alpha-1} \) of \( \mathcal{N} \) is isomorphic to a normal subgroup of \( H_\alpha /H_{\alpha-1}. \) But \( H_\alpha /H_{\alpha-1} \in \mathcal{F}. \) Thus \( N \cap H_\alpha /N \cap H_{\alpha-1} \in \mathcal{F}. \) Also, each factor \( (NH_\alpha /N)/(NH_{\alpha-1}/N) \cong NH_\alpha /NH_{\alpha-1} \) of \( \mathcal{M} \) is the epimorphic image of \( H_\alpha /H_{\alpha-1}. \) Since \( H_\alpha /H_{\alpha-1} \in \mathcal{F} \) and \( \mathcal{F} \) is a formation, then each factor of \( \mathcal{M} \) is a member of \( \mathcal{F}. \)

Let \( M \trianglelefteq G \) and let \( M \) and \( G/M \) possess ascending normal \( \mathcal{F} \)-systems \( \{ K_\alpha \mid 0 \leq \alpha \leq \tau \} \) and \( \{ L_\alpha \mid 0 \leq \alpha \leq \mu \}, \) respectively. Then the ascending normal system determined by \( \{ K_\alpha \} \) and the inverse images of the members of \( \{ L_\alpha \} \) in \( G \) is an ascending normal \( \mathcal{F} \)-system for \( G. \)

For a formation (strong formation) \( \mathcal{F} \) let \( \mathcal{M} (G) \)
denote the characteristic subgroup of $G$ generated by all the normal subgroups of $G$ that are members of $\mathcal{F}$. If $G \in \mathcal{F}$, then $\mathcal{M}(G) = G$. However, it is not necessarily true that $\mathcal{M}(G) \in \mathcal{F}$ for each group $G$.

**DEFINITION 2.5.** Let $G$ be a group, $\mathcal{F}$ a formation (strong formation). An $\mathcal{F}$-Fitting chain for a group $G$ is a collection of subgroups $\{M_\alpha(G) | \alpha \in \mathcal{F}\}$ of $G$ with the following properties:

1. $M_0(G) = E$.
2. For each ordinal $\alpha$ having an immediate predecessor $\alpha - 1 \in \mathcal{F}$, $M_\alpha(G) / M_{\alpha - 1}(G) = \mathcal{M}(G / M_{\alpha - 1}(G))$; otherwise $M_\alpha(G) = \bigcup_{\beta < \alpha} M'_\beta(G)$.

If there exists a $\tau' \in \Delta_G$ such that $M_{\tau}(G) = G$ and $M_{\alpha - 1}(G) \subset M_\alpha(G)$ for all $\alpha$, $\alpha \leq \tau'$, then $\{M_\alpha | 0 \leq \alpha \leq \tau\}$ is called an $\mathcal{F}$-Fitting system for $G$. An $\mathcal{F}$-Fitting system containing only a finite number of subgroups is called an $\mathcal{F}$-Fitting series and the least natural number $k$ such that $M_k(G) = G$ is called the $\mathcal{F}$-Fitting length of $G$.

Let $M_\alpha(G) = M'_\alpha$. It should be noted that although $M_\alpha$ is characteristic in $G$ for all $\alpha \in \mathcal{F}$, one may not conclude that $M_\alpha / M_{\alpha - 1} \in \mathcal{F}$ for each group $G$.

**LEMMA 2.6.** Let $\mathcal{F}$ be a normal (strong normal) formation and let $\{M_\alpha | \alpha \in \mathcal{F}\}$ denote the $\mathcal{F}$-Fitting chain for the group $G$. If $G$ possesses an ascending invariant $\mathcal{F}$-chain $\{H_\alpha | \alpha \in \mathcal{F}\}$, then $H_\alpha \subset M_\alpha$ for each $\alpha \in \mathcal{F}$.
Proof. Proceed by transfinite induction. For \( \alpha = 0 \), the theorem follows trivially. Assume that the theorem is true for all \( \alpha \leq \mu \). If \( \mu \) is a limit ordinal, then \( H_\mu = \bigcup_{\alpha < \mu} H_\alpha \subseteq \bigcup_{\alpha < \mu} M_\alpha = M_\mu \). If \( \mu \) isn't a limit ordinal, then consider the factor group \( H_\mu/H_\mu.I \). By Zassenhaus' lemma,

\[
M_\mu/H_\mu/I = M_\mu/I = M_\mu/I = M_\mu/I.
\]

But \( H_\mu/H_\mu/I = \left[H_\mu/H_\mu/I\right]/\left[H_\mu/I\right] \). Thus since \( H_\mu/H_\mu/I \in \mathcal{F} \) and \( \mathcal{F} \) is a formation, \( M_\mu/H_\mu/I \in \mathcal{F} \). Therefore \( M_\mu/H_\mu/I \in M_\mu \) and hence \( H_\mu \subseteq M_\mu \).

**THEOREM 2.4.** Let \( \mathcal{F} \) be a normal (strong normal) formation. A group \( G \) possesses an ascending invariant \( \mathcal{F} \)-system if and only if \( G \) possesses an \( \mathcal{F} \)-Fitting system.

Proof. Let \( \{ M_\alpha | \alpha \in \mathcal{F} \} \) be the \( \mathcal{F} \)-Fitting chain for \( G \). If \( G \) possesses an ascending invariant \( \mathcal{F} \)-system \( \{ H_\alpha | 0 \leq \alpha \leq \gamma \} \), then by the previous lemma \( H_\alpha \subseteq M_\alpha \) for all \( \alpha, 0 \leq \alpha \leq \gamma \). In particular, \( G = H_\gamma \subseteq M_\gamma \). Therefore the \( \mathcal{F} \)-Fitting chain for \( G \) terminates with \( G \) and hence \( G \) possesses an \( \mathcal{F} \)-Fitting system.

Conversely, let \( G \) possess an \( \mathcal{F} \)-Fitting system \( \{ M_\alpha | 0 \leq \alpha \leq \gamma \} \). Let \( \mathcal{N} = \{ N \mid N \leq G \text{ and } N \in \mathcal{F} \} \). By hypothesis, \( \mathcal{M}(G) \neq \emptyset \) and hence \( \mathcal{N} \) is not empty. Let \( E = N_0, N_1, \ldots, N_\alpha, \ldots, N_\rho, \rho \in \Delta G, \) be a well-ordering of \( \mathcal{N} \). Form the normal subgroups

\[
H_0 = E, \quad H_\alpha = \prod N_\alpha \quad \text{for all } \alpha, 0 \leq \alpha \leq \rho + 1.
\]

After deletion of repetitions, \( \{ H_\alpha | 0 \leq \alpha \leq \rho + 1 \} \) is an ascending invariant system for the jump \( E = M_0, M_1 = \mathcal{M}(G) \). Since
$H_0 = E$ and $M_1 = M(G) = \prod_{\alpha \leq \rho} N_\alpha = H_{\rho+1}$, then \( \{ H_\alpha | 0 \leq \alpha \leq \rho+1 \} \) forms an ascending collection of normal subgroups between $E$ and $M_1$. If $\nu$ is a limit ordinal, then $H_\nu = \prod_{\alpha \leq \nu} N_\alpha = \bigcup H_\alpha$. If $\beta$ is not a limit ordinal, $0 \leq \beta \leq \rho+1$, then

$$H_\beta / H_{\beta-1} = \prod_{\alpha \leq \beta-1} N_\alpha / H_{\beta-1} = (\prod_{\alpha \leq \beta-1} N_\alpha / H_{\beta-1}) N_{\beta-1} / H_{\beta-1} = H_{\beta-1} N_{\beta-1} / H_{\beta-1}.$$ But $H_{\beta-1} N_{\beta-1} / H_{\beta-1} \cong N_{\beta-1} / N_{\beta-1} \cap H_{\beta-1}$ and $N_{\beta-1} \in \mathcal{F}$ imply that $H_\beta / H_{\beta-1} \in \mathcal{F}$.

In a similar manner it can be shown that each jump $M_{\alpha-1}$, $M_\alpha$ of the $\mathcal{F}$-Fitting system \( \{ M_\alpha | 0 \leq \alpha \leq \tau \} \) for $G$ possesses an ascending invariant $\mathcal{F}$-system. The insertion of the ascending invariant $\mathcal{F}$-systems in the $\mathcal{F}$-Fitting system refines the $\mathcal{F}$-Fitting system to an invariant $\mathcal{F}$-

system for $G$. 
DEFINITION 2.6. Let \( \mathcal{F} \) be a formation (strong formation). A normal series \( G = A_0 \triangleright A_1 \triangleright \ldots \triangleright A_n = E \) in which \( A_{i-1}/A_i \in \mathcal{F} \) for all \( i, 1 \leq i \leq n \), is called a normal \( \mathcal{F} \)-series for \( G \) and \( n \) is called the \( \mathcal{F} \)-length of the series. A similar definition can be made for invariant \( \mathcal{F} \)-series.

Applying the results of the previous sections to \( \mathcal{F} \)-series one obtains:

THEOREM 2.5. Let \( \mathcal{F} \) be a strong normal formation.
(a) \( G \in \mathcal{F} \) if and only if \( G \) has \( \mathcal{F} \)-derived length 1.
(b) If \( G \) has a normal (invariant) \( \mathcal{F} \)-series of length \( n \), then \( G \) has an \( \mathcal{F} \)-derived series of length less than or equal to \( n \).
(c) \( G \) has a normal (invariant) \( \mathcal{F} \)-series if and only if \( G \) has an \( \mathcal{F} \)-derived series.
(d) If \( G \) has an invariant \( \mathcal{F} \)-series of length \( n \), then \( G \) has an \( \mathcal{F} \)-Fitting series of length less than or equal to \( n \).
(e) The \( \mathcal{F} \)-Fitting length of \( G \) is less than or equal to the \( \mathcal{F} \)-derived length of \( G \).
(f) If \( G \) possesses a normal \( \mathcal{F} \)-series of \( \mathcal{F} \)-derived length \( n \), then each epimorphic image and each normal subgroup of \( G \) has \( \mathcal{F} \)-derived length less than
or equal to \( n \).

(g) For a subgroup \( N \), \( N \triangleleft G \), if \( G/N \) has \( \mathcal{F} \)-derived length \( j \), then \( F_j \subseteq N \). If \( F_j \subseteq N \), then \( G/N \) has \( \mathcal{F} \)-derived length at most \( j \).

(h) An extension \( G \) of a group \( N \) by a group \( M \), such that \( N \) has \( \mathcal{F} \)-derived length \( n \) and \( M \) has \( \mathcal{F} \)-derived length \( m \), has \( \mathcal{F} \)-derived length less than or equal to \( mn \).

(i) If \( G \) has a normal \( \mathcal{F} \)-series, then each normal subgroup \( N \) of \( G \) is included in a normal \( \mathcal{F} \)-series for \( G \).

Proof. First consider (a). If \( G \in \mathcal{F} \), then \( F_1 = \mathcal{F}(G) = E \) and \( G \) has \( \mathcal{F} \)-derived length 1. If \( G \) has \( \mathcal{F} \)-derived length 1, then \( F_1 = \mathcal{F}(G) = E \). But \( \mathcal{F} \) is a strong formation, thus \( G = G/\mathcal{F}(G) \in \mathcal{F} \).

For (b) apply Lemma 2.4. If \( G = K_0 \supset K_1 \supset \ldots \supset K_n = E \) is a normal \( \mathcal{F} \)-series for \( G \) and \( \{ F_\alpha | \alpha \in \mathcal{A} \} \) is the \( \mathcal{F} \)-derived chain for \( G \), then \( F_i \subseteq K_i \) for all \( i \), \( 0 \leq i \leq n \). Thus since \( K_n = E \), \( F_n = E \) and \( G \) has \( \mathcal{F} \)-derived length less than or equal to \( n \).

Clearly (b) implies the existence of an \( \mathcal{F} \)-derived series for \( G \) whenever \( G \) possesses a normal (invariant) \( \mathcal{F} \)-series. On the other hand, if \( G \) possesses an \( \mathcal{F} \)-derived series \( \{ F_i | 0 \leq i \leq n \} \), then each \( F_i \not\subseteq G \). Since \( \mathcal{F} \) is a strong normal formation \( F_{i-1}/F_i \not\in \mathcal{F} \) for all \( i \), \( 0 \leq i \leq n \). Thus each \( \mathcal{F} \)-derived series for \( G \) is an invariant \( \mathcal{F} \)-series and hence a normal \( \mathcal{F} \)-series.

The proof of (d) requires the application of Lemma 2.6. Let \( E = H_0 \subset H_1 \subset \ldots \subset H_n = E \) be an invariant
\( \mathfrak{F} \)-series for \( G \), \( \{ H_{\alpha} | \alpha \in \mathbb{Z} \} \) the \( \mathfrak{F} \)-Fitting chain for \( G \). Therefore \( H_i \leq M_i \) for all \( i, 0 \leq i \leq n \). But \( H_n = G \) and hence \( M_n = G \). Thus the \( \mathfrak{F} \)-Fitting length of \( G \) is less than or equal to \( n \).

In order to prove (e) just apply (d). Since the \( \mathfrak{F} \)-derived series of a group \( G \) is an invariant \( \mathfrak{F} \)-series for \( G \), the result follows.

Next consider (f). Let \( \{ H_i | 0 \leq i \leq m \} \) be a normal \( \mathfrak{F} \)-series for \( G \). By the method of proof used in Theorem 2.3 it is easily seen that for each normal subgroup \( N \) of \( G \), \( \{ H_i N / N | 0 \leq i \leq m \} \) and \( \{ H_i \cap N | 0 \leq i \leq m \} \), after deletion of repetitions, are normal \( \mathfrak{F} \)-series for \( G/N \) and \( N \), respectively. In particular, the \( \mathfrak{F} \)-derived series for \( G \) is a normal \( \mathfrak{F} \)-series for \( G \). Thus \( N \) and \( G/N \) have normal \( \mathfrak{F} \)-series of length less than or equal to the \( \mathfrak{F} \)-derived length of \( G \).

