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Global Analysis of the Neutron Magnetic Form Factor

Joseph P. Jayne
University of New Hampshire, Durham

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Joseph Jayne

Advisor: Nathaly Santiesteban

University of New Hampshire

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TABLE OF CONTENTS

ABSTRACT

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by

Joseph Jayne

University of New Hampshire, May, 2024

Protons and neutrons, the nucleons, are made up of smaller particles called quarks. Nucleons have three valence quarks, which are the dominant contributors to its properties, as well as sea quarks (quark-antiquark pairs) and gluons. The distribution of the charge and magnetization of these particles is described by the electric and magnetic form factors. These form factors can be measured with scattering experiments since they directly affect the particle's cross section. Neutron form factors are challenging to measure because a pure neutron target will decay very fast and is not viable for experiments. Instead, targets like 2H and 3H are used. However, the proton in these targets introduces large uncertainties. New data for the neutron magnetic form factor has been found using a method that uses mirror nuclei 3H and 3He and relies on canceling uncertainties in the ratio of cross sections of both nuclei. With this new data, we create a new global fit of the neutron magnetic form factor that incorporates both new and old data. This new global fit can support new calculations and theories about neutrons.

CHAPTER 1

Background

1.1 Introduction to Form Factors

Protons and neutrons are made up of smaller particles called quarks. Inside of each proton and neutron are three valence quarks which are the dominant contributors to the properties of the nucleon. In addition to these valence quarks, nucleons also contain sea quarks which are virtual quark-antiquark pairs, and gluons. All of these particles contribute to the electric and magnetic distribution in the nucleon. These spatial distributions are described by the electric and magnetic form factors.

These form factors can be measured using electron-nucleon scattering experiments. When two particles collide, their interactions are described by their cross sections. Classically, these can be thought of as the size of the particle's bisection. The total cross section of a collision can be found with

$$\sigma_{tot} = \frac{\text{number of reactions per unit time}}{\text{number of beam particles per unit time} \cdot \text{number of target particles per unit area}} \tag{1.1}$$

This cross section assumes that every interaction is measured, where in a real experiment only a fraction of interactions are measured. When particle accelerators do a scattering experiment, detectors are set up around the targets so that certain reactions can be measured. It is impossible for the detectors to completely encase a target, so instead a detector placed at a distance r and angle θ from the target are represented using

$$\Delta\Omega = \frac{A_d}{r^2} \tag{1.2}$$

where $\Delta\Omega$ is the solid angle and A_d is the area of the detector. The cross section can then be represented as a derivative of this solid angle

$$\frac{\mathrm{d}\sigma(E,\theta)}{\mathrm{d}\Omega} \tag{1.3}$$

which I will now refer to as the differential cross section. [8] The form factor of a nucleon directly affects the differential cross section of that nucleon as seen in [8].

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{exp.} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{*}_{Mott} \cdot \left|F(q^2)\right|^2 \tag{1.4}$$

The Mott cross section describes electron scattering and the asterisk appended means that this form of the Mott cross section ignores recoil on the target nucleus. The function $F(q^2)$ is the form factor, and q^2 is the momentum transfer between the beam and target. The motivation behind the creation of the form factor comes from a discrepancy between experimentally measured cross sections and the Mott cross section. At high q^2, the beam particle will collide with smaller and smaller elements of the target. This is due to the wavelength of the virtual photon exchanged becoming smaller with a higher q^2 [8]. This causes the cross section to decrease at high q^2, and the form factor accounts for this.

Specifically for nucleons, this entire $|F(q^2)|^2$ term instead becomes

$$\left[\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau}+2\tau G_M^2(Q^2)\tan^2\left(\frac{\theta}{2}\right)\right] \tag{1.5}$$

where G_E and G_M represent the respective electric and magnetic form factor of the nucleon, $Q^2=-q^2$ to avoid negative quantities, θ is still the angle that the electron scatters off of the target, and τ can be found with

$$\tau=\frac{Q^2}{4M^2c^2} \tag{1.6}$$

where M is the mass of the particle and c is the speed of light, although most calculations use units where $c=1$.

