Spring 1968

THE EXTENSION OF A LOCALLY COMPACT LOCAL GROUP TO A GLOBAL GROUP

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COHEN, David Warren, 1940-
THE EXTENSION OF A LOCALLY COMPACT LOCAL GROUP TO A GLOBAL GROUP.

University of New Hampshire, Ph.D., 1968
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
THE EXTENSION OF A LOCALLY
COMPACT LOCAL GROUP TO A GLOBAL GROUP

by

David W. Cohen
B.S., Worcester Polytechnic Institute, 1962

A THESIS

Submitted to the University of New Hampshire
In Partial Fulfillment of
The Requirements for the Degree of

Doctor of Philosophy
Graduate School
Department of Mathematics
June, 1968
This thesis has been examined and approved.

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Dedication

to Doris

and to all others who "knew" before I did
that this requirement would be met.
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ABSTRACT

THE EXTENSION OF A LOCALLY
COMPACT LOCAL GROUP TO A GLOBAL GROUP

by

David W. Cohen
The main purpose of this paper is to prove that a locally compact local group is locally isomorphic to a global group; in fact, such a local group is locally the direct product of a compact group and a local Lie group. A similar structure theorem was obtained for locally compact global groups by V.M. Gluskov [1; Theorem A, p. 56]. The generalization of the global result to the local case is based upon the results and techniques of A.R. Jacoby [5].

Since the main body of the paper consists of an uninterrupted sequence of definitions and theorems, an outline of the structure of the theory is included below in order to show how the theorems interlock to give the main result.

Before proceeding to the outline I wish to acknowledge the debt I owe to A.R. Jacoby; specifically, for his help in connection with this paper, and generally for several years of "intensive care".

For the notation used in this introduction and the remaining text see section 0.

The first three sections are preliminary. They contain the definitions and results required for any discussion of local groups and the key results which are already known or which are immediate consequences of known results. In particular, in section two it is stated that locally compact local groups are generalized local Lie groups.
The first main result is Theorem 4 and its Corollary. If \( L \) is a locally compact local group, and if \( B \) is a compact group in \( L \) contained in a suitably small balanced neighborhood \( U \) which is saturated with respect to \( B \); if \( L/(B, U) \) is a local Lie group, and if \( N \) is the commutator of \( B \) in \( U \) (i.e. \( N = \{ x : x \in U, (b)(b \in B \Rightarrow x \cdot b = b \cdot x) \} \)), then \( B \cdot N \) is a neighborhood of the identity \( e \). Proposition F and its Corollary show that this conclusion is equivalent to the following: if \( Q_1 \) and \( Q_2 \) are neighborhoods of the identity in \( B \) and \( N \) respectively, then \( Q_1 \cdot Q_2 \) is a neighborhood of the identity in \( L \).

It is then shown that \( N \) contains a local Lie group \( M \) such that \( [N \cap B] \cdot M \) is a neighborhood of \( e \) in \( N \). In Theorem 6 a compact subgroup \( A \) of \( B \) and a sublocal Lie group \( M \) of \( N \) are obtained such that \( A \cap M = \{ e \} \) and such that if \( Q_1 \) and \( Q_2 \) are neighborhoods of the identity in \( A \) and \( M \) respectively, then \( Q_1 \cdot Q_2 \) is a neighborhood of \( e \) in \( L \). Thus \( L \) is locally isomorphic to \( (L+A)^X/L \overset{\mathcal{L}}{\times} (L+M) \).

Since every locally compact local group is a generalized local Lie group (Theorem 1), the conditions of Theorem 4 are always met, and so every locally compact local group is locally isomorphic to a global group.
The following notations are used for special sets: "\( \mathbb{N} \)" for the set of natural numbers; "\( \mathbb{N}_n \)" for the set of natural numbers greater than or equal to \( n \); "\( \mathbb{N}_n^m \)" for the set of natural numbers less than or equal to \( m \); "\( \mathbb{N}_n^m \)" for \( \mathbb{N}_n \cap \mathbb{N}_m \); "\( \emptyset \)" for the null set; "\( \mathbb{R} \)" for the set of real numbers, and "\( \mathbb{R}^* \)" for the ordinary topology on \( \mathbb{R} \). The function whose domain is \( A \) and whose value at \( x \) for any element \( x \) in \( A \) is \( f(x) \) is denoted by "\( [x: A; f(x)] \)". Any variable can be used in place of "\( x \)", any term in place of "\( A \)" and any formula in place of "\( f(x) \)". A similar notation is used for a function of two or more variables. For sets \( A, B, C \), the set of all sets of form \( A \cap C \), where \( C \) is an arbitrary member of \( B \) is denoted by "\( A \# B \)"; any terms can be used in place of "\( A \)" and "\( B \)".

If \( \tau_1 \) and \( \tau_2 \) are two topologies, the product topology is denoted "\( \tau_1 \times \tau_2 \)". If \( G_1 \) and \( G_2 \) are two topological groups (local groups, Lie groups or local Lie groups) the product group is denoted "\( G_1 \times G_2 \)"

("\( G_1 \times_\mathcal{L} G_2 \), "\( G_1 \times_\mathcal{G} G_2 \)" or "\( G_1 \times_\mathcal{B} G_2 \)").

The notations "\( f: A \rightarrow B \)", "\( f: A \rightarrow^e B \)", and "\( f: A \rightarrow^i B \)" will denote that \( f \) is a function with domain \( A \) and range a subset of \( B \), and is respectively injective, surjective and bijective.

Occasionally in a long proof certain statements are isolated as claims. The proof of a claim is preceded by "\( \exists \)" and followed by "\( \exists \)". Proofs of subclaims and subsubclaims are indicated in a similar manner by "\( \exists \)", "\( \exists \)" and "\( \exists \)".
SECTION 1

**Definition 1.** A local group is a quintuple $L = (L, e, \cdot, \cdot', \tau)$ satisfying the following:

1) $(L, \tau)$ is a topological space;

ii) $\cdot$ is a function from a subset of $L \times L$ into $L$;

iii) $\cdot'$ is a function from a subset of $L$ into $L$;

iv) there is a subset $0$ of $L$ such that

(a) $e \in 0 \in \tau$
(b) $0 \times 0 \subseteq \text{Domain (\cdot)}$
(c) $0 \subseteq \text{Domain (\cdot')}$
(d) $(a)(b)(c)(a,b,c,a \cdot b, b \cdot c \in 0 \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c)$
(e) $(a)(a \in 0 \Rightarrow a' \in 0, a \cdot e = e \cdot a = a, a \cdot a' = a' \cdot a = e)$
(f) $\cdot |0 \times 0$ is $(0 \neq \tau)^0(0 \neq \tau)$-continuous
(g) $\cdot'|0$ is $(0 \neq \tau)$-continuous;

v) $\{e\}$ is $\tau$-closed.

**Definition 2.** If $L$ is a local group, and if $0$ is a set satisfying iv) above, $0$ will be called a balanced neighborhood in $L$. The set of all balanced neighborhoods in $L$ will be denoted by $\mathcal{BN}(L)$.

**Notations.** A subscript or superscript on "$L$" will signify that the names of the elements in the quintuple are similarly adorned. Thus, "$L_1$ is a local group" will mean the notation is chosen so that $L_1 = (L_1, e_1, \cdot_1, \cdot'_1, \tau_1)$.
Definition 3. If \( L \) is a local group, and \( n \in \mathbb{N} \), and \( C \subseteq L \), then 
\( C \) is \( n^{th} \)-able in \( L \) in case there is a function \( B: \mathbb{N}^n_1 \to 2^L \) such that
\( B(1) = C \), and for every integer \( i \in \mathbb{N}^{n-1}_1 \), \( B(i) \times C \subseteq \text{Domain}(\cdot) \), and \([B(i)] \cdot [C] = B(i + 1)\).

The set of all \( n^{th} \)-able subsets of \( L \) will be denoted by \( \phi_n(L, n) \). If \( C \) is a member of \( \phi(L, m) \), and if \( B \) is the function satisfying the above conditions, and if \( i \) is an integer in \( \mathbb{N}^m_1 \), then we will denote \( B(i) \) by \( C^i \). Finally, \( \phi^0 \) will denote \( \{e\} \).

Definition 4. If \( L \) is a local group, \( n \) is a positive integer, and \( C \)
and \( D \) are subsets of \( L \), then we will say \( C \) is \( n^{th} \)-able in \( D \) in case
\( C \) is a member of \( \phi(L, n) \), and for all integers \( i \in \mathbb{N}^n_1 \), \( C^i \subseteq D \).
When these conditions hold, we will write \( C \in \phi(L, D, n) \).

Finally, if \( n \) is a positive integer, we will define
\( \phi(L, n) = \{C : (\exists 0)(0 \in \text{BN}(L), C \in \phi(L, 0, n))\} \).

Note. \( (0)(n)(0 \in \text{BN}(L), n \in \mathbb{N} \Rightarrow (\exists 0_1)(0_1 \in \text{BN}(L) \cap \phi(L, 0, n))) \).

Definition 5. If \( L_1 \) and \( L_2 \) are local groups, then a local homomorphism from \( L_1 \) to \( L_2 \) is a map \( \phi \) from a subset of \( L_1 \) into \( L_2 \) such that there is an \( 0_1 \) in \( \text{BN}(L_1) \) and an \( 0_2 \) in \( \text{BN}(L_2) \) satisfying the following:
1) \( 0_1 \in \phi(L_1, \text{Domain}(\phi), 2) \);
2) \( \phi|0_1 \) is \( 0_1 \#_1 - 2 \)-continuous ;
3) \( (a)(b)(a, b \in 0_1 \Rightarrow \phi(a), \phi(b) \in 0_2, \phi(a \cdot_1 b) = \phi(a) \cdot_2 \phi(b)) \).

Further, \( \phi \) is a local isomorphism if there are sets \( 0_3 \) and \( 0_4 \)
in \( \text{BN}(L_1) \) and \( \text{BN}(L_2) \) respectively such that:
4) \( 0_3 \in \phi(L_1, \text{Domain}(\phi), 2) \);
v) $\phi|_0 : 0_3 \rightarrow 0_4$;  
vi) $(\phi|_0)^+ \text{ is } (0_4 \# \tau_2) - \tau_1\text{-continuous.}$

Finally, $\phi$ is \textit{locally one to one} if there is an $0$ in $BN(L)$ such that $\phi|_0$ is one to one.

\textbf{Definition 6.} A \underline{subgroup} of a local group $L$ is a subset $B$ of $L$ satisfying the following:  

i) $e \in B \subseteq \phi(L, 2)$;  

ii) $[B]^\cdot \subseteq B$;  

iii) $[B] \cdot [B] \subseteq B$.

The collection of all subgroups of $L$ will be denoted by "$\mathcal{T}L$".

\textbf{Definition 7.} A \underline{sublocal group former} in a local group $L$ is a pair $(H, U)$ satisfying:  

i) $e \in H \subseteq L$;  

ii) $U \in BN(L)$;  

iii) $(a)(b)(a, b \in U \cap H \Rightarrow a \cdot b \in H)$;  

iv) $(c)(c \in U \cap H \Rightarrow c^\cdot \in H)$.

\textbf{Remark.} If $(H, U)$ is a sublocal group former in $L$, $\ddot{\cdot} = \cdot \cap (H \times H) \times H$, and $\ddot{\cdot} = \cdot \cap H \times H$, then $(H, e, \ddot{\cdot}, \ddot{\cdot}, H \# \tau)$ is a local group, which will be denoted by "$L^\tau H$".

\textbf{Definition 8.} $(H, U)$ is \underline{invariant} in $L$ in case  

i) $U \in BN(L) \cap \phi(L, 6)$;  

ii) $(H, U^6)$ is a sublocal group former in $L$;  

iii) $U^2 \sim H \in \tau$;  

iv) $(a)(b)(a \in H \cap U^4, b \in U^2 \Rightarrow b^\cdot (a \cdot b) \in H)$.

The collection of pairs invariant in $L$ will be denoted by "$\mathcal{A}L$".
(H, U) is weakly-invariant in L in case

1) \( H \cup U \in BN(L) \),
2) \( H \in \Gamma L \)
3) \((x)(h) (x \in U, h \in H \Rightarrow h \cdot x \in U, x^{-1} \cdot (h \cdot x) \in H)\).

The collection of pairs weakly-invariant in L will be denoted "W-\( \Delta L \)".

**Definition 9.** If \( L \) is a local group, and \((H, U)\) is a member of \( \Delta L \), then

\[
T'(L, H, U) = \{ x : U^2 : (\{x\} \cdot [H \cap U^4]) \cap U^2 \}.
\]

**Proposition A.** If \( L \) is a local group, \((H, U)\) is a member of \( \Delta L \), and

\[
T_0 = T(L, H, U),
\]

then \((a_1)(a_2)(b_1)(b_2)(a_1, a_2, b_1, b_2 \in U)\),

\[
T_0(a_1) = T_0(a_2), T_0(b_1) = T_0(b_2) \Rightarrow T_0(a_1 \cdot b_1) = T_0(a_2 \cdot b_2),
\]

\[
T_0(a_1^*) = T_0(a_2^*).
\]

Thus there exist uniquely determined functions \( \circ \) and \( \ast \) on \( T_0[U] \times T_0[U] \) and \( T_0[U] \) respectively with ranges in \( T_0[U^2] \) such that

\[
(a)(b)(a, b \in U \Rightarrow T_0(a \ast b) = T_0(a) \circ T_0(b), T_0(a^*) = (T_0(a))^\#).
\]

Further, \((T_0[U^2], T_0(e), \circ, \#, \{T_0[0] : 0 \in U^2 \# \tau\})\) is a local group (denoted by "\( L/(H, U) \)") and \( T_0[U] \) is an element of \( BN(L/(H, U)) \).