For (g) let \( N \triangleleft G \) and suppose that \( G/N \) has \( \mathfrak{F} \)-derived length \( j \). Consider the inverse images of the members of the \( \mathfrak{F} \)-derived series of \( G/N \) in \( G \). These images form part of a descending normal \( \mathfrak{F} \)-chain for \( G \). By Lemma 2.4, \( F_j \leq N \). For the second part note that \( F_j \leq F_{j-1} N \leq \ldots \leq F_0 N = G \) forms part of a descending normal \( \mathfrak{F} \)-chain for \( G \). This gives a normal \( \mathfrak{F} \)-series for \( G/N \) and the proof is completed.

In order to prove (h) set up a normal \( \mathfrak{F} \)-series for \( G \) using the inverse images of the members of the \( \mathfrak{F} \)-derived series for \( M \) in \( G \) and the members of the
$\cong$-derived series for $N$. This series is a normal $\cong$-series for $G$ of length $mn$. Then by part (b), the $\cong$-derived length of $G$ is less than or equal to $mn$.

Consider (i). If $G$ has a normal $\cong$-series, then by part (f) both $N$ and $G/N$ possess normal $\cong$-series. The normal $\cong$-series for $N$ and the inverse images of the terms of the normal $\cong$-series for $G/N$ form a normal $\cong$-series for $G$ containing $N$.

**THEOREM 2.6.** Let $\cong$ be a strong normal formation. If $G$ has $\cong$-derived length $n$, then for each epimorphism $f$ of $G$, $f[F_j(G)] = F_j[f(G)]$, $1 \leq j \leq n$.

Proof. Let $N = \text{Ker}(f)$. If $N \subseteq F_j$, then by Lemma 2.3, $F_j(G/N) = F_j(G)/N$, i.e., $F_j[f(G)] = f[F_j(G)]$.

If $F_j \subseteq N$, then $G/N$ has $\cong$-derived length $\leq j$. Thus $F_j[f(G)] = F_j(G/N) = E$. Suppose that neither of these conditions occur. Proceed by mathematical induction. For $n = 1$, the theorem reduces to Lemma 2.2. Assume that the theorem holds for $j-1$. Then $F_{j-1}(G/N) = F_{j-1}N/N$ and $F_j(G/N) = F[F_{j-1}(G/N)] = F(F_{j-1}N/N)$ by the induction hypothesis. Note that $(F_{j-1}N/N)/(F_jN/N) = F_{j-1}N/F_jN$. But $F_{j-1}N/F_jN$ is an epimorphic image of $F_{j-1}/F_j \in \cong$. Thus $F(F_{j-1}N/N) \subseteq F_jN/N$.

To show the opposite inclusion consider the chain

$$G \supseteq F_1N \supseteq F_2N \supseteq \ldots \supseteq F_{j-1}N \supseteq \cong(F_{j-1}N) \supseteq \ldots$$

This is a normal $\cong$-chain for $G$. By Lemma 2.4, $F_j \subseteq \cong(F_{j-1}N)$ and thus $F_jN \subseteq \cong(F_{j-1}N)N$. Therefore
For each strong formation \( \mathcal{F} \), let \( \mathcal{F}^* \) denote the class of all groups which possess an \( \mathcal{F} \)-derived series; equivalently a normal or invariant \( \mathcal{F} \)-series. Let \( \mathcal{F}_n^* \) represent the class of all groups which possess an \( \mathcal{F} \)-derived series of length less than or equal to \( n \).

**THEOREM 2.7.** Let \( \mathcal{F} \) be a strong normal formation.

(a) \( \mathcal{F}^* \) is a polycrass.

(b) \( \mathcal{F}_n^* \) is a normal formation.

Proof. If \( G \in \mathcal{F}^* \), \( N \triangleleft G \), then by Theorem 2.5(f), \( N \in \mathcal{F}^* \), \( G/N \in \mathcal{F}^* \). By Theorem 2.5(h), for each \( N \triangleleft G \), \( N \in \mathcal{F}^* \) and \( G/N \in \mathcal{F}^* \), imply that \( G \in \mathcal{F}^* \) thus completing the proof of (a).

For (b) let \( G \in \mathcal{F}_n^* \), \( N \triangleleft G \). By Theorem 2.5(f), \( N \in \mathcal{F}_n^* \), \( G/N \in \mathcal{F}_n^* \). Suppose that \( H \triangleleft G \), \( K \triangleleft G \) and \( G/H \in \mathcal{F}_n^* \), \( G/K \in \mathcal{F}_n^* \). By Theorem 2.5(g), there exist integers \( i, j, 0 \leq i, j \leq n \), such that \( F_i \subseteq H \), \( F_j \subseteq K \). Let \( k = \max\{i, j\} \leq n \). Thus \( F_k \subseteq H \cap K \) and hence \( G/H \cap K \in \mathcal{F}_n^* \).

**COROLLARY 2.7.1.** Let \( \mathcal{F} \) be a strong normal formation.

(a) If \( N \triangleleft G \), \( M \triangleleft G \) and \( N \in \mathcal{F}_n^* \), \( M \in \mathcal{F}_n^* \), then \( NM \in \mathcal{F}_n^* \).

(b) Each group \( G \) with the ascending chain condition for normal subgroups contains a unique characteristic
subgroup that is maximal in $G$ with respect to being in $\mathfrak{F}^*$ and to containing each normal subgroup of $G$ which is in $\mathfrak{F}^*$.

Proof. By the previous theorem $\mathfrak{F}^*$ is a poly-class. Hence part (a) holds by Theorem 1.1.

Part (b) follows immediately from part (a).
CHAPTER III

FORMATIONS OF FINITE GROUPS

If only finite groups or groups satisfying the minimal condition for subgroups are considered, then the concepts of formation and strong formation are equivalent. Thus, by placing either of these restrictions on the class of groups under consideration, all theorems proven for strong formations and their corresponding $\mathcal{F}$-series can be restated for formations. Unless otherwise specified, all groups considered in the remainder of this thesis will be finite.

THEOREM 3.1. Let $\mathcal{F}$ be a normal formation. For each group $G$ the following are equivalent:

(a) $G$ possesses an invariant $\mathcal{F}$-series.
(b) $G$ possesses a principal $\mathcal{F}$-series.
(c) $G$ possesses a composition $\mathcal{F}$-series.
(d) $G$ possesses a normal $\mathcal{F}$-series.
(e) $G$ possesses an $\mathcal{F}$-derived series.
(f) $G$ possesses an $\mathcal{F}$-Fitting series.

Proof. Let $G$ possess an invariant $\mathcal{F}$-series

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = E.$$  

Refine this series to a principal series for $G$. Consider any arbitrary link

$$G_i = H_{i0} \supset H_{i1} \supset H_{i2} \supset \ldots \supset H_{ik_i} = G_{i+1}$$

of such a refinement. For all $j$, $0 \leq j \leq k_i$, $H_{ij} \subseteq G$. By hypothesis, $G_i / G_{i+1} \in \mathcal{F}$. Since $\mathcal{F}$ is a normal formation,
then \( G_i / H_{ij} \in \mathcal{F} \), \( 0 \leq j \leq k_i \). Therefore \( H_{i(j-1)}/H_{ij} \leq G/H_{ij} \) implies that \( H_{i(j-1)}/H_{ij} \in \mathcal{F} \). Thus \( G \) possesses a principal \( \mathcal{F} \)-series.

In a similar manner, the equivalence of (c) and (d) can be shown.

By Theorem 2.2, the statements (a), (d) and (e) are equivalent. The statements (a) and (f) are equivalent by Theorem 2.4.

By Theorem 2.7, the class of groups possessing one of these equivalent types of series forms a polyclass. An obvious prototype of this arrangement is the class of abelian groups which gives rise to the polyclass of soluble groups. Another application of this theorem follows the next lemma.

**Lemma 3.1.** The class of elementary groups, i.e., the class of groups \( \mathcal{E} \) having the property that \( G \in \mathcal{E} \) if and only if \( \mathcal{F}(H) = E \) for each subgroup \( H \) of \( G \), is a normal formation.

Proof. Let \( G \) be an elementary group and let \( H \) be any subgroup of \( G \). Since each subgroup of \( H \) is also a subgroup of \( G \), then \( H \) is an elementary group.

Let \( \theta \) be a homomorphism of the elementary group \( G \) and let \( H \) be a subgroup of \( G \Theta \). There exists a subgroup \( K \) of \( G \) such that \( K\Theta = H \). Also, there is a subgroup \( A \) of \( K \) such that \( \mathcal{F}(K\Theta) = [\mathcal{F}(A)]\Theta \). But \( K \) is an elementary group, thus \( \mathcal{F}(A) = E \) and hence \( \mathcal{F}(K\Theta) = [\mathcal{F}(K)]\Theta = E \).
Therefore \( \mathcal{E}(H) = \mathcal{E}(K \mathfrak{G}) = E \). Hence \( G \mathfrak{G} \) is an elementary group.

Let \( G/M \) and \( G/N \) be elementary groups. Without loss of generality, let \( M \cap N = E \). Denote the natural homomorphism from \( G \) to \( G/M \) by \( \alpha \). If \( H \) is any subgroup of \( G \), then \( H \) has an image \( H \alpha \) in \( G/M \). Since \( G/M \) is an elementary group, then \( \mathcal{E}(H \alpha) = E \). But \( \mathcal{E}(H) \alpha \subseteq \mathcal{E}(H \alpha) \). Therefore \( \mathcal{E}(H) \alpha = E \) and hence \( \mathcal{E}(H) \subseteq M \).

Similarly \( \mathcal{E}(H) \subseteq N \). Thus \( \mathcal{E}(H) \subseteq M \cap N = E \) for each subgroup \( H \) of \( G \).

An equivalent definition for the class of elementary groups is the class of groups \( \mathcal{E} \) having the property that \( G \in \mathcal{E} \) if and only if \( \mathcal{E}(P) = E \) for each Sylow subgroup \( P \) of \( G \). If \( \mathcal{E}(P) = E \) for each Sylow subgroup \( P \) of \( G \), then each of the Sylow subgroups is elementary abelian. Thus by a result of Gaschutz \([16]\), \( G \) splits over each of its normal subgroups. In particular, \( G \) splits over \( \mathcal{E}(G) \). But \( G \) splits over \( \mathcal{E}(G) \) if and only if \( \mathcal{E}(G) = E \). A similar argument shows that \( \mathcal{E}(H) = E \) for each subgroup \( H \) of \( G \).

**EXAMPLE 3.1.** Let \( \mathcal{E} \) be the class of elementary groups. Since the class of elementary groups contains the cyclic groups of prime order, then each solvable group has an \( \mathcal{E} \)-composition series. Consider the symmetric group on five symbols, \( S_5 \). Denote the alternating group on five symbols by \( A_5 \). Since \( A_5 = 2^2 \cdot 3 \cdot 5 \), then \( A_5 \) is an
elementary group if and only if its Sylow 2-subgroups are Klein four groups. Let $K$ be a Sylow 2-subgroup of $A_5$. If $K$ is cyclic, then $A_5$ has a normal subgroup of index four. But $A_5$ is a simple group. Hence $K$ is a Klein four group and $A_5 \in \mathcal{C}$. Thus the class of finite solvable groups is properly contained in $\mathcal{C}^*$.

The definitions of normal formations, Fitting formations and polyclasses are all concerned with normal subgroups having the required properties. Actually, for finite groups, all the examples that have been considered have been subgroup inherited. The next question to be answered is whether or not there are well-defined classes of groups which are normal formations but not subgroup inherited. To give an example of one such class of groups, the following lemma due to B. Huppert [24] will be required.

**Lemma 3.2.** If $p$ is a fixed prime and $\mathcal{F}$ is the set of finite groups with abelian Sylow $p$-subgroups that possess no nonidentity $p$-factor groups, then $\mathcal{F}$ is a formation.

**Proof.** First show that $\mathcal{F}$ is closed with respect to epimorphic images. Let $G$ have an abelian Sylow $p$-subgroup $P$ and let $N \triangleleft G$. Then $PN/N \cong P/P \cap N$ is a Sylow $p$-subgroup of $G$. $PN/N$ is abelian since $P$ is abelian. Assume that $G/N$ has a normal subgroup $H/N$ such that $(G/N)/(H/N)$ is a nonidentity $p$-factor group. Since $G/H \cong (G/N)/(H/N)$, then $G$ would also have a nonidentity $p$-factor
group. Thus \( \mathfrak{F} \) is closed with respect to epimorphic images.

Let \( G/N_i \in \mathfrak{F}, \ i = 1,2, \) and \( N_1 \cap N_2 = E. \) If \( R \) is a Sylow \( p \)-subgroup of \( G, \) then \( R' = R' \cap N_1 \cap N_2 = E \) and consequently, \( R \) is abelian. Assume there exists a subgroup \( M \) of \( G \) such that \( M \trianglelefteq G \) and \( |G/M| = p. \) If \( N_i \trianglelefteq M, \ i = 1,2, \) then \( G/N_i \) would have a normal subgroup \( M/N_i \) with the property that \( (G/N_i)/(M/N_i) \cong G/M. \) Since \( G/N_i \in \mathfrak{F}, \) a contradiction arises. Thus \( N_i \not\in M \) and \( G = MN_i. \) Let \( L = (M \cap N_1) \trianglelefteq (M \cap N_2) \) and \( \bar{G} = G/L. \) If \( U \) is a subgroup of \( G, \) then designate \( UL/L \) by \( \bar{U}. \) So \( \bar{G} = \bar{M} \cap \bar{N}_1 = \bar{M} \cap \bar{N}_2 \) and \( |ar{N}_i| = |ar{G}/\bar{M}| = |G/M| = p. \) Since each \( \bar{N}_i \) is prime cyclic, then

\[
(N_1 \cap N_2) \cap \bar{M} \subseteq Z(\bar{G}) \cap \bar{M} \subseteq Z(\bar{M}).
\]

Since \( \bar{M} \) is an epimorphic image of \( G/N_i, \) then \( \bar{M} \) possesses an abelian Sylow \( p \)-subgroup. Hence \( p \nmid |\bar{M} \cap Z(\bar{M})|. \)

If \( p \nmid |\bar{M} \cap Z(\bar{M})|, \) then \( \bar{M} \cap Z(\bar{M}) \) has an element of order \( p \) which is contained in an abelian Sylow \( p \)-subgroup \( \bar{F} \) of \( \bar{M}. \) Hence \( \bar{M} \cap Z(\bar{M}) \cap \bar{F} \neq E \) and a contradiction is reached.

