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Mathematical analysis of blood flow model through channels with flexible walls

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MATHEMATICAL ANALYSIS OF BLOOD FLOW MODEL THROUGH CHANNELS WITH FLEXIBLE WALLS

BY

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B.S., Mathematics, University of New Hampshire, 2009

Submitted to the University of New Hampshire in partial fulfillment of the requirements for the degree of

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This thesis has been examined and approved.

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ABSTRACT

MATHEMATICAL ANALYSIS OF BLOOD FLOW MODEL
THROUGH CHANNELS WITH FLEXIBLE WALLS

by

LISA A. BOZZUTO
University of New Hampshire, September, 2011

A simplified mathematical model of blood flow through flexible arteries is developed and analyzed. The resulting system of non-linear, non-homogeneous PDE’s is analyzed numerically using the Richtmyer Lax-Wendroff method. Numerical and theoretical results show excellent agreement suggesting that in physiologically relevant situations shocks only develop outside the domain of interest. These results suggest that when the model assumptions are satisfied the model provides sufficient regularity to yield a physically reasonable representation of flow through a flexible artery. We conclude with a discussion of future directions for this model.
CHAPTER 1
INTRODUCTION

Mathematical modeling of the circulatory system has greatly increased in popularity in recent years. In fact, the FDA now requires mathematical modeling of human physiology before allowing human or animal testing of vascular devices as stated in [4]. In order to develop helpful, realistic models, several things need to be taken into account. Among those things, the model needs to be simple enough that the resulting problems are computationally feasible. “The study of the blood flow in arteries is a rich field encompassing unsteady flows, varying geometries, turbulence, and secondary structures.” [11]

As Ku states, studying blood flow can allow us to quantify diseases, such as atherosclerosis, stenosis, and thrombosis. Altered flow patterns, which may be detected by simulation, such as flow reversal, and low or oscillatory shear stress, are important factors in developing arterial disease [14]. Once these diseases are understood, a model that incorporates blood flow, the effect of the walls on flow, and the adaptation of vessel walls after surgery can help make clinical decisions. These can include when surgery is necessary, how to develop diagnostic tools, and how to design devices to mimic or alter blood flow ([11], [15]). The complexity of such models depends on the nonlinear interactions in the cardiovascular system, and the vast differences between individual structures [15]. To study these disease states, Ku looks at pulsatile flow on the verge of turbulence [11].

Similarly, Canic has been studying the implementation of stents, and when they fail. She has looked at a method of attaching stents to arterial walls, and the problems
that arise [4]. In order to accurately describe the change between the arterial wall and the stent, Canic and coauthors have obtained a number of results, which have been presented in a series of papers investigating blood flow in an artery with variable radius ([6], [16], [8], [5]), and how to model arterial walls’ response to pulsatile blood flow ([2], [7]).

Several other methods have been used to describe blood flow in diseased conditions. Magnetic resonance angiography has been used to collect data from the carotid artery to develop a computational fluid dynamic model [9]. This model was able to accurately simulate flow through the carotid artery with stenosis. This can possibly be used to enhance image-based diagnostic tools [9]. Another method used fluid-structure interaction modeling to describe the interaction between blood flow and the arterial wall. The fluid-structure interaction model developed in [12] accurately describes blood flow with pressure boundary conditions. When models involving shunts have been developed to sufficient accuracy, different shunt geometries and procedures can be tested numerically before surgery to discover the best approach [14].

Another area of interest deals with drug-eluting stents. Frequently, after stent implantation, cells grow inward and block the stent, known as restenosis. Drug-eluting stents are designed to reduce the recurrence of stenosis within the stent by delivering anti-proliferative drugs. However, the effect of the drug greatly depends on the blood flow through the stent, tissue uptake, and the dosage and timing of the drug delivery [10]. To determine the effects of the drug, a model can be developed using computational fluid dynamics and diffusion-convection principles ([1], [10]). The goal is to determine not only how the drug distributes into the artery, but also how it is absorbed and stored in the arterial wall. Both stages are crucial in balancing the effectiveness of the drug against potential toxicity. These models suggest that different drug delivery strategies have an effect on drug uptake [1].
CHAPTER 2

PROBLEM STATEMENT

The cardiovascular system is so complex, that it is necessary to make simplifying assumptions when attempting to create a mathematical model. The first complication comes from blood itself. Blood is not a uniform fluid, but instead a suspension of several types of particles. Plasma is the liquid in which red blood cells (RBCs), white blood cells (WBCs), and platelets are suspended. These three types of particles are the most common, but there are also proteins, electrolytes, hormones, and nutrients suspended in plasma as well. Red blood cells, or erythrocytes, transport oxygen and carbon dioxide throughout the body. They make up 40 percent of blood by volume, so the properties of red blood cells determine most of the characteristics of blood as a whole. There are significantly fewer white blood cells, or leukocytes, but they are much larger than red blood cells. They control immune responses. Platelets, or thrombocytes, are the smallest particles, and are involved in clotting.

![Figure 2-1: The relative percentages by volume of the four main components of blood are shown.](image)
Red blood cells determine many characteristics of blood. They are semisolid, circular disks whose edges are thicker than the center of the cell. This shape maximizes the surface area, providing more bonding areas for the oxygen and carbon dioxide. Unfortunately, the diameter of red blood cells can be larger than the diameter of capillaries. The unique geometry of the cell allows it to deform to fit through small vessels. This characteristic contributes to the non-Newtonian characteristic of blood.

![Image of red blood cells](image)

Figure 2-2: This is a scanning electron micrograph image of a group of red blood cells. Courtesy of the CDC/Janice Carr.

Blood is a non-Newtonian fluid, which means the viscosity of blood is not constant. It changes based on the temperature of blood, and the volume or concentration of red blood cells. Consequently, the force required to move blood is effected by these factors as well.

