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Generally Covariant Theory of Multipole Moment Conserving Quasiparticles

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Abstract

This report represents the creation of a field theory which is capable of describing quasiparticle excitations that preserve 2^k-pole moments. These quasiparticles exhibit certain 'semi-dynamic' properties such as individual particle immobility but free movement of bound 2^ℓ-tuples. We provide a review of work done on dipole conserving fractons and their dynamics [1] and expand upon it to describe higher moment conserving systems with global quadratic (and higher) phase symmetry. This requires the selection of the temporal and spatial directions. The selection of a temporal direction is done with a foliation defined by an anisotropic scaling of space and time, defining a hypersurface of constant time, the vector tangent to this hypersurface, u^μ, is our temporal direction. The selection of a spatial direction is done by introducing a ghost field, S, characterized by a wrong sign kinematic term. The gradient of this ghost field, s^μ, is our spatial direction. Dynamically selecting these directions and defining a Lagrangian which conserves a particular multipole moment creates a generally covariant theory of multipole conserving quasiparticles.

1 Introduction

1.1 Field Theory Background

Fractons[1] are defined by a complex scalar field Φ that is invariant under global phase rotations, $\Phi \to e^{i\alpha}\Phi$ as well as linear phase rotations, $\Phi \to e^{i\vec{\lambda}\cdot\vec{x}}\Phi$. Via Noether's theorem we know there exists a conserved current $J^\mu = \lambda\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Phi$. To find the Lagrangian of a field with these symmetries, the prototypical example is Pretko [1]. First combine the symmetries with $\alpha(x) = \alpha + \vec{\lambda}\cdot\vec{x}$ so the phase rotation is $\Phi \to e^{i\alpha(x)}\Phi$. Then find the lowest order covariant derivative operator, the operator takes the form:

$$\Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi$$
$$\to e^{i2\alpha(x)}(\Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi + (i\partial_i\partial_j\alpha)\Phi^2) \quad (1)$$

Now because our choice of $\alpha(x)$ is linear, that means $\partial_i\partial_j\alpha = 0$. A similar excercise finds another covariant, the g' term below. In total, this gives us a lowest order Lagrangian density

$$\mathcal{L} = |\partial_t\Phi|^2 - m^2|\Phi|^2 - g|\Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi|^2$$
$$-g'\Phi^{*2}(\Phi\partial^2\Phi - \partial_i\Phi\partial^i\Phi) \quad (2)$$

note that in Pretko's work, $i, j, k \in \{1, 2, 3\}$. With the lowest order Lagrangian and the $\alpha(x)$ symmetry we have a conserved quantity

$$Q^i = \lambda^2 \int d^3x^i\ x|\Phi|^2 \quad (3)$$

this should be recognized as the conservation of the dipole moment.

1.2 Fracton Dynamics

In a theory with only charge conservation, the allowable dipole creation operators take the form $\partial_i\Phi^\dagger\partial_i\Phi$, [1] meaning that positive and negative charges are created in pairs. This gives quadratic terms in the field theory. For an example of this, think of electron-positron production in gamma

[1] As a note to the reader, in the literature the word fracton has been used for two distinct quasiparticles. The dipole conserving quasiparticals [1], as well as phonon behavior in fractal substrates. We will not be referring to the later in this work.

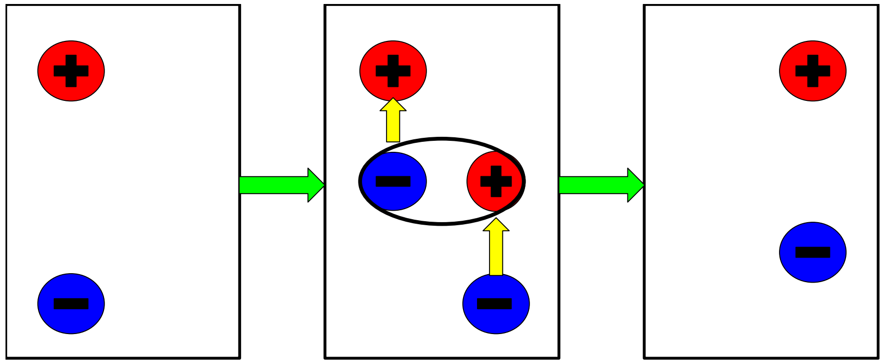

Figure 1: Fracton dipoles can move through the mutual exchange of virtual dipoles. Fractons are able to interact without the aid of a mediating field.