And with this, the equation for the cross section of a nucleon becomes

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)=\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{Mott}\cdot\left[\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau}+2\tau G_M^2(Q^2)\tan^2\left(\frac{\theta}{2}\right)\right] \tag{1.7}$$

Note that along the way, the recoil on the target is accounted for and so the asterisk on the Mott cross section is removed. [8]

1.2 Measurements of the Neutron Magnetic Form Factor

To make measurements of a form factor, particle accelerators accelerate a beam to high energy to collide it with some target. Electrons are often used as beam particles in these experiments. This is because electrons are easy to generate, are charged, and are lightweight and small point-like particles (meaning they have no internal structure). These properties make electrons easy to manipulate in experiments and useful as beam particles since they can probe deep into a target.

As discussed previously, the interaction between the beam and the target is described by the cross section. For a neutron, which is the focus of theis work, this cross section can be described with equation (1.7). Because the Mott cross section can be calculated, the ratio

of the cross sections can be used to find the form factors.

To measure the neutron form factors, the target of the scattering experiment must be neutrons. However, neutrons have a lifetime of roughly 15 minutes on their own. Therefore we need to use isotopes of hydrogen like deuterium (^{2}H) and tritium (^{3}H). The proton in these atoms still interact with the electron beam, which introduces uncertainties into the measurements. Many measurements have been made using similar targets in the past. However, new data has been acquired with a novel method using mirror nuclei ^{3}H and ^{3}He [9]. Quasi-elastic data of ^{3}H and ^{3}He was taken, and assuming that both the proton form factors as well as the neutron electric form factor are known, the neutron magnetic form factor can be extracted. Many systematic uncertainties in this data cancel in the ratio of the cross sections of the proton and neutron.

CHAPTER 2

Methods

With the introduction of new data that has lower uncertainties, a greater understanding of the neutron magnetic form factor can be gained by creating a new global fit. A global fit incorporates multiple sets of data from different experiments and takes into account normalization uncertainties. As I will explain later, each data set has a normalization coefficient to account for differences between experiments.

2.1 Z-Transform

Before creating the fit, a transformation is applied to the data being used for the fit. This transformation, called the z-transform, allows us to apply constraints on the fit we create. The z-transform bounds all Q^2 values to be between 0 and 1, which is easily visible in figure 2.1. z is found from Q^2 with

$$z(t, t_{cut}, t_0) = \frac{\sqrt{t_{cut} - t} - \sqrt{t_{cut} - t_0}}{\sqrt{t_{cut} - t} + \sqrt{t_{cut} - t_0}} \tag{2.1}$$

where

$$t = -Q^2 \ , \ t_{cut} = 4m_\pi^2 \ , \ t_0 = 0 \tag{2.2}$$

where m_π is the mass of a pion. Note that t_0 is chosen to be 0 here, though it can also be found with

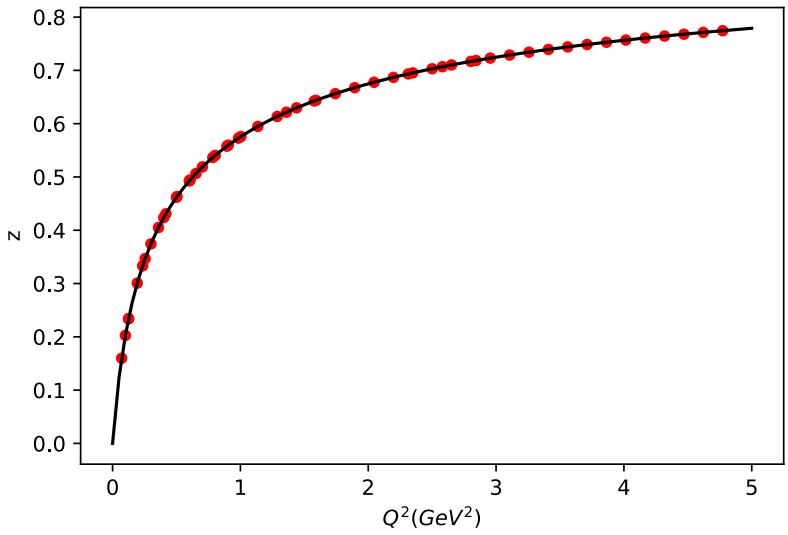

Figure 2.1: Graph of the z-transform. The black line is the functional form of the transform while the red dots are the transforms of the data points later used in the fit. Notice how the the lower Q^2 values are grouped closer together while the larger Q^2 values are spread apart.