In addition to the complexities of blood, the flexibility of arteries adds another layer of complications. A liquid is not generally compressible, so when the heart contracts, another area of the circulatory system needs to expand to accommodate the extra blood that is expelled. If the arterial walls were not flexible, the vessel wall would burst from the sudden increase in pressure from the extra blood. Thus it is extremely important that the vessel walls are indeed flexible, and allow the blood to travel in a realistic manner.
In order to make the problem tractable, several simplifying assumptions must be made. The first assumption is that the system is axisymmetric. This means that the vessel and blood flow is symmetric about the center of the vessel. The next assumption is that blood is an incompressible fluid. This is a common assumption, which allows the density of blood to be considered constant. Using these assumptions, we can use the Navier-Stokes equations and study a simplified, one-dimensional asymptotic model. This model can be used to investigate conditions under which there will be smooth solutions, and numerically represent blood flow under pulsatile boundary conditions.
CHAPTER 3
DERIVATION OF MODEL EQUATIONS

The derivation of the model equations will closely follow the derivation proposed by Canic and Kim in [3]. This model assumes that tangential wall displacements of the vessel due to the fluid are prohibited. In other words, the vessel walls expand and contract, but these changes in cross-sectional area do not travel along the length of the vessel. This behavior is due to an assumption about the motion of the fluid. In order to develop the one dimensional model equations, start with the Navier-Stokes equations in cylindrical coordinates \((x, r, \theta)\), with the velocity components denoted by \(\vec{V} = (V_x, V_r, V_\theta)\). Recall the assumption that the system is axially symmetric, which means all derivatives with respect to \(\theta\) are zero, and the velocity in the theta direction is zero \((V_\theta = 0)\). In addition, forces due to gravity, which may cause the angular velocity to be nonzero, will be ignored. So, the Navier-Stokes system for the two non-trivial components, \(V_x\) and \(V_r\), can be given in the following form:

\[
\begin{align*}
\frac{\partial V_x}{\partial t} + V_r \frac{\partial V_x}{\partial r} + V_x \frac{\partial V_x}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= \nu \left[ \frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r} \frac{\partial V_x}{\partial r} + \frac{\partial^2 V_x}{\partial x^2} \right], \\
\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_x \frac{\partial V_r}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial r} &= \nu \left[ \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} + \frac{\partial^2 V_r}{\partial x^2} \right].
\end{align*}
\]

In (3.1), \(t\) is the time variable, \(r\) is the radial variable, \(x\) is the axial variable, \(p\) is the pressure, and \(\nu\) is the kinematic viscosity, which is the ratio of dynamic viscosity to density. Assuming blood is incompressible (density, \(\rho\), is constant), the continuity
equation reduces to $\nabla \cdot \vec{V} = 0$. Taking into account axial symmetry, the continuity equation reduces further to

$$\frac{\partial V_x}{\partial x} + \frac{1}{r} \frac{\partial (r V_r)}{\partial r} = 0.$$  \hfill (3.2)

### 3.1 Reduced Nondimensional Equations

In order to determine the dominant terms in the previous equations ((3.1),(3.2)), perform an order of magnitude analysis by nondimensionalizing the variables. First identify characteristic values with which the variables will be scaled. Let

$$U_0 \quad \text{be the characteristic radial velocity,}$$

$$V_0 \quad \text{be the characteristic axial velocity,}$$

$$\lambda \quad \text{be the characteristic length,}$$

$$R_0 \quad \text{be the characteristic inner vessel radius.}$$

From these characteristic values, nondimensional variables can be determined.

$$\tilde{r} = \frac{r}{R_0}, \quad \tilde{V}_x = \frac{V_x}{V_0},$$

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{V}_r = \frac{V_r}{U_0},$$

$$\tilde{t} = \frac{V_0}{\lambda} t, \quad \tilde{p} = \frac{p}{\rho V_0^2}.$$  \hfill (3.4)

Note $\varepsilon = \frac{R_0}{\lambda} = \frac{U_0}{V_0}$, or the ratio of the width versus the length of the vessel. For the computations, consider a vessel whose length is much greater than its radius, i.e., $\varepsilon \sim O(10^{-2})$.

First nondimensionalize the incompressibility condition (3.2). Replacing all variables with the nondimensional ones (3.4), obtain a new equation
\[
\frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial (\lambda \tilde{x})} + \frac{1}{R_0 \tilde{r}} \frac{\partial \left( R_0 \tilde{r} V_0 \tilde{V}_r \right)}{\partial \left( R_0 \tilde{r} \right)} = 0,
\]
or
\[
\frac{V_0 R_0}{U_0 \lambda} \frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_r \right) + \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{V}_r \right) = 0.
\] (3.5)

Note \( \varepsilon = \frac{R_0}{\lambda} = \frac{U_0}{V_0} \), so \( \frac{V_0 R_0}{U_0 \lambda} = 1 \). Thus (3.5) becomes

\[
\frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{V}_r \right) + \frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_x \right) = 0.
\] (3.6)

Next nondimensionalize the first momentum equation (equation one from (3.1)). Replacing all variables with the nondimensional ones (3.4), to obtain a new equation.

\[
\frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial \left( \frac{\tilde{t}}{V_0} \right)} + U_0 \tilde{V}_x \frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial \left( R_0 \tilde{r} \right)} + V_0 \tilde{V}_x \frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial \left( \lambda \tilde{x} \right)} + \frac{1}{\rho} \frac{\partial \left( \rho V_0^2 \tilde{p} \right)}{\partial \left( \lambda \tilde{x} \right)}
\]
\[
= \nu \left[ \frac{\partial^2 \left( V_0 \tilde{V}_x \right)}{\partial \left( R_0 \tilde{r} \right)^2} + \frac{1}{R_0^2 \tilde{r}} \frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial \left( R_0 \tilde{r} \right)} + \frac{\partial^2 \left( V_0 \tilde{V}_x \right)}{\partial \left( \lambda \tilde{x} \right)^2} \right],
\] (3.7)

which after simplification becomes

\[
\frac{V_0}{\lambda} \frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial \tilde{t}} + \frac{U_0 V_0}{R_0} \frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial \tilde{r}} + \frac{V_0}{\lambda} \tilde{V}_x \frac{\partial \left( V_0 \tilde{V}_x \right)}{\partial \tilde{x}} + \frac{V_0^2}{\lambda} \frac{\partial \tilde{p}}{\partial \tilde{x}}
\]
\[
= \nu \left[ \frac{V_0}{R_0^2} \frac{\partial^2 \tilde{V}_x}{\partial \tilde{r}^2} + \frac{1}{R_0^2 \tilde{r}} \frac{\partial \tilde{V}_x}{\partial \tilde{r}} + \frac{V_0}{\lambda^2} \frac{\partial^2 \tilde{V}_x}{\partial \tilde{x}^2} \right].
\] (3.8)