rays. However, in a theory with dipole moment conservation, the allowable creation operators are now quadrupolar, $(\Phi^\dagger \partial_x \partial_y \Phi^\dagger)(\Phi \partial_x \partial_y \Phi)$, and dipoles can move. This leads to quartic terms in the field theory.

Since the minimum creation operator became quadrupolar rather than dipolar, individual fractons are completely immobile [2, 3], an individual fracton being the field excitation $\Phi^\dagger$.However, dipoles can move freely as is reflected by the final two terms in the Lagrangian density. The g' term in (2) contains diagonal second derivatives (∂_x^2 but not $\partial_x \partial_y$) this term describes the longitudinal motion (along the direction of the dipole) of fracton dipoles as opposed to the g term which describes their transverse movement. This movement is facilitated by the exchange of a virtual dipole between the fractons, as is shown in figure 1.

1.3 Fracton Systems

Fractons can be found in several condensed matter systems such as tilted optical lattices, where the wavelength number density modulations can be described as a fluid of fractons with a locally conserved dipole moment. In the optical lattice there is also an emergent dipole conservation law at non-zero tilt due to energy conservation. In the limit time, $t \to \infty$ the lattice behavior is exactly described as a fluid of fractons [4].

Quantum crystal systems are dual to fracton gauge theory. Crystalline dislocations can only move along their 'Burgers vector', a vector which describes lattice distortion due to the defect. This motion requires absorption or emission of vacancies or interstitials. There are no local processes to move a single disclination by itself, a clear example of fractonic behavior. The creation of this elastic-fracton dual has lead to testable predictions of fracton behavior and quantum crystal properties. [3]

Excitingly, fracton phases may also be utilized in fault-tolerant quantum computation. Specifically in Majorana based quantum computing, there exist local stabilizers codes that exhibit glassy dynamics and individual excitation immobility [2, 5]. This is exactly the behavior of fractons. They have relaxation times that diverge at low temperatures, T, as a super exponential function $t_r \propto e^{\frac{1}{T^2}}$. States also exist, called 'dynamic scar states', in driven systems, that reach athermal equilibrium. These scar states are a vanishing fraction of states in the limit $T \to \infty$. However, there is large overlap with easily prepared states so they cannot be ignored. [6, 7]

2 Multipole Expansion

Now that we have explained the properties and uses of dipole moment conserving fractons, we will now begin to expand on the theory to allow for higher-pole conservation. First we must note that fracton systems are non-relativistic, they have a prefered frame in which they are not moving. The dipole moment conserving Lagrangian (2) also features a splitting of space and time and treats them differently, whereas relativity treats them the same. This splitting between the properties of time and the properties of space, is known as a foliation.

2.1 Spacetime Foliation

Foliations of spacetime are not a new concept, being introduced in the 1940's in the realm of pure mathematics and began being used in physics in the 1970's. They have been used in the study of singularity prevention [8],early universe gravitation [9], Lorentz violating gravity theories [10], and quantum gravity theories [11].

Roughly speaking, a foliation is the selection of a particular hypersurface on which a certain coordinate or set of coordinates is constant. We could think of this as our universe, at a particular moment in time, being a 3d-hypersurface of constant time. As we move to the next moment in time, we move to a new 'leaf' of the foliation. Another, possibly easier, way to visualize this would be the slicing done by a 3d-printer. It lays down one layer of material at a time, each layer is a leaf of the foliation defined by its height above the printing surface. While the 3d-printer takes flat slices, a foliation could be spherical, flat or shaped like a mountain range. The shape is irrelevant. The important property is that each leaf of the foliation a given coordinate is constant.

