Spring 2011

Proof and reasoning in an inquiry-oriented class: The impact of classroom discourse

Susan D. Generazzo
University of New Hampshire, Durham

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Proof and reasoning in an inquiry-oriented class: The impact of classroom discourse

Abstract
Over the past decade, mathematics educators and researchers have become increasingly aware of the impact of social interactions on students' learning (NCTM, 2000; Bowers & Nickerson, 2001; Forman, 2003). Current research indicates that the classroom environment, including the activities and discussions that take place, can have a significant effect on the ways students make sense of mathematical concepts (Yackel, 2001). Understanding mathematics involves knowing how to make sense of key concepts through the processes of reasoning and justification. Educators and researchers agree on the importance of providing students with opportunities in class to explore, conjecture, and prove in order to promote mathematical understanding beyond procedural knowledge (Lakatos, 1976; Rasmussen & Marrongelle, 2006).

Although there are a number of studies that investigate many different aspects of classroom discourse and students' learning, there remains a need for more understanding (Franke, Kazemi & Battey, 2007). This study is aimed at investigating the nature and impact of social interactions, both teacher-student and student-student, in classroom discourse. In particular, the study seeks to gain understanding of how interactions influence students' engagement in proof and reasoning activities. In addition, the study analyzes students' argumentation schemes as they occurred in classroom discussions and during student group work.

Through the perspective that learning is both a social and an individual activity, this research focuses on the social component of the learning process as it occurs in the classroom. Ethnographic techniques of participant observation and interviews provided methods of data collection, and analysis of discourse and argumentation structures provided a way to interpret the data. This study contributes to the existing research by highlighting certain types of interactions that resulted in students contributing to proof construction and collective reasoning.

Keywords
Education, Mathematics

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PROOF AND REASONING
IN AN INQUIRY-ORIENTED CLASS:
THE IMPACT OF CLASSROOM DISCOURSE

BY

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B.S., University of Massachusetts at Lowell, 1999
M.S., Tufts University, 2003

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

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in
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May, 2011
This dissertation has been examined and approved.

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Karen J. Graham, Professor of Mathematics

Maria Basterra, Associate Professor of Mathematics

4/29/2011
Date
DEDICATION

This dissertation is dedicated to my wife and best friend, Wendy. Your love and support has sustained me in countless ways. Your belief in me never faltered, even when I doubted myself. Your constant encouragement kept me focused on achieving my goal, one page at a time. You patiently listened and offered advice through every phase of the process. I am deeply grateful to have you to share in the joy of my achievement.

This dissertation is also dedicated to my parents, Marjorie and Louis, and to Jill, Carlos, and Joseph. You instilled in me a love of learning, and helped me to realize my lifelong dream. You have been a pillar of support over the years, helping me through difficult times and celebrating with me through joyous times. I will always be grateful for all of you.
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ABSTRACT

PROOF AND REASONING IN AN INQUIRY-ORIENTED CLASS:

THE IMPACT OF CLASSROOM DISCOURSE

by

Susan Generazzo

University of New Hampshire, May, 2011

Over the past decade, mathematics educators and researchers have become increasingly aware of the impact of social interactions on students' learning (NCTM, 2000; Bowers & Nickerson, 2001; Forman, 2003). Current research indicates that the classroom environment, including the activities and discussions that take place, can have a significant effect on the ways students make sense of mathematical concepts (Yackel, 2001). Understanding mathematics involves knowing how to make sense of key concepts through the processes of reasoning and justification. Educators and researchers agree on the importance of providing students with opportunities in class to explore, conjecture, and prove in order to promote mathematical understanding beyond procedural knowledge (Lakatos, 1976; Rasmussen & Marrongelle, 2006).

Although there are a number of studies that investigate many different aspects of classroom discourse and students' learning, there remains a need for more understanding (Franke, Kazemi & Battey, 2007). This study is aimed at investigating the nature and impact of social interactions, both teacher-student and student-student, in classroom discourse. In particular, the study seeks to gain understanding of how interactions influence students' engagement in proof and reasoning activities. In addition, the study analyzes students' argumentation schemes as they occurred in classroom discussions and during student group work.

Through the perspective that learning is both a social and an individual activity, this research focuses on the social component of the learning process as it occurs in the classroom. Ethnographic techniques of participant observation and interviews provided methods of data collection, and analysis of discourse and argumentation structures provided a way to interpret the data. This study contributes to the existing research by highlighting certain types of interactions that resulted in students contributing to proof construction and collective reasoning.
CHAPTER I

INTRODUCTION OF THE RESEARCH PROBLEM

The Classroom Environment

Over the past decade, researchers in mathematics education have placed considerable attention on the importance of the classroom environment and its relation to the learning process (NCTM, 2000; Bowers & Nickerson, 2001; Forman, 2003; Sfard, 2001; Kumpulainen & Wray, 2002; Martin, McCrone, Bower & Dindyal, 2005; Truxaw & DeFranco, 2008). The National Council of Teachers of Mathematics [NCTM], in Principles and Standards for School Mathematics, recommends a classroom environment in which students are actively engaged in complex mathematical tasks, working collaboratively, and participating in class discussions. NCTM highlights social interactions as essential to learning, allowing students to assess their own thinking as well as that of their peers, recognize connections, and reorganize their knowledge. Classroom discourse is important for instruction as well, according to NCTM, providing the teacher with opportunities to realize and respond to students’ developing knowledge (NCTM, 2000).

Classroom discourse includes not only formal and informal discussion that occurs among students and the teacher, but also encompasses behaviors, gestures, and the attitudes and beliefs of the teacher and students (Gee, 2005). Researchers have investigated many different aspects of discourse, including forms of interaction
between teacher and student (Bowers & Nickerson, 2001; Goos, 2004; Martin, et al., 2005), teaching practices that promote discourse (Truxaw & DeFranco, 2008; McCrone, 2005; Morrone, Harkness, D’Ambrosio & Caulfield, 2004), and peer interactions in collaborative group work (Kumpulainen & Mutanen, 2000; Goos, Galbraith & Renshaw, 2002; Megowan & Zandieh, 2005). Other research looks at the relation between students’ participation in mathematics classrooms and students’ motivation (Jansen, 2008). Although the current body of research has made many contributions in identifying aspects of discourse that shape students’ learning, there is still need for further exploration in this area.

Proof and reasoning

Proof and reasoning is an integral part of doing and understanding mathematics. NCTM’s *Principles and Standards* considers proof and reasoning an essential component of mathematics ability that should be incorporated into instruction at all grade levels, describing reasoning mathematically as a “habit of mind” that “must be developed through consistent use in many contexts” (2000, p. 56). Understanding mathematics involves knowing how to make sense of key concepts through the processes of reasoning and justification. The fact that students struggle with proof and proving activities is well documented (Selden & Selden, 1995; Recio & Godino, 2001; Heinze & Reiss, 2009). Educators and researchers agree on the importance of providing students with opportunities in class to explore, conjecture, and prove in order to promote mathematical understanding beyond procedural knowledge of how to obtain correct answers (Lakatos, 1976; Rasmussen & Marrongelle, 2006; Herbst, 2002; NCTM, 2000).
From a student's perspective, knowing mathematics often means getting the answer right, which is usually determined by a higher authority, such as the teacher or a textbook (Lampert, 1990). To better understand the voice of authority, researchers have looked at issues of agency and accountability (Cobb, Gresalfi & Hoge, 2009) and negotiation of sociomathematical norms (Yackel, 2001; Voigt, 1995). Classroom discourse plays a prominent part in establishing the roles of teacher and students, and in developing the expectation of shared responsibility.

**Proving as a social process**

Current research gives evidence that participation in discussions involving reasoning and proof helps strengthen students' abilities to convey, understand and defend their mathematical ideas. Classroom interactions among students while conjecturing and proving allow the exchange of ideas (McCrone, 2005), and multiple perspectives help students clarify their thinking, particularly when trying to convince others with conflicting results or opinions (McCrone, 2005; Cazden, 1988; Nussbaum & Bendixen, 2003). Group discussions have also been found empirically to foster students' reasoning abilities by reflecting on aspects of their own explanations (Yackel, 2001; Weber, Maher, Powell & Lee, 2008). Thus, classroom discussion is a valuable tool for developing proving and reasoning habits of mind. The climate of the classroom, the activities, and the social interactions that take place can have a significant impact on the ways students make sense of mathematical concepts.
The Problem

Many educators have responded to suggestions for emphasizing exploration, conjecturing and students' engagement in proving activities in the classroom, and many have also increased opportunities for student interactions. Even so, classroom environments that promote exploration and interaction are rarely found in college level mathematics classrooms. Furthermore, simply encouraging discussion in class does not guarantee mathematical understanding (Weber, et al., 2008), and structuring group work into class time does not immediately result in effective collaboration (Goos, 2004). Also, theorems and proofs are typically presented in finished form, and students often do not have opportunities in class to participate in reasoning and proving. These issues lead to several questions about appropriate classroom practices, discourses and resources that are aligned with the topics discussed above. How are proving activities incorporated into class discussion and group work? What is the instructor's role in establishing dialogue among students while progressing mathematically? What are the students' responsibilities in contributing to class discussions and activities? What roles do students take on when working together in small groups, and are they different from students' roles during whole class discussions? Although there are a number of studies that investigate various aspects of classroom discourse and students' learning, there is a need for further research that will expand upon the body of current knowledge of these topics.
Research Aims

The goal of the proposed study is to investigate the classroom environment of an inquiry-based college-level geometry class, and to gain insight into how various components of the environment influence students' abilities to conjecture, justify and prove. Interactions that occur during classroom instruction can illuminate students' forms of reasoning, including discussions among students and discussions between students and the instructor. A close look at the dialogue that occurs during whole class discussions can reveal ways the instructor promotes or inhibits students' engagement in conjecturing and proving activities. Observation of small group work can highlight dynamics of successful collaboration, and which factors are evident in unsuccessful collaboration. Classroom discussions mitigated by students' use of dynamic geometry software may also be revealing, since the computer adds a sophisticated visual component to the learning process, expanding on students' means of reasoning (Yackel, Rasmussen & King, 2000). Other components of the classroom, such as the activities or tasks involved and the mathematical resources that are utilized, are also key considerations of this study. These topics are reflected in the following research questions.

The Questions

The central research question of this study is:

**How does the classroom environment shape students' abilities to reason and prove in an inquiry-based, undergraduate geometry classroom?**

To address this question it was appropriate to focus on a single mathematics classroom, and an instructor who was experienced at creating a classroom climate
in which students participate regularly in tasks and discussions involving proof and reasoning. A college level geometry class in which the instructor utilized dynamic geometry software and encouraged group work was a suitable choice for this study. In order to investigate more specific aspects of this question, it was useful to consider the following sub-questions:

1. What is the nature of participants' interactions as they engage in proof and reasoning?
2. What resources and mathematical constructs do students call upon while exploring, conjecturing, and justifying? How do students, and the classroom as a whole, determine whether or not these resources and constructs are valid, sufficient, or appropriate?
3. What kinds of mathematical activities does the class engage in? How do these activities foster or inhibit students' engagement in conjecturing, reasoning and proving? How do these activities influence the students' mathematical conceptions about proof, both individually and collectively?

**Rationale**

This section addresses some of the key concepts of the study, such as classroom discourse and inquiry-based learning, and how they are seen to relate to students' abilities to conjecture and prove. The goal of the study is to see how classroom interactions and activities in a college level mathematics class enable or constrain students' reasoning and proving abilities, and how they affect students'
Conceptions of proof. Research in mathematics education indicates some ways in which classroom interactions can play a prominent part in the process of learning.

**Sociomathematical norms**

An essential component of learning and doing mathematics is the ability to provide reasoning, justification, or validation for a mathematical claim. In the classroom, this can have many forms, particularly when students are attempting to explain their mathematical activity to each other or to the teacher. Yackel and Cobb (1996) developed the notion of sociomathematical norms, or norms pertaining specifically to mathematical activity, to describe how students develop mathematical constructs or understandings, such as what constitutes an acceptable explanation, or why one solution is more elegant than another. In a student-centered classroom, where the teacher is not considered the sole source of authority and knowledge, these norms are continually negotiated between the students and teacher.

Yackel and her colleagues found that in a class where these sociomathematical norms had been established, students regularly offered explanations for their own reasoning, considered the arguments of their peers, and contributed alternative solutions, often without the instructor's prompting (Yackel, et al., 2000). Bowers & Nickerson (2001) identified a relation between sociomathematical norms and mathematical practices that resulted in a collective shift from a procedural to a conceptual orientation. Thus, sociomathematical norms contribute to the forms of reasoning used by students, and they can also affect
students' mathematical conceptions. Although these and other studies shed light on the importance of establishing sociomathematical norms, they do not fully address certain aspects. How are sociomathematical norms established, and how are they sustained in a university mathematics course? In what ways do sociomathematical norms influence students' proving abilities and students' conceptions about what constitutes justification?

Inquiry-based instruction

Inquiry-based classrooms can provide opportunities for rich mathematical exchanges, where the negotiation of sociomathematical norms is a key contributor to students' developing autonomy in mathematics (Cobb, et al., 2009; Yackel & Cobb, 1996). Characteristics of inquiry-based classrooms include the engagement of students in exploration of open-ended or unfamiliar problems, conjecturing, making mathematical claims and defending those claims (Wilkins, 2008; NCTM, 2000). Inquiry oriented instruction emphasizes discussion, collaboration, and the consideration of other students' mathematical ideas (Goos, 2004; Lampert, 1990). Social interactions in the classroom are a key component of learning through inquiry, and students are encouraged to reason, provide mathematical arguments, and convince themselves and each other of the likelihood of a mathematical statement (Yackel & Cobb, 1996). Many studies give evidence that classes which incorporate students' exploration and inquiry can provide a variety of learning opportunities; yet there are features of this type of instruction that have not been fully addressed. What kinds of interactions in this type of setting lead to students'
advancement in conjecturing and constructing proofs? How are activities structured to enhance collaboration? What kinds of resources are utilized? How is the voice of authority determined or shared?

**Conceptual framework: An overview**

The emergent perspective, which views both social and individual aspects of learning as equally important, aligns well with the research questions guiding this study. The central question considers equally the importance of discourse, tasks, and resources. The emergent perspective addresses each of these, providing a framework of three fundamental components of a learning environment: social norms, sociomathematical norms, and classroom practices. All three components can be related to various aspects of the research questions. The evolving social and sociomathematical norms in particular are directly related to students' participation in making conjectures and validating claims, and these norms help determine the mathematical resources used by the class. Classroom mathematical practices can be linked to the activities that students engage in while exploring, conjecturing and proving, and to the developing mathematical conceptions of both individual students and the class as a whole about conjecturing and proving. Investigation of each of these components through the emergent perspective framework will inform the central research question of this study, by contributing to an understanding of the overall classroom environment.
Summary

Exploring, making conjectures, and defending one's thinking can provide students with a sense of autonomy, and teach them to look at their own reasoning and logic to gauge the accuracy of their mathematical claims. College-level geometry is an ideal setting for research on inquiry-based instruction and students' understanding of proof for many reasons. As young adults, these students have stronger linguistics with which to articulate their thinking, and they are working with a more advanced set of mathematical structures. College-level geometry offers a rich context for exploration and development of mathematical concepts, in both the familiar field of Euclidean geometry, and less known arenas such as spherical and hyperbolic geometry. Proof tends to be more rigorous at the college level, so the distinction between reasoning and proof can be more easily detected: a formal proof in mathematics is usually presented as a complete, efficient set of logically flawless statements from hypothesis to conclusion, while reasoning often occurs less formally in the process of convincing oneself or one's classmates that a statement is true, and can be messier. In a college-level geometry class where exploration is encouraged, there will be ample opportunity to observe students as they navigate between less formal reasoning to ascertain for themselves the truth of a statement, and construct formal proofs for the purpose of establishing a claim within the framework of axioms.
Overview of the Research

In this chapter, I introduced the questions guiding the study, discussed the rationale for investigating these questions, and outlined the theoretical framework that provides the basis of underlying assumptions. Chapter II provides the theoretical framework, and gives a review of some of the current literature on discourse, inquiry-based instruction and issues around the teaching and learning of proof. The literature review highlights important findings in research that helped to inform this study. This chapter also points out those areas that are underexposed by existing research, and which this study intends to address. Chapter III gives an in-depth description of the methods used to gather and transform data, including various tools of analysis selected for the study. This chapter also discusses the benefits of choosing a qualitative research design, and touches on reasons why particular choices were made throughout the data collection phase of the study, in order to provide the reader with the fullest description possible. Chapter IV presents episodes from the data, followed by discussion, analysis and interpretation. Chapter V discusses conclusions of the study, implications for the field of mathematics education, and suggests avenues for further research.
CHAPTER II

LITERATURE REVIEW

Introduction

Chapter I describes the recent interest of the mathematics education community in the climate of the classroom, discusses the importance of social interactions in the development of knowledge, and touches on some of the current research on issues surrounding mathematical proof and student learning. Chapter I also describes the area of focus for this study, which is to better understand the features of an inquiry structured learning environment that contribute to students' knowledge about proof and justification. In this chapter I first provide the theoretical perspective, which supports my belief that learning occurs as a result of participation in a social environment, and which guided my decisions about the setting, focal points, and data collection and analysis processes. Following the theoretical perspective is a review of some of the literature in mathematics education that helped inform this study. Topics discussed in the literature review include studies on classroom discourse and inquiry oriented instruction, students' proof schemes and strategies, and use of technology in the classroom.
Theoretical Framework

The underlying epistemological perspective for this study is the emergent perspective as posited by Cobb & Yackel (1995). The emergent perspective resolves two opposing theories about learning: the psychological perspective and the sociocultural perspective. Constructivist theory, the psychological perspective formalized by Piaget, holds that students construct knowledge individually while making sense of their own experiences (Piaget, 1970). The sociocultural perspective, which originated with Vygotsky, perceives learning primarily as a social activity that is influenced by historical, cultural and social conventions (Vygotsky, 1986). While each of these perspectives contributes explanatory value to some aspect of learning, each one also has limitations in that each ignores important features of the other perspectives.

The emergent perspective, which incorporates strengths of each of these perspectives, provided a useful framework for this study, since it considers several aspects of the learning environment as essential to a student’s developing knowledge. This study focuses on conjecturing, reasoning and proving activities in the socially situated context of the classroom, and learning is viewed as the act of participation in these activities. Therefore, the emergent perspective allows a framework for viewing the learning process in this environment. This perspective is described in more detail in the paragraphs that follow.

The emergent perspective sees learning as a reflexive relation that occurs between individual, psychological constructs and interactional, social domains. This reflexivity is based on the premise that neither construct (psychological or
interactional) is more prominent; they are both of equal importance. More importantly, the reflexive nature suggests each construct contributes to the development of the other. An individual's beliefs and mental constructs affect that person's contributions socially, and the overall social environment shapes the learner's developing knowledge. There are three main social constructs of the emergent perspective: social norms; sociomathematical norms; and classroom mathematical practices. These social constructs are paired with psychological constructs that describe students' individual behavior in the social culture of the classroom (see Figure 1).

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<th>Psychological Perspective</th>
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<td>Sociomathematical norms</td>
<td>Mathematical beliefs and values</td>
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<td>Classroom mathematical practices</td>
<td>Mathematical conceptions</td>
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Figure 1: Emergent perspective

Social norms are the general norms of the classroom, such as the expectations of the teacher and of the students, and what is considered appropriate behavior. Social norms are not formed solely by either the teacher or the student. Rather, they are jointly negotiated by both the teacher and the students. From the
psychological constructivist view, social norms correspond to an individual’s beliefs about one’s own role in the classroom, others’ roles, and the nature of mathematical activity. The reflexivity of social norms and an individual’s beliefs is seen in the continuous process of negotiation: “Social norms are seen to evolve as students reorganize their beliefs and conversely, the reorganization of these beliefs is seen to be enabled and constrained by the evolving social norms” (Cobb & Yackel, 1995, p. 8). Examples of social norms include the expectation that a student explain or justify his or her conclusions, that other points of view should be taken into account, and that alternative interpretations and conclusions should be considered. These norms are not specific to mathematics classes, since presumably, students would also be expected to explain their reasoning or challenge each other’s thinking in history or chemistry classes.

A second component of the social constructs in the emergent perspective is sociomathematical norms. These norms differ from social norms in that they are norms specific to mathematical concepts. Psychologically, sociomathematical norms correlate to one’s mathematical beliefs and values. Examples of sociomathematical norms include what constitutes a solution as being different, a proof as being elegant, or an explanation as being valid. As with social norms, sociomathematical norms are continuously evolving as meanings and interpretations change. When it is evident from classroom discussions and activities that students’ interpretations of a particular concept or action are aligned, this common understanding is referred to as “taken-as-shared” (Yackel, 2001).
The third social component, classroom mathematical practices, has to do with the mathematical conceptions and activities of the classroom community. Examples of classroom mathematical practices include the different representations used for mathematical concepts and the ways in which those concepts are developed. The psychological correlate to this is individual students' mathematical conceptions, and they are reflexively related to the classroom community's mathematical conceptions. As students reorganize their mathematical conceptions, this affects how they participate in class discussion and activities. Conversely, classroom activities and dialogues influence how individual students reorganize their mathematical conceptions.

The emergent perspective has provided a framework for researchers on a broad range of topics. Investigations on classroom discourse have relied on the emergent perspective to understand how the negotiation of meaning occurs between the teacher and students. Studies on discourse have also used the emergent perspective to identify patterns in classroom discussions, and to analyze how those patterns can shape students' learning. Furthermore, researchers have focused on the nature of mathematical tasks, and how those tasks influence classroom mathematical practices. These topics are discussed in more detail in the following sections.
**Classroom discourse**

Communication is recognized as an essential component of learning in all disciplines. Vygotsky and other psychologists believe cognitive functions are dependent on the social contexts in which they occur (Wertsch & Toma, 1995). Cazden (1988) asserts, “the basic purpose of school is achieved through communication” (Cazden, 1988, p. 2). According to Cazden, communication is important in any educational institution for several reasons: (1) it is the medium through which teachers convey ideas and students convey understanding; (2) it is necessary for maintaining a common purpose and focus among all participants; and (3) it enables students to express their individual identities, resolve their differences, and understand their diverse backgrounds (Cazden, 1988). Gee sees the primary function of communication as twofold: supporting the performance of individuals’ social activities and social identities; and supporting individuals’ affiliation within social institutions (Gee, 1999). Learning environments are certainly arenas in which all of these are desirable characteristics. However, communication in mathematics classrooms has not always been about students sharing ideas and discussing issues with an instructor.

Historically, the predominant method of mathematics instruction has involved a teacher standing at the front of the classroom, delivering information in a monologue lecture style. Students, on the other hand, were traditionally expected to listen attentively, take notes, and raise their hands only if they had a question. After several decades of efforts at changing the manner in which mathematics is taught, as evidenced by such documents as *Everybody Counts* (National Research Council,
1989), the publication of a series of documents by the National Council of Teachers of Mathematics (NCTM) contributed to a nationwide mathematics reform movement. Most influential of these documents was the publication in 2000 of NCTM’s *Principles and Standards for School Mathematics*, which sets forth a vision of the ideal mathematics classroom, emphasizing among other things the importance of communication of mathematical ideas. In the decade since the publication of *Principles and Standards*, there has been an increasing interest in the field of mathematics education on classroom discourse (Elbers, 2003; Engle & Conant, 2002; Forman & Ansell, 2001; McCrone, 2005; Morrone, Harkness, D'Ambrosio, & Caulfield, 2004; Speer & Wagner, 2009; Wilkins, 2008). Although there is a considerable body of research on classroom discourse in mathematics at the elementary, middle, and secondary levels, focus on discourse in collegiate mathematics is not as abundant. This might be in part because college mathematics classes that are highly participatory are not as prevalent as in lower grade levels. Especially at the university level, mathematics classes that encourage students to contribute to mathematical dialogue are hard to find; traditional forms of instruction still dominate. The importance of thoughtful discussion in learning mathematics is not, however, being overlooked by mathematics educators.

Yackel, Rasmussen and King (2000) propose that the processes of explaining one’s thinking to the class and considering other students’ reasoning can enable a student to make significant mathematical meanings and connections. Speaking about mathematics can create opportunities for reflection on what was just said, allowing the advancement of mathematical ideas. Furthermore, discussions among
students allow the exchange of ideas, which can help students clarify their own thinking. Discussions present multiple perspectives, which can expand not only a particular student’s understanding, but also the mathematical development of the classroom as a community. The communication of ideas benefits the learner in other ways; it enables the teacher to have a better sense of the student’s understanding, allowing the teacher to respond accordingly.