Next multiply (3.8) by \( \frac{\tilde{t}}{V_0^2} \) to simplify coefficients to get

\[
\frac{\tilde{V}_x}{\partial \tilde{t}} + \frac{U_0 \lambda}{V_0 R_0} \frac{\tilde{V}_x}{\partial \tilde{r}} + \tilde{V}_x \frac{\partial \tilde{V}_x}{\partial \tilde{x}} + \frac{\partial \tilde{p}}{\partial \tilde{x}} = \nu \frac{\lambda}{V_0 R_0^2} \left[ \frac{\partial^2 \tilde{V}_x}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{V}_x}{\partial \tilde{r}} + \frac{R_0^2}{\lambda^2} \frac{\partial^2 \tilde{V}_x}{\partial \tilde{x}^2} \right].
\] (3.9)
Since $\frac{U_0 \lambda}{V_0 R_0} = 1$, and $\frac{R_0^2}{\lambda^2} = \varepsilon^2 \sim O(10^{-4})$, observe that the last term on the right hand side of (3.9) is extremely small compared to the other terms, so it can safely be dropped. Now multiply by $\tilde{r}$ to have

$$\tilde{r} \frac{\partial \tilde{V}_x}{\partial t} + \tilde{r} \tilde{V}_r \frac{\partial \tilde{V}_x}{\partial \tilde{r}} + \tilde{r} \tilde{V}_x \frac{\partial \tilde{V}_x}{\partial \tilde{x}} + \tilde{r} \frac{\partial \tilde{p}}{\partial \tilde{x}} = \frac{\nu \lambda}{V_0 R_0^2} \left[ \tilde{r} \frac{\partial^2 \tilde{V}_x}{\partial \tilde{r}^2} + \frac{\partial \tilde{V}_x}{\partial \tilde{r}} \right]$$

(3.10)

Taking into account that

$$\frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{V}_r \tilde{V}_x \right) = \tilde{V}_x \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{V}_r \right) + \tilde{r} \tilde{V}_r \frac{\partial \tilde{V}_x}{\partial \tilde{r}},$$

$$\frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_x^2 \right) = \tilde{V}_x \frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_x \right) + \tilde{r} \tilde{V}_x \frac{\partial \tilde{V}_x}{\partial \tilde{x}},$$

and using the incompressibility condition, $\frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{V}_r \right) = -\frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_z \right)$, transform (3.10) into

$$\frac{\partial}{\partial t} \left( \tilde{r} \tilde{V}_x \right) + \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{V}_r \tilde{V}_x \right) + \frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_x^2 \right) + \frac{\partial}{\partial \tilde{\bar{r}}} \left( \tilde{r} \tilde{\bar{p}} \right) = \frac{\nu \lambda}{V_0 R_0^2} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{V}_x}{\partial \tilde{r}} \right).$$

(3.11)

Similarly, nondimensionalize the second momentum equation (equation 2 from (3.1)). Replacing all variables with the nondimensional ones (3.4), to obtain a new equation.

$$\frac{\partial}{\partial \left( \frac{\lambda \tilde{t}}{V_0^2} \right)} \left[ \frac{\partial}{\partial \left( \frac{\lambda \tilde{r}}{V_0} \right)} \left( U_0 \tilde{V}_r \right) + U_0 \tilde{V}_r \frac{\partial}{\partial \left( \frac{R_0 \tilde{r}}{R_0} \right)} \left( U_0 \tilde{V}_r \right) + V_0 \tilde{V}_x \frac{\partial}{\partial \left( \frac{\lambda \tilde{x}}{V_0} \right)} \left( U_0 \tilde{V}_r \right) + \frac{1}{\rho} \frac{\partial}{\partial \left( \frac{\lambda \tilde{r}}{R_0} \right)} \left( \rho V_0^2 \tilde{\bar{p}} \right) \right]$$

$$= \nu \left[ \frac{\partial^2}{\partial \left( \frac{\lambda \tilde{r}}{R_0} \right)^2} \left( U_0 \tilde{V}_r \right) + \frac{1}{\rho} \frac{\partial}{\partial \left( \frac{\lambda \tilde{r}}{R_0} \right)} \left( \rho V_0^2 \tilde{\bar{p}} \right) - \frac{U_0 \tilde{V}_r}{\frac{R_0^2 \tilde{r}^2}{R_0^2 \tilde{r}^2}} + \frac{\partial^2}{\partial \left( \frac{\lambda \tilde{x}}{V_0} \right)^2} \left( U_0 \tilde{V}_r \right) \right].$$
Simplifying this equation, since density is constant, obtain

\[ \frac{V_0}{\lambda} \frac{\partial (U_0 \tilde{V}_r)}{\partial t} + \frac{U_0}{R_0} \tilde{V}_r \frac{\partial (U_0 \tilde{V}_r)}{\partial \tilde{r}} + \frac{V_0}{\lambda} \tilde{V}_r \frac{\partial (U_0 \tilde{V}_r)}{\partial \tilde{x}} + \frac{V_0^2}{R_0} \frac{\partial \tilde{p}}{\partial \tilde{r}} = \nu \left[ \frac{U_0}{R_0^2} \frac{\partial^2 \tilde{V}_r}{\partial \tilde{r}^2} + \frac{U_0}{R_0} \frac{1}{\tilde{r}} \frac{\partial \tilde{V}_r}{\partial \tilde{r}} - \frac{U_0 \tilde{V}_r}{R_0^2} \frac{\partial \tilde{V}_r}{\partial \tilde{r}} + \frac{U_0 \partial^2 \tilde{V}_r}{\lambda^2 \frac{\partial \tilde{V}_r}{\partial \tilde{r}}} \right]. \tag{3.12} \]

Now multiply (3.12) by \( \frac{R_0}{r_0^2} \) to simplify the coefficients, to obtain

\[ \frac{R_0 U_0}{\lambda V_0} \frac{\partial \tilde{V}_r}{\partial t} + \frac{U_0^2}{V_0^2} \tilde{V}_r \frac{\partial \tilde{V}_r}{\partial \tilde{r}} + \frac{U_0}{\lambda V_0} \tilde{V}_r \frac{\partial \tilde{V}_r}{\partial \tilde{x}} + \frac{\partial \tilde{p}}{\partial \tilde{r}} = \nu \left[ \frac{U_0}{V_0^2 R_0} \frac{\partial^2 \tilde{V}_r}{\partial \tilde{r}^2} + \frac{U_0}{V_0^2} \frac{1}{R_0} \frac{\partial \tilde{V}_r}{\partial \tilde{r}} - \frac{U_0 \tilde{V}_r}{V_0^2 R_0} \frac{\partial \tilde{V}_r}{\partial \tilde{r}} + \frac{U_0 R_0 \partial^2 \tilde{V}_r}{V_0^2 \lambda^2 \frac{\partial \tilde{V}_r}{\partial \tilde{r}}} \right]. \tag{3.13} \]