Foliations are inconsistent with relativistic field theories and their particles, such as electrons in electromagnetic theory. Electrons need no preferred frame as relativity treats space and time the same. Foliations, are necessary for the preservation of dipole moment charges like fractons. Their necessity comes from the fact that an individual fracton, a single field excitation $\Phi^\dagger$, is immobile in a preferred frame. This is in contrast to relativity where all frames are equally valid and our physics is the same regardless.

While fractons are incapable of moving individually, this is only true for space-like dimensions. Individual fractons move through time normally; they do not treat space and time equivalently. While foliations are not mentioned by Pretko in the derivation of (2) [1], he has one hiding in his theory due to the fact that time and space are treated differently. $(i, j, k \in \{1, 2, 3\})$.

Using the idea of a foliation it is possible to recreate Pretko's Lagrangian. We will now switch to Greek indices $\mu, \nu, etc \cdots \in \{0, 1, 2, 3\}$, to indicate space-time and we will keep Latin indices to indicate purely spatial objects. We begin by taking a foliation which creates leaves of constant time across the 3d-hypersurface; we then can take a vector normal to this surface, u^μ, to be the direction of time. This allows us to modify terms in the Lagrangian (2)

$$|u^\mu \partial_\mu \Phi|^2 - g|\Phi(g^{\mu\nu} + u^\mu u^\nu)\partial_\mu \partial_\nu \Phi$$
$$-(g^{\mu\nu} + u^\mu u^\nu)\partial_\mu \Phi \partial_\nu \Phi|^2 + \dots \tag{4}$$

Simply $u^\mu \partial_\mu$ selects a derivative along the direction of u^μ and $(g^{\mu\nu} + u^\mu u^\nu)$ supplies all other derivatives. This creates the same fractonic behavior but more readily illustrates exactly what is happening and the difference between spatial and temporal derivatives.

With the use of foliations established we are now ready to begin generalizing fractonic behavior to higher poles. To do this, our phase angle, $\alpha(x)$, will, rather than being constant or linear, be quadratic.

2.2 Quadratic Phase Transformation

Pretko had introduced a vector, $\vec{\lambda}$, along which the fracton dipole would be oriented. In similar fashion, we introduce a quadrupole tensor, γ_{ij}. Clearly this can be extended upwards to octupole, and higher tensors. With our quadrupole tensor established, we can now define a quadratic phase transformation as

$$\Phi \to e^{ix^i \gamma_{ij} x^j} \Phi \tag{5}$$

Via Noether's theorem we get a charge of the form $Q^{ij} = c\int dx^3 x^i x^j |\Phi|^2$. This should be recognized as a quadrupole moment, though notably without a dipole or monopole contribution. The latter property of which will be discussed in section 2.2.1. In order to model this higher order charge, we introduce a new scalar field S, the gradient of which, s^μ, will select a spatial direction. The need for a particular spatial direction to be selected is due to the fact that a quadrupole exists on a 2d surface, the vector normal to this surface is s^μ. The difference between s^μ and u^μ that is relevant for now, is that the spatial vector is defined as the gradient of a scalar field rather than the time-like foliation from before. How exactly we define our spatial and temporal vectors dynamically are different and will be discussed in sections 3.2 and 3.3.

2.2.1 Restrictions on Multipole Tensor

While our goal is the preservation of the quadrupole moment, we look at a subsector of the quadrupolar space. While the multipole moment tensor is known to be transverse $(\nabla_{i\ldots(j-1)}\gamma^{ij} = 0)$ and traceless $g_{ij\ldots}\gamma^{ij\cdots} = 0$ we add one more restriction. One particular transverse traceless form has the restriction that $\gamma^{ii\cdots} = 0$. This additional restriction simplifies our algebra considerably while still allowing us to prove the existence of quarupole moment conserving excitations.