**Proof.** Omitted. See 15, Theorem 31, p. 441.

**Notation.** "\( \tau(L/(H, U)) \)" and "\( e(L/(H, U)) \)" will denote respectively the topology of \( L/(H, U) \) and the identity of \( L/(H, U) \).

**Remark.** We will say that local group \( L \) is locally compact when there is an element \( 0 \) in \( BN(L) \) such that \( 0 \# \tau \) is locally compact.
SECTION 2

Definition 10. A local group $L$ is a local Lie group in case there is an element $0$ in $BN(L)$, and a map $\phi$ mapping $0$ homeomorphically onto an open subset of some finite dimensional Euclidean space such that the function

$$[(x, y): \phi[0] x \phi[0]: \phi(\phi^- (x) \cdot \phi^- (y))]$$

is real analytic at $\phi(e)$.

Definition 11. A locally compact local group $L$ is a generalized local Lie group if for every element $0$ in $BN(L)$ there are elements $H$ and $W$ in $IL$ and $BN(L)$ respectively such that $H < W < O$, $H$ is $c$-compact, $(H, W)$ is a member of $\Delta L$, and $L/(H, W)$ is a local Lie group.

Proposition B. If $L$ is a locally compact local group, $(H, U)$ is an element of $\Delta L$, then $L/(H, U)$ is a locally compact local group.

Proof. [5, Theorem 33, p. 44].

Definition 12. A local group $L$ is without small subgroups if there is an $0$ in $BN(L)$ such that the only member of $IL$ which is a subset of $0$ is $\{e\}$.

Proposition C. If $L$ is a locally compact local group, then $L$ is a local Lie group if and only if $L$ is without small subgroups.

Proof. [5, Theorem 96, p. 62].

Definition 13. If $L$ is a local group, then $V_0$ is special in $BN(L)$ provided:

1) $V_0 \in BN(L) \land \phi(L, 6)$

1i) $(V)(V \in BN(L) \Rightarrow (\exists B)(\exists W)(B \in IL, B$ is $c$-compact, $W \in BN(L)$,
Proposition D. If \( L \) is a locally compact local group, there is a \( V_0 \) which is special in \( BN(L) \).

Proof. \([5, \text{Theorem 101, p. 66}]\). ▲

Proposition E. If \( L \) is a locally compact local group and \( 0 \) is an element of \( BN(L) \), then there is a \( U \) in \( BN(L) \) and a \( B \) in \( rL \) such that \( B \) is \( \tau \)-compact, \( (B, U) \) is an element of \( AL \), \( B \cdot U = U \cdot B = U \cdot 0 \), and such that the only subsets of \( U \) which are members of \( rL \) are subgroups of \( B \).

Proof. There is a \( V_0 \) special in \( BN(L) \). There is a \( \tau \)-compact element \( B \) in \( rL \) and a \( W \) in \( BN(L) \) such that \( B < W < V_0 \cap 0 \), \( (B, V_0) \) is an element of \( AL \) and such that the only subsets of \( W \) which are members of \( rL \) are subgroups of \( B \). There is an element \( V \) in \( BN(L) \cap \phi(L, W, 2) \) such that \( V \cdot B \cup B \cdot V < W \) \([5, \text{Theorem 15, p. 41}]\).

Let \( U = V \cdot B \cap B \cdot V \). Then \( 0 \supset U \in BN(L) \).

Suppose \( x \in U \cdot B \). Then

\[
(\exists v_1)(\exists b_1)(\exists v_2)(\exists b_2)(\exists b_3)(\exists b_4) (v_1, v_2, v_3, v_4 \in V, b_1, b_2, b_3, b_4 \in B, x = v_1 \cdot b_1 \cdot b_3 = b_2 \cdot v_2 \cdot b_4). \text{ Since } x = v_1 \cdot b_1 \cdot b_3, x \in V \cdot B. \text{ Since } x = b_2 \cdot v_2 \cdot b_4 \cdot v_2, x \in B \cdot V. \text{ So } x \in U. \text{ Thus } U \cdot B < U.
\]

Similarly, \( B \cdot U < U \):

Since \( U < V_0 \), \( (B, U) \in AL \). Finally, if \( K \) is a member of \( rL \) and \( K \subset U \), then \( K \subset W \), so \( K \) is a subgroup of \( B \). ▲
Theorem 1. Every locally compact local group is a generalized local Lie group.

Proof. Suppose $L$ is a locally compact local group.

Suppose $0 \in \mathcal{BN}(L)$. By Proposition E there is a $U$ in $\mathcal{BN}(L)$ and a $B$ in $\mathcal{IL}$ such that $B$ is $\tau$-compact, $(B, U)$ is an element of $\mathcal{IL}$, $B \subseteq B \cdot U = U \cdot B = U \subseteq 0$, and such that the only elements of $\mathcal{IL}$ which are subsets of $U$ are subgroups of $B$.

Let $L_1 = L/(B, U)$, $T_1 = T(L, B, U)$, and $U_1 = T_1 \{U\}$. Suppose $K_1$ is an element of $\mathcal{IL}_1$ and $K_1 \subseteq U_1$. Let $K = T_1^+ [K_1]$.

Claim. $K$ is an element of $\mathcal{IL}$ and $K \subseteq U$.

Suppose $k \in K$. Then since $K_1 \subseteq U_1$, there is an element $u$ in $U$ such that $T_1(u) = T_1(k)$. So $u^{-1}k \in B$. Hence $k \in U \cdot B = U$. So $K \subseteq U$.

Since $T_1$ is a homomorphism on $U$, it is obvious that $K$ is an element of $\mathcal{IL}_1$.

Thus $K \subseteq B$. Hence $K_1 = \{e_1\}$. Consequently, $L_1$ is without small subgroups, and so by Proposition C, $L_1$ is a local Lie group. △
SECTION 3

**Lemma 1.** If $L_1$ is a locally compact local group, $L_2$ is a local Lie group, $\phi$ is a locally one to one homomorphism from $L_1$ into $L_2$, then $L_1$ is a local Lie group.

**Proof.** Apply Proposition C.

**Lemma 2.** If $L$ is a generalized local Lie group, $(H, U)$ is a sublocal group former of $L$, $V$ is an element of $\mathbb{B}W(L)$ and a subset of $U$, and $V \sim H$ is a member of $\tau$, then $L+H$ is a generalized local Lie group.

**Proof.** Since $L+H$ is a locally compact local group, this result follows from Theorem 1.

**Lemma 3.** If $L$ is a local group, $(H_1, U_1)$ and $(H_2, U_2)$ are elements in $\Delta L$, and $H_1 \cap (U_1 \cap U_2)^H = H_2 \cap (U_1 \cap U_2)^H$, then $L/(H_1, U_1)$ is locally isomorphic to $L/(H_2, U_2)$.

**Proof.**

**Lemma 4.** If $L$ is a locally compact local group and $(H, U)$ is a member of $\Delta L$, then $L/(H, U)$ is a generalized local Lie group.

**Proof.** Proposition B and Theorem 1.

**Lemma 5.** If $L$ is a locally compact local group, $(H, U)$ is a member of $W-\Delta L$; $H$ is a $\tau$-compact, commutative member of $RL$, 


and $U$ is $\tau$-connected, then $(g)(h)(g \in U, h \in H \Rightarrow g \cdot h = h \cdot g)$.

**Proof.** Let $X$ be the character group of $H$, and $\mathcal{D}$ the discrete topology on $X$. For every element $g$ in $U$ and every $\chi$ in $X$, 

$$\chi_g = [h: H: \chi(g^{-1} \cdot (h \cdot g))]$$

is a member of $X$. Further, for every member $\chi$ of $X$ the function $\phi_\chi = [g: U: \chi_g]$ is $(U \# \tau)$-$\mathcal{D}$-continuous. ($H$ is $\tau$-compact, so every $\chi$ is uniformly continuous). Since $\mathcal{D}$ is discrete, for every $\chi$ in $X$ we have $\phi_\chi[U] = \{\chi\}$. Thus, $(h)(g)(\chi)$

$(h \in H, g \in U, \chi \in X \Rightarrow \chi_g(h) = \chi(g^{-1} \cdot (h \cdot g)) = \chi(h))$. Since $X$ is a sufficient system of characters, this implies that $g^{-1} \cdot (h \cdot g) = h$ for all $g$ in $U$, and all $h$ in $H$. ▲

**Lemma 6.** If $L$ is a locally compact local group, $B$ is a $\tau$-compact member of $\Gamma L$, $(B, U)$ is a member of $\Delta L$, $B \cdot U = U \cdot B = U$, 

$N = \{g: g \in U, (b)(b \in B \Rightarrow g \cdot b = b \cdot g)\}$, and $\phi = [g: U: [b: B: g^{-1} \cdot (b \cdot g)]]$

then $(x)(g)(x \in B, g \in U, \phi(x) = \phi(g) \Rightarrow x \cdot g = g \cdot x, g \cdot x^{-1} \in N)$.

**Proof.** Suppose $g$ is an element of $U$, $x$ is an element of $B$ and $\phi(x) = \phi(g)$. Then $x^{-1} \cdot g$ is a member of $B \cdot U = U$. Since $g^{-1} \cdot x \cdot g = x^{-1} \cdot x \cdot x = x$, $x \cdot g = g \cdot x$.

If $b$ is any element in $B$, $g^{-1} \cdot b \cdot g = x^{-1} \cdot b \cdot x$, so $b \cdot g \cdot x^{-1} = g \cdot x^{-1} \cdot b$. Thus $g \cdot x^{-1} \in N$. ▲

**Corollary.** If $L$, $B$, $U$, $N$ and $\phi$ satisfy the hypotheses of Lemma 6, and if $V$ is a subset of $U$ and a member of $BN(L)$, then 

$B \cdot [N \cap V] = \{g: g \in U, (\exists x)(x \in B, x^{-1} \cdot g \in V, \phi(x) = \phi(g))\}$, and $\phi[B \cdot [N \cap V]] = \phi[B]$. 

Proof. Suppose $b_0 \in B$, $n \in N \cap V$. Let $g = b_0 \cdot n$. Then $g \in B \cdot U \subset U$.

Let $x = b_0$. Then $x^{-1} \cdot g = n \in V$. Further, $(b)(b \in B \Rightarrow g \cdot (b \cdot g) = (n \cdot b_0^{-1})(b \cdot (b_0 \cdot n)) = b_0^{-1} \cdot (b \cdot b_0))$.

Conversely, suppose $g \in U$, $x \in B$, $\phi(x) = \phi(g)$, $x^{-1} \cdot g \in V$. Then since $g^{-1}(x \cdot g) = x^{-1}(x \cdot x) = x$, we have $g = (x^{-1} \cdot g) \cdot x$ and $x^{-1} \cdot g$ is a member of $N$ by lemma 6.

The second conclusion follows immediately from the first. ▶

Proposition 2. If $L$ is a locally compact local group, $B$ is a $\tau$-compact member of $\Gamma L$, $(B, U)$ is a member of $\Delta L$, $U \cdot B = B \cdot U = U$, and $N = \{g : g \in U, (b)(b \in B \Rightarrow g \cdot b = b \cdot g)\}$, then

$$e \in \tau\text{-interior}(B \cdot N) \iff (V)(U \supset V \in BN(L) \Rightarrow e \in \tau\text{-interior}(B \cdot [N \cap V]))$$

Proof. Let "I(B)" denote the inner automorphism group of $B$ with compact open topology, $C$. Let "A(B)" denote the automorphism group of $B$ with compact open topology, $C_o$. Then $I(B) \subseteq A(B)$, and $I(B) \neq C$ = $C$.

Suppose $e \in \tau\text{-interior}(B \cdot N)$, $U \supset V \in BN(L)$. Select $V_o$ in $BN(L) \cap \phi(L, V, 2)$. Let $\phi = \{g : U \subseteq B : g \cdot (b \cdot g)\}$. Then $\phi$ is $(U \neq \phi)$ $C_o$-continuous, and so $\phi|B$ is $(B \neq \phi)$ $C$-continuous. Further, $B$ is $B$ a $\tau$-compact, and $\phi|B$ is a homomorphism into the $C$-compact Hausdorff topological group $I(B)$. So by [3, theorem 5.20, p. 42], $\phi|B$ is a $B \neq \phi$ $C$-open map. Thus $(\phi|B)[V_o \cap B] \subseteq C$.

There exists an $0$ in $C_o$ such that $(\phi|B)[V_o \cap B] = \phi[V_o \cap B] = 0 \cap I(B) = 0 \cap \phi[B]$. There is a $W$ in $BN(L) \cap \phi(L, V_o, 2)$ such that $W \subseteq B \cdot N$, and $\phi[W] \subseteq 0$. Hence, using Corollary to Lemma 6

$\phi[W] \subseteq \phi[B \cdot N] \cap 0 = \phi[B] \cap 0 = \phi[V_o \cap B]$. Since $V_o \cdot W \subseteq V_o^2 \subseteq V$, we have $(g)(g \in W \Rightarrow (\exists x)(x \in V_o \cap B, x^{-1} \cdot g \in V, \phi(g) = \phi(x)))$. 
So by Corollary to Lemma 6, \( W \subseteq B \cdot [N \cap V] \).