Since \( \varepsilon = \frac{R_0}{\lambda} = \frac{U_0}{V_0} \),

\[ \varepsilon^2 = \frac{R_0 U_0}{\lambda V_0} = \frac{U_0^2}{V_0^2} = \frac{U_0}{V_0^2 R_0}, \]

\[ \varepsilon^4 = \frac{U_0}{V_0^2 R_0} \frac{R_0^2}{\lambda^2} = \frac{U_0 R_0}{V_0^2 \lambda^2}. \]

Thus all of the terms other than the pressure term in (3.13) are of order \( \varepsilon^2 \) or smaller. Therefore, asymptotically (3.13) becomes quite simple, i.e.,

\[ \frac{\partial \tilde{p}}{\partial \tilde{r}} = 0, \tag{3.14} \]

which means that the nondimensional pressure is a function of the nondimensional axial position \( (\tilde{p}(\tilde{x})) \), only.

### 3.2 Averaged Equations

The two equations are still very complex, so in an effort to simplify them further, the mathematical step of averaging the equations across the cross-sectional area will
be taken. This is a purely mathematical step to take for simplification purposes, so the validity of the resulting equations will have to be evaluated to determine if this is an acceptable simplification. First, a few new terms need to be introduced. Let $\tilde{R}$ be the inner vessel radius. Then $\tilde{U}$ is the average axial velocity defined by

$$\tilde{U} = \frac{1}{\tilde{R}^2} \int_0^{\tilde{R}} 2\tilde{V}_x \tilde{r} d\tilde{r}, \quad (3.15)$$

and $\alpha$ is the “Coriolis coefficient” or the correction term which takes into account that momentum equation will represent averaged momentum, and not the actual momentum,

$$\alpha = \frac{1}{\tilde{R}^2 \tilde{U}^2} \int_0^{\tilde{R}} 2\tilde{V}_x^2 \tilde{r} d\tilde{r}. \quad (3.16)$$

Notice that when $\tilde{V}_x$ is independent of $x$, $\alpha$ is a constant.

Now assume that the streamline condition holds

$$\tilde{V}_r \bigg|_{\tilde{r} = \tilde{R}} = \frac{\partial \tilde{R}}{\partial \tilde{x}} \left[ \tilde{V}_x \right]_{\tilde{r} = \tilde{R}} + \frac{\partial \tilde{R}}{\partial \tilde{t}}, \quad (3.17)$$

which means that the fluid velocity is tangent to the surface of the wall. This also means that tangential wall displacements are prohibited.

By integrating the incompressibility condition (3.6), obtain

$$\int_0^{\tilde{R}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{V}_r \right) d\tilde{r} + \int_0^{\tilde{R}} \frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_x \right) d\tilde{r} = 0. \quad (3.18)$$

Using the Leibniz Formula, interchanging integration and differentiation in the second term in (3.18)

$$\tilde{r} \tilde{V}_r \bigg|_{\tilde{r} = \tilde{R}} + \frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{R}} \tilde{r} \tilde{V}_x d\tilde{r} - \left[ \tilde{r} \tilde{V}_x \right]_{\tilde{r} = \tilde{R}} \frac{\partial \tilde{R}}{\partial \tilde{x}} = 0,$$

$$\tilde{R} \left[ \tilde{V}_r \right]_{\tilde{r} = \tilde{R}} + \frac{\partial}{\partial \tilde{x}} \left( \frac{\tilde{R}^2}{2} \tilde{U} \right) - \tilde{R} \frac{\partial \tilde{R}}{\partial \tilde{x}} \left[ \tilde{V}_x \right]_{\tilde{r} = \tilde{R}} = 0. \quad (3.19)$$
Substitute the streamline condition (3.17) into the first term of (3.19) and simplify to

\[
\frac{\partial \tilde{R}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left( \frac{\tilde{R}^2}{2} \tilde{U} \right) = \frac{\partial}{\partial \tilde{t}} \left( \frac{\tilde{R}^2}{2} \right) + \frac{\partial}{\partial \tilde{x}} \left( \frac{\tilde{R}^2}{2} \tilde{U} \right) = 0.
\]

Finally, we obtain

\[
\frac{\partial}{\partial \tilde{t}} \left( \tilde{R}^2 \right) + \frac{\partial}{\partial \tilde{x}} \left( \tilde{R}^2 \tilde{U} \right) = 0. \tag{3.20}
\]

Now integrate the First Momentum Equation (3.11) to obtain

\[
\int_0^\tilde{R} \frac{\partial}{\partial \tilde{t}} \left( \tilde{r} \tilde{V}_x \right) \, d\tilde{r} + \left[ \tilde{r} \tilde{V}_x \tilde{V}_x \right]_{\tilde{r}=\tilde{R}} + \int_0^\tilde{R} \frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{V}_x^2 \right) \, d\tilde{r} + \int_0^\tilde{R} \frac{\partial}{\partial \tilde{x}} \left( \tilde{r} \tilde{\rho} \right) \, d\tilde{r} = \frac{\nu \lambda}{V_0 \tilde{R}_0^2} \tilde{R} \left[ \frac{\partial \tilde{V}_x}{\partial \tilde{r}} \right]_{\tilde{r}=\tilde{R}}.
\]

Again, use the Leibniz Formula to interchange integration and differentiation for the second and third integrals, and have

\[
\frac{\partial}{\partial \tilde{t}} \int_0^\tilde{R} \tilde{r} \tilde{V}_x \, d\tilde{r} - \tilde{R} \left[ \tilde{V}_x \right]_{\tilde{r}=\tilde{R}} \frac{\partial \tilde{R}}{\partial \tilde{t}} + \tilde{R} \left[ \tilde{V}_x \tilde{V}_x \right]_{\tilde{r}=\tilde{R}} + \frac{\partial}{\partial \tilde{x}} \int_0^\tilde{R} \tilde{r} \tilde{V}_x^2 \, d\tilde{r} - \tilde{R} \left[ \tilde{V}_x^2 \right]_{\tilde{r}=\tilde{R}} \frac{\partial \tilde{R}}{\partial \tilde{x}} + \int_0^\tilde{R} \tilde{r} \tilde{\rho} \, d\tilde{r} = \frac{\nu \lambda}{V_0 \tilde{R}_0^2} \tilde{R} \left[ \frac{\partial \tilde{V}_x}{\partial \tilde{r}} \right]_{\tilde{r}=\tilde{R}}.
\]