Since \( |\bar{N}_1 \cap \bar{N}_2| = p^2, \) if \( (\bar{N}_1 \cap \bar{N}_2) \cap \bar{M} = E, \) then \( \bar{G} \cong (\bar{N}_1 \cap \bar{N}_2) \cap \bar{M} \) and \( |ar{G}/\bar{M}| = |G/M| = p^2. \) This contradicts the choice of \( M. \) Thus \( (\bar{N}_1 \cap \bar{N}_2) \cap \bar{M} \) is a nontrivial \( p \)-group. Therefore it follows that \( (\bar{N}_1 \cap \bar{N}_2) \cap \bar{M} \not\subseteq \bar{M}'. \) Since \( (\bar{N}_1 \cap \bar{N}_2) \cap \bar{M} \subseteq Z(\bar{M}), \) if \( (\bar{N}_1 \cap \bar{N}_2) \cap \bar{M} \subseteq \bar{M}', \) then \( p \mid |Z(\bar{M}) \cap \bar{M}'| \) and a contradiction arises. But \( \bar{M} \) has an abelian Sylow \( p \)-subgroup and \( p \nmid |\bar{M}'|. \) Thus \( p \nmid |\bar{M}/\bar{M}'|. \) Then \( \bar{G}' = \bar{M}' \cap \bar{N}_1 = \bar{M}' \) implies that \( \bar{G}/\bar{G}' = \bar{M}/\bar{M}' \cap \bar{N}_1 \bar{M}/\bar{M}' \cong \bar{M}/\bar{M}' \cap \bar{N}_1. \) Hence
Therefore

\[ \frac{G}{G'} \cong \frac{M}{M'} \cong \mathfrak{S}_p \cong (S_1 \oplus S_2 \oplus \ldots \oplus S_k) \oplus \mathfrak{N}_1 \]

where \( S_p \cong (S_1 \oplus S_2 \oplus \ldots \oplus S_k) \) is a decomposition of the abelian groups \( \frac{M}{M'} \) into a direct product of cyclic groups. Since \( p \mid |M/M'| \), then one of the factors, denoted by \( S_p \), is a \( p \)-group. Denote \( S_1 \oplus S_2 \oplus \ldots \oplus S_k \) by \( K \).

Hence \( G/N_1LKG' \cong S_p \). Thus \( G/N_1 \) has a normal subgroup \( N_1LKG'/N_1 \) such that \( (G/N_1)/(N_1LKG'/N_1) \cong G/N_1LKG' \cong S_p \).

This contradicts \( G/N_1 \notin \mathfrak{F} \). Therefore \( G \in \mathfrak{F} \) and the result follows.

EXAMPLE 3.2. Let \( p = 3 \) and let \( \mathfrak{F} \) be the formation of the previous lemma. The symmetric group on three symbols, \( S_3 \), is a member of \( \mathfrak{F} \). \( S_3 \) has a normal subgroup, \( C_3 \), which fails to belong to \( \mathfrak{F} \). Thus \( \mathfrak{F} \) is not a normal formation. Let \( \mathfrak{D} = \{ G \mid \text{for each } N \leq G, N \in \mathfrak{F} \} \). By Lemma 1.1, \( \mathfrak{D} \) is a normal formation. \( C_5 \in \mathfrak{D} \) implies that \( \mathfrak{D} \neq \{ \mathfrak{E} \} \). \( \mathfrak{D} \) is not subgroup inherited since \( A_5 \in \mathfrak{D} \) but \( C_3 \) is not in \( \mathfrak{D} \).

Let \( \mathfrak{D}^* \) be the polyclass generated by \( \mathfrak{D} \). Once again \( A_5 \in \mathfrak{D}^* \) but \( C_3 \notin \mathfrak{D}^* \). Thus \( \mathfrak{D}^* \) is not subgroup inherited.

In the next chapter, a better known and much more widely studied example of a normal formation which is not subgroup inherited will be considered.

The next lemma by Durbin [13] will be used to find necessary and sufficient conditions for a group to be a
member of the polyclass generated by a given normal formation.

**Lemma 3.3.** Let \( \mathfrak{R} \) be a class of groups possessing the property that factor groups and subnormal subgroups of members of \( \mathfrak{R} \) are again in \( \mathfrak{R} \). Let \( G \) be a group having the property that for each nonidentity element \( x \in G \) there exist subgroups \( K \triangleleft H \triangleleft G \), with \( K \triangleleft G \), such that \( x \in H \setminus K \) and \( H/K \in \mathfrak{R} \). Then

(a) each epimorphic image of \( G \) also has the above mentioned property and

(b) each minimal normal subgroup of \( G \) is in \( \mathfrak{R} \).

**Proof.** For (a), assume that \( G \) has the required property and that \( N \) is a normal subgroup of \( G \). Let \( \overline{x} \) be a nonidentity member of \( G/N \); \( \overline{x} \) denotes the image of \( x \) under the natural mapping from \( G \) onto \( G/N \). By assumption, there exist subgroups \( K \triangleleft H \triangleleft G \) such that \( K \triangleleft G \), \( x \in H \setminus K \) and \( H/K \in \mathfrak{R} \). Application of the Theorem of Zassenhaus shows that \( HN/KN \) is an epimorphic image of \( H/K \). Hence \( HN/KN \in \mathfrak{R} \). Thus \( H/K \cong HN/KN \) implies that \( H/K \in \mathfrak{R} \).

Since \( x \in H \), then \( \overline{x} \in H = HN/N \). If \( \overline{x} \not\in K = KN/N \), then \( H \) and \( K \) are the required subgroups of \( G/N \). Assume that \( \overline{x} \in K \). Let \( K_0 = K, H_0 = H \), and proceed inductively to form a chain of normal subgroups of \( G \),

\[
H_0 \supseteq K_0 \supseteq H_1 \supseteq K_1 \supseteq H_2 \supseteq K_2 \supseteq \ldots
\]

with the property that \( \overline{x} \in H_i = H_i N/N \) and \( H_i/K_i \in \mathfrak{R} \) for each \( i \). Suppose \( \overline{x} \in K_i = K_i N/N \) for some \( i \); such is the
case for $i = 0$. Then $x = kn$ with $k \in K_i \setminus N$, $n \in N$.

By hypothesis, there exist subgroups $B \triangleleft A \triangleleft G$, with $B \triangleleft G$, such that $k \in A \setminus B$ and $A/B \in \mathcal{R}$. Let $H_{i+1} = A \cap K_i$, $K_{i+1} = B \cap K_i$. Then $K_{i+1} \triangleleft H_{i+1} \triangleleft G$, $K_{i+1} \triangleleft G$, and $k \in H_{i+1}/K_{i+1}$. Since $H_{i+1}/K_{i+1} \cong B(A \cap K_i)/B \triangleleft A/B$, then $H_{i+1}/K_{i+1} \in \mathcal{R}$. Thus for $H_{i+1}/K_{i+1}$, which is an epimorphic image of $H_{i+1}/K_{i+1}$, $H_{i+1}/K_{i+1} \in \mathcal{R}$. Hence $K_{i+1}$ and $H_{i+1}$ are the required subgroups of $G/N$, unless $x \notin K_{i+1}$. Also, $H_i \supset K_i \supseteq H_{i+1} \supseteq K_{i+1}$. Since this chain must terminate after a finite number of terms, say $H_m = K_m$, then $K_{m-1}$ and $H_{m-1}$ will always be a pair of subgroups that will satisfy the condition.

Consider (b). Let $N$ be a minimal normal subgroup of $G$, $x$ a nonidentity element of $N$. By hypothesis, there exist subgroups $K \triangleleft H \triangleleft G$, with $K \triangleleft G$, such that $x \in H \setminus K$ and $H/K \in \mathcal{R}$. Since $N$ is minimal normal in $G$, then $K \cap N = \{e\}$ and $H \cap N = N$. Thus $N = N\cap N \cong KN/K \triangleleft H/K$. But $H/K \in \mathcal{R}$ and hence $N \in \mathcal{R}$.

THEOREM 3.2. Let $\mathfrak{F}$ be a normal formation. For each group $G$ the following are equivalent:

(a) $G \in \mathfrak{F}^*$.

(b) For each nonidentity element $x \in G$, there exist subgroups $K \triangleleft H \triangleleft G$, with $K \triangleleft G$, such that $x \in H \setminus K$ and $H/K \in \mathfrak{F}$.

(c) For each epimorphic image $H$ of $G$ and for each minimal normal subgroup $N$ of $H$, $N \in \mathfrak{F}$.

Proof. If $G \in \mathfrak{F}^*$, then $G$ possesses an invariant
that clearly satisfies condition (b). Thus (a) implies (b).

Assume that $G$ satisfies condition (b). Let $H$ be any epimorphic image of $G$. Since $G$ satisfies (b), then $H$ also does by the previous lemma. Thus, by the second part of this lemma, for each minimal normal subgroup $N$ of $H$, $N \in \mathfrak{F}$. Therefore (b) implies (c).

Let condition (c) hold. Let $N_1$ be a minimal normal subgroup of $G$. Since $G$ is an epimorphic image of itself, $N_1 \in \mathfrak{F}$ by hypothesis. If $N_1 = G$, then the required normal $\mathfrak{F}$-series for $G$ has been constructed. Thus $G \in \mathfrak{F}^*$. If $N_1 \neq G$, consider the epimorphic image $G/N_1$ of $G$. Let $N_2/N_1$ be any minimal normal subgroup of $G/N_1$. By hypothesis, $N_2/N_1 \in \mathfrak{F}$. If $N_2 = G$, then $G \in \mathfrak{F}^*$. If $N_2 \neq G$, then continue the above process. But $G$ is a finite group, so $N_n = G$ for some positive integer $n$. Therefore $E = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_n = G$ is a normal $\mathfrak{F}$-series for $G$ and $G \in \mathfrak{F}^*$. Thus (a) follows and the proof is completed.

**COROLLARY 3.2.1.** Let $\mathfrak{F}$ be a normal formation.

(a) If $G \in \mathfrak{F}^*$ and $N$ is a minimal normal subgroup of $G$, then $N \in \mathfrak{F}$.

(b) If $G \in \mathfrak{F}^*$, then $\text{Soc}(G) \in \mathfrak{F}$.

(c) If $G \in \mathfrak{F}^*$ and $\mathfrak{F}(G) = E$, then $\text{Fit}(G) \in \mathfrak{F}$.

Proof. Statement (a) is obvious from the previous theorem.

Next, consider (b). According to Scott [31, p.168], the sockel of $G$, $\text{Soc}(G)$, is the direct product of minimal
normal subgroups of $G$. From (a), every minimal normal subgroup $N$ of $G$ belongs to $\mathfrak{F}$. Since the direct product of members of $\mathfrak{F}$ is again a member of $\mathfrak{F}$, then $\text{Soc}(G) \in \mathfrak{F}$.

As for (c), Scott has proven [3, p. 169] that $\text{Soc}(G)$ is the direct product of its abelian sockel and its non-abelian sockel. The abelian sockel is a direct product of abelian minimal normal subgroups of $G$. Thus the abelian sockel is a member of $\mathfrak{F}$. If $\mathfrak{E}(G) = E$, then $\text{Fit}(G)$, the Fitting group of $G$, is the abelian sockel of $G$ [3, p. 170]. Consequently, $\text{Fit}(G) \in \mathfrak{F}$.

Letting $\mathfrak{F}$ be the class of abelian groups in Theorem 3.2 one immediately deduces:

**COROLLARY 3.2.2.** For each group $G$ the following are equivalent:

(a) $G$ is solvable.

(b) For each nonidentity element $x \in G$ there exist subgroups $K \triangleleft H \triangleleft G$, with $K \triangleleft G$, such that $x \in H \setminus K$ and $H/K$ is abelian.

(c) For each epimorphic image $H$ of $G$ and for each minimal normal subgroup $N$ of $H$, $N$ is abelian.

Next, the relationship between the $\mathfrak{F}$-Fitting length and the $\mathfrak{F}$-derived length of a group $G$ is considered.

**Lemma 3.4.** Let $\mathfrak{F}$ be a normal formation. If
the group \( G \) has \( \mathcal{F} \)-derived length \( n \), then

(a) \( G \) has \( \mathcal{F} \)-Fitting length less than or equal to \( n \).

(b) \( F_{n-i} \leq M_i \) for all \( i, i = 0,1,\ldots,n \).

Proof. Let \( G \) have \( \mathcal{F} \)-derived length \( n \), i.e.,
\[
G = F_0 \supset F_1 \supset \ldots \supset F_n = E
\]
is the \( \mathcal{F} \)-derived series for \( G \).

By Theorem 3.1, \( G \) possesses an \( \mathcal{F} \)-Fitting series
\[
E = M_0 \leq M_1 \leq \ldots \leq M_k = G.
\]
The \( \mathcal{F} \)-derived series for \( G \) is an invariant \( \mathcal{F} \)-series. Moreover, \( F_{n-i} \leq M_i \) for all \( i, 0 \leq i \leq n \), by Lemma 2.4. In particular, \( G = F_0 \leq M_n \). Thus \( G \) has \( \mathcal{F} \)-Fitting length less than or equal to \( n \).

**THEOREM 3.3.** Let \( \mathcal{F} \) be a Fitting formation. If the group \( G \) has \( \mathcal{F} \)-derived length \( n \), then

(a) \( G \) has \( \mathcal{F} \)-Fitting length \( n \).

(b) \( F_{n-i} \leq M_i \) for all \( i, i = 0,1,\ldots,n \).

Proof. Let \( G \) have \( \mathcal{F} \)-Fitting length \( k \). All that remains to be proven is that \( n \) is less than or equal to \( k \). Since \( \mathcal{F} \) is a Fitting formation, each factor of the \( \mathcal{F} \)-Fitting series for \( G \) is in \( \mathcal{F} \). Thus the \( \mathcal{F} \)-Fitting series for \( G \) is an invariant \( \mathcal{F} \)-series for \( G \).

By Theorem 2.5(b), \( n \leq k \), and the theorem is proven.

**COROLLARY 3.3.1 [23, p. 279].** Let \( G \) be a solvable group. Define characteristic subgroups \( K_i \) and \( R_i \) of \( G \) recursively as follows:

(1) Let \( K_0 = E \) and \( K_i/K_{i-1} = \text{Fit}(G/K_{i-1}) \).