2.2.2 Example of Using Foliations

In order to better illuminate the usefulness of foliations, I will give a simplified and idealized example. Let us presume we are in a flat Minkowski space with the (-,+,+,+) signature. Let us say that we wanted to write the spatial Laplacian, $\nabla^2\Phi = \partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi$ covariantly. We create a foliation splitting space and time, and take the vector normal to this surface, u^μ. In this case, it looks like $u^\mu = [1,0,0,0]$. We then combine the metric and time vector like,

$$\eta^{\mu\nu} + u^\mu u^\nu = diag(0,1,1,1)$$

and finally we see that

$$(\eta^{\mu\nu} + u^\mu u^\nu)\partial_\mu\partial_\nu\Phi = \partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi \quad (6)$$

This can also be used in tandem with s^μ to produce things like $\partial_x^2\Phi + \partial_y^2\Phi$.

2.2.3 Phase Covariant Derivative

In order to find the phase covariant derivative[2] for our transformation (eq:5), we begin by taking all possible spatial derivatives of the field up to order $n = 2^k$ where k is the order of $\alpha(x)$. In 3+1d space we see 3 first order derivatives, 6 second order derivatives, 10 third order derivatives. Ultimately these follow the shifted triangular numbers $(\frac{(k+1)(k+2)}{2})$. This was implemented in Mathematica by David Mattingly as follows;

```
dT1=DeleteCases[Flatten[Table[If[a+b+c==1,
D[g[x, y, z],{x,a},{y,b},{z,c}],0],{a,0,1},
{b,0,1},{c,0,1}]],0]
.....
dTn=DeleteCases[Flatten[Table[If[a+b+c==1,
D[g[x, y, z],{x,a},{y,b},{z,c}],0],{a,0,n},
{b,0,n},{c,0,n}]],0]
```

We then multiply these in all possible combinations that lead to all possible terms of order n in derivatives. For the case of k=1 (n=2), the dipole moment conservation Pretko studies, there are 15 terms. For the case of k=2 (n=4), the quadrupole moment conservation we are discussing, we have 216 terms. If we were to discuss octupole moments, k=3 (n=8), there would be 28,689 terms and a hexadecapole moment would have over 260 million terms. This pattern has been discussed before in the Online Encyclopedia of Integer Sequences [14]. For this reason, we will be discussing only the cases of k=1 (n=2) and k=2 (n=4) and simply acknowledging that this process can be expanded to higher moments. I show this implemented in Mathematica for k=2;

```
Combo1=Apply[Times,Tuples[{dT1,dT1,dT1,dT1}],{1}];
Combo12=Apply[Times,Tuples[{dT1,dT1,dT2}],{1}];
Combo13=Apply[Times,Tuples[{dT1,dT3}],{1}];
Combo22=Apply[Times,Tuples[{dT2,dT2}],{1}];
Combo4=dT4;
```

After defining all possible terms, all that is left is to eliminate all of the terms which are not covariant under our transformation. It may be possible to define a method of doing this efficiently, however, that was out of the scope of

[2]The term 'phase covariant derivative' is used to mean a derivative which only changes by multiples of the phase transformation. It is distinct from the covariant derivative, which is discussed later.

this study, and elimination by hand was effective for our 216 terms. We add all of the terms in our combos together, each with its own coefficient, and call this our 'total derivative operator'. However, we only want the terms which are covariant under our quadratic phase shift.

Now define g[x,y,z] as our transformed field. Without loss of generality, we can rotate our quadrupole such that it is in the X-Y plane; this rotation eliminates the XZ and YZ terms. Then collect by powers of γ_{xy}, x, y, z, and Φ. Set all of the unique powers of these to zero; thus eliminates everything which is not covariant under our phase shift and simplifying the coefficients which are covariant.

We are then left with only those derivative terms which transform covariantly. These terms are

