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Bimodule categories and monoidal 2-structure

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Bimodule categories and monoidal 2-structure

Abstract
We define a notion of tensor product of bimodule categories and prove that with this product the 2-category of $C$-bimodule categories for fixed tensor $C$ is a monoidal 2-category in the sense of Kapranov and Voevodsky ([KV91]). We then provide a monoidal-structure preserving 2-equivalence between the 2-category of $C$-bimodule categories and $Z(C)$-module categories (module categories over the center of $C$). The (braided) tensor structure of $C_1 \boxtimes D C_2$ for (braided) fusion categories over braided fusion $D$ is introduced. For a finite group $G$ we show that de-equivariantization is equivalent to the tensor product over $\text{Rep}(G)$. The fusion rules for the Grothendeick ring of $\text{Rep}(G)$-module categories are derived and it is shown that the group of invertible $\text{Rep}(G)$-module categories is isomorphic to $H^2(G, k_x)$, extending results in [ENO09].

Keywords
Mathematics, Physics, Theory
BIMODULE CATEGORIES AND MONOIDAL 2-STRUCTURE

BY

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ABSTRACT

BIMODULE CATEGORIES AND MONOIDAL 2-STRUCTURE:

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Advisor: Dr. Dmitri Nikshych

We define a notion of tensor product of bimodule categories and prove that with this product the 2-category of $C$-bimodule categories for fixed tensor $C$ is a monoidal 2-category in the sense of Kapranov and Voevodsky ([KV91]). We then provide a monoidal-structure preserving 2-equivalence between the 2-category of $C$-bimodule categories and $Z(C)$-module categories (module categories over the center of $C$). The (braided) tensor structure of $C_1 \boxtimes_D C_2$ for (braided) fusion categories over braided fusion $D$ is introduced. For a finite group $G$ we show that de-equivariantization is equivalent to the tensor product over $\text{Rep}(G)$. The fusion rules for the Grothendeick ring of $\text{Rep}(G)$-module categories are derived and it is shown that the group of invertible $\text{Rep}(G)$-module categories is isomorphic to $H^2(G, k^x)$, extending results in [ENO09].
INTRODUCTION

0.1 Generalities

Over the last century it has become evident that the study of algebraic structures from a module theoretic perspective is effective and powerful. The essential paradigm hinges on the observation that one structure may "act" on another and that in studying such actions one may learn something about the structures involved. The application of this basic notion has led to the development of a vast machinery of techniques and methods. In the theory of group representations, for example, one defines the action of a group on a vector space by specifying an association between elements of a group and linear transformations on a fixed space. Much can be understood about groups by making observations about the sorts of linear transformations which can arise by this process and in particular the traces of these linear maps (character theory). To study Lie algebras one defines an associative algebra as a certain quotient of the tensor algebra and then studies modules over this algebra. This notion also occurs naturally in more physical contexts, such as Boundary Conformal Field Theory (see for example [Car], [JF03], [VP01]). To any CFT is associated a ring-like object which acts on boundary conditions in a higher-dimensional space. Considerations about how similar constructions can be deformed have helped lead to the development of the theory of quantum groups, Hopf algebras and algebraic category theory, and have
deep applications in theoretical physics (see [Maj02], [Str07], [FMS99] among many others).

It is beneficial to consider the ways in which modules interact. The collection of modules defined over a given structure will generally form a category with extra algebraic structure allowing the application of extended classical results and constructions from ring and group theory. In precisely this fashion one moves from a "one-dimensional theory" to an enriched categorical theory with analogous but subtler structures yielding analogous but more refined results. Thus the classical picture acts as a cartography for the new theory and provides a narrative over which it develops.

The categories under study are required to satisfy axioms making something akin to linear algebra possible (so called abelian categories). Fusion categories are defined to be abelian categories equipped with a monoidal structure (multiplication) that behaves nicely with respect to other important operations. The notion of "monoidal category" is an abstraction of the notion of a ring and is intended to capture ring-like properties on an axiomatic level. Similarly we may define symmetric braided tensor categories as abstractions of a commutative ring, and module category as an abstracted module. Module categories, introduced by Bernstein in [Ber95] and studied in [Ost03], [EO04] among many others, form the basic objects of study in this thesis. The definition of a module category involves describing the action of a monoidal category just as classical modules describe actions of rings. Because we are dealing with more abstract structures the new axioms take the form of commuting diagrams whose vertices are objects and whose edges consist of appropriately defined maps.
which form part of the definition of the action. These maps dictate the appropriate associativity and unit constraints, in the categorical context, that one would see expressed in equations such as $(xy)z = x(yz)$ and $1x = x$ in algebra. As with rings and modules, one would like some meaningful way by which to relate pairs of module categories. There we have functions preserving module structure (linearity) and here we have functors preserving module category structure, so called module functors. One primary difference in the categorical setting is that here we have a way of relating pairs of module functors. Since module functors themselves are required to satisfy certain axioms (again taking the form of commuting diagrams) we may define module transformations as transformations preserving this structure in the appropriate fashion.

Just as modules over a fixed ring form a category, module categories over a fixed monoidal category form an appropriately enriched structure, called a 2-category. Just as in certain circumstances the category of modules over a fixed ring may itself have the structure of a monoidal category (under tensor product of modules) so may the associated 2-category of module categories have under certain conditions a monoidal structure making it a monoidal 2-category. As we move from monoidal category to monoidal 2-category the basic data expressed in diagrams (2-dimensional versions of equations in the lower-dimensional case) are replaced by 3-dimensional diagrams, polytopes, which represent restrictions on the ways cells of various levels are allowed to interact. Now instead of just 0-cells (objects) and 1-cells (morphisms) we have 2-cells (morphisms between morphisms). A priori there is no reason why the theory should fail to continue beyond level two yielding 3-cells, 4-cells etc. Although it is possible to
define higher level structures leading to \( n \)-categories, and even \( \infty \)-categories, we leave this to future endeavor.

The basic "nice" condition allowing us to define a tensor product between module categories occurs when we require that module categories are really *bimodule* categories. One instance in which this happens arises naturally when we stipulate that the underlying monoidal category is *braided*, a notion generalizing the idea of ring commutativity in an appropriately categorical way. In such a case we can define a tensor product of bimodule categories in a way reminiscent of the definition of the tensor product of modules; by stipulating an object universal for certain types of functors. If the bimodule categories in question are taken over a fixed monoidal category \( C \) we denote this new tensor product \( \boxtimes_C \). As the notation suggests \( \boxtimes_C \) reduces to a well known product for abelian categories developed by Deligne in [Del90]: in the case that \( C = \text{Vec} \), the category of vector spaces, we have \( \boxtimes_C = \boxtimes \).

A major part of this thesis has focused on asking and answering basic theoretical questions about \( \boxtimes_C \) and the associated monoidal 2-category of bimodule categories. It turns out that \( \boxtimes_C \) shares, in categorical analogue, many properties of the classical module theoretic tensor product, e.g. weak associativity, Frobenius reciprocity, and unitality with respect to the underlying monoidal category. These results, as in the classical case, provide powerful tools required for difficult calculations and form a basic starting point from which to develop algebraic aspects of the theory.
0.2 Thesis outline

First steps in defining this extended product involve defining balanced functors from the Deligne product of a pair of module categories. This approach mimics the classical definition of tensor product of modules as universal object for balanced or middle linear morphisms. Tensor product of module categories is then defined in terms of a universal functor factoring balanced functors. In Theorem 2.3.1 we prove that the tensor product exists; explicitly we prove that, for $\mathcal{M}$ a right $\mathcal{C}$-module category and $\mathcal{N}$ a left $\mathcal{C}$-module category there is a canonical equivalence $\mathcal{M} \boxtimes_\mathcal{C} \mathcal{N} \simeq \text{Fun}_\mathcal{C}(\mathcal{M}^{\text{op}}, \mathcal{N})$ where the category on the right is the appropriate category of $\mathcal{C}$-module functors.

In order to apply the tensor product of module categories we provide results in §2.3 giving 2-category analogues to classical formulas relating tensor product and hom-functor. In this setting the classical horn functor is replaced by the 2-functor $\text{Fun}_\mathcal{C}$ giving categories of right exact $\mathcal{C}$-module functors.

In §4.1 we prove

Theorem 0.2.1. For any monoidal category $\mathcal{C}$ the associated 2-category $\mathcal{B}(\mathcal{C})$ of $\mathcal{C}$-bimodule categories equipped with the tensor product $\boxtimes_\mathcal{C}$ becomes a (non-semistrict) monoidal 2-category in the sense of [KV91].

In Chapter 5 we discuss the tensor product for a special class of module categories. Here we assume our module categories are equipped with the structure of fusion categories and that their centers contain a faithful image of some fixed braided fusion category $\mathcal{D}$ (such categories are said to be tensor over $\mathcal{D}$, see Definition 5.0.7). Under these circumstances the tensor product itself has the structure of a fusion category.
If the module categories are braided the tensor product is braided. We describe these
structures explicitly.

As an immediate application we prove in Chapter 6 that de-equivariantization of
a tensor category can be represented as a tensor product over Rep(G), the category of
finite dimensional representations of a finite group G. Let A be the regular algebra in Rep(G). For tensor category \( C \) over Rep(G) the de-equivariantization \( C_G \) is defined
to be the tensor category of \( A \)-modules in \( C \). This definition was given in [DGNO10]
and studied extensively there. We prove

**Theorem 0.2.2.** There is a canonical tensor equivalence \( C_G \simeq C \boxtimes_{Rep(G)} Vec \) such
that the canonical functor \( C \to C \boxtimes_{Rep(G)} Vec \) is identified with the canonical (free
module) functor \( C \to C_G \).

In §7 we introduce the notion of the center of a bimodule category generalizing
the notion of the center of a monoidal category. We then prove a monoidal-structure
preserving 2-equivalence between the monoidal 2-category of \( C \)-bimodule categories,
denoted \( B(C) \), and \( Z(C) \)-Mod, module categories over the center \( Z(C) \):

**Theorem 0.2.3.** There is a canonical monoidal equivalence between 2-categories \( B(C) \)
and \( Z(C) \)-Mod.

In §8 we give a second application of the monoidal structure in \( B(C) \). To be precise
we show that, for arbitrary finite group \( G \), fusion rules for Rep(G)-module categories
over \( \boxtimes_{Rep(G)} \) correspond to products in the twisted Burnside ring over \( G \) (see e.g.
[OY01] and [Ros07]). As a side effect we show that the group of indecomposable
invertible $\text{Rep}(G)$-module categories is isomorphic to $H^2(G, k^\times)$ thus generalizing results in [ENO09] given for finite abelian groups.
CHAPTER I

PRELIMINARIES, BACKGROUND

Very little in this section is new. Where it seemed necessary sources have been indicated. In most cases what is included here has become standard and so we omit references (suggested general references: [Mac00], [BK01], [Kas95] along with those already given in the introduction).

1.1 Abelian categories

As mentioned in the introduction we are interested in studying an enriched, categorified version of the theory of rings and modules. The proper context in which to do this should provide tools and structures allowing us to do something akin to linear algebra in this extended region of discourse. In this section we will outline the basic sorts of categories with which we will have occasion to work in later sections.

Definition 1.1.1. An additive category is a category $C$ satisfying the following.

i) Every hom set has the structure of an abelian group with respect to which composition of morphisms is a group homomorphism.

ii) $C$ has a zero object $0$ with the property that $\text{Hom}(0,0) = 0$.

iii) (Existence of direct sums.) for any objects $X_1, X_2 \in C$ there exists an object $Z := X_1 \oplus X_2 \in C$ and morphisms $j_i : X_i \to Z, p_i : Z \to X_i$ for $i = 1, 2$ such
that \( p_i \circ j_i = id_{X_i} \) and \( j_1 \circ p_1 + j_2 \circ p_2 = id_Z \) and \( Z \) is unique object up to a unique isomorphism having this property.

The object \( Z \) in \((iii)\) is called the direct sum of \( X_1 \) and \( X_2 \) and is denoted \( X_1 \oplus X_2 \). A functor \( F : C \rightarrow D \) between additive categories is said to be an additive functor if the associated functions \( \text{Hom}_C(X, Y) \rightarrow \text{Hom}_D(F(X), F(Y)) \) are group homomorphisms.

**Definition 1.1.2.** Let \( k \) be any field. An additive category is \( k \)-linear if each hom set has the structure of a vector space over \( k \) with respect to which composition of morphisms is bilinear. A functor between \( k \)-linear categories is a \( k \)-linear functor if the associated functions between hom sets are linear transformations.

Let \( C \) be an additive category, and \( f : X \rightarrow Y \) a morphism in \( C \). Then the kernel of \( f \) (if it exists) is the unique (up to a unique isomorphism) object \( K \) together with a morphism \( \kappa : K \rightarrow X \) such that \( f \circ \kappa = 0 \) and if \( \kappa' : K' \rightarrow X \) is any other morphism with this property there is a morphism \( j : K' \rightarrow K \) with \( \kappa \circ j = \kappa' \). Typically we denote the kernel of \( f \) by \( \ker(f) \). Similarly one defines the cokernel of \( f \) to be an object \( \text{coker}(f) \) and a morphism \( c : Y \rightarrow \text{coker}(f) \) with the property that \( c \circ f = 0 \) and which is universal with respect to this property in a way analogous to the universality defining \( \ker(f) \). If \( \ker(f) = 0 \) \( f \) is said to be injective, and surjective if \( \text{coker}(f) = 0 \).

In the case that \( f \) is injective we call \( X \) a subobject of \( Y \) and if \( f \) is surjective we call \( Y \) a quotient object of \( X \). In an additive category there is no guarantee that kernels and cokernels exist. We will require that they do.

**Definition 1.1.3.** Let \( C \) be an additive category. Then \( C \) is an abelian category if it satisfies the further property that for any morphism \( f : X \rightarrow Y \) there is a composition
ker(\(f\)) \xrightarrow{\xi} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{\zeta} \text{coker}(f) \text{ with } j \circ i = f \text{ and } \text{coker}(\kappa) = I = \text{ker}(\epsilon). \text{ The object } I \text{ is called the image of } f. \text{ In particular, kernels and cokernels exist in an abelian category.}

**Definition 1.1.4.** In an abelian category \(C\) an object is said to be simple if its only subobjects are itself and 0. If there are only finitely many isomorphism classes of simple objects then \(C\) is called finite. If an object \(Y\) can be written as the direct sum of simple objects \(Y\) is called semisimple, and \(C\) is called semisimple if all of its objects are semisimple.

Any category for which the class of objects form a set will be called small. An important theorem of Mitchell shows that the category of modules over a fixed ring is the typical example of an abelian category. We include it here without proof for the sake of completeness. See [Fre64] and [Mit64] for a more thorough discussion.

**Theorem 1.1.5** (Mitchell). Every small abelian category is equivalent to a full subcategory of the category of left modules over an associative unital ring. If the category is \(k\)-linear then the ring is a \(k\)-algebra.

We end this section on abelian categories with a few definitions familiar from topology, the theory of modules, and representation theory which will be of importance to us in the sequel.

**Definition 1.1.6.** An exact sequence in an abelian category is a diagram of the form

\[
\cdots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_i} X_i \xrightarrow{f_{i+1}} X_{i+1} \xrightarrow{f_{i+2}} \cdots
\]
where \( \ker(f_{i+1}) \) is the image of \( f_i \) for every \( i \). That is \( f_{i+1}f_i = 0 \). If all but finitely many of the \( X_i \) are 0 then this is called a **finite** exact sequence.

**Definition 1.1.7.** Functor \( F : \mathcal{A} \to \mathcal{B} \) is said to be **right exact** if \( F \) takes short exact sequences \( 0 \to A \to B \to C \to 0 \) in \( \mathcal{A} \) to exact sequences \( F(A) \to F(B) \to F(C) \to 0 \) in \( \mathcal{B} \). Similarly one defines left exact functors. Denote by \( \text{Fun}(\mathcal{A}, \mathcal{B}) \) the category of right exact functors \( \mathcal{A} \to \mathcal{B} \).

### 1.2 Monoidal and fusion categories

In the rest of this thesis all categories are assumed to be abelian and \( k \)-linear, have finite-dimensional hom spaces, and all functors are assumed to be additive and \( k \)-linear. Even though most of what we do here is valid over fields of positive characteristic we assume at the outset that \( k \) is a fixed field of characteristic 0.

**Definition 1.2.1.** A **monoidal category** \( \mathcal{C} \) consists of the following: a category \( \mathcal{C} \) containing an object \( 1 \) called the **unit** of \( \mathcal{C} \), an exact-in-both-variables bifunctor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \), natural isomorphisms \( a : \otimes(\otimes \times \text{id}) \to \otimes(\text{id} \times \otimes) \), \( r_X : X \otimes 1 \simeq X \), \( \ell_X : 1 \otimes X \simeq X \) whenever \( X \in \mathcal{C} \), required to satisfy the following commutative diagrams:

\[
\begin{align*}
((WX)Y)Z & \xrightarrow{aw_{w,x,y,z}} (WX)(YZ) \\
(W(XY))Z & \xrightarrow{aw_{w,x,y,z}} W((XY)Z) \\
(WX)(YZ) & \xrightarrow{aw_{w,x,y,z}} W(X(YZ))
\end{align*}
\]
for any objects $W, X, Y, Z \in C$. Here, as in the sequel, we may abbreviate tensor products as juxtaposition in an effort to save space. The natural isomorphism $\alpha$ is called an *associativity constraint* and $\ell, r$ *unit constraints* of $C$. The monoidal category $C$ is said to be *strict* if all the natural isomorphisms $\alpha, \ell, r$ are identity.

**Remark 1.2.2.** Denote by $\boxtimes$ the product of abelian categories introduced in [Del90].
This is an object in the category of abelian categories universal for right-exact in both variables bifunctors from the cartesian product category $C \times D$. If $(C, \otimes, 1, \alpha, \ell, r)$ and $(D, \otimes, 1', \alpha', \ell', r')$ are monoidal categories then $C \boxtimes D$ has the structure of a monoidal category as follows: $(X_1 \boxtimes X_2) \otimes (Y_1 \boxtimes Y_2) = (X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2)$ with associativity constraint $\alpha \boxtimes \alpha'$, unit object $1 \boxtimes 1'$ and unit constraints $\ell \boxtimes \ell'$, $r \boxtimes r'$.

**Definition 1.2.3.** Let $C = (C, \otimes, 1, \alpha, \ell, r, 1), D = (D, \otimes, 1', \alpha', \ell', r', 1')$ be monoidal categories. A functor $F : C \rightarrow D$ is said to be a *monoidal functor* if it comes with natural isomorphisms $f_{X,Y} : F(X \otimes Y) \simeq F(X) \otimes F(Y)$ and $u : F(1) \simeq 1'$ satisfying the following hexagon and squares for every $X, Y, Z \in C$.

\[
\begin{align*}
F(XY)F(Z) & \xrightarrow{f_{XY,Z}} F((XY)Z) \xrightarrow{F(\alpha_{X,Y,Z})} F(X(YZ)) \\
F(XF(Y))F(Z) & \xleftarrow{f_{X,Y \otimes id}} (F(X)F(Y))F(Z) \xrightarrow{F(\alpha_{X,Y} \otimes id)} F(X)(F(Y)F(Z)) \\
F(X)F(1) & \xrightarrow{id \otimes f_{Y,Z}} F(X)(F(Y)F(Z)) \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} F(X)(F(YZ)) \\
F(X) & \xrightarrow{F(\ell_Z)} F(X) \xrightarrow{F(\ell_X \otimes id)} F(X \otimes 1) \xleftarrow{f_{X,1}} F(X) \otimes F(1) \\
F(1 \otimes X) & \xrightarrow{F(\ell_X)} F(X) \xrightarrow{u \otimes id} 1' \otimes F(X) \xrightarrow{F(1 \otimes \alpha)} F(1 \otimes X) \xrightarrow{F(t_X)} F(X)
\end{align*}
\]
In order to emphasize or designate the linearity constraint $f$ for a functor $F$ we may on occasion write $(F, f)$. The functor is said to be a strict monoidal functor if the natural isomorphism $f$ is identity.

Following terminology from regular category theory we will say that two monoidal categories are monoidally equivalent if there is a monoidal functor between them which is an equivalence. As it turns out, by MacLane's famous "strictness theorem," we may justifiably assume all monoidal categories to be strict. We include the statement here for completeness and because we will use it extensively in what follows.

**Theorem 1.2.4** (MacLane strictness theorem). Any monoidal category is monoidally equivalent to a strict one.

A nice proof of Theorem 1.2.4 may be found in Joyal and Street's 1993 paper on braided tensor categories [JS93]. In what follows we will assume monoidal categories strict unless stated otherwise. The primary benefit of having such a theorem is that it provides notational convenience simplifying diagrams and calculations. For example it allows us to replace the expressions $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ with the now unambiguous expression $X \otimes Y \otimes Z$, and allows us to dispense with structural constraints.