The expectation that a student be prepared to elaborate on how they arrived at a conclusion, or why they think a particular result is wrong, are examples of social norms in which the student is expected to communicate her or his mathematical ideas. Research shows that once students are accustomed to these norms, they often contribute mathematical ideas that are unsolicited. That is, students feel more freedom to partake in the mathematical conversation of the class, rather than waiting to merely respond to a particular question from the teacher. This has been found to happen even in classes where the students had little or no prior experience with these expectations in a class. For instance, Yackel, Rasmussen and King (2000) conducted a classroom teaching experiment in an undergraduate differential equations class in which the instructor’s intent was to foster the development of social norms of (a) providing explanations for students’ thinking and (b) making sense of other students’ reasoning. These norms were developed through the structure of a typical class, which generally began with students working collaboratively in groups on problem solving for a fixed amount of time. While the students worked, the instructor moved from group to group, inquiring the students about their approaches and providing guidance when needed. The group problem
solving sessions were followed by a class discussion, in which students presented their approaches and interpretations. The researchers found the classroom discussions to be a significant source of meaningful mathematical conversations in which students were actively engaged in listening to each other's ideas, explaining their reasoning, and either expressing agreement or challenging their peers' statements. The instructor promoted these discussions by encouraging all students to participate, and facilitated the mathematical advancement of the class by emphasizing important ideas and methods of analysis. The resulting outcome was that students offered explanations and alternative reasoning, often with no prompting from the instructor. These results were particularly interesting since the students past experiences were limited to traditional forms of instruction.

The work of Yackel, et al. (2000) discussed above, as well as other studies described in the sections to follow, shows that the establishment of certain norms in a college level classroom can lead to students freely providing justifications for their thinking and challenging each other. Establishing the expectation that students regularly engage in mathematical discussions is therefore largely determined by the ways in which the teacher interacts with the students. The purpose of the current study is to expand on these ideas by investigating the kinds of norms that developed in a college geometry class with an emphasis on proof, and how those norms influenced students' proof and justification competencies. In particular, I address the questions: What forms of interaction encouraged students' engagement in proving activities? How did the professor contribute to the development of norms leading to these interactions, and how did the students contribute? The studies
described in the next section provide a link between the instructor’s interactions with students and the creation of environments conducive to classroom discussions.

**Whole class discussions and teacher-student interactions**

Several studies have found that the development of meaningful mathematical discussions depends largely on the teachers’ pedagogical choices, with regard to the way the instructor interacts with students. In particular, when a student offers a mathematical idea or explanation, the way the teacher responds to the student can directly impact the nature of what follows. In one pattern of teacher-student interactions identified by researchers as *initiation-response-evaluation* (IRE), the teacher initiates the interaction with a question or other prompt, the student responds with an answer, and the teacher evaluates the response (Forman & Ansell, 2001). In this pattern, the verbal exchange does not generally elicit an elaboration from the student as to how they arrived at their answer or why they think that answer is correct. The answer from the student may be very brief, and the evaluative response from the instructor often conveys merely “right” or “wrong”. Furthermore, this pattern of interaction does not necessarily call on students to reflect on other students’ ideas. For these reasons, interactions of this form can be limiting in terms of the nature of conversation that results. The IRE pattern has been the most dominant pattern of discourse that remains prevalent today in mathematics classrooms in the United States and elsewhere (Franke, Kazemi & Battey, 2007).
Bowers and Nickerson (2001) expanded on the IRE pattern in a study that investigated emerging discourse structures in a college level class for prospective mathematics teachers. These researchers identified a pattern (ERE) of teacher-student interactions and student-class interactions in which the teacher elicited information, a student responded, and the teacher elaborated on the student’s response. The instructor's response of elaborating on a student’s idea, rather than simply evaluating it and moving on, helped to create more thoughtful discussions. Initially, the instructor provided all elaborations, but gradually the students began to offer longer responses that described each student’s thinking process better so that others could understand it. As the ways of communicating continued to evolve, a second trend began to appear, called a proposition-discussion (PD) pattern. In this pattern, either the instructor or a student would make a proposition, and the class would then discuss it. In the ERE pattern, the students made valuable contributions to discussions, but they were typically in direct response to a prompt made by the teacher. Furthermore, the ERE pattern did not tend to promote interactions in which students responded to each other. The PD pattern did produce occasional interactions between students, and students sometimes offered a proposition without a direct prompt from the instructor. The research by Bowers and Nickerson described here illustrates valuable forms of discussion in which students play an active part in discourse. I intend to expand on this research by examining patterns of interaction in both whole class discussions and small group interactions in which students are engaged in proof and reasoning activities.
The studies described above point to the significance of the teacher’s pedagogical choices regarding the manner in which they respond to students’ contributions during classroom discussions. In particular, through certain types of teacher utterances, social norms can evolve and an atmosphere can be created in which students give conceptual reasons for their mathematical statements (rather than play-by-play procedural accounts of how they arrived at solutions, for instance). Several researchers have analyzed specific types of utterances spoken by the instructor, which have been found to contribute to students’ participation in classroom dialogue that provide opportunities for conceptual reasoning.

*Revoicing* is one form of teacher utterance that is characteristic of classroom discourse in which students engage in mathematical discussions. Revoicing is defined as a response to a student’s explanation in which the teacher repeats, rephrases, elaborates on, or translates students’ statements (Forman & Ansell, 2001; Martin, McCrone, Bower, & Dindyal, 2005). Revoicing is seen as a way of legitimizing a student’s ideas to the student as well as to the rest of the class (Forman & Ansell, 2001). Martin, et al. (2005) expanded on the notion of revoicing, defining *rebounding* as a response to a student’s question in which the teacher repeats or rephrases a question, returning the question back to the students. Another important form of teacher utterance was described by Martin, et al. (2005) as *coaching*, in which the teacher values students’ ideas by acknowledging their contributions, pursuing strategies offered by students, praising and encouraging them to continue participating. One study that focused on a high school geometry teacher’s pedagogical choices (Martin, et al., 2005), found that through revoicing,
rebounding, and coaching, the teacher was able to engage the students in verbal reasoning. Through these forms of interactions, the teacher enabled students to construct sequences of justification in contributing to proof constructions. An analysis of several other studies found that through revoicing of students' strategies, the teacher was able to transfer some of the authority to the students (Forman & Ansell, 2001). The study by Martin, et al. (2005) discussed above is particularly relevant to the current study, since it involve students' participation in proof and reasoning as a social process. Revoicing, rebounding and coaching were useful in coding and analysis of the current study, particularly for whole class discussions. Although the students made valuable contributions to discussions in both of the studies mentioned above, all the contributions were directed to the teacher. There was very little student-to-student discourse in either the study by Forman & Ansell (2001) or the study by Martin, et al. (2005). The current study investigates the nature of discourse between students as well as teacher-student interactions.

While the studies described in the sections above illustrate ways the teacher responded to students' contributions, they do not reveal how the teacher elicited those contributions in the first place. The study discussed earlier by Yackel, et al. (2000) conveyed how the social norms of students giving explanations and considering the ideas of other students resulted from a class structured around small group work followed by a discussion of the students' work. The discourse patterns recognized in the work of Forman & Ansell (2001) and Bowers & Nickerson (2001), although they describe certain forms of verbal exchanges between the teacher and the students, are limited in that they do not directly lead to
class discussions in which the students make significant contributions. Many educators can attest to the fact that it is not necessarily easy to extract a student’s thoughts during class, especially a mathematics class. Merely posing a question to a class can result in a sea of blank faces and dead silence. The purpose of the current study is, in part, to understand how the instructor initiated and orchestrated classroom discussions to engage students in proof and reasoning. The work of Blanton, Stylaniou & David (2009), discussed in the following section, helps to inform these questions.

Prompting Student Discussion

In an attempt to understand the factors that shape classroom discourse, a study by Blanton and her colleagues focused on individual utterances made by the teacher and students during classroom discourse (Blanton, et al., 2009). In a one-year teaching experiment that studied an undergraduate discrete mathematics class, Blanton and colleagues analyzed discourse of whole class discussions. These researchers found that utterances can either act as a catalyst for discussion or a hindrance. Blanton and her colleagues identified four main classifications of utterances spoken by the instructor:

- **transactive prompts**: requests for explanation, justification, clarification, elaboration, critiques, and strategies;
- **facilitative utterances**: comments that repeat or rephrase a student’s ideas, or comments that serve to structure the conversation by summarizing, setting the pace, or redirecting focus;
• **directive utterances**: provide corrective feedback or specific information towards solving a mathematical problem;

• **didactive utterances**: statements on the nature of mathematical knowledge.

Transactive and facilitative utterances invite students’ participation, both explicitly and implicitly, whereas directive and didactive statements are seen as non-negotiable, following the more traditional instructor paradigm of “teacher-as-teller”. Blanton and her colleagues also classified five types of students’ utterances that they considered either metacognitive or transactive in nature. Building on the work of Goos, et al. (2002), Blanton et al. (2009) define *metacognitive* acts to be cases where students offered new information or assessed their own ideas or the ideas of other students. Transactive utterances were defined in the same way as for the teacher’s utterances. The five categories for students’ utterances were:

• **Proposal of a new idea**: A student offers new information that may or may not be useful in solving the problem at hand. This can include noticing a connection, suggesting a new form of representation, or elaborating on an idea in a different direction;

• **Proposal of a new strategy**: A student presents a new course of action, plan, or strategy that may or may not be useful in solving the problem at hand;

• **Contribution to or development of an idea**: an extension of an existing idea, often provided by different students than the one who offered the initial idea;

• **Transactive questions**: requests for elaboration, justification, clarification, critique or explanation of peers’ ideas;
Transactive responses: responses to transactive questions that provide elaboration, justification, clarification, critique or explanation.

A pattern of teacher utterances was identified by Blanton, et al. (2009) in class discussions involving proof construction, in which the teacher began the discussion with a facilitative statement followed by a transactive prompt. A student then responded by sharing her or his thoughts, which the teacher responded to with a facilitative statement followed by a transactive prompt. This pattern repeated, with each student response followed by a facilitative/transactive pair of utterances from the teacher. In this way, these researchers observed a trend toward increasing student participation in dialogue. The teacher's use of primarily transactive and facilitative utterances were found to effectively transfer responsibility from the instructor to students, creating opportunities for students to contribute to proof constructions in class. These classifications of utterances were useful, in addition to other codes, for the analysis of the current study. In particular, facilitative and transactive utterances were useful for the current study in analyzing the instructor's role in guiding whole class discussions involving proof and reasoning. Metacognitive and transactive student utterances also provided an important tool for coding students' participation in both whole class discussions and small group discussions.

The studies in this section provide insight into understanding how meaningful whole class discussions can be created and sustained. The studies helped to inform this dissertation by illuminating particular types of utterances that were found to be more effective in establishing a classroom environment in which students contributed thoughtful explanations and ideas, and also regarded ideas.
from their peers. The current study also seeks to understand how students learn proof and reasoning skills from these social interactions. In addition, this study takes a close look at peer interactions while students work collaboratively, and how these interactions influence students' reasoning and proving competencies. The next section presents a perspective on how an individual's learning can be advanced through interactions with others. One theory, called the zone of proximal development, has been applied to both teacher-student interactions and student-student exchanges. In particular, the work based on the zone of proximal development provides a way to analyze peer interactions and collaborative learning, and highlights characteristics of these kinds of exchanges.

Zone of proximal development

The zone of proximal development (ZPD), a theory proposed by Vygotsky, is one way to describe the increased learning potential of a student through social interactions (Goos, Galbraith & Renshaw, 2002). The ZPD is commonly acknowledged as "the distance between learners' independent performance and the higher level that can be achieved under the guidance of a more expert partner, such as an adult or more capable peer" (Goos, et al., 2002, p. 196). The learning opportunities afforded by these interactions are also dependent on other necessary features, such as a student’s possession of an adequate base of knowledge upon entering into the interaction, and the nature and goal of the activities involved (Hiebert & Grouws, 2007). The zone of proximal development describes the learning that is possible, given the appropriate supportive conditions.
A hypothetical example of the zone of proximal development is provided by Hiebert & Grouws (2007), in which a calculus lesson is taught to a group of first graders. It is not realistic or likely that the first graders will learn much about calculus, since they presumably will not have the necessary background knowledge, and so will not be able to engage in a meaningful way in any discussion or activities during the lesson. However, they may learn something about sitting still and passively listening to someone talk about something that has little meaning to them. Thus, the zone of proximal development in this example may include the potential to learn certain ways of behaving politely while someone is speaking to them, the space in this case within which these first graders could be expected to learn.

Based on the notion that learning within the ZPD is reliant on guidance from an individual possessing a higher level of knowledge, such as a teacher or peer tutor, researchers in mathematics education have used the concept of ZPD to analyze ways students advanced mathematically during classroom discussion through scaffolding instruction (Blanton, et al., 2009; Morrone, Harkness, D’Ambrosio & Caulfield, 2004; Goos, et al., 2002). The scaffolding process is defined as “involving mutual adjustment and appropriation of ideas rather than a simple transfer of information and skills from teacher to learner” (Goos, et al., 2002, p. 195). Building on the work of Goos and her colleagues (2002), Blanton and her colleagues (2009) analyzed teacher-student interactions during whole class discussions to determine how classroom discourse influenced students’ access to their ZPDs. These researchers found that through the use of primarily transactive and facilitative utterances, the teacher was able to effectively scaffold students towards proof construction.
Analysis of student utterances during these proof constructions revealed that in response to the teacher’s transactive requests and facilitative comments, students provided ideas of their own, and developed strategies that built on the ideas of other students. This study provides evidence that through this scaffolding discourse, some students accessed their ZPDs and advanced mathematically (Blanton, et al., 2009). The work of Blanton and her colleagues (2009) provides evidence that through certain forms of utterances, meaningful mathematical conversations can be created in which students contribute significantly to proof construction. The study by Blanton et al. (2009) also includes a brief discussion of students working in small groups, claiming that through public negotiation of ideas during whole class discussions, students internalized these forms of argumentation, extending them to small group discussions. The study presented by Blanton et al. (2009) claims that students’ forms of argumentation became more sophisticated over time, when compared to students’ attempts to prove prior to instruction. However, it could be argued that simply by virtue of exposure, students became more fluent in the language of proving. Furthermore, the episode cited as evidence focuses only on the types of utterances made by students during the discussion, but does not consider the mathematical content or mathematical legitimacy of their statements. It is the intent of the current study to extend the ideas presented by Blanton and Goos through further investigation of both whole class discussions and small group discussions. In particular, the current study aims to focus in part on students’ forms of reasoning during small group discussion, and how those forms of reasoning relate to successful collaboration.
Peer interactions and the collaborative ZPD

Piaget believed peer interactions to be critical to the development of new ideas, particularly in adulthood (Cazden, 1988). Discussions between students are most commonly found in classrooms where students are given opportunities to work collaboratively on tasks. Small group work is believed by some researchers to present learning opportunities for students that may not likely have formed otherwise (Goos, 2004; Vidakovic & Martin, 2004; Goos, et al., 2002). However, simply putting students together in groups to work collaboratively does not automatically result in successful learning. The view of scaffolding as a mutual, bi-directional exchange of ideas, together with the ZPD, has been modified to apply to studies of collaborative group work. In one study, a variation of the definition of ZPD, in which Vygotsky extended the idea to include interactions among peers of roughly equal status, provided a framework with which to study students’ interactions as they worked in small groups (Goos, et al., 2002). Drawing on the work of Forman and McPhail, Goos and her colleagues describe the ‘collaborative ZPD’ of peer group work, to refer to the two-way appropriation of knowledge as students negotiated alternate views to make mathematical advances (Goos, et al., 2002). In a three-year study of senior secondary school mathematics students, Goos and her colleagues (2002) observed students working together on problem solving to identify characteristics of successful collaboration and individual students’ mathematical progress as a result of the group interactions. Cases of successful collaboration indicated student exchanges containing two descriptors: these
exchanges were both transactive and metacognitive in nature. These findings echo the results of Blanton, et al. (2009), who claimed that it was through transactive reasoning in which students offered and developed ideas, and assessed the legitimacy of those ideas, and that students accessed their ZPDs and learned to engage in abstract, symbolic forms of argumentation. Metacognition refers to students' self-awareness and self-regulation of their cognitive processes. When students are working together on mathematical tasks, it is through their transactive exchanges with each other that these metacognitive thoughts become apparent to both themselves and their peers. The zone of proximal development provides a way to measure the extent to which learning takes place through metacognitive and transactive exchanges. Through the metacognitive monitoring of one's own thought processes, a student learns to critique and assess her or his ideas; that is, to rely on one's own sense of judgment, rather than the authority of the teacher. Through exchanges with peers in which ideas are exposed and evaluated, students learn to defend their own ideas and consider and evaluate feedback from their peers, developing a shared sense of responsibility for determining the legitimacy of ideas. While the teacher as the expert is most naturally the final authority in any mathematics classroom, it is important for students to learn to think for themselves and be able to determine the validity of their mathematical ideas, particularly at the college level. Classroom norms play an essential role in establishing the expectation of shared responsibility.
Classroom norms and authority

Social norms exist in every classroom; that is, normative patterns of behavior are created and established throughout the course of any class, in which there is a common understanding of what sort of behavior is expected from the students, and what can be expected from the teacher. Consider a class in which the teacher does almost all of the talking, rarely asks questions of the students, or asks them questions that are determined solely by the teacher to be either right or wrong. In a class like the one just described, a social norm likely to be formed might be that the students are only expected to give short answers to questions, and are not required to elaborate or evaluate their own ideas. In this case the teacher is clearly the authority. On the other hand, consider a class in which a social norm is established of regularly providing reasons behind students’ ideas and critiquing both their own ideas and the contributions of their peers. In a class such as this, there is a transfer of some of the authority from the teacher on to the students. This shift in responsibility from the teacher to the student is described by Blanton and her colleagues as “a continuum between authoritarian and internally persuasive dimensions” (2009, p. 298). The establishment of this type of norm sends the important message to the students that they are responsible for making sense of their ideas and evaluating the soundness of their mathematical statements (Martin et al., 2005). Cobb, Gresalfi & Hodge (2009) support this idea in defining authority in the classroom: “Authority concerns the degree to which students are given opportunities to be involved in decision making about the interpretation of tasks, the reasonableness of solution methods, and the legitimacy of solutions. Authority is
therefore about 'who's in charge' in terms of making mathematical contributions” (p. 44). Since the current study focuses on students' engagement in proof and justification, the ability to interpret, assess, and make decisions and choices about strategies or results is highly relevant. The idea of shared authority was one focal point of this dissertation, as I sought to answer several related questions:

1) Where in the spectrum of shared authority does this class fall? How is it negotiated?

2) How are students given opportunities to be involved in making mathematical decisions during whole class discussions?

3) Where does authority fall when students are working in groups? How is it negotiated?

The studies discussed above provide much information in understanding how certain forms of interaction between participants can result in mathematical discussions in which students are contributing key ideas and assessing the mathematical legitimacy of those ideas. Developing a sense of shared authority can create opportunities for those kinds of discussions. Another critical factor in establishing a classroom in which students engage in thoughtful mathematical reasoning is the mathematical practices of the classroom. This relates to the kinds of mathematical tasks, problems and activities in which the class takes part. It also includes the way concepts are introduced and developed, and the various forms of representation and other tools, such as technology, used in class. The nature of
classroom activity plays a key role in the development of the overall learning environment.

**Inquiry-Based Instruction**

Forms of participation such as active engagement in solving challenging problems, collaboration, and meaningful mathematical conversations only take place in a climate that promotes this type of behavior (NCTM, 2000; Rasmussen, 2006; Lampert, 1990). Inquiry-oriented classrooms provide such opportunities, giving students a chance to discover mathematical connections or results on their own, and attempt to decipher the truth or falsehood of their mathematical claims. Inquiry-based mathematics instruction is characterized by students' participation in meaningful mathematical problems and activities that involve conjecturing, investigating, collecting and analyzing data, reasoning, making conclusions, and communicating mathematical ideas (NCTM, 2000). Inquiry-based classrooms value discussion and collaboration; open-ended questions or unfamiliar problems are posed; students are expected to explore mathematical relations, defend mathematical claims, and consider fellow students' mathematical ideas (Goos, 2004). Complex problems and activities may be integrated with other subjects, which often reflect the real-world messiness and uncertainty of mathematical problem-solving. The kinds of tasks described above not only provide opportunities for exploratory learning; they also create opportunities for conversation in the classroom.

Many researchers have drawn on the view of learning that emphasizes
students' engagement in classroom discussion and activities as essential to learning, describing the classroom as a community of learners with its own relationships, values, and social conventions (van Oers, 2001; Goos, 2004; Forman & Ansell, 2001). Goos (2004) points out that every classroom can be viewed as a community of practice, but the kinds of practices that are established may be very different from one classroom to the next: “Teaching methods that foster learning mathematics by memorization and reproduction of procedures can be contrasted with the more open approaches in reform-oriented mathematics classrooms, where quite different learning practices such as discussion and collaboration are valued in building a climate of intellectual challenge” (Goos, 2004, p. 259). Instruction that is characterized by open-ended tasks is sometimes met with resistance, however; some parents of school children have been found to believe procedural instruction in mathematics is best (Forman & Ansell, 2001). Nevertheless, studies of elementary and middle school mathematics classrooms that have incorporated such non-traditional activities have found these environments to be effective in developing students’ abilities to reason about and make sense of their mathematical ideas.

Sociomathematical norms

Several studies have investigated inquiry-oriented classrooms to understand how social and sociomathematical norms arise in this environment (Yackel, 2001; Hershkowitz & Schwarz, 1999; Goos, 2004). Yackel documented the evolving norms of one second-grade class, in which the teacher, through explicit conversations with the children, communicated the expectation that students’ explanations and justifications were mathematically based. For instance, when one student offered
her solution and the teacher asked the rest of the class whether or not they agreed, the child misinterpreted the teacher's question and retracted her answer on the assumption that she must have been wrong. The teacher used this case as a model to communicate his expectation that students' explanations are grounded in mathematics rather than social factors. Over the course of the school year, some children came to use other students' explanations as objects of reflection, and verbally challenged the adequacy of those explanations (Yackel, 2001).

Hershkowitz and Schwarz (1999) found in a study of middle school classrooms that sociomathematical norms were developed not only from verbal interactions among students, but also from students' interactions with software tools and from their engagement in multi-phased activities. The classes in this study were part of a large scale educational project called the CompuMath Project, which incorporated collaborative group work, open-ended tasks, and the use of technology in the classroom. The use of graphing calculators and other software challenged some students' initial hypotheses, which led to the norm of testing mathematical conjectures with data obtained from computational tools. The graphing calculators enabled the students to use different representations, including numerical values, graphs and algebraically defined functions, which constituted evidence that either supported or refuted their conjectures. This study also found that the sociomathematical norm of what constitutes a good hypothesis was established through class discussions and students' engagement in various phases of tasks. For instance, one student's hypothesis was believed by her to be good at first, because it was close to the correct answer. After a class discussion in which several groups
compared the strategies they used to arrive at their own hypotheses, it was collectively established that it was the grounds on which a hypothesis was based that made it good. This discussion introduced another norm to the class: it was expected that students would have different strategies and results, that not all of those would be accurate, and that it was acceptable to discuss why some strategies or results were wrong.

**Approaches to inquiry-oriented learning**

Inquiry-based instruction has gained popularity among mathematics educators in the past two decades. Gravemeijer and his colleagues outline several methods of instruction alternative to the traditional approach, including an exploratory technology-based design, an expressive method, and a Realistic Mathematics Education approach (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). These researchers define the exploratory approach in general as one in which “students explore conventional mathematical symbolizations in experientially real settings” (p. 228). The intention of the exploratory model described in this context is to help close the gap between formal mathematics and everyday experiences. Technology-based models for exploratory instruction are designed to behave exactly according to a specific set of laws, such as a falling object that obeys Newton’s laws of motion. Exploratory models are designed with a certain endpoint in mind: in this case, that the student will make conjectures, and then test those conjectures through experimentation with the software.

One model of instruction that follows the exploratory approach, described by Gravemeijer, et al. (2000), is a program designed by Kaput called MathCars. The
model provides a simulated driving experience, including a full dashboard display, and computer-generated graphs and tables that record details such as velocity, distance and time traveled. Kaput's model was based on his view that traditional mathematics instruction kept mathematical symbols and algebraic functions isolated from students' real world experiences. For example, Gravemeijer and his colleagues cite one study in which a class of seventh graders struggled with a problem presented to them, in which a car was driving at a constant speed of 50 kilometers, since this idea did not match their everyday experiences of a car's speed naturally fluctuating over the course of the trip. Kaput emphasized that more research was needed to determine whether the activities incorporated in MathCars generated knowledge that could be applied widely to other branches of mathematics, and that the development of hypothetical learning trajectories1 was needed. In addition, Kaput's epistemological stance was primarily psychological, so his model did not take into account any learning as a result of social activity.