$$\alpha(s^\mu\partial_\mu\Phi)^4+\beta(s^\mu\partial_\mu\Phi)^2(s^\nu s^\sigma\partial_\nu\partial_\sigma\Phi)$$
$$+\gamma s^\mu s^\nu s^\sigma s^\omega\partial_\mu\partial_\nu\partial_\sigma\partial_\omega\Phi-\delta(s^\mu s^\nu\partial_\mu\partial_\nu\Phi)^2$$
$$-\epsilon|s^\mu\partial_\mu(g^{\nu\sigma}+u^\nu u^\sigma-s^\nu s^\sigma)\partial_\nu\Phi|^2$$
$$+\zeta s^\mu\partial_\mu\Phi[s^\nu\partial_\nu[(g^{\sigma\omega}+u^\sigma u^\omega)\partial_\sigma\partial_\omega\Phi+$$
$$-\eta\{(s^\mu s^\nu\partial_\mu\partial_\nu\Phi)s^\sigma\partial_\sigma(g^{\omega\zeta}+u^\omega u^\zeta-s^\omega s^\zeta)\partial_\omega\Phi$$
$$-s^\mu s^\nu\partial_\mu\partial_\nu(g^{\sigma\omega}+u^\sigma u^\omega-s^\sigma s^\omega)\partial_\sigma\Phi]]\} \qquad (7)$$

The η term is a boundary term and thus does not contribute to the equations of motion, therefore we will be dropping that term from further discussion. It is now straight-forward to construct a Lagrangian that respects monopole and quadrupole charge conservation laws. At lowest order, it takes the form,

$$\mathcal{L}_q=|u^\mu\partial_\mu\Phi|^2-m^2|\Phi|^2-\alpha\Phi^{\dagger 4}(s^\mu\partial_\mu\Phi)^4$$
$$+\beta\Phi^{\dagger 3}(s^\mu\partial_\mu\Phi)^2(s^\nu s^\sigma\partial_\nu\partial_\sigma\Phi)$$
$$+\gamma\Phi^\dagger s^\mu s^\nu s^\sigma s^\omega\partial_\mu\partial_\nu\partial_\sigma\partial_\omega\Phi+\delta\Phi^{\dagger 2}(s^\mu s^\nu\partial_\mu\partial_\nu\Phi)^2$$
$$+\epsilon\Phi^{\dagger 2}|s^\mu\partial_\mu(g^{\nu\sigma}+u^\nu u^\sigma-s^\nu s^\sigma)\partial_\nu\Phi|^2$$
$$+\zeta\Phi^{\dagger 2}s^\mu\partial_\mu\Phi[s^\nu\partial_\nu[(g^{\sigma\omega}+u^\sigma u^\omega)\partial_\sigma\partial_\omega\Phi]] \qquad (8)$$

Here we have accomplished the goal of generalizing fractonic dynamics to quadrupole moment conservation with a global quadratic phase symmetry. We have also explained how it is possible to extend this upwards to higher order moments.

The theory as laid out thus far is covariant with respect to quadratic phase transformations but still relies on the particular definitions of the u^μ and s^μ directions.We did not allow for u^μ and s^μ to change at all with respect to position, thus forcing our foliation and S to be flat planes. We will expand the theory to allow for this in the following section.

3 General Covariantization

If a theory is generally covariant, that means that the form of its governing equations remain the same regardless of any possible coordinate transformations. That means all quantities must be dynamical and have their own equations of motion. This is the case for theories like general relativity or electrodynamics, but is not the case for Newtonian mechanics and the like.

Newtonian mechanics as an example is covariant under Galilean transforms, like $x'=x-vt$ where v is the relative velocity of the two reference frames. This can be shown with:

$$F'_x=m'\partial^2_{t'}x'=m\partial_t(\partial_t(x-vt))$$

$$=m\partial_t(v_x-v)=ma_x=F_x$$

However, it is easily shown that Newton's equations are not generally covariant (meaning not covariant under all transformations) if we look at $x'=\kappa x$,

$$F'_x=m'\partial^2_{t'}x'=m\partial_t(\partial_t(\kappa x))=\kappa ma_x\neq F_x$$

The above shows that while Newton's second law is covariant under translation (translational covariance), but the law is not covariant under scaling (scale covariance).This means Newton's second law is not generally covariant. General covariance is the key to theories which work in any reference frame. The key to general covariance is the covariant derivative.