Remembering that pressure (\( \tilde{\rho} \)) is independent of radius (\( \tilde{r} \)), modify this equation

\[
\frac{\partial}{\partial \tilde{t}} \int_0^\tilde{R} \tilde{r} \tilde{V}_x \, d\tilde{r} + \frac{\partial}{\partial \tilde{x}} \int_0^\tilde{R} \tilde{r} \tilde{V}_x^2 \, d\tilde{r} + \frac{\partial \tilde{\rho}}{\partial \tilde{x}} \int_0^\tilde{R} \tilde{r} \, d\tilde{r} + \tilde{R} \left[ \tilde{V}_x \right]_{\tilde{r}=\tilde{R}} \left[ \frac{\partial \tilde{R}}{\partial \tilde{t}} \right]_{\tilde{r}=\tilde{R}} - \frac{\partial \tilde{R}}{\partial \tilde{t}} \left[ \tilde{V}_x \right]_{\tilde{r}=\tilde{R}} - \tilde{R} \left[ \frac{\partial \tilde{R}}{\partial \tilde{x}} \right]_{\tilde{r}=\tilde{R}} = \frac{\nu \lambda}{V_0 \tilde{R}_0^2} \tilde{R} \left[ \frac{\partial \tilde{V}_x}{\partial \tilde{r}} \right]_{\tilde{r}=\tilde{R}}.
\]

Note that the last term on the left hand side of the previous equation is equal to zero due to the Streamline Condition (3.17).
\[ \frac{\partial}{\partial t} \int_0^R \tilde{r} V_x \, d\tilde{r} + \frac{\partial}{\partial \tilde{x}} \int_0^R \tilde{r} \tilde{V}_x^2 \, d\tilde{r} = \nu \lambda \frac{R^2}{V_0 R_0^3} \tilde{R} \frac{\partial \tilde{V}_x}{\partial \tilde{r}} \bigg|_{\tilde{r}=\tilde{R}}. \] 

Therefore the averaged, nondimensionalized, reduced equations are

\[ \frac{\partial}{\partial t} \left( \frac{\tilde{R}^2}{2} \tilde{U} \right) + \frac{\partial}{\partial \tilde{x}} \left( \frac{\tilde{R}^2 \tilde{U}^2}{2} \alpha \right) + \frac{\tilde{R}^2}{2} \frac{\partial \tilde{p}}{\partial \tilde{x}} = \frac{\nu \lambda}{V_0 R_0^3} \tilde{R} \frac{\partial \tilde{V}_x}{\partial \tilde{r}} \bigg|_{\tilde{r}=\tilde{R}}. \] 

(3.21)

3.3 Reduced, Averaged Equations in Dimensional Form

Rewrite system (3.22) in the dimensional variables. Note \( \tilde{R} = \frac{R}{R_0} \). Thus

\[ U = \frac{1}{R^2} \int_0^R 2r V_x \, dr, \quad U = V_0 \tilde{U}, \]

(3.23)

\[ \alpha = \frac{1}{R^2 U^2} \int_0^R 2V_x^2 r \, dr. \]

Then the equations in dimensional variables are

\[ \frac{\partial \left( \frac{R^2 U}{R_0^2} \right)}{\partial \left( \frac{U}{V_0} \right)} + \frac{\partial \left( \frac{R^2 U}{R_0^2} V_0 \right)}{\partial \left( \frac{V_0}{U} \right)} = 0, \]

\[ \frac{\partial}{\partial \left( \frac{U}{V_0} \right)} \left( \frac{R^2 U}{2R_0^2 V_0} \right) + \frac{\partial}{\partial \left( \frac{V_0}{U} \right)} \left( \frac{R^2 U^2}{2R_0^2 V_0^2} \alpha \right) + \frac{R^2}{2R_0^2} \frac{\partial \left( \frac{\rho \tilde{V}_x}{\tilde{r}} \right)}{\partial \left( \frac{\tilde{r}}{R_0} \right)} = \frac{\nu \lambda}{V_0 R_0^3} \frac{R}{R_0} \frac{\partial \left( \frac{V_0}{V_0} \right)}{\partial \left( \frac{r}{R} \right)} \bigg|_{\tilde{r}=\tilde{R}}. \]

Simplifying, this system reduces to

\[ \frac{\partial R^2}{\partial t} + \frac{\partial (R^2 U)}{\partial x} = 0, \]

\[ \frac{\partial}{\partial t} (R^2 U) + \frac{\partial}{\partial x} (R^2 U^2 \alpha) + \frac{R^2}{\rho} \frac{\partial p}{\partial x} = 2\nu R \frac{\partial V_x}{\partial r} \bigg|_{r=R}. \] 

(3.24)
3.4 The Viscous Term

Assuming that $V_x$ is independent of $x$, then $a$ is constant. A typical modified Poiseuillian velocity profile can be represented by the following empirical formula

$$V_x = \frac{\gamma + 2}{\gamma} U \left[1 - \left(\frac{r}{R}\right)^\gamma\right],$$

(3.25)

where $\gamma$ is a parameter describing the bluntness of the velocity profile. Substituting (3.25) into the second equation of (3.23), one obtains

$$a = \frac{1}{R^2 U^2} \int_0^R 2V_x^2 r \, dr$$

$$= \frac{1}{R^2 U^2} \int_0^R 2 \left(\frac{\gamma + 2}{\gamma}\right)^2 U^2 \left[1 - \left(\frac{r}{R}\right)^\gamma\right]^2 r \, dr$$

$$= \frac{2}{R^2} \left(\frac{\gamma + 2}{\gamma}\right)^2 \int_0^R \left(1 - \left(\frac{r}{R}\right)^\gamma\right)^2 r \, dr.$$

To complete integration, let $\xi = \frac{r}{R}$. Then

$$a = 2 \left(\frac{\gamma + 2}{\gamma}\right)^2 \int_0^1 (\xi - 2\xi^{\gamma+1} + \xi^{2\gamma+1}) \, d\xi$$

$$= 2 \left(\frac{\gamma + 2}{\gamma}\right)^2 \left(\frac{\gamma^2}{2(\gamma + 2)(\gamma + 1)}\right)$$

$$= \frac{\gamma + 2}{\gamma + 1}.$$

(3.26)

Thus there is a relationship between $\gamma$ and $a$, where $\gamma = \frac{2 - a}{a - 1}$.