(2) Let \( R_0 = G \) and let \( R_i \) be the smallest
normal subgroup of \( R_{i-1} \) with nilpotent factor group.
If \( K_{n-1} \subseteq K_n = G \) and \( R_{m-1} \supseteq R_m = E \), then \( m = n \) and \( R_i \subseteq K_{n-i} \).

Proof. Let \( \mathcal{F} \) be the class of nilpotent groups in the preceding theorem.

COROLLARY 3.3.2. For any solvable group \( G \) the Fitting length of \( G \) is less than or equal to the derived length of \( G \).

Proof. Let \( \mathcal{N} \) represent the class of nilpotent groups. If \( G \) has derived length \( n \), then the derived series for \( G \) is an invariant \( \mathcal{N} \)-series for \( G \). Hence by Theorem 2.5(b), \( G \) has \( \mathcal{N} \)-derived length less than or equal to \( n \). Therefore the Fitting length of \( G \) is less than or equal to the derived length by the previous theorem.

One could have just as easily started with an \( \mathcal{F} \)-Fitting series for \( G \) and shown that the corresponding \( \mathcal{F} \)-derived series had equal length. This is stated without proof.

THEOREM 3.4. Let \( \mathcal{F} \) be a Fitting formation. If the group \( G \) has \( \mathcal{F} \)-Fitting length \( r \), then

(a) \( G \) has \( \mathcal{F} \)-derived length \( r \).

(b) \( F_i \subseteq M_{r-i} \) for all \( i, i = 0, 1, \ldots, r \).

Another generalization of a property of the class of nilpotent groups is given next.

THEOREM 3.5. Let \( \mathcal{D} \) be a Fitting class. If \( A \)
is subnormal in $G$ and $A \in \mathcal{D}$, then there exists a normal subgroup $B$ of $G$ such that $B \in \mathcal{D}$ and $A \subseteq B$.

Proof. Let $B$ be a maximal subnormal subgroup of $G$ such that $B \in \mathcal{D}$ and $A \subseteq B$. If $B \triangleleft G$, then the proof is completed. Assume that $B \not\triangleleft G$. Since $B \triangleleft G$, there exist subnormal subgroups $B_1$ and $B_2$ of $G$ such that $B \subseteq B_1 \subseteq B_2$, where $B \triangleleft B_1$, $B_1 \triangleleft B_2$ but $B \not\triangleleft B_2$. Hence there exists a subgroup $C$ of $B_1$ which is conjugate to $B$ but different from $B$. But $\mathcal{D}$ is a Fitting class. Thus $BC \triangleleft B_1$ and $BC \in \mathcal{D}$. So $BC \triangleleft G$ and $A \subseteq B \subseteq BC \in \mathcal{D}$. This is a contradiction to the choice of $B$. Consequently, $B \triangleleft G$ and the result follows.

COROLLARY 3.5.1. (Ito) If $A$ is a nilpotent subnormal subgroup of a group $G$, then there exists a normal nilpotent subgroup $B$ of $G$ that contains $A$.

COROLLARY 3.5.2. Let $\mathcal{D}$ be a Fitting class. If $A$ is subnormal in $G$, then $\mathfrak{m}(A) \subseteq \mathfrak{m}(G)$.

Proof. Since $A \triangleleft G$ and $\mathfrak{m}(A)$ is characteristic in $A$, then $\mathfrak{m}(A) \triangleleft G$. By the previous lemma, there exists an $N \triangleleft G$ such that $N \in \mathcal{D}$ and $\mathfrak{m}(A) \subseteq N$. But $N \subseteq \mathfrak{m}(G)$ by definition. So $\mathfrak{m}(A) \subseteq \mathfrak{m}(G)$.

THEOREM 3.6. Let $\mathcal{F}$ be a Fitting formation. Let $S$ be a subnormal subgroup of $G$ with $\mathcal{F}$-derived length $n$. If $\{M_\alpha(G) | \alpha \in \mathcal{F}\}$ is the $\mathcal{F}$-Fitting chain for $G$, then $S \subseteq M_n(G)$.

Proof. By Theorem 3.3, if $S$ has $\mathcal{F}$-derived
length \( n \), then \( S \) has \( \mathcal{F} \)-Fitting length \( n \). Hence
\[
E = M_0(S) \subseteq M_1(S) \subseteq \ldots \subseteq M_n(S) = S
\]
is the \( \mathcal{F} \)-Fitting series for \( G \). Proceed by induction on the \( \mathcal{F} \)-derived length, equivalently the \( \mathcal{F} \)-Fitting length, of \( S \). If \( n = 0 \), then the result trivially follows.

Assume that for each subnormal subgroup \( R \) of \( G \) of \( \mathcal{F} \)-derived length \( n-1 \), \( R \subseteq M_{n-1}(G) \). Consider \( \mathcal{F}(S) \). Since \( S \) has \( \mathcal{F} \)-derived length \( n \), \( \mathcal{F}(S) \) has \( \mathcal{F} \)-derived length \( n-1 \). Using the induction hypothesis, \( \mathcal{F}(S) \subseteq M_{n-1}(G) \).

Consider \( G/M_{n-1}(G) \). \( S \) subnormal in \( G \) implies that
\[
S^{M_{n-1}(G)/M_{n-1}(G)} \subseteq G/M_{n-1}(G).
\]
By Corollary 3.5.2,
\[
\mathcal{M}(S^{M_{n-1}(G)/M_{n-1}(G)}) \subseteq \mathcal{M}(G/M_{n-1}(G)) = M_n(G)/M_{n-1}(G).
\]
By a basic isomorphism theorem,
\[
S^{M_{n-1}(G)/M_{n-1}(G)} = S/S \cap M_{n-1}(G).
\]
But \( \mathcal{F}(S) \subseteq S \) and \( \mathcal{F}(S) \subseteq M_{n-1}(G) \). Thus \( \mathcal{F}(S) \subseteq S \cap M_{n-1}(G) \).

Hence \( S/S \cap M_{n-1}(G) \in \mathcal{F} \). Therefore \( S^{M_{n-1}(G)/M_{n-1}(G)} \in \mathcal{F} \).

So \( \mathcal{M}(S^{M_{n-1}(G)/M_{n-1}(G)}) = S^{M_{n-1}(G)/M_{n-1}(G)} \). Therefore
\[
S^{M_{n-1}(G)} \subseteq M_n(G).
\]
This implies that \( S \in M_n(G) \).

The next theorem tells when a normal formation is the same as the polyclass associated with it.

**THEOREM 3.7.** Let \( \mathcal{F} \) be a normal formation.

\( \mathcal{F} = \mathcal{F}^\ast \) if and only if \( \mathcal{F} \) is a polyclass.

Proof. By Theorem 2.7, \( \mathcal{F}^\ast \) is a polyclass for each normal formation \( \mathcal{F} \). Thus if \( \mathcal{F} = \mathcal{F}^\ast \), then \( \mathcal{F}^\ast \) is a polyclass.

Conversely, let \( \mathcal{F} \) be a polyclass. If \( G \in \mathcal{F}^\ast \),
then $G$ has an $\mathcal{F}$-derived series and hence, by Theorem 3.3, an $\mathcal{F}$-Fitting series of the same length. Let $n$ be the $\mathcal{F}$-Fitting length of $G$ and let

$$E = M_0 \leq M_1 \leq \ldots \leq M_n = G$$

be the $\mathcal{F}$-Fitting series for $G$, where $M_i/M_{i-1} = \mathfrak{m}(G/M_{i-1})$. By definition, $M_1 = \mathfrak{m}(G)$ is the maximal normal subgroup of $G$ which is contained in $\mathfrak{F}$. Assume $M_1 \neq G$. $M_2$ is defined so that $M_2/M_1 = \mathfrak{m}(G/M_1) \in \mathfrak{F}$. Since $\mathfrak{F}$ is a polyclass, $M_1 \in \mathfrak{F}$ and $M_2/M_1 \in \mathfrak{F}$ imply that $M_2 \in \mathfrak{F}$. But $M_1 \leq M_2$. This contradicts the choice of $M_1$. Thus $\mathfrak{m}(G) = M_1 = G$ and $G \in \mathfrak{F}$. Therefore $\mathfrak{F}^* \subseteq \mathfrak{F}$ and since it is always valid that $\mathfrak{F} \subseteq \mathfrak{F}^*$, then it follows that $\mathfrak{F} = \mathfrak{F}^*$.

**LEMMA 3.5.** Let $\mathfrak{F}$ and $\mathfrak{D}$ be normal formations.

(a) If $\mathfrak{F} = \mathfrak{D}^*$, then $\mathfrak{F} = \mathfrak{F}^*$. In particular, $(\mathfrak{F}^*)^* = \mathfrak{F}^*$.

(b) If $\mathfrak{F} \subseteq \mathfrak{D}$, then $\mathfrak{F}^* \subseteq \mathfrak{D}^*$.

(c) If $\mathfrak{F} \subseteq \mathfrak{D} \subseteq \mathfrak{F}^*$, then $\mathfrak{D}^* = \mathfrak{F}^*$.

**Proof.** By Theorem 2.7, $\mathfrak{D}^*$ is a polyclass. Thus $\mathfrak{F} = \mathfrak{D}^*$ implies that $\mathfrak{F}$ is a polyclass. Hence $\mathfrak{F} = \mathfrak{F}^*$ by the previous theorem.

For (b), let $\{A_i \mid 0 \leq i \leq n\}$ be a normal $\mathfrak{F}$-series for $G$. Since $\mathfrak{F} \subseteq \mathfrak{D}$, for each factor $A_{i-1}/A_i$, then $A_{i-1}/A_i \in \mathfrak{F}$ implies that $A_{i-1}/A_i \in \mathfrak{D}$. Thus $\{A_i\}$ is a normal $\mathfrak{F}$-series for $G$. So $\mathfrak{F}^* \subseteq \mathfrak{D}^*$.

Consider (c). $\mathfrak{F} \subseteq \mathfrak{D}$ implies that $\mathfrak{F}^* \subseteq \mathfrak{D}^*$, by part (b). $\mathfrak{D} \subseteq \mathfrak{F}^*$ implies that $\mathfrak{D}^* \subseteq (\mathfrak{F}^*)^* = \mathfrak{F}^*$. 


by part (a). Thus $D^* = X^*$. 

**THEOREM 3.8.** Given a normal formation $X$, any polyclass containing $X$ also contains $X^*$. 

Proof. Assume there exists a polyclass $B$ such that $X \leq B \leq X^*$. By the previous lemma, $B^* = X^*$. But $B$ is a polyclass, thus $B = B^*$. Hence $B = X^*$. 

**EXAMPLE 3.3.** If $A$ is the class of abelian groups, then $A^* = I$, where $I$ is the class of solvable groups. Also, for the class of nilpotent and supersolvable groups, denoted by $N$ and $SL$, respectively, $A \leq N \leq SL \leq I$ and hence $A^* = N^* = SL^* = I$. 

CHAPTER IV

SATURATION

DEFINITION 4.1. A nonempty class $\mathcal{R}$ of groups is called saturated if and only if for each group $G$, $G/\mathcal{E}(G) \in \mathcal{R}$ implies that $G \in \mathcal{R}$.

In this chapter, the question of whether or not each polyclass is saturated is considered. To begin this investigation, consider the polyclass generated by a subgroup inherited formation.

THEOREM 4.1. If $\mathcal{F}$ is a subgroup inherited formation, then $\mathcal{F}^*$ is saturated.

Proof. Assume that $G/\mathcal{E}(G) \in \mathcal{F}^*$ and $G \in \mathcal{F}$. Thus $\mathcal{E}(G) \not\in \mathcal{F}^*$; otherwise $G \in \mathcal{F}^*$ by the extension property of $\mathcal{F}^*$. Then $\mathcal{E}(G)$ possesses a composition series in which at least one of the composition factors is not a member of $\mathcal{F}$. But $\mathcal{E}(G)$ is nilpotent and hence solvable. Therefore $\mathcal{E}(G)$ has a prime cyclic composition factor, $C_p$, which is not in $\mathcal{F}$. But $p|\mathcal{E}(G)|$ implies $p|G/\mathcal{E}(G)|$. Thus $G/\mathcal{E}(G)$ has a composition factor, $H/K$, with the properties that $p|H/K|$ and $H/K \in \mathcal{F}$. So $H/K$ has a subgroup of order $p$. But $\mathcal{F}$ is subgroup inherited. Thus $C_p \in \mathcal{F}$. This contradiction implies that $G \in \mathcal{F}^*$ and $\mathcal{F}^*$ is saturated.

COROLLARY 4.1. If $\mathcal{F}$ is a formation of nilpotent groups, then $\mathcal{F}^*$ is saturated.
Proof. By a result of P. Neumann [30], every formation of nilpotent groups is subgroup inherited. The required result follows from the previous theorem.

Using a method of proof similar to that used in Theorem 4.1, the validity of the following theorem is easily established.

THEOREM 4.2. Let $\mathcal{F}$ be a normal formation, and let $G$ be a solvable group. $G \in \mathcal{F}^*$ if and only if $G/\Phi(G) \in \mathcal{F}^*$.

THEOREM 4.3. Let $\mathcal{B}$ be a polyclass. If $C_p \in \mathcal{B}$ for all primes $p$, then $\mathcal{B}$ is saturated.

Proof. Let $G/\Phi(G) \in \mathcal{B}$. Using the properties of $p$-groups and the extension property of $\mathcal{B}$, it is readily seen that every $p$-group is in $\mathcal{B}$. Since $\Phi(G)$ is nilpotent, it is the direct product of its Sylow subgroups. But $\mathcal{B}$ is a polyclass and hence closed with respect to direct products. Thus $\Phi(G) \in \mathcal{B}$. By the extension property of $\mathcal{B}$, $G/\Phi(G) \in \mathcal{B}$ and $\Phi(G) \in \mathcal{B}$ imply that $G \in \mathcal{B}$.

Example 3.2 shows that there exist polyclasses which fail to contain all the cyclic groups of prime order. Thus this theorem does not answer the question of whether or not each polyclass is saturated.