**Definition 1.2.5.** Let $(F, f), (G, g) : C \rightarrow D$ be two monoidal functors. A monoidal natural transformation $\eta : F \rightarrow G$ is a natural transformation satisfying the rectangle

$$
\begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{\eta_{X \otimes Y}} & G(X \otimes Y) \\
\downarrow f_{X,Y} & & \downarrow g_{X,Y} \\
F(X) \otimes F(Y) & \xrightarrow{\eta_X \otimes \eta_Y} & G(X) \otimes G(Y)
\end{array}
$$
for any $X, Y \in C$.

As in many familiar classical situations ($\text{Rep}(G)$, $R\text{-Mod}$ for commutative $R$, pointed topological spaces, etc) there is a natural notion of duality. The following definition gives a categorical axiomatization of this concept.

**Definition 1.2.6.** Let $C$ be a monoidal category, and let $X$ be an object of $C$. An object $Y$ is said to be a *right dual* of $X$ if there are morphisms $\text{ev}_X : Y \otimes X \to 1$ and $\text{coev}_X : 1 \to X \otimes Y$, called *evaluation* and *coevaluation*, such that both of the compositions

$$X = 1 \otimes X \xrightarrow{\text{coev}_X} X \otimes Y \otimes X \xrightarrow{\text{ev}_X} X \otimes 1 = X$$

$$Y = Y \otimes 1 \xrightarrow{\text{coev}_X} X \otimes Y \otimes X \xrightarrow{\text{ev}_X} 1 \otimes Y = Y$$

are equal to identity. Similarly one defines a *left dual* of $X$ to be an object $V$ together with morphisms $\text{ev}'_X : X \otimes V \to 1$ and $\text{coev}'_X : 1 \to V \otimes X$ making both of the compositions

$$X = X \otimes 1 \xrightarrow{\text{coev}'_X} X \otimes V \otimes X \xrightarrow{\text{ev}'_X} 1 \otimes X = X$$

$$V = 1 \otimes V \xrightarrow{\text{coev}'_X} V \otimes X \otimes V \xrightarrow{\text{ev}'_X} V \otimes 1 = V$$

identity.

It is well known that if $X$ possesses any left (right) dual then it is unique up to a unique isomorphism. In this case the left (right) dual object of $X$ is denoted $^*X$ (resp. $X^*$). Furthermore this process of associating to an object its duals, should such dual objects exist, extends to morphisms. Explicitly, if $f : X \to Y$ is a morphism between objects $X, Y$ possessing right duals then define the right dual $f^* : Y^* \to X^*$ of $f$ by
the composition

\[ Y^* = Y^* \otimes 1 \xrightarrow{coev_X} Y^* \otimes X \otimes X^* \xrightarrow{id \otimes f \otimes id} Y^* \otimes Y \otimes X^* \xrightarrow{ev_Y} 1 \otimes X^* = X^*. \]

Similarly one defines the left dual \( ^*f : ^*Y \rightarrow ^*X \).

**Definition 1.2.7.** A monoidal category is said to be **rigid** if every object possesses both a right and a left dual object.

**Definition 1.2.8.** Let \( C \) be an abelian \( k \)-linear monoidal category having finite dimensional hom spaces with respect to which the bifunctor \( \otimes \) is bilinear. \( C \) is called a **tensor category** if it is finite, rigid and has a simple unit object \( 1 \). \( C \) is called a **fusion category** if it is tensor and semisimple.

Also of interest is the notion of invertible object in a tensor category.

**Definition 1.2.9.** An object \( X \) is **invertible** if there is an object \( Y \) such that \( X \otimes Y \simeq 1 \simeq Y \otimes X \). If every simple object is invertible the category is said to be **pointed**.

1.2.1 Braiding, center.

The definitions given thus far in §1.2 describe basic categorical analogues to the objects of study in the classical theory of rings. The next definition describes the categorical version of a commutative ring.

**Definition 1.2.10.** A monoidal category \( C \) is said to be **braided** if it is equipped with a class of natural isomorphisms

\[ c_{V,W} : V \otimes W \rightarrow W \otimes V \]
satisfying the pair of hexagons

for all objects $U, V, W \in C$.

When $C$ is strict these reduce to commuting triangles giving equations

$$c_{U,V \otimes W} = (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W)$$

$$c_{U \otimes V,W} = (c_{U,W} \otimes id_V)(id_U \otimes c_{V,W}).$$

In any braided monoidal category the isomorphisms $c_{X,Y}, c_{Y,X}$ are composable. We adapt the following definition from [Mug00], [Mug03].

**Definition 1.2.11.** Two objects $X, Y$ in a braided monoidal category are said to centralize each other if $c_{X,Y} c_{Y,X} = id_{Y \otimes X}$.

Let $D$ be a fusion subcategory of a braided fusion category $C$. Following [DGNO10] we make the following definition.

**Definition 1.2.12.** The centralizer $D'$ of $D$ is the full subcategory of objects of $C$ that centralize each objects of $D$. The centralizer $C'$ is sometimes called the Müger center of $C$.

In the next two examples $G$ is a finite group.
Example 1.2.13. Rep(G), the category of finite dimensional representations of $G$, is a braided tensor category with the usual tensor product.

Example 1.2.14. The category $\text{Vec}_G^\omega$ of finite dimensional $G$-graded vector spaces twisted by $\omega \in H^3(G, k^*)$ is a rigid monoidal category. Simple objects are given by $k_g$ ($g^{th}$ component $k$, 0 elsewhere) with unit object $k_1$. Associativity is given by $a_{k_g, k_h, k_m} = \omega(g, h, m)id$ on simple objects, tensor product is defined by

$$(V \otimes W)_g = \bigoplus_{hk=g} V_h \otimes W_k$$

and $(V^*)_g = (V_g)^{-1}$. In general $\text{Vec}_G^\omega$ is not braided.

Definition 1.2.15. The center $Z(C)$ of a monoidal category $C$ is the category having as objects pairs $(X, c)$ where $X \in C$ and for every $Y \in C$ $c_Y : Y \otimes X \to X \otimes Y$ is a family of natural isomorphisms satisfying the hexagon

$$\begin{array}{c}
\xymatrix{ (X \otimes Y) \otimes Z \ar[r]^{c_{X,Y,Z}} & Z \otimes (X \otimes Y) \\
X \otimes (Y \otimes Z) \ar[r]_{a_{X,Y,Z}^{-1}} & (X \otimes Y) \otimes Z \\
X \otimes (Z \otimes Y) \ar[r]_{a_{X,Z,Y}^{-1}} & (X \otimes Z) \otimes Y \\
\ar[u]^{c_{X,Z,Y}} & (Z \otimes Y) \otimes X \ar[u]_{a_{Z,Y,X}^{-1}} & (Z \otimes X) \otimes Y \ar[u]_{a_{Z,X,Y}^{-1}} & (Z \otimes X) \otimes Y \ar[u]_{c_{Z,X,Y}} & (Z \otimes X) \otimes Y \ar[u]_{c_{Z,X,Y}}}
\end{array}$$

for all $Y, Z \in C$. Here $a$ is the associativity constraint for the monoidal structure in $C$. A morphism $(X, c) \to (X', c')$ is a morphism $f \in \text{Hom}_C(X, X')$ satisfying the equation $c'_Y(f \otimes id_Y) = (id_Y \otimes f)c_Y$ for every $Y \in C$.

The center $Z(C)$ has the structure of a monoidal category as follows. Define the
tensor product \((X, c) \otimes (X', c') = (X \otimes X', \bar{c})\) where \(\bar{c}\) is defined by the composition

\[
Y \otimes (X \otimes X') \xrightarrow{\alpha_{Y,X,X'}^{-1}} (Y \otimes X) \otimes X' \xrightarrow{\alpha_Y} (X \otimes Y) \otimes X' \xrightarrow{\delta_Y} (X \otimes X') \otimes Y \xrightarrow{\alpha_{X,X',Y}^{-1}} X \otimes (X' \otimes Y) \xleftarrow{\bar{c}_Y} X \otimes (Y \otimes X')
\]

If \(r\) and \(\ell\) are the right and left unit constraints for the monoidal structure in \(\mathcal{C}\) then the unit object for the monoidal structure in \(Z(\mathcal{C})\) is given by \((1, r^{-1}\ell)\) as one may easily check. Suppose now that \(\mathcal{C}\) is rigid and \(X \in \mathcal{C}\) has right dual \(X^*\) (recall Definition 1.2.6). Then \((X, c) \in Z(\mathcal{C})\) has right dual \((X^*, \bar{c})\) where \(\bar{c}_Y := (c_{Y^*})^*\) and \(Y^*\) is the left dual of \(Y\). One may also check that \(Z(\mathcal{C})\) is braided by \(c_{(X,c) \otimes (X',c')} := \bar{c}_X\).

There is a canonical inclusion of monoidal category \(\mathcal{C}\) into its center given by \(X \mapsto (X, c_X)\). It is well known that the center \(Z(\mathcal{C})\) is in some sense "larger" than \(\mathcal{C}\). This differs from the classical analogue in which a ring contains its center. We generalize the notion of center in §7.

1.2.2 Pre-metric groups

Everything in this subsection may be found in [DGNO10]. We refer the reader to [DGNO10], [Kas95] and [BK01] for definitions and other information relating to braided fusion categories.

Recall that a quadratic form on an abelian group \(G\) having values in an abelian group \(B\) is a map \(q : G \to B\) such that \(q(g^{-1}) = q(g)\) and the symmetric function \(b(g, h) := \frac{q(gh)}{q(g)q(h)}\) is bimultiplicative. We call \(b : G \times G \to B\) the bimultiplicative form associated to \(q\). If \(B = k^*\) we call \(b\) the bicharacter associated to \(q\).
Definition 1.2.16. A pre-metric group is a pair \((G, q)\) where \(G\) is a finite abelian group and \(q : G \rightarrow k^\times\) is a quadratic form. A morphism of pre-metric groups \((G_1, q_1) \rightarrow (G_2, q_2)\) is a homomorphism \(\varphi : G_1 \rightarrow G_2\) such that \(q_2 \circ \varphi = q_1\).

The set of isomorphism classes of the simple objects of any pointed braided fusion category \(C\) form a group \(G\). For \(g \in G\) denote by \(q(g) \in k^\times\) the braiding \(c_{X,X} \in \text{Aut}(X \otimes X)\) where \(X\) is in \(g\). Then \(g \mapsto q(g)\) is a quadratic form \(G \rightarrow k^\times\). In this way \(C\) determines the pre-metric group \((G, q)\).

Conversely every pre-metric group \((G, q)\) determines a pointed braided fusion category \(C(G, q)\) as follows. As a fusion category \(C(G, q)\) is \(\text{Vec}_G\), the category of (finite-dimensional) \(G\)-graded vector spaces. For \(X\) homogeneous object of degree \(g\) define the twist \(\theta_X = q(g)\). Then the braiding \(c_{X,Y} : X \otimes Y \rightarrow Y \otimes X\) satisfies

\[ c_{X,Y}c_{Y,X} = b(g,h)id_{Y \otimes X} \tag{1} \]

where \(b\) is the bicharacter determined by \(q\). In the special case that \(q\) comes from a bicharacter \(\beta : G \times G \rightarrow k^\times\) via the equation \(q(x) = \beta(x, x)\), the associated braiding is \(c_{X,Y} = \beta(g,h)\tau\) for \(\tau\) the linear twist. These two constructions define reciprocal equivalences between the category of pre-metric groups and the (truncated 2-) category of pointed braided fusion categories.

1.3 Module categories

In §1.2 we described the basic objects of study for a categorical version of classical ring theory. In this section we will define the categorical analogue of the classical
theory of modules. The first definition is crucial for this thesis.

**Definition 1.3.1.** Let $C$ be a monoidal category. A *left $C$-module category* $(M, \mu)$ is a category $\mathcal{M}$ together with an exact bifunctor $\otimes : C \times \mathcal{M} \to \mathcal{M}$ and a family of natural isomorphisms $\mu_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$, $\ell_M : 1 \otimes M \to M$ for $X, Y \in C$ and $M \in \mathcal{M}$ subject to the coherence diagrams

\[
\begin{align*}
((WX)Y)M & \xrightarrow{\mu_{W,Y,M}} (WXY)M \\
(W(1)M) & \xrightarrow{\mu_{W,1,M}} WXYM \\
W((XY)M) & \xrightarrow{W \otimes \mu_{X,Y,M}} W(XYM)
\end{align*}
\]

Similarly one defines the structure of *right* module category on $\mathcal{M}$. If the structure maps are identity we say $\mathcal{M}$ is *strict* as a module category over $C$.

**Example 1.3.2.** Any monoidal category $C$ is a module category over itself with action given by monoidal structure. This is referred to as the *regular* module category structure on $C$.

**Example 1.3.3.** Let $G$ be a finite group with subgroup $H$. For 2-cocycle $\mu \in H^2(H, k^\times)$ the category $\text{Rep}_\mu(H)$ of projective representations of $H$ corresponding to Schur multiplier $\mu$ constitutes a $\text{Rep}(G)$-module category with module category structure defined by $W \otimes V := \text{res}_H^G(W) \otimes V$ whenever $W \in \text{Rep}(G), V \in \text{Rep}_\mu(H)$ and $\text{res} : \text{Rep}(G) \to \text{Rep}(H)$ is the restriction functor.

For any $X \in C$ we get a functor $L_X : \mathcal{M} \to \mathcal{M}$ given by $M \mapsto X \otimes M$ (left multiplication by $X$). It is natural to ask about the existence of adjoints of $L_X$. The
following definition introduces a convenient technical tool for dealing with module categories. For the next definition assume \( \mathcal{M} \) is a \( \mathcal{C} \)-module category for semisimple \( \mathcal{C} \). Denote by \( \text{Vec} \) the braided tensor category of finite dimensional vector spaces.

**Definition 1.3.4.** For \( M, N \in \mathcal{M} \) the *internal hom* \( \text{Hom}(M, N) \) is defined to be the object in \( \mathcal{C} \) representing the functor \( \text{Hom}_\mathcal{M}(- \otimes M, N) : \mathcal{C} \to \text{Vec} \). That is, for any object \( X \in \mathcal{C} \) we have

\[
\text{Hom}_\mathcal{M}(X \otimes M, N) \cong \text{Hom}_\mathcal{C}(X, \text{Hom}(M, N))
\]

naturally in \( \text{Vec} \). It follows from Yoneda’s Lemma that \( \text{Hom}(M, N) \) is well defined up to a unique isomorphism and \( \text{Hom}(-, -) \) is a bifunctor.

**Definition 1.3.5.** For \( \mathcal{M}, \mathcal{N} \) left \( \mathcal{C} \)-module categories a functor \( F : \mathcal{M} \to \mathcal{N} \) is said to be a \( \mathcal{C} \)-module functor if \( F \) comes equipped with a family of natural isomorphisms \( f_{X,M} : F(X \otimes M) \to X \otimes F(M) \) satisfying the coherence diagrams

\[
\begin{array}{ccc}
F((XY)M) & \xrightarrow{F(\mu_{X,Y,M})} & F((XY)M) \\
F(X(YM)) & \xrightarrow{f_{XY,M}} & (XY)F(M) \\
F(XYM) & \xrightarrow{f_{X,YM}} & X \otimes F(M) \\
X \otimes F(M) & \xrightarrow{X \otimes f_{YM}} & X(YF(M))
\end{array}
\]

\[
\begin{array}{ccc}
F(1M) & \xrightarrow{F(\ell_M)} & F(M) \\
F(1M) & \xrightarrow{f_{1,M}} & 1F(M) \\
F(1M) & \xrightarrow{F(\ell_M)} & F(M)
\end{array}
\]

whenever \( X, Y \in \mathcal{C} \) and \( M \in \mathcal{M} \). We may write \((F, f)\) when referring to such a functor. A natural transformation \( \tau : F \Rightarrow G \) for bimodule functors \((F, f), (G, g) :\)
\( \mathcal{M} \to \mathcal{N} \) is said to be a module natural transformation whenever the diagram

\[
\begin{array}{ccc}
F(X \otimes M) & \xrightarrow{\tau_{X \otimes M}} & G(X \otimes M) \\
\downarrow_{f_{X,M}} & & \downarrow_{g_{X,M}} \\
X \otimes F(M) & \xrightarrow{id_X \otimes \tau_{X \otimes M}} & X \otimes G(M)
\end{array}
\]

commutes for all \( X \in \mathcal{C} \) and \( M \in \mathcal{M} \).

In what follows we will have occasion to deal with categories of module functors. We fix notation now.

**Definition 1.3.6.** The category of left \( \mathcal{C} \)-module functors from \( \mathcal{M} \to \mathcal{N} \) with morphisms given by module natural transformations will be denoted \( \text{Func}(\mathcal{M}, \mathcal{N}) \). The subcategory of right-exact \( \mathcal{C} \)-module functors (recall Definition 1.1.7) will be denoted \( \text{Func}^c(\mathcal{M}, \mathcal{N}) \).

It is known that the category \( \text{Func}(\mathcal{M}, \mathcal{N}) \) is abelian. Furthermore if \( \mathcal{M}, \mathcal{N} \) are semisimple then so is \( \text{Func}(\mathcal{M}, \mathcal{N}) \) (see [ENO05] for details).

In much of this thesis we will be concerned with categories for which there are left and right module structures which interact in a consistent and predictable way. In the next subsection we will discuss this in more detail and for now simply give a definition.

**Definition 1.3.7.** Let \( \mathcal{C}, \mathcal{D} \) be monoidal categories. \( \mathcal{M} \) is said to be a \((\mathcal{C}, \mathcal{D})\)-bimodule category if \( \mathcal{M} \) is a \( \mathcal{C} \boxtimes \mathcal{D}^{\text{op}} \)-module category. If \( \mathcal{M} \) and \( \mathcal{N} \) are \((\mathcal{C}, \mathcal{D})\)-bimodule categories call \( F : \mathcal{M} \to \mathcal{N} \) a \((\mathcal{C}, \mathcal{D})\)-bimodule functor if it is a \( \mathcal{C} \boxtimes \mathcal{D}^{\text{op}} \)-module functor.
Recall MacLane's strictness theorem for monoidal categories stating that every monoidal category is equivalent to a strict one (Theorem 1.2.4). Next we prove a generalized version for module categories which reduces to the monoidal strictness theorem in the regular module case. Our proof mimics the proof of the monoidal strictness theorem found in [JS93].

**Theorem 1.3.8.** Any module category is module equivalent to a strict module category.

**Proof.** Let \((\mathcal{M}, \mu, r)\) be a right \(C\)-module category for some strict monoidal category \(C\). The strategy is to show that \(\mathcal{M}\) is module equivalent to a \(C\)-module category \(\mathcal{M}'\) which is defined to be a category of functors on which \(C\) acts by functor composition and which is therefore strict. We begin by recalling that \(C\) is monoidally equivalent to the category of \(C\)-module endofunctors \(\text{Func}_C(C,C)\) with equivalence given by \(X \mapsto F^X\).

\(F^X : C \to C\) is the functor sending \(1 \mapsto X\) (1 is unit object in \(C\)): \(F^X(Y) = X \otimes Y\).

Define \(\mathcal{M}'\) to have objects given by pairs \((F, f)\) where \(F\) is a functor \(C \to \mathcal{M}\) and \(f_{X,Y} : F(X) \otimes Y \to F(X \otimes Y)\) is a natural isomorphism in \(\mathcal{M}\) satisfying the diagram

\[
\begin{array}{ccc}
F(X) \otimes (Y \otimes Z) & \xrightarrow{f_{X,Y} \otimes z} & F(X \otimes Y \otimes Z) \\
\mu_{F(X),Y,Z} & & \downarrow f_{X \otimes Y,Z} \\
(F(X) \otimes Y) \otimes Z & \xrightarrow{f_{X,Y} \otimes \text{id}} & F(X \otimes Y) \otimes Z
\end{array}
\]

for every triple \(X, Y, Z \in C\). In short, \(F\) is a right \(C\)-module functor with module linearity given by \(f\). A morphism \(\theta : (F^1, f^1) \to (F^2, f^2)\) in \(\mathcal{M}'\) is defined to be a natural transformation \(\theta : F^1 \to F^2\) satisfying the diagram in Definition 1.3.5 making
it a module natural transformation. Composition in $\mathcal{M}'$ is vertical composition of natural transformations.

Now note that $\mathcal{M}'$ is a right $\mathcal{C}$-module category: for $X \in \mathcal{C}$ and $(F, f) \in \mathcal{M}'$ define $(F, f) \otimes X := (F \circ F^X, f^X)$ where $f^X$ is defined by

$$FF^X(Y) \otimes Z = F(X \otimes Y) \otimes Z \xrightarrow{\mu_{X,Y,Z}} F(X \otimes Y \otimes Z) = FF^X(Y \otimes Z).$$

Note that the action of $\mathcal{C}$ on $\mathcal{M}'$ is strict since composition of functors is strictly associative and $F^1 = id$. We show that $\mathcal{M}$ is module equivalent to $\mathcal{M}'$.