$$t_0^{opt} = t_{cut}\left(1 - \sqrt{1 + \frac{Q^2_{max}}{t_{cut}}}\right) \tag{2.3}$$

however it is not used in this work. [4]

One of the strongest motivations to use the z-transform is that in many fits, there is a trade off between using too many parameters or too few parameters. A fit with a large number of parameters will match the data too well and it will become difficult to extrapolate. In this case, the fit will be accurate to the data, but will not be suitable for predictions. A fit with a small number of parameters will struggle to match the data and may not capture important aspects of the data. In this case, the fit will be inaccurate. The z-transform guarantees that a finite number of parameters can sufficiently describe the form factor to a given precision with a fit of the form [4]

$$G_M^n(q^2) = \sum_{k=0}^{\inf} a_k z(q^2)^k \tag{2.4}$$

Another motivation is that since all values of z are between 0 and 1, larger order parameters are less dominant in high q^2, which increases predictive power.

2.2 Fitting with the Least Squares Method

In creating the fit a routine called the least squares method was used. The least squares method is a routine that varies parameters in order to minimize a cost function. The elements of the cost function we seek to minimize are called residuals [7]. In our case, our goal is to fit a function of the form (2.4).

2.2.1 Residuals

Although the function we are fitting to is a simple power series, our residual function is not. Residuals typically are found with

$$\text{Residuals} = \left[\frac{F(p, x_i) - y_i}{\sigma_{y_i}}\right]^2 \text{ for each data point } i \tag{2.5}$$

where $F(p, x_i)$ is the function we are fitting to with parameters p and data points x_i, y_i is the data that we fit the function to, and σ_{y_i} is the uncertainty on each data point y_i. With a function like this, the intuition for how it behaves is clear. If $F(p, x_i)$ is very close to y_i, then the resulting residual is small, meaning the fit is good. If the uncertainty σ_{y_i} on the data point is large, the residual becomes smaller meaning it is weighted as less important than other data points.

Since we are making a global fit, we are including data points from many different sources. Due to the nature of the experiments and measurements there are some uncertainties in each data set as a whole, which we approximate using a new parameter in our residual function. This new parameter is a normalization coefficient, which multiplies each G_m^n value to shift each data set. Using these normalization coefficients, our residuals now look like this

$$\text{Residuals} = \left[\frac{F(p, x_i) - N_j y_i}{N_j \sigma_{y_i}}\right]^2 \text{ for each data point } i \text{ in each data set } j \tag{2.6}$$

where N_j is the normalization coefficient for data set j. However, in its current state, (2.6) allows the data sets to shift by any amount necessary to minimize the residuals. This could result in a fit that is overfitted and is less useful. To encourage the normalization coefficients to stay small, we introduce the normalization coefficient for each data set as a new data point. Our residuals then become

$$\text{Residuals} = \left[\frac{F(p, x_i) - N_j y_i}{N_j \sigma_{y_i}}\right]^2 \text{ for each data point } i \text{ in each data set } j \text{ ,} \quad \left[\frac{N_j - 1}{\sigma_{N_j}}\right]^2 \text{ for each data set } j \tag{2.7}$$

with σ_{N_j} being uncertainties on the data sets themselves. This way, the improvement in minimization from changing N_j must be more significant than $\frac{N_j - 1}{\sigma_{N_j}}$, and so normalization coefficients stay relatively near 1.