Next, a method of constructing saturated formations is given. This method is due to Gaschütz [18].
DEFINITION 4.2. For each prime $p$, let $\mathcal{F}(p)$ denote either a formation or the null set. Let $\mathcal{F}$ be the class of groups $G$ which satisfy the following conditions:

1. If $\mathcal{F}(p) = \emptyset$, then $p \nmid |G|$.
2. Each $p$-principal factor $H/K$ is $\mathcal{F}$-central in $G$, i.e., $\mathcal{F}_G(H/K) \subseteq G/\mathcal{C}_G(H/K) \in \mathcal{F}(p)$, if $\mathcal{F}(p) \neq \emptyset$. $\mathcal{F}$ is said to be locally defined by $\{\mathcal{F}(p)\}$.

For each principal series $E = G_0 \leq G_1 \leq \ldots \leq G_s = G$ of $G$, let $\mathcal{S}_{p',p}(G)$ denote the intersection of all the $p$-principal factors of $G$.

LEMMA 4.1. If $\mathcal{F}$ is locally defined by $\{\mathcal{F}(p)\}$, then the following are equivalent:

(a) $G \in \mathcal{F}$.
(b) For $\mathcal{F}(p) = \emptyset$, $p \nmid |G|$ and $G/\mathcal{C}_{p',p}(G) \in \mathcal{F}(p)$ for $\mathcal{F}(p) \neq \emptyset$.

Proof. If $G \in \mathcal{F}$ and $\mathcal{F}(p) \neq \emptyset$, then for each $p$-principal factor $H/K$ of $G$, $G/\mathcal{C}_G(H/K) \in \mathcal{F}(p)$. Since $\mathcal{F}(p)$ is a formation, $G/\cap \mathcal{C}_G(H/K) = G/\mathcal{C}_{p',p}(G) \in \mathcal{F}(p)$.

Conversely, let $G/\mathcal{C}_{p',p}(G) \in \mathcal{F}(p)$. Since $G/\mathcal{C}_G(H/K)$ is an epimorphic image of $G/\mathcal{C}_{p',p}(G)$, then $G/\mathcal{C}_G(H/K) \in \mathcal{F}(p)$.

LEMMA 4.2. If $\mathcal{F}$ is a nonempty class of groups that is locally defined by $\{\mathcal{F}(p)\}$, then $\mathcal{F}$ is a saturated formation.
Proof. Clearly, epimorphic images of members of $\mathcal{F}$ are in $\mathcal{F}$. Assume that $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$.

Without loss of generality, let $M \cap N = E$. The principal factors of $G$ are the principal factors of $G$ in $G/M$ and $M$. Moreover, $M$ is isomorphic to the normal subgroup $MN/N$ of $G/N$ and the result follows.

Assume that $G/\mathcal{E}(G) \in \mathcal{F}$. Then $p|G/\mathcal{E}(G)|$ for $\mathcal{F}(p) = \emptyset$ and $G/\mathcal{E}(G)/C_{p,p}(G/\mathcal{E}(G)) \in \mathcal{F}(p)$ for $\mathcal{F}(p) \neq \emptyset$.

Note that $p|G|$ implies that $p|G/\mathcal{E}(G)|$. Thus $p|G|$ for $\mathcal{F}(p) = \emptyset$. Since $C_{p,p}(G/\mathcal{E}(G)) = C_{p,p}(G)/\mathcal{E}(G)$, then $G/C_{p,p}(G) \cong G/\mathcal{E}(G)/C_{p,p}(G/\mathcal{E}(G)) \in \mathcal{F}(p)$.

Gaschütz and Lubeseder [19] have shown that the converse of Lemma 4.2 is also valid for finite solvable groups. This result is stated without proof.

**THEOREM 4.4.** Every saturated formation of solvable groups can be locally defined.

The next example, due to T. Hawkes [22], answers the following question: Is every saturated formation subgroup inherited?

**EXAMPLE 4.1.** Let $\mathcal{F}$ be the formation locally defined by $\{\mathcal{F}(p)\}$ where $\mathcal{F}(2) =$ the smallest formation containing $S_3$, $\mathcal{F}(3) =$ the smallest formation containing $S_4$, and $\mathcal{F}(p) = \{E\}$ for all other primes $p$. Hawkes has shown that the smallest formation containing $S_4$ is the class of groups satisfying:
(1) G has a normal subgroup K which, if non-trivial, is the direct product of minimal normal subgroups of G of order 4.

(2) G/K has a normal subgroup H/K such that G/H is an elementary abelian 2-group and such that H/K is the direct product of eccentric minimal normal subgroups of G/K of order 3.

Let L = C₃⁻S₄ be the wreath product of a cyclic group of order 3 by S₄ using its natural representation of degree 4. Let B = ⟨x₁⟩ ⊕ ⟨x₂⟩ ⊕ ⟨x₃⟩ ⊕ ⟨x₄⟩ be the base group and let D be the Sylow 2-subgroup of S₄ containing the permutation (1324). Write Z = ⟨x₁x₂x₃x₄⟩, the normal subgroup of L generated by the element x₁x₂x₃x₄ of B, and put G = L/Z. Then every principal factor of G is -central and so G ∈ Φ. Consider the subgroup W = DB/Z of G. W has a minimal normal subgroup N of order 9. W(N) ≅ W(C₈(N)) ≅ D₈, the dihedral group of order 8. Since D₈ ∈ Φ(3), then W ∈ Φ.

LEMMA 4.3 [12]. Let Φ be locally defined by {Φ(p)}. If each of the formations Φ(p) is a normal formation, then Φ is a normal formation.

Proof. Let G ∈ Φ. Then G/Φ₂(G) ∈ Φ(p) for each prime p for which Φ(p) ≠ ∅. Let N be a normal subgroup of G. N/Φ₂(G) ∩ N = N/Φ₂(G) ∩ N ∈ Φ₂(G). Thus N/Φ₂(G) ∩ N ∈ Φ(p). But Φ₂(G) ∩ N ⊆ Φ₂(G). Therefore N/Φ₂(G) ∩ N ∈ Φ(p) and N ∈ Φ.
THEOREM 4.5. Let $\mathcal{F}$ be a locally defined normal formation. Then $\mathcal{F}^*$ is saturated.

Proof. Let $\{\mathcal{F}(p)\}$ locally define $\mathcal{F}$. Assume that $G/\mathcal{F}(G) \in \mathcal{F}^*$ but $G \notin \mathcal{F}^*$. Thus $\mathcal{F}(G) \notin \mathcal{F}^*$. By an argument similar to that given in Theorem 4.1, there exists a prime $q$ such that $C_q \notin \mathcal{F}$ and $q \mid |G/\mathcal{F}(G)|$.

Since $\mathcal{F}$ is locally defined, $C_q \notin \mathcal{F}$ implies that $\mathcal{F}(q)$ is the null set. Also, $q \mid |G/\mathcal{F}(G)|$. Consider a principal series for $G/\mathcal{F}(G)$. Since $G/\mathcal{F}(G) \in \mathcal{F}^*$, $G/\mathcal{F}(G)$ has a principal factor $H/K$ such that $q \mid |H/K|$ and $H/K \notin \mathcal{F}$. This is a contradiction since, by definition of $\mathcal{F}$, no group containing $q$ as a factor can be a member of $\mathcal{F}$. Thus $\mathcal{F}^*$ is saturated.

EXAMPLE 4.2. Let $\mathcal{D}$ be the formation defined in Lemma 3.2 for $p = 2$. Let $\mathcal{F} = \{G \mid$ for each $N \in \mathcal{D}, N \in \mathcal{D}\}$. By Lemma 1.1, $\mathcal{F}$ is a normal formation. Let $\mathcal{F}^*$ be the polyclass associated with $\mathcal{F}$. Consider the special linear group $SL(2,5)$. Denote the center of $SL(2,5)$ by $Z_0$. $SL(2,5)/Z_0 \cong PSL(2,5) \cong A_5$. By [23, Theorem 6.10, p. 181], $SL(2,5)$ is its own commutator subgroup. Since for any group $G$, $Z(G) \cap G \subseteq \mathcal{F}(G)$ [17, Theorem 4], then $Z_0 \subseteq \mathcal{F}(SL(2,5))$. For any homomorphism $\theta$ of a group $G$, $[\mathcal{F}(G)] \theta \subseteq \mathcal{F}(G) \theta$ [17, Theorem 3]. Thus $\mathcal{F}(SL(2,5)) \subseteq Z_0$ and hence $\mathcal{F}(SL(2,5)) = Z_0$. Therefore $A_5 \subseteq \mathcal{F}^*$ and $SL(2,5) \notin \mathcal{F}^*$ imply that $\mathcal{F}^*$ is not saturated.
CHAPTER V

APPLICATIONS AND EXAMPLES

In this chapter examples are considered which illustrate the material in the preceding sections and clarify the relationships between the classes of groups previously examined.

Solvable K-groups

An example of a widely studied class of groups that is a normal formation but is not subgroup inherited is presented in this section.

DEFINITION 5.1. A solvable group having the property that each normal subgroup has a complement in the group is called a solvable K-group.

An equivalent definition, given by G. Zacher [34] who initiated the investigation of solvable K-groups, is stated without proof.

THEOREM 5.1. A solvable group \( G \) is a K-group if and only if \( G \) has a series \( E = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_n = G \) such that \( N_{i+1}/N_i \) is a maximal nilpotent normal subgroup of \( G/N_i \) and the Frattini subgroup of \( G/N_i \), \( \Phi(G/N_i) \), is the identity for \( i = 0, 1, \ldots, n-1 \).

LEMMA 5.1. Each homomorphic image of a solvable K-group is a solvable K-group.
Proof. Let $H/N$ be a normal subgroup of $G/N$.
Since $G$ is a solvable $K$-group, then $H$ has a complement $L$ in $G$. Thus $LN/N$ is the required complement of $H/N$ in $G/N$ and hence $G/N$ is a solvable $K$-group.

Using the above results, Bechtell [3] has given a simplified proof of the following result due to Gross [20].

**Lemma 5.2.** Each normal subgroup of a solvable $K$-group is a solvable $K$-group.

**Proof.** Let $G$ be a solvable $K$-group. For each normal subgroup $N$ of $G$, $N$ is a solvable group. Hence $N$ possesses a series $E = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_n = N$ where each $N_{i+1}/N_i$, $0 \leq i \leq n$, is a maximal nilpotent subgroup of $N/N_i$. Since the Frattini subgroup of each normal subgroup is contained in the Frattini subgroup of the group,

\[ \Phi(N_{i+1}/N_i) \subseteq \Phi(N/N_i) \subseteq \Phi(G/N_i). \]

Lemma 5.1 implies that $G/N_i$ is a solvable $K$-group. Thus $G/N_i$ splits over every normal subgroup. In particular, $G/N_i$ splits over $\Phi(G/N_i)$. So $\Phi(G/N_i) = E$ and hence $\Phi(N_{i+1}/N_i) = E$. Therefore $N$ has a series satisfying the conditions of Theorem 5.1 and $N$ is a solvable $K$-group.

That the property of being a solvable $K$-group is not a subgroup inherited property is seen by considering the symmetric group on four symbols, $S_4$. This group is a solvable $K$-group that contains the dihedral group of order 8 as a subgroup. The center of this dihedral group is a cyclic group of order 2; it is contained in each of the subgroups of order 4. Thus the dihedral
group of order 8 fails to split over its center and hence is not a solvable K-group.

To show that the class of solvable K-groups is indeed a normal formation the following lemma is required. The proof is by M. Kotzen [25].

**LEMMA 5.3.** A group $G$ is a solvable K-group if and only if $G$ is a subdirect product of a finite collection of solvable K-groups, each of which possesses a unique minimal normal subgroup.

Proof. Proceed by induction on the order of $G$. Suppose that $G$ is a subdirect product of $\bigoplus_{i=1}^{n} H_i$ where each $H_i$ is a solvable K-group possessing a unique minimal normal subgroup. Furthermore, suppose that the projections $\varpi_i$ exist, $G \varpi_i = H_i$, with kernels $A_i$ such that $\bigcap_{i=1}^{n} A_i = E$. Since each $H_i$ is a solvable K-group, then $\mathfrak{F}(H_i) = E$ for all $i$. By a result of Gaschütz [17], $\mathfrak{F}(G) \varpi_i \subseteq \mathfrak{F}(H_i) = E$. Also, $\mathfrak{F}(N) \subseteq \mathfrak{F}(G) = E$ for each normal subgroup $N$ of $G$. In particular, $\mathfrak{F}($Fit($G$)) = $E$ and hence Fit($G$) is elementary abelian. Thus by [17, Theorem 7], there exists a subgroup $C$ of $G$ such that $G = $Fit($G$)$C$ and Fit($G$)$\cap C = E$. Since each $H_i$ is a solvable K-group, then $\mathfrak{F}(H_i) = E$. Hence Fit($G$) is equal to the abelian socle of $H_i$, i.e. Fit($G$) is a product of some abelian minimal normal subgroups of $G$. But by hypothesis, each $H_i$ has a unique minimal normal subgroup. Therefore Fit($H_i$) is the unique minimal normal subgroup of $H_i$ for each $i$. 

Clearly \( H_i = (\text{Fit}(G) \Pi_i)(C \Pi_i) \). If
\[
\text{Fit}(G) \Pi_i \cap C \Pi_i = E,
\]
then \( C \Pi_i \cong H_i / \text{Fit}(G) \Pi_i \) implies that \( C \Pi_i \) is a solvable K-group, by Lemma 5.1. Suppose that
\[
\text{Fit}(G) \Pi_i \cap C \Pi_i = B_i \neq E.
\]
Then \( B_i \leq \text{Fit}(H_i) \) and \( B_i \neq E \) implies that \( B_i = \text{Fit}(H_i) \).
But \( B_i \leq C \Pi_i \). Therefore
\[
\text{Fit}(G) \Pi_i \leq \text{Fit}(H_i) = B_i \leq C \Pi_i
\]
and hence \( C \Pi_i = H_i \). Since for each \( i, 1 \leq i \leq n, C \Pi_i \)
is a solvable K-group, it follows that each \( C \Pi_i \) is a subdirect product of solvable K-groups having precisely one minimal normal subgroup. By combining the \( H_i \) for which \( C \Pi_i = H_i \) with the direct factors in the subdirect products involved with those \( C \Pi_i \) for which \( C \Pi_i \neq H_i \), using composition of mappings where necessary, a direct product can be formed for which \( C \) is a subdirect product of solvable K-groups containing precisely one minimal normal subgroup. Inductively, \( C \) is a solvable K-group. This implies that \( G = [\text{Fit}(G)] \) has a series of subgroups \( E = N_0 \leq N_1 \leq \ldots \leq N_r = G \) such that \( N_{i+1} / N_i \) is a maximal normal nilpotent subgroup of \( G / N_i \) and \( \phi(G / N_i) = E \). By Theorem 5.1, this is sufficient for \( G \) to be a K-group. Since \( G \) is evidently solvable, then \( G \) is a solvable K-group.