In contrast to the exploratory approach, the expressive approach does not have a particular endpoint as a goal, instead allowing students to invent their own symbolization and develop their own models to describe observed phenomena. The expressive approach also may consider the social roles of the teacher and the other students. Finally, although there need not be a hypothetical learning trajectory1 in place, the teacher in one case guided the class as the classroom community developed taken-as-shared goals. Where the exploratory approach begins with the

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1 A hypothetical learning trajectory is defined as "the goal for students' learning, the mathematical tasks that will be used to promote student learning, and hypotheses about the process of the students' learning" (Simon & Tzur, 2004, p. 3).
introduction of formal symbolization, and the expressive approach begins with students’ ways of formalizing, the two methods are likened to invention- versus discovery-based instruction (Gravemeijer, et al., 2000).

The Realistic Mathematics Approach (RME), on the other hand, is based on the work of Freudenthal, who believed the process of guided reinvention should be the primary focus of mathematics educators. Freudenthal believed that by engaging in the exploration and discovery of guided reinvention, students could come closer to experiencing mathematics the way it was developed historically (Gravemeijer, et al., 2000). As with the expressive approach, RME creates opportunities for students to invent their own mathematical symbols as students formalize their understandings. RME has as its foundation problem solving activities based on real life situations, and emphasizes generalization, developing certainty by making and testing conjectures, exactness and brevity (Gravemeijer, et al., 2000).

The work of Freudenthal and his colleagues has inspired many, and has sparked a flurry of research, primarily in relation to teaching mathematics to children. Although more sparse, there are some studies that have analyzed college level RME classes. One study of a college level abstract algebra class (Larsen & Zandieh, 2008) investigates students’ initial conjectures and use of examples and counter-examples. Following a method put forth by Lakatos that engages students in guided mathematical discovery through proofs and refutations, Larsen and Zandieh found evidence that an undergraduate mathematics class can successfully reinvent key concepts. Another study focusing on an undergraduate differential equations course looks at the factors influencing college teachers’ ability to conduct
inquiry-based lessons, and the forms of pedagogical knowledge needed for success in reform-oriented teaching (Wagner, Speer & Rossa, 2007). The study by Wagner and his colleagues highlights some of the challenges instructors may encounter when trying to implement a discovery-based approach for the first time. One challenge faced by the instructor in this study was how much guidance to give students working collaboratively, when the intent was for them to discover the mathematics on their own. Other difficulties included finding the best way to assess the content learned during group activities, both in terms of what was learned and the extent to which they learned it. A final source of struggle concerned the overall map of the course, in terms of which activities would lead to the development of what major concepts, and which ideas or discussions should be elaborated upon as opposed to treated lightly. Although the instructor in this study had many years’ experience teaching college level differential equations from a traditional approach, and the curriculum he followed was highly structured, these limitations suggest several possible reasons why discovery-based instruction is not dominant in college level classrooms. However, all of the challenges listed revolved primarily around the instructor’s inability to predict how students would respond to each situation, given that this was his first time teaching a discovery-based class. Wagner, et al. (2007) suggest that in order to successfully guide a mathematical discussion, an instructor should have specific objectives about the outcomes of the discussion, some expectations about the ideas students are likely to pose, and plans for a course of action in the event that the students do not come forth with the necessary ideas.
The studies discussed above support the notion that a learning environment of guided inquiry provides an atmosphere conducive of students engaging in making predictions, testing conjectures, and defending mathematical claims. While there is a whole spectrum of different types of instruction classified as guided inquiry, depending on the nature and extent of the instructor’s guidance, it is evident that classrooms possessing the characteristics of inquiry-based learning can lead to students’ participation in rich discussion involving mathematical reasoning. The purpose of the current study is to investigate a classroom having many of the characteristics of inquiry learning, to understand how the classroom mathematical practices engage students, and shape students’ competencies in proof and reasoning. The next section discusses research exposing students’ difficulties in learning proof.

Proof and reasoning

The fact that students struggle with reasoning, sense-making and proving activities is well documented in current research (Harel & Sowder, 2009; Weber, 2001; Hoyles & Kuchemann, 2002; Selden & Selden, 2003). Recent studies indicate that students lack fundamental understanding of the role of proof and of what constitutes a proof (McCrone & Martin, 2009). Chazan (1993) found that students believe empirical evidence such as measurements constitutes a proof, and that proof by deductive reasoning merely gives evidence for a single case; these students did not understand the power of the general case in the proof. Other studies have found
students think the purpose of a proof is merely to explain why a statement is true, not that it is required to convince oneself or others of the validity of a statement (Selden & Selden, 1995). A second category of research indicates that students lack ability to use deductive reasoning (Recio & Godino, 2001).

Proof writing skills

Students’ difficulties with proof writing can be due to many different aspects of proofs. Selden and Selden (2009) describe these aspects as proof structures, and introduce three main structures of proof writing: 1) a hierarchical structure; 2) a construction path; 3) formal-rhetorical and problem-centered parts. The hierarchical structure describes the logical structure of the proof, and includes subproofs and lemmas. The construction path is the means by which the proof is created. The formal-rhetorical part of proof writing refers to the need for introduction of rhetorical objects into some proofs. For instance, if a theorem says, "for all real numbers ...” then the proof should include the introduction of an arbitrary real number: “let x be a real number...” (Selden & Selden, 2009, p. 343). The construction path also includes symbol manipulations within the body of the proof. The problem-centered part of proof writing includes recognizing key ideas and connecting aspects of the proof.

The proposed study aimed to investigate students’ conceptions about proof and what constitutes proof, and how classroom activities and discussions shape those understandings. The remainder of this chapter presents two useful models for examining students’ conceptual understanding of proof and justification. The first,
Toulmin’s model, provides a way of analyzing students’ forms of reasoning. The second model defines two types of mathematical activities viewed as evidence of students’ mathematical advancement. The chapter concludes with a brief overview of the literature on the use of technology in the classroom, and the impact of technology on the process of proving.

**Toulmin’s model**

Toulmin’s model (Krummheuer, 2007) provides a useful tool for assessing students’ conceptions about proof and argumentation. The model gives a template for decomposing a proof or argument into its main components, highlighting the intended role of each statement in the overall structure of the proof. The major components of this model are a conclusion (claim/conjecture); data to support the claim; a warrant, which provides the reasoning for why the data support the claim; and backing, which provides further support for the warrant (see Figure 2). The arrows in Figure 2 may be portrayed in either direction, or they may be bidirectional. The direction of the arrows generally indicates the direction of support, but can also be used to show which component was established in what order. A bidirectional arrow, for instance, might imply that the order is not necessarily relevant to the argument. Researchers in mathematics education have used Toulmin’s model in several different ways to analyze students’ conceptions of proof. For instance, Toulmin’s model has been helpful in analyzing what students, both individually and collectively, take as sufficient evidence for their mathematical statements (Yackel, 2001). Alternately, Toulmin’s framework has been useful in
identifying structures of argumentation used by students (Knipping, 2008) and learning opportunities created from the use of warrants during classroom proving activities (Weber, Maher, Powell & Lee, 2008). Toulmin’s model has been an effective tool for analysis of the logical structure of formal proofs as well (Pedemonte, 2007).

![Toulmin's model](image)

**Figure 2: Toulmin’s model**

The distinction between argumentation and proof is considered by many mathematicians to be about the lack or presence of formal logic (Krummheuer, 1995). Toulmin defined *analytic argumentation* as a series of formal, logical deductions; in contrast, he defined *substantial argumentation* as not necessarily containing formal deductions, but as a collection of statements that support a conclusion by means of relations, qualifiers, and other forms of justification (Krummheuer, 1995). Krummheuer points out Toulmin’s belief that “a substantial argumentation should not be subordinated or related to an analytic one in the sense
that the latter is the ideal type of arguing and that one can always identify in substantial arguments the logical gulf in comparison to an analytic one. Substantial argumentation has a right by itself. By substantial argumentation a statement or decision is gradually supported" (Krummheuer, 1995, p. 236).

From a social constructivist perspective of learning, substantial argumentation is viewed as interactional, since in any form of argumentation, the statements that comprise that argument are interdependent and ineffective if taken apart (Krummheuer, 1995). Symbolic interaction, which originates from Mead, Dewey, and others, is a theoretical lens that is compatible with the emergent perspective, as it considers both an individual's cognitive constructs and social processes as components of learning (Yackel, 2001). From the symbolic interactional view of argumentation, the meaning of a mathematical argument arises from each individual's communication of his or her own ideas and interpretation of the ideas of others, as the individual attempts to understand the meanings of a peer's actions, and realigns his or her ideas accordingly (Yackel, 2001). The interactional view of argumentation together with the basic components of Toulmin's model was a useful tool in this study for viewing both individual students' conceptions of what constitutes a justification and the collective view that evolved from classroom mathematical practices. In particular, the extent to which warrants were used to validate data supporting mathematical claims was examined during class discussions and in students' work, and was key in understanding the mathematical meanings students formed as they engaged in mathematical discussions and activities involving proof and reasoning.
Horizontal and vertical mathematizing

The process of constructing a proof as a collective activity by both teacher and students is complex and multi-faceted, and the argument structures that are developed do not typically follow the logical flow of a complete, polished proof (Knipping, 2008). Engaging students in the process of proving can result in students presenting ideas that may seem illogical, but they are key components of students' understanding. Rasmussen, Zandieh, King, and Teppo (2005) developed the concepts of horizontal mathematizing and vertical mathematizing, as a tool to illuminate students' mathematical progression while engaging in different mathematical activities.

*Horizontal mathematizing* is considered any type of activity that helps to formulate a mathematical situation so that it may be analyzed. This may include, but is not limited to, experimenting, conjecturing, classifying and organizing. *Vertical mathematizing* is both grounded in and builds on horizontal activities. This may include reasoning about abstract mathematical structures, generalizing, and formalizing. The two are seen as reflexively related, where horizontal mathematizing can lead to vertical mathematizing, which can create a new mathematical reality that provides a basis for further horizontal mathematizing. In this way, each builds off the other to create a sequence of mathematical progressions. Rasmussen and his colleagues (2005) found that through vertical and horizontal mathematizing, students' mathematical abilities advanced in activities involving symbolizing, algorithmatizing, and defining.
In one undergraduate differential equations course being conducted as a teaching experiment for the study (Rasmussen, et al., 2005), the instructor followed a Realistic Mathematics Education curriculum. In this class, a phase line was initially developed, but remained unnamed, by the instructor as a response to students' mathematical reasoning. This form of symbolizing was seen as horizontal mathematizing, since the purpose of the activity was to formulate the problem situation symbolically. In a later episode, a student on an exam responded to a task the class had not yet experienced, in which he used a series of phase lines to depict multiple solutions to a differential equation problem. This student's use of phase lines was different from the instructor's, since the exam problem involved more than one solution function. Additionally, the student upon being interviewed revealed that he had an 'epiphany' when finding his solution. The researchers concluded that this student's use of phase lines could not have been a result of memorizing a procedure, but rather was the result of vertical mathematizing in which the student built on the horizontal mathematizing utilized earlier by the instructor.

In another case, students in the differential equations class were given a task involving finding solutions to a population growth problem, without being given any algorithmic approach. The students made tables and graphs of their calculations, attempting to organize the information they were gathering towards the solution. This form of algorithmatizing was perceived as horizontal mathematizing, since it served to help them formulate the problem mathematically. The students were later asked to describe their approach in a way that might help another student.
understand how to find an approximate solution to this type of problem. This task
presented the students with an opportunity to reflect on their work and try to
generalize their procedures. This form of algorithmatizing was viewed as vertical
mathematizing, since the students were building on their initial horizontal work,
advancing their knowledge by engaging in the process of developing a formal
algorithm.

In another teaching experiment of an undergraduate geometry class,
students were asked to define several geometric concepts, including a triangle. The
students discussed several possible definitions, debating over whether to include
extreme and trivial types of triangles, and whether their definition was as minimal
as possible. This form of defining was an example of horizontal mathematizing, as
students organized and clarified their criteria. The students were then presented
with the task of constructing a definition for a less familiar object, by interpreting
their definition of a planar triangle to a definition of a spherical triangle. Building on
their previous activity, the students again discussed the criteria and examined
possible cases, creating a generalized, more abstract definition. This form of defining
was classified as vertical mathematizing.

Building on the work of Tall (1992), in which he discusses the concept of
advanced mathematical thinking, Rasmussen, et al. (2005) argue for the word
‘advancing’ rather than ‘advanced’, to steer away from the evaluative nature of the
word ‘advanced’. This is built on the premise that one’s learning is never complete,
but is a continual process. More importantly, the use of the word ‘advancing’
highlights students’ progression and evolving reasoning abilities, which are
characterized by students’ total activity, not just the final stage. The use of the word ‘activity’ in place of ‘thinking’ reflects the authors’ beliefs that mathematical learning is characterized by acts of participation in different activities, in a variety of settings.

These ideas are consistent with the emergent perspective, since they view learning as an act of engagement in certain activities. The concepts of vertical and horizontal mathematizing were utilized in the current study as a way of viewing students’ mathematical progression while engaging in proving activities. In this context, proving was perceived as a form of mathematical activity that possessed both horizontal and vertical aspects.

The literature review closes with an overview of some of the research on technology in classrooms. Use of technology is often a feature of inquiry-oriented classrooms, and in spite of the ever-increasing dominance of technology in our world, incorporating computers in classrooms continues to be a subject of much debate among mathematics educators.

Technology and proof

Although the focus of this study is not centered on the role of technology in the classroom, it is an important component of the geometry class chosen as a site for the research. Dynamic geometry software (DGS) was utilized as a tool for conjecturing and exploratory activities in the class on a regular basis. The use of DGS has raised some controversy among mathematicians and mathematics educators, especially with regard to reasoning and formal proof. Some critics of DGS claim that students have difficulty recognizing the need for deductive proofs when exploring
empirical situations with the software (Yerushalmy, Chazan, & Gordon, 1993; De Villiers, 1997, 1998; Goldenberg, Cuoco, & Mark, 1998). On the contrary, a series of studies contest the idea that the use of a DGS in a geometry class reduces the need for proof. Rather, these papers provide empirical evidence that suggests tasks involving a DGS support a variety of proving activities. A few of these studies are summarized next.

One study (Mariotti, 2000) looks at the role played by the dynamic geometry software on the process of teaching and learning proof. This study finds students' views of geometry change from an intuitive one, where properties make sense based on prior knowledge or visual displays, to a theoretical one, based on proof of relevant statements. The use of the DGS contributes to this transition, according to Mariotti, by providing a 'semiotic mediation'. Although this study contributes to the current knowledge of the impact of using DGS on students' understanding of proof, it does not look specifically at how students' interactions with each other while using the DGS influence their proving abilities.

A study by Olivero (2003) looked at the interactions between students, and among students and the geometric tool they used (Cabri), during the processes of conjecturing and proving. Olivero points out that students assigned to work in groups do not all work together in the same way, and that students must develop and maintain shared language, activities, and knowledge while working on a problem together. Olivero's study found that the students' individual understandings intersected at various points, which led to a shared understanding. An interesting finding from this study was that these intersections did not
necessarily stem from well-formed, logical statements, but that the DGS acted as a medium allowing for construction of shared knowledge.

Yackel, et al. (2000) found that in a college differential equations classroom in which social norms of explaining one's thinking and making sense of other students' reasoning were established, explanations enhanced by technology were grounded in a conceptual understanding of derivative as a rate of change. Hershkowitz and Schwarz (1999) found that in an inquiry-based class in which software is utilized, social norms are constituted not only by verbal interactions between participants, but also through non-verbal interactions with the software.

The studies discussed above indicate that the use of appropriate technological tools can create further opportunities for student interaction, and can lead to the development of shared knowledge. Additionally, computers can enhance students' conceptual reasoning. The current study builds on these results by looking at the ways in which students construct shared knowledge about reasoning and proof through their interactions with Geometer's Sketchpad while working in small groups.

**Conclusion**

Although there is a considerable body of research in each of the areas of inquiry-based instruction, classroom discourse, and the process of conjecturing and proving, there is relatively little existing research that focuses on the intersection of these, particularly at the college level. Especially in light of the complex and subtle nature of human interactions and their impact on learning, there is always more to be gleaned from research along these lines. The existing research on classroom
discourse highlights significant characteristics of the instructor's role in creating meaningful classroom discussions, and provides useful tools for analyzing utterances of participants and identifying patterns of interaction. Research on inquiry-oriented learning emphasizes the importance of the types of activities and tasks utilized during class, and the ways in which concepts are introduced and developed, as influencing the development of classroom norms. Current studies on proof and reasoning indicate that proving and reasoning socially can provide students with learning opportunities; however, more research is needed to better understand how engagement in proof and reasoning advances students' abilities to reason mathematically.

The following chapter gives an in-depth account of the research approach, setting, and methods of data collection. It also provides a detailed description of the data analysis process, including documentation methods, coding categories, and a description of the process of transforming data into text.
CHAPTER III

METHODOLOGICAL FRAMEWORK

Introduction

The purpose of the present study is to understand the ways in which the complex environment of the classroom affects students’ developing abilities to conjecture, reason and prove. The central research questions together with the theoretical framework guiding this study called for a qualitative stance that would allow me to observe, record, reflect on, describe, and analyze these events. The study was conducted over eight weeks in a college level geometry class where inquiry and collaboration were promoted. After conducting a pilot study, I was able to make a more informed decision about the type of class that would be most appropriate for this study. In the following chapter I discuss the research design and setting in more detail. Also, I describe the data collection and analysis processes.

Research Approach

The underlying theoretical perspective for this study assumes that learning is a result of both social interactions and individual constructions. The central research question looks at how the classroom environment influences students’ developing reasoning abilities. Thus, a qualitative stance was a natural fit for investigation into the social world of the classroom. The guiding assumptions of the
qualitative researcher are: (1) social processes are best understood through personal experiences in natural settings; (2) engagement with others impacts what we consider as meaningful knowledge for our research; (3) research inquiries on topics that are social in nature demand sensitivity to context; (4) research inquiries into the social world require attentiveness to particulars; (5) qualitative inquiry is fundamentally interpretive; and (6) qualitative research is an inherently selective process (Schram, 2006). These assumptions helped confirm my choice of research design, and guided me in structuring the data collection techniques and data analysis strategies.

In the context of the classroom, several components must be considered: the attitudes, beliefs, and expectations of both instructor and students, the active engagement in activities and discussions of students with instructor, historical traditions, even the physical surroundings of the classroom itself, such as the placement of students and instructor and the arrangement of desks (Agar, 2006). Ethnographic methods provided a way to witness and make sense of these dynamics as they occurred within the classroom, offering a framework appropriate for this study (Emerson, Fretz, & Shaw, 1995). Schram (2006) defines ethnographic fieldwork as:

the process by which a researcher comes to discern patterns and regularities of behavior in human social activity. The process embraces multiple techniques ... and requires deep appreciation for the characteristic ethnographic tension of holding together corroborative, contrasting, and even incompatible perspectives as a necessary condition for documenting what is actually going on (p. 95).
The predominant techniques borrowed from ethnographic tradition were participant observations and interviews. Participant observation is defined as “a methodology that assumes immersion in a setting” and “requires that the researcher ... take some part in the daily activities among the people whom he or she is studying, and reconstruct their activities through the processes of inscription, transcription, and description in field notes made on the spot or soon thereafter” (Schwandt, 2007, p. 219). The purpose of participant observations is to provide a close look at, and participate in, the everyday activities and experiences of the research subjects, in order to provide a “thick description” of the setting, people and events (Emerson, Fretz, & Shaw, 1995). The purpose of interviews is in “understanding the lived experience of other people and the meaning they make of that experience” (Seidman, 2006, p. 9). Interviews provided another medium for interpreting students’ understanding as they described their experiences. Each of these techniques is described in more detail in the next section. Although ethnographic methods cannot provide a complete, undisputable description of any object of study, participant observations and interviews allowed me to immerse myself in the class, and to obtain detailed descriptions of observed activity, in order to grasp what the students experienced as meaningful and important.

**Research Setting**

The class selected as the research site for this study was a college geometry course given in the fall semester of 2009. Although I considered the possibility of conducting the study at the high school level, the college environment seemed to
lend itself well to the research topic. College level mathematics is generally more sophisticated than secondary level mathematics, and presents many opportunities for students to make connections and develop a more mature understanding than is typical of earlier grades. Geometry also seemed to be a suitable choice, because the subject matter contains many proofs that are accessible to students. The instructor created an engaging, collaborative climate in which students frequently participated in class discussion, and structured the course in a way that encouraged students' exploration, conjecturing, and validation of geometry concepts. Therefore, this class provided a fertile background for examining how students develop conjecturing and proving abilities through social interactions.

Although there is a significant amount of existing literature in the areas of students' understanding of proof, the nature/influence of classroom discourse, and inquiry-oriented learning, there is insufficient research that focuses on the intersection of these areas, particularly at the college level. The proposed study will contribute to existing research by considering this intersection. Thus, some of the key findings from current research in these areas will be used to frame a new study that will deepen our knowledge of the impact of classroom interactions, students' understanding of proof, and inquiry-based instruction in a college geometry class.

The students enrolled in the course were primarily undergraduate, pre-service mathematics teachers, in their third or fourth year of college. The chosen class was suitable for the purposes of this study for multiple reasons. Although many of the students were mathematics education majors, the emphasis of the instruction was on mathematical content versus pedagogy. The use of dynamic
geometry software facilitated students' activities, and helped to provide portraits of students' work. The arrangement of students seated three-to-four at a table, with a computer terminal on each table, fostered collaborative efforts. Having taught this course several times, the instructor was adept at getting students to engage in discussions, ask questions, and propose ideas, so there were ample opportunities for collection of rich data.

Researcher's Presence

Qualitative data are produced from social interactions and relationships created in a particular community, and the role of the researcher in observing that community is dynamic and complex. Whether a researcher chooses to engage with participants or simply observe, the researcher needs to be aware of her or his role in that community (Schram, 2006). Although I chose to be a participating observer, the level of participation was gradual. Initially, I was self-conscious about my presence in the classroom, and chose to merely observe the class from the back of the room, without interacting with students. As I became more comfortable with my role of observer, I began interacting with students by asking general questions about what they were working on or how they were progressing. As I developed rapport with the students, the level of interaction increased, and they began asking me questions occasionally.

Although I had received consent from every student to record, I also was sensitive to the use of the video camera at first, and would ask a group of students if they minded being recorded. Early on, some students made comments that
suggested they were nervous or slightly uncomfortable about being recorded, but as they grew familiar with my daily presence in the classroom, there was a noticeable shift in their attitudes, and they appeared increasingly comfortable with me observing and recording them at work. They began to banter with me, and at times seemed to even enjoy the fact that they were being observed. As I earned their trust, I developed a relationship with the students, and established myself as a member of the classroom community.

The decision of whether or not to interact with the students during observations, and to what extent, required some deliberation. The choice to be involved served two purposes: to enable me to describe, in as minute detail as possible, the ways in which students discussed, debated, and worked collaboratively during class; and to help build a relationship with the students. "(Participant observation) simply codes the assumption that the raw material of ethnographic research lies out there in the daily activities of the people you are interested in, and the only way to access those activities is to establish relationships with people, participate with them in what they do, and observe what is going on" (Agar, p. 31).

**Data Collection**

The primary techniques of data collection for the study were classroom observations and formal interviews with groups of students. Beginning at approximately three weeks into the semester, I observed classes regularly for nearly eight consecutive weeks, excluding exam days. The class met three times per week, and was eighty minutes in length. Initially I would choose a seat at one of the tables
and remain there for the duration of the class, monitoring the camera and taking notes. As I became familiar with the dynamics of the class, I gradually began engaging informally with students during class, asking them how they arrived at a particular conclusion, or to elaborate on their work. After the first few observations, I began changing my position in the room, periodically walking around during class, to gain different perspectives of the classroom or to get a closer look at students' work. Over the course of the remaining weeks, I gradually increased my involvement with the students during observations.

Data was gathered during observations by means of field notes, audio-recordings of students working in small groups, and video-recordings of the entire class. Formal interviews were also conducted with selected groups of focus students, and these were video-recorded as well. I began interviewing students after several weeks of observations, and interviewed each focus group twice. Collection of selected students' work added another component through which to gain understanding of students' proving abilities, and included homework and exams.

**Participant Observation and Focus Groups**

Classroom observations provided an essential means of gathering data in several different forms. Since it was impossible for me as sole observer to notice and document all the discussions and activities occurring simultaneously during class, it was necessary to video-record each class. I also periodically placed an audio-recorder on a table to capture dialogue of groups of students as they worked together. During observations, I took field notes by hand. My objectives in taking
field notes were twofold: first, to supplement the video-recordings with detailed
descriptions of my observations; and second, to help guide me as to what I should
pay special attention to upon viewing video- and audio-recordings later. The field
notes included jottings about the mathematical topics being explored each day, the
tasks being given to the students, and the nature of the discussion and activities.
Field notes helped answer my research questions by highlighting key incidents
involving conjecturing and proving, and by supplementing the data collected
through the recordings.