The implication in the other direction is obviously true. \( \blacksquare \)

**Corollary.** If \( L \) is a local group, \( U \) is an element of
\[ B \cdot N \cap V \]
\( B \cdot [N \cap V] \) is a neighborhood of \( e \), then for all \( V \) and \( W \) in \( B \cdot N \cap V \)
\[ [B \cap W] \cdot [N \cap V] \] is a neighborhood of \( e \).

**Proof.** Suppose \( V, W \) are elements of \( B \cdot N \cap V \). There is an \( X \) in
\[ B \cdot N \cap V \] \( B \cdot [N \cap V] \), \( B \cdot [N \cap X] \), \( 2 \).

Suppose \( x \in X \). Then \( (\exists b)(\exists n)(b \in B, n \in N \cap X, b \cdot n = x) \). Then
\[ b \cdot x \cdot n \in X \cap W \]. So \( x \in [B \cdot N \cap W] \cdot [N \cap V] \). Thus \( x \in [B \cdot N \cap W] \cdot [N \cap V] \). \( \blacksquare \)
SECTION 4

**Definition 14.** If \( L \) is a local Lie group, then \( L \) is a one parameter
sublocal group of \( L \) if there exists a subset \( I \) of \( \mathbb{R} \) such that either
\( I = [0, \varepsilon) \) for some \( \varepsilon > 0 \), or \( I = \{ t : t > 0 \} \); and there exists a
\( V \) in \( BW(L) \) such that:

1. \( \ell : I \rightarrow V ; \)
2. \( \ell \) is \((I \neq \mathbb{R})\)-\( \tau \)-continuous ;
3. \( (t_1)(t_2)(t_1, t_2, t_1 + t_2 \in I \)

\[ \Rightarrow \ell(t_1) \cdot \ell(t_2) = \ell(t_1 + t_2). \]

The set of all one parameter sublocal groups of \( L \) is denoted "SL".

**Definition 15.** If \( L \) is a local Lie group,

\[ \varepsilon_L = \{ (\ell_1, \ell_2) : \ell_1, \ell_2 \in SL, \ell_1 \mid (\text{Domain}(\ell_1) \cap \text{Domain}(\ell_2)) \}
\]

\[ = \ell_2 \mid (\text{Domain}(\ell_1) \cap \text{Domain}(\ell_2))). \]

Remark. Obviously \( \varepsilon_L \) is a symmetric, reflexive relation. If \( \ell_1, \ell_2 \) are
elements in \( SL \), and if there is an \( \varepsilon > 0 \) such that \( \ell_1 \mid [0, \varepsilon) = \ell_2 \mid [0, \varepsilon) \),
then \( \ell_1 \equiv_L \ell_2 \). Thus, \( \varepsilon_L \) is an equivalence relation.

**Notation.** If \( L \) is a local Lie group and \( \ell \) is an element in \( SL \), then "\( |\ell|_L \)"
denotes \( \{ \ell_1 : \ell_1 \in SL, \ell_1 \equiv_L \ell \} \).

**Definitions 16-18.** If \( L \) is a local Lie group, define:

1. \( +_L = \{ ((|x|_L, |y|_L, |z|_L) : x, y, z \in SL,

   \exists \varepsilon)(\varepsilon > 0, |0, \varepsilon) \subset \text{Domain}(x) \cap \text{Domain}(y),

   z = [t : |0, \varepsilon) : \tau\lim_{n \to \infty}(x(t_n) \cdot y(t_n))^{n}]) \} ; \)
11) \( \mathfrak{L}_L = \{((r, |x|_L), |y|_L) : r > 0, x, y \in SL, \]
\[ y = \{ t : \{ t : t > 0, rt \in \text{Domain}(x) \} : x(rt) \} \} \]
\[ U \{ ((r, |x|_L), |y|_L) : r < 0, x, y \in SL, \]
\[ y = \{ t : \{ t : t > 0, -rt \in \text{Domain}(x) \} : (x(-rt)) \} \}; \]

iii) \[ L, L = \{ ((|x|_L, |y|_L), |z|_L) : x, y, z \in SL, \]
\[ (\exists \varepsilon)(\varepsilon > 0, [0, \varepsilon) \subset \text{Domain}(x) \cap \text{Domain}(y), \]
\[ z = \{ t : [0, \varepsilon) : \lim_{n \to \infty}(x(\frac{\sqrt{\varepsilon}}{n}) \cdot y(\frac{\sqrt{\varepsilon}}{n}) \cdot x(-\frac{\sqrt{\varepsilon}}{n}) \cdot y(-\frac{\sqrt{\varepsilon}}{n})) \} \}. \]

Note. \( (SL, +_L, \times_L, [, ]_L) \) is a Lie algebra called "the Lie algebra of \( L \)."

**Definition 19.** If \( L \) is a local Lie group,
\[ \exp_L = \{(|x|_L, x(1)) : x \in SL, 1 \in \text{Domain}(x) \}. \]

**Definition 20.** If \( L_1, L_2 \) are local Lie groups, and if \( \psi \) is a local homo-
morphism from \( L_1 \) to \( L_2 \), define
\[ (d_{L_1}, L_2)(\psi) = \{(|x|_{L_1}, |\psi \cdot x|_{L_2}) : x \in SL_1, \psi \cdot x \in SL_2 \}. \]
Theorem 2. If $L$ is a locally compact local group, $B$ is a $\tau$-compact element in $\mathfrak{N}$, $L+B$ is a connected Lie group, $U$ is a $\tau$-connected element in $\mathcal{E}(L)$, $B \cdot U = U \cdot B = U$, $(B, U)$ is weakly-invariant in $L$, $N = \{ x: x \in U, (b)(b \in B \Rightarrow x \cdot b = b \cdot x) \}$, then $B \cdot N = U$.

Proof. There is a maximum semi-simple connected normal subgroup $S$ of $B$. Then $B = S\cdot Z$, where $Z$ is the center of $B$. [4, Theorem 1.3, p. 144]. Both $S$ and $Z$ are invariant under all automorphisms of $B$. Then $(S, U), (Z, U) \in \mathcal{W}$-$\mathcal{L}$. By Lemma 5, $(g)(z)(g \in U, z \in Z \Rightarrow z = g^\ast(z \cdot g))$.

Claim. $(g)(g \in U \Rightarrow (\exists s_0)(s_0 \in S, (s)(s \in S \Rightarrow s_0^\ast(s \cdot s_0) = g^\ast(s \cdot g)))$.

Let $\phi = [g: U: [s: S: g^\ast(s \cdot g)]]$. Since $(S, U)$ is a member of $W-L$, $\phi$ is a map from $U$ to $A(S)$, the automorphism group of $S$. Further, if "C" denotes the compact open topology on $A(S)$, $\phi$ is $(U \neq \tau)$-C-continuous, and so $\phi[ U]$ generates a C-connected subgroup of $A(S)$. Such a subgroup is necessarily a subgroup of $T(S)$, the inner automorphism group of $S$. [2, Chapter II, Section 5]

Suppose $g$ is in $U$. Then there exists an $s_0$ in $S$ such that for every $s$ in $S$, $g^\ast(s \cdot g) = s_0^\ast(s \cdot s_0)$. Suppose $b$ is an element in $B$. Then there are elements $s$ and $z$ of $S$ and $Z$ respectively such that $b = s \cdot z$. Further, $s_0^\ast(b \cdot s_0) = s_0^\ast((s \cdot z) \cdot s_0)$

$= (s_0^\ast(s \cdot s_0)) \cdot (s_0^\ast(z \cdot s_0)) = (g^\ast(s \cdot g)) \cdot (s_0^\ast(s_0^\ast(z \cdot s_0))) = (g^\ast(s \cdot g)) \cdot ((g^\ast(g) \cdot z) = (g^\ast(s \cdot g)) \cdot (g^\ast(z \cdot g)) = g^\ast((s \cdot z) \cdot g) = g^\ast(b \cdot g)$. So by the fact that $b$ was arbitrary in $B$, and by
Corollary to Lemma 6 we have $U = B \cdot N$. △

Lemma 7. If $L$ is a locally compact local group, $(B, U)$ is a member of $\Delta L$, $B$ is a $\tau$-compact $\tau$-connected member of $\Gamma L$, $B \cdot U = U \cdot B = U$, $L_1 = L/(B, U)$, $T_1 = T(L, B, U)$, $C_0$ is a $\tau_1$-connected member of $\tau_1$, and $C = T_1^+[C_0]$, then $C$ is $\tau$-connected.

Proof. Suppose $0_1, 0_2$ are members of $\tau$, $0_1 \cup 0_2 = C$, and $0_1 \cap 0_2 = \emptyset$.

Suppose $x \in U^2$. Since $B \cdot U = U \cdot B = U$, we have $(x \cdot B) \cap C = \emptyset$ or $x \cdot B \subset C$.

If $(x \cdot B) \cap C = \emptyset$, then $(x \cdot B) \cap 0_1 = (x \cdot B) \cap 0_2 = \emptyset$. If $x \cdot B \subset C$, then since $x \cdot B$ is $\tau$-connected, $(x \cdot B) \cap 0_1 = \emptyset$ or $(x \cdot B) \cap 0_2 = \emptyset$. So $x \in U^2 \Rightarrow (x \cdot B) \cap 0_1 = \emptyset$ or $(x \cdot B) \cap 0_2 = \emptyset$. If $y$ is a member of $T_1[0_1] \cap T_1[0_2]$, then there are elements $x_1$ and $x_2$ of $0_1$ and $0_2$ respectively such that $T(x_1) = y = T(x_2)$. So $x_1 \cdot x_2 \in B$, $x_2 \in (x_1 \cdot B) \cap 0_2$, $x_1 \in (x_1 \cdot B) \cap 0_1$. Since this is impossible, we conclude that $T_1[0_1] \cap T_1[0_2] = \emptyset$. Since $T_1[0_1]$ and $T_1[0_2]$ are members of $\tau_1$, and $T_1[0_1] \cup T_1[0_2] = C_0$, we conclude that $T_1[0_1] = \emptyset$ or $T_1[0_2] = \emptyset$. So $0_1 = \emptyset$ or $0_2 = \emptyset$. △

Lemma 8. If $G$ is a compact abelian Lie group, there is a finite subgroup $H$ of $G$ which is invariant under all automorphisms of $G$, and is such that $G/H$ is isomorphic to an $n$-dimensional torus (a toroidal group).

Proof. By [3, Theorem 9.5, p. 89], since $G$ is without small subgroups, there are subgroups $L$ and $K$ of $G$ such that $K$ is finite, $L$ is isomorphic to an $n$-dimensional torus, and $G \cong L \oplus K$.

Let $\ell$ be the order of $K$, and let $H = \{x : x \in G, \text{order}(x) | \ell\}$.

Then $H$ is a finite group and $K \subset H$. Since $G/K \cong H/K \cong G/H$ and $G/K \cong L$,
we have that \( G/H \) is a connected Lie group. Since it is also compact and abelian, it is isomorphic to a toroidal group, again by [3, Theorem 9.5, p. 89].

Finally, if \( \alpha \) is any automorphism of \( G \), and if \( x \) is an element in \( H \), then \( \alpha(x) \) has order which divides \( \ell \), and thus \( \alpha(x) \) is a member of \( H \). \( \diamond \)
Theorem 3. If $L$ is a locally compact local group, $B$ is a $\tau$-compact element in $\Gamma L$, $U$ is an element in $BN(L)$, $B \cdot U = U \cdot B = U$, $(B, U)$ is weak-invariant in $L$, $L+B$ is a Lie group, $B_0$ is the identity component of $L+B$, $\theta$ is a $\tau$-connected element in $BN(L)$, $B_0 \cdot \theta = \theta \cdot B_0 = \theta \subset U$.

$$N = \{n: n \in \theta, (b)(b \in B \Rightarrow n \cdot b = b \cdot n)\},$$

$$N_0 = \{n_0: n_0 \in \theta, (b_0)(b_0 \in B_0 \Rightarrow n_0 \cdot b_0 = b_0 \cdot n_0)\},$$

then $\theta = N_0 \cdot B_0 = N \cdot B_0$.

Proof. Let $\equiv = \{(x_1, x_2): x_1 \neq x_2 \in \theta, (\exists b_0)(b_0 \in B_0, (b)(b \in B \Rightarrow x_1 \cdot (b \cdot x_1) = x_2 \cdot (b \cdot x_2)))\}$.

Claim 1. $\equiv$ is an equivalence relation.

It is obviously reflexive. If $x_1 \equiv x_2$, then

$$x_1 \cdot ((b_0 \cdot b \cdot b_0) \cdot x_1) = x_2 \cdot ((b_0 \cdot (b_0 \cdot b_0) \cdot b_0) \cdot x_2) = x_2 \cdot (b \cdot x_2).$$

So $\equiv$ is symmetric. If $b_0$ and $b_1$ are elements in $B_0$ respectively, making $x_1 \equiv x_2$ and $x_2 \equiv x_3$, then for all $b$ in $B$, $x_1 \cdot (b \cdot x_1) = x_2 \cdot (b_0 \cdot b_0) \cdot x_2 = x_3 \cdot ((b_0 \cdot b_1) \cdot b \cdot (b_0 \cdot b_1)) \cdot x_3$.

So $(b_0 \cdot b_1)$ makes $x_1 \equiv x_3$.

Claim 2. The equivalence classes of $\theta$ determined by $\equiv$ are $\theta \neq \tau$-closed.

Since $B_0$ is $\tau$-compact, this follows by a standard net argument.

For each $x$, an element in $\theta$, define $\mu_x = [b \in B: b \cdot (x \cdot (b \cdot x))]$.