Using (3.26) we can examine the right hand side of the first momentum equation (3.24), and have for $a$,

$$2\nu R \left[\frac{\partial V_x}{\partial r}\right]_{r=R} = 2\nu R \left[\frac{\gamma + 2}{\gamma} U \left(-\frac{\gamma}{R} \left(\frac{r}{R}\right)^{\gamma-1}\right)\right]_{r=R}$$

$$= 2\nu R \left[\frac{\gamma + 2}{\gamma} U \left(-\frac{\gamma}{R}\right)\right]$$

$$= -2\nu U (\gamma + 2).$$

(3.27)
Note $\gamma + 2 = \frac{2 - \alpha}{\alpha - 1} + 2 = \frac{\alpha}{\alpha - 1}$.

### 3.5 Dimensional Equations in terms of Conserved Quantities

Let $A = R^2$ be the (scaled) cross-sectional area, and $m = AU$ be the momentum based on the averaged velocity. Then the incompressibility condition and the first momentum equation (3.24), with (3.27) become

$$\frac{\partial A}{\partial t} + \frac{\partial m}{\partial x} = 0,$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\alpha m^2}{A} \right) + \frac{A}{\rho} \frac{\partial p}{\partial x} = -2 \left( \frac{\alpha}{\alpha - 1} \right) \nu \frac{m}{A}. \quad (3.28)$$

Assume the blood vessels are modeled by the “independent ring model.” This model assumes that the force exerted on the vessel wall is the pressure of the blood. In other words, the effect of shear stress is ignored. Therefore the vessel contracts and expands in rings that do not move along the length of the vessel. [13] Thus the pressure can be represented as a function of the cross-sectional area of the vessel.

$$p(A) = G_0 \left( \left( \frac{A}{A_0} \right)^{\frac{1}{2}} - 1 \right), \quad (3.29)$$

where $A_0$ is the characteristic (unstressed) cross-sectional area and $G_0$ is the elasticity coefficient. To take into account that the vessel radius changes slower at higher pressures (nonlinear response) introduce $\beta$.

$$p(A) = G_0 \left( \left( \frac{A}{A_0} \right)^{\frac{\beta}{2}} - 1 \right) \quad (3.30)$$

where $\beta > 1$ describes nonlinear stress-strain response, $\beta \to \infty$ describes stiff walls, and $\beta = 2$ is a good fit for experimental data. Assume $G_0, \beta, A_0$ are constant. Also the domain is $D = \{(x, t) : 0 < x < \infty, t \geq 0\}$. 
Therefore, the system (3.28) with (3.30) becomes

\[
\begin{pmatrix}
\frac{\partial A}{\partial t} \\
\frac{\partial m}{\partial t}
\end{pmatrix} + \left[ \begin{array}{cc} 0 & 1 \\ -\frac{\alpha m^2}{A^2} + \frac{G_0 \beta}{2 \rho} \left( \frac{A}{A_0} \right)^{\frac{\beta}{2}} & \frac{2 \alpha m}{A} \end{array} \right] \begin{pmatrix}
\frac{\partial A}{\partial x} \\
\frac{\partial m}{\partial x}
\end{pmatrix} = \begin{pmatrix} 0 \\ -2 \left( \frac{\alpha}{\alpha-1} \right) \nu \frac{m}{A} + \frac{G_0 \beta}{2 \rho} \left( \frac{A}{A_0} \right)^{\frac{\beta}{2} + 1} \frac{\partial A_0}{\partial x} \end{pmatrix}.
\]

(3.31)
CHAPTER 4
NUMERICAL ANALYSIS

4.1 The System in Conservation Law Form

If any characteristics of a system cross, then a discontinuous solution develops. In fact, due to the pulsatile nature of blood flow, a unique, global $C^1$ solution does not exist. We use numerical analysis to approximate the solution in order to solve the system (3.31), where shock waves can develop.

The system (3.31) can be written in the conservation law form as

$$\frac{\partial}{\partial t} U + \frac{\partial}{\partial x} F = S, \quad (4.1)$$

where

$$U = \begin{bmatrix} A \\ m \end{bmatrix}, \quad F(U) = \begin{bmatrix} m \\ \frac{am^2}{A} + \frac{G_0\beta}{\rho(\beta+2)} \left( \frac{A}{A_0} \right)^{\frac{\beta}{2}+1} A_0 \end{bmatrix}, \quad (4.2)$$

and

$$S(U) = \begin{bmatrix} 0 \\ -2\frac{\alpha}{\alpha-1} \nu \frac{m}{A} + \frac{G_0\beta}{\rho} \left( \frac{A}{A_0} \right)^{\frac{\beta}{2}+1} \frac{\partial A_0}{\partial x} \end{bmatrix}. \quad (4.3)$$

This is an initial-boundary value problem on $D = [0, L]$. The initial conditions are $A(x, 0) = A_0(x)$ and $m(x, 0) = 0$. The boundary conditions are $U$ corresponding to a pulsatile velocity profile at $x = 0$, and a transparent boundary at $x = L$. This allows the numerics to run on a finite interval in $x$, but acts like the vessel is infinite in length, because there is nothing impeding the flow at $x = L$. 

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In order to determine an appropriate numerical method, investigate the behavior of the system. The eigenvalues of the matrix in (3.31) are

\[
\lambda = \frac{\alpha m}{A} \pm \sqrt{\alpha (\alpha - 1) \left( \frac{m}{A} \right)^2 + \frac{G_0 \beta}{2 \rho} \left( \frac{A}{A_0} \right)^{\frac{\beta}{2}}}.
\] (4.4)

It is clear that the eigenvalues are real, because \( A \) and \( m \) are real valued. Also, initially one eigenvalue is negative, and the other is positive. Due to this, in characteristic variables, there would be one wave propagating to the left and another moving to the right. Therefore, a simple upwinding method cannot capture the correct dynamics. Thus a method that uses information from both sides of a point is necessary. An example of this type of method is the two-step Lax-Wendroff method.