After the first few observations, I chose four groups of students to focus on
during observations, and for subsequent interviews. The groups ranged in size from
two to four students, and they were groups that shared a table and worked together
regularly during class. Since a primary focus of the study is on students' interactions, I chose those students that were demonstrably more likely to talk and engage with each other during class. The focus students displayed a range of abilities and personalities. Once the focus groups had been selected, I generally sat at one of their tables during observations, and during group work I concentrated video- and audio-recordings on these groups.

Focus of Observations

The first topical question guiding the study investigates the nature of students' interactions with each other and with the instructor. To capture the essence of these interactions as fully as possible required me witnessing them as they occurred naturally in the classroom setting. In particular, I noted more closely
those discussions revolving around exploration, conjecturing, and proving activities. These included both small group and whole class discussions. As I observed small groups, I noticed who was working together and to what extent, how the various groups interacted among themselves and with other groups, as well as how students interacted with the instructor. I was interested in the different roles students took on while working together, such as the role of idea-generator and question-asker. I wanted to know if one student seemed to have more authority in the eyes of the students than others, and how they tried to convince one another that their claims were valid. During whole class discussions, I identified different ways the instructor engaged the class in proof and reasoning, and students’ responses to the instructor’s prompts.

The second topical question looks at the resources and mathematical constructs used by students, and how they determine whether their forms of reasoning are valid. Since I was primarily interested in how these were socially negotiated, this became another focal point of the observations. During small group discussions and activities, I took note of what kinds of resources students made use of while working together, and which of those resources were deemed most reliable by students. For example, did they look to certain students as more knowing than others? Did they rely on the computer or the instructor for absolute certainty? I also looked for types of constructs students used, such as mathematical definitions or axioms, and in what ways they deemed these constructs appropriate. During class discussions, I observed the ways in which the class determined which mathematical constructs were necessary, sufficient or appropriate.
The third topical question inquires about the nature of the mathematical activities the students engage in, and how these activities influence students’ reasoning abilities. During observations, I noted the various tasks given to the class by the instructor, which included exploration and conjecturing with Geometer’s Sketchpad, brainstorming for ideas on a proof, completing a proof that the instructor set up, and presenting students’ proofs on the board. I paid closer attention to those activities involving students’ exploration, conjecturing, and proving. During proving activities, I looked for the strategies students came up with, the ways they were able to make progress, and their general approaches to proving.

Interviews

Another important component of data was obtained from formal interviews with the selected focus groups of students. According to Seidman (2006), “Interviewing provides access to the context of people’s behavior and thereby provides a way for researchers to understand the meaning of that behavior” (p. 10). The interviews were conducted by me, and were of two different types. Class-based interviews were structured based on a particular classroom observation, and took place during weeks five and six of classroom observations. Task-based interviews were based on specific activities I gave to the students, and took place during weeks seven and eight of classroom observations. I chose to interview students in groups to encourage them to talk to each other as well as to me, and to work together during the task-based interviews. Also, I kept the members of each group the same for the interviews as they had naturally occurred in class; in other words, the groups
that tended to sit together and work together during class were the same groups I selected to interview. This helped to ensure consistency, and it maintained the comfort level these groups of students had developed in working with each other. Each group of students was interviewed twice, once for each type of interview. Both types of interviews are described in more detail in sections that follow below. Interview protocols are included in the appendix.

Class-based interviews. Class-based interviews with groups of students served to clarify, and create a more complete picture of, students' experiences in class. To help accomplish this, I shared relevant video clips with the focus students to remind them of certain conversations and activities I had observed. The purpose of these interviews was also to gain insight into students' conceptions of mathematical proof. The interview questions aimed at uncovering students’ conceptions about proof in general, and also revolved around certain notable situations occurring in class that I found of interest. An example of a ‘notable situation’ was a case where the group came up with a key idea (Raman, 2003) for a proof that they then were able to construct during class. The kinds of things I inquired about in this case were how they came up with the big idea, how working together helped or hindered, and what impact working with Geometer's Sketchpad had. After having observed the class for several weeks, I began going through my data looking for a notable incident for each focus group, and directed some of the interview questions towards the selected incident. Given that the time constraints of the class and other factors restricted the amount of information I was able to obtain
while observing, these interviews served to complement, complete, or clarify any understanding I had taken away from the incident.

**Task-based interviews.** Task-based interviews with groups of students provided another opportunity for students to interact with each other and with geometry situations. This form of interview gave me a closer look at how students use discourse and other resources in the process of conjecturing and justifying claims, in a different way than with class-based interviews. The point of task-based interviews was to present the students from the focus group with a fresh problem that they presumably had not seen before, enabling me to witness the entire problem-solving process as it occurred. Focus students were encouraged to work together and to use Geometer’s Sketchpad, in order to simulate the classroom experience as closely as possible. While each group worked on the given task, I occasionally interjected with questions and feedback. The interviews were video-recorded in order to capture as much detail and verbatim dialogue as possible. Each group was given two different tasks; both of which involved exploring with Geometer’s Sketchpad, making a conjecture based on their observations, and proving that conjecture.

The level of difficulty of each task was typical of something they were likely to work on in class or on homework. The focus was not only on whether they were able to prove the claim adequately, but also on the forms of reasoning they used, how they determined whether their reasoning was valid, and how they engaged as a group while reasoning. For example, I was interested in seeing if the discussion
enabled them to reflect on their reasoning, or if it helped them consolidate their thinking, or expand their argument. The task-based interviews were designed to build on concepts familiar to the students from class, and served to shed light on how they understand and develop aspects of conjectures and proofs within the frame of the geometry situations presented to them.

Data Analysis

Analysis of data consisted of several distinct phases. First, all video and audio recordings from classroom observations and interviews were transcribed. During the process of transcribing, I frequently referred to my field notes for additional details. Once the transcriptions were completed, I coded the data using an open coding method. Open coding is a qualitative method of coding data by asking relevant questions of the data, and then creating codes according to what is found in the data. Open coding, followed by reexamining and reflecting on data, highlights important aspects of the data to the researcher, and provides new lines of inquiry that help inform the research questions (Emerson, Fretz, & Shaw; 1995). As I coded data, I found new codes to implement into the coding scheme, and made revisions on initial codes, until the coding system was robust enough to encompass the key aspects of the data. A complete list of all codes used for this study, including a brief description and source, is given in Table 1.

Once coding was completed, the next phase was a line-by-line analysis of coded data to look for emergent themes. The research questions guided me in finding pertinent themes, and these themes often overlapped and were interrelated.
As these themes developed, I collected excerpts to be included in the text. The excerpts were chosen not necessarily because they were the most interesting examples, but because they introduced more specific themes or identified significant variations in themes.

The final phase of the study was transforming the data into written text. Analysis of data continued simultaneously with the writing process. According to Emerson, Fretz, & Shaw (1995): “events and actions become meaningful in light of an emerging meaningful whole” (p. 168). As the text was being developed, I frequently returned to my data, codes, and the emerging themes to refine the analysis and verify preliminary conclusions with additional data. Miles and Huberman (1994) refer to this process as a steady movement among data collection, data reduction, data display and conclusions.

The first research question looks at the nature of social interactions, and ways students' and the instructor’s participation influenced those interactions. The examination of naturally occurring conversations in a specific context is commonly referred to as discourse analysis (Schwandt, 2007). Discourse analysis was therefore an important piece of the analytic framework. Open coding, as described by Emerson, Fretz & Shaw (1995), is a process of examining the data to look for emerging patterns, and creating codes or categories based on those patterns. These codes are then examined and analyzed for more general categories, which are scrutinized further for emerging themes. Open coding proved helpful in highlighting patterns and relationships among recurring types of interactions, and identifying ways in which students worked with each other and with the instructor.
Utterances

While analyzing classroom dialogue, specific types of utterances were found in the data that were in close alignment with frameworks developed by Blanton, Stylianou and David (2009) and Goos, et al. (2002). The classifications developed by Blanton, Goos, and their colleagues, which were discussed in detail in the previous chapter, were insightful for this study in identifying the ways students interact in pairs or groups, with the instructor, and collectively as a class. These codes were also helpful in finding specific patterns of interaction that are more conducive to advancing students' mathematical conceptions than others.

Forms of Reasoning and Proof Schemes

The second research question looks at the mathematical constructs and resources used by students, and the ways in which students assess their validity. Coding schemes that were useful in analyzing data related to this question focus on students' forms of reasoning and argumentation (Chazan, 1993; Tirosh & Stavy, 1999; Weber & Alcock, 2005; Weber, et al., 2008) and proof schemes (Harel & Sowder, 1998).

When looking at various forms of reasoning used by students, codes arose from instances where the students used various forms of reasoning to convince themselves of the validity of their conjectures (Chazan, 1993; Tirosh & Stavy, 1999). Toulmin's framework has been used by many researchers when analyzing students' argumentation, which in some cases includes implicit reference to warrants (Weber & Alcock, 2005), and in other cases explicit references (Weber, et al., 2008). Another
type of coding therefore focused on use of warrants and whether students explicitly cited warrants in their arguments. These codes helped to illuminate students’ conceptions of what constitutes evidence.

When analyzing students’ construction of proofs, one form of coding classified students’ proof schemes, which are organized into three different categories: external, empirical, and analytical (Harel & Sowder, 1998). These schemes have provided ways for researchers to analyze students’ conceptions of what constitutes proofs. Each category represents an intellectual stage of mathematical development of the student. These codes were useful in looking at students’ development of proofs, both in class and in task-based interviews, by providing a structure that frames students’ conceptions of proof.

Mathematizing

The third research question focuses on the mathematical activities in which the students were engaged, and how their participation in those activities reflected and influenced students’ mathematical conceptions. A form of coding for analyzing data related to this question is the concept of horizontal and vertical mathematizing (Rasmussen, et al., 2005). The concepts described by Rasmussen and his colleagues were useful in identifying ways students’ conceptions and abilities involving proof and conjecture were advanced through horizontal and vertical mathematizing.
Transforming the data

The analysis was performed simultaneously while transcribing and coding the data. The code schemes helped me to make connections and see relations between students' interactions and their understanding of conjecturing and proof. Data from transcribed field notes, recordings, and interviews was used to corroborate conclusions by triangulation methods as discussed by Miles & Huberman (1994).

Each class observation was transcribed by viewing the video-recording, reading over my handwritten field notes of that day, and typing a detailed description of the day’s events. Transcribing was a selective process in which I focused mainly on classroom events that involved whole class discussion, small group discussion, or group work. Discussions were transcribed verbatim, and included descriptions of the relevant activities. For example, if I was focusing on a particular group while they tried to prove a theorem, I noted what theorem they were working on. If the instructor came over and gave the group a hint, I included this in the transcription. For those portions of class when the instructor was predominantly lecturing, I either transcribed verbatim or simply summarized the event, including the topic being presented by the instructor and the activity she was engaged in (proving a theorem, introducing a new topic, describing examples, etc.)

Once I had a significant amount of transcribing completed, I began coding all relevant classroom episodes. I started the coding process with the framework developed by Blanton, et al. (2009). As I worked with these codes, new questions and new ways of looking at the data set emerged. This prompted me to create
additional codes that would help inform these questions. The coding schemes are given in Table 1, Table 2, and Table 3. Although there is some overlap in each of the research questions, codes given in Table 1 primarily correspond to the nature of interactions, those in Table 2 to the resources and mathematical constructs, and those in Table 3 to mathematical activities.

**Table 1: Nature of interactions**

<table>
<thead>
<tr>
<th>Codes for instructor utterances</th>
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<tbody>
<tr>
<td><strong>Transactive Prompts</strong></td>
</tr>
<tr>
<td>These are requests for critique, explanation, justification, clarification, elaboration, and strategies (Blanton, et al., 2009).</td>
</tr>
<tr>
<td><em>Example:</em> How could you convince me that that is true?</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Facilitative Utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>These are statements that guide discussion through revoicing, confirmation, and summarizing, or structure discussion by setting the pace and redirecting focus (Blanton, et al., 2009).</td>
</tr>
<tr>
<td><em>Example:</em> So you think we should use the equation of a line.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Directive Utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>These are statements that provide students with corrective feedback or specific information towards solving a problem (Blanton, et al., 2009).</td>
</tr>
<tr>
<td><em>Example:</em> The Pythagorean Theorem is the main idea behind this proof.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Didactive Utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>These are statements on the nature of mathematical knowledge, such as axioms and fundamental principles (Blanton, et al., 2009).</td>
</tr>
<tr>
<td><em>Example:</em> There can be more than one way of defining something in mathematics.</td>
</tr>
</tbody>
</table>
**Codes for student utterances**

**Proposal of a New Idea**
This classification applies when a student brings new information relevant to the proof being attempted, and can include a new concept that links to existing ideas, a different form of representation, or an extension of an idea that leads in a new direction (Blanton, et al., 2009).

*Example*: We know that the sides are all equal.

**Proposal of a New Plan**
This occurs when a student offers a plan or strategy that may or may not be useful in the development of the proof. This type of utterance is differentiated from the first type, proposal of a new idea, since it specifically represents a course of action (Blanton, et al., 2009).

*Example*: What if we drop a perpendicular down and show that we have congruent triangles?

**Contribution to or development of a New Idea**
This form of utterance builds on existing ideas, often made by other students, and indicates acceptance of the existing ideas by the student who makes the utterance. (Blanton, et al., 2009)

**Transactive Questions**
These are prompts for more information, such as clarification, justification, elaboration, critique, and explanation (Blanton, et al., 2009)

**Transactive Responses**
These are direct or indirect responses to explicit or implicit transactive questions. These responses serve to elaborate, justify, clarify, critique or explain the student's thinking. (Blanton, et al., 2009)

**Metacognitive Utterances**
These are statements or questions that reveal metacognitive activity, and include new information and assessments. (Goos, et al., 2002)

**Assessment**
These are statements or questions that express assessment, or request assessment, of procedures, strategies, and results.

**Role Sharing**
This code was used for interactions in which students alternated metacognitive roles such as idea-generator, calculation-checker, and procedural-assessor (Goos, et al., 2002)
Self-Disclosure
This code was used for peer interactions during group work, when students clarified, elaborated, and justified their New Ideas for the benefit of their peers (Goos, et al., 2002).

Feedback Request
This code was used for peer interactions during group work, when students sought feedback on New Ideas they proposed, and also when they asked their peers for help in finding errors by inviting critique of strategies and results (Goos, et al., 2002).

Other-monitoring
This code was used for peer interactions during group work, when students attempted to understand their peers' thinking by offering critiques, elaborating on peers' ideas, or requesting explanations (Goos, et al., 2002).

Table 2: Resources and mathematical constructs

Resources
A set of codes were used to identify any resources used by students, and included the instructor, other students, the textbook, class notes, Geometer's Sketchpad, the internet, and manipulatives.

Proof schemes and validation

External conviction
This code was used when a student's argument was built on external sources, such as ritual or form of appearance, word of authority, or a symbolic manipulation with no reference to the meaning of the symbol (Harel & Sowder, 1998).

Empirical
This code applied when students' reasoning was either inductive, i.e., involved examples, or perceptual, using rudimentary mental images (Harel & Sowder, 1998).

Analytical
This code was given when students applied logical deduction in their reasoning. This proof scheme was of two types: transformational, which is a goal-oriented operation on mathematical objects, or axiomatic, which rests on statements that are accepted as known facts (Harel & Sowder, 1998).
**Claim**  
This code applied to a statement, assertion or conclusion explicitly made by a student or the professor, or one which was implicit in the context of an argument, according to Toulmin’s model (Rasmussen and Stephan, 2008).

**Data**  
This code was given to facts or procedures provided by a student that were seen to provide evidence of a mathematical claim, according to Toulmin’s model (Rasmussen and Stephan, 2008).

**Warrant**  
This code applied to any information provided by either a student or the professor that made a connection between and the data given as evidence of the claim, according to Toulmin’s model. This included both elaborations on procedures as well as theorems, axioms, and definitions that provided support as to why the data lead to the claim (Rasmussen and Stephan, 2008).

**Backing**  
This code was given to any statement that justified why a warrant links data to a claim, or explained why an argument is valid, according to Toulmin’s model (Rasmussen and Stephan, 2008).

**Table 3: Mathematical activities**

<table>
<thead>
<tr>
<th>Horizontal mathematizing</th>
<th>Vertical mathematizing</th>
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</thead>
<tbody>
<tr>
<td>This refers to any form of activity that aids in formulating a mathematical situation so that it may be analyzed (Rasmussen, et al., 2005). Based on the work of Rasmussen and his colleagues, I created a set of codes specific to proving activities that could be considered horizontal mathematizing:</td>
<td></td>
</tr>
<tr>
<td>This form of activity builds on horizontal mathematizing, and includes reasoning about abstract structures, generalizing and formalizing. Codes were created for proving activities that were deemed to have vertical mathematizing aspects:</td>
<td></td>
</tr>
<tr>
<td>1. Drawing a picture, constructing/manipulating a figure using Geometer’s Sketchpad, or constructing/manipulating a figure using manipulatives</td>
<td></td>
</tr>
<tr>
<td>2. Listing or identifying given information needed for a proof</td>
<td></td>
</tr>
<tr>
<td>3. Determining what needs to be shown for a proof</td>
<td></td>
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<tr>
<td>4. Recalling definitions</td>
<td></td>
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<tr>
<td>5. Justifying through self-evident facts or basic properties</td>
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<tr>
<td>6. Conjecturing</td>
<td></td>
</tr>
<tr>
<td>7. Classifying or organizing</td>
<td></td>
</tr>
<tr>
<td>8. Experimenting</td>
<td></td>
</tr>
</tbody>
</table>
1. Choosing a strategy;
2. Revising a strategy;
3. Applying deductive logic;
4. Assessing progress or results;
5. Justifying by using a known theorem or previously proven result;
6. Justifying using different forms of representation or notation;
7. Choosing an appropriate definition;
8. Generalizing.

Validity

There has been much discussion among researchers over what constitutes credibility in qualitative studies (Schram, 2003; Freeman, et al., 2007; Emerson, et al., 1995). Although there is not one agreed upon set of standards, these scholars summarize several key commonalities in standards of practice among qualitative research communities, which were incorporated into this study. These standards are outlined below.

Detailed descriptions and documentation

Detailed descriptions of the research process, including difficulties encountered and decision making, demystifies the process and presents the researcher's thinking as comprehensible (Freeman, deMarrais, Preissle, Roulston, & St. Pierre, 2007). Systematic and careful documentation of all procedures (including reflection and peer review) serves to represent the relationship between data and claims, and support researcher's interpretations and assertions (Freeman, et al., 2007).
Engagement in research setting

Immersion and involvement in the research setting allows the researcher to experience events in a closer approximation to how participants experience them (Emerson, Fretz & Shaw, 1995). Maintaining a balance of authentic engagement with participants and commitment to the research agenda is essential (Schram, 2003).

Quality of video transcriptions

Transcribing of data was done entirely by me, to avoid misinterpretations and to ensure the most accurate possible representation of events. Transcriptions included not just verbatim dialogue, but also gestures, intonations, and other subtle nuances of speech such as brief hesitations or longer pauses. Although the process of video recording is necessarily selective and guided by the theoretical perspective and research questions of the study (Powell, Francisco, & Maher, 2003), the quality of transcriptions described here was as exact and genuine as possible, providing a valid representation of interactions and events observed.

During the coding phase, I occasionally viewed video recordings a second time to clarify meaning conveyed in transcripts. As discusses in earlier sections of this document, a robust set of data was collected that did not rely solely on video recordings, but also included field notes from classroom observations, collection of students’ work, and interviews. While re-viewing video recordings, I also re-viewed corresponding field notes to ensure the most accurate interpretation possible.
Consideration of strengths and limitations of the study attends to ethical
carens, issues about relationships of the researcher and participants, and the
roles of the researcher in the study (Freeman, et al., 2007). The following chapter
draws on the literature discussed in Chapter II, as well as the methods described
above, investigating the discourse of participants while engaged in activities and
discussions involving proof and reasoning. The chapter is presented in two parts:
the first part looks at whole class discussions, and the second part analyzes small
groups of students working together. Several episodes of each type of discussion are
included, followed by an analysis of the discourse, mathematical constructs, and
tasks involved.
CHAPTER IV

RESULTS AND ANALYSIS

Introduction

The purpose of this chapter is to present relevant episodes from the data, and through description and analysis investigate the central research questions of this study. Throughout the chapter, both individual utterances and whole episodes are treated as units of analysis; the utterances give a finer grain analysis and the episodes give a broader perspective. The chapter is organized into two parts. Part one illuminates whole class discussions and activities, and part two looks at students working in small groups as they engage in mathematical tasks. This manner of organization was used because each of these two group structures informed the research questions in different ways. Part one addresses each of the research questions by looking at the interactions between the professor and the students, the resources and norms of the class, and the activities in which the participants were engaged during whole class discussions. Part two analyzes predominantly student-student interactions, and the resources, norms, and activities in which students were engaged while working in small groups.

Utterances as units of analysis

In order to describe the nature of classroom discourse in general, and to analyze the influence of professor-student interactions on students’ mathematical development, a framework developed by Blanton, et al. (2009), that was influenced
by the work of Goos, et al. (2002), provided ways to categorize utterances made by
the instructor and by students. Through the perspective that learning occurs as a
result of participation in social behavior, speech plays a key role in the process of
learning in general (Blanton et al., 2009, p. 291). Thus, the utterance as a unit of
analysis allows a close inspection of the interplay between participants, and
provides a way to understand students’ developing proof and reasoning abilities.
Based on Vygotsky’s theory of the zone of proximal development, Blanton and her
colleagues established a classification system of four types of instructor utterances:
transactive prompts, facilitative utterances, didactive utterances, and directive
utterances.

As discussed in chapter II, transactive prompts are requests made by the
instructor for clarification, justification, elaboration, strategies, and critiques.
Transactive prompts play an important role in discourse, as they encourage
students’ reasoning and argumentation through explicit and specific requests for
more information. Through transactive prompts, the instructor places some
responsibility for mathematical ideas on the students.

Facilitative utterances are utterances in which the instructor guides
discussion through confirmation or re-voicing, or structures discussion by
redirecting, summarizing, or pacing. Through facilitative utterances, the instructor
accepts partial responsibility, but also places partial responsibility on to the
students. By confirming or re-voicing, the instructor is indicating approval or
acceptance of the students’ statements, but withholds from offering any further
information. These types of utterances are also a critical part of discourse, as they
promote students’ engagement in class discussion. They differ from transactive prompts in that they are not direct requests for specific types of information, but create an implicit expectation for participation.

Directive utterances are statements made by the instructor that present specific information or feedback towards a mathematical solution. In these types of statements, the instructor holds all responsibility, and there is no direct expectation that students respond. Didactive utterances are statements on the nature of mathematical knowledge. These types of utterances may be about the nature of mathematical proofs or definitions, historically established concepts, or principles and axioms, and are not negotiable.

**Episodes as units of analysis**

Episodes, or vignettes, presented in the following text were selected on the basis that they were representative of a typical day in the class, and also provided insights into the lines of inquiry of this study by highlighting recurring themes. According to Miles and Huberman (1994, p. 81), “A vignette is a focused description of a series of events taken to be representative, typical, or emblematic in the case you are doing.” Emerson, Fretz and Shaw (1995) advise that a researcher may select excerpts “because they aptly illustrate recurring patterns of behavior or typical situations in that setting” (p. 175). The episodes presented in part one of this chapter illustrate mathematical discussions during class in which the instructor and several students were involved. They were chosen because they highlight the ways discussions were typically developed during class, and the roles taken on by both
the professor and the students during discussions. The episodes also illustrate how class discourse enabled students to contribute significantly to the construction of proofs. The episodes in part two illustrate the nature of peer interactions while students worked collaboratively. Some of the episodes in part two were taken from group work that was assigned during class, and other episodes were taken from task-based interviews. The episodes in part two were selected because they reveal the ways students made mathematical advances through peer interactions.

**Part I:**

**Whole class discussions**

**Episode 1: Justifying the midpoint formula (October 19)**

The first twenty minutes of class, the professor had been talking to the students about finding distances using both a skewed coordinate system and a rectangular grid. Eventually settling on the rectangular grid, the professor had just written the following statement on the board, leaving the conclusion blank:

*Given A(x1, y1) and B(x2, y2), then the midpt. of AB has coordinates____________.*

A student quickly offered the conclusion of the statement: “X one plus x two divided by two, y one plus y two divided by two.” The professor wrote this on the board, filling in the blank at the end of the statement: \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \).