Claim 3. If $x \in \theta$, $\mu_x: B \rightarrow B_0$. 
For $b \in B$ define $F_b = [y: 0; b \cdot (y \cdot (b \cdot y))]$. Then $F_b$ is $(0 \neq 1)$-r-continuous, so $F_b[0]$ is a $r$-connected subset of $B$ containing $e$. So $F_b[0] \subseteq B_0$. Thus, for $x$, an element of $0$ and $b$ in $B$, $u(x)(b) = F_b(x)$ is an element of $B_0$.

Claim 4. If $n_0$ is an element in $N_0$, $\mu_{n_0}$ is constant on $B_0$-cosets of $B$.

If $n_0, b, b_0$ are elements in $N_0$, $B, B_0$ respectively, $\mu_{n_0}(b \cdot b_0) = (b_0 \cdot b) \cdot (n_0 \cdot ((b_0 \cdot b) \cdot n_0)) = (b \cdot b_0) \cdot (n_0 \cdot ((b_0 \cdot b) \cdot n_1))$

$= (b' \cdot b_0') \cdot (n_0 \cdot ((b_0' \cdot b_0') \cdot n_0)) = (b' \cdot b_0') \cdot ((b_0' \cdot (n_0 \cdot b)) \cdot n_0)$

$= ((b' \cdot b_0') \cdot (b_0' \cdot (n_0 \cdot b))) \cdot n_0 = (b' \cdot (n_0 \cdot b)) \cdot n_0 = b' \cdot ((n_0 \cdot b) \cdot n_0 = \mu_{n_0}(b)$

Let $M$ be a complete (finite) set of representatives of $B_0$-cosets of $B$. Let $n$ be its cardinal. Let $Z = \text{center}(B_0)$ and $Z_1$ be a finite characteristic subgroup of $Z$ such that $Z/Z_1$ is a toroidal group (Lemma 8). Let $Z_2 = \{z: z \in Z, z^n \in Z_1\}$, and let $A = \{f: f: B \to Z_2, f \text{ is constant on } B_0\text{-cosets of } B\}$. Then $A$ is finite.

Claim 5. If $n_0 \in N_0$, $\mu_{n_0}: B \to Z$.

Suppose $n_0, b$ are elements in $N_0$ and $B$ respectively. Then $\mu_{n_0}(b) = \mu_{n_0}(b \cdot b_0) = (b \cdot b_0) \cdot (n_0 \cdot ((b \cdot b_0) \cdot n_0)) = b \cdot (\mu_{n_0}(b) \cdot b_0)$. So $b \cdot \mu_{n_0}(b) = \mu_{n_0}(b) \cdot b_0$.

Claim 6. $(x)(x \in 0 \Rightarrow (\exists n_0)(n_0 \in N_0, n_0 \equiv x))$.

Suppose $x \in 0$. By Theorem 2, $0 = B_0 \cdot N_0$. So

$(\exists n_0)(\exists b_0)(n_0 \in N_0, b_0 \in B_0, x = b_0 \cdot n_0)$. Then for any $b$ in $B$,

$x \cdot (b \cdot x) = (n_0 \cdot b_0) \cdot (b \cdot (b_0 \cdot n_0)) = (n_0 \cdot b_0) \cdot ((b \cdot b_0) \cdot n_0)$
\[ ((n_0 \cdot b_0^\cdot (b \cdot b_0)) \cdot n_0 = (n_0 \cdot (b_0^\cdot (b \cdot b_0))) \cdot n_0 = n_0^\cdot ((b_0^\cdot b_0) \cdot n_0). \]

So \( x = n_0. \)

**Claim 7.** \( (x)(x \in 0 \Rightarrow (\exists n_1)(n_1 \in N_0, n_1 = x, \mu_{n_1} \in A)). \)

Suppose \( x \in 0. \) Then there is an \( n_0 \) in \( N_0 \) such that
\[ n_0 = x \text{ (Claim 6).} \]
For \( b_1, \) an element of \( B, \)
\[ \prod_{b \in M_0} \mu_{n_0}(b) = \prod_{b \in M_0} \mu_{n_0}(b \cdot b_1) = \prod_{b \in M_0} (b_1 \cdot b^\cdot) \cdot (n_0^\cdot ((b \cdot b_1) \cdot n_0)) \]
\[ = \prod_{b \in M_0} (b_1 \cdot (\mu_{n_0}(b) \cdot (n_0^\cdot (b_1 \cdot n_0)))) = \prod_{b \in M_0} (b_1 \cdot (\mu_{n_0}(b) \cdot (b_1 \cdot \mu_{n_0}(b_1)))) \]
\[ = b_1^\cdot (\prod_{b \in M_0} \mu_{n_0}(b) \cdot (b_1 \cdot (\mu_{n_0}(b_1) \cdot b_1^\cdot))) \cdot b_1 \]
\[ = b_1^\cdot ((\prod_{b \in M_0} \mu_{n_0}(b)) \cdot b_1 \cdot (\mu_{n_0}(b_1))^n). \] So \( (\mu_{n_0}(b_1))^n \)
\[ = b_1^\cdot (\prod_{b \in M_0} \mu_{n_0}(b)) \cdot b_1 \cdot (\prod_{b \in M_0} \mu_{n_0}(b)). \]

There exist elements \( r \) and \( q \) in \( Z_1 \) and \( Z \) respectively such that
\[ (q)^n = (\prod_{b \in M_0} \mu_{n_0}(b)) \cdot r. \]
Then for \( b_1 \) in \( B, \)
\[ (\mu_{n_0}(b_1))^n = b_1^\cdot ((q)^n \cdot r) \cdot b_1 \cdot r^\cdot (q)^n \]
\[ = (b_1^\cdot q \cdot b_1)^n \cdot (b_1 \cdot r \cdot b_1) \cdot r^\cdot (q)^n = (b_1^\cdot q \cdot b_1 \cdot q)^n \cdot b_1 \cdot r \cdot b_1 \cdot r^\cdot. \]
So \( (\mu_{n_0}(b_1))^n = (b_1^\cdot ((q \cdot n_0^\cdot) \cdot (b_1 \cdot (n_0^\cdot q^\cdot))))^n \]
\[ = ((b_1^\cdot q \cdot b_1) \cdot (b_1^\cdot n_0^\cdot \cdot b_1 \cdot n_0^\cdot) \cdot q^\cdot)^n = (b_1^\cdot n_0^\cdot \cdot b_1 \cdot n_0^\cdot) \cdot ((b_1^\cdot q \cdot b_1 \cdot q^\cdot))^n \]
\[ = (b_1^\cdot r \cdot b_1) \cdot r^\cdot \in Z_1. \] So \( \mu_{n_0} \cdot q^\cdot \in A. \)

**Subclaim.** \( n_0 = q^\cdot n_0. \)

Clearly \( q^\cdot n_0 \in Z \cdot 0 \subseteq B \cdot 0 = 0. \) For \( b \in B, \)
\[ (q^\cdot n_0)^\cdot \cdot (b \cdot (q^\cdot n_0)) \]
\[ = (n_0 \cdot q) \cdot ((b \cdot q^\cdot) \cdot n_0) = n_0^\cdot (q \cdot ((b \cdot q^\cdot) \cdot n_0)) = n_0^\cdot ((q \cdot (b \cdot q^\cdot)) \cdot n_0) \]

Clearly, \( q^\cdot n_0 \in N_0. \) Let \( n_1 = q^\cdot n_0. \)
Claim 8. \((x)(y)(x, y \in 0, \mu_x = \mu_y \Rightarrow x = y)\).

If \(b \in B\), \(b^- \cdot (x^- \cdot (b \cdot x)) = b^- \cdot (y^- \cdot (b \cdot x))\). So \(x^- \cdot (b \cdot x) = y^- \cdot (b \cdot y)\).

Thus, the number of equivalence class in 0 is finite. So every equivalence class is open in 0 as well as being \((0 \# \tau)\)-closed. Since 0 is \(\tau\)-connected, this implies that there is only one equivalence class. So \((x)(x \in 0 \Rightarrow x = e)\). Hence

\[(x)(x \in 0 \Rightarrow (\exists b_o)(b_o \in B_o, (b)(b \in B \Rightarrow x^- \cdot (b \cdot x) = b_o^- \cdot (b \cdot b_o))))\).

Since \(b_o \cdot x^-\) is in 0, this shows that \(b_o \cdot x^-\) is an element in N. So \(x \in B_o \cdot N\). Thus \(0 < B_o \cdot N\). △
Theorem 4. If $L$ is a locally compact local group, $B$ is a $\tau$-compact element in $\mathfrak{M}(L)$, $(B, U)$ is an element in $\mathcal{N}(L, B, U) = U$, $\mathfrak{O}$ is an element in $\mathfrak{M}(L)$, $0 \cdot B = B \cdot 0 = 0 \subset U$, then $(B, U)$ is $\tau(L, B, U)$-connected and locally connected, and $N = \{n : n \in 0, (b)(b \in B \implies b \cdot n = n \cdot b)\}$, then $B \cdot N = 0$.

Proof. Let $T_1 = T(L, B, U)$, $L_1 = L/(B, U)$. Let $\theta$ be an $m$-dimensional continuous linear representation of $B$ with character $\chi$. Let $A_\theta = \{g : g \in U, (b)(b \in B \implies \chi(b) = \chi(g*(b \cdot g)))\}$.

Claim 1. $0 \subset A_\theta$.

We know there is an element $W$ in $\mathfrak{M}(L)$ such that $W \subset A_\theta$.

[1, p. 60]. If $g$ is an element of $A_\theta$, then there is an element $W_0$ of $\mathfrak{M}(L)$ such that $W_0 \subset W$, and $g \cdot W_0 \subset U$. If $x = g \cdot w$ is an element of $g \cdot W_0$, we have $\chi(x^\ast \cdot (b \cdot x)) = \chi(w^\ast \cdot (g^\ast \cdot (b \cdot g)) \cdot w) = \chi(g^\ast \cdot (b \cdot g)) = \chi(b)$. So $g \cdot W_0 \subset A_\theta$. This proves that $A_\theta$ is a member of $\tau$.

Further, $A_\theta \cdot B \subset U \cdot B \subset U$, and if $x \cdot b$ is an element of $A_\theta \cdot B$, we see that for every element $b_1$ in $B$, we have $\chi(b^\ast \cdot (x^\ast \cdot (b_1 \cdot (x \cdot b)))) = \chi(x^\ast \cdot (b_1 \cdot x)) = \chi(b_1)$. So $A_\theta \cdot B \subset A_\theta$.

Thus, $A_\theta$ is an element of $\tau$ which is saturated with respect to $B$. Also, $A_\theta \cap 0$ is $(0 \# \tau)$-closed; for if $(S, >)$ is a net in $A_\theta \cap 0$ converging in $0$ to $x$, then for every element $b$ in $B$, we have, by continuity of $\chi$, $\chi(x^\ast \cdot (b \cdot x)) = \chi(b)$. So $x \in A_\theta \cap 0$. 
Hence, since $T_1[0]$ is $\tau_1$-connected, $0 \subset A_\theta$.]

Let $R_\theta$ be the kernel of $\theta$. Observe, since $\theta$ is equivalent to a unitary representation, an element $b$ of $B$ belongs to $R_\theta$ if and only if $\chi(b) = m$. Since $0 \subset A_\theta$, $(R_\theta, 0)$ is an element of $W\Delta L$. Let $T_\theta = T(L, R_\theta, 0)$, and let $B_\theta = T_\theta[B]$. Then since $B_\theta$ is mapped one to one continuously into the automorphism group of $W^m$ by the map induced on $B_\theta$ by $\theta$ in the natural way, we have that $B_\theta$ is a compact Lie subgroup of the local group $L_\theta = L/(R_\theta, U)$.

Let $O_\theta = T_\theta[0]$. Then $(B_\theta', O_\theta) \in W\Delta L_\theta$.

Let $C_\theta$ be the component in $B_\theta$ of $e_\theta$. Let $V_\theta$ be the $\tau_\theta$-component in $O_\theta$ of $e_\theta$. Clearly, $V_\theta \cdot C_\theta = C_\theta \cdot V_\theta = V_\theta$.

Claim 2. $V_\theta \in \tau_\theta$.

Suppose $x \in V_\theta$. There is a $Y$ in $\mathcal{B}(L_\theta)$ such that $x \cdot Y \subset O_\theta$. Let $Y_\theta = C_\theta \cdot Y = Y \cdot C_\theta$. Let $C_\theta^\ast = T_\theta^+[C_\theta]$. Then $(C_\theta^\ast, U) \in \Delta L$.

Let $L_\theta^\ast = L/(C_\theta^\ast, U)$, $T_\theta^\ast = T(L, C^\ast, U)$, and $\alpha, \beta$ and $\gamma$ be the natural maps from $L_\theta$ onto $L_\theta^\ast$; from $L_\theta^\ast$ onto $L_1$; and from $L_\theta$ onto $L_1$ respectively. Since $T_\theta^+[B] = B_\theta/C_\theta$ is finite, $B$ is a local isomorphism. There exists a $\tau_\theta^\ast$-connected open subset $D_\theta^\ast$ of $\alpha[Y_\theta]$.

Let $D_\theta = \alpha^+[D_\theta^\ast]$. Since the kernel of $\alpha$ is the $\tau_\theta$-connected set $C_\theta$, $D_\theta$ is $\tau_\theta$-open and connected (imitate proof of Lemma 7). Since $D_\theta \subset \gamma^\prime Y_\theta$, $x \cdot D_\theta \subset O_\theta$. Hence $x \cdot D_\theta \subset V_\theta$.