![Graph](image)

Figure 4-1: The curve on the left with negative slope corresponds to the negative eigenvalue, while the curve on the right represents the positive eigenvalue. Therefore, for an inflow boundary condition on the left, an inflow value can be determined at each grid point from the backward characteristic.

### 4.2 The Two-Step Lax-Wendroff Method

The two-step Lax-Wendroff method, or Richtmyer Lax-Wendroff method, can be used to numerically approximate the solution to systems of hyperbolic conservation...
laws. It is a three point method, meaning it uses information from the given point and the points to the left and right to determine the value at the next time step, as seen in Figure 4-2.

![Figure 4-2: In order to determine the value at point (2,2), the values at points (1,1), (2,1), and (3,1) are used, along with approximations in between each whole grid point.](image)

In addition to using the grid points, it also approximates the flux \( F(U) \) and the source \( S(U) \) in between each grid point. This approximation between grid points is what separates it from the Lax-Friedrichs method, and allows the Richtmyer Lax-Wendroff method to be second order accurate in both time and space. Consequently, the scheme is very good at numerically approximating smooth solutions, as seen in Figure 4-3a. Due to numerical diffusion, oscillations develop whenever the exact solution is non-smooth. Therefore, where discontinuities develop the scheme does not approximate the solution accurately, as seen in Figure 4-3b.

### 4.3 The Conservative System using Lax-Wendroff

Assume the grid is uniform, with \( \Delta x \) representing the distance between grid points in the \( x \) direction, and \( \Delta t \) the time step. Define \( U_m^n \) to be the approximation of the solution \( U(m\Delta x, n\Delta t) \). Then the solution is found using
(a) The solution of a first order wave equation with an initial condition of a gaussian pulse is represented in red, while the numerical solution is in green.

(b) The solution of a first order wave equation with an initial condition of a unit step function is represented in red, while the numerical solution is in green.

Figure 4-3: Richtmyer Lax-Wendroff solutions to the wave equation.

\[ U_{m+1}^n = U_m^n - \frac{\Delta t}{\Delta x} \left( F(U_{m+\frac{1}{2}}^n) - F(U_{m-\frac{1}{2}}^n) \right) + \frac{\Delta t}{2} \left( S(U_{m+\frac{1}{2}}^n) + S(U_{m-\frac{1}{2}}^n) \right) \quad (4.5) \]

where

\[ U_j^{n+\frac{1}{2}} = \frac{U_j^n + U_{j-\frac{1}{2}}^n}{2} + \frac{\Delta t}{2} \left( - \frac{F(U_{j+\frac{1}{2}}^n) - F(U_{j-\frac{1}{2}}^n)}{\Delta x} + \frac{S(U_{j+\frac{1}{2}}^n) + S(U_{j-\frac{1}{2}}^n)}{2} \right) \quad (4.6) \]

for \( j = m + \frac{1}{2} \) and \( j = m - \frac{1}{2} \).

This method is stable when the CFL condition holds:

\[ \max |\lambda_{1,2}| \frac{\Delta t}{\Delta x} = \max \left| \frac{\alpha m}{A} \pm \sqrt{\alpha (\alpha - 1) \left( \frac{m}{A} \right)^2 + \frac{G_0 \beta}{2 \rho} \left( \frac{A}{A_0} \right)^2} \right| \frac{\Delta t}{\Delta x} < 1. \quad (4.7) \]

In order to explicitly determine the boundary condition at \( x = 0 \), use the first equation in the system (4.1). Approximate the time derivative using a simple forward difference, and the spatial derivative using a modified forward difference:
\[ A_t + m_x = 0, \quad (4.8) \]

\[ \frac{A_j^{n+1} - A_j^n}{\Delta t} + \frac{4m_j^{n+1} - m_j^{n+2} - 3m_j^n}{2\Delta x} = 0. \quad (4.9) \]

Therefore, the update scheme to determine the value of \( A \) at the boundary points at subsequent time steps is

\[ A_j^{n+1} = A_j^n - \frac{\Delta t}{2\Delta x} \left( 4m_j^{n+1} - m_j^{n+2} - 3m_j^n \right). \quad (4.10) \]

Note the truncation error for the time derivative is \( O(\Delta t) \) while the truncation error for the spatial derivative is \( O(\Delta x^3) \). Decreasing the temporal truncation error would mean storing the values at previous time steps, which would significantly slow down the computations.

The momentum is found using a representation of the velocity profile of pulsatile blood flow \( u \) and the relationship

\[ m = A \ast u. \quad (4.11) \]

Using the method of characteristics, it is possible to determine when a shock will develop in the numerical solution. Canic has determined a formula for this system in [3]:

\[ t_s = \xi + \frac{U(t) + \sqrt{\frac{C_0 A}{\rho A_0}}}{\frac{3}{2} U'_{pul}(t)} \quad (4.12) \]

where \( t_s \) is the time at which the shock forms, \( \xi \) is the time parameter from the method of characteristics, and \( U'_{pul}(t) \) is the maximum of the derivative of the inflow, pulsatile velocity. The maximum of the derivative happens very close to the beginning of the cycle, so \( t \approx 0, \xi = 0, \) and \( U(t) = \frac{m}{A} = 0 \) from the initial condition. Thus

\[ t_s \approx \frac{\sqrt{40000/1050}}{1.5 \times 10} = 0.411 \text{s}, \quad (4.13) \]
and

\[ x_s = t_s \lambda_2 = t_s \sqrt{\frac{G_0}{\rho}} = 0.478 \times \sqrt{\frac{40000}{1050}} = 2.54m. \tag{4.14} \]

Clearly this is beyond the physical domain of relevance, so the numerical solution will be smooth in the relevant area of concern. If the vessel walls become stiffer \((G_0\) increases) the shock will develop later, whereas if the vessel walls are less rigid the shock will form sooner. This is a snapshot of the numerical solution where it is clear that a shock has developed.

![Figure 4-4: It is clear in both plots, that a shock wave is forming. The oscillations just behind the shock are due to the Richtmyer Lax-Wendroff method not approximating the shock accurately.](image-url)
CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH AREAS

The model developed using dimensional analysis and the independent ring model appears to incorporate the elasticity of the vessel walls well. The model appears to be well behaved in the physically relevant domain of interest.