Rather than simply confirming that the student’s answer was correct and moving on, Professor Williams then asked a question, which sparked a discussion.
Prof: If I asked you to justify that, what would you do? (Pause) How can you justify that this is really the midpoint of A and B? [transactive prompt - request for justification]

Cheryl: The average of the two different points? [proposal of a new idea]

Prof: Yeah, it’s an average ... so? [transactive - request for justification]

Cheryl: Well, the average is (given?) ... an average is basically the... [transactive response]

Prof: Are we sure that it’s actually on the segment AB? [transactive-request for justification]

Cheryl: Yeah. [general confirmation]

Prof: Oh. How do you know? [transactive-request for justification, explanation]

Cheryl: Because it’s between the two... it’s between the two y’s. [transactive response]

Prof: How could you do it more rigorously? [transactive-request for justification] - (pauses for a few seconds, writes on board) Can you find a way to, um, rigorously defend that claim, Cheryl? [transactive-request for justification] And I’m not really asking you to do it. Anyone else, too. What would you do to convince me that this point, x one plus x two divided by two, y one plus y two divided by two, is really the midpoint of segment AB? [transactive-request for justification, strategies]

Rachel: Could you um, like, plot the points... (inaudible)... you could draw a right triangle? [proposal of a new plan]

Prof: Okay... so, here’s A, and this is x1, y1, here’s B, x2, y2... (drawing and labeling appropriate points on board) [facilitative – revocing]

Rachel: If you were to draw a right triangle... [transactive response]

Prof: Ok, let’s just do that (draws two lines and a right angle) so here’s B. Yup? [facilitative, transactive prompt - elaboration]

Rachel: If you add x1 and x2 ... to the other... [developing an idea]
Prof: (Pauses) So if I add x1 and x2, so x1 is this distance, x2 is ...
(draws a point x2 distance up from x1, looks over at the student)

Rachel: Yeah. [general confirmation] Doesn’t that prove it?
[transactive question]

Figure 2: Drawing based on Rachel’s suggestion

The figure the instructor drew on the board based on Rachel’s suggestion is shown in Figure 2. Cheryl responded to the instructor’s prompts with several attempts to provide a convincing argument, but she was limited to thinking about the formula as an average of the two points. Even after repeated requests from the professor for further justification, Cheryl was unable to elaborate on her thinking or find a way to more rigorously describe the mathematical situation. By asking for a more rigorous explanation, the professor was explicitly communicating that Cheryl’s ideas were not adequate justification. The transactive question (line 13), “Are we sure that it’s actually on segment AB?” was a request for a more specific form of justification, directing students towards making the connection that the point with the given coordinates is on the segment. By extending the invitation to participate to the whole class, she let the class know that it was everyone’s responsibility (lines 26-27): “And I’m not really asking you to do it. Anyone else, too.” The next
contribution, to formulate a triangle and add the components, may have eventually
led to some sort of coordinate proof, but Rachel seemed unable to pursue her idea
further, and thought her idea was sufficient, although not confident about this as
expressed by her question to the instructor in line 49. What follows is a continuation
of the discussion about justifying the midpoint formula. A few other students took
up the challenge and offered the following ideas.

Mike: Um, could you just use the distance formula? From the midpoint
to A and from the midpoint to B, equals from A to B? [proposal of a
new strategy]

Prof: Okay, so if we take this, so we could - let's call this M. So if I use,
um, find the distance from A to M, and the distance from B to M, and
add them together, then it should be the same as the distance from A
to B? [facilitative - rephrasing] That sounds like a good start!
[facilitative - coaching] Bruce?

Bruce: Well that wouldn't really...That right there wouldn't really
show that it's the midpoint, though, would it? Because M could be
closer to A, and AM plus BM would still be equal to AB. As long as M is
on that line. [transactive response - critique]

Prof: Right! [facilitative - confirming] So we could have two different
distances here that add up to AB. [facilitative - revoicing]

Bruce: But you have...you could check to see that AM = BM. [building
on a suggested idea]

Prof: Ok, so you want to check this, and, you want to check that AM is
equal to BM? So if they're the same, and they add up to AB. [facilitative
we need? [transactive prompt - request for further justification] -
(Waits several seconds) So [if you] tell me that I have these lengths
that add up to AB, and those lengths are the same... [facilitative -
revoicing]

Sam: Do you need to prove that the point's actually on AB?
[transactive question - critique]
Prof: Yeah, how do we know that that point is really on AB? Could it be somewhere else? Could it be up here? [facilitative – confirming, revoicing; transactive – request for explanation] (Draws a point above the triangle) Bruce?

Bruce: Well, I mean, I don’t know if this is what you’re looking for, but couldn’t you just come up with the formula for a line, between A and B and see if that point M, when you plug it in, if that, uh, point’s on that line? [proposal of a new plan]

The students were unsure about whether they had provided enough or whether more was needed, so several more ideas were discussed. After one student offered the triangle inequality, the professor acknowledged that his idea worked, and then reminded the class that the definition of betweenness and collinearity also explained why they had everything they needed. The professor then concluded the discussion, summarizing each student’s suggestion, and ending with the following comments.

Prof: So, if it’s between A and B, if this is the case, meaning it’s collinear with A and B. [directive] So we really had it all here. [directive] We need to show that AM plus BM is equal to AB, and that AM is equal to BM. [directive] So it’s all right here. [directive]

**Analysis of Episode 1**

**Professor utterances in episode 1**

Episode 1 illustrates a typical class discussion in which the professor called upon the class to participate in the process of proving. The discussion began with the professor soliciting ideas through a *transactive prompt for justification*. Throughout the exchange between the professor and Cheryl, Professor Williams persisted with several more transactive prompts for justification, such as in line 8:
“Yeah, it’s an average... so?” and in line 13: “Are we sure that it’s actually on the segment AB?” and line 18: “Oh. How do you know?” As other students contributed their ideas, the professor encouraged them to proceed by using facilitative revoicing and facilitative coaching. After Mike (line 51) and Bruce (line 69) had proposed using the distance formula to show the two sub-segments were equal in length and added up to the length of the whole segment, the professor again provided facilitative coaching and revoicing: “Ok, so you want to check this, and you want to check that AM is equal to BM? So if they’re the same and they add up to AB. That’s a good start!” (lines 72-74).

At that point, Professor Williams prompted: “What else do we need?” By asking this question, she was shifting responsibility to the class to determine whether they had all the necessary components of the proof outline. By posing this question, the professor was modeling an important step in the process of proof construction: determining how much was sufficient to adequately prove the theorem. When Sam asked whether they needed to show the point was actually on the line AB, the professor rebounded the question to the class: “Yeah, how do we know that that point is really on AB?” (line 83). This was another way the professor transferred responsibility back to the students. The discussion was concluded when the instructor issued the first directive statement in the entire episode: “So we really had it all here.” (line 93).
Student utterances in episode 1

The students’ utterances in this episode consisted of *transactive responses*, *transactive questions*, and *proposals of new ideas and strategies*. Although the first several contributions made by the first student were in response to repeated transactive prompts from the professor, the students seemed to need less prompting further on in the episode. For instance, it was following a facilitative coaching utterance from the professor that Bruce challenged Mike’s idea (lines 61-64): “Well that wouldn’t really... that right there wouldn’t really show that it’s the midpoint, though, would it? Because M could be closer to A, and AM plus BM would still be equal to AB. As long as M is on that line.” Bruce’s criticism was that they had not provided sufficient argumentation to support the claim. Again following a facilitative comment from the professor, Bruce responded to his own critique and built on Mike’s idea by suggesting that they also show the two distances, AM and BM, are equal (line 69). In this way, Bruce built on Mike’s ideas, and the students collectively contributed the key elements of the proof outline. In all, six different students shared their ideas with the class in this episode.

Mathematical constructs in episode 1

In order to analyze the kinds of mathematical constructs that were used during class discussions involving proof and reasoning, Toulmin’s model was useful. The episode presented above was coded to determine the structure of the argumentation, and the extent to which data and warrants were provided by either
the instructor or the students. As each student contributed to the conversation, several different arguments were formed.

The first argument was formed by Cheryl. After the conclusion was provided by one student at the start of the episode, Cheryl provided data that she believed was evidence of the conclusion. Cheryl’s claim was that the midpoint formula gives the midpoint, and the data she gave to support this claim was that it is the average of the two points. This was coded as data, since it was perceived by Cheryl to be evidence of the claim. The professor prompted her for a warrant (line 8): “Yeah, it’s an average... so?” and then (line 13), “Are we sure that it’s actually on the segment AB?” and (line 18), “How do you know?” Cheryl’s response (line 21), “Because it’s between the two y’s,” was coded as a warrant, since it was the connection she made between her data and the claim. Cheryl’s argumentation scheme is depicted in Figure 3 below.

![Figure 3: Cheryl's argument scheme](image)

Figure 3: Cheryl’s argument scheme
The professor's response to Cheryl's ideas conveyed that Cheryl's scheme was not sufficient (lines 24-29). These questions advanced the argumentation by the professor's insistence on a more rigorous explanation. The professor's questions, together with Mike's idea (lines 51-52), became a pivotal point in the collective reasoning of the class. Mike's statement was a point in the discussion where the focus of argumentation shifted.

Previously, the discussion had centered on the claim that the midpoint formula gives the coordinates of the midpoint. Following a revoicing comment by the professor, Bruce assessed Mike's suggestion with a qualifier (lines 61-64). Bruce's statement was classified as a qualifier, since he had found conditions under which Mike's idea would hold, that AM plus BM equals AB, but it would not be true that M was equidistant from A and B. Following a facilitative utterance from the professor encouraging him to pursue his idea, Bruce continued by responding to his own critique (line 69). Bruce's argumentation scheme is depicted in Figure 4 below.

Data 1
Use distance formula to show AM+MB=AB

Data 2
Show AM=BM

Claim:
M is the midpoint of AB

Figure 4: Bruce's scheme
At this point, the class had collectively contributed all the essential components of the proof, although no one had offered an explicit warrant to explain why the above data would sufficiently lead to the implication. The professor’s next response encouraged the progress of the class (lines 72-74), and then prompted them for additional information (lines 74-75). Although the students had everything they needed at this point, the instructor did not directly tell the class this; rather, she wanted the class to realize on their own that they were finished. However, the class did not catch on to this, and instead continued suggesting additional ideas. It was the final student’s contribution, followed by the instructor’s comments, in which two different warrants were given. Although neither warrant was necessary, they provided the explanation for why the two data sufficiently inferred the claim.

By the triangle inequality, the point M is on the segment AB if and only if the length of segment AM plus the length of segment BM are equal to the length of segment AB. The definition of betweenness similarly guarantees points A, M and B are collinear when the lengths of the smaller segments add up to the length of the total segment. Therefore, the first data guarantees M is on segment AB and the second data guarantees M is equidistant from both A and B. The resulting argumentation scheme is shown in Figure 5 below.
Summary of episode 1

Through the use of transactive prompts for justification, the instructor elicited responses from students in which they offered strategies and defended their reasoning. Through facilitative revoicing, rephrasing and coaching, the professor encouraged students to elaborate on their own ideas, and consider, critique, and build on one another's ideas. By asking key questions and rebounding students' own questions, the professor transferred responsibility onto the students to assess their results through their own reasoning. By prompting the class for strategies, soliciting verification of students' ideas, and drawing on students to determine what was sufficient for the proof, the professor provided scaffolding for students as they engaged in the process of proving. The episode presented above is fairly representative of a typical class discussion. Although this demonstrates the
professor's commitment to encouraging discussion and supporting student participation in the process of proving, this analysis of utterances does not necessarily demonstrate students' abilities to correctly formulate a valid mathematical argument. Toulmin's analysis provided a lens through which to view the construction of students' argumentation, and to identify the role of each component in the overall proof.

Through the analysis of students' mathematical constructs, under the lens of Toulmin's model, it is evident that students' initial ideas lacked mathematical rigor, but as the discussion continued the students began providing more substantial reasoning to support their ideas. This was a result of both explicit requests from the professor for more rigorous arguments, and of explicit responses from the students in which they identified gaps or deficiencies in the overall argument. The episode illustrates how students used the resources of the classroom, which in this case were largely the instructor's prompts and the reasoning of their peers, to create a substantial collective argument.

One of the professor's goals in this episode was to construct an outline of the key elements of the proof through class discussion and interaction. This activity occurred frequently throughout the semester in various forms. On most occasions, discussions of this nature were followed by the process of formally writing down the proof, either by the professor leading at the board, or by students working individually or in groups. On this particular day, the class did not write out the details of the proof; Professor Williams instead told the class they could fill in the
details on their own. The emphasis of the class discussion in this episode was to collectively formulate an outline of the proof.

**Episode 2: How much is sufficient? (November 2)**

The class had been working in small groups on proving that reflections, translations, and rotations are isometries. One group of students had been discussing possible approaches, including an analytic approach, for proving a reflection is an isometry. As they constructed the geometric situation with the dynamic geometry software, they debated about how much they needed to show, eventually calling the professor over to their table. The group of students, consisting of Michelle, Sarah and Amy, asked Professor Williams whether they could prove the theorem by only showing one segment is mapped to a congruent segment, or if all three sides of a triangle were needed. In other words, the students wanted to know if it was necessary to show the image of a triangle under the reflection is a congruent triangle. Rather than directly answering the question, the professor encouraged the group to try to think more about how much they needed to show, and why. The students were given some time to work together, and then a few of them were selected to write their proofs on the board. Three groups had written their proofs on the board, one for each of the three different theorems: reflections, rotations and translations. Once the selected groups had their proofs on the board, Professor Williams raised a question to the class. The students’ proofs are included in Appendix A.
Prof: You know what I notice about the two on the back board is that for the translation, Connor showed that triangle ABC is congruent to triangle A'B'C', and for the rotation Sam only showed one side of the triangle [maps to a congruent side], right? That A - that BC is congruent to B'C'. (pause) So, my question would be, [what] does the angle of the center of rotation necessarily have to do with the original triangle BCD...?

[transactive prompt – explanation]

Sam: Well, there wasn't an original triangle, there was just a ...
[transactive response - explanation]

Prof: There was just a segment, so you started with just a segment.
[directive, facilitative – rephrasing]

Sam: Yeah, cause there's two points B and C, and I just proved that they preserve their distance. [transactive response – elaboration]

Prof: So I guess my question would be, what's necessary? [transactive prompt for justification] Do we need to show that a triangle goes to a congruent triangle? [transactive prompt for justification] Or do we have to show that a segment goes to a congruent segment?
[transactive prompt for justification]

Michelle: I feel like when you have that fixed angle, it's ok to show that BC is congruent to B'C'? Then there's only two side lengths that will make that a triangle? But, if you're not dealing with a rotation you probably can't do that. [transactive response – explanation]

Prof: Do you think it's not enough? [transactive prompt – clarification]

Michelle: No, I think it's enough because we have that fixed angle. [transactive response – clarification]

Prof: It's enough for a rotation, but it's not enough in general. [facilitative – revoicing]

Michelle: Yeah.

Owen: Don't you at least wanna show it's true for three non-collinearpoints? [transactive question – critique]

Prof: Why? [transactive prompt – justification]

Owen: Cause, you might... you might be able to get that for just the segment but when you throw in the additional points not on that line
it may or may not be preserved distance-wise. [transactive response – explanation]

Sam: But if you throw in another point, you can just use that point (inaudible) and prove the same thing. [transactive response – critique]

So I just proved you can take any two points, it’s preserved. So therefore, any two other points will also (inaudible). [transactive response – elaboration]

Prof: What do other people think about that? (pauses several seconds)
Do we need to show that three non-collinear points ... when we do one of these transformations that we end up with a congruent triangle? [transactive prompt for justification] Or is it ok to just choose two [arbitrary points] like Sam did? [transactive prompt – justification, facilitative rebounding]

Amy: Are you asking in general if all three of the transformations...? [transactive request for clarification]


Owen: By definition, you’re only choosing two, not three... You only need to show A and B... [transactive response – justification]

Prof: So that it preserves distance, right? [facilitative – rephrasing] The definition of isometry that I wrote on the board says that it preserves distance, so it takes two points B and C, to their image points B', C', so that we have congruent segments. [directive, facilitative – revoicing, confirming] (pauses) Maybe we’ll come back to this?

Owen: It’s easier for us to visualize triangles... (inaudible)

Prof: Mm hm.
Analysis of episode 2

Professor utterances of episode 2

The episode began with a transactive prompt from the professor to determine the necessary amount of information needed to prove the statements, and whether that amount was different for each of the statements. By pointing out that the two different proofs demonstrated different things and asking the class, “what’s necessary?” (line 113), the professor was explicitly requesting that the students determine what would be sufficient to prove the theorems. This form of questioning created an opportunity for the students to engage in argumentation revolving around defending their particular approaches to the proofs. The intention of the instructor in doing this was that the students, through reflection and discussion, would expand their understanding of what constitutes an efficient proof. Through facilitative revoicing (line 128) and rebounding Owen’s question (lines 133-134) with, “Why?” (line 136) the professor repeatedly shifted the authority back to the class to try to decide what is sufficient. She encouraged other students to get involved in the debate (line 149): “What do other people think about that?”

Student utterances of episode 2

The students’ utterances were primarily transactive responses in which they provided clarification, explanation, elaboration, and justification. There were also several utterances in which students critiqued each other’s ideas. While most of the student utterances were in direct response to an instructor’s utterance, there was an instance of a student-student exchange. Sam’s critique (lines 143-146) of
Owen's idea was in direct response to Owen. Owen appeared to completely reverse his position, from his initial claim that three non-collinear points are needed (lines 133-134), to his final claim that only two points are necessary (lines 163-164). This is evidence that Owen reconstructed his understanding of what was necessary.

### Mathematical constructs of episode 2

Sam's argument consisted of his claim that a triangle is not necessary (line 104), which he backed up with the data that 1) only two points were given; and 2) he proved that their distance is preserved (lines 110-111). Sam's argument scheme is given in Figure 6 below.

![Data: Two points were given. Proved their distance is preserved. Claim: Triangle is not necessary. (No triangle was given.)](image)

**Figure 6: Sam's argument scheme**

Michelle's claim was that for a rotation it was sufficient to show for only one segment that distance is preserved, but otherwise it probably was not enough. Michelle's reasoning (line 118-121) was coded as data, since it was the factual evidence Michelle gave to support her claim. Michelle's argumentation scheme is given in Figure 7 below:
Owen’s question (lines 133-134), “Don’t you at least wanna show it’s true for three non-collinear points?” was coded as a claim, since he was asserting that it was necessary to show distance was preserved for three points, and the reason he gave was coded as data (lines 138-140): “... you might be able to get that for just the segment but when you throw in the additional points not on that line it may or may not be preserved distance-wise.” Owen’s argumentation scheme is depicted in Figure 8 below:

Sam responded directly to Owen, arguing (lines 143-144), “But if you throw in another point, you can just use that point ... and prove the same thing.” Sam’s refutation (lines 145-146) of Owen’s reasoning became the data that Owen used to support his own strategy (lines 163-164). Owen’s claim was that two points are sufficient, and the data he gave was that only two points were given. His last
statement (line 163): “By definition, you’re only choosing two, not three...” was also
coded as a warrant, since it is by the definition of isometry that only two points are
needed. Owen’s argumentation scheme is shown in Figure 9 below:

![Diagram of argumentation scheme]

**Figure 9: Owen’s second scheme**

**Summary of episode 2**

This episode provides an example of how the professor took a question that
arose spontaneously from a group of students, and used it as a springboard for a
class discussion (Yackel, et al., 1990). In the resulting discussion, several students
debated how much was sufficient in general to prove the given set of theorems. The
episode highlights how participating students challenged one another’s statements
and provided justification for their claims. The analysis of mathematical constructs
reveals how the students’ argumentation became more rigorous throughout the
discussion. This is evident from the fact that Owen supported his final claim that
two points were sufficient by using the definition of isometry, whereas previous
claims by Sam and Michelle lacked mathematical reasons. Another characteristic
that distinguishes episode 2 from episode 1 is the patterns of interaction. The first
episode demonstrates the teacher-student pattern of interaction, which is the most
commonly occurring pattern of interaction in classroom discourse. In episode 2 there are two instances of a student-student exchange, in which one student responded directly to another student. Lastly, this episode differs from the previous one in that the class discussion in this episode took place after the students had presented their written proofs to the class.

**Episode 3: Is it different? (November 9)**

Professor Williams had been leading the class through the proof of the following theorem: In the Euclidean plane, every isometry is completely determined by the images of three non-collinear points. The proof was divided into two different cases: the first case supposed an arbitrary point $X$ was on a given triangle $ABC$, and the second case supposed $X$ was not on the triangle $ABC$. Professor Williams had completed the first case with class contributions, and had just begun case two. Figure 10 illustrates the diagram Professor Williams drew on the board.

![Figure 10: Case two](image-url)
Prof: Case two is we have some other X; it's not on the triangle, it's not A, B, or C. [directive] Alright, so suppose X is not an element of triangle ABC. So... do we know anything about where the image of X is going to go? (Pauses) What's the image of X - what's its relation, um, to A'? The image of X. What's its relation to A'. [transactive prompt – explanation]

Michelle: Would it be the same distance – would X' be the same distance from X as A' is from A? [transactive response, proposal of new idea]

Prof: Alright. Would X' be the same distance from X as A' is from A. [facilitative – revoicing] Do you have an answer to that? [facilitative – rebounding]

Dylan: But isn't the distance from A to X gonna be the same as A' to X'? [transactive response - critique, proposal of a new idea]

Prof: Is that different from what Michelle said? [transactive prompt – explanation]

Owen: That is different from what she said. [transactive response – explanation]

Sam: Yeah, she said X to X', right? And A to A'? [transactive response – clarification]

Prof: Yeah. (pause) Yeah, so when we have a reflection... here's A, and the reflection is over here... this is for Michelle and everyone... say this is X, and this is X', so the distance from X to X' is not the same as the distance from A to A'. [directive]

**Analysis of episode 3**

**Professor utterances of episode 3**

At the professor’s initial transactive prompt (lines 178-179), one student offered an erroneous suggestion (lines 181-182). Rather than taking the role of authority immediately and correcting her, the professor rebounded Michelle’s question and transferred authority to the class to determine whether the student’s
idea was correct (lines 185-186). This generated some discussion among a few students, and another student offered the correct interpretation. Again, rather than assuming the role of authority and confirming the second student’s idea, Professor Williams returned the second student’s idea with a question (line 192), asking if the second student’s idea was different from the first. By asking this question, the professor again shifted responsibility on to the class to determine whether the two statements were mathematically different. The professor closed the discussion with a directive utterance (lines 201-204), confirming that Michelle’s idea was incorrect.

Student utterances of episode 3

The students in this episode provided transactive responses to the instructor’s prompts, proposing new ideas (lines 181-182) and critiquing each other’s ideas (line 189). In all, four students participated in this brief exchange. Dylan, Owen and Sam collectively determined that Michelle’s statement was incorrect, contributed the correct interpretation, and clarified how the two statements were different.

Mathematical constructs of episode 3

In this episode, Michelle’s initial suggestion indicated that she was confused about the definition of isometry. By sharing her idea with the class and through the professor’s response, an opportunity was provided for Michelle and the rest of the class to rethink the definition of isometry. Dylan’s response provided an example that demonstrated the falsity of Michelle’s statement by correctly using the
definition, and the professor provided reinforcement. Together, Dylan’s example and the professor’s concluding statement highlighted what could be taken from the definition, and what could not.

Summary of episode 3

This episode illustrates how the professor turned a student’s erroneous statement into a class discussion. Through facilitative revoicing and rebounding, the professor gave the students a chance to reflect on and evaluate the first student’s idea. In this way, the professor provided a learning opportunity for the class. The students accepted shared responsibility for determining whether the idea was valid, and for determining whether two students’ ideas were mathematically different. The professor assumed the final voice of authority, concluding the discussion with a directive utterance.

Episode 4: Saccheri Quadrilaterals (November 20)

On this day, the class had begun the study of neutral geometry, which does not assume Euclid’s Fifth Postulate. Professor Williams gave a brief history of various people’s attempts to prove Euclid’s Fifth Postulate, also known as the Parallel Postulate, to introduce the topic. The discussion led to the Saccheri Quadrilateral, which is a quadrilateral in neutral geometry defined by two adjacent right angles, and the two segments stemming from those right angles as being congruent. The side of the quadrilateral containing the two right angles is commonly referred to as the base and its opposite side is the summit, while the pair of
congruent sides are called the legs of the quadrilateral. The professor then constructed a figure, which she drew on the board, and claimed it was a Saccheri Quadrilateral. The construction began with an arbitrary triangle, then the formation of a midline (a line containing the midpoints of two sides of the triangle), and finally the construction of a quadrilateral formed by dropping two perpendiculars from the base vertices of the triangle, B and C, to corresponding points P and Q on the midline. The Saccheri Quadrilateral construction and accompanying claim written on the board are shown in Figure 11 below.