**THEOREM 5.2.** The class of solvable K-groups is a normal formation.

Proof. Let \( G/N \) and \( G/M \) be solvable K-groups.
Consider $G/M \cap N$. It may be assumed, without loss of generality, that $M \cap N = E$. Thus $G$ is a subdirect product of $G/M \oplus G/N$. By Lemma 5.3, $G/M$ is a subdirect product of $\bigoplus_{i=1}^{m} H_i$ where each $H_i$ is a solvable $K$-group possessing precisely one minimal normal subgroup. Similarly $G/N$ is a subdirect product of $\bigoplus_{i=1}^{n} K_i$ where each $K_i$ is a solvable $K$-group possessing precisely one minimal subgroup. Consider the direct product $D$ of these associated direct products. $G$ is a subdirect product of $D$ and hence $G$ is a solvable $K$-group.

That the class of solvable $K$-groups has none of the other properties that have been previously discussed is seen in the following example.

**EXAMPLE 5.1.** Let $G = \langle a, b | a^4 = b^2 = e, b^{-1}ab = a^3 \rangle$, i.e. $G$ is the dihedral group of order 8. The center of $G$, $\langle a^2 \rangle$, is contained in each subgroup of order 4. Hence $G$ is not a solvable $K$-group. Let

$$H = \langle a^2 \rangle \oplus \langle b \rangle, \quad K = \langle ab \rangle, \quad M = \langle a^2 \rangle \oplus \langle ab \rangle.$$ 

Then $H$, $K$ and $M$ are each solvable $K$-groups. Since $G = [H]K$ and $G$ isn't a solvable $K$-group, the class of solvable $K$-groups fails to have the extension property. Also, $G = MH$ where $M$ and $H$ are both normal subgroups of $G$. Thus the class of solvable $K$-groups fails to be a Fitting formation. $G/\mathcal{Z}(G) = G/\langle a^2 \rangle$ and hence $G/\mathcal{Z}(G)$ is a solvable $K$-group. Therefore the class of solvable $K$-groups is not a saturated formation.
The remainder of this section will be devoted to a proof by H. Bechtell [4] which generalizes a result of C. Christensen [11] concerning the complements of the next to last term in the derived series of a solvable K-group.

**Lemma 5.4.** Let $\mathcal{F}$ be a normal formation. If a solvable group $G$ having $\mathcal{F}$-derived length $n+1$ splits over $F_n$ and $F_n$ is a minimal normal subgroup of $G$, then the complements of $F_n$ are conjugate in $G$.

Proof. Suppose that $G = [F_n]A = [F_n]B$. Form $N = \cap A^g$, for all elements $g \in G$. If $N \notin B$, then $G = NB$. Since $B$ has $\mathcal{F}$-derived length $n$ by Lemma 2.3, then $G/N \cong B/N \cap B$ has $\mathcal{F}$-derived length $\leq n$ by Theorem 2.5(f).

By Theorem 2.5(g), $F_n \leq N$. But $N \leq A$ and a contradiction arises. So $N \subseteq B$. As is known, $G/N$ is isomorphic to a primitive permutation group (on the conjugate class of $A$). Moreover, $G/N = [F_nN/N][A/N] = [F_nN/N][B/N]$. Since $F_nN/N$ is a minimal normal subgroup of $G/N$, then $A/N$ and $B/N$ are conjugate in $G/N$ [Theorem 3.2(f), p. 159, [23]]. Hence $A$ and $B$ are conjugate in $G$.

**Theorem 5.3.** Let $\mathcal{F}$ be a normal formation. If a solvable K-group $G$ has $\mathcal{F}$-derived length $n+1$ and $F_n$ is abelian, then the complements of $F_n$ are conjugate in $G$.

Proof. As is known, $F_n$ is a direct product of a collection of minimal normal subgroups of $G$. Suppose that $F_n = \bigoplus_{j=1}^{k} N_j$ where each $N_j$ is a minimal normal subgroup of $G$. Proceed by induction on the number of
subgroups in the direct product. For \( k = 1 \), the statement reduces to Lemma 5.4. Let \( k \geq 2 \) and consider \( G/N_1 \).

By Lemma 2.3, \( F_n(G/N_1) = F_n/N_1 \). Let \( A \) and \( B \) be two complements of \( F_n \) in \( G \), i.e. \( G = \left[F_n\right]A = \left[F_n\right]B \). It follows that \( G/N_1 = (F_n/N_1)(N_1B/N_1) \) and also that \( F_n \cap N_1B = N_1(F_n \cap B) = N_1 \). Hence \( G/N_1 = \left[F_n/N_1\right](N_1B/N_1) \).

Similarly \( G/N_1 = \left[F_n/N_1\right](N_1A/N_1) \). Since \( G = \left[F_n\right]A \), then \( G = \left[N_1\right]((N_2 \oplus \ldots \oplus N_k)A) \). Then by Lemma 5.1, \( H = (N_2 \oplus \ldots \oplus N_k)A \) is a K-group having \( F_n(H) = N_2 \oplus \ldots \oplus N_k \).

Each \( N_j \), for \( j \geq 2 \), is a minimal normal in \( H \). Therefore \( F_n(G/N_1) = N_1(\oplus_{j=2}^k N_j)/N_1 \) is a direct product of minimal normal subgroups of \( G/N_1 \). Inductively there exists a \( g \in G \) such that \( (N_1B)^g = N_1B^g = N_1A \). By Theorem 2.5, \( F_n(N_1A) \subseteq N_1 \). If \( F_n(N_1A) \subseteq N_1 \) properly, then for \( L = F_n(N_1A), F_n(G) \subseteq N_2 \oplus \ldots \oplus N_k \oplus L \neq F_n \). Hence \( F_n(N_1A) = N_1 \). But by Lemma 5.1, \( G/N_1 \) is a K-group. Thus \( N_1A \) is a K-group. Moreover, \( N_j \) is a minimal normal subgroup of \( N_1A \). By Lemma 5.4, there exists an \( h \in N_1A \) such that \( A = (B^g)^h = B^gh \). So \( A \) and \( B \) are conjugate in \( G \).

**COROLLARY 5.3.** Let \( G \) be a solvable K-group of solvability length \( n \). Then all the complements of \( G^{(n-1)} \) are conjugate in \( G \).

**Proof.** Let \( \mathcal{A} \) be the class of abelian groups in Theorem 5.3.
Hall \((\mathcal{F}, \mathfrak{P})\)-subgroups and \((\mathcal{F}, \mathfrak{P})\)-separable groups

Let \(\mathfrak{P}\) denote a set of prime numbers and let \(\mathfrak{P}'\) be the complement of \(\mathfrak{P}\) in the set of all primes. When not specifically designated, \(\mathcal{F}\) will denote a class of finite groups.

**Definition 5.2.** A subgroup \(H\) of \(G\) is called a **Hall \((\mathcal{F}, \mathfrak{P})\)-subgroup** of \(G\) if and only if \(H\) is a Hall \(\mathfrak{P}\)-subgroup, i.e., \(|H| \in \mathfrak{P}\) and \((G:H) \in \mathfrak{P}'\), and \(H \in \mathcal{F}\).

**Definition 5.3.** A group \(G\) is called **\((\mathcal{F}, \mathfrak{P})\)-separable** if and only if \(G\) has a principal series with the index of each principal factor either in \(\mathfrak{P}\) or \(\mathfrak{P}'\) and each principal factor with index in \(\mathfrak{P}\) is a member of \(\mathcal{F}\).

If \(\mathcal{F}\) is the polyclass of all finite groups, then the Hall \((\mathcal{F}, \mathfrak{P})\)-subgroups are just Hall \(\mathfrak{P}\)-subgroups and the \((\mathcal{F}, \mathfrak{P})\)-separable groups are the \(\mathfrak{P}\)-separable groups as defined in [23]. If \(\mathcal{F}\) is the class of solvable groups, then the class of \((\mathcal{F}, \mathfrak{P})\)-separable groups becomes the class of \(\mathfrak{P}\)-solvable groups. These are but two examples of well known classes of groups that are special cases of the classes of groups that will be examined in this section. Thus, in considering the \((\mathcal{F}, \mathfrak{P})\)-separable groups and Hall \((\mathcal{F}, \mathfrak{P})\)-subgroups, the properties of many important classes of groups are being studied simultaneously.
LEMMA 5.5. Let $\mathcal{F}$ be a normal formation. Each normal subgroup and each epimorphic image of an $(\mathcal{F},\mathcal{T})$-separable group is $(\mathcal{F},\mathcal{T})$-separable.

Proof. Let $N \triangleleft G$ and let

$$G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_s = E$$

be a principal series for $G$. Consider the normal subgroups $N_i = G_i \cap N$ of $N$. For each $i$,

$$N_i / N_{i+1} = N \cap G_i / N \cap G_{i+1} = G_{i+1} (N \cap G_i) / G_{i+1} \subseteq G_i / G_{i+1}.$$ 

Since $N_i / N_{i+1} \subseteq G_i / G_{i+1}$ and each $\mathcal{T}$-principal factor $G_i / G_{i+1} \in \mathcal{F}$, each corresponding $\mathcal{T}$-factor $N_i / N_{i+1} \in \mathcal{F}$. Thus $\{N_i \mid 0 \leq i \leq s\}$ might not form a principal series for $N$. Consider a refinement of $\{N_i \}$ to a principal series for $N$. For any $\mathcal{T}$-factor $N_i / N_{i+1}$ let

$$N_i = N_{i0} \supseteq N_{i1} \supseteq \ldots \supseteq N_{ip} = N_{i+1}$$

be an arbitrary link in such a refinement. Then $N_i / N_{i+1} = N_{i0} / N_{ip} \in \mathcal{F}$. For all $j$, $0 \leq j \leq p$, $N_{ij} / N_{ip} \subseteq N_{i0} / N_{ip}$ and hence $N_{ij} / N_{ip} \in \mathcal{F}$. But for every $j$, $N_{ij} / N_{i(j+1)}$ is an epimorphic image of $N_{ij} / N_{ip}$. Thus $N_{ij} / N_{i(j+1)} \in \mathcal{F}$. Now consider any $\mathcal{T}'$-factor $N_k / N_{k+1}$. By a similar argument, any refinement of this link to a principal series consists of factors which are $\mathcal{T}'$-groups. Thus $N$ is $(\mathcal{F},\mathcal{N})$-separable.

Consider the principal series

$$G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_s = N \supseteq G_{s+1} \supseteq \ldots \supseteq G_t = E$$

for $G$ that passes through $N$. Then the section of the series between $G$ and $N$ gives rise to a principal series for $G/N$ in the obvious way. Each factor $(G_i / N) / (G_{i+1} / N)$ of this principal series for $G/N$ is isomorphic to the
factor group $G_{i}/G_{i+1}$. Thus the factors in this principal series for $G/N$ are either $\mathfrak{V}$-groups or $\mathfrak{P}^l$-groups and the $\mathfrak{V}$-factors are $\mathfrak{F}$.

**LEMMA 5.6.** Let $\mathfrak{F}$ be a Fitting formation. Let $M$ be the subgroup generated by any finite collection $A_1, A_2, \ldots, A_n$ of subnormal subgroups of $G$. If $A_i \in \mathfrak{F}$, $i = 1, 2, \ldots, n$, then $M \in \mathfrak{F}$.

Proof. Consider the case when $n = 2$. By Corollary 3.5.2, $A \triangleleft \mathcal{M}(A) \subseteq \mathcal{M}(G)$ and $B \triangleleft \mathcal{M}(B) \subseteq \mathcal{M}(G)$. Thus since $\langle A, B \rangle \triangleleft \mathcal{M}(G)$ and $\mathfrak{F}$ is a Fitting formation, $\langle A, B \rangle \in \mathfrak{F}$. The theorem follows by simple induction.

**THEOREM 5.4.** Let $\mathfrak{F}$ be a Fitting formation. The class of $(\mathfrak{F}, \mathfrak{V})$-separable groups is a polyclass.

Proof. As a result of Lemma 5.5, all that remains to be shown is that the class of $(\mathfrak{F}, \mathfrak{V})$-separable groups satisfies the extension property. Let $N \triangleleft G$ and let $N$ and $G/N$ be $(\mathfrak{F}, \mathfrak{V})$-separable groups. Let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_n = E$ be a principal series for $G$ containing $N$. Since $G/N$ is $(\mathfrak{F}, \mathfrak{V})$-separable, each factor $G_i/G_{i+1}$ is either a $\mathfrak{V}$-group or a $\mathfrak{P}^l$-group and the $\mathfrak{V}$-factors are in $\mathfrak{F}$. Consider the factor groups $N_j/N_{j+1}$. $N_{n-1}$ is minimal normal in $G$. Let $K$ be a minimal normal subgroup of $N$ contained in $N_{n-1}$, i.e., $E \subseteq K \subseteq N_{n-1}$. Since $N$ is $(\mathfrak{F}, \mathfrak{V})$-separable, $K$ is either a $\mathfrak{V}$-group or a $\mathfrak{P}^l$-group. If $K$ is a $\mathfrak{V}$-group, then $K \in \mathfrak{F}$. Consider $M = \langle K^g \mid g \in G \rangle$. 
Then $E C M \subseteq N_{n-1}$. But $M$ is normal in $G$. Therefore $N_{n-1}$ minimal normal in $G$ implies that $M = N_{n-1}$. Since $K \trianglelefteq G$, $M$ is either a $\mathfrak{T}$-group or a $\mathfrak{T}'$-group. If $M$ is a $\mathfrak{T}$-group, then $M = N_{n-1} \in \mathfrak{F}$. Similarly it can be shown that each factor $N_j/N_{j+1}$ is either a $\mathfrak{T}$-group or a $\mathfrak{T}'$-group and that each $\mathfrak{T}$-factor is a member of $\mathfrak{F}$. Thus the class of $(\mathfrak{F}, \mathfrak{T})$-separable groups is a polyclass.