For $M \in \mathcal{M}$ define functor $L_M : \mathcal{C} \rightarrow \mathcal{M}$ by left $M$ multiplication in $\mathcal{C}$, i.e. $X \mapsto M \otimes X$. This allows us to define functor $L : \mathcal{M} \rightarrow \mathcal{M}'$ by

$$L(M) := (L_M, \mu_{M,-,-}).$$

It is evident that $L(M)$ is an object in $\mathcal{M}'$: the diagram required of $\mu_{M,-,-}$ is precisely the pentagon in the definition of module category. We show that $L$ is both essentially surjective and fully faithful.

To see essential surjectivity observe that any $(F, f) \in \mathcal{M}'$ is isomorphic to $L_{F(1)}$. Indeed $f_{1,X} : L_{F(1)}(X) = F(1) \otimes X \simeq F(X)$ for any $X \in \mathcal{C}$, and $f_{1,-}$ is natural.

Next let $\theta : L_M \rightarrow L_N$ be a morphism in $\mathcal{M}'$ for $M, N \in \mathcal{M}$. Define the morphism $\varphi : M \rightarrow N$ in $\mathcal{M}$ by the composition

$$\varphi := M \xrightarrow{\cdot 1_M} M \otimes 1 \xrightarrow{\theta_1} N \otimes 1 \xrightarrow{r_N} N.$$
We claim that for all $Z \in C$ one has $\theta_Z = \varphi \otimes Z$, whence $\theta = L(\varphi)$ and $L$ is thereby full. To see this consider the following diagram.

\[
\begin{array}{cccccc}
M \otimes Z & \xrightarrow{r^{-1}_M \otimes Z} & (M \otimes 1) \otimes Z & \xrightarrow{\mu_{M,1,Z}} & M \otimes (1 \otimes Z) & \xrightarrow{M \otimes \ell_Z} & M \otimes Z \\
\varphi \otimes Z & \downarrow & \theta_1 \otimes Z & \downarrow & \theta_1 \otimes Z & \downarrow & \theta_Z \\
N \otimes Z & \xrightarrow{r^{-1}_N \otimes Z} & (N \otimes 1) \otimes Z & \xrightarrow{\mu_{N,1,Z}} & N \otimes (1 \otimes Z) & \xrightarrow{N \otimes \ell_Z} & N \otimes Z
\end{array}
\]

Rectangle on the left is definition of $\varphi$, middle rectangle commutes since $\theta$ is a morphism in $\mathcal{M}'$, right rectangle commutes on naturality of $\theta$. Top and bottom horizontal compositions are identity (two applications of the triangle axiom which forms a part of the definition of module category). Thus perimeter is identical to the equation $\varphi \otimes Z = \theta_Z$ and $L$ is full. On the other hand if $L(f) = L(g)$ for any morphisms $f, g \in \mathcal{M}$ then the square of naturality for $r$ implies $f = g$, and $L$ is also therefore faithful. This completes the proof that $L$ is an equivalence.

We now show that $L$ is a module functor and finish the proof of the theorem. Define natural isomorphism

\[
J_{M,Y} := \mu_{M,Y,-} : (L(M \otimes Y), \mu_{M \otimes Y,-,-}) \rightarrow (L(M) \otimes Y, \mu_{M,Y \otimes -,-}).
\]

Pentagon in the definition of module category implies that $J_{M,Y}$ is a morphism in $\mathcal{M}'$ and that $J$ is a module functor. We are done. \qed
1.3.1 Bimodule categories

For right $C$-module category $\mathcal{M}$ having module associativity $\mu$ define $\tilde{\mu}_{X,Y,M} = \mu_{M \cdot Y \cdot X}$. Then $\mathcal{M}^{op}$ has left $C$-module category structure given by $(X, M) \mapsto M \otimes X$ with module associativity $\tilde{\mu}^{-1}$. Similarly, if $\mathcal{M}$ has left $C$-module structure with associativity $\sigma$ then $\mathcal{M}^{op}$ has right $C$-module category structure $(M, Y) \mapsto Y^* \otimes M$ with associativity $\tilde{\sigma}^{-1}$ for $\tilde{\sigma}_{M,X,Y} := \sigma_{Y^* \cdot X^* \cdot M}$.

**Proposition 1.3.9.** These actions determine a $(D,C)$-bimodule structure

$$(Y \otimes X, M) \mapsto X^* \otimes M \otimes Y$$

on $\mathcal{M}^{op}$ whenever $\mathcal{M}$ has $(C,D)$-bimodule structure. If $\gamma$ are the bimodule coherence isomorphisms for the left/right module structures in $\mathcal{M}$ (see Proposition 1.3.10), then $\tilde{\gamma}_{Y,M,X} = \gamma_{X^*,M,Y}$ are those for $\mathcal{M}^{op}$.

In the sequel whenever $\mathcal{M}$ is a bimodule category $\mathcal{M}^{op}$ will always refer to $\mathcal{M}$ with the bimodule structure described in Proposition 1.3.9.

**Proposition 1.3.10.** Let $\mathcal{C}, \mathcal{D}$ be strict monoidal catgories. Suppose $\mathcal{M}$ has both left $\mathcal{C}$-module and right $\mathcal{D}$-module category structures $\mu^l, \mu^r$ and a natural family of isomorphisms $\gamma_{X,M,Y} : (X \otimes M) \otimes Y \to X \otimes (M \otimes Y)$ for $X$ in $\mathcal{C}$, $Y$ in $\mathcal{D}$ making
the pentagons

\[
\begin{align*}
((XY)M)Z & \xrightarrow{\tau} (XY)(MZ) & (XM)(YZ) & \xrightarrow{\gamma} X(MYZ) & (1M) & \xrightarrow{\gamma} 1(M1) \\
\mu \otimes \text{id} & & \mu & & \ell_M & & \rho_M \\
(XYM)Z & \xrightarrow{\gamma} & (XM)YZ & \xrightarrow{\gamma} & (YM)Z & \xrightarrow{\gamma} & M & \xleftarrow{\ell_M} & 1M \\
X((YM)Z) & \xrightarrow{\gamma} & X(Y(MZ)) & & & & & & \\
\end{align*}
\]

commute. Then \( \mathcal{M} \) has canonical \((\mathcal{C}, \mathcal{D})\)-bimodule category structure.

Proof. Throughout abbreviate \( \overline{X} := X_1 \otimes X_2 \) in \( \mathcal{C} \otimes \mathcal{D}^{op} \). Suppose given \( \mu^l, \mu^r \) and \( \gamma \) as in the statement of the proposition. Observe that \( \mathcal{C} \otimes \mathcal{D}^{op} \) acts on \( \mathcal{M} \) by 

\[
(X \otimes Y) \otimes' M := (X \otimes M) \otimes Y
\]

where the \( \otimes \) on the right are the given module structures assumed for \( \mathcal{M} \). For \( M \in \mathcal{M} \) define natural isomorphism \( \mu : \otimes'(\text{id}_{\mathcal{C} \otimes \mathcal{D}^{op}} \times \otimes') \to \otimes'(\otimes' \times \text{id}_M) \) by the composition

\[
\mu_{\overline{X}, \overline{Y}, M} = (\gamma_{X_1, Y_1, M, Y_2} \otimes \text{id}_{Y_2})(\mu^l_{X_1, Y_1, M} \otimes \text{id}_{Y_2, X_2})(\mu^r_{X_1 Y_1 M, Y_2, X_2}).
\]

Thus \( \mu_{\overline{X}, \overline{Y}, M} : ((X_1 \otimes Y_1) \otimes M) \otimes (Y_2 \otimes Y_1) \to (X_1 \otimes ((Y_1 \otimes M) \otimes Y_2)) \otimes X_2 \) in the language of left and right module structures.

Consider the partitioned diagram below whose periphery is the appropriate diagram for \( \mu \) written as the composition which defines it. To save space we elide identity morphisms, morphism subscripts, and objects occurring at internal vertices. Label the subdiagrams \( Di \).
Diagrams $D1$, $D4$ are the associativity diagrams for $\mu^r$, $\mu^l$, diagrams $D2$, $D3$, $D5$, $D7$, $D9$, $D10$ are naturality diagrams for either $\mu^l$ or $\gamma$, and diagrams $D6$ and $D8$ are the second and first diagrams given at the beginning of this remark.

**Remark 1.3.11.** For bimodule structure $(M, \mu)$, $\gamma$ is given by $\gamma_{X,Y,M} = \mu_{XZ1,XY,M}$ over the inherent left and right module category structures. In this way we get the converse of Proposition 1.3.10: every bimodule structure gives separate left and right module category structures and the special constraints described therein in a predictable way.

**Remark 1.3.12.** We saw in Proposition 1.3.10 that bimodule category structure can be described separately as left and right structures which interact in a predictable fashion. We make an analogous observation for bimodule functors. Let $F : (M, \gamma) \rightarrow (N, \delta)$ be a functor with left $C$-module structure $f^l$ and right $D$-module structure $f^r$, where $(M, \gamma)$ and $(N, \delta)$ are $(C, D)$-bimodule categories with bimodule consistency
isomorphisms $\gamma, \delta$ as above. Then $F$ is a $(C, D)$-bimodule functor iff the hexagon

\[
\begin{array}{c}
F(X \otimes (M \otimes Y)) \xrightarrow{F_{X,M,Y}} F((X \otimes M) \otimes Y) \xrightarrow{f_{X \otimes M,Y}} F(X \otimes M) \otimes Y \\
\downarrow f_{X,M \otimes Y} \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow f_{X,M \otimes Y} \\
X \otimes F(M \otimes Y) \xrightarrow{X \otimes f_{M,Y}} X \otimes (F(M) \otimes Y) \xleftarrow{\delta_{X,F(M),Y}} (X \otimes F(M)) \otimes Y 
\end{array}
\]

commutes for all $X$ in $C$, $Y$ in $D$, $M$ in $M$. The proof is straightforward and so we do not include it.

1.3.2 Exact module categories

It is desirable to restrict the general study of module categories in order to render questions of classification tractable. In their beautiful paper [EO04] Etingof and Ostrik suggest the class of exact module categories as an appropriate restriction intermediary between the semisimple and general (non-semisimple, possibly non-finite) cases. Let $P$ be an object in any abelian category. We say $P$ is projective if the functor $\text{Hom}(P, -)$ is exact.

**Definition 1.3.13** ([EO04]). A module category $\mathcal{M}$ over tensor category $C$ is said to be exact if for any projective object $P \in C$ and any $M \in \mathcal{M}$ the object $P \otimes M$ is projective.

It turns out that module category exactness is equivalent to exactness of certain functors. We will not require the general formulation here but give the next lemma for exact module categories because exactness ensures adjoints for module functors.
Lemma 1.3.14. For $\mathcal{M}, \mathcal{N}$ exact left $\mathcal{C}$-module categories the association

$$\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \overset{\text{ad}}{\to} \text{Fun}_\mathcal{C}(\mathcal{N}, \mathcal{M})^{\text{op}}$$

sending $F$ to its left adjoint is an equivalence of abelian categories. If $\mathcal{M}, \mathcal{N}$ are bimodule categories then this equivalence is bimodule.

Proof. By Lemma 3.21 in loc. cit. such adjoints exist and since adjoints are unique up to isomorphism the association is bijective on isomorphism classes of objects (functors). For $F : \mathcal{M} \to \mathcal{N}$ linearity of $F^{\text{ad}}$ over $\mathcal{C}$ comes from that for $F$ via the composition

$$\alpha_R := \text{Hom}_\mathcal{M}(F^{\text{ad}}(X \otimes N), R) \simeq \text{Hom}_\mathcal{N}(X \otimes N, F(R))$$

$$\simeq \text{Hom}_\mathcal{N}(N, X^* \otimes F(R)) \simeq \text{Hom}_\mathcal{N}(N, F(X^* \otimes R))$$

$$\simeq \text{Hom}_\mathcal{M}(F^{\text{ad}}(N), X^* \otimes R) \simeq \text{Hom}_\mathcal{M}(X \otimes F^{\text{ad}}(N), R)$$

for $X \in \mathcal{C}, N \in \mathcal{N}, R \in \mathcal{M}$. The third $\simeq$ is linearity of $F$. Define

$$\alpha_{F^{\text{ad}}(X \otimes N)}(id) : X \otimes F^{\text{ad}}(N) \xrightarrow{\sim} F^{\text{ad}}(X \otimes N)$$

The diagrams required to show that $\alpha$ gives $\mathcal{C}$-linearity for $F^{\text{ad}}$ in $\text{Fun}_\mathcal{C}(\mathcal{N}, \mathcal{M})^{\text{op}}$ are not difficult to draw but tedious and non-enlightening and so we omit them. Now assume that the module categories involved are bimodule. Define left and right
$C$-module action on $F^{ad}$ by the equations

$$X \otimes F^{ad} := (F \otimes *X)^{ad}, \quad F^{ad} \otimes X := (*X \otimes F)^{ad}.$$ 

This defines bimodule action $(X \boxtimes Y) \otimes F^{ad} := (*Y \otimes (F \otimes *X))^{ad}$ with bimodule coherence the adjoints of those for $F$: If $\gamma_{X,Y} : X \otimes (F \otimes Y) \to (X \otimes F) \otimes Y$ are those for $F$ then those for $F^{ad}$ are given by $\gamma'_{X,Y} = (\gamma_{Y,X})^{ad}$. 

**Note 1.3.15** (Notation). For $C$ and $D$ finite tensor categories we define a new category whose objects are exact $(C,D)$-bimodule categories with morphisms $(C,D)$-bimodule functors. Denote this category $B(C,D)$. When $C = D$ this is the category of exact bimodule categories over $C$, which we denote $B(C)$. For $\mathcal{M}$ and $\mathcal{N}$ in $B(C,D)$ denote by $\text{Fun}_{C,D}(\mathcal{M},\mathcal{N})$ the category of $(C,D)$-bimodule functors from $\mathcal{M}$ to $\mathcal{N}$. It is evident that for exact $(C,D)$ bimodule category $\mathcal{M}$ and $(C,E)$ bimodule category $\mathcal{N}$ the category of module functors $\text{Fun}_{C}(\mathcal{M},\mathcal{N})$ has the structure of a $(D,E)$ bimodule category with action $(X \boxtimes Y) \otimes F := F(\cdot \otimes X) \otimes Y$. For finite exact module categories $\mathcal{M}, \mathcal{N}$ the category of functors $\text{Fun}_{C}(\mathcal{M},\mathcal{N})$ is known to be an exact module category over the tensor category $\text{Fun}_{C}(\mathcal{N},\mathcal{N})$ with action given by composition of functors (Lemma 3.30 loc. cit.).

**1.3.3 Dominant functors**

Let $F : A \to B$ be an additive functor between abelian categories and define its image $\text{Im}(F)$ to be the full subcategory of $B$ having objects given by all subquotients of objects of the form $F(X)$ for any $X \in A$. 

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Definition 1.3.16. The functor $F$ is said to be dominant if $\text{Im}(F) = B$.

It is an easy exercise to show that $\text{Im}(F)$ is itself an abelian category. Furthermore if $A, B$ are tensor categories and $F$ a tensor functor then $\text{Im}(F)$ is a tensor subcategory of $B$. Indeed if $A_1, A_2$ are quotients of subobjects $Z_1, Z_2$ of $F(X_1), F(X_2)$ for $X_i$ objects of $A$ then exactness of tensor structure $\otimes$ of $B$ implies that $A_1 \otimes A_2$ is a quotient of $Z_1 \otimes Z_2$ which is a subobject of $F(X_1) \otimes F(X_2) \simeq F(X_1 \otimes X_2)$. Hence $A_1 \otimes A_2$ is a subquotient of $F(X_1 \otimes X_2)$ and is therefore an object of $\text{Im}(F)$. The unit object $1$ is contained in $\text{Im}(F)$ because it is a trivial subobject of $F(1)$, and constraints come from those in $B$.

It is also evident that if $A, B$ are semisimple then dominance of $F$ means that any object of $B$ is actually a subobject of $F(X)$ for some $X \in A$.

1.4 2-categories and monoidal 2-categories

Recall that a 2-category is a generalized version of an ordinary category where we have cells of various degrees and rules dictating how cells of different degrees interact. There are two ways to compose 2-cells $\alpha, \beta$: vertical composition $\beta\alpha$ and horizontal composition $\beta \ast \alpha$ as described by the diagrams below.

\[
\begin{align*}
A \xrightarrow{f} B & \Rightarrow A \xrightarrow{f \alpha} B, \\
A \xrightarrow{f} B \xrightarrow{h} C & \Rightarrow A \xrightarrow{f \alpha \beta} C.
\end{align*}
\]

It is required that $\alpha \ast \beta = (\beta \ast h)(f' \ast \alpha) = (h' \ast \alpha)(\beta \ast f)$ where $\ast$ signifies composition between 1-cells and 2-cells giving 2-cells (see [Lei04] for a thorough treatment of higher
category theory and [Ben67], [Kel82] for theory of enriched categories). For fixed monoidal category $C$ we have an evident 2-category with 0-cells $C$-module categories, 1-cells $C$-module functors and 2-cells monoidal natural transformations.

**Example 1.4.1.** The category of rings defines a 2-category with 0-cells rings, 1-cells bimodules and 2-cells tensor products.

A *monoidal* 2-category is essentially a 2-category equipped with a monoidal structure that acts on pairs of cells of various types. For convenience we reproduce, in part, the definition of monoidal 2-category as it appears in [KV91].

**Definition 1.4.2.** Let $A$ be a strict 2-category. A *(lax)* monoidal structure on $A$ consists of the following data:

M1. An object $1 = 1_A$ called the unit object

M2. For any two objects $A, B$ in $A$ a new object $A \otimes B$, also denoted $AB$

M3. For any 1-morphism $u : A \to A'$ and any object $B$ a pair of 1-morphisms

$$u \otimes B : A \otimes B \to A' \otimes B \quad \text{and} \quad B \otimes u : B \otimes A \to B \otimes A'$$

M4. For any 2-morphism

and object $B$ there exist 2-morphisms

$$A \otimes B \xrightarrow{u \otimes B} A' \otimes B \quad \text{and} \quad B \otimes A \xrightarrow{B \otimes u} B \otimes A'$$

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M5. For any three objects $A$, $B$, $C$ an isomorphism $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$

M6. For any object $A$ isomorphisms $l_A : 1 \otimes A \to A$ and $r_A : A \otimes 1 \to A$

M7. For any two morphisms $u : A \to A'$, $v : B \to B'$ a 2-isomorphism

\[
\begin{array}{c}
A \otimes B \xrightarrow{A \otimes u} A \otimes B' \\
\downarrow u \otimes B \quad \Theta_{u,v} \quad \downarrow u' \otimes B' \\
A' \otimes B \xrightarrow{A' \otimes u} A' \otimes B'
\end{array}
\]

M8. For any pair of composable morphisms $A \xrightarrow{u} A' \xrightarrow{u'} A''$ and object $B$ 2-isomorphisms

\[
\begin{array}{c}
A \otimes B \xrightarrow{(u' \circ u) \otimes B} A'' \otimes B \\
\downarrow u \otimes B \quad \Theta_{u,u',B} \quad \downarrow u' \otimes B \\
A' \otimes B \xrightarrow{u' \otimes B} B \otimes A'
\end{array}
\]

\[
\begin{array}{c}
B \otimes A \xrightarrow{B \otimes (u' \circ u)} B \otimes A'' \\
\downarrow B \otimes u \quad \Theta_{B,u,u'} \quad \downarrow B \otimes u' \\
B \otimes A' \xrightarrow{B \otimes u'} B \otimes A'
\end{array}
\]

M9. For any four objects $A, B, C, D$ a 2-morphism

\[
\begin{array}{c}
A \otimes (B \otimes (C \otimes D)) \xrightarrow{A \otimes \otimes_{A,B,C,D}} A \otimes ((B \otimes C) \otimes D) \\
\downarrow \otimes_{A,B,C,D} \quad \Theta_{A,B,C,D} \quad \downarrow \otimes_{A,B,C,D} \\
((A \otimes B) \otimes (C \otimes D)) \xrightarrow{\otimes_{A,B,C,D}} (A \otimes (B \otimes C)) \otimes D
\end{array}
\]

M10. For any morphism $u : A \to A'$, $v : B \to B'$, $w : C \to C'$ 2-isomorphisms

\[
\begin{array}{c}
A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C \\
\downarrow w \otimes (B \otimes C) \quad \Theta_{u,B,C} \quad \downarrow (w \otimes B) \otimes C \\
A' \otimes (B \otimes C) \xrightarrow{\alpha_{A',B,C}} (A' \otimes B) \otimes C
\end{array}
\]

\[
\begin{array}{c}
A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C \\
\downarrow A \otimes (v \otimes C) \quad \Theta_{A,B,v,C} \quad \downarrow A \otimes (v \otimes C) \\
A \otimes (B' \otimes C) \xrightarrow{\alpha_{A,B',C}} (A \otimes B') \otimes C
\end{array}
\]

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M11. For any two objects $A, B$ 2-isomorphism

\[
\begin{align*}
A \otimes (B \otimes 1) & \xrightarrow{\rho_{A,B}} A \otimes B \\
(A \otimes B) \otimes 1 & \xrightarrow{\alpha_{A,B,1}} (A \otimes B) \otimes 1
\end{align*}
\]

\[
\begin{align*}
1 \otimes (A \otimes B) & \xrightarrow{l_{A\otimes B}} A \otimes B \\
(1 \otimes A) \otimes B & \xrightarrow{a_{1,A,B}} (1 \otimes A) \otimes B
\end{align*}
\]

\[
\begin{align*}
A \otimes (1 \otimes B) & \xrightarrow{\lambda_{A,B}} A \otimes B \\
(A \otimes 1) \otimes B & \xrightarrow{\mu_{A,B}} (A \otimes 1) \otimes B
\end{align*}
\]

M12. For any morphism $u : A \rightarrow A'$ 2-isomorphisms

\[
\begin{align*}
1 \otimes A & \xrightarrow{l_A} 1 \otimes A' \\
A & \xrightarrow{u_A} A'
\end{align*}
\]

\[
\begin{align*}
A \otimes 1 & \xrightarrow{w_1} A' \otimes 1 \\
A & \xrightarrow{r_A} A'
\end{align*}
\]

M13. A 2-isomorphism $\epsilon : r_1 \Rightarrow l_1$.