Some next steps would include determining a way of representing changes in the vessel wall elasticity along the length of the vessel. This could incorporate stents and plaque buildup. Also, it would be interesting to see how a more accurate representation of the inflow velocity would affect the overall model. In addition, the model is capable of handling vessels that are not straight in their relaxed state. It would be interesting to see how the model changes if the vessel narrows or has different areas of narrowing and widening.


% MATLAB script to use Lax-Wendroff difference
% scheme to approximate solutions of the one-way wave equation:
% \( u_t + F(u)_x = 0 \)
% with initial conditions \( u(x,0) = f(x) \)

%Initialize avifile to record plot
aviobj=avifile('Thesis.avi');
%Define the area to be recorded
hf=figure;
rect = get(hf,'Position');
rect(1:2) = [0 0];

%Physical data shared with "F", "S", "boundary"
global rho nu R0 GO beta alpha AO epsilon ;
rho = 1050; %blood density
nu = 3.2*10^-6; %viscosity
R0=0.0082; %unstressed radius
GO = 4*10^-4; %elasticity coefficient
beta = 2;
alpha = 1.1;
AO = R0^2; %unstressed cross sectional area
L = 6.0; % length of x-axis interval
T = 1.6; % time interval
M = 200; % number of grid points on x-axis
N = 300; % number of time steps
h = L/M; % delta x
k = 0.4*T/N; % delta t
lambda = k/h;
x = [0:h:L] ;

% Calculate Initial Data
u_init = zeros(2,size(x,2)) ;
u_init(1,:) = AO;
u_init(2,:) = 0;
subplot(2,1,1) ; plot( x,sqrt(u_init(1,:)),'k',x,-sqrt(u_init(1,:)),'k','LineWidth',1.9 ) ;
title('Radius of Vessel','FontSize',14) ;
xlabel('Distance along vessel','FontSize',12) ;
ylabel('Radius','FontSize',12) ;
ylim([-0.015 0.015]);
grid on
subplot(2,1,2) ; plot( x,u_init(2,:)/u_init(1,:),x',"r","LineWidth",1.5) ;
title('Velocity','FontSize',14);
xlabel('Distance along vessel','FontSize',12);
ylabel('Velocity','FontSize',12);
ylim([-0.5 1]);
grid on
G=getframe(hf,rect);
aviobj=addframe(aviobj,G);
fprintf('Strike any key to continue

') ; pause

% Initialize appropriate vectors and constants
unew_rlw = zeros(2,size(u_init,2)) ; % rlx scheme whole step
uhalf_rlw = zeros(2,size(u_init,2)) ; % rlx scheme half step
Fhalf = zeros(2,size(u_init,2)) ;
Shalf = zeros(2,size(u_init,2)) ;
uold_rlw = u_init ; % rlx scheme old values
Fnew = zeros(2,size(u_init,2)) ;
Snew = zeros(2,size(u_init,2)) ;
nsteps = fix(2*L / h) ;

for q=1:nsteps
    t=k*q ;
    % calculate half-step using Richtmyer-Lax-Wendroff scheme.
    for j=1:size(x,2)
        Fhalf(:,j) = F( uold_rlw(:,j) ) ;
        Shalf(:,j) = S( uold_rlw(:,j) ) ;
    end
    uhalf_rlw(:,1:end-1) = 0.5*( uold_rlw(:,1:end-1) + uold_rlw(:,2:end) ) - lambda*(Fhalf(:,2:end) - Fhalf(:,1:end-1)) + k*( Shalf(:,2:end) + Shalf(:,1:end-1))/2 ;
    % calculate whole-step using Richtmyer-Lax-Wendroff scheme
    for j=1:size(x,2)
        Fnew(:,j) = F( uhalf_rlw(:,j) ) ;
        Snew(:,j) = S( uhalf_rlw(:,j) ) ;
    end
    unew_rlw(:,2:end-1) = uold_rlw(:,2:end-1) - lambda.*( Fnew(:,2:end-2) - Fnew(:,1:end-3) ) + k*(Snew(:,2:end-1) + Snew(:,1:end-2))/2 ;
    unew_rlw(:,1) = boundary(h,k,t,x,uold_rlw); % boundary condition
    unew_rlw(:,end) = unew_rlw(:,end-1); % boundary condition
subplot(2,1,1);
    plot(x,sqrt(real(unew_rlw(1,:))),'k',x,-sqrt(real(unew_rlw(1,:))),'k-','LineWidth',1.9);
title('Radius of Vessel','FontSize',14);
xlabel('Distance along vessel','FontSize',12);
ylabel('Radius','FontSize',12);
ylim([-0.015 0.015]);
grid on
subplot(2,1,2) ;
plot(x,real(unew_rlw(2,:))./real(unew_rlw(1,:)),'r -','LineWidth',1.8);
title('Velocity','FontSize',14);
xlabel('Distance along vessel','FontSize',12);
ylabel('Velocity','FontSize',12);
ylim([-0.5 1]);
grid on
G=getframe(hf,rect);
aviobj=addframe(aviobj,G);
pause(.001);
uold_rlw = unew_rlw;       % update uold_rlw to new time level
end

aviobj=close(aviobj);
function [Unew]=boundary(h,k,t,x,Uold)

% Shared Constants
global rho nu R0 G0 beta alpha A0 epsilon

% Ratio of delta t to delta x
lambda2 = k/(2*h);

% Update Scheme
Unew(1) = Uold(1,1) - lambda2*(4*Uold(2,2) - Uold(2,3) - 3*Uold(2,1));

% Inflow Velocity Pulse
if (t < pi/10)
    u=sin(10*(t+k));
else
    u = 0.0;
end

% Calculate m from velocity and A
Unew(2)=u*Unew(1);
function [out]=F(U)

% Shared Constants
global rho nu r0 G0 beta alpha A0 ;

out(1)=U(2);

out(2)=alpha.*(U(2)./U(1) + G0*A0*beta/(rho*(beta+2)) ./((U(1)./A0)^(beta/2+1)) ;
end
APPENDIX D
MATLAB SOURCE FUNCTION

function [out]=S(U)

% Shared Constants
global rho nu r0 G0 beta alpha A0;

% Derivative of area is A0'(x)
% Here we are considering a straight vessel
dA0=0;

out(1)=0;

out(2)=-2*nu*alpha/(alpha-1).*U(2)./U(1) + G0*beta/(rho*(beta+2)).*
(U(1)./A0)^(beta/2+1).*dA0;

end