The following notation, which is adapted from the notation used by P. Hall in [21], will be used in the remainder of this section. Let $E^\mathfrak{F}_\mathfrak{T}$ and $C^\mathfrak{F}_\mathfrak{T}$ be the propositions about a group $G$ for which

- $E^\mathfrak{F}_\mathfrak{T}$: $G$ has at least one Hall $(\mathfrak{F}, \mathfrak{T})$-subgroup and
- $C^\mathfrak{F}_\mathfrak{T}$: $G$ satisfies $E^\mathfrak{F}_\mathfrak{T}$ and any two Hall $(\mathfrak{F}, \mathfrak{T})$-subgroups are conjugate in $G$.

**LEMMA 5.7.** Let $\mathfrak{F}$ be a normal formation. If $N \trianglelefteq G$ and $K$ is a Hall $(\mathfrak{F}, \mathfrak{T})$-subgroup of $G$, then $KN/N$ and $K \cap N$ are Hall $(\mathfrak{F}, \mathfrak{T})$-subgroups of $G/N$ and $N$, respectively.

**Proof.** The fact that $KN/N$ is a Hall $\mathfrak{T}$-subgroup of $G$ is well known. Also, $KN/N \cong K/K \cap N$. Since $\mathfrak{F}$ is a normal formation, $K/K \cap N \in \mathfrak{F}$ and hence $KN/N \in \mathfrak{F}$.

It is also known that $K \cap N$ is a Hall $\mathfrak{T}$-subgroup of $N$. But $N \cap K \trianglelefteq K \in \mathfrak{F}$. Thus $N \cap K \in \mathfrak{F}$.

**THEOREM 5.5.** Let $\mathfrak{F}$ be a polyclass. Each $(\mathfrak{F}, \mathfrak{T})$-separable group is a member of the class $C^\mathfrak{F}_\mathfrak{T}$.

**Proof.** Proceed by induction on the order of $G$. 
Let $N$ be a minimal normal subgroup of $G$. If $N = G$, then either $|G| \in \pi$ or $|G| \in \pi'$. If $|G| \in \pi$, then $G$ is its own Hall $(\pi, \pi)$-subgroup. If $|G| \in \pi'$, then $E$ is the required Hall $(\pi, \pi)$-subgroup of $G$. Assume that $N \subseteq G$. By the induction hypothesis, there exists a Hall $(\pi, \pi)$-subgroup $H/N$ of $G/N$. From the definition of Hall $(\pi, \pi)$-subgroups, it is known that $(H:N) \in \pi$, $H/N \in \mathcal{F}$, and $(G/N:H/N) \in \pi'$. If $|N| \in \pi$, then $|H| \in \pi$. Also, $N \in \mathcal{F}$ and $H/N \in \mathcal{F}$ imply that $H \in \mathcal{F}$. Thus $H$ is the required Hall $(\pi, \pi)$-subgroup of $G$. If $|N| \in \pi'$, then $((H:N),|N|) = 1$. Hence by the Theorem of Zassenhaus, $H = [N]B$ for some subgroup $B$ of $H$. Note that $B \cong H/N \in \mathcal{F}$, $|B| = (H:N) \in \pi$ and $(G:B) = (G:H)(H:B) = (G:H)|N| \in \pi'$.

Therefore $B$ is a Hall $(\pi, \pi)$-subgroup of $G$.

For the conjugacy proof let $L$ and $M$ be Hall $(\pi, \pi)$-subgroups of $G$. By Lemma 5.7, $LN/N$ and $MN/N$ are Hall $(\pi, \pi)$-subgroups of $G/N$. By the induction hypothesis, there exists a $g \in G$ such that $(LN)^g = MN$. Hence $L^g \subseteq MN$. If $|N| \in \pi$, then $|MN| = |M||N|/|M \cap N|$ is an element of $\pi$. But $M$ is a Hall $(\pi, \pi)$-subgroup of $G$; thus a maximal $\pi$-subgroup of $G$. Therefore $M = MN$. Then $L^g \subseteq M$ and $|L^g| = |M|$ imply that $L^g = M$.

If $|N| \in \pi'$, then $MN/N \cong M/M \cap N$ implies that $(MN:N)$ is in $\pi$. Thus $((MN:N),|N|) = 1$. By the result of Feit-Thompson, either $MN/N$ or $N$ is solvable. Application of the Theorem of Zassenhaus shows that any complements of $N$
in MN are conjugate. Hence there exists an \( r \in MN \) such that \( M = L^r \). Thus \( M \) and \( L \) are conjugate in \( G \).

Letting \( \mathcal{F} \) be the polycategory of finite groups, one obtains the following well-known theorem of P. Hall.

**Corollary 5.5.1.** If \( G \) is a \( \mathcal{T} \)-separable group, then \( G \) possesses Hall \( \mathcal{T} \)-subgroups and all the Hall \( \mathcal{T} \)-subgroups are conjugate in \( G \).

**Corollary 5.5.2.** Let \( \mathcal{F} \) be a polycategory. If \( \mathcal{C}_p \in \mathcal{F} \) for each \( p \in \mathcal{T} \), then \( \mathcal{C}_p^\mathcal{F} \) contains the class of solvable groups.

Proof. Each solvable group \( G \) has a principal series with elementary abelian factors. Since \( \mathcal{F} \) is closed with respect to direct products, \( G \) is \((\mathcal{F}, \mathcal{T})\)-separable and hence a member of \( \mathcal{C}_p^\mathcal{F} \) by the previous theorem.

From Lemma 5.7 it is evident that for all \( N \triangleleft G \), \( G \in \mathcal{E}_N^\mathcal{F} \) implies that \( N \in \mathcal{E}_N^\mathcal{F} \) and \( G/N \in \mathcal{E}_N^\mathcal{F} \). This raises the question: Under what conditions will \( \mathcal{E}_N^\mathcal{F} \) be a polycategory? This investigation uses the method of proof developed by W. Brauer \([8]\).

**Lemma 5.8.** Let \( \mathcal{F} \) be a polycategory. If \( N \) is a normal \((\mathcal{F}, \mathcal{T})\)-subgroup of \( G \), then \( G \in \mathcal{E}_N^\mathcal{F} \) if and only if \( G/N \in \mathcal{E}_N^\mathcal{F} \).

Proof. Assume that \( G \in \mathcal{E}_N^\mathcal{F} \). By Lemma 5.7, \( G \in \mathcal{E}_N^\mathcal{F} \) implies that \( G/N \in \mathcal{E}_N^\mathcal{F} \).

Conversely, let \( G/N \in \mathcal{E}_N^\mathcal{F} \). Thus \( G/N \) has a Hall
$(\mathfrak{X}, \pi)$-subgroup $U/N$. $N \in \mathfrak{X}$ and $U/N \in \mathfrak{X}$ imply that $U \in \mathfrak{X}$. Also, $|U| = (U:N)|N| \in \pi$ and $(G:U) = (G/N:U/N)$.

Hence $(G:U) \in \pi'$. Therefore $U$ is the required Hall $(\mathfrak{X}, \pi)$-subgroup of $G$.

**THEOREM 5.6.** Let $N$ be a normal subgroup of $G$ and let $N \in \mathfrak{C}_\pi$. Then $G \in \mathfrak{E}_\pi$ if and only if $G/N \in \mathfrak{E}_\pi$.

Proof. Let $G$ be a group of minimal order having $\mathfrak{C}_\pi$-normal subgroup $N$ and $\mathfrak{E}_\pi$-factor group $G/N$, but $G \not\in \mathfrak{E}_\pi$. Since $N \in \mathfrak{C}_\pi$, $N$ possesses a Hall $(\mathfrak{X}, \pi)$-subgroup $K$. If $(G:N) \in \pi'$, then the Hall $(\mathfrak{X}, \pi)$-subgroups of $N$ are precisely the Hall $(\mathfrak{X}, \pi)$-subgroups of $G$ and the theorem is proven. Consider the case where $(G:N)$ is divisible by prime numbers from $\pi$.

First, let $K = E$. Each Hall $(\mathfrak{X}, \pi)$-subgroup of $G/N$ is of the form $U/N$ for some subgroup $U$ of $G$ containing $N$. Since $K = E$, $N$ is a normal Hall $\pi'$-subgroup of $U$. Thus, by the Theorem of Zassenhaus, $U$ splits over $N$, i.e. there exists a subgroup $M$ of $U$ such that $U = [N]M$. Therefore $M \not\subset U/N \in \mathfrak{X}$, $|M| \in \pi$ and

$$(G:M) = (G:U)(U:M) = (G:U)|N| \in \pi'.$$

Hence $M$ is a Hall $(\mathfrak{X}, \pi)$-subgroup of $G$ and $G \in \mathfrak{E}_\pi$.

Next, suppose that $K \neq E$ and $K \leq G$. By definition, $K$ is a Hall $(\mathfrak{X}, \pi)$-subgroup of $N$. Hence $(N:K) \in \pi'$. Thus $N/K \in \mathfrak{C}_\pi$. But $G/N \not\cong (G/K)/(N/K) \in \mathfrak{E}_\pi$. By the induction hypothesis, $G/K \in \mathfrak{E}_\pi$. According to Lemma 5.8, $G \in \mathfrak{E}_\pi$ since $K$ is an $(\mathfrak{X}, \pi)$-subgroup of $G$.

Consider the remaining case, i.e. when $L = \mathfrak{N}_G(K)$
and $\mathcal{N}_G(K) \neq G$. For each $g \in G$, $N$ normal in $G$ implies that $K^g$ is a Hall $(\mathcal{F}, \Pi)$-subgroup of $G$ which is contained in $N$. Thus there exists an $n \in N$ such that $K^{gn} = K$.

Then $gn \in L$ and hence $g \in LN$. Therefore $G = LN$ and $G/N \cong LN/N \cong L/L \cap N \in \mathcal{E}_{\Pi}$. Since $N \in C_\Pi$, $L \cap N \in \mathcal{E}_{\Pi}$ by Lemma 5.8. But $L \cap N$ contains $K$ as a normal Hall $(\mathcal{F}, \Pi)$-subgroup. Thus $L \cap N \in C_\Pi$. By the induction hypothesis, $L \in \mathcal{E}_{\Pi}$. Since $K \leq L \cap N$ and $(N:K) \in \Pi'$, then $(N:LN)$ is in $\Pi'$. Therefore $(G:L) = (N:LN) \in \Pi'$ implies that each Hall $(\mathcal{F}, \Pi)$-subgroup of $L$ is also a Hall $(\mathcal{F}, \Pi)$-subgroup of $G$.

COROLLARY 5.6. If $\mathcal{F}$ is a polyclass and $\mathcal{A}$ is the class of solvable groups, then $\mathcal{E}_{\Pi} \cap \mathcal{A}$ is a polyclass.

Proof. Let $\mathcal{B} = \mathcal{E}_{\Pi} \cap \mathcal{A}$ and let $G \in \mathcal{B}$. The properties of solvable groups and Lemma 5.8 imply that for each normal subgroup $N$ of $G$, $N \in \mathcal{B}$ and $G/N \in \mathcal{B}$.

Let $M \in \mathcal{B}$, $G/M \in \mathcal{B}$. Since $M$ is solvable, all the Hall $\Pi$-subgroups of $M$ are conjugate and hence $M \in C_\Pi$. By the previous theorem, $M \in C_\Pi$ and $G/M \in \mathcal{E}_{\Pi}$ imply that $G \in \mathcal{E}_{\Pi}$. Since the extension of a solvable group by a solvable group is a solvable group, the result follows.
(\mathcal{F}, \pi)-closed groups

As demonstrated in the previous section, the class of groups possessing a Hall (\mathcal{F}, \pi)-subgroup is generally neither a polyclass nor a normal formation. In this section, the properties of those groups in which the Hall (\mathcal{F}, \pi)-subgroups are normal subgroups are considered.

**DEFINITION 5.4.** A group G is said to be (\mathcal{F}, \pi)-closed if and only if it has a normal Hall (\mathcal{F}, \pi)-subgroup.

**THEOREM 5.7.** Let \mathcal{F} be a normal formation. The class of (\mathcal{F}, \pi)-closed groups is a normal formation.

**Proof.** By Lemma 5.7, if G has a normal Hall (\mathcal{F}, \pi)-subgroup, then for each N \triangleleft G, N and G/N have normal Hall (\mathcal{F}, \pi)-subgroups. By an argument similar to that given in Lemma 5.7, if a group H possesses a normal Hall \pi-subgroup, then each subgroup of H possesses a normal Hall \pi-subgroup. If H = A \ast B and A, B contain normal Hall \pi-subgroups A', B', respectively, then H contains a normal Hall \pi-subgroup A' \ast B'. Let G/M, G/N possess normal Hall (\mathcal{F}, \pi)-subgroups. Without loss of generality, assume that M \cap N = E. G \cong G/M \cap N is isomorphic to a subgroup of G/M \ast G/N. Hence G possesses a normal Hall \pi-subgroup K. All that remains to be proven is that K \in \mathcal{F}. Since K is a normal Hall \pi-subgroup of G, it is the only Hall \pi-subgroup of G. Thus MK/M, NK/N are the unique normal Hall \pi-subgroups of G/M, G/N,
respectively. Therefore \( MK/M \in \mathcal{F} \), \( NK/N \in \mathcal{F} \) and \( MK/M \cong K/M \cap K/N \cong K/N \) imply that \( K/M \cap K/N \in \mathcal{F} \).

But \( \mathcal{F} \) is a normal formation. Hence \( K/(M \cap K) \cap (N \cap K) \cong K \in \mathcal{F} \).

The following theorem is a stronger version of Theorem 5.4; it shows that for any normal formation \( \mathcal{F} \), the class of \((\mathcal{F}, \pi)\)-separable groups is a polyclass.

**Theorem 5.8.** Let \( \mathcal{F} \) be a normal formation and let \( \mathcal{B} \) be the class of \((\mathcal{F}, \pi)\)-closed groups. Then \( \mathcal{B}^* \) is the class of \((\mathcal{F}, \pi)\)-separable groups.