Claim: (Quadrilateral) PBCQ is a Saccheri Quadrilateral.

![Diagram of a triangle with a Saccheri Quadrilateral](image)

Figure 11: Triangle with associated Saccheri Quadrilateral

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Prof: How would we show that [PBCQ is a Saccheri Quadrilateral]?  
[transactive prompt – strategies] (Pauses for several moments as the class studies the figure on the board)

Mike: Drop a perpendicular from M [midpoint of segment AB] to BC.  
[proposal of new idea]
Prof: From M to BC... (draws as in Figure 11 below) \([facilitative - revoicing]\)

Mike: and reflect that point over BM \([contribution to a new idea]\)

Prof: Reflect this point (indicates the point where the new perpendicular intersects side BC, later labeled point L) ... where? \([transactive prompt - clarification]\)

Mike: Over BM? \([transactive response - clarification]\)

Prof: Up here? Oh, no, over BM. Reflect it... over here? \([transactive prompt - clarification]\)

Mike: Maybe not..

Prof: (Pauses, looks back and forth from student to figure on board) Where are you thinking it would go? This way? \([transactive prompt - elaboration]\)

Mike: Yeah... \([general confirmation]\)

Prof: Or up? \([transactive prompt - clarification]\)

Mike: Not sure.

Prof: Other ideas? What do we need to show for it to be a Saccheri Quadrilateral? \([transactive prompt]\)

Several students: PB is congruent to QC? \([transactive response]\)

Prof: That's all we need, right? \([directive, facilitative - confirmation]\) PB congruent to QC. We have the right angles already, so our base is up here, the summit is... BC. So we just need congruent segments. \([facilitative - rephrasing]\) (erases board) Ideas? \([transactive prompt]\) Bill?

Bill: If we go so far as to say that PB is congruent to M... L... \([proposal of a new idea]\)

Prof: Whatever the segment is... (laughs and writes 'L' at the end of the segment originally introduced by Mike, as shown in Figure 11)

Bill: Yeah. Then can we continue that for any ML? Or [inaudible]? \([contribution to a new idea]\)
Figure 12: Mike’s suggested construction as drawn by Prof. Williams

Prof: So if I chose some other point (draws another line on figure) [facilitative - rephrasing]

Bill: Yeah. M1 or [referring to the segment formed by ‘some other point’ the professor refers to above] - [general confirmation]

Prof: Would it always be... the same... [facilitative] (steps back, looks at board) I think it’s going to be challenging, because um, the fact that parallel lines are equidistant is a result of the parallel postulate. [directive] If you’re up for the challenge, I invite you to give that a try. Can we take advantage of the fact that M and N are midpoints? [facilitative - redirecting] (erases extra lines she had drawn on the figure, pauses, looks at class) Sam?

Sam: What if you drop a line from A down to PQ, the perpendicular to BC... [proposal of a new idea]

Prof: So do one more perpendicular? [transactive prompt – clarification]

Sam: Yeah, from A down. [transactive response – clarification] (Professor Williams draws a perpendicular line from A down to midsegment PQ.) Is it – okay, so you’ve got PMB... is similar to RA – or RMA. [contribution to a new idea]

Prof: (Steps back from the board) I’m giving everyone else a chance to take a look at what we have now. (Begins writing what they want to
prove, and what they need to show. Several students are talking quietly among themselves.) Some ideas swimming around in your heads now? [transactive prompt – ideas] Alan?

Alan: Um, I was just looking at that diagram, and I'm looking at vertical angles? [contribution to a new idea] PMB and AMR, and I'm also noticing one for ANR and CNQ? [contribution to a new idea] (Prof. Williams marks each pair of vertical angles congruent on the board) Right there? Let's see, I have... angles -

Prof: Okay? I'm sorry – keep going. Do you want to keep thinking in your head?

Alan: Yes, I want to...

Prof: Dylan?

Dylan: Yeah, you can show that PMB is congruent to AMR. [contribution to a new idea]

Prof: Angles, or triangles? [transactive prompt – clarification]

Dylan: Triangles. [transactive response – clarification] And then triangle ANR is congruent to QNC... so... and then you can show that AR is congruent to PB is congruent to QC. [contribution to a new idea]
Prof: (pauses several seconds) Alan, where were you going with yours? [transactive prompt – elaboration]

Alan: Well, I was thinking the same way, but the first problem would be to - which reason...

Jen: yeah...

Alan: would make - would be valid to show congruent. [transactive question]

Prof: For congruent triangles. [transactive, facilitative - rephrasing]

Alan: Yes. Cause I know we have an angle and a side by an angle and – [transactive statement – justification]

Marc: Just... we have an angle. [transactive statement – critique]

(Others in that group talk among themselves.) Angle angle side. [proposal of a new idea]

Alan: Oh – yeah. We have an angle and a side, right there. [transactive response]

Prof: Okay. Are people starting to see this? So it looks like the idea... (writing) Dylan’s idea is to show that these triangles, AMR and B... let me get this right... BMP are congruent, and ANR and CNQ are congruent? (writes this on board) And then from these we would have AR is congruent to BP, AR congruent to CQ... transitivity? [directive, facilitative – revoicing] Do I have that the way that you said it, kind of (inaudible)? Yeah? Ok, so if we can show these two pairs of triangles congruent, then we can show their corresponding parts congruent, and use transitivity. [facilitative – rephrasing] Okay. That’s the idea. Let’s see if we can write it down. [facilitative – coaching]

The work of the class continued as Professor Williams recorded the “formal” proof of the claim on the board, taking student contributions and affirmations along the way.
**Analysis of episode 4**

**Professor utterances of episode 4**

As was characteristic of many class discussions, the episode began with a transactive prompt from the professor for proof strategies. As students offered suggestions, the professor encouraged their continued participation with facilitative revoicing and transactive prompts for clarification and elaboration. Certain facilitative utterances and transactive prompts also served to scaffold instruction, modeling key steps in the process of proving.

What do we need to show for it to be a Saccheri Quadrilateral?

By asking a specific question (lines 238-239), Professor Williams modeled the first step in the construction of a proof. The prompt for what was needed to show it was a Saccheri Quadrilateral was an attempt to direct the students’ attention to the specific goal of what was necessary for the proof outline. Several students responded simultaneously, which was evidence that many students, at least all those who responded, were now focused on the agreed-upon task at hand.

The use of a directive utterance (lines 262-264) discouraged one student from continuing with his idea, effectively steering the focus of the class away from a strategy that the professor was anticipating would not be fruitful. A directed question coded as facilitative (lines 265-266) redirected the students’ focus to the given information. In doing this, the professor was scaffolding the process of proving, by encouraging the students to use the underlying assumptions of the hypothesis.
When the third student (Sam) proposed a new idea (lines 270-271), the professor responded by marking the figure with the appropriate segment, and then stepping away from board. This action, combined with her next statement (lines 281-282), “take a look at what we have now…”, was significant. Although her delivery of that statement was subtle, it hinted at the possibility that this last student’s idea was promising. The students immediately began talking quietly among themselves while the professor summarized what they had agreed upon so far, writing on the board what they were trying to prove, and what they needed to show. This pause in the discussion provided a learning opportunity for all the students, allowing them to consider Sam’s idea and try to move forward with it. In this way, the professor was again scaffolding the proof construction, guiding the class toward coming up with the next step.

As the proof construction advanced with each student’s additional contribution, the professor’s utterances were primarily transactive prompts for clarification or elaboration, and facilitative revoicing and rephrasing, as students contributed ideas. The professor concluded the discussion by summarizing the key ideas that provided the outline for the proof, and writing the plan on the board.

**Student utterances of episode 4**

Many students participated in this class discussion, by proposing new ideas and elaborating on ideas in response to instructor’s transactive prompts. Although most interactions followed the teacher-student pattern, there was also an instance of a student-student interaction (lines 311-316, lines 321-328). Students also...
contributed to one another’s ideas (lines 286-290, line 299, lines 304-306) and critiqued each other’s ideas (line 324).

A pivotal point of the episode was directly following the professor’s prompt for the class to think about Sam’s construction (lines 281-282). Alan provided a statement (lines 286-290) that was based on the diagram that included Sam’s construction. This is evidence that the specific prompt from the professor provided an opportunity for a student to successfully build on another student’s idea. For the remainder of the discussion, students continued to build on each other’s ideas, but with less direct prompting from the professor. The students also challenged each other’s ideas and provided justifications for each other’s statements without any prompting from the professor.

Mathematical constructs

Under Toulmin’s scheme, the segment Sam proposed was coded as data, since it provided one of the triangles (RMA) he claimed were similar. The statement that PMB is similar to RMA was classified as a claim. Alan made the next contribution (Lines 286-287): “Um, I was just looking at that diagram, and I’m looking at vertical angles? PMB and AMR, and I’m also noticing one for ANR and CNQ?” Alan’s statement was coded as a claim, since he was making the assertion that the two pairs of angles were vertical angles. Since Alan’s contribution provided data for the next student’s contribution, Alan’s statement was also coded as data. Dylan’s contribution, which built on Alan’s statement, was the claim (Line 299): “Yeah, you can show that PMB is congruent to AMR.” Dylan’s idea was similar to
Sam's original idea; Dylan correctly identified the pair of triangles as congruent (not similar). In addition, Dylan extended this claim in his next statement to include another pair of triangles (Line 305): "And then triangle ANR is congruent to QNC..." and made another claim based on those two statements (Line 306): "so... and then you can show that AR is congruent to PB is congruent to QC." The first two statements made by Dylan provided data for his claim that the three segments were congruent, so they were double-coded as both claims and data.

Dylan did not provide warrants for his final claim, which would be 1) Corresponding Parts of Congruent Triangles are Congruent (CPCTC); and 2) Transitivity. It was typical by this point in the semester that these kinds of warrants often were not explicitly stated. The most likely reason for this is that they had been used so frequently that they became taken-as-shared by the class, and they were understood. However, warrants such as these were frequently included in written-up proofs presented on the board, on homework assignments and on exams. The professor provided two warrants for the argumentation: she stated that the pairs of vertical angles Alan identified are congruent (lines 334-335), and she mentioned transitivity (line 335).

The final warrant needed to support the whole argument was provided by Marc in response to Alan, who wondered aloud why the pairs of triangles were congruent (lines 311-312 and 316): "Well, I was thinking the same way, but the first problem would be to – which reason... would make – would be valid to show congruent." Alan attempted to provide justification (line 321): "Cause I know we have an angle and a side by an angle and –" to which Marc critiqued and corrected
(Lines 324-325): “Just... we have an angle. Angle angle side.” The entire collective argumentation scheme is presented in Figure 15 below. The bold numbers indicate the sequence of arguments as they occurred in the discussion.

Figure 14: Collective scheme

Summary of episode 4

Transactive prompts and scaffolding characterized the professor’s interactions with students in this episode. The initial transactive prompt for proof strategies opened the class to discussion. Through the use of certain transactive prompts, the professor modeled key strategies of proof construction. By asking, “What do we need to show for it to be a Saccheri Quadrilateral?” (lines 238-239), the
professor focused the students on the first important step of a proof construction, determining what must be shown. By asking, "Can we take advantage of the fact that M and N are midpoints?" (lines 265-266), she redirected the students' attention on given information that would be useful in finding a proof, modeling another important step of proof construction: making a connection between given mathematical constructs and statements to be proved. Finally, by pausing the discussion and inviting the class to look at the construction Sam had suggested, she created an opportunity for the entire class to assess this plan and try to build from it. These three moves were pivotal points in the class discussion, scaffolding the proof construction by emphasizing key components and strategies.

Throughout the episode, the students proposed several new ideas, built on one another's ideas, and challenged each other's statements. Students provided justifications for one another's ideas without prompting from the instructor, as in line 325. Although the most frequently occurring pattern of interaction was teacher-student, there were also several instances of student-student interactions in this episode. This episode features a number of valuable contributions that were made from several different students, almost all of which were necessary components of the proof. The key ideas of this proof did not come from any one student; rather, they were the result of a collective effort in which each student either built on the ideas of a previous student, or challenged another student's idea and then offered an alternate idea.
Concluding Analysis of Part I

The individual analyses of the four episodes presented above illustrate the ways in which the professor and students interacted, and the ways mathematical meaning was negotiated through those interactions. The episodes were then analyzed on a more global level, to illuminate emerging themes and patterns. This global analysis found similarities in the nature of classroom discourse and patterns of interaction. On further scrutiny, the global analysis also revealed changes in classroom discourse that occurred over time. These findings are presented in the sections to follow.

Similarities across episodes

In each of the episodes above, the initial *transactive prompt* from the professor was a catalyst for the discussion that followed. The initial transactive utterance used by the professor in this way was a prevalent feature in every episode, and had multiple effects on the classroom discourse. First, it placed an emphasis on reasoning. Second, it placed the responsibility on the students to provide reasons for mathematical claims. Third, the consistent use of this form of prompting contributed to the shared understanding that the students be active participants in the processes of proof construction and collective reasoning. The professor utilized various forms of transactive prompts as a means for modeling the process of constructing a proof. She provided scaffolding for students using key questions to prompt them to determine: 1) what exactly they needed to show; 2) how they could use the given information; and 3) whether they had the sufficient
components of the proof outline to prove the given theorem. In episodes one and four, in which the class discussed the structure of the arguments they needed to make, transactive prompts enabled students to collectively provide all or most of the significant components of the proof outline. In episode two, the professor used transactive prompts to engage the students in reasoning about the methods they had chosen for the proofs they had developed, and about what was sufficient in general for proving all four of the isometry theorems. In episode three, a transactive prompt from the professor sparked a brief discussion in which several students identified a misconception and corrected a mathematical statement. This is evidence that the professor's use of transactive prompts successfully engaged the students in the process of reasoning, and navigated them through the challenge of constructing proof outlines.

A second type of utterance by the professor that was a recurring characteristic of many class discussions was facilitative revoicing. Many studies in mathematics education research have examined the role of revoicing in mathematics classrooms, but there has been little evidence to indicate when aspects of revoicing might lead to productive engagement of the students, and when it may result in disengagement (Franke, Kazemi & Battey, 2007). Overall analysis of the episodes presented above indicates that specific forms of revoicing provided a framework that supported students as they engaged in class discussions, and allowed them to advance their mathematical ideas.

One form of revoicing, in which the professor would restate a student's utterance, encouraged the student to continue to develop her or his mathematical
idea. When a student's statement is repeated by a teacher, who is in a position of higher authority and power, this has the effect of legitimizing a student's thoughts (Forman & Ansell, 2001). By restating a student's ideas, the professor is also redirecting the contribution to a different audience. The student who makes the original statement is typically directing it to the teacher. By revoicing a student's thoughts, the teacher is presenting the student's idea to the class. This legitimizes a student's idea not only for the student, but also for the whole class. Particularly when revoicing was used in conjunction with a coaching utterance that provided positive feedback, students responded by elaborating on their own ideas, or by using the original student’s idea as a base from which to formulate a new idea. Revoicing therefore proved to be a powerful tool for engaging students in elaborating on and extending their own ideas or the ideas of others.

Revoicing also created opportunities for students to reflect on and evaluate the statements of their peers. For instance, a revoicing was often used in conjunction with a transactive request to assess a student’s contribution, as in episode two. In many of these cases, the students responded by adding contributions that expanded on a student’s idea. One form of facilitative utterance similar to revoicing that was utilized frequently was rebounding. In episode 3, facilitative rebounding of students’ ideas and questions played a key role in presenting opportunities for students to consider each other’s ideas, and assess their validity. These kinds of interactions helped to establish the expectation that students use each other as resources, to work collaboratively towards a common goal. They also contributed to the development of the classroom norm that the
students shared responsibility for determining what was mathematically valid, sufficient, or appropriate.

Thus, the professor’s consistent use of transactive and facilitative utterances demonstrated to students that she expected the students to participate. Through these forms of utterances, the professor also demonstrated what participation should look like, and resulted in students’ participation. Finally, the discourse provided a template for students’ further argumentation during student-student interactions.

**Differences across episodes**

Although the initial analysis found mainly similarities in whole class discussions, upon closer examination some differences were revealed. In particular, analysis with an eye toward changes over a span of time uncovered subtle changes in several aspects of classroom discourse. One aspect was an increase in student-student interactions. In the earlier episodes, the dominant pattern of interaction was Professor-Student. For instance, in Episode 1 this was the only pattern of interaction throughout the entire episode. As time progressed, more frequent instances of Student-Student exchanges were observed. For example, in Episode 4 there were several occasions in which a student responded directly to another student.

Another difference that emerged was that students in later episodes needed less prompting from the professor, and offered contributions more willingy, often with no solicitation from the professor. In addition, students voluntarily provided
more justifications with their statements in later episodes. In Episode 1, several initial exchanges between the professor and the student were repeated requests and attempts at providing justification for the student’s ideas. In Episode 4, many of the students’ statements were in response to facilitative prompts from the professor. Furthermore, some justifications that were provided by students were direct responses to other students’ statements. For instance, directly following the professor’s suggestion that the students consider Sam’s construction, Alan provided a justification for Sam’s idea. In the remainder of the discussion, there was not a single transactive prompt for justification from the professor. In fact, throughout the entire discussion, the only transactive prompt was the characteristic initial one, in which the professor solicited strategies for the proof outline.

Throughout the course of the semester, the number of student utterances during class discussions increased. Students’ statements also contained more mathematical justifications over time. Finally, the number of student-student interactions increased throughout the semester. These observations are summarized in Table 4 below. A frequency count for a sample class was taken from October, November and December to illustrate these findings. Although the counts varied from class to class, a steady increase was apparent over time in student utterances, student justifications, and student-student interactions. The proportions shown in the table are not appropriate for a statistical analysis; rather, they are offered as an additional means of viewing the changes in utterances over time.
Table 4: Frequency of Student Utterances

<table>
<thead>
<tr>
<th>Date of class</th>
<th>10/21/09</th>
<th>11/9/09</th>
<th>12/7/09</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number of student utterances</td>
<td>32</td>
<td>73</td>
<td>81</td>
</tr>
<tr>
<td>Student utterances containing justifications</td>
<td>3</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>Proportion of student utterances containing justification</td>
<td>.094</td>
<td>.137</td>
<td>.160</td>
</tr>
<tr>
<td>Total number of student-student interactions</td>
<td>1</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>Proportion of student-student interactions</td>
<td>.031</td>
<td>.137</td>
<td>.173</td>
</tr>
</tbody>
</table>

Collective reasoning

Collective activity is defined as “the normative ways of reasoning of a classroom community, (...) that develop as learners solve problems, explain their thinking, represent their ideas, etc.” (Rasmussen & Stephan, 2008, p. 195.) This definition was useful for the current study because it provided a way to examine the learning opportunities afforded to students through participation in collective reasoning. These researchers view learning as “conceptual shifts that occur as a person participates in and contributes to the meaning that is negotiated in a series of interactions with other individuals” (p. 195). Drawing on Toulmin's model provided a way to analyze the structure of argumentation, and to link collective activity to the development of mathematical ideas. As outlined in previous sections of the text, the basic components of Toulmin's model are defined as follows:

- **Data**: Facts or procedures that lead to a conclusion;
- **Claim**: A statement, assertion, or conclusion;
• **Warrant:** Information that makes a connection between the data and the conclusion;

• **Backing:** Justifies why the warrant is valid.

In classifying discourse into one of the four categories above, close attention was paid to the intended *function* of each contribution. The same statement could be identified as data or a warrant, for instance, depending on how it is used in the context of the argument.

Collective reasoning was identified by the following characteristics: 1) a student challenging another student's ideas; 2) a student providing a warrant or backing for another student's claim; 3) students building on one another's ideas to collectively form key argumentation. Collective reasoning was a regularly occurring feature of the classroom discourse. For instance, in Episode 1, Bruce challenged Mike's idea, arguing that Mike's suggestion was not sufficient to prove the theorem. Bruce then extended Mike's argument by providing a necessary second component. Sam provided a third component, and then Bruce and Lucas each offered different warrants to support the overall argument. In episode 2, the main purpose of the discussion was for students to reflect on their chosen methods of proving the theorems, and to determine whether it was the minimal amount needed or whether it contained extraneous information. In this episode, students provided justifications to support their beliefs, refuted each other's positions in one case, and changed their claims in another case based on the collective argumentation.

The most notable instance of collective reasoning is Episode 4. In this episode, several components of the overall argument were double-coded, as they
served multiple functions in the structure of the argument. For instance, Alan offered a claim that also provided data for Sam’s claim, making a connection between Sam’s statement and his original data. Dylan’s claim became data for his next claim, and Marc provided the warrant for the overall argumentation scheme. Thus, collective reasoning was a central theme in the classroom that played a significant role in students’ construction of proof outlines.

In the episodes presented above, the role of the professor was one of orchestrating discourse. Through the pedagogical moves outlined in previous sections, the professor created a dynamic classroom environment in which students regularly took part in class discussions, critically evaluated mathematical progress, and collaboratively contributed to the development of proofs and other forms of reasoning. The need for a better understanding of how this kind of classroom environment is created is considered as an important one:

“Although researchers have long recognized the potential of teacher practices to foster meaningful conversation and student learning in classrooms, researchers have only begun to study ways of changing classroom discourse from traditional recitation patterns in which the teacher dominates classroom exchanges to more balanced and student-centered communication in which students take a more active role in classroom discourse” (Franke, Kazemi & Battey, 2007, p. 233).

The evidence provided in this section offers further insights into the ways a teacher can establish the kind of classroom in which students’ ideas are encouraged. It also demonstrates how students’ participation can be used a basis for classroom activities. Finally, the episodes provided above highlight the ways certain forms of
discourse can provide opportunities for students to engage in the processes of proving and collective reasoning.
Part II:

Peer interactions and group work

This section of Chapter 4 examines students working in small groups. The focus of data presentation and analysis is on the focus groups that were selected as discussed in Chapter 3. The episodes include sessions of group work that occurred during class as well as group work that occurred during task-based interviews. Since the nature of interactions between students is fundamentally different than interactions between students and the instructor, a slightly different coding scheme was used for small group discussions. As discussed in Chapter 2, the work of Goos, et al. (2002) provides a framework for identifying collaborative metacognitive activity among peers working in groups. Based on Vygotsky's notion of the zone of proximal development (ZPD), the concept of a collaborative zone of proximal development refers to the learning potential of peers of roughly equal expertise, in which each student has some knowledge and skill, but requires contributions from the others to advance. The collaborative ZPD is an expansion of the original theory of ZPD, with the main distinction being equal-status interactions versus expert-novice interactions (Goos, et al., 2002).

The codes used are very similar to the codes described in Blanton's framework, since Blanton and her colleagues based their framework on the work of Goos. Goos and her colleagues (2002) identify two types of acts as metacognitive in nature: those involving a New Idea, and those involving an Assessment. The category of New Idea includes any utterance that brings new information or a
contribution to new information, and may or may not be useful towards solving the problem. Assessment can be a question, statement, or a response that involves evaluating a strategy, a result, or one's knowledge or understanding. Transactive reasoning is classified in a similar way as was previously defined, as characterized by elaboration, explanation, justification, critique, and clarification. Goos, et al. (2002) found that in episodes considered to be instances of successful collaboration, a predominant feature was many interactions that had double codes: they were classified as both metacognitive and transactive in nature.

**Episode 5: Lucas and Sam (December 16)**

The following episode is an excerpt from a task-based interview with one of the focus groups. The students (Lucas and Sam), through guided exploration with Geometer's Sketchpad, had just made the conjecture that the midsegment\(^1\) quadrilateral of an isosceles trapezoid is a rhombus, and they had begun working on the proof. Using Geometer's Sketchpad, they had constructed the figure and referred to it as they discussed the proof. The students also made sketches by hand, based on the computer sketch, as they considered the proof. The first hand sketch, which they refer to in the discussion below, includes segments HF and EG (and corresponding labels), which were not included in the computer sketch. Their sketch is shown in Figure 15 below.