These data are further required to satisfy a series of axioms given in the form of commutative polytopes listed by Kapranov and Voevodsky. As well as describing the sort of naturality we should expect (extending that appearing in the definition of 2-cells for categories of functors) these polytopes provide constraints on the various cells at different levels and dictates how they are to interact. For the sake of brevity we do not list them here but will refer to the diagrams in the original paper when
needed. In [KV91] these polytopes are indicated using hieroglyphic notation. The Stasheff polytope, for example, (which they signify by $(\bullet \otimes \bullet \otimes \bullet \otimes \bullet \otimes \bullet)$, pg. 217) describes how associativity 2-cells and their related morphisms on pentuples of 0-cells interact. In the sequel we will adapt their hieroglyphic notation without explanation.

We digress briefly to explain what is meant by "commuting polytope." This notion will be needed for the proof of Theorem 0.2.1 Our discussion is taken from loc. cit.. In a strict 2-category $\mathcal{A}$ algebraic expressions may take the form of 2-dimensional cells subdivided into smaller cells indicating the way in which the larger 2-cells are to be composed. This procedure is referred to as pasting. Consider the diagram below left.

Edges are 1-cells and faces (double arrows) are 2-cells in $\mathcal{A}$; $T : gh \Rightarrow dk$, $V : ek \Rightarrow be$, $U : fd \Rightarrow ae$. The diagram represents a 2-cell $fgh \Rightarrow abc$ in $\mathcal{A}$ as follows. It is possible to compose 1-cell $F$ and 2-cell $\alpha$ obtaining new 2-cells $F * \alpha, \alpha * F$ whenever these compositions make sense. If $\alpha : G \Rightarrow H$, these are new 2-cells $FG \Rightarrow FH$ and $GF \Rightarrow HF$, respectively. Pasting of diagram above left represents the composition

$$fgh \xRightarrow{f \ast T} fdk \xRightarrow{U \ast k} aek \xRightarrow{a \ast V} abc.$$ 

For 2-composition abbreviated by juxtaposition the pasting is then $(a \ast V)(U \ast k)(f \ast T)$. 

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In case the same external diagram is subdivided in different ways a new 3-dimensional polytope may be formed by gluing along the common edges. Thus the two 2-dimension diagrams can be combined along the edges $fgh$ and $abc$ to form the new 3-dimensional polytope

We have labeled only those edges common to the two original figures. As an aid to deciphering polytope commutativity we will denote the boundary with bold arrows as above. To say that the polytope commutes is to say that the results of the pastings of the two sections of its boundary agree. In such a case we say that the pair of diagrams composing the figure are equal: the 2-cells they denote in $A$ coincide.
CHAPTER II

TENSOR PRODUCT OF BIMODULE CATEGORIES

The next few chapters contain a description of the data giving the 2-category of
$C$-bimodule categories for a fixed monoidal category $C$ the structure of a monoidal
2-category. In the rest of this thesis all categories are assumed to be abelian and $k$-
linear, have finite-dimensional hom spaces, and all functors are assumed to be additive
and $k$-linear. Even though most of what we do here is valid over fields of positive
characteristic, we assume at the outset that $k$ is a fixed field of characteristic 0.

2.1 Preliminary definitions and first properties

Recall definition of right exactness (Definition 1.1.7).

Definition 2.1.1. Suppose $(M, \mu)$ right, $(N, \eta)$ left $C$-module categories. A functor
$F : M \boxtimes N \to A$ is said to be $C$-balanced if there are natural isomorphisms $b_{M,X,Y} :$
$F((M \otimes X) \boxtimes N) \simeq F(M \boxtimes (X \otimes N))$ satisfying the pentagon

\[
\begin{array}{ccc}
F((M \otimes (X \otimes Y)) \boxtimes N) & \xrightarrow{b_{M,X\otimes Y,N}} & F(M \boxtimes ((X \otimes Y) \otimes N)) \\
\downarrow^{\mu_{M,X,Y}} & & \downarrow^{\eta_{X,Y,N}} \\
F(((M \otimes X) \otimes Y) \boxtimes N) & & F(M \boxtimes (X \otimes (Y \otimes N))) \\
\downarrow^{b_{M@X,Y,N}} & & \downarrow^{b_{M,X,Y@N}} \\
F((M \otimes X) \boxtimes (Y \otimes N)) & & \\
\end{array}
\]
whenever $X, Y$ are objects of $C$ and $M \in \mathcal{M}$.

Of course Definition 2.1.1 can be extended to functors from the Deligne product of more than two categories.

**Definition 2.1.2.** Let $F : \mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \cdots \boxtimes \mathcal{M}_n \to \mathcal{N}$ be a functor of abelian categories and suppose that, for some $i$, $1 \leq i \leq n-1$, $\mathcal{M}_i$ is a right $C$-module category and $\mathcal{M}_{i+1}$ a left $C$-module category. Then $F$ is said to be balanced in the $i^{th}$ position if there are natural isomorphisms $b^i_{X,M_1,M_2,\ldots,M_n} : F(M_i \otimes \cdots \otimes (M_i \otimes X) \boxtimes M_{i+1} \boxtimes \cdots \boxtimes M_n) \simeq F(M_1 \boxtimes \cdots \boxtimes M_i \boxtimes (X \otimes M_{i+1}) \boxtimes \cdots \boxtimes M_n)$ whenever $M_i$ are in $\mathcal{M}_i$ and $X$ is in $C$. The $b^i$ are required to satisfy a diagram analogous to that described in Definition 2.1.1.

One may also define multibalanced functors $F$ balanced at multiple positions simultaneously. We will need, and so define, only the simplest nontrivial case.

**Definition 2.1.3.** Let $\mathcal{M}_1$ be right $C$-module, $\mathcal{M}_2$ ($C, D$)-bimodule, and $\mathcal{M}_3$ a left $D$-module category. The functor $F : \mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \mathcal{M}_3 \to \mathcal{N}$ is said to be completely balanced (or 2-balanced) if for $X \in C, Y \in D, N \in \mathcal{M}_2, M \in \mathcal{M}_1$ and $P \in \mathcal{M}_3$ there are natural isomorphisms

$$b^1_{M,X,N,P} : F((M \otimes X) \boxtimes N \boxtimes P) \simeq F(M \boxtimes (X \otimes N) \boxtimes P)$$

$$b^2_{M,N,Y,P} : F(M \boxtimes (N \otimes Y) \boxtimes P) \simeq F(M \boxtimes N \boxtimes (Y \otimes P))$$
satisfying the balancing diagrams in Definition 2.1.1 and the consistency pentagon

\[
\begin{array}{c}
F((M \otimes X) \boxtimes (N \otimes Y) \boxtimes P) \xrightarrow{b^\otimes_{M,X,N,Y,P}} F((M \otimes X) \boxtimes N \boxtimes (Y \otimes P)) \\
\downarrow b^\gamma_{M,X,N,Y,P} \\
F(M \boxtimes (X \otimes (N \otimes Y)) \boxtimes P) \\
\downarrow \gamma^\otimes_{X,N,Y} \\
F(M \boxtimes ((X \otimes N) \otimes Y) \boxtimes P) \xrightarrow{b^\otimes_{M,X,Y@N}} F(M \boxtimes (X \otimes N) \boxtimes (Y \otimes P)) \\
\end{array}
\]

Here \( \gamma \) is the family of natural isomorphisms associated to the bimodule structure in \( \mathcal{M}_2 \) (see Remark 1.3.10). Whenever \( F \) from \( \mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \cdots \boxtimes \mathcal{M}_n \) is balanced in "all" positions call \( F \) \((n - 1)\)-balanced or completely balanced. In this case the consistency axioms take the form of commuting polytopes. For example the consistency axiom for 4-balanced functors is equivalent to the commutativity of a polytope having eight faces (four pentagons and four squares) which reduces to a cube on elision of \( \gamma \)-labeled edges. With this labeling scheme the 1-balanced functors are the original ones given in Definition 2.1.1.

**Definition 2.1.4.** The tensor product of right \( C \)-module category \( \mathcal{M} \) and left \( C \)-module category \( \mathcal{N} \) consists of an abelian category \( \mathcal{M} \boxtimes_C \mathcal{N} \) and a right exact \( C \)-balanced functor \( B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes_C \mathcal{N} \) universal for right exact \( C \)-balanced functors from \( \mathcal{M} \boxtimes \mathcal{N} \).

**Remark 2.1.5.** In [Tam01] constructions similar to these were defined for \( k \)-linear categories as part of a program to study the representation categories of Hopf algebras and their duals. Balanced functors appeared under the name bilinear functors, and the tensor product there is given in terms of generators and relations instead of
the universal properties used here. The tensor product was defined and applied extensively by [ENO09] in the study of semisimple module categories over fusion $C$.

**Remark 2.1.6.** Universality here means that for any right exact $C$-balanced functor $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ there exists a unique right exact functor $\overline{F}$ such that the diagram on the left commutes.

\[
\begin{array}{ccc}
\mathcal{M} \boxtimes \mathcal{N} & \xrightarrow{F} & \mathcal{A} \\
\downarrow B_{\mathcal{M},\mathcal{N}} & & \downarrow \overline{F} \\
\mathcal{M} \boxtimes_C \mathcal{N} & & \\
\end{array}
\quad \quad
\begin{array}{ccc}
\mathcal{M} \boxtimes \mathcal{N} & \xrightarrow{U} & \mathcal{U} \\
\downarrow B_{\mathcal{M},\mathcal{N}} & & \downarrow F' \\
\mathcal{M} \boxtimes_C \mathcal{N} & \xrightarrow{F} & \mathcal{A} \\
\end{array}
\]

The category $\mathcal{M} \boxtimes_C \mathcal{N}$ and the functor $B_{\mathcal{M},\mathcal{N}}$ are defined up to a unique equivalence. This means that if $U : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{U}$ is a second right exact balanced functor with $F = F'U$ for unique right exact functor $F'$ there is a unique equivalence of abelian categories $\alpha : \mathcal{U} \to \mathcal{M} \boxtimes_C \mathcal{N}$ making the diagram on the right commute.

**Remark 2.1.7.** The definition of balanced functor may be easily adapted to bifunctors from $\mathcal{M} \times \mathcal{N}$ instead of $\mathcal{M} \boxtimes \mathcal{N}$. In this case the definition of tensor product becomes object universal for balanced functors right exact in both variables from $\mathcal{M} \times \mathcal{N}$ (Remark 1.2.2). This is the approach taken by Deligne in [Del90]. One easily checks that our definition reduces to Deligne's for $C = \text{Vec}$. This provides some justification for defining the relative tensor product in terms of right-exact functors as opposed to functors of some other sort.

**Lemma 2.1.8.** Let $\mathcal{M}, \mathcal{N}$ be right, left $C$-module categories for $C$ a monoidal category. Then the universal balanced functor $B_{\mathcal{M},\mathcal{N}}$ from Definition 2.1.4 is dominant
Proof. Let $F$ be any balanced functor from $\mathcal{M} \boxtimes \mathcal{N}$, and let $\overline{F}$ be the unique functor from $\mathcal{M} \boxtimes \mathcal{N}$ with $\overline{F}B_{\mathcal{M},\mathcal{N}} = F$. For the inclusion $i : \text{Im}(B_{\mathcal{M},\mathcal{N}}) \to \mathcal{M} \boxtimes \mathcal{N}$ define $F' := \overline{F}i$. Then it is obvious that $F'B_{\mathcal{M},\mathcal{N}} = F$, hence $F$ factors through $\text{Im}(B_{\mathcal{M},\mathcal{N}})$ uniquely. As a consequence of the universality of the relative tensor product $\mathcal{M} \boxtimes \mathcal{N} = \text{Im}(B_{\mathcal{M},\mathcal{N}})$.

The following lemma is a straightforward application of the tensor product universality from Definition 2.1.4. We list it here for reference in the sequel.

**Lemma 2.1.9.** Let $F, G$ be right exact functors $\mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ such that $FB_{\mathcal{M},\mathcal{N}} = GB_{\mathcal{M},\mathcal{N}}$. Then $F = G$.

**Proof.** In the diagram

\[
\begin{array}{ccc}
\mathcal{M} \boxtimes \mathcal{N} & \xrightarrow{B_{\mathcal{M},\mathcal{N}}} & \mathcal{M} \boxtimes \mathcal{C} \mathcal{N} \\
\downarrow B_{\mathcal{M},\mathcal{N}} & & \downarrow \alpha \\
\mathcal{M} \boxtimes \mathcal{C} \mathcal{N} & \xrightarrow{T} & \mathcal{A}
\end{array}
\]

for $T = FB_{\mathcal{M},\mathcal{N}} = GB_{\mathcal{M},\mathcal{N}}$ the unique equivalence $\alpha$ is $id_{\mathcal{M} \boxtimes \mathcal{C} \mathcal{N}}$. □

**Definition 2.1.10.** For $\mathcal{M}$ a right $\mathcal{C}$-module category and $\mathcal{N}$ a left $\mathcal{C}$-module category denote by $F_{\text{uni}}^{\text{bal}}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{A})$ the category of right exact $\mathcal{C}$-balanced functors. Morphisms are natural transformations $\tau : (F, f) \to (G, g)$ where $f$ and $g$ are balancing
isomorphisms for $F$ and $G$ satisfying, whenever $M \in \mathcal{M}$ and $N \in \mathcal{N}$,

$$
\begin{array}{ccc}
F((M \otimes X) \otimes N) & \xrightarrow{\tau_{M \otimes X, N}} & G((M \otimes X) \otimes N) \\
\downarrow f_{M, X, N} & & \downarrow g_{M, X, N} \\
F((M \otimes (X \otimes N)) & \xrightarrow{\tau_{M \otimes (X \otimes N)}} & G(M \otimes (X \otimes N))
\end{array}
$$

for $X$ in $\mathcal{C}$. Call morphisms in a category of balanced functors balanced natural transformations. Similarly we can define $\text{Fun}^{\text{bal}}_{i}(\mathcal{M}_{1} \otimes \cdots \otimes \mathcal{M}_{n}, \mathcal{A})$ to be the category of right exact functors “balanced in the $i$th position” requiring of morphisms a diagram similar to that above.

### 2.2 Module category theoretic structure of tensor product

In this section we examine functoriality of $\boxtimes_{\mathcal{C}}$ and discuss module structure of the tensor product.

For $\mathcal{M}$ a right $\mathcal{C}$-module category, $\mathcal{N}$ a left $\mathcal{C}$-module category, universality of $B_{\mathcal{M}, \mathcal{N}}$ implies an equivalence between categories of functors

$$
\mathcal{Y} : \text{Fun}^{\text{bal}}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{A}) \xrightarrow{\sim} \text{Fun}(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \mathcal{A})
$$

(2)

sending $F \mapsto \overline{F}$ (here overline is as in Definition 2.1.4). Quasi-inverse $\mathcal{W}$ sends $G \mapsto GB_{\mathcal{M}, \mathcal{N}}$ with balancing $G \ast b$, $b$ the balancing of $B_{\mathcal{M}, \mathcal{N}}$. On natural transformations $\tau$, $\mathcal{W}$ is defined by $\mathcal{W}(\tau) = \tau \ast B_{\mathcal{M}, \mathcal{N}}$ where $\ast$ is the product of 2-morphism and 1-morphism: components are given by $\mathcal{W}(\tau)_{M \boxtimes N} = \tau_{B_{\mathcal{M}, \mathcal{N}}(M \boxtimes N)}$. One easily checks that $\mathcal{Y}\mathcal{W} = \text{id}$ so that $\mathcal{W}$ is a strict right quasi-inverse for $\mathcal{Y}$. Let $J : \mathcal{W}\mathcal{Y} \rightarrow \text{id}$
be any natural isomorphism. Then components of $J$ are balanced isomorphisms $J_{(F,f)} : (F, F \ast b) \to (F, f)$ where $f$ is balancing for functor $F$. Being balanced means commutativity of the diagram

\[
\begin{array}{ccc}
F(M \otimes X \otimes N) & \xrightarrow{F(b_{M,X,N})} & F(M \otimes X \otimes N) \\
\downarrow_{J_{M \otimes X \otimes N}} & & \downarrow_{J_{M \otimes X \otimes N}} \\
F(M \otimes X \otimes N) & \xrightarrow{f_{M,X,N}} & F(M \otimes X \otimes N)
\end{array}
\]

for any $M \in \mathcal{M}, X \in \mathcal{C}, N \in \mathcal{N}$. Hence any balancing structure $f$ on the functor $F$ is conjugate to $F \ast b$ in the sense that

\[
f_{M,X,N} = J_{M \otimes X \otimes N} \circ F(b_{M,X,N}) \circ J_{M \otimes X \otimes N}^{-1}.
\] (3)

**Remark 2.2.1.** Let $F, G : \mathcal{M} \otimes \mathcal{N} \to \mathcal{A}$ be right exact $C$-balanced functors. To understand how $\mathcal{Y}$ acts on balanced natural transformation $\tau : F \to G$ recall that to any functor $E : S \to T$ we associate the comma category, denoted $(E, T)$, having objects triples $(X, Y, q) \in S \times T \times \text{Hom}_T(E(X), Y)$. A morphism $(X, Y, q) \to (X', Y', q')$ is a pair of morphisms $(h, k)$ with the property that $k \circ q = q' \circ E(h)$. For $E$ right exact and $S, T$ abelian, $(E, T)$ is abelian ([FGR75]).

Let $\overline{F}$ be the unique right exact functor having $\overline{F}B_{\mathcal{M}, \mathcal{N}} = F$ and consider the comma category $(\overline{F}, A)$. Natural balanced transformation $\tau$ determines a functor $S_{\tau} : \mathcal{M} \otimes \mathcal{N} \to (\overline{F}, A), X \mapsto (B_{\mathcal{M}, \mathcal{N}}(X), G(X), \tau_X)$ and $f \mapsto (F(f), G(f))$. It is evident that $S_{\tau}$ is right exact and inherits $C$-balancing from that in $B_{\mathcal{M}, \mathcal{N}}$, $G$ and $\tau$. Thus we have a unique functor $\overline{S_{\tau}} : \mathcal{M} \otimes \mathcal{C} \mathcal{N} \to (\overline{F}, A)$ with $\overline{S_{\tau}}B_{\mathcal{M}, \mathcal{N}} = S_{\tau}$. Write
\( \overline{S}_r = (S_1, S_2, \sigma) \). Using Lemma 2.1.9 one shows that \( S_1 = id_{\mathcal{M} \boxtimes \mathcal{N}} \) and \( S_2 = \overline{G} \). Then \( \sigma(Y) : \overline{F}(Y) \to \overline{G}(Y) \) for \( Y \in \mathcal{M} \boxtimes \mathcal{N} \). This is precisely \( \overline{\tau} : \overline{F} \to \overline{G} \).