**Proof.** Let \( G \) be \((\mathcal{F}, \pi)\)-separable. Thus \( G \) has a principal series where each principal factor is either a \( \pi \)-group or a \( \pi' \)-group and the \( \pi \)-groups are members of \( \mathcal{F} \). Since \( E \in \mathcal{F} \), each of the principal factors are \((\mathcal{F}, \pi)\)-closed and therefore in \( \mathcal{B} \). Hence the class of \((\mathcal{F}, \pi)\)-separable groups is contained in \( \mathcal{B}^* \).

Conversely, let \( G \in \mathcal{B}^* \). By Theorem 3.1, \( G \) possesses an invariant \( \mathcal{B} \)-series.

\[
G = H_0 \supset H_1 \supset H_2 \supset \ldots \supset H_n = E
\]
such that the factors are \((\mathcal{F}, \pi)\)-closed. Since \( H_i/H_{i+1} \) is \((\mathcal{F}, \pi)\)-closed, it possesses a characteristic Hall \((\mathcal{F}, \pi)\)-subgroup \( A_i/H_i+1 \). Thus \( A_i \leq G \). \( A_i \) has the property that \( H_i/A_i \) is a \( \pi' \)-group and \( A_i/H_i+1 \) is a \((\mathcal{F}, \pi)\)-group. Insert the \( A_i \) in the \( \mathcal{B} \)-series to get the series

\[
G = H_0 \supset A_0 \supset H_1 \supset A_1 \supset \ldots \supset H_{n-1} \supset A_{n-1} \supset H_n = E.
\]

Consider the refinement of the above series to a principal
series for G. Consider an arbitrary link $H_i = H_{i0} \supset H_{i1} \supset \cdots \supset H_{in_i} = A_i = A_{i0} \supset A_{i1} \supset \cdots \supset A_{imi} = H_{i+1}$ of this refinement. Since $H_i/A_i$ is a $\mathfrak{T}'$-group, then $H_i/H_{ij}$ is a $\mathfrak{T}'$-group for each $j$, $0 \leq j \leq n_i$. But $H_i(j-1)/H_{ij} \leq H_i/H_{ij}$. Thus $H_i(j-1)/H_{ij} \in \mathfrak{T}'$ for all $j$. Thus each of the factors in this part of the series are $\mathfrak{T}'$-groups. Next, consider the other part of the link. $A_i/H_{i+1}$ is an $(\mathfrak{F},\mathfrak{F})$-group. By an argument similar to that for the $\mathfrak{T}'$-factors, each factor in this part of the link is an $(\mathfrak{F},\mathfrak{F})$-group. Thus G has a principal series whose factors are either $\mathfrak{F}$-groups or $\mathfrak{T}'$-groups and the $\mathfrak{F}$-groups are members of $\mathfrak{F}$. Hence the $(\mathfrak{F},\mathfrak{F})$-separable groups contain $\mathfrak{B}^*$ and the theorem is proven.

COROLLARY 5.8. If $\mathfrak{B}$ is the class of $\mathfrak{F}$-closed groups, then $\mathfrak{B}^*$ is the class of $\mathfrak{F}$-separable groups.

Proof. Let $\mathfrak{F}$ be the class of finite groups in the previous theorem.

The previous theorem now makes it possible to give a number of conditions which imply saturation for the class of $(\mathfrak{F},\mathfrak{F})$-separable groups. The proof of this theorem will be omitted as it would closely parallel the proof of the corresponding theorems in Chapter IV.

THEOREM 5.9. Let $\mathfrak{F}$ be a normal formation. Each of the following conditions implies saturation for the class of $(\mathfrak{F},\mathfrak{F})$-separable groups.

(a) $C_p \in \mathfrak{F}$ for every $p \in \mathfrak{F}$.
(b) $\mathcal{F}$ is subgroup inherited.
(c) $\mathcal{F}$ is locally defined.
(d) $\mathcal{F}$ contains only solvable groups.

**Theorem 5.10.** If $\mathcal{F}$ is a Fitting formation, then the class of $(\mathcal{F}, \mathcal{N})$-closed groups is a Fitting formation.

**Proof.** By Theorem 5.7, it is known that the class of $(\mathcal{F}, \mathcal{N})$-closed groups is a normal formation. Let $H$ and $K$ be normal subgroups of $G$ possessing normal Hall $(\mathcal{F}, \mathcal{N})$-subgroups $H'$ and $K'$, respectively. Since $H'$ is a normal Hall $(\mathcal{F}, \mathcal{N})$-subgroup of $H$ and Hall $\mathcal{N}$-subgroups are maximal $\mathcal{N}$-subgroups of a group, then $H'$ is characteristic in $H$. Similarly, $K'$ is characteristic in $K$. Thus $H'$ and $K'$ are both normal in $G$. Hence $P = H'K'$ is a normal $\mathcal{N}$-subgroup of $G$. Since $\mathcal{F}$ is a Fitting formation, $P \in \mathcal{F}$. Also, $HK/P = (HP/P)(KP/P)$. Note that
\[
HP/P \cong H/H \cap P \cong (H/H')/(H \cap P/H') \in \mathcal{N}'.
\]
Similarly, $KP/P \in \mathcal{N}'$. Thus $HK/P \in \mathcal{N}'$. Consequently, $P = H'K'$ is a normal Hall $(\mathcal{F}, \mathcal{N})$-subgroup of $HK$.

Upon application of this theorem to the class of $p$-nilpotent groups, i.e., the class $\mathcal{G}$ of groups having the property that $G \in \mathcal{G}$ if and only if $G$ possesses a normal subgroup $N$ of order relatively prime to a fixed prime $p$ such that $G/N$ is a $p$-group, one obtains:

**Corollary 5.10.** The class of $p$-nilpotent groups is a Fitting formation.
Proof. Let $\mathfrak{P} = \{ q \mid q$ is a prime, $q \neq p \}$ and let $\mathfrak{F}$ be the class of all finite groups. The class of groups having normal Hall $(\mathfrak{F}, \mathfrak{P})$-subgroups is precisely the class of $p$-nilpotent groups. Hence by the previous theorem, the class of $p$-nilpotent groups is a Fitting formation.

That the class of $(\mathfrak{F}, \mathfrak{P})$-closed groups does not have the extension property is seen in the following example.

EXAMPLE 5.2. Let $\mathfrak{F}$ be the class of all finite groups and let $\mathfrak{P} = \{ 2 \}$. Consider $S_3$. Let $N$ be the normal subgroup in $S_3$ of order 3. Since $|N| = 3$, $N$ has a normal Hall $\mathfrak{P}$-subgroup $E$. Also, $(S_3:N) = 2$. Thus $S_3/N$ is a normal Hall $\mathfrak{P}$-subgroup of itself. Therefore $N$ and $S_3/N$ are $\mathfrak{P}$-closed, but $S_3$ fails to have a normal Hall $\mathfrak{P}$-subgroup.

Since the extension of an $(\mathfrak{F}, \mathfrak{P})$-closed group by an $(\mathfrak{F}, \mathfrak{P})$-closed group is in general not $(\mathfrak{F}, \mathfrak{P})$-closed, letting $\mathfrak{F}$ be a polycategory still only yields that the class of $(\mathfrak{F}, \mathfrak{P})$-closed groups is a Fitting formation. This fact is used to provide an example of a Fitting formation that fails to be subgroup inherited.

EXAMPLE 5.3. Let $\mathfrak{D}^*$ be the polycategory associated with the formation given in Example 3.2. Let $\mathfrak{P} = \{ 2, 3, 5 \}$ and let $\mathfrak{F}$ be the class of $(\mathfrak{D}^*, \mathfrak{P})$-closed groups. By Theorem 5.10, $\mathfrak{F}$ is a Fitting formation. $A_5 \in \mathfrak{F}$ but $C_3 \notin \mathfrak{F}$. Thus $\mathfrak{F}$ is not subgroup inherited.
Although the extension property does not hold in general for the class of \((\mathfrak{A}, \mathfrak{N})\)-closed groups, it can be shown to hold if all groups under consideration are taken to be \(\mathfrak{N}\)-closed.

**Lemma 5.9.** Let \(\mathfrak{A}\) be a polyclass. If \(N\) is a normal subgroup of the \(\mathfrak{N}\)-closed group \(G\) and if \(N, G/N\) are \((\mathfrak{A}, \mathfrak{N})\)-closed, then \(G\) is \((\mathfrak{A}, \mathfrak{N})\)-closed.

**Proof.** Since \(G\) is \(\mathfrak{N}\)-closed, \(G\) possesses a normal Hall \(\mathfrak{N}\)-subgroup \(K\) which turns out to be the unique Hall \(\mathfrak{N}\)-subgroup of \(G\). Then \(N \cap K\) and \(KN/N\) are the unique normal Hall \(\mathfrak{N}\)-subgroups of \(N\) and \(G/N\), respectively. Thus \(N \cap K \in \mathfrak{A}\) and \(KN/N \in \mathfrak{A}\). But \(KN/N \cong K/KN\). Therefore \(N \cap K \in \mathfrak{A}\) and \(K/KN \in \mathfrak{A}\) imply that \(K \in \mathfrak{A}\). Hence \(G\) is \((\mathfrak{A}, \mathfrak{N})\)-closed.

**Theorem 5.11.** Let \(\mathfrak{A}\) be a polyclass containing \(C_p\) for every prime \(p \in \mathfrak{N}\). A group \(G\) is \((\mathfrak{A}, \mathfrak{N})\)-closed if and only if \(G/\mathfrak{E}(G)\) is \((\mathfrak{A}, \mathfrak{N})\)-closed.

**Proof.** Since \(G/\mathfrak{E}(G)\) is an epimorphic image of \(G\), then if \(G\) has a normal Hall \((\mathfrak{A}, \mathfrak{N})\)-subgroup, it follows that \(G/\mathfrak{E}(G)\) also is \((\mathfrak{A}, \mathfrak{N})\)-closed.

Suppose that \(G/\mathfrak{E}(G)\) is \((\mathfrak{A}, \mathfrak{N})\)-closed. Then \(G/\mathfrak{E}(G)\) is \(\mathfrak{N}\)-closed. Hence by a result of Baer [1], \(G\) is \(\mathfrak{N}\)-closed. Consider \(\mathfrak{E}(G)\). \(\mathfrak{E}(G) = P_1^{a_1} \circ P_2^{a_2} \circ \cdots \circ P_n^{a_n}\), the direct product of its Sylow subgroups. For all \(p \in \mathfrak{N}\), \(C_p \in \mathfrak{A}\) and \(\mathfrak{A}\) a polyclass imply that the corresponding Sylow subgroups are in \(\mathfrak{A}\). Also, the direct product \(K\)
of Sylow subgroups for the $p \in \mathcal{N}$ is a member of $\mathfrak{N}$. 
$K$ is the required normal Hall $(\mathfrak{N}, \mathfrak{M})$-subgroup of $\mathfrak{H}(G)$. 
Therefore $G$ is $(\mathfrak{N}, \mathfrak{M})$-closed by the previous theorem.
This thesis has been concerned with a discussion of the relationships between a number of classes of groups. Theorem 1.1 gives the basic relationship between polyclasses, Fitting formations, and normal formations. The relationships between these three classes of groups and the other general classes of groups which have been discussed is illustrated in the following diagram.

The abelian and nilpotent groups and the classes of groups constructed in the examples provide the necessary counterexamples to show that this diagram is complete for finite groups.

It has been shown that most of the structural properties of the class of finite solvable groups, which don't depend upon the fact that the minimal normal subgroup of a solvable group is elementary abelian, carry over to the classes of groups possessing normal \( \mathfrak{F} \)-series. Thus the theory of normal formations and their associated polyclases provides a method of investigating the structure of a large
number of distinct classes of groups simultaneously.

In addition to answering a number of questions about the relationships between the various classes of groups examined, this thesis also poses some new problems. Theorem 2.7, when restricted to finite groups, shows that each normal formation \( \mathcal{F} \) gives rise to a polyclass \( \mathcal{F}^* \). This raises the question: For every polyclass \( \mathcal{A} \) is there a normal formation \( \mathcal{F} \) which is properly contained in \( \mathcal{A} \) such that \( \mathcal{A} = \mathcal{F}^* \)? If this is not true in general, then what properties must be placed on \( \mathcal{A} \) to insure that it is the polyclass associated with some proper normal formation?

Also, Theorem 2.7 shows that between a normal formation \( \mathcal{F} \) and its associated polyclass \( \mathcal{F}^* \), there exists an infinite number of normal formations. But Theorem 3.8 shows that there are no polyclasses between \( \mathcal{F} \) and \( \mathcal{F}^* \). This suggests the possibility of investigating questions concerning the number of subformations of a normal formation, the number of polyclasses contained in a given polyclass and the number of polyclasses between two given polyclasses. This type of investigation would appear to lead in the direction of work being done in varieties of groups.

For finite groups, the classes of abelian, nilpotent and solvable groups have the property that \( A \preceq \mathcal{N} \preceq A \). The question presents itself as to whether or not there is a class of groups corresponding to the
class of nilpotent groups, i.e., a Fitting formation, which is contained between each formation and its associated polyclass. An affirmative answer would help to clarify the position of the nilpotent groups in the class of finite groups.

For the particular case of the class \( \mathcal{F} \) of solvable \( K \)-groups, Theorem 5.2 demonstrates that this class of groups is a normal formation. Thus \( \mathcal{F} \) has a unique characteristic subgroup \( \mathcal{F}(G) \) with the property that \( G/N \) is a solvable \( K \)-group if and only if \( \mathcal{F}(G) \triangleleft N \). Since \( G/E(G) \) is an elementary group and hence a solvable \( K \)-group, \( \mathcal{F}(G) \subseteq E(G) \) where \( E(G) \) is the elementary commutator subgroup of \( G \), i.e., the normal subgroup of least order containing \( \mathcal{E}(H) \) for all subgroups \( H \subseteq G \). This raises the questions: Under what conditions is \( \mathcal{F}(G) = E(G) \)? What relationship exists between \( \mathcal{F}(G) \) and \( E(G) \)? Is it necessary that \( \mathcal{F}(G) \) is always nilpotent?
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