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\(^1\) A midsegment is defined as the segment formed by joining the midpoints of two segments. Thus, the midsegment quadrilateral is the quadrilateral formed by joining all four midpoints of each segment of the isosceles trapezoid.
Researcher: So how would you think about proving that first one? The midsegment quadrilateral is a rhombus? [transactive prompt]

Lucas: There’d be a lot of triangles. [meta-new idea]

Researcher: So what would you need to show? [transactive prompt-elaboration]

Lucas: Sides are the same? Or parallel. Or both. [meta-new idea]

Sam: Uh... the sides of what? [transactive question-clarification]

Lucas: Of the triangle ... shape that you just made (laughs). [transactive response-clarification] Well, you could show that these two triangles [EAH & GDH] are congruent... pretty easily. And then you’d have to show that um, those two [segments EH & FG] or those two [segments EF & HG] are parallel. [meta-development of new idea; transactive response-elaboration] And that would work. [meta-assessment of strategy]

Sam: I’m not sure which ones you’re pointing to. [transactive question-clarification]

Lucas: This side and this side [segments EH & FG] – or this side and this side [segments EF & HG] - but if you show this triangle here, this corner triangle [EAH] and this corner triangle [GDH] here are the same, then you’ll get the two sides [EH & HG] are the same. [transactive response-clarification, elaboration]

Sam: Well, we know – um, yeah, okay. We know this angle [EAH] is the same as that angle [GDH], cause it’s the definition of isosceles trapezoid. [meta-contribution to a new idea, transactive statement-justification]

Lucas: Yeah - they’re congruent triangles.
Sam: This [H] is the midpoint [of segment AD], so this is congruent to that 
[segments AH & HD], and this [E] is the midpoint [of segment AB], so this is 
congruent to that [segments AE & DG]. So you have side angle side gives us this 
triangle congruent to that triangle [EAH & GDH]. Therefore this side [EH] is 
congruent to this side [HG]. [meta-contribution to a new idea; transactive 
statement-elaboration, justification]

Sam: But that only proves this and this are congruent [segments EH & HG]. 
[meta-assessment; transactive-critique]

Lucas: Right [general confirmation]

Sam: Similarly we prove this and this are congruent [segments EF & FG], [meta-
contribution to a new idea] but for a rhombus we prove they're all congruent. 
[meta-assessment, transactive-critique]

Lucas: Or that two sides are parallel. [proposal of a new idea]

Sam: Congruent does – yeah, but we haven’t proven that this is parallel to this 
yet [segments EH & FG], or that this is parallel to this [segments EF & HG]. 
[meta-assessment; transactive-critique]

Lucas: Right. If we prove one of those we’re set. [meta-assessment]

Sam: Right. Um... since this (E) is a midpoint, and this (G) is a midpoint, doesn’t 
that make CB parallel to BG – or, what is that? EG – yeah, this is parallel, and this 
[segment HF] would be perpendicular [to segment EG]. [meta-proposal of a new 
idea, transactive statement-justification]

Lucas: Right. [general confirmation]

Sam: What does that give us? [meta-assessment; transactive question-critique]

Lucas: A bunch of little triangles that look the same. [meta-contribution to new 
idea] That we don’t know are the same. [meta-assessment]

Sam: We know...

Lucas: Oh, actually, that does. [meta-assessment] Because we know that the 
point [I] in the middle of this [segment HF], and this, which means we know that 
this length is the same as this length [segments HI & IF] – [meta-contribution to 
a new idea, transactive statement-justification]

Sam: Right [general confirmation]
Lucas: And this length is the same as that length [segments EI & IG], and we know the corner [I], so we'd get side angle side. [meta-contribution to a new idea, transactive statement-justification]

Sam: That would give us four congruent triangles [EIH, HIG, GIF, & FIE]. [meta-contribution to new idea] Right? [transactive-critique]

Lucas: Right. And we're set. That was actually a better direction. [meta-assessment; transactive-critique]

Analysis of episode 5

Utterances of episode 5

This exchange illustrates how, through transactive utterances, Lucas and Sam negotiated a mutual understanding of what was needed and developed a strategy to prove the theorem (lines 8-46). It also highlights how these students utilized metacognitive utterances by proposing new ideas (lines 57-59), developing one another’s ideas (lines 71-73, lines 78-79), and frequently evaluating results (lines 51-52, lines 66-67, line 71, line 85). Transactive prompts served to clarify Lucas’ initial idea (lines 13-15), and allowed Sam to contribute to Lucas’ idea (lines 27-29, 33-37). Sam frequently assessed their progress (lines 40, 46, 51-52), as did Lucas (lines 55, 66-67), which served to advance their collective argumentation.

Mathematical constructs of episode 5

The initial argumentation scheme constructed jointly by Sam and Lucas is shown in Figure 16 below.
Lucas made the initial claim that the two upper triangles are congruent, and then continued outlining the rest of his plan, concluding with a confident assessment that "...that would work." (lines 16-17). However, Lucas did not offer any data or warrants for his claim. Instead, Sam provided the justification of Lucas' claim, providing data and warrants to complete the argument. The fact that Sam convinced himself of Lucas' claim suggests that the two shared knowledge that was mutually agreed upon: that the two upper triangles are congruent (lines 27-36). This allowed them to advance the mathematical argument, when Sam extended the argument to the two lower triangles, and then pointed out that this would not give them four congruent sides; only two pairs (line 46): "Similarly we prove this and this are congruent, but for a rhombus we prove they're all congruent."

The component labeled Data 2 shown in Figure 16 was not fully stated by either student. When Sam proposed that because of the midpoints, the pairs of segments were congruent (lines 33-37), he was referring to the fact that since both
E and G were midpoints, segments AE and DG were congruent. The implicit warrants in this case are: 1) segments AB and DC are congruent by definition of an isosceles trapezoid, and 2) since points E and G are midpoints of each of the respective segments, it follows that segments AE and DG are congruent by transitivity and segment addition. Since this argument was not pursued, the implicit warrants have been left out of the argumentation scheme diagram.

Sam's evaluation of their progress (lines 51-52) was mutually agreed upon by Lucas (line 55). By assessing their plan, the students were able to realize its limitations and thus form a new plan. Thus, a piece of taken-as-shared knowledge was formed: that the first argumentation scheme would not yield the necessary results. This was a pivotal point in the overall proof construction, advancing the argument by taking a new direction that ultimately was successful. The second argumentation scheme formed by Lucas and Sam is given in Figure 17 below. Although several of the necessary warrants were not yet explicitly stated by the students, Lucas and Sam collaboratively produced an outline of the key ideas needed for the proof. Both students seemed satisfied that they had found all the necessary components. Their second scheme was more successful than the first one, since by showing all four triangles are congruent, it followed directly that corresponding segments of those four triangles were also congruent. This would sufficiently prove the quadrilateral was a rhombus, by showing the four midsegments (EH, HG, EF & FG) were congruent.
Figure 17: Second argumentation scheme

Summary of episode 5

This episode is an example of successful collaboration among students working together toward the common goal of constructing a proof outline. Through both metacognitive and transactive exchanges, Sam and Lucas were able to figure out what they needed to show in order to prove the theorem, and to produce and revise their strategies as they constructed shared knowledge. One significant feature of the excerpt presented above is role-sharing. Each student alternately contributed in different ways toward the overall formation of their argument, with shared responsibility in making claims, providing supporting data and warrants, and assessing progress. In other words, neither student assumed sole responsibility for generating ideas, or for determining the validity of those ideas. Also, neither student was considered to be the authority of the pair. The analysis using Toulmin’s argumentation framework shows that, as a result of their interactions, the students...
provided many of the necessary claims and supporting data and warrants to create the framework of the proof.

**Episode 6: Sarah, Michelle & Amy (December 7)**

In the following excerpt from a task-based interview, a group of students (Sarah, Michelle, and Amy) had been working on proving their conjecture that the midsegment quadrilateral of an isosceles trapezoid is a rhombus. After discussing what it was they needed to show and recalling different ways of defining a rhombus, they decided to first try to prove opposite sides are parallel. They began with the isosceles trapezoid ABCD constructed using Geometer’s Sketchpad. The diagram is shown in Figure 18.

![Figure 18: Midsegment Quadrilateral](image)

87 Michelle: We know that AD is parallel to BC. *[meta-proposal of a new idea]*

88 Sarah: Can we... can we use the diagonals? *[meta-proposal of a new plan]* I mean, I was thinking about the project we just did, how we used ...
At this point, Sarah added diagonals to the figure, as shown in Figure 19.

![Figure 19: Addition of Diagonals](image)

Michelle: The midline? [meta-contribution to a new idea, transactive request for elaboration]

Sarah: Yeah, like if EH is parallel to BD, then... [meta-contribution to a new idea, transactive response for elaboration]

Michelle: Oh yeah. [general confirmation]

Sarah: You see what I mean? [meta-assessment]

Michelle: Yeah. [general confirmation]

Amy: Yeah, but it'd have to be the midsegment. [meta-assessment; transactive-critique]

Michelle: It is. (Points to screen) E and H are midsegments of triangle ABD [transactive response: elaboration]

Sarah: Right. [general confirmation]

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2 Here Michelle is using the terms midline and midsegment interchangeably; she seems to be discussing the midsegment.

3 Michelle should have said that since both E and H are midpoints, segment EH is a midsegment.
Michelle: So EH is parallel to BD... [transactive response-elaboration]

Sarah: (both Sarah and Amy point to screen) - and then the same goes for that one [meta-development of new idea]

Michelle: By transitivity EH is parallel to GF, yeah, that works. [meta-development of a new idea, transactive-justification]

Researcher: So that gives you that that's parallel, so ... would you be able to prove that, if you needed to? Where does that come from? [transactive prompt – justification]

Michelle: We did prove that once, didn't we, in class?

Amy & Sarah: I think we did, yeah

Amy: that a midsegment [of the triangle formed by the diagonal] is parallel to the base of either triangle... in the quadrilateral... [meta-new idea; transactive response - clarification]

Michelle: I think we did that with... congruent triangles, right? (points to screen) Cause, uh... the triangle created with the midsegment is congruent to the triangle that it's the midsegment of, right? [meta-proposal of a new idea]

Researcher: or, similar? [transactive question – clarification]

Michelle: Or similar, because the sides are exactly half the length, and they share that angle. [transactive response – clarification, justification]

Researcher: Right, okay. Yeah! You've got it. Alright, so you've got that they're – these pairs are parallel by that diagonal and these pairs are parallel by that diagonal...

Michelle: mm hm

Researcher: So that they're parallel to the –

Sarah: I just had a light bulb – sorry – I just had a light bulb moment! (laughter) Um, don't we also know that EH is half of BD? And that, is there like a relationship between - [meta-proposal of a new idea]

Michelle: Oh yeah! You're right. The midsegment is half the length. [meta-contribution to new idea]
Sarah: Is there—right. Is there a relationship with the diagonals and isosceles trapezoids? Like they're the same? [meta-development of a new idea]

Amy: The diagonals? Is that what you're saying? [transactive question-clarification]

Sarah: Yeah. Isn't that—

Amy: (looking more closely at computer screen) If BD is the same as AC? Is that what you think? [transactive question—clarification]

Sarah: Yeah. Isn't that, like... because then you could do congruent sides. [meta-development of a new idea]

Amy: Like, I could see if they're all the same. [meta-contribution to new idea]

Michelle: Can you repeat that again? [transactive question—explanation]

Sarah: Yeah. If these (pointing to screen)—I'm just, I just thought of something that we knew. BD and AC would be congruent, because it's an isosceles trapezoid, right? [transactive response—explanation]

Michelle: Mm hm [general confirmation]

Sarah: And then, so EH would be one half BD. Because of the midsegment thing. [transactive response—explanation]

Michelle: Oh yeah [general confirmation]

Sarah: And then FG would be one half of BD, cause that's the midsegment of that. So if they're both one half of BD, then— [transactive response—explanation]

Michelle: they equal each other [meta-contribution to new idea; transactive statement—explanation]

Sarah (simultaneously): they'd be the same [meta-contribution to new idea; transactive statement—explanation]

Michelle: Yeah [general confirmation]

Researcher: and of course you could do the same thing the other way
Utterances of episode 6

In this episode, Sarah proposed the initial idea (line 90) on which the rest of the argumentation was ultimately based. Amy (line 105) challenged the idea with a critique, and Michelle (lines 108-109) provided an explanation. These first few lines of dialogue illustrate how the contributions of these three students wove together to create the fabric of their initial argument. Although it was Sarah’s idea to use the diagonals, Michelle indicated that she understood exactly how to use the diagonals by her next statement (line 93): “The midline?” Amy indicated that she understood the idea proposed by Sarah and Michelle by asking the question (line 105): “Yeah, but it’d have to be the midsegment.” Through both transactive and metacognitive utterances, these students created a basis of communication in which they established a taken-as-shared piece of knowledge before proceeding with the argumentation.

Mathematical constructs of episode 6

In the first part of the proof, Michelle made the first claim: that segment AD is parallel to segment BC. The data/warrant for her claim was never explicitly stated, but Michelle’s claim was ultimately not needed for the proof outline. Sarah made the next claim, that segment EH is parallel to BD. Amy solicited justification, which was provided by Michelle: that segment EH is the midsegment of triangle ABD (although
she does not state this correctly, it is evident from the context that this is what she meant to say). Although the next claim came from Sarah (lines 115-116): "...and then the same goes for that one," the moment Sarah spoke she and Amy had both excitedly pointed to the screen, seeing the next move simultaneously. Sarah was referring to the triangle BCD, extending the same argument to the midsegment FG. Michelle made the final statement (line 118) that provided both the final claim and a warrant which linked previous claims to the final claim: "By transitivity EH is parallel to GF, yeah, that works." In one statement, Michelle contributed an idea that built directly from Sarah's claim, provided a warrant, and assessed the results. The overall argument is given in Figure 20 below. The initial statement made by Michelle is not included in Figure 20, since the students did not use it in their eventual proof.

**Figure 20: Michelle, Sarah & Amy's argumentation scheme**
Summary of episode 6

Episode 6 highlights how these students developed a shared understanding of how the diagonals relate to the subsequent triangles, and used it to collectively formulate an argument. Throughout other parts of the group discussion not included in the episode above, the students alternately took on different roles, at times “teaching” one another, challenging each other, and contributing to each other's ideas. Through the analysis of the utterances and Toulmin’s analysis, this episode shows that the students collaborated successfully, constructing the key elements of the proof.

Episode 7: Caitlin, Jen & Marc (November 20)

The class that day had begun the study of neutral geometry, which does not assume Euclid’s Fifth Postulate (also known as the Parallel Postulate). Following a class discussion guided by the professor in which they developed a proof, the class had been given the task of working in groups to prove a theorem about a Saccheri Quadrilateral using neutral geometry. As defined previously, a Saccheri Quadrilateral is a quadrilateral in neutral geometry defined by two adjacent right angles, and the two opposite segments stemming from those right angles as being congruent. Caitlin, Jen and Marc were working together to prove the theorem. The professor wrote the following theorem and diagram on the board, shown in Figure 7 below. At the end of the class, the group presented their proof to the class. The full written proof is included in Appendix B. What follows is the discussion of the geometry situation that led Caitlin, Jen and Marc to the key ideas needed to write the
proof. The figure drawn by the group, which they refer to in the discussion below, is shown in Figure 21 below.

\[\text{Figure 21: Original diagram to accompany Saccheri Quadrilateral proof}\]

\begin{center}
\begin{tikzpicture}
    \fill (0,0) circle (1pt) node[below] {B} -- (5,0) circle (1pt) node[below] {C} -- (5,4) circle (1pt) node[above] {A} -- (0,4) circle (1pt) node[above] {D} -- (0,0);
    \draw (0,0) -- (5,0) -- (5,4) -- (0,4) -- cycle;
    \draw (1,2) circle (1pt) node[above] {N} -- (2,2) circle (1pt) node[above] {M} -- (3,2) circle (1pt) node[above] {M};
    \end{tikzpicture}
\end{center}

\textbf{Given:} SQ [Saccheri Quadrilateral] as shown, M and N are midpoints of $\overline{BC}$ and $\overline{AD}$ [the base and summit] respectively

\textbf{Prove:} $\overline{MN} \perp AD$ and $\overline{MN} \perp BC$

\textbf{Figure 21: Original diagram to accompany Saccheri Quadrilateral proof assignment}

206 Caitlin: We have, we have side angle side [postulate]... [meta-proposal of a new idea]

207

208

209 Jen: Mm hm. (pauses, thinking)

210

211 Caitlin: Oh. I think we do it, like, the same way. [She is referring to a technique the group had used for a previous proof that had been assigned for them to work on in groups.] So we prove this is um, these two [triangles ABM & DCM] are congruent by side angle side [postulate], then we prove that these [segments AM & DM] are congruent, then we can construct this line [segment NM] - [meta-development of a new idea]

218

219 Jen: We can’t do side angle side [postulate] because (inaudible). [meta-assessment; transactive - critique]

220

221

222 Caitlin: No, I really see it.
Caitlin: So, I'm assuming we drop a perpendicular, we just draw this [segment NM] - \[meta-development of new plan; transactive-elaboration\]

Jen: Mhm.

Caitlin: So we have side... no wait, we just have side - side. Cause these are congruent [segments AN & ND], this is the midpoint [N]. \[meta-development of new plan; transactive-explanation, justification\]

Jen: Oh, I get it. \[general confirmation\]

Caitlin: And that [segment NM] is reflexive, so side side side [triangles ANM & DNM are congruent by side side side postulate]. Therefore this angle [ANM] is congruent to that angle [DNM]. \[meta-development of new plan; transactive-elaboration, justification\] I don't know how to prove (perpendicular?).

The perpendicular segment and two diagonal segments Caitlin constructed, which she referred to above, are shown in Figure 22 below.

![Figure 22: Caitlin's constructions](image)

Jen: Well - doesn't that prove it? If you have a congruent (inaudible)? \[meta-assessment; transactive-critique\]
Marc: Yeah, it should.

Caitlin: Yeah.

Jen: Ok. Ready? So, we'll call these [triangles] AMN and DMN. These are congruent - um... [meta-contribution to a new idea]

Marc: How do you know that? [meta-assessment; transactive – justification, critique]

Jen: Because it's the midpoint. [transactive – justification]

Marc: Oh yeah.

Jen: This is reflexive. And then wait - Caitlin, how do you say AM and DM are congruent? Cause of these triangles [ANM & DNM] are congruent? Well we, yeah, we know these two are congruent [segments AM & DM] because these triangles are congruent. So then we have these angles congruent [ANM & DNM]. Right there, those two. And they're congruent supplementaries [angles ANM & DNM] so they have to be perpendicular. Okay. So that's the top of the summit. [transactive-explanation]

Marc: So we have side angle side. [transactive prompt-clarification]

Jen: No, we have side side side. Cause these two triangles - here, let me- [transactive response – clarification, explanation]

Marc: Or, side angle side – [transactive-critique]

Jen: No, cause -

Marc: Side, you got the same angle here [points to angles ANM & DNM] – [transactive-justification]

Jen: No, cause we're proving it. (pause) We want to show that these two angles [ANM & DNM] are congruent. [transactive statement-clarification]

Marc: Why? [transactive question-explanation]

Jen: Because we want to show that their supplements are congruent to show they're perpendicular. [transactive response-explanation]
Marc: We know its - we know this angle and this angle [ANM & DNM] are supplementary. We dropped a perpendicular. [transactive statement-critique]

Jen: No, no - we're doing number one.

Caitlin: You can't drop it -

Marc: Yeah, that's what I'm saying.

Caitlin: We can't drop a perpendicular. [transactive statement-explanation]

Marc: It's [segment NM] perpendicular to both the summit and base. [transactive-critique]

Jen: The line joining the midpoints of the summit and base is perpendicular. That's what we're proving - it is perpendicular. [transactive-explanation]

Marc: Oh, that's what we're proving? We're not given that? [transactive question-clarification]

Jen: Yeah. Right. (pause)

Caitlin: So does that prove that it's perpendicular to the summit? Or does - [transactive question-assessment]

Jen: That's just the summit [angles]. Wouldn't you think so? [transactive response; assessment]

Caitlin: Yeah, I would agree. [general confirmation]

Jen: But, I mean, we could probably use this proof to show that the bottom angles [BMN & CMN] are congruent, if you want. Or, we could just redo it. It's the same formula- [contribution to an idea]

Caitlin: Well, then can't we just say then we know this angle [BMA] is congruent to that angle [CMD], and that angle [AMN] is congruent to that angle [DMN], by CPCTC, [angle] addition ... [angle BMA plus AMN is equal to angle CMD plus DMN] - [meta-proposal of a new plan]

Jen: Yeah, and by constru- yeah. We could do that. By construction, these [angles BMA & AMN] are that angle [BMN], and these [angles CMD & DMN] are that angle [CMN]. You know what I mean? So we did it! [meta-contribution to new plan; assessment]
Utterances of episode 7

This episode illustrates how, through metacognitive and transactive utterances, one student (Caitlin) led the group in the development of the proof. In response to Jen’s critique of Caitlin’s initial idea, Caitlin convinced Jen that her initial plan would work through transactive explanation, elaboration and justification (lines 222, 226-227, 232-233). This created a learning opportunity for Jen, as evidenced by the next several exchanges. Her initial response (line 236): “Oh, I get it,” was coded as merely a general confirmation, so does not necessarily indicate her mathematical meaning was reconstructed. However, her contribution to Caitlin’s idea (lines 250-251, lines 260-266) and transactive response to Marc with a justification (line 271-272) show that she appropriated the argument to the extent that she was able to rephrase it.

Marc challenged the group’s plan several times, with transactive prompts for clarification (line 269), critique (lines 253, 274, 290-291, 303), justification (line 278), and finally a transactive request for explanation (line 285). It became clear after both Caitlin and Jen offered explanations that Marc did finally realize his idea would not work (line 310): “Oh, that’s what we’re proving? We’re not given that?” Marc had mistakenly been trying to use the fact that they needed to prove. He did not understand that the line Caitlin had drawn onto the figure could not be assumed to be a perpendicular line, and that that in fact was what they were trying to show. It is difficult to know exactly how much Marc got out of the discussion, since he did not contribute much for the remainder of the episode. Caitlin and Jen continued building on their ideas, extending the argument to show the bottom angles are congruent.
Mathematical constructs of episode 7

Initially, Caitlin proposed the strategy: "So we prove this is um, these two are congruent [triangles] by side angle side [postulate], then we prove that these [segments] are congruent, then we can construct this line" (lines 215-217). The figure Caitlin drew is shown again for convenience in Figure 23 below. The triangles Caitlin first proposed are congruent are triangles ABM and DCM. The line she was referring to is segment NM in the figure. The other two segments, AM and DM, were also drawn in by her. The second pair of congruences to which Caitlin referred next are segments AM and DM.

![Figure 23: Caitlin's diagram](image)

Jen challenged Caitlin's claim with the rebuttal (line 219): "We can't do side angle side because (inaudible)". Jen's critique appears to have caused Caitlin to change her claim (lines 232-233): "So we have side... no wait, we just have side-side. Cause these are congruent, this is the midpoint." However, when the final proof scheme appears it is evident that Caitlin could "see" the whole proof outline in her
head, and that both the Side Angle Side Postulate (SAS) and the Side Side Side Theorem (SSS) were necessary components. Caitlin filled in more details of her proof strategy (line 238): “And that is reflexive, so side side side. Therefore this angle is congruent to that angle.” Later, Jen also filled in some of the details when they began writing the proof down, explaining as she wrote as shown in lines 260-266. The first two arguments of the overall proof are shown in Figures 24 and 25 below. Although none of the three students ever verbally identified the data necessary to support the first claim, it was included in the proof they eventually wrote on the board (see Appendix B). This indicates that the data were taken-as-shared by all three students as self-evident. This is not particularly surprising, since two of the necessary data were given information. The third data followed directly from the segments AM and DM that Caitlin drew in, and the fact that M was assumed to be a midpoint.

![Figure 24: First part of argumentation scheme](image)

Data (Implicit)
1) Segments AB & DC are congruent (given)
2) Angles B & C are congruent (given)
3) Segments BM & MC are congruent (M is midpoint)

Claim 1: (Caitlin)
Triangles ABM and DCM are congruent

Warrant 1: (Caitlin)
Side Angle Side
Summary of episode 7

This episode is an example of successful collaboration, in which the group members challenged each other’s ideas and convinced each other through argumentation. Although Caitlin provided most of the key ideas, the other two students also contributed to the development of the proof. The analysis using Toulmin’s argumentation theory shows that through transactive (justifications, for instance) and metacognitive (new ideas, for example) utterances, the students were prompted to provide data and warrants to support their argumentation scheme.

Marc is seen more as the novice in this episode, in the expert-novice framework of the ZPD, since his contributions were not appropriate in the proof, in that they were based on his misunderstanding of the problem statement. However, overall the group engaged in collective reasoning and demonstrated successful collaboration.
Concluding analysis of part II

The analysis of the episodes presented above illustrates the nature of interactions of several students working in groups. Each of the episodes highlights a group of students who succeeded in working collaboratively, as they engaged in the process of proving as a social activity. The next level of analysis examines the episodes for common themes across all the small group episodes. Although each group of students had very different individual personalities and social relationships, some similarities were found across the small group episodes characteristic of successful collaboration. These results are presented in the remaining sections of this chapter, in which the predominant features of successful collaboration are identified and discussed.