There is a finite subset $M$ of $B_\theta$ such that $B_\theta = \bigcup_{b \in M} C_\theta \cdot b$.

So $V_\theta \cdot B_\theta = \bigcup_{b \in M} V_\theta \cdot b$ is open and closed in $O_\theta$.

and is saturated with respect to $B_\theta$. Then $\gamma[V_\theta \cdot B_\theta]$ is open and closed in $\gamma[O_\theta] = T_1[0]$ which is $\tau_1$-connected. So $\gamma[V_\theta \cdot B_\theta] = T_1[0]$, and hence $V_\theta \cdot B_\theta = O_\theta$. 
Let \( N_0 = \{ z : z \notin V_0, (b) (b \cdot e : B_0 \mapsto z \cdot b = b \cdot z) \} \). By Theorem 3, \( V_0 = N_0 \cdot e \cdot c_0 \). So \( \theta_0 = (N_0 \cdot e \cdot c_0) \cdot e B_0 = N_0 \cdot e B_0 \). Thus, for every element \( g \) in \( \theta \), there is an element \( \beta \) in \( B \) such that \( T_\theta(\beta) \) and \( T_\theta(g) \) induce the same automorphism on \( B_\theta \).

Define a function \( M \) on the set of pairs \( (g, \theta) \) where \( g \) is an element of \( \theta \), and \( \theta \) is a finite dimensional continuous linear representation of \( B \) as follows:

\[
M(g, \theta) = \{ \beta : \beta \text{ is an element of } B, \text{ and } T_\theta(g) \text{ and } T_\theta(\beta) \text{ induce the same automorphism on } B_\theta \}.
\]

For each \( (g, \theta) \), \( M(g, \theta) \) is \( (B \neq \emptyset) \)-closed, and hence \( \emptyset \)-compact. Further, \( M(g, \theta) \neq \emptyset \).

**Claim 3.** If \( \theta_1, \ldots, \theta_n \) are representations of \( B \), then \( \bigcap_{i=1}^{n} M(g, \theta_i) \neq \emptyset \).

There is a representation \( \theta \) such that \( R_\theta = \bigcap_{i=1}^{n} R_{\theta_i} \) (e.g. the direct sum of the representations \( \theta_1, \ldots, \theta_n \)). Then \( \beta \in M(g, \theta) \iff (b) (b \in B \Rightarrow T_\theta(g) \cdot e (T_\theta(b), e T_\theta(\beta)) = T_\theta(\beta) \cdot e (T_\theta(b), e T_\theta(\beta))) \iff (b) (b \in B \Rightarrow (\beta \cdot b \cdot \beta) \cdot (g \cdot b \cdot g) \in \bigcap_{i=1}^{n} R_{\theta_i}) \Rightarrow (b) (b \in B \Rightarrow (\beta \cdot b \cdot \beta) \cdot (g \cdot b \cdot g) \in \bigcap_{i=1}^{n} R_{\theta_i}).

The last statement is equivalent to the statement that for any element \( b \) in \( B \), and for any \( i \) in \( \mathbb{N} \), \( T_{\theta_i}(g) \) and \( T_{\theta_i}(b) \) induce the same automorphism on \( B_{\theta_i} \). So \( M(g, \theta) \subset \bigcap_{i=1}^{n} M(g, \theta_i) \).

So there is an element \( d \) in \( \bigcap_{\theta} M(g, \theta) \). Thus, for every \( \theta \), \( T_\theta(g) \) and \( T_\theta(d) \) induce the same automorphism on \( B_\theta \). By the completeness
of the system of representations, for any element $b$ in $B$, we have $g^{-1}(b \cdot g) = d^{-1}(b \cdot d)$. Since $d$ is an element in $B$, we have shown $0 \subseteq B \cdot N$. (See Corollary to Lemma 6). △

Corollary. If $L$ is a locally compact local group, $(B, U)$ is an element in $\mathfrak{I}L$, $B$ is a $\tau$-compact element in $\mathfrak{I}L$, $B \cdot U = U \cdot B = U$, $L_1 = L/(B, U)$ is a local Lie group, and $N = \{g : g \in U, (b \in B \Rightarrow b \cdot g = g \cdot b)\}$, then $e$ is in the $\tau$-interior of $B \cdot N$.

Proof. △
Lemma 9. If \( L \) is a locally compact local group, \((B, U)\) is a member of \( \Delta L \), \( B \) is a locally compact member of \( \Gamma L \), \( B \cdot U = U \cdot B = U \),
\[
L_0 = L/(B, U)
\]
is a local Lie group,
\[
N = \{x: x \in U, (b)(b \in B \Rightarrow x \cdot b = b \cdot x)\}, \text{ and } Z = N \cap B,
\]
then \( L+N \) is a locally compact local group, and if \( V^+ \) is a member of \( BN(L) \cap \phi(L, U, \{I\}) \), then \((Z, V^+ \cap N)\) is an element of \( \Delta(L+N) \).
\[
(L+N)/(Z, V^+ \cap N)\text{ is a local Lie group locally isomorphic to } L_0.
\]

Proof. The first two assertions are obvious.

Let \( T_0 = T(L, B, U) \), and \( T_1 = T(N, Z, V) \). Then the map
\[
\phi = \{(T_1(x), T_0(x)): x \in V^2\}
\]
is a locally one to one continuous local homomorphism from locally compact \( N/(Z, V) \) into \( L_0 \). □

Definition 21. \( L_0 \) is a sublocal group of \( L \) if there is a set \( V \) such that \((L_0, V)\) is a sublocal group former of \( L \).

Definition 22. If \( L \) is a local group, \( V \) is an element in \( BN(L) \), and \( \alpha \) is a positive real number, then \( \text{Arc}(L, V, \alpha) \) is the set of all \(([0, \alpha] \not= R)\) \( \tau \)-continuous functions with range a subset of \( V \).
Definition 23. If $L$ is a local group, $V$, $W$ are elements in $BW(L)$, $\alpha$, $t_1$ and $t_2$ are real numbers and $\ell$ is an element in $\text{Arc}(L, W, \alpha)$, then $\theta$ is an $(L, t_1, t_2, V)$-deformation of $\ell$ in $W$ if the following are satisfied:

1) $0 < t_1 < t_2 < \alpha$;
2) $\theta$ is a $\tau$-deformation of $\ell$ with range in $W$;
3) for all $s$, if $0 < s < \alpha$, then for $t$ in $(t_1, t_2)$, $(\ell(t_1))^\tau \cdot \theta(t, s)$ is an element of $V$, and for $t$ in $[0, 1] \sim (t_1, t_2)$, $\theta(t, s) = \ell(t)$.
Theorem 5. If $L$ is a locally compact local group, $(Z_0, U)$ is a member of $\Delta L$, $U$ is an element of $\Phi(L, 18)$, $Z_0$ is a $\tau$-compact member of $\Pi L$, $Z_0 \cdot U = U \cdot Z_0 = U$, and if every member of $\Pi L$ which is a subset of $U$ is a subgroup of $Z_0$, and if for every $x$ a member of $U$, and every $z$ a member of $Z_0$, $x \cdot z = z \cdot x$, then there is a Lie group $L^*$ and a locally one to one continuous local homomorphism $\xi^*$ from $L^*$ into $L$ such that range $(\xi^*) \subset U$, and for any $\tau$-neighborhood $M^*$ of $e^*$, $Z_0 \cdot \{\xi^*[M^*]\}$ is a $\tau$-neighborhood of $e$.

Proof. Let $T_1 = T(L, Z_0, U)$. Let $H_1$ be the Lie algebra of $L_1 = L/(Z_0, U)$, \{h_1, \ldots, h_n\} be a basis for $H_1$. There exists a positive number $\delta_1$,

one parameter sublocal groups $g_{1}^{(1)}, \ldots, g_{n}^{(1)}$ of $L_1$, and

one parameter sublocal groups $g_1, \ldots, g_n$ of $L$ such that for every $i$ in $\mathbb{N}_1^n$ and every $t$ in $(-\delta_1, \delta_1)$, $g_i(t)$ is a member of $U$ and $T_1(g_i(t)) = g_i^{(1)}(t) = \exp_{L_1}(t h_i)$. [6, p. 192].

Let $p = n(\frac{n-1}{2})$. There is an $(n + p)$-dimensional real vector space $H_0$ having $H_1$ as a subvector space and a basis \{a_1, \ldots, a_{p+n}\} such that for $i$ in $\mathbb{N}_1^n$, $a_i = h_i$. Consider the function $\ell$ defined by

\[
\ell_{ij} = a_n + \frac{(j-1)(j-2)}{2} + 1 \quad \text{if} \quad i, j \in \mathbb{N}_1^n, \quad i < j
\]

\[
\ell_{ij} = -\ell_{ji} \quad \text{if} \quad i, j \in \mathbb{N}_1^n, \quad j < i
\]

Let $H$ be the Lie algebra obtained from $H_0$ by defining

\[
[h_i, h_j] = h_{ij} + \ell_{ij} \quad \text{if} \quad i, j \in \mathbb{N}_1^n, \quad i < j
\]

\[
[h_i, \ell_{ij}] = 0 \quad \text{if} \quad i, j, k \in \mathbb{N}_1^n, \quad i < j
\]

\[
[\ell_{ij}, \ell_{km}] = 0 \quad \text{if} \quad i, j, k, m \in \mathbb{N}_1^n, \quad i < j, \quad k < m.
\]
There is a Lie group $\overline{L}$ such that there is a Lie algebra isomorphism $\Lambda$ from $\overline{H}$, the Lie algebra of $\overline{L}$, to $H$. Let $\text{Exp}_{\overline{L}} = \exp_{\overline{L}} \circ \Lambda^+.$

There is a Lie homomorphism $\psi$ from $H$ into $H_1$ such that if $i$, $j$ and $k$ are in $\mathbb{H}_n$ and $j < k$, then $\psi(h_1) = h_1$ and $\psi(\ell_{jk}) = 0.$

Then $\psi \circ \Lambda$ is a Lie homomorphism from $\overline{H}$ to $H_1$, and thus determines a continuous local homomorphism $\psi$ from $\overline{L}$ to $L_1$ such that $d_{\overline{L}}(\psi) = \psi \circ \Lambda.$

There is an element $Q$ of $BN(L) \cap \Phi(L, U, 2)$ such that $Q \cdot Z_0 = Z_0 \cdot Q = Q$. Let $R_0$ be a connected, simply connected member of $BN(\overline{L}) \cap \Phi(\overline{L}, 3)$ such that $(R_0)^3 \subseteq \text{Domain}(\psi)$, and $\psi[R_0^3] \subseteq T_1 [Q]$.

There is a connected, simply connected member $R$ of $BN(\overline{L}) \cap \Phi(\overline{L}, R_0, 2)$ such that for all elements $x$ and $y$ of $R$, $\psi(x \cdot y) = \psi(x) \cdot \psi(y)$.

There is a positive number $\delta < \delta_1$ such that for $i$ in $\mathbb{H}_n$ and $t$ in $(-\delta, \delta)$, $\text{Exp}_{\overline{L}}(th_1)$ is an element of $R$, and such that $\mathfrak{g}_1 = \{ t: (-\delta, \delta) : \text{Exp}_{\overline{L}}(th_1) \}$ is a member of $S\overline{L}$, and for $t$ in $(-\delta, \delta)$, $\psi(\mathfrak{g}_1 (t)) = \mathfrak{g}_1^{(1)} (t)$.

Suppose $Z^-$ is a $\tau$-compact subgroup of $Z_0$. Then $(Z^-, U)$ is a member of $\Delta L$. Further, $Q \cdot Z^- = Z^- \cdot Q = Q$. Suppose that $L^- = L/(Z^-, U)$ is a local Lie group. Let $T^- = T(L, Z^-, U)$ and $\phi^-$ be the natural local homomorphism from $L^-$ to $L_1$.

**Claim 1.** $(\phi^-)^* [ T^- [Q] ] = T^- [Q].$

If $x$ is a member of $U^2$ and $\phi^-(T^-(x)) = T^-_1 (x)$ is an element of $T^-_1 [Q]$, then there is an element $u$ of $Q$ such that $u \cdot x$ is an element of $Z_0$, which implies that $x$ is an element of $Q \cdot Z_0 = Q$, and thus that $T^- (x)$ is an element of $T^- [Q]$. So
\[(\psi^*)^+ [T_1[Q]] < T^{-1[Q]}. \text{ The other inclusion is obvious.}\]

For \( i \) an element of \( \mathbb{N}_1^n \) define \( g_i^- = T^* g_i \). Let \( L^- \) be the Lie algebra of \( L^\pm \). Let \( h_1^-, \ldots, h_n^- \) be the vectors in \( H^- \) determined by the one parameter sublocal groups \( \xi_1, \ldots, \xi_n \).

**Claim 2.** There is a function \( \psi^- \) uniquely determined by the following conditions:

1. \( \psi^- : R^2 \to T^{-1[Q]} \)
2. \( \psi^-|R \) is \( (R \neq \overline{1}) - \tau^- \)-continuous
3. \( (x)(y) \in R \Rightarrow \psi^-(x \cdot y) = \psi^-(x) \cdot \psi^-(-y) \)
4. \( (i)(t) \in \mathbb{N}_1^n, t \in (-\delta, \delta) \Rightarrow \psi^-(g_i(t)) = T^-(g_i(t)) \)

Further, the \( \psi^- \) uniquely determined by these conditions satisfies:

5. \( \psi^- : R^2 \to T^{-1[Q]} \)
6. \( \psi^- \circ \psi^- = \psi|R^2. \)

**Subclaim 1.** There is a uniquely determined Lie algebra homomorphism \( \psi^- \) from \( H \) to \( H^- \) such that for \( i \) an element of \( \mathbb{N}_1^n \), \( \psi^-(h_i) = h_i^- \).