Given right exact right \( C \)-module functor \( F : \mathcal{M} \to \mathcal{M}' \) and right exact left \( C \)-module functor \( G : \mathcal{N} \to \mathcal{N}' \) note that \( B_{\mathcal{M}', \mathcal{N}'}(F \boxtimes G) : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M}' \boxtimes \mathcal{N}' \) is \( C \)-balanced. Thus the universality of \( B \) implies the existence of a unique right exact functor \( F \boxtimes_c G := B_{\mathcal{M}', \mathcal{N}'}(F \boxtimes G) \) making the diagram

\[
\begin{array}{ccc}
\mathcal{M} \boxtimes \mathcal{N} & \xrightarrow{F \boxtimes G} & \mathcal{M}' \boxtimes \mathcal{N}' \\
B_{\mathcal{M}, \mathcal{N}} & & B_{\mathcal{M}', \mathcal{N}'} \\
\mathcal{M} \boxtimes_c \mathcal{N} & \xrightarrow{F \boxtimes_c G} & \mathcal{M}' \boxtimes_c \mathcal{N}'
\end{array}
\]

commute. One uses Lemma 2.1.9 (see the next diagram) to show that \( \boxtimes_c \) is functorial on 1-cells: \( (F' \boxtimes_c E')(F \boxtimes_c E) = F'F \boxtimes_c E'E \).

Thus the 2-cells in M7. of Definition 1.4.2 are identity. If we define \( F \otimes \mathcal{N} := F \boxtimes_c id_{\mathcal{N}} \) (Definition 3.1.5) then the 2-cells in M8. are identity as well.

**Remark 2.2.2.** Next we consider how \( \boxtimes_c \) can be applied to pairs of module natural transformations. Apply \( B_{\mathcal{M}, \mathcal{N}'} \) to the right of the diagram for the Deligne product of
\( \tau \) and \( \sigma \\

\[
\begin{array}{ccc}
F \otimes E & \xrightarrow{B_{N,N'}} & G \otimes H \\
\otimes & \downarrow & \otimes \\
\mathcal{M} \otimes \mathcal{N} & \xrightarrow{B_{N,N'}} & \mathcal{N}' \otimes \mathcal{N}'
\end{array}
\]

giving natural transformation

\[
(\tau \otimes \sigma)' := B_{N,N'} \ast (\tau \otimes \sigma) : B_{N,N'}(F \otimes E) \Rightarrow B_{N,N'}(G \otimes H)
\]  \hspace{1cm} (4)

having components \( B_{N,N'} \ast (\tau \otimes \sigma)_{AB} = B_{N,N'}(\tau_{A} \otimes \sigma_{B}) \). Here \( \ast \) indicates composition between cells of different index (in this case a 1-cell and a 2-cell with the usual 2-category structure in \( \text{Cat} \)).

It is easy to see that this is a balanced natural transformation, i.e. a morphism in the category of balanced right exact functors \( \text{Fun}^{\text{bal}}(\mathcal{M} \otimes \mathcal{N}, \mathcal{M'} \otimes \mathcal{N'}) \). Using comma category \( (F \otimes_{C} F', \mathcal{M'} \otimes_{C} \mathcal{N'}) \) we get

\[
\tau \otimes_{C} \sigma := (\tau \otimes \sigma)' : F \otimes_{C} F' \Rightarrow G \otimes_{C} G'.
\]  \hspace{1cm} (5)

Note also that \( \otimes_{C} \) is functorial over vertical composition of 2-cells: \( (\tau' \otimes_{C} \sigma')(\tau \otimes_{C} \sigma) = \tau' \tau \otimes_{C} \sigma' \sigma \) whenever the compositions make sense. Though we do not prove it here observe also that \( \otimes_{C} \) preserves horizontal composition \( \bullet \) of 2-cells:

\[
(\tau' \bullet \tau) \otimes_{C} (\sigma' \bullet \sigma) = (\tau' \otimes_{C} \sigma') \bullet (\tau \otimes_{C} \sigma).
\]

For the following proposition recall that, for left \( C \)-module category \( \mathcal{M} \), the functor \( L_{X} : \mathcal{M} \rightarrow \mathcal{M} \) sending \( M \mapsto X \otimes M \) for \( X \in C \) fixed is right exact. This follows from
the fact that \( \text{Hom}(X^* \otimes N, \_\) is left exact for any \( N \in \mathcal{M} \).

**Proposition 2.2.3.** Let \( \mathcal{M} \) be a \((\mathcal{C}, \mathcal{E})\)-bimodule category and \( \mathcal{N} \) an \((\mathcal{E}, \mathcal{D})\)-bimodule category. Then \( \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N} \) is a \((\mathcal{C}, \mathcal{D})\)-bimodule category and \( B_{\mathcal{M}, \mathcal{N}} \) is a \((\mathcal{C}, \mathcal{D})\)-bimodule functor.

**Proof.** For \( X \) in \( \mathcal{C} \) define functor \( L_X : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes \mathcal{N} : M \boxtimes N \mapsto (X \otimes M) \boxtimes N \). Then there is a unique right exact \( \overline{L_X} \) making the diagram on the left commute; bimodule consistency isomorphisms in \( \mathcal{M} \) make \( L_X \) balanced.

Similarly, for \( Y \) in \( \mathcal{D} \) define endofunctor \( R_Y : M \boxtimes N \mapsto M \boxtimes (N \otimes Y) \). Then there is unique right exact \( \overline{R_Y} \) making the diagram on the right commute; bimodule consistency isomorphisms in \( \mathcal{N} \) make \( R_Y \) balanced. \( \overline{L_X} \) and \( \overline{R_Y} \) define left/right module category structures on \( \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N} \). Indeed for \( \mu \) the left module associativity in \( \mathcal{M} \) note that \( B_{\mathcal{M}, \mathcal{N}}(\mu_{X,Y,M} \boxtimes id_N) : \overline{L_X L_Y B_{\mathcal{M}, \mathcal{N}}} \simeq \overline{L_X \otimes Y B_{\mathcal{M}, \mathcal{N}}} \) is an isomorphism in \( \text{Fun}^{\text{kal}}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}) \) so corresponds to an isomorphism \( \overline{L_X L_Y} \simeq \overline{L_X \otimes Y} \) in \( \text{End}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}) \) which therefore satisfies the diagram for left module associativity in \( \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N} \). Composing diagonal arrows we obtain the following commutative diagram.
Note then that

\[ \overline{L_X} \overline{R_Y} B_{M,N} = \overline{R_Y} \overline{L_X} B_{M,N} \]

and since \( \overline{R_Y} \overline{L_X} B_{M,N} \) is balanced Lemma 2.1.9 implies \( \overline{R_Y} \overline{L_X} = \overline{L_X} \overline{R_Y} \). Suppose \( Q \in \mathcal{M} \boxtimes_{\varepsilon} \mathcal{N} \). Then \( (X \boxtimes Y) \otimes Q := \overline{L_X} \overline{R_Y} Q = \overline{R_Y} \overline{L_X} Q \) defines \((C, \mathcal{D})\)-bimodule category structure on \( \mathcal{M} \boxtimes \mathcal{N} \). Note also that since the bimodule consistency isomorphisms in \( \mathcal{M} \boxtimes \mathcal{N} \) are trivial the same holds in \( \mathcal{M} \boxtimes_{\varepsilon} \mathcal{N} \). As a result \( B_{M,N} \) is a \((C, \mathcal{D})\)-bimodule functor.

In the sequel we may use \( L_X \) to denote left action of \( X \in C \) in \( \mathcal{M} \boxtimes \mathcal{N} \) and for the induced action on \( \mathcal{M} \boxtimes_{\varepsilon} \mathcal{N} \). Similarly for \( R_X \).

**Remark 2.2.4.** The above construction is equivalent to defining left and right module category structures as follows. For the right module structure

\[ \otimes : (\mathcal{M} \mathcal{N}) \boxtimes C \xrightarrow{\alpha^1_{\mathcal{M},\mathcal{N},C}} \mathcal{M}(\mathcal{N} \boxtimes C) \xrightarrow{id_{\otimes}} \mathcal{M} \mathcal{N} \]

where \( \alpha^1 \) is defined in Lemma 3.1.1 and where tensor product of module categories has been written as juxtaposition. The left action is similarly defined using \( \alpha^2 \) and left module structure of \( \mathcal{M} \) in second arrow.

**Proposition 2.2.5.** Let \( \mathcal{M} \) be a \((C, \mathcal{D})\)-bimodule category. Then there are canonical \((C, \mathcal{D})\)-bimodule equivalences \( \mathcal{M} \boxtimes_{\varepsilon} \mathcal{D} \simeq \mathcal{M} \simeq C \boxtimes_{\varepsilon} \mathcal{M} \).

**Proof.** Observing that the \( \mathcal{D} \)-module action \( \otimes \) in \( \mathcal{M} \) is balanced let \( l_{\mathcal{M}} : \mathcal{M} \boxtimes_{\varepsilon} \mathcal{D} \to \mathcal{M} \) denote the unique exact functor factoring \( \otimes \) through \( B_{\mathcal{M},\mathcal{D}} \). Define \( U : \mathcal{M} \to \)
$\mathcal{M} \otimes \mathcal{D}$ by $M \mapsto M \otimes 1$ and write $U' = B_{\mathcal{M},\mathcal{D}}U$. We wish to show that $l_{\mathcal{M}}$ and $U'$ are inverses.

Note first that $l_{\mathcal{M}}U' = id_{\mathcal{M}}$. Now define natural isomorphism $\tau : B_{\mathcal{M},\mathcal{D}} \Rightarrow U' \otimes$ by $\tau_{M,X} = b_{M,X,1}^{-1}$ where $b$ is balancing isomorphism for $B_{\mathcal{M},\mathcal{D}}$. As a balanced natural isomorphism $\tau$ corresponds to an isomorphism $\overline{\tau} : B_{\mathcal{M},\mathcal{D}} = id_{\mathcal{M} \otimes \mathcal{D}} \Rightarrow U' \otimes$ in the category $\text{End}(\mathcal{M} \otimes \mathcal{D} \mathcal{D})$. Commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{M} \otimes \mathcal{D} & \otimes & \mathcal{M} \\
B_{\mathcal{M},\mathcal{D}} \downarrow & l_{\mathcal{M}} & \downarrow U' \\
\mathcal{M} \otimes \mathcal{D} \mathcal{D} & \overline{U' \otimes} & \mathcal{M} \otimes \mathcal{D} \mathcal{D}
\end{array}
\]

implies $U'l_{\mathcal{M}} = \overline{U' \otimes}$ so that $id_{\mathcal{M} \otimes \mathcal{D} \mathcal{D}} \simeq U'l_{\mathcal{M}}$ via $\overline{\tau}$. In proving $C \otimes \mathcal{M} \simeq \mathcal{M}$ one lifts the left action of $C$ for an equivalence $r_{\mathcal{M}} : C \otimes \mathcal{M} \simeq \mathcal{M}$. Strict associativity of the module action on $\mathcal{M}$ implies that both $r_{\mathcal{M}}$ and $l_{\mathcal{M}}$ are trivially balanced.

**Corollary 2.2.6.** Let $(F,f) : \mathcal{M} \to \mathcal{N}$ be a morphism in $B(C)$ where $f$ is left $C$-module linearity for $F$. Then there is a natural isomorphism $Fr_{\mathcal{M}} \simeq r_{\mathcal{N}}(id_C \otimes_C F)$ satisfying a polytope version of the diagram for module functors in Definition 1.3.5. A similar result holds for the equivalence $l$.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
C \otimes \mathcal{M} & \overset{id_C \otimes F}{\longrightarrow} & C \otimes \mathcal{N} \\
\downarrow B_{C,\mathcal{M}} & \downarrow B_{C,\mathcal{N}} & \downarrow B_{C,\mathcal{N}} \\
CM & \overset{id_C \otimes_C F}{\longrightarrow} & CN \\
\downarrow r_M & \downarrow r_N & \downarrow r_N \\
\mathcal{M} & \overset{F}{\longrightarrow} & \mathcal{N}
\end{array}
\]
The top rectangle is definition of $id_C \boxtimes_C F$, right triangle definition of functor $r_N$, and left triangle definition of $r_M$. The outer edge commutes up to $f$. We therefore have natural isomorphism $f : Fr_M B_{C,M} \rightarrow r_N(id_C \boxtimes_C F)B_{C,N}$. Now observe that, using the regular module structure in $C$ we have the following isomorphisms.

$$Fr_M B_{C,M}(XY \boxtimes M) = F((XY)M) = F(X(YM)) = Fr_M B_{C,M}(X \boxtimes YM),$$

$$r_N(id_C \boxtimes_C F)B_{C,N}(XY \boxtimes M) = (XY)F(M) \simeq XF(YM) = r_N(id_C \boxtimes_C F)B_{C,N}(X \boxtimes YM)$$

Here $X, Y \in C$, $M \in M$ and $\sim$ is $id_X \otimes f_Y^{-1}$. Using the relations required of the module structure $f$ described in Definition 1.3.5 one sees that the second isomorphism constitutes a $C$-balancing for the functor $r_N(id_C \boxtimes_C F)B_{C,N}$. Thus both functors are balanced. Using the relations for $f$ from Definition 1.3.5 a second time shows that $f$ is actually a balanced natural isomorphism $Fr_M B_{C,M} \rightarrow r_N(id_C \boxtimes_C F)B_{C,N}$. Hence we may descend to a natural isomorphism $r_F := \overline{f} : Fr_M \rightarrow r_N(id_C \boxtimes_C F)$. The associated polytopes are given in Polytope 4.1.2, Chapter 4. The result for $l$ is similar.

Corollary 2.2.6 shows, predictably, that functoriality of $l, r$ depends on module linearity of the underlying functors. In particular, if $F$ is a strict module functor $l_F$ and $r_F$ are both identity. As an example note that the associativity is strict as a module functor (this follows from Proposition 3.1.6) and so $r_{a_{M,N},p} = id$ for the
relevant module categories. Similarly for $l$. Thus polytopes of the form $(1 \otimes \cdots \otimes \cdot)$ (pg. 222 in [KV91]) describing interaction between $a$, $l$ and $r$ commute trivially.

**Remark 2.2.7.** $r_M : C \boxtimes_C M \to M$ is itself a strict left $C$-module functor as follows. Let $X \in C$ and let $L_X$ be left $C$-module action in $C \boxtimes M$. Replacing $L_X$ with $id \boxtimes F$ in the diagram given in the proof of Corollary 2.2.6 and chasing around the resulting diagram allows us to write the equation

$$L'_X r_M B_{C,M} = r_M \overline{L}_X B_{C,M}$$

where $L'_X$ is left $X$-multiplication in $M$ and $\overline{L}_X$ the induced left $X$-multiplication in $C \boxtimes_C M$. Thus $L'_X r_M = r_M \overline{L}_X$, which is precisely the statement that $r_M$ is strict as a $C$-module functor. Thus Corollary 2.2.6 implies that $r_M = id$ for any $C$-module category $M$. If $M$ is a bimodule category it is evident that $r_M$ is also a strict right module functor and hence strict as a bimodule functor.

**Proposition 2.2.8.** For $(C, D)$-bimodule category $M$ and $(C, E)$-bimodule category $N$ the category of right exact $C$-module functors $Fun_C(M, N)$ has canonical structure of a $(D, E)$-bimodule category.

**Proof.** $((X \boxtimes Y) \otimes F)(M) = F(M \otimes X) \otimes Y$ defines $D \boxtimes E_{rev}$-action on $Fun_C(M, N)$. Right exactness of $(X \boxtimes Y) \otimes F$ comes from right exactness of $F$ and of module action in $M, N$. $D \boxtimes E_{rev}$ acts on the module part $f$ of $F$ by

$$((X \boxtimes Y) \otimes f)_{Z,M} = \gamma_{Z,F(M \otimes X),Y} f_{Z,M \otimes X} F(\gamma_{Z,M,X})$$
where \( \gamma \) is the bimodule consistency for the left and right module structures in \( \mathcal{M} \), \( \mathcal{N} \) (Proposition 1.3.10). The required diagrams commute since they do for \( f \).

Next let \( \tau : F \Rightarrow G \) be a natural left \( \mathcal{C} \)-module transformation for right exact left \( \mathcal{C} \)-module functors \( (F,f),(G,g) : \mathcal{M} \rightarrow \mathcal{N} \). Define action of \( X \otimes Y \) on \( \tau \) by

\[
((X \otimes Y) \otimes \tau)_M = \tau_{M \otimes X} \otimes \text{id}_Y : ((X \otimes Y) \otimes F)(M) \rightarrow ((X \otimes Y) \otimes G)(M).
\]

Then \( (X \otimes Y) \otimes \tau \) is a natural left \( \mathcal{C} \)-module transformation. Indeed the diagram

\[
\begin{array}{ccc}
F((Z \otimes M) \otimes X) \otimes Y & \xrightarrow{\tau_{(Z \otimes M) \otimes X} \otimes \text{id}_Y} & G((Z \otimes M) \otimes X) \otimes Y \\
| & & | \\
F(Z \otimes (M \otimes X)) \otimes Y & \xrightarrow{\tau_{Z \otimes (M \otimes X)} \otimes \text{id}_Y} & G(Z \otimes (M \otimes X)) \otimes Y \\
| & & | \\
\gamma & \xrightarrow{\text{id}_Z \otimes \tau_{M \otimes X} \otimes \text{id}_Y} & \gamma \\
| & & | \\
Z \otimes (F(M \otimes X) \otimes Y) & \xrightarrow{\tau_{M \otimes X} \otimes \text{id}_Y} & Z \otimes (G(M \otimes X) \otimes Y)
\end{array}
\]

commutes. The top rectangle is the rectangle of naturality for \( \tau \). The middle rectangle expresses the fact that \( \tau \) is a natural left \( \mathcal{C} \)-module transformation. The bottom rectangle is the rectangle of naturality for \( \gamma \). Perimeter is the diagram expressing that \( (X \otimes Y) \otimes \tau \) is a module natural transformation. \( \square \)

**Remark 2.2.9.** \( \mathcal{Y} \) in equation (2) at the beginning of this section is an equivalence of \( (\mathcal{D}, \mathcal{F}) \)-bimodule categories

\[
\text{Fun}_\mathcal{C}^{bal}(\mathcal{M} \otimes \mathcal{N}, \mathcal{S}) \rightarrow \text{Fun}_\mathcal{C}(\mathcal{M} \otimes \mathcal{E}, \mathcal{N}, \mathcal{S}) \quad (6)
\]

whenever \( \mathcal{M} \in \mathcal{B}(\mathcal{C}, \mathcal{E}), \mathcal{N} \in \mathcal{B}(\mathcal{E}, \mathcal{D}), \mathcal{S} \in \mathcal{B}(\mathcal{C}, \mathcal{F}) \). If balanced right exact bimodule
functor \( u : \mathcal{M} \Box N \to U \) is universal for such functors from \( \mathcal{M} \Box N \) then \( \mathcal{M} \Box \_ \simeq U \) as bimodule categories.

To see the first claim let \( F \) be \( \mathcal{E} \)-balanced left \( \mathcal{C} \)-module functor \( \mathcal{M} \Box N \to \mathcal{S} \) with module linearity \( f \) and balancing \( t \). For \( X \in \mathcal{C} \) denote by \( L_X : \mathcal{M} \Box N \to \mathcal{M} \Box N \) left action of \( X \), and define natural isomorphisms \( f_X : F L_X \simeq L_X F \) by \( (f_X)_A = f_{X,A} \) whenever \( A \in \mathcal{M} \Box N \). Note that \( L_X F \) has balancing \( \text{id}_X \Box t \) and that \( F L_X \) is balanced by

\[
t_X \otimes M, Y, N F(\gamma^{-1}_{X, M, Y} \otimes \text{id}_N) : (F L_X)((M \otimes Y) \Box N) \simeq (F L_X)(M \Box (Y \otimes N))
\]

whenever \( M \in \mathcal{M} \), \( Y \in \mathcal{E} \) and \( N \in \mathcal{N} \). Using Lemma 2.1.9 one verifies that \( \overline{F L_X} = F \circ \overline{B L_X} \) and \( \overline{L_X F} = L_X \overline{F} \). Note that \( \overline{B L_X} \) is the induced left action of \( X \) in \( \mathcal{M} \Box \_ \mathcal{N} \) which we will also denote \( L_X \). Naturality of \( f \) implies that \( f_X \) is balanced hence and application of \( \mathcal{Y} \) gives \( \overline{f_X} : \overline{F L_X} \simeq L_X \overline{F} \) naturally in \( \text{Fun}(\mathcal{M} \Box \_ \mathcal{N}, \mathcal{S}) \).

One checks that \( \overline{F} \) is bimodule functor with module linearity \( \overline{f_{X,Q}} = (\overline{f_X})_Q \) whenever \( Q \in \mathcal{M} \Box \_ \mathcal{N} \) (\( \overline{f} \) satisfies required diagrams because \( f \) does).