Equation (2.7) handles most of our residual points used for the fit, but not quite all. We place additional residuals of the form $\frac{p}{5}$ as data points to place soft bounds on the fit parameters. The 5 in these residuals are somewhat arbitrary; 5 was chosen as a value that we did not expect the parameters to need to reach in order to make a good fit. The motivation for placing these soft bounds on the parameters rather than a hard bound is so that covariance matrix for the fit knows that these parameters are bounded, which is further explained later. Using all of these residuals on data sets from references [9, 6, 1, 2, 3, 5] gets us the fit shown in figure 2.3.

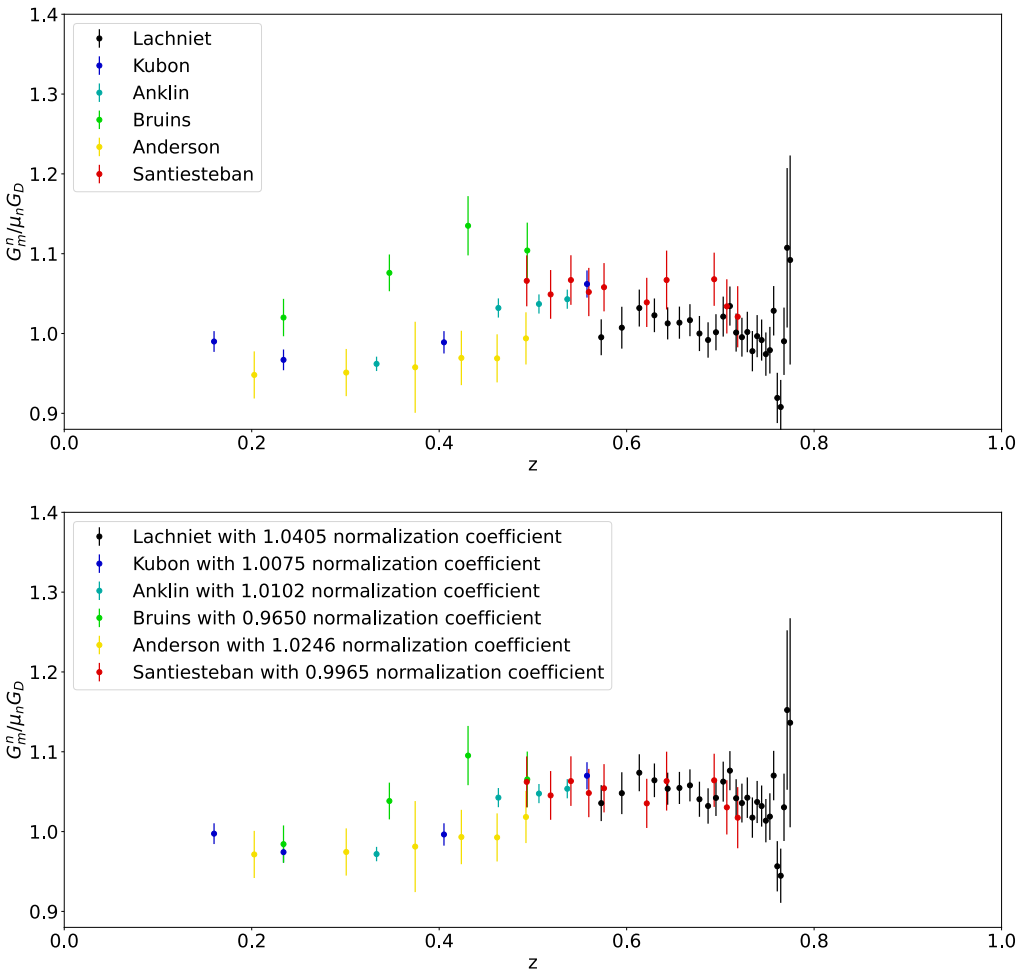

Figure 2.2: Showcase of the normalization coefficients. Each dataset has its applied normalization coefficient listed in the legend. Note that although the fit has been created in the process of finding the normalization coefficients, it is omitted here.

$$\text{Residuals} = \left[\frac{F(p,x_i) - N_j y_i}{N_j \sigma_{y_i}}\right]^2 \text{ for each data point } i \text{ in each data set } j \text{ ,}$$
$$\left[\frac{N_j - 1}{\sigma_{N_j}}\right]^2 \text{ for each data set } j,$$
$$\left[\frac{p_m}{5}\right]^2 \text{ for each fit parameter m} \quad (2.8)$$