3.1 Covariant Derivative

Given any coordinate functions on a manifold, x^μ, we can describe any vector tangent to the manifold in terms of its components in the basis $e_\mu = \frac{\partial}{\partial x^\mu}$ [15]. We can then define the covariant derivative by specifying the covariant derivative of each basis vector, e_μ, along the others, e_ν. Essentially saying how much of e_0 is in the direction of e_1 and etc.. This is looks like $\nabla_{e_\nu} e_\mu = \nabla_\nu e_\mu = \Gamma^\sigma_{\mu\nu} e_\sigma$. Here $\Gamma^\sigma_{\mu\nu}$ are the Christoffel symbols, defined in terms of the metric as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\zeta}(\partial_\zeta g_{\mu\nu} + \partial_\nu g_{\mu\zeta} - \partial_\mu g_{\nu\zeta})$$

This means if we take the covariant derivative of a general vector field, $\ell = \ell^\mu e_\mu$, with respect to one of our basis vectors e_ν we get

$$\nabla_{e_\nu} \ell = \nabla_\nu \ell^\mu e_\mu = e_\mu \nabla_\nu \ell^\mu + \ell^\mu \nabla_\nu e_\mu$$

$$= e_\mu \partial_{x^\nu} \ell^\mu + \ell^\sigma \Gamma^\mu_{\sigma\nu} e_\mu = (\partial_{x^\nu} \ell^\mu + \ell^\sigma \Gamma^\mu_{\sigma\nu}) e_\mu$$

We can insert this into our Lagrangian (Eq: 7) with some slight rearrangement. While we can rearrange partial derivatives like

$$\ell^\mu \ell^\nu \partial_\mu \partial_\nu = \ell^\mu \partial_\mu \ell^\nu \partial_\nu$$

this is not true for the covariant derivative. In the above equation if ℓ^ν has a component in the direction of e_μ, the left and right hand sides would no longer be equivalent. This means our Lagrangian then becomes

$$\begin{aligned} \mathcal{L}_q = |u^\mu \nabla_\mu \Phi|^2 - m^2|\Phi|^2 - \alpha \Phi^{\dagger 4}(s^\mu \nabla_\mu \Phi)^4 \\ +\beta \Phi^{\dagger 3}(s^\mu \nabla_\mu \Phi)^2 (s^\nu \nabla_\nu s^\sigma \nabla_\sigma \Phi) \\ +\gamma \Phi^\dagger s^\mu \nabla_\mu s^\nu \nabla_\nu s^\sigma \nabla_\sigma s^\omega \nabla_\omega \Phi + \delta \Phi^{\dagger 2}(s^\mu \nabla_\mu s^\nu \nabla_\nu \Phi)^2 \\ +\epsilon \Phi^{\dagger 2} |s^\mu \nabla_\mu (g^{\nu\sigma} + u^\nu u^\sigma - s^\nu s^\sigma) \nabla_\nu \Phi|^2 \\ +\zeta \Phi^{\dagger 2} s^\mu \nabla_\mu \Phi [s^\nu \nabla_\nu [(g^{\sigma\omega} + u^\sigma u^\omega) \nabla_\sigma \nabla_\omega \Phi]] \quad (9) \end{aligned}$$

The covariant derivative means that we are able to take any shaped foliation or S field we please, meaning that u^μ and s^μ must have dynamics and change with respect to position. Our Lagrangian is not yet generally covariant, because those dynamics have not yet been included. Previously, our temporal and spatial vectors were both used similarly, to select particular derivatives. However, they have different dynamics and result from different underlying physics. Both of which will now be explored.

3.2 Defining Time, u^μ

3.2.1 Holography

We will begin the process of defining our time foliation by looking at an idea from string theories and properties of many theories of quantum gravity (QG). Holography, or the holographic principle, states that a description of a volume of space can be encoded on a lower-dimensional boundary of the region.

Holography is useful due to a property known as Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. This is a conjectured connection between the AdS fields present in QG theories and the CFT field used in quantum field-theories. The usefulness is in the fact that this process allows strongly coupled QFT problems to be effectively translated into weakly coupled QG problems where the math is more amenable. In fact, one model of fractons using a Yang-Mills CFT, was used by Yan [16] to show the correspondence of certain condensed matter systems and QG.