**Uniqueness.** If such a Lie homomorphism \( \psi^- \) exists,

\[
\psi^-(\xi_{ij}) = \psi^-([h_i, h_j]) - [h_i, h_j]_1
= \psi^-(h_i, h_j)_1 - \psi^-(h_i, h_j)_1
= [h_i^-, h_j^-] - \psi^-([h_i, h_j]_1)
\]

Since \( \{h_1, \ldots, h_n, \xi_{ij} : i < j \in \mathbb{N}_1^n \} \) is a basis for \( H, \)
this shows that \( \Psi^- \) is determined completely by its values on \( H_1^- \).

Existence. There is a positive number \( \delta^- < \delta \) such that for \( i \) an element of \( \mathbb{N}_1^n \) and \( t \) an element of \( (-\delta^-, \delta^-) \),

\[ \exp_{L^-}(t h_i) = g_i(t) \text{ and } \phi_i(g_i(t)) = g_i(1)(t). \]

So by the definition of \( d_{L^-}, l_1 \phi^-, \) if \( i \) is an element of \( \mathbb{N}_1^n \),

\[ (d_{L^-}, l_1)(\phi^-) = h_1^- \]

Thus, since \( \{h_1^-, \ldots, h_n^-\} \) is linearly independent, so is \( \{h_1^-, \ldots, h_n^-\} \).

Next we will show there is a basis

\( \{h_1^-, \ldots, h_n^-, \delta_1, \ldots, \delta_m\} \) for \( \mathbb{H}^- \) such that for \( i \) in \( \mathbb{N}_1^m \), \( \delta_1 \) is an element in the center of \( \mathbb{H}^- \). Suppose \( \delta \) is an element of \( \mathbb{H}^- \). Suppose \( \sigma \) is a one parameter sublocal group of \( L^- \), and \( \delta \) is a positive real number such that for \( t \), a member \( (-\delta, \delta) \), \( \exp_{L^-}(t \delta) = \sigma(t) \) is an element of \( T^- \gamma U \). Then

\( \phi^- \circ \sigma \) is a one parameter sublocal group of \( L_1^- \). Suppose the vector in \( H_1^- \) determined by \( \phi^- \circ \sigma \) is

\[ \sum_{i=1}^{n} f_i h_i^- \]

Let \( \delta_0 = \delta - \sum_{i=1}^{n} f_i h_i^- \). Let \( k = \sum_{i=1}^{n} f_i h_i^- \).

Subsubclaim 1. \(( \exists \delta_0 \) \( \delta_0 \) > 0, \( \forall t \) \((t+ \delta_0, \delta_0) \Rightarrow

\[ \exp_{L^-}(t \delta_0) \in T^- \gamma Z_0 \)) \).

Since \( (d_{L^-}, l_1 \phi^-)(\delta) = \sum_{i=1}^{n} f_i h_i = (d_{L^-}, l_1 \phi^-)(k) \),

\( \exists t_1 \) \((t < \delta_4, \delta, (t+ \delta_1, \delta_1) \Rightarrow \phi^- (\exp_{L^-}(t \delta)) \)

\[ = \phi^- (\exp_{L^-}(t k)), \exp_{L^-}(t k) \in T^- \gamma U \)). \)
So for any element \( t \) in \((-t_1, t_1)\),
\[
\exp_L(\langle t \rangle) \cdot \exp_L(-\langle t \rangle) \text{ is a member of } T^-[Z],
\]
and hence commutes with all elements of \( T^-[U] \).

Thus for small \( t \),
\[
\exp_L(\langle t \rangle) \cdot \exp_L(-\langle t \rangle) = \exp_L(\langle t \rangle).
\]
So \( \exp_L(\langle t \rangle) = \exp_L(-\langle t \rangle) \).

Then using [8, Theorem 6.5.1, p. 127], there is a \( \delta_0 \) such that for \( t \), an element of \((-\delta_0, \delta_0)\),
\[
\exp_L(\langle t \rangle) = \exp_L(-\langle t \rangle).
\]

Subsubclaim 2. \( \delta_0 \) is an element of the center of \( H^- \).

Suppose \( q \in H^- \). (\( 3 \ t_2 \))(0 < t_2 < \delta_0, \ (t)(t \in (-t_2, t_2))
\Rightarrow \exp_L(\langle t \rangle) \in T^-[V] \). Then \( t(t \in (-t_2, t_2)) \)
\Rightarrow \exp_L(\langle t \rangle) \cdot \exp_L(-\langle t \rangle) = \exp_L(-\langle t \rangle) \cdot \exp_L(\langle t \rangle).
\[
[\exp_L(\langle t \rangle)] = 0. \]

It is now clear that there is a basis for \( H^- \) consisting of
\( h_1^-, \ldots, h_n^- \) and elements of the center of \( H^- \).

For elements \( i \) and \( j \) in \( \mathbb{N} \),
there are uniquely determined numbers \( f \) and \( c \) such that
\[
\sum_{k=1}^{n} d_{ij}^{(k)} \ h_k^- + \sum_{k=1}^{m} f_{ij}^{(k)} \ s_k, \text{ and } \ [h_i^-, h_j^-] = \sum_{k=1}^{n} c_{ij}^{(k)} \ h_k^-.
\]
Let $\psi^-$ be the linear transformation from $H$ to $H^-$ such that for $i, j, r$ in $\mathbb{N}_1^n$, if $i < j$ then $\psi^-(h_r) = h_r^-$. And $\psi^-(L_{ij}) = \sum_{k=1}^{m} f_{ij}^{(k)} h_k^-$. Since $(d_L^- L_1^-)(h_k^-) = 0$ for each $k$ in $\mathbb{N}_1^m$, we have $(d_L^- L_1^-)([h_1^-, h_j^-]) = [h_1^-, h_j^-]_1$.

Thus $\psi^-$ is readily verified to be a Lie homomorphism.

There exists a continuous local homomorphism $\overline{\psi}$ from $L_1$ to $L_1^-$ such that $(d_L^- L_1^-)(\overline{\psi}) = \psi^* \Lambda$.

There is an element $W$ of $BN(\overline{L}_1) \cap \phi(\overline{L}_1, R, 2)$ such that $W$ is $\overline{\psi}$-connected, $W^2 \subset \text{Domain}(\overline{\psi}^-) \cap R$, $\phi|W = \phi^- \circ (\overline{\psi}^-|W)$, and for all elements $a$ and $b$ in $W$, $\overline{\psi}^- (a \cdot b) = \overline{\psi}^- (a) \cdot \overline{\psi}^- (b)$.

Suppose $\ell \in \text{Arc}(\overline{L}_1, R_o, 1)$, $\ell(\overline{0}) = \overline{\ell}^o$.

Subclaim 2. $(\exists! \ell_o^o)(\ell_o^o \in \text{Arc}(L_1^-, T^o[I^o], 1)$;

1) $\ell_o^o(0) = e$;

ii) $(\exists \alpha)(0 < \alpha, (t_1^o(t_2^o)(t_1, t_2 \in \text{Domain}(\ell_o^o))$ $|t_1 - t_2| < \alpha \Rightarrow (\ell(t_1))^2 \cdot \ell(t_2) \in W,$ $(\ell_o^o(t_1))^2 \cdot \ell_o^o(t_2) = \overline{\psi}^- (\ell(t_1))^2 \cdot \ell(t_2)));$

iii) $(t)(t \in \text{Domain}(\ell_o^o) \Rightarrow \phi^- (\ell_o^o(t)) = \psi(\ell(t)))$. 
Uniqueness. Suppose \( \ell_o^{(1)}, \ell_o^{(2)} \) are elements of \( \text{Arc}(L^-, T^- [Q, 1), and both satisfy conditions i) - iii}), and \( \alpha_1, \alpha_2 \) are positive real numbers satisfying condition ii) for \( \ell_o^{(1)} \) and \( \ell_o^{(2)} \) respectively. Let 
\[ a = \min(\alpha_1, \alpha_2). \]
For \( t \), an element of \([0, a]\), we have 
\[ \ell_o^{(1)}(t) = \ell_o^{(2)}(t), \]
and by continuity we have 
\[ \ell_o^{(1)}(a) = \ell_o^{(2)}(a). \]
If \( t \) is a real number, and 
\[ a < t < 2a, \]
then \( (t - a) < a. \) So by (ii) applied to \( t - a \) and \( t, \) 
\[ \ell_o^{(1)}(t) = \ell_o^{(2)}(t). \]
Thus \( \ell_o^{(1)}(2a) = \ell_o^{(2)}(2a). \)
Induction completes the proof.

Existence. There is an element \( Y \) in \( BN(L) \cap (L, W, 2) \) such that \( \overline{\psi} [Y] \subset T^- [Q]. \) Further, \( \exists n \in \mathbb{N}, (t_1)(t_2) \)
\[ (t_1, t_2 \in [0, 1], |t_1 - t_2| < \frac{1}{n} \Rightarrow (\ell(t_1)) \overline{\psi} \ell(t_2) \in Y). \)
Let \( a = \frac{1}{n}. \) We will use induction and show that if \( m \) is in \( \mathbb{N} \)
and \( (m + 1)a < 1, \) and if \( \ell_o \) is an element of \( \text{Arc}(L^-, T^- [Q], a) \)
satisfying conditions i) - iii), then there is an element \( \ell_o \) in \( \text{Arc}(L^-, T^- [Q], (m + 1)a) \)
satisfying conditions i) - iii).

If \( m = 1, \) let \( \ell_o = (\overline{\psi} \circ \ell)[0, a]. \) Then \( \ell_o \) satisfies
conditions i) - iii).

Suppose \( m > 1 \) and \( (m + 1)a < 1, \) and \( \ell_o \) is a member of \( \text{Arc}(L^-, T^- [Q], a) \)
satisfying conditions i) - iii) (with \( a = \frac{1}{n}. \) For \( n, \) an element of \([0, a], \) define 
\[ \ell_o^+(m \alpha + n) = \ell_o(m \alpha) \cdot \overline{\psi}((\ell(m \alpha)) \overline{\psi} \ell(m \alpha + n)). \]
We will show that if \( n \) is an element in \([0, \alpha]\), then
\[
\phi^\prime(\ell_0^+(m \alpha + n)) = \psi(\ell(m \alpha + n))
\]
and \( \ell_0^+(m \alpha + n) \) is an element of \( T_1[Q] \). Since \( \phi^\prime(\ell_0(m \alpha)) = \psi(\ell(m \alpha)) \), for \( n \), an element of \([0, \alpha]\), we have
\[
\phi^\prime(\ell_0^+(m \alpha + n)) = \phi^\prime(\ell_0(m \alpha) \cdot \phi^\prime(\ell(m \alpha))^n \cdot \ell(m \alpha + n))
\]
\[
= \phi^\prime(\ell_0(m \alpha) \cdot \ell_0^+(m \alpha + n))
\]
\[
= \psi(\ell(m \alpha)) \cdot \psi((\ell(m \alpha))^n) \cdot \psi(m \alpha + n) = \psi(m \alpha + n).
\]
Since \( \psi(m \alpha + n) \) is an element of \( T_1[Q] \), it follows from Claim 1 that \( \ell_0^+(m \alpha + n) \) is an element of \( T_1[Q] \).

Let \( \overline{\ell}_0 = \ell_0 \cup \ell_0^+ \). Then \( \overline{\ell}_0 \) is an element of \( \text{Arc}(\ell^-, T^-[Q], (m + 1) \alpha) \), and condition iii) is satisfied for \( \overline{\ell}_0 \). If \( t_1 \) and \( t_2 \) are elements in \([0, m \alpha]\), then
\[
(\overline{\ell}_0(t_1)) \cdot \cdot \cdot \overline{\ell}_0(t_2) = (\ell_0(t_1)) \cdot \cdot \cdot \ell_0(t_2) = \phi^\prime((\ell(t_1))^n \cdot \ell(t_2))
\]
by the induction assumption. If \( t_1 \) is an element of \([0, m \alpha]\) and \( t_2 \) is an element of \((m \alpha, (m + 1) \alpha] \), let \( n_2 = t_2 - m \alpha \).

Then
\[
(\overline{\ell}_0(t_1)) \cdot \cdot \cdot \overline{\ell}_0(t_2) = (\ell_0(t_1)) \cdot \cdot \cdot \ell_0(m \alpha) \cdot \phi^\prime((\ell(m \alpha))^n) \cdot \phi^\prime((\ell(m \alpha))^n) \cdot \ell(m \alpha + n_2)
\]
\[
= \phi^\prime((\ell(t_1))^n \cdot \ell(m \alpha + n_2)).
\]
Finally, if \( t_1, t_2 \) are elements in \((m \alpha, (m + 1) \alpha], \) let \( n_1 = t_1 - m \alpha \) and \( n_2 = t_2 - m \alpha \).

Then 
\[
(\overline{\ell}_0(m \alpha + n_1)) \cdot \cdot \cdot \overline{\ell}_0(m \alpha + n_2) = (\ell_0(m \alpha) \cdot \phi^\prime((\ell(m \alpha))^n) \cdot \ell(m \alpha + n_2))
\]
\[
= \phi^\prime((\ell(m \alpha + n_1))^n \ell(m \alpha + n_2)).
\]
Thus, condition ii) is satisfied by \( \overline{\ell}_0 \).