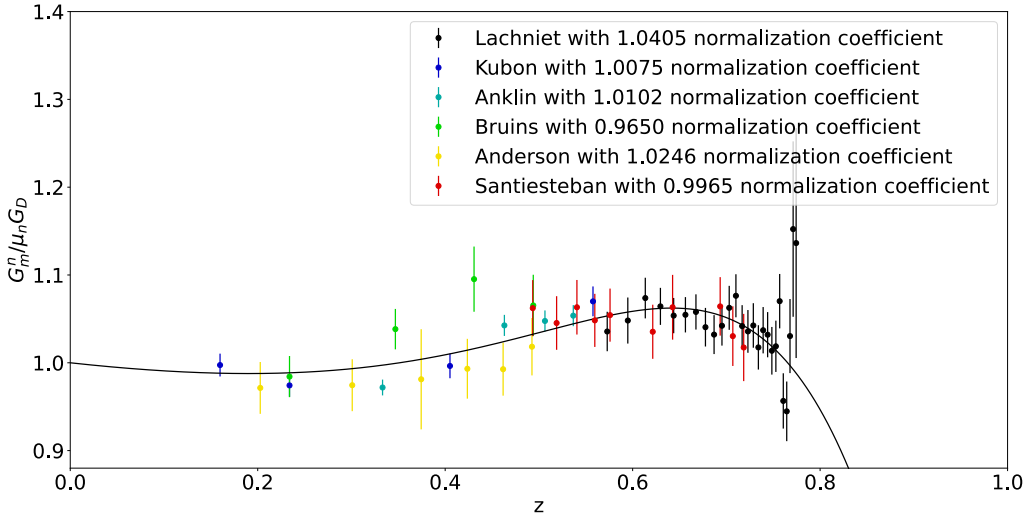

Figure 2.3: Fit found from Scipy least squares method using the residuals discussed in this section. The black line is the resulting fit.

2.2.2 Uncertainties on the Fit

In order to find the uncertainty on fit, a few things are needed. First is the jacobian matrix of the fit function, which in our case is (2.4). The jacobian for this is calculated by taking the partial derivative of the fit function with respect to each fit parameter, for each data point.

$$\text{Fit Function Jacobian} = \begin{pmatrix} \frac{\partial F(p,x)}{\partial p_1}\big|_{x_1} & \cdots & \frac{\partial F(p,x)}{\partial p_m}\big|_{x_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial F(p,x)}{\partial p_1}\big|_{x_n} & \cdots & \frac{\partial F(p,x)}{\partial p_m}\big|_{x_n} \end{pmatrix} \text{ for n data points and m fit parameters} \tag{2.9}$$

We also need the jacobian for the residuals function, which is considerably more complicated. This one is calculated by taking the partial derivative of the residuals function (2.8) with respect to each parameter (this time including the normalization parameters) at each data point (including the data points from normalization parameters and fit parameters). This creates