We may therefore write \( (\overline{F}, \overline{f}) = \overline{(F, f)} \) for the functor in \( \text{Fun}_{\mathcal{C}}(\mathcal{M} \Box \_ \mathcal{N}, \mathcal{S}) \).

We now show that \( \mathcal{Y} \) respects the bimodule structure in the functor categories. For \( Y \in \mathcal{D}, Z \in \mathcal{F} \) and \( Q \in \mathcal{M} \Box \_ \mathcal{N} \) one checks easily that

\[
\mathcal{Y}(Y \otimes F)(Q) = \overline{F R_Y}(Q) = \overline{F} \circ \overline{B R_Y}(Q) = \overline{F}(Q \otimes Y) = (Y \otimes \mathcal{Y}(F))(Q)
\]

and similarly that \( \mathcal{Y}(F \otimes Z)(Q) = (\mathcal{Y}(F) \otimes Z)(Q) \) making \( \mathcal{Y} \) a bimodule functor.
For the second claim, universality of both $B_{\mathcal{M}, \mathcal{N}}$ and $u$ gives unique equivalence $\alpha$ of abelian categories making the diagram

$$
\begin{array}{c}
\mathcal{M} \boxtimes \mathcal{N} \\
\downarrow u \\
\mathcal{U} \longrightarrow \mathcal{M} \boxtimes \mathcal{N}
\end{array}
\xrightarrow{B_{\mathcal{M}, \mathcal{N}}} 
$$

commute. Thus $\alpha$ is the unique exact functor factoring $B_{\mathcal{M}, \mathcal{N}}$, and since the latter is a balanced bimodule functor $\alpha$ inherits this property by the first part of the proposition.

2.3 Relative tensor product as category of functors

The purpose of this section is to prove an existence theorem for the relative tensor product by providing a canonical equivalence with a certain category of module functors. Let $\mathcal{M}, \mathcal{N}$ be exact right, left module categories over tensor category $\mathcal{C}$, and define $I : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \text{Fun}_\mathcal{C}(\mathcal{M}^{op}, \mathcal{N})$ by

$$
I : M \boxtimes N \mapsto \text{Hom}_\mathcal{M}(-, M) \otimes N
$$

where $\text{Hom}_\mathcal{M}$ means internal hom for right $\mathcal{C}$-module structure in $\mathcal{M}$ (Definition 1.3.4). Using the formulas satisfied by internal hom for right module category structure we see that images under $I$ are indeed $\mathcal{C}$-module functors:

$$
I(M \boxtimes N)(X \otimes M') = \text{Hom}_\mathcal{M}(X \otimes M', M) \otimes N = \text{Hom}_\mathcal{M}(M', X \otimes M) \otimes N
$$

$$
= X \otimes \text{Hom}_\mathcal{M}(M', M) \otimes N = X \otimes I(M \boxtimes N)(M').
$$
Using similar relations one easily shows that $I$ is $C$-balanced. Hence $I$ descends to a unique right-exact functor $\bar{I} : \mathcal{M} \otimes_C \mathcal{N} \to \text{Fun}_C(\mathcal{M}^{\text{op}}, \mathcal{N})$ satisfying $\bar{I}B_{\mathcal{M}, \mathcal{N}} = I$.

In the opposite direction define $J : \text{Fun}_C(\mathcal{M}^{\text{op}}, \mathcal{N}) \to \mathcal{M} \otimes \mathcal{N}$ as follows. For $F$ a $C$-module functor $\mathcal{M}^{\text{op}} \to \mathcal{N}$ let $J(F)$ be the object representing the functor $M \otimes N \mapsto \text{Hom}(N, F(M))$, that is $\text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, J(F)) = \text{Hom}_\mathcal{N}(N, F(M))$.

Now denote by $J' : \text{Fun}_C(\mathcal{M}^{\text{op}}, \mathcal{N}) \to \mathcal{M} \otimes \mathcal{N}$ the composition $B_{\mathcal{M}, \mathcal{N}}J$.

**Theorem 2.3.1.** Let $C$ be a rigid monoidal category. For $\mathcal{M}$ a right $C$-module category and $\mathcal{N}$ a left $C$-module category there is a canonical equivalence

$$\mathcal{M} \otimes_C \mathcal{N} \simeq \text{Fun}_C(\mathcal{M}^{\text{op}}, \mathcal{N}).$$

If $\mathcal{M}, \mathcal{N}$ are bimodule categories this equivalence is bimodule.

**Proof.** In order to prove the theorem we simply show that $\bar{I}$ and $J'$ defined above are quasi-inverses. This will follow easily if we can first show that $I, J$ are quasi-inverses, and so we dedicate a separate lemma to proving this.

**Lemma 2.3.2.** $I, J$ are quasi-inverses.

**Proof.** Let us first discuss internal homs for the $C$-module structure in $\mathcal{M} \otimes \mathcal{N}$ induced by $X \otimes (M \otimes N) := (X \otimes M) \otimes N$. Let $X$ be any simple object in $C$. Then one shows, using the relations for internal hom in $\mathcal{M}$ and $\mathcal{N}$ separately, that the internal hom in $\mathcal{M} \otimes \mathcal{N}$ is given by

$$\text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, S \otimes T) = \text{Hom}_\mathcal{M}(M, S) \otimes \text{Hom}_\mathcal{N}(N, T)$$

(7)
where the $\otimes$ is of course that in $C$. Using this and the definitions of $I$ and $J$ we have

$$
\text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, JJ(S \otimes T)) = \text{Hom}_\mathcal{N}(N, \text{Hom}_M(M, S) \otimes T)$$

$$= \text{Hom}_\mathcal{C}(1, \text{Hom}_\mathcal{N}(N, \text{Hom}_M(M, S) \otimes T))$$

$$= \text{Hom}_\mathcal{C}(1, \text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, S \otimes T))$$

$$= \text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, S \otimes T).$$

The third line is an application of (7). The first and the last line imply that the functor $M \otimes N \mapsto \text{Hom}_\mathcal{N}(N, \text{Hom}_M(M, S) \otimes T)$ is represented by both $S \otimes T$ and $JJ(S \otimes T)$, and these objects must therefore be equal up to a unique isomorphism, hence $JJ \simeq \text{id}$.

Next we show that $IJ \simeq \text{id}$. Let $F$ be any functor $\mathcal{M}^{\text{op}} \to \mathcal{N}$. From the first part of this proof we may write the following equation (up to unique linear isomorphism):

$$\text{Hom}_\mathcal{N}(N, IJ(F)(M)) = \text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, JJ(F))$$

$$= \text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, J(F)) = \text{Hom}_\mathcal{N}(N, F(M)).$$

Thus both $IJ(F)(M)$ and $F(M)$ are representing objects for the functor $N \mapsto \text{Hom}_{\mathcal{M} \otimes \mathcal{N}}(M \otimes N, J(F))$ for each fixed $M \in \mathcal{M}$. Thus $IJ(F)(M) = F(M)$ up to a unique isomorphism. The collection of all such isomorphisms gives a natural isomorphism $IJ(F) \simeq F$, and therefore $IJ \simeq \text{id}$. This, with the first part of this proof, is equivalent to the statement that $J$ is a quasi-inverse for $I$, proving the lemma. \(\square\)

Now we are ready to complete the proof of Theorem 2.3.1. Using the definition
of \( J' \) and \( \bar{I} \) write \( J'\bar{I}B_{M,N} = B_{M,N}JI \simeq B_{M,N} \). By uniqueness (Lemma 2.1.9) it therefore follows that \( J'\bar{I} \simeq id \). Also \( \bar{I}J' = \bar{I}B_{M,N}J = IJ \simeq id \), and we are done. \( \square \)

As an immediate corollary to Theorem 2.3.1 and associativity of relative tensor product (equation 9, given below) we are able to prove a module category theoretic version of a theorem which appears in many places, notably as Frobenius reciprocity for induced representations of finite groups ([Ser77, §3.3]) and generally as a classical adjunction in the theory of modules.

**Corollary 2.3.3** (Frobenius Reciprocity). Let \( \mathcal{M} \) be a \((\mathcal{C}, \mathcal{D})\)-bimodule category, \( \mathcal{N} \) a \((\mathcal{D}, \mathcal{F})\)-module category, and \( \mathcal{A} \) a \((\mathcal{C}, \mathcal{F})\)-module category. Then there is a canonical equivalence

\[
\text{Fun}_\mathcal{C}(\mathcal{M} \boxtimes_\mathcal{D} \mathcal{N}, \mathcal{A}) \simeq \text{Fun}_\mathcal{D}(\mathcal{N}, \text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{A}))
\]

as \((\mathcal{E}, \mathcal{F})\)-bimodule categories.

*Proof.* To see this we will first use Lemma 1.3.14 to describe the behaviour of the tensor product under \( op \). Observe that

\[
(\mathcal{M} \boxtimes_\mathcal{D} \mathcal{N})^{op} \simeq \text{Fun}_\mathcal{D}(\mathcal{M}^{op}, \mathcal{N})^{op} \simeq \text{Fun}_\mathcal{D}(\mathcal{N}, \mathcal{M}^{op}) \simeq \mathcal{N}^{op} \boxtimes_\mathcal{D} \mathcal{M}^{op}
\]

applying Theorem 2.3.1 twice (first and third) and Lemma 1.3.14 for the second step.
Now we may write

\[ \text{Fun}_c(\mathcal{M} \boxtimes_D \mathcal{N}, A) \cong (\mathcal{M} \boxtimes_D \mathcal{N})^{op} \boxtimes_c A \cong (\mathcal{N}^{op} \boxtimes_D \mathcal{M}^{op}) \boxtimes_c A \]
\[ \cong \mathcal{N}^{op} \boxtimes_D (\mathcal{M}^{op} \boxtimes_c A) \]
\[ \cong \text{Fun}_c(\mathcal{N}, \text{Fun}_D(\mathcal{M}, A)). \]

\[\Box\]

Theorem 2.3.3 states that functor \( \mathcal{M} \boxtimes_D - : \mathcal{B}(\mathcal{D}, \mathcal{E}) \to \mathcal{B}(\mathcal{C}, \mathcal{E}) \) is left adjoint to functor \( \text{Fun}_c(\mathcal{M}, -) : \mathcal{B}(\mathcal{C}, \mathcal{E}) \to \mathcal{B}(\mathcal{D}, \mathcal{E}) \).
CHAPTER III

ASSOCIATIVITY AND UNIT

CONSTRAINTS FOR $\mathcal{B}(\mathcal{C})$

3.1 Tensor product associativity

In this section we discuss associativity of tensor product. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be tensor categories. Let $\mathcal{A}$ be a right $\mathcal{C}$-module category, $\mathcal{M}$ a $\mathcal{C}\mathcal{D}$-bimodule category, $\mathcal{N}$ a $\mathcal{D}\mathcal{E}$-bimodule category and $\mathcal{P}$ a left $\mathcal{E}$-module category. In an effort to save space we will at times abbreviate tensor product by juxtaposition.

Lemma 3.1.1. $A \boxtimes (\mathcal{M} \boxtimes \mathcal{D} \mathcal{N}) \simeq (A \boxtimes \mathcal{M}) \boxtimes \mathcal{D} \mathcal{N}$ and $(\mathcal{M} \boxtimes \mathcal{D} \mathcal{N}) \boxtimes \mathcal{A} \simeq \mathcal{M} \boxtimes \mathcal{D} (\mathcal{N} \boxtimes \mathcal{A})$ as abelian categories.

Proof. Let $F : A \boxtimes \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{S}$ be totally balanced (Definition 2.1.3). For $A$ in $\mathcal{A}$ define functor $F_A : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{S}$ by $M \boxtimes N \mapsto F(A \boxtimes M \boxtimes N)$ on simple tensors and $f \mapsto F(id_A \boxtimes f)$ on morphisms. Note that functors $F_A$ are balanced since $F$ is totally balanced. Thus for any object $A$ there is a unique functor $\overline{F_A} : \mathcal{M} \boxtimes \mathcal{D} \mathcal{N} \to \mathcal{S}$ satisfying the diagram below left. The $\overline{F_A}$ allow us to define functor $F' : A \boxtimes (\mathcal{M} \boxtimes \mathcal{D} \mathcal{N}) \to \mathcal{S} : A \boxtimes Q \mapsto \overline{F_A}(Q)$ whenever $Q$ is an object of $\mathcal{M} \boxtimes \mathcal{D} \mathcal{N}$. 

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giving the commutative upper right triangle in the diagram on the right.

\[
\begin{array}{ccc}
\mathcal{M} \otimes \mathcal{N} & \xrightarrow{B_{\mathcal{M}, \mathcal{N}}} & \mathcal{M} \otimes_{D} \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{M} \otimes_{D} \mathcal{N} & \xrightarrow{F} & \mathcal{S}
\end{array}
\quad
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{M} \otimes \mathcal{N} & \xrightarrow{B_{\mathcal{M}, \mathcal{N}}} & \mathcal{A} \otimes (\mathcal{M} \otimes_{D} \mathcal{N}) \\
\downarrow & & \downarrow \\
\mathcal{A} \otimes \mathcal{M} \otimes_{D} \mathcal{N} & \xrightarrow{F} & \mathcal{S}
\end{array}
\]

Since the functors \(B_{\mathcal{M}, \mathcal{N}}, B_{\mathcal{M}, \mathcal{N}}, F\) and \(F'\) are unique by the various universal properties by which they are defined, both \(\mathcal{A} \otimes (\mathcal{M} \otimes_{D} \mathcal{N})\) and \((\mathcal{A} \otimes \mathcal{M}) \otimes_{D} \mathcal{N}\) are universal factorizations of \(F\) and must therefore be connected by a unique equivalence

\[
\alpha_{\mathcal{A}, \mathcal{M}, \mathcal{N}}^{2} : \mathcal{A} \otimes (\mathcal{M} \otimes_{D} \mathcal{N}) \sim (\mathcal{A} \otimes \mathcal{M}) \otimes_{D} \mathcal{N}
\]

(perforated arrow in diagram). One obtains natural equivalence \(\alpha_{\mathcal{M}, \mathcal{N}, \mathcal{A}}^{1} : (\mathcal{M} \otimes_{D} \mathcal{N}) \otimes \mathcal{A} \sim \mathcal{M} \otimes_{D} (\mathcal{N} \otimes \mathcal{A})\) by giving the same argument "on the other side," i.e. by first defining \(F_{\mathcal{N}} : \mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{S}\) for fixed \(N \in \mathcal{N}\) and proceeding analogously.

Remark 3.1.2. For bimodule category \(\mathcal{A}\) Remark 2.2.9 implies that \(\alpha^{1}\) are bimodule equivalences.

Lemma 3.1.3. For \(\alpha^{1}\) in Lemma 3.1.1 \((\mathcal{A} \otimes_{\mathcal{C}} B_{\mathcal{M}, \mathcal{N}})\alpha_{\mathcal{A}, \mathcal{M}, \mathcal{N}}^{1} : (\mathcal{A} \otimes_{\mathcal{C}} \mathcal{M}) \otimes \mathcal{N} \rightarrow \mathcal{A} \otimes_{\mathcal{C}} (\mathcal{M} \otimes_{D} \mathcal{N})\) is balanced.

Proof. Treat \(\mathcal{M}\) as having right \(\mathcal{C}\)-module structure coming from its bimodule structure, and similarly give \(\mathcal{N}\) its left \(\mathcal{C}\)-module structure. Recall, as above, we define \(R_{X} : \mathcal{M} \rightarrow \mathcal{M}\) and \(L_{X} : \mathcal{N} \rightarrow \mathcal{N}\) right and left action of \(X \in \mathcal{C}\) on \(\mathcal{M}, \mathcal{N}\) respectively. We will use superscripts to keep track of where \(\mathcal{C}\)-action is taking place, e.g. \(R_{Y}^{M}\) means right action of \(Y\) in \(\mathcal{M}\). Recall also that we have right \(D\)-action.
Consider the following diagram:

Leftmost rectangle is (definition of $R_X \otimes id_N$, top rectangle is tautologically $B \otimes L_X$, upper right and lower left triangles are definition of $\alpha^1$, lower right rectangles definition of $id_A \otimes C B_{M,N}$ and $b$ is $id_A \otimes$ (balancing isomorphism for $B_{M,N}$). An application of Lemma 2.1.9 then gives

$$\left( id_A \otimes_C B_{M,N} \right) \alpha^1_{A,M,N} (R_X \otimes id_N) \simeq \left( id_A \otimes_C B_{M,N} \right) \alpha^1_{A,M,N} (id_A \otimes_C M \otimes L_X)$$

Since $b$ satisfies the balancing axiom (Definition 2.1.1) for $B_{M,N}$ it satisfies it here. This is precisely the statement that $(A \otimes_C B_{M,N}) \alpha^1_{A,M,N}$ is balanced.

**Proposition 3.1.4.** If $A$ and $N$ are bimodules we have $(A \otimes_C M) \otimes_D N \simeq A \otimes_C (M \otimes_D N)$ as bimodule categories.

**Proof.** We plan to define the stated equivalence as the image of the functor $(A \otimes_C$
$B_{M,N} \alpha^1_{A,M,N} : (A \boxtimes C M) \boxtimes N \to A \boxtimes C (M \boxtimes_D N)$ under $\mathcal{Y}$ (equation (2)). Lemma 3.1.3 implies that indeed $\mathcal{Y}$ is defined there. With notation as above define $a^1$ and $a^2$ using the universality of $B$ by the following diagrams.

\[
\begin{array}{ccc}
(A \boxtimes C M) \boxtimes N & \xrightarrow{\alpha^1_{A,M,N}} & A \boxtimes C (M \boxtimes N) \\
B_{A,M,N} \downarrow & & \downarrow \text{id}_A \boxtimes B_{M,N} \\
(A \boxtimes C M) \boxtimes_D N & \xrightarrow{\alpha^1_{A,M,N}} & A \boxtimes C (M \boxtimes_D N) \\
\end{array} \quad \begin{array}{ccc}
A \boxtimes (M \boxtimes_D N) & \xrightarrow{\alpha^2_{A,M,N}} & (A \boxtimes M) \boxtimes_D N \\
B_{A,M,N} \downarrow & & \downarrow \text{id}_A \boxtimes B_{M,N} \\
A \boxtimes C (M \boxtimes_D N) & \xrightarrow{\alpha^2_{A,M,N}} & (A \boxtimes C M) \boxtimes_D N \\
\end{array}
\]

$\alpha^i$ are defined in Lemma 3.1.1. To see that $a^1$ and $a^2$ are quasi-inverses consider the diagram

\[
\begin{array}{ccc}
A \boxtimes (MN) & \xrightarrow{id_A \boxtimes B_{M,N}} & A \boxtimes M \boxtimes N \\
B_{A,M,N} \downarrow & & \downarrow B_{A,M,N} \\
(A \boxtimes M)N & \xrightarrow{B_{A,M,N}} & (AM) \boxtimes N \\
\end{array} \quad \begin{array}{ccc}
A \boxtimes (MN) & \xrightarrow{B_{A,M \boxtimes N}} & A(M \boxtimes N) \\
B_{A,M,N} \downarrow & & \downarrow \text{id}_A \boxtimes B_{M,N} \\
(A \boxtimes M)N & \xrightarrow{B_{A,M,N}} & (AM) \boxtimes N \\
\end{array}
\]

The triangles in upper left and right are those defining $\alpha^2$, $\alpha^1$ respectively. The central square is the definition of $B_{A,M \boxtimes_D id_N}$, and the left and right squares those
defining $a^2$ and $a^1$. Thus the perimeter commutes, giving

$$a^1a^2B_{A,M|N}(id_A \boxtimes B_{M|N}) = (id_A \boxtimes_c B_{M|N})B_{A,M\boxtimes N}$$

$$\Rightarrow a^1a^2B_{A,M|N}(id_A \boxtimes B_{M|N}) = B_{A,M|N}(id_A \boxtimes B_{M|N})$$

$$\Rightarrow a^1a^2B_{A,M|N}(a^2)^{-1}B_{A\boxtimes M,N} = B_{A,M|N}(a^2)^{-1}B_{A\boxtimes M,N}$$

$$\Rightarrow a^1a^2B_{A,M|N} = B_{A,M|N}$$

$$\Rightarrow a^1a^2 = id_{A|M|N}$$

where the first implication follows from the square defining $id_A \boxtimes_c B_{M|N}$, the second by the definition of $a^2$, the third by Lemma 2.1.9 (for $B_{A\boxtimes M,N}$, $B_{A,M|N}$, resp.). Using a similar diagram one derives $a^2a^1 = id_{(A|M)_N}$ hence the $a^i$ are equivalences and by Remark 2.2.9 they are bimodule equivalences.