While AdS/CFT correspondence is still a conjecture, it is undeniably true that certain QG systems are dual to condensed matter and other strongly coupled systems. We will use the idea of translating between condensed matter and quantum gravity systems to define our time foliation.

3.2.2 Hořava-Lifshitz Quantum Gravity

Hořava-Lifshitz Quantum Gravity was introduced in the early 2000's [10, 12, 13], it is built off of the idea that Lorentz invariance may not be a fundamental property but rather an emergent one at large distances and low energies. This emergent Lorentz invariance is done by the application of an anisotropic scaling, meaning a different scaling of space and time. Hořava takes the scaling $x \to bx$ and $t \to b^z t$, where z measures the degree of anisotropy between space and time. z is a free parameter that can be chosen.

For example in the case of z=2, the dimension where the gravitational coupling constant is dimensionless shifts to 2+1. That makes the system a suitable to describe a volume of space-time on a bosonic membrane [10].

For a Lorentz invariant theory like loop quantum gravity, we find a graviton propagator that scales with the 4-momentum, $K_\mu = (\omega, k)$, like $(K^\mu g_{\mu\nu} K^\nu)^{-1}$. However, with an anisotropic scaling. We find a graviton propagator that scales like

$$\frac{1}{\omega^2 - c^2k^2 - G(k^2)^z}$$

Where G is a coupling constant. At low energy, the theory acts like z=1, and we recover the Lorentz invariant propagator. But at higher energies, and thus smaller distances, Lorentz invariance is broken and space and time split into distinct, mathematically distinguishable objects. In particular for a 3+1d space-time, z=3.

The anisotropic scaling gives time a special role, which will be encoded by assuming our space-time, $\mathcal{M}$, also has a codimension-1 foliation[3], $\mathcal{F}$. The gradient of this foliation provides a unit time-like vector u^μ known as the "æther". This is not the luminiferous æther of Michelson-Morley, this is a locally preferred frame at every point in space. This is the minimal structure required to split space and time. The dynamics of this time-like vector have been explored before by Jacobson and Mattingly [17, 18, 19]. The most general Lagrangian, that is generally covariant, that defines the dynamics of u^μ is

$$\mathcal{L}_{æ} = -R - \lambda(u^\mu u_\mu - 1) - K^{\mu\nu}_{\sigma\omega}\nabla_\mu u^\sigma \nabla_\nu u^\omega \quad (10)$$

where λ is a Lagrange multiplier that forces u^μ to be unit length, and where

$$K^{\mu\nu}_{\sigma\omega} = c_1 g^{\mu\nu} g_{\sigma\omega} + c_2 \delta^\mu_\sigma \delta^\nu_\omega + c_3 \delta^\mu_\omega \delta^\nu_\sigma + c_4 u^\mu u^\nu g_{\sigma\omega} \quad (11)$$

with c_i being arbitrary constants and R, the Ricci scalar, is defined as

$$R = g^{\mu\nu}(\nabla_\sigma \Gamma^\sigma_{\mu\nu} - \nabla_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\omega_{\mu\nu}\Gamma^\sigma_{\sigma\omega} - \Gamma^\omega_{\mu\sigma}\Gamma^\sigma_{\nu\omega}) \quad (12)$$

This effectively fixes the length and dynamics of u^μ. By adding $\mathcal{L}_{æ}$ to $\mathcal{L}_q$ in equation 8, we take one step closer to general covariance. All that is left is to define the dynamics of the scalar field S and thus s^μ.

3.3 Defining Directionality, s^μ

A ghost condensate is a kind of fluid which can fill the universe. It is similar to a cosmological constant, Λ, because it does not dilute as the universe expands, and because the ghost condensate and Λ equations of state $p = -\rho$ are the same [20, 21].

Therefore, ghost condensation can drive de Sitter inflation of the universe, even if the cosmological constant vanishes. However it is unlike Λ, a ghost condensate is described by a scalar field with a negative kinetic energy term around $\nabla_\mu S = 0$, and it is this difference that makes ghost condensation useful to us. A negative kinetic energy means that being stationary is no longer our lowest energy state.