$$\text{Residual Function Jacobian} = \begin{pmatrix} \frac{\partial}{\partial p_1}\left(\frac{F(p,x)}{N_x\sigma_x}\right)|_{x_1} & \cdots & \frac{\partial}{\partial p_{m-l}}\left(\frac{F(p,x)}{N_x\sigma_x}\right)|_{x_1} & \frac{\partial}{\partial N_1}\left(\frac{F(p,x)-N_x y_x}{N_x\sigma_x}\right)|_{x_1} & \cdots & \frac{\partial}{\partial N_l}\left(\frac{F(p,x)-N_x y_1}{N_x\sigma_x}\right)|_{x_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial p_1}\left(\frac{F(p,x)}{N_x\sigma_x}\right)|_{x_i} & \cdots & \frac{\partial}{\partial p_{m-l}}\left(\frac{F(p,x)}{N_x\sigma_x}\right)|_{x_i} & \frac{\partial}{\partial N_1}\left(\frac{F(p,x)-N_x y_x}{N_x\sigma_x}\right)|_{x_i} & \cdots & \frac{\partial}{\partial N_l}\left(\frac{F(p,x)-N_x y_1}{N_x\sigma_x}\right)|_{x_i} \\ \frac{\partial}{\partial p_1}\left(\frac{p}{5}\right)|_{p_1} & \cdots & \frac{\partial}{\partial p_{m-l}}\left(\frac{p}{5}\right)|_{p_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial p_1}\left(\frac{p}{5}\right)|_{p_{m-l}} & \cdots & \frac{\partial}{\partial p_{m-l}}\left(\frac{p}{5}\right)|_{p_{m-l}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial}{\partial N_1}\left(\frac{N-1}{\sigma_N}\right)|_{N_1} & \cdots & \frac{\partial}{\partial N_l}\left(\frac{N-1}{\sigma_N}\right)|_{N_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{\partial}{\partial N_1}\left(\frac{N-1}{\sigma_N}\right)|_{N_l} & \cdots & \frac{\partial}{\partial N_l}\left(\frac{N-1}{\sigma_N}\right)|_{N_l} \end{pmatrix} \tag{2.10}$$

which is a $(m+l\ ,\ m+l+i)$ matrix, where m is the number of fit parameters, l is the number of data sets, and i is the total number of data points from the data set. The covariance matrix is defined using the residual function jacobian with

$$\text{Covariance Matrix} = [J_{Residual}^T J_{Residual}]^{-1} \tag{2.11}$$

The diagonals of the covariance matrix are the squared uncertainties of the fit parameters. The covariance can also be used to find the total fit uncertainty with

$$\text{Fit Uncertainty} = \sqrt{J_{Fit} M_{Cov} J_{Fit}^T} \tag{2.12}$$

where M_{Cov} is the covariance matrix and J_{Fit} is the jacobian of the fit function defined in (2.9). Additionally, the entries in the covariance matrix from normalization data points or fit parameter data points need to be removed for this calculation to work. The result is a 1 dimensional array with each entry corresponding to the uncertainty in the fit at each x.

CHAPTER 3

Results

3.1 The Global Fit

Using the techniques discussed in previous sections, I wrote some code in Python using SciPy's least squares implementation [10] to calculate the fit. The resulting fit has been shown already in 2.3, but the complete fit with uncertainties is shown in 3.1 and 3.2. This final fit used the fit function (2.4) with 12 parameters. A χ^2 test of the fit can easily be done by summing the squares of the residuals and dividing by the degrees of freedom, which for us is the number of data points minus the number of parameters, which is $NDF = 42$. This results in a $\frac{\chi^2}{NDF} = 1.437$, which indicates a strong fit. The 12 parameters determined during the fit are shown in table 3.1.

3.1.1 Conclusion

In the introduction I showed how the neutron magnetic form factor directly affects the cross section of the neutron. To intimately understand this form factor, and thus the cross section of the neutron, will illuminate how the neutron interacts with other particles. With a more accurate global fit for the neutron magnetic form factor, other calculations involving this form factor can be made to a higher accuracy as well.

Parameter	Value	Uncertainty
a_1	-0.1159	± 0.2330
a_2	0.2107	± 1.5231
a_3	0.2742	± 3.4677
a_4	0.2092	± 3.9191
a_5	0.0717	± 4.1658
a_6	-0.1161	± 4.2579
a_7	-0.3666	± 4.4082
a_8	-0.8367	± 4.504
a_9	-1.2350	± 4.5099
a_{10}	0.0502	± 4.4727
a_{11}	0.3345	± 4.4460
a_{12}	0.4989	± 4.4563

Table 3.1: Parameters found from a 12th order fit to equation (2.4)