Let \( A = \{ \ell : \ell \in \text{Arc}(I, R_0, 1), \ell(0) = \overline{\ell} \} \). Let \( k \) be the function from \( A \) to \( \text{Arc}(\ell^-, T^-[Q], 1) \) such that for
every $\ell$ in $A$, $k(\ell)$ satisfies conditions i) - iii) of Subclaim 2.

Suppose $\ell^{(1)}$ and $\ell^{(2)}$ are elements in $\text{Arc}(\bar{I}, R_0, 1)$, $R_1$ is an element in $\text{BN}(\bar{I}) \cap \Phi(\bar{I}, W, 2)$, $a_1$ and $a_2$ are real numbers satisfying condition ii) of Subsubclaim 1 for $\ell^{(1)}$ and $\ell^{(2)}$ respectively, and $t_1$ and $t_2$ are real numbers with $0 < t_1 < t_2 < 1$ and $|t_1 - t_2| < \min(a_1, a_2)$. Suppose $\theta$ is an $(\bar{I}, t_1, t_2, R_1)$-deformation of $\ell^{(1)}$ into $\ell^{(2)}$ in $R_0$. Define the function $\theta_0$ as follows:

$$\theta_0(t, s) = \begin{cases} (k(\ell^{(1)}))(t) ; & s \in [0, 1], \ t \in [0, 1] - (t_1, t_2) \\ (k(\ell^{(1)}))(t_1) \cdot \left(1 - \frac{t_1}{t_2} \right) \cdot \frac{t_2 - s}{t_2 - t_1} \cdot \theta(t, s) ; & s \in [0, 1], \ t_1 < t < t_2 \end{cases}$$

Then $\theta_0$ is an $(\bar{I}^{-}, t_1, t_2, T^{-}[Q])$-deformation of $k(\ell^{(1)})$ into $k(\ell^{(2)})$ in $T^{-}[Q]$. Consequently $(k(\ell^{(1)}))(1) = (k(\ell^{(2)}))(1)$.

Now suppose $\ell^{(1)}$ and $\ell^{(2)}$ are elements in $\text{Arc}(\bar{I}, R_0, 1)$, and $\theta$ is any deformation of $\ell^{(1)}$ into $\ell^{(2)}$ in $R_0$. Suppose $a_1$ and $a_2$ satisfy condition ii) of Subclaim 1 for $\ell^{(1)}$ and $\ell^{(2)}$ respectively, and $R_1$ is an element of $\text{BN}(\bar{I}) \cap \Phi(\bar{I}, W, 2)$. Then there is a sequence $\ell_1, \ldots, \ell_n$ of elements in $\text{Arc}(\bar{I}, R_0, 1)$, a sequence $\left(q_1, r_1\right), \ldots, \left(q_n, r_n\right)$ of pairs of real numbers, and a sequence $\theta_1, \ldots, \theta_n$ of maps satisfying the
following:

1) for $i$ in $\mathbb{N}_1^n$, $0 < q_1 < r_1 < 1$ and $|q_1 - r_1| < \min(a_1, a_2)$;

ii) for $i$ in $\mathbb{N}_1^{n-1}$, $\theta_i$ is an $(L, q_1, r_1, R_1)$-deformation of $\ell_1$ into $\ell_{i+1}$ in $R_0$;

iii) $\ell_1 = \ell^{(1)}$ and $\ell_n = \ell^{(2)}$.

Consequently $(k(\ell^{(1)}))(1) = (k(\ell^{(2)}))(1)$. 

So we have a function
\[ \psi = \{(x, (k(\ell))(1)) : x \in \mathbb{R}^2, \ell \in \text{Arc}(L, R, 1), \ell(0) = \bar{e}, \ell(1) = x\}. \]

**Subclaim 3.** \( \psi^*|W = \psi^*|W. \)

Suppose \( x \in W. \) There is an \( \ell \) in \( \text{Arc}(L, W, 1) \) such that \( \ell(0) = \bar{e} \) and \( \ell(1) = x. \) Let \( \ell_0 = \bar{\psi} \circ \ell. \) Since \( \ell_0 \) satisfies conditions i) - iii) of Subclaim 1, 
\( \ell_0 = k(\ell), \) hence \( \psi(x) = \ell_0(1) = \bar{\psi}(\ell(1)) = \bar{\psi}(x). \]

This establishes the fact that \( \psi^*|W \) is \( (W \# \tau \tau \tau) \)-continuous.

**Subclaim 4.** \( (a)(b)(a, b \in R \Rightarrow \psi^-(a \tau b) = \psi^-(a) \cdot \psi^-(b)). \)

Suppose \( a, b \in R. \) There are members \( \ell_1, \ell_2 \) of \( \text{Arc}(L, R, 1) \) such that \( \ell_1(0) = \ell_2(0) = \bar{e} \) and \( \ell_1(1) = a, \ell_2(1) = b. \)

Let \( m = \left[ t : \left[ 0, \frac{1}{2} \right] : \ell_1(2t) \right] \cup \left[ t : \left[ \frac{1}{2}, 1 \right] : a^2\ell_2(2t - 1) \right] \).

Then \( m(0) = \bar{e}, m(1) = g(1) = a^2\ell_2(1) = a^2b. \)

It can be checked that 
\[ k(m) = \left[ t : \left[ 0, \frac{1}{2} \right] : (k(\ell_1))(2t) \right] \cup \left[ t : \left[ \frac{1}{2}, 1 \right] : (k(\ell_1))(1) \cdot (k(\ell_2))(2t - 1) \right]. \] Thus \( \psi^-(a^2b) = (k(m))(1) \)
\[ = \psi^-(a) \cdot \psi^-(b). \]

These last two subclaims establish the fact that 
\( \psi^*|R \) is \( (R \# \tau \tau \tau) \)-continuous.

Finally, by condition iii) of Subclaim 2, we have for every \( i \) in \( \mathbb{N} \) and every \( t \) in \( (-\delta, \delta), \) 
\[ \psi^-(\bar{\varepsilon}_1(t)) = T^\tau(\varepsilon_1(t)). \]
Claim 3. There is a function $\xi$ uniquely determined by the following conditions:

1) $\xi: R^2 \rightarrow U$;

2) $\xi|_R$ is $(R \neq \tau)$-continuous;

3) for elements $x, y \in R$, $\xi(x \cdot y) = \xi(x) \cdot \xi(y)$;

4) for $i \in \mathbb{N}$ and $t \in (-\delta, \delta)$, $\xi(\tilde{\xi}_i(t)) = g_i(t)$.

Further, the $\xi$ uniquely determined by these conditions satisfies:

5) $\xi: R^2 \rightarrow Q$

6) $T_1 \circ \xi = \psi|_{R^2}$.

Let $Z$ be the set of $\tau$-compact subgroups $Z$ of $Z_0$ such that $L/(Z, U)$ is a local Lie group. For each $Z$, an element of $Z$, let $L_{Z} = L/(Z, U)$, $T_{Z} = T(L, Z, U)$, and let $\psi_Z$ be the function uniquely determined by conditions 1) - 4) of Claim 2.

Let $\xi_o = \left\{ x: R^2: \bigcap_{Z \in Z} (T_{Z})^* \left[ \{ \psi_{Z}(x) \} \right] \right\}$.

Since $\bigcap_{Z \in Z} Z = \{ e \}$, $\xi_o(x)$ is a singleton for every $x$ in $R^2$. Let $\xi$ be the function from $R^2$ into $U$ such that for every $x$ in $R^2$,

$\{\xi(x)\} = \xi_o(x)$

There is a member $R_1$ of $BN(L) \cap \phi(L, R, \delta)$. If $K = \{ x: x \in R^2, \xi(x) = e \}$, then $(K, R_1)$ is an element of $\Delta L$, and $L^* = L/(K, R_1)$ is a local Lie group.

Let $T^* = T(L, K, R_1)$ and $R^*_1 = T^*[R_1]$.

Let $\xi^* = \{(T^*(x), \xi(x)) : x \in R^2_1\}$.

Then $\xi^*$ is a $\tau^*$-$\tau$-continuous one to one function which is a homomorphism on $R^*_1$. Further, there is a positive number $\delta$
such that for \( i \in \mathbb{N}_1^n \) and \( t \in (-\delta, \delta) \), \( \bar{e}_1(t) \) is a member of \( R_1 \) and \( T_1(\xi^*(T^*(\bar{e}_1(t)))) = e_1(t) \). Thus, since \( T_1[\xi^*[R_1]] \) is a sublocal group of \( L_1 \), \( e_1 \) is an element of the \( t_1 \)-interior of \( T_1[\text{range}(\xi^*)] \). ▲
Theorem 6. If $L$ is a locally compact local group, $V_0$ is special in $BN(L)$, $U$ is an element in $BN(L) \cap \Phi(L, V_0, 36)$, $B$ is a $\tau$-compact element of $\Gamma L$, $(B, U)$ is an element of $\Delta L$, $B \cdot U = U \cdot B = U$, $N = \{x : x \in U, (b)(b \in B \Rightarrow x \cdot b = b \cdot x)\}$, and if every member of $\Gamma L$ which is a subset of $U$ is a subgroup of $B$, then there is a $\tau$-compact subgroup $A$ of $B$ and a sublocal group former $(M, W)$ of $L \cdot N$ such that $L \cdot M$ is a local Lie group and $L$ is locally isomorphic to $(L \cdot A) \times (L \cdot M)$.

Proof. Let $N^* = L \cdot N$. There is an element $F_1$ in $BN(L) \cap \Phi(L, U, 4)$. Let $F = (F_1 \cdot B)$. Then $B \subseteq F$, $F$ is an element of $BN(L) \cap \Phi(L, U, 4)$ and $F \cap N$ is an element of $BN(N^*) \cap \Phi(N^*, 2)$.

Then $(F \cap N)^2 \subseteq (F \cdot N) \subseteq BN(N^*)$. There is an element $V_1$ in $BN(L) \cap \Phi(L, F, 18)$. Let $Z_0 = B \cap N$ and $V = Z_0 \cdot [V_1 \cap N]$. Then $V$ is an element of $BN(N^*) \cap \Phi(N^*, 18)$. $V^{18} = Z_0 \cdot [V_1 \cap N]$.

$Z_0 \cdot [V^{18} \cap N] \subseteq Z_0 \cdot [F \cap N]$, $Z_0 \cdot V = V \cdot Z_0 = V$, and $Z_0$ is an $(N \neq \tau)$-compact member of $\Gamma N^*$. Further, $(x)(z)(x \in V, z \in Z_0 \Rightarrow x \cdot z = z \cdot x)$. So $(Z_0, V) \in \Delta N^*$. Also if $H$ is any member of $\Gamma N^*$ and a subset of $V$, then $H$ is a member of $\Gamma L$ and a subset of $U$. Hence $H \subseteq B$, and thus $H \subseteq N \cap B = Z_0$. Consequently $L_0 = N^*/(Z_0, V)$ is a local Lie group.

Then by Theorem 5 there is a Lie group $L^*$ and a locally one to one continuous local homomorphism $\xi^*$ from $L^*$ into $N^*$ such that image $(\xi^*) \subseteq V$ and for any neighborhood $M^*$ of $e^*$, $Z_0 \cdot [\xi^* \cdot [M^*]]$ is an $(N \neq \tau)$-neighborhood of $e$.

There is an element $R$ in $BN(L^*) \cap \Phi(L^*, Dom(\xi^*), 2)$ such that
\( \xi^*|R^2 \) is \((R^2 \# \tau^*)-(N \# \tau)\)-continuous and one to one, and is such that for all elements \( x \) and \( y \) in \( R \), \( \xi^*(x \cdot y) = \xi^*(x) \cdot \xi^*(y) \).

There is an element \( M^* \) in \( BN(L^*) \cap \varphi(L^*, R, 3) \) such that the \( \tau^*\)-closure of \((M^*)^2\) is \( \tau^*\)-compact, and such that the only element of \( M^* \) which is a subset of \( M^* \) is \{\( e^* \}\}. Then the only element of \( M^* \) which is a subset of \( \xi^*[M^*] \) is \{\( e \}\).

Let \( M = \xi^*[M^*] \). Then \( e \) is in the \((N \# \tau)\)-interior of \( Z_O \cdot M \), so by Corollary to Theorem 4 and Proposition F

\[ e \text{ is in the } \tau \text{-interior of } B^*(Z_O \cdot M) = B \cdot M. \]

**Claim 1.** \( (0)(0 < M^*, 0 \in BN(L^*) \Rightarrow (\exists W)(W \in BN(L), W \subseteq U, W \cap M^2 < \xi^*[0]) \).\]

Suppose \( 0 \) is a member of \( BN(L^*) \) and \( 0 \subseteq M^* \). Then \( C^* = (\tau^*\)-closure \((M^*)^2 \sim 0) \) is a \( \tau^*\)-compact subset of \( R \), so \( C = \xi^*[C^*] \) is \((N \# \tau)\)-compact, and hence \( C \) is \( \tau \)-compact.