Characteristics of successful collaboration

In the episodes highlighted above, each group of students collectively contributed most or all of the essential components needed to prove the theorem. In episode 5, after jointly assessing their progress the students were able to revise their strategy and formulate a plan that would lead to a successful proof. In episode 6, one student (Michelle) provided an explanation of another student’s idea (Sarah) to the third student (Amy), enabling all three students to continue to build off the initial idea, collectively contributing the essential elements of the proof. In episode 7, although Caitlin had the initial idea for the outline of the proof, Jen also contributed to its construction by providing additional explanations and warrants, and the final claim. In each of the episodes, the group accessed their collective zone
of proximal development, with each student having some knowledge that enabled them to make contributions to the group's progress and learn from the other students' contributions.

**Small group interactions**

The interactions observed in each of the episodes described above displayed a public exchange of ideas and negotiation of mathematical assertions. This is evidenced by each group's discourse, which is characterized as being multivocal. Cobb (1995) describes a discourse as being *univocal* if the perspective of one student dominates, and as being *multivocal* when multiple perspectives are shared. In the univocal case, the dominant student makes the judgment that the other students do not understand or have made a mistake, and the other students accept this judgment without question. In the multivocal case, every student attempts to advance their own perspectives by explaining their thoughts and challenging the ideas of their peers.

The characteristic of the discourse as being multivocal leads to the concept of *shared authority*. Univocal discourse can lead to an imbalance of power, in which peers are compelled to adapt to one student's ideas in order for the group activity to be effective (Cobb, 1995). Analysis of the episodes previously discussed shows that students interacting within a multivocal discourse accepted shared responsibility for making sense of their mathematical ideas: 1) through metacognitive monitoring of both one's own thoughts and the thoughts of one's peers; and 2) through transactive exchanges in which students' ideas were displayed and examined.
Collective reasoning

The episodes presented above show that students, through collaborative interactions, regularly provided data and warrants as they constructed substantial sequences of argumentation in proving activities. In addition, these components of their collective argumentation were often in response to transactive and metacognitive prompts. That is, students were prompted to provide additional information to support the structure of their collaborative arguments in order to convince each other of the validity of their mathematical claims. In some instances, data and warrants were not explicitly stated by students; however in these cases these components were usually included in the formal written proof. This shows that the data or warrants had been established as taken-as-shared knowledge. For instance, in episode 7, Caitlin and Jen based their first argument on data that was never explicitly stated, but that they later included in the written proof.

The analysis of students working in groups highlights discourse structures in which students developed strategies, presented ideas, defended mathematical claims, and assessed validity of arguments. Characteristics of successful collaboration included metacognitive monitoring of the group’s progress as well as transactive forms of justifying and critiquing one another’s ideas. Other characteristics included multi-vocal discussions in which the group shared authority. These discourse structures resulted in students providing strategies and supporting data and warrants, and constructing proofs through the establishment of taken-as-shared knowledge.
Concluding summary of chapter IV

In analyzing both the whole class discussions presented in Part I, and the small group work presented in Part II, several common themes emerged. One feature that was displayed in all the episodes was that students contributed significantly to the development of substantial argumentation. In whole class discussions, students' utterances were primarily in response to professor's prompts; however, over time an increase was observed in students responding directly to other students during class discussions. This is seen as a result of the professor transferring responsibility to the class, through transactive prompts for justification and through facilitative revoicing and rebounding. In small group work, students displayed evidence of shared authority through multi-vocal discussions. This is evident by the role-sharing of students as they alternately generated new ideas, critiqued each other's ideas, and evaluated the group's progress. Thus the roles that students and the professor played in discussions turned out to have a considerable impact on students' mathematical advancements.

During class discussions, the professor modeled the process of proof construction, by highlighting three major steps: 1) Determining what is needed for the proof; 2) Using the given information; and 3) Evaluating the results to decide whether sufficient information has been produced. During small group work, students displayed evidence of successfully developing proof outlines by following the three major steps. During whole class discussions, the professor placed an emphasis on collective reasoning, through transactive prompts for justification. During small group work, students provided appropriate justification for most of
their mathematical contributions as they filled in proofs. These justifications were frequently in response to their partners’ prompts for explanation or justification. This shows that the students had internalized the process of constructing an outline of the key components needed for a proof, and providing necessary data and warrants to support their arguments.

The next chapter addresses the central research question of this study, by summarizing the significant findings presented in this chapter. Chapter V also discusses the implications of these findings to the research community as well as the limitations and biases of the study, and finally suggests possible avenues for future research.
In the previous chapter, analysis of discourse and mathematical structures yielded several findings related to types and patterns of interactions that emerged in the discourse, as well as to the structure of mathematical argumentation that was developed. This chapter provides a summary of the findings presented in Chapter IV in relation to the central research question of this study. The central research question is: How does the classroom environment shape students' abilities to reason and prove in an inquiry-based, undergraduate geometry classroom? In order to address this question, I will first discuss findings as related to each of three topical questions. I will then draw upon the insights revealed by reflecting on the topical questions to answer the central question. Finally, I will discuss the significance and implications of these findings to the research community, the limitations and biases of the study, and possible directions for future research.

**Summary of Findings**

In addressing the central question of this study, analysis focused on three aspects of the classroom environment: 1) the nature of interactions between participants; 2) the mathematical resources and constructs of the classroom; and 3) the types of activities in which the class engaged. While summarizing the findings related to each of these aspects, particular attention will be paid to how they influenced the development of students' understanding, and what features played
the most prominent role in facilitating learning. From a sociocultural perspective, understanding is defined as “participating in a community of people who are becoming adept at doing and making sense of mathematics as well as coming to value such activity” (Hiebert & Grouws, 2007; p. 382). This definition will be useful in viewing students' understanding as expressed through their contributions to class discussions and their ability to participate in the assigned activities.

Nature of interactions

Analysis of professor utterances revealed that during class discussions focused on proving activities, the most predominant utterances were transactive and facilitative. During whole class discussions, the professor:

- emphasized reasoning and encouraged participation through transactive prompts for justification, explanation, clarification, elaboration and proof strategies;
- supported students in developing ideas and guided structure of arguments through facilitative revoicing and redirecting.

The use of these utterances during class discussions resulted in the creation of a public forum through which ideas were displayed, analyzed, and elaborated upon. Analysis of students’ utterances found that during class discussions, students participated in proof construction by:

- proposing new ideas and strategies;
- defending claims;
- considering, critiquing, and expanding on other students ideas;
• evaluating results.

Students provided justifications for their own statements as well as those of their peers, and offered reasoning for why a particular strategy would not be sufficient, or why a mathematical statement did not make sense.

Findings presented in Part I of Chapter IV revealed certain patterns of interactions in the ways the professor structured class discussions, and in the ways students engaged in those discussions. One prominent pattern during proving activities illustrated how the professor orchestrated a class discussion through a series of utterances. In this pattern, she solicited ideas and justifications through transactive prompts, guided and supported the development of the discussion through facilitative utterances, and concluded the discussion with a directive statement. This pattern is shown in Figure 26 below.

![Pattern of interaction](image-url)

**Figure 26: Pattern of interaction**
Through this form of discussion, the professor modeled the process of constructing a proof by highlighting essential components in writing a proof. This was done by placing emphasis on:

- determining what is to be shown;
- using given information;
- making connections among given constructs;
- assessing strategies and results to determine what is necessary and sufficient.

In this way, the professor scaffolded instruction by determining, on the basis of students' contributions, how to stimulate further advancements or realign their collective understanding.

Findings also show that through the forms of utterances and patterns of interactions previously discussed, the professor and students engaged in an exchange of ideas in which many students contributed. But the most significant finding of whole class interactions was that students contributed all or most of the essential components of each proof construction, by developing and building on each other's ideas publicly.

Furthermore, this carried over into small group discussions. Findings of small group work displaying evidence of successful collaboration revealed students often shared roles, and the discourse was multivocal. Thus, no one student was considered to be the authority of a group. Even in cases when one student appeared to lead the group, responsibility was shared as students challenged one another's
reasoning, defended their own ideas, and often convinced themselves of their peer’s statements by providing justification.

**Mathematical resources and constructs**

Analysis of the structure of argumentation presented by students in whole-class and small-group settings using Toulmin’s model revealed that through the discourse, *data and warrants were introduced* into the developing argumentation in order to create a shared basis of knowledge. That is, it was as a result of collaborative effort of the classroom participants that many of the necessary components of a proof were offered. In whole-class discussions, necessary data and warrants were often provided in response to the professor’s transactive prompts for justification. Analysis of whole-class argumentation often displayed several different argument streams, each constructed by one or more students. Each of these streams was analyzed both individually and in relation to the global argument. By analyzing the arguments in this way, a pattern was observed, particularly in earlier episodes, of increasing mathematical sophistication of a justification throughout the discussion. This is evidenced, for instance, in episode one, where the initial attempts to provide justifications (by Cheryl and Rachel) are mathematically weak, but subsequent contributions (from Mike and Bruce) are mathematically stronger.

In small group work, a justification was often provided in response to a peer’s critique, that served to support an argument. In some cases during small group work, one student’s conclusion became the data for another student’s claim. In other cases, a datum or warrant was implicitly used in a verbal argument, but
later was explicitly stated in the written proof. In this way, students readjusted their conceptions as they considered the contributions of their peers, through a process of constructing shared understanding.

Activities

The class engaged in many different activities throughout the semester. The most typical activities observed were:

- constructing a proof or an outline of a proof through a class discussion;
- working in groups to construct proofs or outlines of proofs;
- presenting written proofs to the class;
- working in groups using the dynamic geometry software or manipulatives to explore and make conjectures;
- class discussion in which students reflected on proof strategies and proof structures.

All of these tasks led to opportunities for students to engage in collective and individual reasoning, and to make sense of their conclusions. However, certain activities played a more significant role in shaping students' abilities to reason and prove.

Class discussions centered on proof construction were a dominant feature of the class, occurring frequently throughout the semester. Analysis of utterances showed that through interactions between the professor and the students during class discussions, a dynamic exchange was created and sustained in which the students were encouraged and supported as their proof and reasoning abilities
were publicly displayed through the discourse. Class discussions involving proof construction also provided a way for the professor to model the process of developing a proof.

Small group work also played a significant role in students' developing proof and reasoning abilities. Analysis of student-student discourse during small group work revealed similarities to aspects of whole-class discussions, such as students challenging and critiquing each other's ideas, and students building on one another's contributions. Thus, small group work provided an opportunity for students to negotiate ideas on a more intimate level, and continue developing verbal and written proof and reasoning skills while applying the techniques modeled during class discussions.

The other activities listed above provided a variety of tasks for students, and offered multiple perspectives through alternative representations and approaches. However, these activities did not appear as frequently in class and so did not seem to contribute as significantly as other activities. For instance, the use of dynamic geometry software or manipulatives was occasional. Furthermore, analysis of classes in which software or manipulatives were utilized did not produce significant findings on students' increased reasoning abilities related to these tools.

Overall environment

The answer to the central research question, How does the classroom environment shape students' abilities to reason and prove in an inquiry-based, undergraduate geometry classroom, is found by first describing the overall
environment of the classroom in this study, and then by discussing how the environment shaped students' learning. During whole-class discussions, the overall environment was one in which *participation was encouraged*. As noted earlier, through frequent use of transactive and facilitative utterances, the professor set the expectation that students would contribute, and also demonstrated what types of contributions were expected. As a result, students regularly and freely shared ideas, whether they were useful, correct or appropriate, and did not appear to be afraid of being wrong. The professor provided a supportive environment, often following students' leads by drawing their suggestions on the board and encouraging them to continue. The professor also consistently allotted ample time for these discussions to develop, providing opportunities for many different students to share their thoughts and ideas. Analysis of students' interactions in whole class discussions found students contributed more to class discussions over time. Analysis also revealed that the occurrence of student-student interactions increased over time. These observations are evident in Table 4 of Chapter IV (Part 1).

Another important feature of the classroom environment was an *emphasis on collective reasoning*. Through transactive prompts for justification and activities that focused on proving and reasoning activities, the professor encouraged students to justify their mathematical statements. As a result, students responded to prompts for justification by providing reasoning for their own statements as well as those of their peers. Findings also revealed that the number of student utterances that contained justification during class discussions increased over time, as evidenced in Table 4 of Chapter IV (Part 1). Activities centered on proving often formed the basis
for a class discussion, and frequently included significant contributions from students. This shows that all of the components of the classroom environment mentioned above played a role in students' participation in reasoning and proof construction.

This was also evidenced by students' group work. Through successful collaboration, students constructed proofs, providing most or all of the necessary data and warrants to support their arguments. This shows that they understood what was needed to adequately construct proofs. This also shows that they internalized the proof-modeling techniques used by the instructor, determining what they needed to show, how much was sufficient, and making logical connections from what was given to the final conclusions. Students identified key ideas needed for proofs, provided justifications, and assessed their progress. As a result, students made mathematical advancements as they participated in the social activity of proving.

**Significance and Implications**

This study contributes to the field of mathematics education by adding to the emerging body of research on classroom discourse and proving as a social activity, in particular at the college level, where existing research is inadequate. Most of the existing studies on proof and discourse have not considered the mathematical content or structure of argumentation formed through discussions. The present study contributes to the field of mathematics education by expanding on existing findings related to classroom discourse and proving. One such finding, the pattern of
discourse in which the professor solicited ideas through transactive prompts, facilitated discussion through revoicing and rebounding, and concluded the discussion with a directive prompt, illustrates a form of discussion that created an opportunity for students to make significant contributions to proof constructions. A finer-grain analysis of this pattern also revealed that the professor used transactive and facilitative prompts, including specific questions that redirected students' focus, in order to scaffold instruction by modeling essential steps of proof construction.

The present study also adds to existing research on the transfer of authority, showing that transferring authority onto students provided opportunities for the students to engage in collective reasoning and create shared knowledge, allowing them to challenge or expand on one another's arguments. Analysis of student group work indicated that in cases in which the group demonstrated successful collaboration, a key feature was that of shared authority. This shows that the students internalized the forms of reasoning collectively in classroom discussions, and used these forms of reasoning in order to advance their collective argumentation during small group work.

The present study expands on the work of others by using Toulmin's model in conjunction with discourse analysis, to analyze argumentation as it develops in the context of a college classroom discussion. Weber, et al. (2008) assert that certain necessary conditions are essential for a learning environment which supports students' engagement in reasoning: "... learning environments where student contributions are encouraged and not judged, sense making is encouraged and students are arbiters of what makes sense, and extended time is granted for
investigations and discussion will invite students to attend to and challenge the arguments of others...” (p. 260). The present study shows how these conditions were created in one college level geometry class. The present study revealed that collective reasoning found in the classroom carried over into student group work, in the form of collaborative proof construction.

**Implications to field of mathematics education**

Most researchers in the mathematics education community appear to agree that classrooms in which students are encouraged to explore mathematical ideas, develop various forms of reasoning, and participate in mathematical discussions are valuable to students’ learning. However, existing research shows that in the majority of mathematics classrooms in the U.S., “traditional” forms of instruction still dominate (Franke, Kazemi & Battey, 2007). There are likely many reasons this is the case: Engaging students in meaningful mathematical conversations and providing students with opportunities to participate significantly in the construction of proofs is no easy task, and what is needed to support such environments is not well defined in the research literature (Franke, Kazemi & Battey, 2007). This study shows how one professor created such an environment, while maintaining her teaching goals and her ultimate position as the expert, in a college geometry class.

Research also shows that university level students’ proof competencies are weak (Harel & Sowder, 2007). Many students exhibit lack of strategic knowledge, unsure of when to use different proof techniques and strategies (Weber, 2001). The
current study shows how through classroom discourse in which proof-construction was modeled and in which student participation was encouraged, students were able to collectively produce proofs and proof outlines, including necessary justifications. This shows that class discussions involving collaborative proving activities and emphasizing collective reasoning are effective ways of engaging students in participating in the construction of proofs.

Limitations and biases

One limitation of this study was in the collection of data, due to the complex and multi-faceted nature of a classroom. As sole researcher of this study, it was my responsibility to capture and record as much relevant data as possible, and to include as many details as my methods allowed. Use of a video camera and audio recorder helped tremendously with this task; however, it was impossible to pay attention to the multiple discussions and activities that frequently occurred simultaneously, especially during small group work. Selecting focus groups helped me in placing a recorder near a particular group; even so, by choosing one group I had to ignore the others. To compensate for this, I frequently had multiple recorders going at the same time. Fully capturing and deciphering discussions of students while working together also proved to be a challenge. Much of the student-student dialogue I captured during small group work was inaudible; I discovered that some students tended to speak very softly and did not speak in full sentences. Furthermore, they often did not clearly articulate which mathematical concepts or
symbols they were referring to. For these reasons, I had a limited amount of data that could be accurately understood and interpreted.

Another limitation was in my choice of focus groups. After one or two informal observations, I selected focus groups by choosing those groups that seemed to interact more with their peers during structured group work. The choice to observe more interactive students was a natural one, based on the focus of this study. However, in doing so it neglected those students who were quieter in nature, and chose to work more independently. In addition, I only presented cases of successful collaboration in the final text, although there were a few instances of unsuccessful collaboration as well. The reason I chose not to include the unsuccessful cases was primarily that they did not significantly contribute to the analysis of the study. In the cases of unsuccessful collaboration, it was difficult to identify a common characteristic that explained why they were unsuccessful.

A possible bias of the study is the influence of my presence on the instructor and students during observations. Although it is my assumption that the instructor conducted the class in a way that was typical of her nature, she was aware of the purpose of the study, and this may have influenced her pedagogical choices. However, this would not influence the integrity of the study, since the intention was in part to describe the ways in which the professor created the environment described above. The possible influence on students of being observed and recorded also cannot be overlooked. As was noted in Chapter III, a slight change was visible in students' reactions to me, as they displayed signs of being more comfortable as research subjects over the first couple of weeks. However, there were occasions
when a group of students appeared excited or nervous when they saw the camera focused on them, and this may have influenced their actions and dialogue.

**Generalizability**

Qualitative inquiry by nature focuses on context and particulars: "... it proceeds from the assumption that ideas, people, and events cannot be fully understood if isolated from the circumstances in which and through which they naturally occur" (Schram, 2006, p. 9). There has been much debate among researchers about the generalizability of qualitative research. Some researchers claim it is possible to generalize results of a qualitative study, others claim it is not, and still others maintain that it is irrelevant (Schram, 2006). By providing a sufficiently detailed account of the context and specifics of the classroom environment presented in this study, I claim that I have provided the reader with enough information to make an informed judgment about the applicability of my findings to other, similar classroom situations. My findings are not generalizable as descriptions of what will happen with other teachers and classes. Rather, they are generalizable as descriptions of what other educators might do, and might see, given that they are engaging in a set of similar circumstances and goals. That being said, an elementary or middle school teacher should take caution in applying the ideas presented here, and should adjust them accordingly to the base of knowledge and learning goals that would realistically be set for students at that level.
Possible avenues for future research

The results of this study suggest that collaborative discussions centered on proving activities in which students are involved can enable students to become more competent at proving. An interesting extension of this study would be to simultaneously analyze students’ individual work in comparison to students’ construction of knowledge as evidenced by their participation in group discussions, to gain more insight into how discussions and proving as a social activity affect students’ individual conceptions and abilities. This might be accomplished by adding individual interviews to the methodology in conjunction with written assessments.

Another interesting direction to explore would be to analyze students conceptions about proof and how those conceptions change over time in a class in which proof as a social activity is a regular feature. It might also be informative to follow pre-service teachers in their own classrooms during internship or student teaching to see how the pedagogy of this course has influenced their own teaching strategies.

Final thoughts

Although the class-based interviews with students did not provide useful data in addressing the research questions of this study, they offered insights into students’ conceptions on such things as working collaboratively and the purpose of proofs. A few of the students’ responses are included below.
During the interviews, students expressed their thoughts on working together:

Like when you come up with an argument in your own head, you might think it's, you know, fool proof – like I've covered everything. But then if you explain it to someone else who had a different idea, and then they'll say, well, what about this? And it makes you realize that, okay, mine isn't as, you know, well organized as I thought, and I gotta rethink it, or you know, they're right, and I was wrong, based off how strong their arguments were. (Bruce)

Students also expressed their thoughts on the purpose or value of proving:

Once you're in the mindset of proving things, then you always want to know why. Like, why is that true? Like, what makes it right? (Jen)

And you know, we're all in teacher and math education... so obviously we're going to be faced with questions of why. Um, you could introduce a new concept on the board, and some kid says, well, why is that always the case? ... if you understand the proof of it, and understand how I got, how mathematicians got to this rule or law, or theorem, then you have an answer for the kid. (Dylan)
APPENDICES

APPENDIX A

Connor’s group’s proof that a translation is an isometry: (Nov. 2)

Proof:

By definition of a translation, $\overline{AA'} \parallel \overline{BB'} \parallel \overline{CC'}$

Observe the quadrilateral $ABB'A'$ has a pair of $\cong$ and $\parallel$ sides, $(\overline{AA'}$ and $\overline{BB'})$ and is $\therefore$ a parallelogram.

$\therefore AB \cong A'B'$ by definition of parallelogram

Also observe that quadrilateral $BCC'B'$ & quadrilateral $ACC'A'$ are also parallelograms by similar reasoning, and so $\overline{AC} \cong \overline{A'C'}$ & $\overline{BC} \cong \overline{B'C'}$.

Since $\overline{AC} \cong \overline{A'C'}$, $\overline{AB} \cong \overline{A'B'}$ & $\overline{BC} \cong \overline{B'C'}$, $\triangle ABC \cong \triangle A'B'C'$ by SSS, so $\therefore$ a translation is an isometry.
Proof:

Let A be the center of rotation.

\[ BA = B'A \quad \text{definition of rotation} \]
\[ CA = C'A \quad \text{definition of rotation} \]

1) \( \angle BAB' \equiv \angle CAC' \quad \text{definition of rotation} \]
2) \( \angle BAB' = \angle BAC + \angle CAB' \quad \text{angle addition} \]
3) \( \angle CAC' = \angle B'AC' + \angle CAB' \quad \text{angle addition} \]
4) \( \angle BAC + \angle CAB' = \angle B'AC' + \angle CAB' \)
5) \( \angle BAC = \angle B'AC' \quad \text{(subtract } \angle CAB' \text{ from both sides)} \]
6) Create \( BC \) and \( B'C' \) \quad \text{Euclid} \]
7) \( \triangle BAC \equiv \triangle B'AC' \quad \text{(SAS, A, B, 5)} \]
8) \( BC = B'C' \quad \text{CPCTC} \]

\[ \therefore \text{Distance is preserved.} \]
Proof:
Use distance formula to show

a) \[|AB| = |A'B'|\]
b) \[|BC| = |B'C'|\]
c) \[|AC = |A'C'||\]

1) \[AB = \sqrt{(-b) - (-a)^2 + (d - 0)^2}\]
   \[= \sqrt{(b - a)^2 + d^2}\]
\[A'B' = \sqrt{(b - a)^2 + (d - 0)^2}\]
   \[= \sqrt{(b - a)^2 + d^2}\]

\[\therefore |AB| = |A'B'|\]

repeat for b) and c)
APPENDIX B

Caitlin, Jen and Marc's proof (Nov. 20)

Given SQ quadrilateral ABCD with ∠B and ∠C right angles, and \( AB \equiv DC \)

Construct midpoints of AD and BC, label M & N

Construct \( AM \) and \( MD \).

\( \triangle ABM \equiv \triangle DCM \) by SAS \( \frac{AB}{DC} \) Given

\( BM \equiv MC \) by midpoint

\( AM \equiv MD \) by CPCTC

Construct midsegment \( MN \)

\( MN \equiv MN \) by reflexive

\( AN \equiv ND \) by midpoint

\( \triangle ANM \equiv \triangle DNM \) by SSS

\( \angle ANM \equiv \angle DNM \) and supplementary

\( \therefore NM \perp \) to summit

\( \angle AMN \equiv \angle DMN \) by CPCTC

\( \angle DMC \equiv \angle AMB \) by CPCTC of \( \triangle ABM \) and \( \triangle DCM \)

\( \angle NMA + \angle AMB = \angle NMB \)

\( \angle NMD + \angle DMC = \angle NMC \)

angle addition, supplementary angles

\( \therefore \angle NMB \equiv \angle NMC \) and \( NM \perp \) to base
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14-Oct-2009

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IRB #: 4684
Study: Conjecturing and Proving in an Inquiry-Based Class
Approval Date: 14-Oct-2009

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Exempt as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 101(b). Approval is granted to conduct your study as described in your protocol.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, Responsibilities of Directors of Research Studies Involving Human Subjects. (This document is also available at http://www.unh.edu/osr/compliance/irb.html.) Please read this document carefully before commencing your work involving human subjects.

Upon completion of your study, please complete the enclosed Exempt Study Final Report form and return it to this office along with a report of your findings.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Manager

cc: File
    McCrone, Sharon