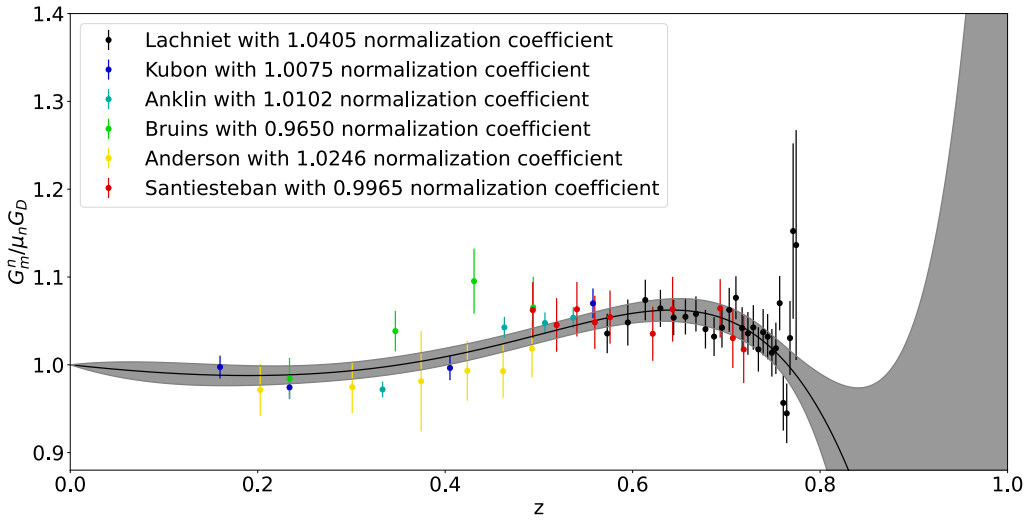

Figure 3.1: Final fit using SciPy's least squares implementation. The shaded region surrounding the fit is the uncertainty.

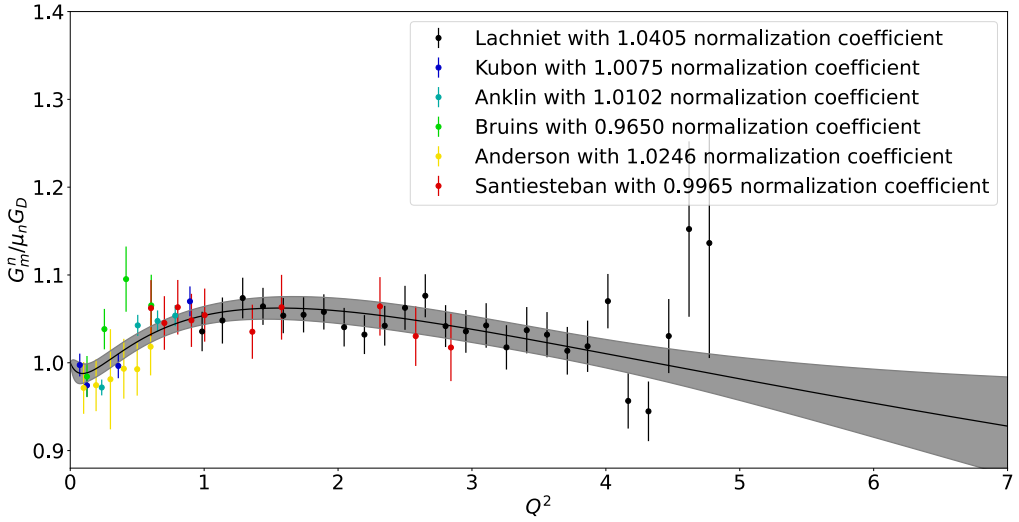

Figure 3.2: Same fit as 3.1, but shown in terms of Q^2 instead of z. As mentioned earlier, the fitting is done in terms of z, but the fit in terms of Q^2 is more useful.

3.1.2 Future Work

While the majority of the work for this project is complete, certain aspects are not yet finalized. For example, the optimal number of parameters of equation (2.4) has yet to be determined. The number of parameters used needs to be optimized to prepare this work for publication. Additionally, more datasets may be incorporated into the fit before it is finalized.

3.2 Acknowledgements

First and foremost I would like to thank Professor Nathaly Santiesteban for giving me the opportunity to work on this project. I would also like to thank John Arrington and Tyler Hague for their help and guidance with this project.

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