In what follows denote

$$a_{A,M,N} := a^1_{A,M,N} : (A \boxtimes_c \mathcal{M}) \boxtimes_D N \simeq A \boxtimes_c (\mathcal{M} \boxtimes_D N). \quad (9)$$

In order to prove coherence for $a$ (Proposition 3.1.8) we will need a couple of simple technical lemmas together with results about the naturality of $a$. In the monoidal category setting associativity of monoidal product is required to be natural in each of its indices, which are taken as objects in the underlying category. In describing monoidal structure in the 2-category setting we also require associativity though
stipulate that it be natural in its indices up to 2-isomorphism (see M.10 in Definition 1.4.2). For us this means, in the first index,

$$a_{F,M,N} : a_{B,M,N}(FM)N \xrightarrow{\sim} F(M \otimes_D N)a_{A,M,N}$$

for bimodule functor $F : A \to B$. Similarly we need 2-isomorphisms for $F$ in the remaining positions. The content of Proposition 3.1.6 is that all such 2-isomorphisms are actually identity. Before stating it we give a definition to introduce a notational convenience.

**Definition 3.1.5.** For right exact right $C$-module functor $F : A \to B$ define 1-cell $FM := F \otimes_C id_M : A \otimes_C M \to B \otimes_C M$ and note that $FM$ is right exact. Similarly we can act on such functors from the right.

**Proposition 3.1.6 (Associativity "2-naturality").** We have

$$a_{B,M,N}(FM)N = F(MN)a_{A,M,N}.$$

Analogous relations hold for the remaining indexing valencies of $a$.

**Proof.** We will prove the stated naturality of $a$ for 1-cells appearing in the first index. A similar proof with analogous diagrams gives the others. Recall $\alpha^1$ defined in Lemma
3.1.1. Consider the diagram:

The top, bottom and center rectangles follow from Definition 3.1.5 and definition of tensor product of functors. Commutativity of all other subdiagrams is given in proof of Proposition 3.1.4. External contour is the stated relation. □

**Remark 3.1.7.** Observe that the proof of Proposition 3.1.6 also gives 2-naturality of $\alpha^1$: the center square with attached arches gives the equation

$$\alpha^1_{B, MN}(F(\mathcal{M}) \boxtimes \text{id}_N) = F(\mathcal{M} \boxtimes \mathcal{N}) \alpha^1_{A, MN}.$$  

(10)

**Lemma 3.1.8.** The hexagon

$$\begin{array}{ccc}
\mathcal{A}(\mathcal{M} \nabla \mathcal{N}) \boxtimes \mathcal{P} & \overset{a_{A, MN}}{\longrightarrow} & (\mathcal{A} \mathcal{M}) \nabla \mathcal{N} \boxtimes \mathcal{P} \\
\alpha^1_{A, MN, P} \downarrow & & \downarrow B_{(\mathcal{A} \mathcal{M}) N, P} \\
\mathcal{A}((\mathcal{M} \nabla \mathcal{N}) \mathcal{P}) & \overset{a_{A, MN}}{\longrightarrow} & (\mathcal{A} \mathcal{M} \nabla \mathcal{N}) \mathcal{P}
\end{array}$$
commutes.

Proof. The arrow $B_{A(MN),P}$ drawn from the upper-left most entry in the hexagon to the lower-right most entry divides the diagram into a pair of rectangles. The upper right rectangle is the definition of $a_{A,M,N} \boxtimes_C id_P$ and the lower left rectangle is the definition of $a_{A,MN,P}$.

In the case of monoidal categories the relevant structure isomorphisms are required to satisfy axioms which take the form of commuting diagrams. In the 2-monoidal case we make similar requirements of the structure morphisms but here, because of the presence of higher dimensional structures, it is necessary to weaken these axioms by requiring only that their diagrams commute up to some 2-morphisms. Above we have defined a 2-associativity isomorphism $a_{M,N,P} : (MN)P \rightarrow MN(P)$. In the definition of monoidal 2-category $a$ is required to satisfy the pentagon which appears in the lower dimensional monoidal case, but only up to 2-isomorphism. The content of Proposition 3.1.9 is that, in the 2-category of bimodule categories, the monoidal structure $\boxtimes_C$ strictly satisfies the associated hexagon just as in the monoidal category setting. For us this means that the 2-isomorphism $a_{A,M,N,P}$ (see M9. Definition 1.4.2) is actually identity for any bimodule categories $A, M, N, P$ for which the relevant tensor products make sense.
Proposition 3.1.9 (2-associativity hexagon). The diagram of functors commutes.

\[
\begin{array}{c}
((AM)N)\mathcal{P} \xrightarrow{\alpha_{AM,N,P}} (AM)(NP) \\
\downarrow a_{AM,N,\mathcal{P}} \\
(A(MN))\mathcal{P} \xrightarrow{\alpha_{AM,N}} (AM)(NP) \\
\downarrow a_{AM,\mathcal{P}} \\
A((MN)\mathcal{P}) \xrightarrow{\alpha_{AM,N,\mathcal{P}}} A(M(NP)) \\
\end{array}
\]

Proof. Consider the diagram below. We first show that the faces peripheral to the embedded hexagon commute and then show that the extended perimeter commutes.

The top rectangle is the definition of \(a_{AM,N,\mathcal{P}}\), the rectangle on the right is naturality of \(a\) as in Proposition 3.1.6, the bottom rectangle the definition of \(a\) tensored on the left by \(A\), and the hexagon is Lemma 3.1.8. To prove commutativity of the extended
perimeter subdivide it as indicated below.

\[(AM)N \boxtimes P \xrightarrow{\alpha_{AM,N,P}} (AM)(N \boxtimes P)\]

\[\xrightarrow{a_{AM,N} \otimes B_{AM,N}} \]

\[\xrightarrow{\alpha_{A(AM),N \boxtimes P}} \]

\[\xrightarrow{a_{A(AM),N \otimes P}} \]

\[\xrightarrow{\alpha_{A(AM),N \otimes P}} \]

\[\xrightarrow{a_{A(AM),N \otimes P}} \]

The upper and lower triangles are the definitions of \(\alpha_{AM,N,P}^1\) and \(A \boxtimes \ast\) (definition of \(\alpha_{A,M,N,P}^1\)), respectively (Lemma 3.1.1). Right rectangle is definition of \(a_{A,M,N \otimes P}\). Upper left rectangle is (definition of \(a_{A,M,N} \otimes P\), and the lower left rectangle is explained in Remark 3.1.7. The central triangle commutes as follows. Using the definition of \(\alpha^1\) given in the proof of Proposition 3.1.4 we can draw the diagram

\[A \boxtimes M \otimes N \boxtimes P \xrightarrow{B_3} A(\otimes M) \boxtimes N \otimes P \xrightarrow{B_1} A(\otimes M \otimes N \boxtimes P) \xrightarrow{B_2} A(\otimes M \otimes N) \boxtimes P\]

where we have abbreviated the various \(\alpha^1\) appearing in the statement of the lemma
by \( \alpha_i \) and associated functors \( B \) occurring in their definitions \( B_i \) in such a way that

\[
\alpha_1 B_1 = B_3, \quad \alpha_3 B_2 = B_3, \quad \alpha_2 B_1 = B_2.
\]

These equations imply \( B_3 = \alpha_3 \alpha_2 \alpha_1^{-1} B_3 \) where \( B_3 = B_{A,M \otimes N \otimes P} \). Apply Lemma 2.1.9 to write \( \text{id} = \alpha_3 \alpha_2 \alpha_1^{-1} \). Now equating paths in the large diagram allows us to write

\[
a_{A,M,N \otimes P}(\alpha_{A,M,N,P}^1)(B_{A,M,N} \otimes P) = (A \otimes_{A,M,N,P} \alpha_{A,M,N,P}^1)(a_{A,M,N \otimes P})B_{A,M,N} \otimes P
\]

and a final application of Lemma 2.1.9 gives the relation expressing commutativity of outer pentagon.

Let \( \mathcal{M}_i \) be a \( (C_{i-1},C_i) \)-bimodule category tensor categories \( C_i \) \( 0 \leq i \leq n + 1 \). Then one extends the arguments above to completely balanced functors (Definition 2.1.2) of larger index to show that any meaningful arrangement of parentheses in the expression \( \mathcal{M}_1 \otimes_{C_1} \mathcal{M}_2 \cdots \otimes_{C_{n-1}} \mathcal{M}_n \) results in an equivalent bimodule category.

**Remark 3.1.10.** Proposition 3.1.9 implies that the 2-morphism described in M9 of Definition 1.4.2 is actually identity. The primary polytope associated to associativity in the monoidal 2-category setting is the Stasheff polytope which commutes in this case. It is obvious that the modified tensor product \( \hat{\otimes} \) with associativity ([KV91] §4) is identity and that nearly every face commutes strictly. The two non-trivial remaining faces (one on each hemisphere) agree trivially. We refer the reader to the original paper for details and notation.
3.2 Unit constraints

Recall from Proposition 2.2.5 the equivalences \( l_M : \mathcal{M} \otimes D \simeq \mathcal{M} \) and \( r_M : C \otimes C \mathcal{M} \simeq \mathcal{M} \). This section's first proposition explains how \( l, r \) interact with 2-associativity.

**Proposition 3.2.1.** \((id_M \otimes_D l_N) a_{M,N,E} = l_{M \otimes_D N}, \ r_{M \otimes_D N}(a_{C,M,N}) = r_M \otimes_D id_N.\)

Also the triangle

\[
\begin{array}{ccc}
(M \otimes_D D) \otimes_D N \\
\downarrow a_{M,D,N} \\
M \otimes_D (D \otimes_D N) \end{array}
\]

commutes up to a natural isomorphism.

**Proof.** The first two statements follow easily from definitions of \( \alpha^1 \) (Lemma 3.1.1), module structure in \( \mathcal{M} \otimes_D N \) and those of \( l \) and \( r \). This means that the 2-isomorphisms \( \rho \) and \( \lambda \) in M11 of Definition 1.4.2 are both trivial.

The diagram below relating \( l \) and \( r \) commutes only up to balancing isomorphism \( b \) for \( B_{M,N} \) where we write \( b : B_{M,N}(\otimes \otimes id_N) \Rightarrow B_{M,N}(id_M \otimes \otimes). \)
Top triangle is definition of $\alpha^1$, rectangle is definition of $id_M \boxtimes_D B_{D,N}$, lower right triangle is $M \boxtimes_D (\text{definition of } r_N)$, triangle on left is (definition of $l_M) \boxtimes_D N$, and central weakly commuting rectangle is definition of balancing $b$ for $B_{M,N}$. The perimeter is a diagram occurring in the proof of Proposition 3.1.4 (we have been sloppy with the labeling of the arrow across the top). Since all other non-labeled faces commute we may write, after chasing paths around the diagram,

$$l_M \boxtimes_D id_N(B_{M,D} \boxtimes_D id_N)B_{M\boxtimes_D N} \cong (id_M \boxtimes_D r_N)a_{M,D,N}(B_{M,D} \boxtimes_D id_N)B_{M\boxtimes_D N}.$$

Applying Lemma 2.1.9 twice we obtain a unique natural isomorphism

$$\mu_{M,N}: l_M \boxtimes_D id_N \rightarrow (id_M \boxtimes_D r_N)a_{M,D,N} \quad (11)$$

having the property that $\mu_{M,N} \ast ((B_{M,D} \boxtimes_D id_N)B_{M\boxtimes_D N}) = b$, the balancing in $B_{M,N}$. \qed
CHAPTER IV

PROOF OF THEOREM 0.2.1

4.1 The commuting polytopes

In this section we finish verifying that the list of requirements given in the definition of monoidal 2-category ([KV91]), Definition 1.4.2 of this thesis, are substantiated by the scenario where we take as underlying 2-category $B(C)$. Recall that for a fixed monoidal category $C$ the 2-category $B(C)$ is defined as having 0-cells $C$-bimodule categories, 1-cells $C$-bimodule functors and 2-cells monoidal natural transformations. M1-M11 are evident given what we have discussed so far; explicitly, and in order, these are given in Proposition 2.2.5, Proposition 2.2.3, Definition 3.1.5, Remark 2.2.2 (take one of the 2-cells to be identity transformation on identity functor), Equation 9, Proof of Proposition 2.2.5, Definition 3.1.5 (trivial, composition with id commutes), Polytope 4.1.3, Proposition 3.1.9 (trivial), Proposition 3.1.6 ($a_{F,M,N} = id$ for bimodule functor $F$), Proof of Proposition 3.2.1. Commutativity of the Stasheff polytope follows from Proposition 3.1.9 (see Remark 3.1.10).

The data introduced throughout are required to satisfy several commuting polytopes describing how they are to interact. Fortunately for us only a few of these require checking since many of the structural morphisms above are identity. Because
of this we prove below only those verifications which are not immediately evident. Recall (Definition 3.1.5) that we define action $\mathcal{M}F$ of bimodule category $\mathcal{M}$ on module functor $F$.

**Polytope 4.1.1.** For $\mathcal{M}, \mathcal{N}, \mathcal{P} \in B(C)$, the pastings

\[
\begin{array}{ccc}
((CM)N)\mathcal{P} & \overset{a_{CM,N,P}}{\longrightarrow} & (CM)(N)\mathcal{P} \\
\downarrow a_{CM,N,P} & & \downarrow a_{CM,N,P} \\
(C(MN))\mathcal{P} & \overset{(r_{MN})\mathcal{P}}{\longrightarrow} & (MN)\mathcal{P} \\
\downarrow a_{CM,N,P} & & \downarrow a_{M,N,P} \\
C((MN)\mathcal{P}) & \overset{a_{CM,N,P}}{\longrightarrow} & C((M)N)\mathcal{P} \\
\downarrow a_{CM,N,P} & & \downarrow \rho_{MN} \mathcal{P} \\
C(M(N)\mathcal{P}) & \overset{r_{MN} \mathcal{P}}{\longrightarrow} & C(M(N)\mathcal{P}) \\
\end{array}
\]

\[
\begin{array}{ccc}
((CM)N)\mathcal{P} & \overset{a_{CM,N,P}}{\longrightarrow} & (CM)(N)\mathcal{P} \\
\downarrow \rho_{MN} \mathcal{P} & & \downarrow \rho_{MN} \mathcal{P} \\
(C(MN))\mathcal{P} & \overset{(r_{MN})\mathcal{P}}{\longrightarrow} & (MN)\mathcal{P} \\
\downarrow \rho_{MN} \mathcal{P} & & \downarrow \rho_{MN} \mathcal{P} \\
C((MN)\mathcal{P}) & \overset{a_{CM,N,P}}{\longrightarrow} & C(M(N)\mathcal{P}) \\
\downarrow \rho_{MN} \mathcal{P} & & \downarrow a_{M,N,P} \\
C(M(N)\mathcal{P}) & \overset{r_{MN} \mathcal{P}}{\longrightarrow} & C(M(N)\mathcal{P}) \\
\end{array}
\]

correspond to the same 2-cell.

**Proof.** Note that every face commutes (all labeling 2-cells are identity) except for $r_{a_{M,N,P}}$ in the second diagram. Thus the pastings give the same 2-cell if we can show $r_{a_{M,N,P}}$ is also identity. By comments following the proof of Corollary 2.2.6 this is equivalent to showing that $a_{M,N,P}$ is a strict module functor, i.e., that the module linearity $w$ associated to $a$ is identity. For $X \in C$ note that for simple tensor

\[
(MN)P = B_{MN,P}(B_{M,N} \otimes \mathcal{P})(M \otimes N \otimes \mathcal{P})
\]

\[
a_{M,N,P}(X \otimes (MN)P) = (X \otimes M)(NP) = X \otimes a_{M,N,P}((MN)P)
\]

so by two applications of Lemma 2.1.9 $w = id.$

The remaining four polytopes describe 2-naturality of the action of the unit object.
in our monoidal 2-category (recall that the unit object in \( \mathcal{B}(C) \) is \( C \) itself). The first concerns 2-naturality of \( \mu, \lambda \) and \( \rho \).

**Polytope 4.1.2.** For \( F : \mathcal{M} \to \mathcal{M}' \) a morphism in \( \mathcal{B}(C) \) and any \( C \)-bimodule category \( \mathcal{N} \) the polytopes

\[
\begin{align*}
\text{(MC)} & \xrightarrow{a} \text{M(CN)} \\
\text{(FC)} & \xrightarrow{f \cdot \mu_{M,N}} \text{MN} \\
\text{(NC)} & \xrightarrow{\text{id}_N \cdot \text{id}_F} \text{N(CF)}
\end{align*}
\]

\[
\begin{align*}
\text{(M'C)} & \xrightarrow{a} \text{M'(CN)} \\
\text{(FC')} & \xrightarrow{f \cdot \mu_{M',N'}} \text{MN'} \\
\text{(NC')} & \xrightarrow{\text{id}_N \cdot \text{id}_{F'}} \text{N(CF')}
\end{align*}
\]

commute. Similarly there are commuting prisms for upper left vertex corresponding to the remaining four permutations of \( \mathcal{M}, \mathcal{C}, \mathcal{N} \) with upper and lower faces commuting up to either \( \lambda \) or \( \rho \).

In [KV91] these triangular prisms are labeled \( (- \to \otimes 1 \otimes \bullet), (1 \otimes - \to \otimes \bullet), \) etc.

**Proof.** We verify commutativity of the second polytope. Commutativity of the other prisms is proved similarly. Denote by \( * \) mixed composition of cells. Commutativity of polytope on the right is equivalent to the equation

\[
(id_N \boxtimes_C \bar{f})(id_N \boxtimes_C F) * \mu_{N,M} = \mu_{N,M'} * (id_{N \boxtimes_C F} \boxtimes_C F)
\]

(12)

where \( f \) is module structure of \( F \) and \( \bar{f} = r_F \) (recall **Corollary 2.2.6**). Let LHS and RHS denote the left and right sides of (12). Then one easily shows that both
LHS*((B_{N,C,M},id_M)B_{N,C,M})_{M\times \mathbb{N}} and RHS*((B_{N,C,M},id_M)B_{N,C,M})_{M\times \mathbb{N}} for N \in \mathbb{N}, X \in C, M \in \mathcal{M}, are equal to b'_{N,X,F(M)} where b' is the balancing for B_{N,M'}. Two applications of Lemma 2.1.9 now imply that LHS=RHS. □

The next polytope concerns functoriality of the 2-cells $l_{F}, r_{F}$.

**Polytope 4.1.3.** Let $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{G} \mathcal{P}$ be composable 1-morphisms in $\mathcal{B}(C)$. Then the prisms

![Diagram](image)

commute.

**Proof.** We prove commutativity of the first prism. Commutativity of the second follows similarly. It is obvious that $\otimes_{C,F,G}$ is trivial (it is just composition of functors).

First polytope is the condition $r_{GF} = (G * r_{F})(r_{G} * CF)$. Let $f$ be left $C$-linearity for $F$, $g$ that for $G$. Then $(G,g)(F,f) := (GF,g \cdot f)$ where $(g \cdot f)_{X,M} = g_{X,F(M)}G(f_{X,M})$ is left $C$-linearity for $GF$. One checks directly that

$$(G * r_{F})(r_{G} * CF) * B_{C,M} = (g \cdot f)^{-1}.$$  

$r_{GF}$ is defined as the unique 2-isomorphism for which $r_{GF} * B_{C,M} = (g \cdot f)^{-1}$ so Lemma 2.1.9 gives the result. □
Polytope 4.1.4. For any 2-cell $\alpha : F \Rightarrow G$ in $B(C)$ the cylinders commute.

Proof. Again we check commutativity of the first polytope. The first cylinder is the condition $(\alpha \ast r_M)r_F = r_G(r_N \ast C\alpha)$ where $C\alpha$ is the 2-cell defined by $id_C \otimes_C \alpha$ and $id_C$ means natural isomorphism $id : id_C \Rightarrow id_C$. One verifies this directly using the bimodule condition on $\alpha$. Again one checks first that components after right $\ast$-composing with the appropriately indexed universal functor $B$ agree. Thus for $X \in C$ and $M \in \mathcal{M}$ we have

$$(\alpha \ast r_M)r_F \ast B_{C,\mathcal{M}})_{X \otimes M} = \alpha_{X \otimes M} f_{X,M}^{-1}$$

$$(r_G(r_N \ast C\alpha) \ast B_{C,\mathcal{M}})_{X \otimes M} = g_{X,M}^{-1}(id_X \otimes \alpha_M)$$

and since $\alpha$ is a natural module transformation the compositions on the right agree.