Since \( e \) is not an element of \( C \) there is a \( W \) in \( BN(L) \) such that \( W \subseteq U \) and \( C \cap W = \emptyset \). Since \( (M^2 \sim \xi^*[0]) \subseteq C \),

\[ W \cap M^2 < \xi^*[0]. \]

Suppose \( M_o^* \in BN(L^*) \cap \varphi(L^*, M^*, 2) \).

\[ \text{Claim 2. } M_o^* = \xi^*[M_o^*] \text{ is an element of } M \# \tau \text{ and } \xi^*[M_o^*] \text{ is an } (M_o^* \# \tau^*)-(M \# \tau)-open \text{ map.} \]

Suppose \( 0^* \subseteq M_o^* \) and \( 0^* \) is an element of \( \tau^* \). Then if \( x \) is an element of \( 0 = \xi^*[O^*] \), then there is an element \( x^* \) in \( O^* \) and element \( \Omega^* \) of \( BN(L^*) \) such that \( x^* \cdot \Omega^* \subseteq O^* \) and \( \xi^*(x^*) = x \). By Claim 1 there is an element \( W \) in \( BN(L) \) such that \( W \subseteq U \)
and \( W \cap M^2 = \xi^* [\eta^*] \). Then \((x \cdot W) \cap M \subset \emptyset \).

Thus \( \xi^* |_{M_0^*} \) is an \((M_0^* \# \tau^*)-(M_0 \# \tau)\)-homeomorphism. Further, by Claim 1 there is an element \( W_1 \) in \( BN(L) \) such that
\( W_1 \cap M^2 \subset \xi^* [M_0^*] \), and there is an element \( W \) in \( BN(L) \cap \phi(L, U \cap W_1, 2) \). Then \((W \cap M)^2 \subset M \) and so \((M, W)\) is a sublocal group former in \( N^* \). Since \( W \sim M \) is an element of \( \tau \), \( L \cdot M \) is a locally compact local group which is locally isomorphic to \( L^* \). Let \( M^* = L \cdot M \).

Since \( W \subset V_0 \) there is a subset \( A_1 \) of \( W \) which is a \( \tau \)-compact element of \( L \) and is such that \((A_1, U_6)\) is an element of \( A L \), and \( L_1 = L/(A_1, U_6) \) is a local Lie group. Since \( A_1 \subset U \), \( A_1 \subset B \). Further, if \( x, y \) are members of \( A_1 \cap M \), then \( x, y \) are in \( W \cap M \) and hence \( x \cdot y \) is in \( A_1 \cap M \). This shows that \( A_1 \cap M \) is a member of \( A L \). So \( A_1 \cap M = \{e\} \).

Let \( T_1 = T(L, A_1, U_6) \), \( B_1 = T_1 \cdot B \), \( U_1 = T_1 \cdot U \), \( N_1 = T_1 \cdot N \), \( M_1 = T_1 \cdot M \). Since \( B_1 \) is a \( \tau_1 \)-compact element of \( A L \), and \( L_1 \) is a local Lie group, \( L_1 \cdot B_1 \) is a Lie group. Its identity component \( B_0 \) is an element of \( B_1 \# \tau_1 \).

Claim 3. \((Q)(Q \in BN(M^*) \Rightarrow e \in \tau \text{-interior } (Q \cdot T_1^+ [B_0]))\).

Suppose \( Q \in BN(M^*) \). Since \( e \in \tau \text{-interior } (Q \cdot B) \), \( e_1 \in \tau_1 \text{-interior } (T_1 \cdot Q \cdot B_1) \). There is an element \( S \) in \( BN(L_1) \cap \phi(L_1, T_1 \cdot Q \cdot B_1, 2) \) such that \((S^2 \cap B_1) \subset B_0 \).

There is an element \( 0^+ \) in \( BN(L) \) such that \( 0^+ \subset U \), \( 0 = (0^+ \cap M) \subset Q \), and \( T_1 \cdot 0^+ \subset S \). Then \( e \) is an element of the \( \tau \text{-interior of } 0 \cdot B \), so \( e_1 \) is an element of the \( \tau_1 \text{-interior of } T_1 \cdot 0 \cdot B_1 \).
Suppose \( s \in S \cap \tau_1\text{-interior} (T_1 \cdot B_1) \). Then 
\[(\exists x)(\exists y)(x \in T_1 \cdot B_1, y \in B_1, x \cdot y = s). \]
Then since \( x \) is an element of \( S \), \( y = x^{-1} \cdot s \) is an element of \( (S^2 \cap B_1) \subset B_0 \).
Thus \( (T_1 \cdot B_0) \supset T_1 \cdot B_1 \supset S \cap \tau_1\text{-interior} (T_1 \cdot B_1) \).
Hence \( e \) is in the \( \tau \)-interior of 
\[T_1^+ [B_0] \cdot Q \cdot A_1 = T_1^+ [B_0] \cdot Q.\]

Every ascending chain of closed connected normal subgroups of \( B_0 \) is finite. \( [ \text{If } D_1, D_2 \text{ are closed connected normal subgroups of } B_0 \text{ with } D_1 \not\subset D_2 \text{ then } D_1 \text{ and } D_2 \text{ determine subvector spaces } H_1, H_2 \text{ of the Lie algebra of } B_0 \text{ with } \dim(D_1) < \dim(D_2) ] \).
Hence there is an \( A_0 \), a closed connected normal subgroup of \( B_0 \), such that there is an element \( P \) in \( BN(\mathcal{M}^-) \) with 
\[T_1^+ [P] \cap A_0 = \{e\} \] and such that \( A_0 \) is maximal with these properties. Let \( A = T_1^+ [A_0] \). Then \( A \) is a \( \tau \)-compact normal subgroup of \( T_1^+ [B_0] \).

There is an element \( X \) in \( BN(L) \) such that \( X^2 \subset (T_1^+ [B_0] \cdot M_0) \cap U \).
Let \( X_1 = T_1 [X] \). Then \( (A_0, X_1) \) is an element of \( \Delta L_1 \), and consequently \( (A_0, B_0 \cdot X_1) \) is an element of \( \Delta L_1 \).
Let \( U_2 = U \cap T_1^+ [B_0 \cdot X_1] \). Then \( T_1^+ [B_0] \subset U_2 \). If \( x \) is an element of \( U_2^2 \), then \( x = x_1 \cdot x_2 \) with \( x_1, x_2 \) elements of \( U_2 \) and since \( T_1 \) is a homomorphism on \( U^6 \),
\[T_1 [x \cdot A \cdot x] = (T_1(x))^2 \cdot [A_0] \cdot T_1(x) \subset A_0.\] This shows that \( (A, U_2) \) is an element of \( \Delta L \).

Let \( L_2 = L/(A, U_2) \), \( T_2 = T(L, A, U_2) \), \( \phi \) be the natural projection map from \( L_1 \) to \( L_2 \) [i.e. \( \phi = \{(T_1(x), T_2(x)) : x \in U_2^2\} \)],
and let $B_2 = \phi[B_0]$.  

Suppose $P_0$ is an element in $BN(M^*)$ with $T_1[P_0] \cap A_0 = \{e_1\}$. Let $P_2 = T_2[P_0]$.

Claim 4. Every non-trivial closed connected normal subgroup of $B_2$ intersects $P_2$ non-discretely.

Suppose $C_2$ is a non-trivial closed connected normal subgroup of $B_2$ intersecting $P_2$ discretely. Then there is an element $Q$ in $BN(M^*)$ with $Q \subset P_0$ such that $T_2[Q] \cap C_2 = \{e_2\}$. Let $C = T_2^+[C_2]$. Then $C$ is a $\tau$-compact subgroup of $T_1^+[B_0]$.

So $C_1 = T_1[C]$ is a closed normal subgroup of $B_0$ properly containing $A_0$. Further, $Q \cap C \subset Q \cap A \subset P_0 \cap A \subset A_1$. So $C_1 \cap T_1[Q] = \{e_1\}$. We will show that $C_1$ is $\tau_1$-connected.

If $D_1$ and $D_2$ are disjoint non-null ($C_1 \neq \tau_1$)-closed subsets of $C_1$ whose union is $C_1$, then $\phi[D_1]$ and $\phi[D_2]$ are ($C \neq \tau_2$)-closed subsets of $C$ whose union is $C$. Since $C$ is $\tau$-connected, there exists an $x$ in $\phi[D_1] \cap \phi[D_2]$. So

$$\exists x_1)(\exists x_2)(x_1 \in D_1, x_2 \in D_2, x = \phi(x_1) = \phi(x_2)).$$

So

$$\exists a_1)(\exists a_2)(a_1, a_2 \in U_2, T_1(a_1) = x_1, T_1(a_2) = x_2, a_1 \cdot a_2^{-1} \in A).$$

Thus $x_1 \cdot x_2^{-1} \in A_0$. But $A_0 \cdot x_2$ is $\tau_1$-connected, so $A_0 \cdot x_2 \subset D_2$. But $x_1$ is an element of $A_0 \cdot x_2$ which implies that $x_1$ is an element of $D_2$.

Now $P_2$ and $B_2$ are elementwise permutable so $P_2 \cap B_2 \subset \text{center}(B_2)$.

Claim 5. $B_2$ contains no non-trivial closed connected normal subgroups with discrete center.

Suppose $H$ is a closed non-trivial connected normal subgroup of
Claim 6. Connected semi-simple Lie groups have discrete centers.

Suppose $G$ is a semi-simple Lie group with Lie algebra $H$. Let $K$ be the ideal in $H$ corresponding to the identity component of the center of $G$. Since $K$ is an abelian ideal and $H$ is semi-simple, $K = \{0\}$. So the identity component of the center of $G$ is a singleton, and hence the center is discrete.

These two claims show that $B_2$ contains no non-trivial connected semi-simple normal subgroups which implies that $B_2$ is commutative [4, theorem 1.3, p. 144].

Claim 7. $e_2 \in \tau_2$-interior $(P_2)$.

Since $B_2$ is a $(B_2 \neq \tau_2)$-compact commutative Lie group, there is an $n$ in $\mathbb{N}$ such that $B_2 = \bigvee_{i=1}^{n} C_1$ with $C_1$ isomorphic to the circle group for each $i$ in $\mathbb{N}$. For each such $i$, $C_1$ is a non-trivial closed connected normal subgroup of $B_2$, and so by claim 4 there is an $0_1$, an element of $BN(L_2)$, such that $0_1 \cap C_1 \subset P_2$. Hence there is an element $Q$ in $BN(L_2)$ such that $Q \cap B_2 \subset P_2$ and $(Q \cap P_2)^2 \subset P_2$. There is an element $D$ in $BN(L_2) \cap \Phi \langle L_2, Q, 2 \rangle$, and there is an element $E$ in $BN(M^+) \cap \Phi \langle M^+, U_2 \cap P_0, 1 \rangle$ such that $E_2 = T_2 \{ E \} \subset D$. Since $e$ is an element of the $\tau$-interior of $E \cdot T_1^+[B_0]$ [Claim 3], $e_2$
is an element of the $\tau_2$-interior of $E_2 \cdot B_2$ \([E \text{ and } T_1 \cdot [B_0]\) are subsets of $U_2\]$. Then there is an element $Y$ in $BN(L_2)$ such that $Y \subset D \cap (E_2 \cdot B_2)$.

**Subclaim 1.** $Y \subset P_2$.

Suppose $y \in Y$. Then $\exists x)(\exists b)(x \in E_2, b \in B_2, y = x \cdot b)$.

Then $b = (x^2 \cdot y) \in (E_2 \cdot Y) \cap B_2 \subset D_2 \cap B_2 \subset \mathcal{Q} \cap B_2 \subset P_2$.

So $y \in E_2 \cdot [P_2 \cap \mathcal{Q}] \subset [D \cap P_2] \cdot [P_2 \cap \mathcal{Q}] \subset (P_2 \cap \mathcal{Q})^2 \subset P_2$.

The claims 4-7 show that if $P_0$ is any element in $BN(M)$ with $T_1 \cdot [P_0] \cap A_0 = \{e_1\}$, then $e_2$ is in the $\tau_2$-interior of $T_2 \cdot [P_0]$, and consequently $e$ is in the $\tau$-interior of $A \cdot A'$. Further, since $P_0 \cap A \subset A_1$, we have $A \cap P_0 = A \cap P_0 \cap M \subset A_1 \cap M = \{e\}$.

Thus there is an element $P_3$ in $BN(M)$ such that $A \cap P_3^2 = \{e\}$, and such that if $Q_1$ and $Q_2$ are any elements in $BN(L+P_3)$ and $BN(L+A)$ respectively, then $e$ is an element of the $\tau$-interior of $Q_1 \cdot Q_2$ [Corollary to Prop. F]. There is an element $0$ in $BN(L)$ such that $0^2 \subset A \cdot P_3$.

Let $G = (L+A) \times (L^*)$ and let $0_1 = \{(a, p) : a \in A, p \in P_3, a \cdot p \in 0\}$, and define $\psi = [(x, y) : 0^2 \psi : x \cdot y]$. Let $U = (A \neq \tau) \psi \tau^*$.

Then $0_1$ is an element of $U$, and $\psi$ is an $(0_1 \neq U)\tau$-continuous open map which is one to one and is such that for all elements $(a_1, b_1)$, $(a_2, b_2)$ of $0_1$, $\psi((a_1, b_1) \cdot G(a_2, b_2)) = \psi((a_1, b_1)) \cdot \psi((a_2, b_2))$.

**Corollary A.** If $L$ is a locally compact local group, there is a $\tau$-compact element $A$ of $\Gamma L$ and a sublocal group $M$ of $L$ such that $L+M$ is a local Lie group and $L$ is locally isomorphic to $(L+M)^{\times}_{1G} (L+A)$.
Proof. A

**Corollary B.** Every locally compact local group is locally isomorphic to a global group.

Proof. A
BIBLIOGRAPHY