If we engineer the Lagrangian such that our field S has a negative kinetic energy, and so it will take the shape of the mexican hat potential. Then our spatial vector $s^\mu = \nabla^\mu S$ will fall over from the unstable peak at zero, to the trough of the hat. This gives a meaningful and dynamical definition of our spatial vector. This Lagrangian takes the form

$$\mathcal{L}_g = (u^i\nabla_i S)^2 - |(g^{ij} + u^i u^j)\nabla_i S|^2 + |(g^{ij} + u^i u^j)\nabla_i S|^4 \quad (13)$$

This selection of a spatial direction spontaneously violates Lorentz invariance, in much the same way the cosmic microwave background radiation violates Lorentz invariance. The ghost condensate forces a preferred frame in which S is spatially isotropic.

Adding $\mathcal{L}_q$, $\mathcal{L}_{æ}$, and $\mathcal{L}_g$ produces a generally covariant Lagrangian which respects charge and quadrupole moment conservation with global

[3] Hořava explains [10] that a codimension-q foliation, $\mathcal{F}$, on a d-dimension manifold, $\mathcal{M}$, is defined with a coordinate system (y^a, x^i) $a = 1, \ldots q; i = 1, \ldots d-q$ and transformations are restricted to $(y'^a, x'^i) = (y'^a(y^b), x'^i(y^b, x^j))$. We take $q = 1$ here because it will seperate time by itself. Taking q=2 would seperate temporal and spatial direction, but those would still require separation and so there is no advantage to that approach.

symmetries. As we attempt to conserve higher poles, we are required to define more spatial vectors. This is done through the addition of more ghost condensate fields.

4 Conclusions

A foliation, $\mathcal{F}$, created through anisotropic scaling defined the direction of time, u^μ (Section 3.2.2). A ghost condensate, S, allowed us to meaningfully define a direction of space, s^μ (Section 3.3). Using these two vectors, we were able to define a generally covariant Lagrangian $\mathcal{L} = \mathcal{L}_q + \mathcal{L}_g + \mathcal{L}_æ$ which respects charge and quadrupole moment conservation, entirely with global phase transformations. We have discussed how this theory can be extended to higher pole conservation.

For completeness, the following equation is the full Lagrangian derived in this work.

$$\begin{gathered}\mathcal{L} = |u^\mu \nabla_\mu \Phi|^2 - m^2|\Phi|^2 - \alpha \Phi^{\dagger 4}(\bar{s}^\mu \nabla_\mu \Phi)^4 \\ +\beta \Phi^{\dagger 3}(\bar{s}^\mu \nabla_\mu \Phi)^2(\bar{s}^\nu \nabla_\nu \bar{s}^\sigma \nabla_\sigma \Phi) \\ +\gamma \Phi^\dagger \bar{s}^\mu \nabla_\mu \bar{s}^\nu \nabla_\nu \bar{s}^\sigma \nabla_\sigma \bar{s}^\omega \nabla_\omega \Phi + \delta \Phi^{\dagger 2}(\bar{s}^\mu \nabla_\mu \bar{s}^\nu \nabla_\nu \Phi)^2 \\ +\epsilon \Phi^{\dagger 2}|\bar{s}^\mu \nabla_\mu (g^{\nu\sigma} + u^\nu u^\sigma - \bar{s}^\nu \bar{s}^\sigma)\nabla_\nu \Phi|^2 \\ +\zeta \Phi^{\dagger 2}\bar{s}^\mu \nabla_\mu \Phi[\bar{s}^\nu \nabla_\nu[(g^{\sigma\omega} + u^\sigma u^\omega)\nabla_\sigma \nabla_\omega \Phi]] \\ -R - \lambda(u^\mu u_\mu - 1) - K^{\mu\nu}_{\sigma\omega}\nabla_\mu u^\sigma \nabla_\nu u^\omega + (u^i \nabla_i S)^2 \\ -|(g^{ij} + u^i u^j)\nabla_i S|^2 + |(g^{ij} + u^i u^j)\nabla_i S|^4 \quad (14)\end{gathered}$$

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