Applying Lemma 2.1.9 for $B_{C,\mathcal{M}}$ gives the result. $\square$
Polytope 4.1.5. For \( \mathcal{M} \) in \( \mathcal{B}(\mathcal{C}) \), the pastings

\[
\begin{array}{c}
\text{(CC)M} \quad \xrightarrow{ac_{C,M}} \quad C(\text{CM}) \\
\downarrow r_{C,M} \quad \quad \quad \downarrow r_{C,M} \\
\text{CM} \quad \xrightarrow{r_{C,M}} \quad \mathcal{M}
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{c}
\text{(CC)M} \quad \xrightarrow{ac_{C,M}} \quad C(\text{CM}) \\
\downarrow r_{C,M} \quad \quad \quad \downarrow r_{C,M} \\
\text{CM} \quad \xrightarrow{id} \quad \mathcal{M}
\end{array}
\]

give the same 2-isomorphism. Each of the remaining two orderings of the multiset \( \{C, C, \mathcal{M}\} \) determines an analogous pair of pastings, and hence a unique 2-isomorphism.

Remark 4.1.6. Note that the pair of diagrams is determined by the order of the objects in the upper left vertex. Keeping parentheses fixed, there are related pairs of diagrams for the remaining two orderings of the multiset \( \{C, C, \mathcal{M}\} \). Each pair determines a pair of pastings, and each such pair of pastings similarly determines a unique 2-isomorphism.

Proof. We give proof in the diagrammed case. The other two are similar. Bimodule linearity for \( r_{\mathcal{M}} \) is trivial (since \( r_{\mathcal{M}} \) is strict à la Remark 2.2.7). The equation

\[
\begin{equation}
 r_{\mathcal{M}} \ast \mu_{C,\mathcal{M}} = id 
\end{equation}
\]

is therefore the content of Polytope 4.1.5. To see this choose natural isomorphism \( J : r_{\mathcal{M}} B_{C,\mathcal{M}} = \otimes \rightarrow \otimes = r_{\mathcal{M}} B_{C,\mathcal{M}} \) having components \( J_{X \otimes M} := r_{\mathcal{M}}(b_{1,X,\mathcal{M}}) \) where \( b \) is the balancing for \( B_{C,\mathcal{M}} \). According to the definition of \( \mu_{C,\mathcal{M}} \) in 11 we see that
\((r_M \ast \mu_{C,M}) \ast (B_{C,C} \otimes_C id_M) B_{C \boxtimes C,M} = r_M \ast b\). Now using the fact that \(r_M\) is trivially balanced (proof of Proposition 2.2.5) the natural isomorphism \(J\) is balanced: that is, we have commutativity of the diagram

\[
\begin{array}{c}
r_M B_{C,M}(X \otimes Y \otimes M) = (XY)M \xrightarrow{(r_M \ast b)_{X,Y,M}} r_M B_{C,M}(X \otimes Y \otimes M) = X(YM) \\
J_{XY \otimes M} = (r_M \ast b)_{1,XY,M} \\
r_M B_{C,M}(X \otimes Y \otimes M) = (XY)M \xrightarrow{J_X \otimes YM = (r_M \ast b)_{1,XY,M}} r_M B_{C,M}(X \otimes Y \otimes M) = X(YM)
\end{array}
\]

This follows from the balancing diagram satisfied by \(b\). Using the relations given in the balancing diagram for \(b\) we derive the relations \(b_{1,XY,M} = b_{1,XYM}b_{X,Y,M} \) and \(b_{X,1,M} = id\) which together imply \(b_{1,XY,M} = id\) for any \(X, Y \in C, M \in M\). Thus the vertical arrows in the diagram above are identity hence \(r_M \ast b = id\). On an application of the uniqueness of the descended 2-cells (Lemma 2.1.9) we must have \(r_M \ast \mu_{C,M} = id\), which is (13). \(\square\)

This completes verification of the polytopes required for monoidal 2-category structure, and therefore completes the proof of Theorem 0.2.1.
CHAPTER V

TENSOR PRODUCT OF FUSION CATEGORIES OVER BRAIDED FUSION CATEGORIES

In this chapter we are interested in examining the relative tensor product of monoidal categories. That is, if monoidal categories $C_1, C_2$ also happen to be module categories over a fusion category $\mathcal{D}$ when is $C_1 \boxtimes_{\mathcal{D}} C_2$ monoidal? When can we give $C_1 \boxtimes_{\mathcal{D}} C_2$ a braided structure? What is its center? Clearly it is possible to formulate many interesting questions. We hope to answer some of them here. We will need the following definition.

**Definition 5.0.7 ([DGNO10]).** Let $\mathcal{C}$ be a monoidal category. Then $\mathcal{C}$ is said to be tensor over braided fusion category $\mathcal{D}$ if there is a braided tensor functor $\varphi : \mathcal{D} \to Z(\mathcal{C})$.

Typically we will identify $\mathcal{D}$ with its image in the center $Z(\mathcal{C})$. Evidently this gives $\mathcal{C}$ the structure of a $\mathcal{D}$-bimodule category: if $X \in \mathcal{C}$ and $D \in \mathcal{D}$ define $D \otimes X := \varphi(D) \otimes X$ where $\otimes$ on the right is in $Z(\mathcal{C})$ and where we identify $\varphi(D) \otimes X$ with its image under the canonical surjection $Z(\mathcal{C}) \to \mathcal{C}$. Right $\mathcal{D}$-module category structure is given by $X \otimes \varphi(D)$. 

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5.1 Tensor product of monoidal categories

Unless otherwise noted assume all tensor categories are semisimple. Let \( D \) be a braided fusion category and let \( \varphi_i : D \to Z(C_i) \), \( i = 1, 2 \), be braided inclusions so that \( C_i \) are tensor over \( D \). Further assume that the compositions \( \pi_i \varphi_i \) are fully faithful functors (\( \pi_i : Z(C_i) \to C_i \) are the canonical surjections). We may thus consider \( D \) as a braided fusion subcategory of both \( C_i \).

5.1.1 Monoidal structure of \( C_1 \boxtimes_D C_2 \)

Let \( C_i \) be monoidal categories over braided fusion category \( D \). Let \( \tau : C_1 \boxtimes C_2 \to C_2 \boxtimes C_1 \) be the functor \( X \boxtimes Y \mapsto Y \boxtimes X \), and denote by \( B_{1,2} : C_1 \boxtimes C_2 \to C_1 \boxtimes_D C_2 \) the universal balanced functor described in Definition 2.1.4.

**Proposition 5.1.1.** \( C_1 \boxtimes_D C_2 \) has canonical structure of a monoidal category with respect to which \( B_{1,2} \) is a strict monoidal functor.

**Proof.** Denote by \( \Gamma \) the composition of functors

\[
\Gamma := C_1 \boxtimes C_2 \boxtimes C_1 \boxtimes C_2 \xrightarrow{\tau(23)} C_1 \boxtimes C_1 \boxtimes C_2 \boxtimes C_2 \xrightarrow{(\otimes_1, \otimes_2)} C_1 \boxtimes C_2
\]

and define \( \Lambda = B_{1,2} \circ \Gamma \). It is evident that \( \Lambda \) is balanced. Thus we get unique functor \( \overline{\Lambda} \) making the diagram

\[
\begin{array}{ccc}
C_1 \boxtimes C_2 \boxtimes C_1 \boxtimes C_2 & \xrightarrow{B_{1,2}^{1,2}} & C_1 \boxtimes C_2 \\
(C_1 \boxtimes_D C_2) \boxtimes (C_1 \boxtimes_D C_2) & \xrightarrow{\Lambda} & C_1 \boxtimes_D C_2 \\
\end{array}
\]
commute. Here \( B_{c_1,c_2}^{1,3} = B_{1,2} \boxtimes B_{1,2} \) is the universal functor for right exact functors balanced in positions 1, 3 (Definition 2.1.3) from the abelian category at the apex.

Associativity for \( \Lambda \) is verified as follows. Abbreviate functors

\[
\Lambda_\ell := \Lambda(\Gamma \boxtimes id_{c_1 \boxtimes c_2}) : (c_1 \boxtimes c_2) \Box^{33} \rightarrow c_1 \boxtimes_D c_2
\]
\[
\Lambda_r := \Lambda(id_{c_1 \boxtimes c_2} \boxtimes \Gamma) : (c_1 \boxtimes c_2) \Box^{33} \rightarrow c_1 \boxtimes_D c_2.
\]

We leave verification that \( \Lambda_\ell, \Lambda_r \) are balanced in positions 1, 3 and 5 to the motivated reader. One checks easily that \( \Lambda_\ell = \Lambda(\Lambda \boxtimes id_{c_1 \boxtimes c_2})B_{1,2}^{33} \) and \( \Lambda_r = \Lambda(id_{c_1 \boxtimes c_2} \boxtimes \Lambda)B_{1,2}^{33} \).

Thus by uniqueness of \( \Lambda_\ell, \Lambda_r \) we must have

\[
\Lambda_\ell = \Lambda(\Lambda \boxtimes id_{c_1 \boxtimes c_2}) \\
\Lambda_r = \Lambda(id_{c_1 \boxtimes c_2} \boxtimes \Lambda).
\]

Next let \( a^1 \) be associativity constraints in \( c_1 \). Then \( B_{1,2} \ast a^1 \boxtimes a^2 : \Lambda_\ell \rightarrow \Lambda_r \) is a balanced natural transformation and we thus get a unique natural isomorphism

\[
B_{1,2} \ast a^1 \boxtimes a^2 : \Lambda(\Lambda \boxtimes id_{c_1 \boxtimes c_2}) \sim \Lambda(id_{c_1 \boxtimes c_2} \boxtimes \Lambda).
\]

This is precisely the associativity diagram required of \( \Lambda \) evincing it a bona fide tensor structure on \( c_1 \boxtimes_D c_2 \). Observe that unit object for \( \Lambda \) comes from identity objects of \( c_1 \) in the obvious way: \( 1 = B_{1,2}(1 \boxtimes 1) \).

Tensor strictness of \( B_{1,2} \) follows from the fact that monoidal structure in \( c_1 \boxtimes c_2 \)
is defined by the functor $\Gamma$. Indeed if $U := X \boxtimes Y$ and $V := X' \boxtimes Y'$ are objects in $C_1 \boxtimes C_2$ we have, from the definition of $\Lambda$ and $\Gamma$,

$$B_{1,2}\Gamma(U \boxtimes V) = \Lambda(B_{1,2} \boxtimes B_{1,2})(U \boxtimes V).$$

The LHS is $B_{1,2}$ evaluated on the tensor product $U \otimes V$ in $C_1 \boxtimes C_2$ and the RHS is the tensor product $B_{1,2}(U) \otimes B_{1,2}(V)$ in $C_1 \boxtimes_{D} C_2$. It is clear that both sides equal $B_{1,2}((X \otimes X') \boxtimes (Y \otimes Y'))$. □

5.1.2 Functors over $D$

In this subsection we are interested in studying the (the as yet undefined) monoidal 2-category of tensor categories over a fixed braided fusion category. The next definition is an essential step in this direction.

**Definition 5.1.2.** Suppose $C_1, C_2$ are tensor categories over braided fusion category $D$, and denote by $\psi_i$ the the compositions $D \hookrightarrow Z(C_i) \rightarrow C_i$. A tensor functor $F : C_1 \rightarrow C_2$ is said to be a **functor over $D$** if $F\psi_1 = \psi_2$.

Definition 5.1.2 stipulates that functors over $D$ are precisely those respecting the relevant braided injections. We require one further definition to form the functorial counterpart to Proposition 5.1.1.

**Definition 5.1.3.** Suppose $B, C, D$ are tensor categories and let $F : C \rightarrow B$ and $G : D \rightarrow B$ be tensor functors with tensor structures $f, g$ respectively. A relative **braiding** for the pair $F, G$ is a family of natural isomorphisms $c_{X,Y} : F(X) \otimes G(Y) \rightarrow$
$G(Y) \otimes F(X)$ satisfying the pentagons

\[
\begin{array}{ccc}
F(XY)G(Z) & \xrightarrow{f_{X,Y}} & F(X)G(Y)G(Z) \\
\downarrow^{c_{X,Y,Z}} & & \downarrow^{c_{Y,Z}} \\
G(Z)F(XY) & \xrightarrow{f_{X,Y}} & G(Z)F(X)F(Y)
\end{array} \quad \begin{array}{ccc}
F(X)G(VZ) & \xrightarrow{g_{V,Z}} & F(X)G(V)G(Z) \\
\downarrow^{c_{X,V}} & & \downarrow^{c_{X,Z}} \\
G(V)F(X)G(Z) & \xrightarrow{g_{V,Z}} & G(V)G(Z)F(X)
\end{array}
\]

for all $X, Y \in C$ and $V, Z \in D$.

Assume the category $B$ in Definition 5.1.3 is braided. Then any pair of tensor functors into $B$ are related by a relative braiding having components given by the braiding indexed by objects in the images of $F$ and $G$. This follows from naturality of the tensor structures $f, g$ and the braiding hexagon.

**Proposition 5.1.4.** Let $C_1, C_2, A$ be tensor categories over braided fusion category $D$ and let $F_i : C_i \to A$ be tensor functors over $D$ related by a relative braiding. Then $F_1, F_2$ determine a unique tensor functor $C_1 \boxtimes_D C_2 \to A$.

**Proof.** Let $t, t'$ be tensor structures for $F_1, F_2$ respectively and let $c$ denote the relative braiding as in Definition 5.1.3. Denote by $c^i$ the braided structure in $Z(C_i)$. The functors $F_i$ determine a tensor functor $F : C_1 \boxtimes C_2 \to A$ defined by sending $X \boxtimes Y \mapsto F_1(X) \otimes F_2(Y)$. We show that $F$ is a tensor functor below.

The proof of Proposition 5.1.4 divides into three parts. First we show that $F$ is $D$ balanced. Then we show that the functors $F \otimes$ and $\otimes(F \otimes F)$ are both tensor and balanced, and finally that tensor structure $f : F \otimes \sim \otimes(F \otimes F)$ is a balanced natural isomorphism. This will imply that all these structures descend to the relative
product.

1. *F balancing.* Denote by $b_{X,D,Y}$ the composition

$$F_1(X \otimes D) \otimes F_2(Y) \xrightarrow{t_{X,D}} F_1(X) \otimes F_1(D) \otimes F_2(Y) \xrightarrow{t_{D,E}^{-1}} F_1(X) \otimes F_2(D \otimes Y).$$

for $X \in C_1$, $Y \in C_2$ and $D \in \mathcal{D}$. This composition makes sense because $F_1(D) = F_2(D)$ thanks to Definition 5.1.3. We show that $b$ satisfies the balancing diagram, thus balancing for $F$. This is a straightforward calculation: simply observe that the diagram commutes for $D, E \in \mathcal{D}$:

$$
\begin{array}{c}
F_1(XDE)F_2(Y) \\
\downarrow t_{X,DE} \quad t_{X,D} \quad t_{E,Y}^{-1} \\
F_1(X)F_1(D)F_2(Y) \\
\downarrow t_{D,E}^{-1} \quad t_{D,EY}^{-1} \\
F_1(X)F_2(DEY) \\
\downarrow \quad \downarrow \\
F_1(X)F_1(D)F_2(EY)
\end{array}
$$

Note that $t_{D,E} = t_{D,E}^{-1}$ via Definition 5.1.3. The rectangles are therefore diagrams required of tensor structures for $F_1, F_2$ and triangle commutes trivially. Perimeter is the balancing diagram for $b$.

2. *Tensor structure of $F$.* In what follows we will be required to draw diagrams having vertices labeled by sextuples of objects. In order to simplify and condense notation let us adopt the following convention: write $F_1(X)F_2(Y) := (X)(Y)$ for the tensor product of the images of $X \in C_1, Y \in C_2$ in $\mathcal{A}$. Since all monoidal categories are assumed strict we lose nothing by so doing. Denote objects of $C_1 \boxtimes C_2$ using overline
and subscripts: $X = X_1 \boxtimes X_2$ etc. Define natural isomorphisms $f_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$ by the composition

$$F(X \otimes Y) = (X_1Y_1)(X_2Y_2) \xrightarrow{t \otimes t'} (X_1)(Y_1)(X_2)(Y_2) \xrightarrow{c_{Y_1,Y_2}} (X_1)(X_2)(Y_1)(Y_2) = F(X)F(Y).$$

We now show that $f$ provides $F$ with the structure of a tensor functor. This will require the defining diagrams for the relative braiding.

All subdiagrams are either relative braiding diagram or diagrams for tensor structure in the $F_i$. Perimeter is tensor diagram for $(F, f)$.

3. **Balancing of $F \otimes$.** In this part of the proof we show that the composition $F \otimes : (C_1 \boxtimes C_2)^{S_2} \to \mathcal{A}$ of $F$ with the monoidal structure in $C_1 \boxtimes C_2$ is $\mathcal{D}$ balanced in positions 1 and 3. This is necessary for $F$ to descend to a functor from the relative product in a way which respects monoidal structure. For $D \in \mathcal{D}$ define natural isomorphism
\[ b^1_{X_1, D, X_2, Y} : F \otimes (X_1 D \boxtimes X_2 \boxtimes Y) \simeq F \otimes (X_1 \boxtimes DX_2 \boxtimes Y) \] by the composition

\[
(X_1 DY_1)(X_2 Y_2) \xrightarrow{c^{b,y}_D} (X_1 Y_1 D)(X_2 Y_2) \xrightarrow{t_{X_1 Y_1, D}} (X_1 Y_1)(D)(X_2 Y_2) \xrightarrow{t_{D, X_2 Y_2}^{-1}} (X_1 Y_1)(DX_2 Y_2)
\]

where we continue to use notation introduced in part 2 of this proof. Note that the third expression is not ambiguous because \( F_1, F_2 \) agree on \( D \). Let \( D, E \in D \). The following diagram shows that \( b^1 \) provides a balancing of \( F \otimes \) in the first position.

Every subdiagram is either braiding hexagon, naturality of \( t \) or tensor structure for \( t, t' \). One similarly defines \( b^2_{X, Y_1, D, X_2} : F \otimes (X \boxtimes Y_1 D \boxtimes Y_2) \simeq F \otimes (X \boxtimes Y_1 \boxtimes DY_2) \) giving balancing in position 2 over \( D \).

4. **Balancing of \( \otimes (F \boxtimes F) \).** In this part of the proof we show that the composition \( \otimes (F \boxtimes F) : (C_1 \boxtimes C_2)^{\otimes 2} \to A \) of the monoidal structure in \( A \) with \( F \boxtimes F \) is \( D \) balanced in positions 1 and 3. Begin by defining natural isomorphism \((id \otimes t_{D, X_2}^{-1})(t_{X_1, D} \otimes id) : F_1(X_1 D)F_2(X_2)F(Y) \to F_1(X_1)F_2(DX_2)F(Y) \). Using the diagrams required of \( t, t' \)
it is easy to show this satisfies the diagram required of a balancing for $\otimes(F \otimes F)$ in position 1. Doing so for position 3 is just as easy.

5. Balancing of $f : F \otimes \rightarrow \otimes(F \otimes F)$. Recall the definition of the tensor structure $f$ for $F$ from Part 1 of this proof. We show that satisfies the diagram required of a balanced natural transformation in both positions 1 and 3. In position 1 this means showing that $(b_{D,X} \otimes id)f_{X,D,Y} = f_{D,Y}b_{X,D,Y}$ where $b^T$ is balancing of $F \otimes$ in position 1 as in Part 3 of this proof and $b$ is balancing of $F$. To show this consider the diagram

Every subdiagram is either naturality or tensor diagrams for $t, t'$ or relative braiding of $F_1, F_2$. Since the composition of morphisms across the top is $t_{X,D,X} \otimes t'_{Y,Y}$ the perimeter is the balancing diagram for $f$ in position 1. Showing that $f$ is balanced in position 3 requires a similar diagram and is just as easy.

Parts 1-5 above imply that there are unique functors and natural isomorphism $\overline{f} : \overline{F} \otimes \rightarrow \otimes(F \otimes F) : (C_1 \otimes_D C_2)^{\otimes 2} \rightarrow A$ satisfying $\overline{f} * B_{1,2}^{\otimes 2} = f$. Using basic properties of the functor $B_{1,2}$ and the definition of $A$ from Proposition 5.1.1 one
shows that $F \otimes = F \Lambda$ and $\otimes(\otimes F \otimes F) = \otimes(\otimes F \otimes F)$ where $F : C_1 \boxtimes_D C_2 \to A$ is the unique functor with $FB_{1,2} = F$. Thus $\bar{f}$ provides $F$ with the structure of a tensor functor in a canonical way. The proposition is proved. \hfill \Box

Proposition 5.1.4, though perhaps interesting in its own right, is of immediate value in that it implies closure over relative product $\boxtimes_D$ of the class of functors over $\mathcal{D}$. This we prove in Proposition 5.1.10.

5.1.3 The fusion category $C_1 \boxtimes_D C_2$

In this section we show that the relative product of two fusion categories over braided $\mathcal{D}$ inherits the structure of a fusion category over $\mathcal{D}$. Notation is retained from the previous section.

**Theorem 5.1.5.** Let $C_i$, $i = 1, 2$ be fusion categories over $\mathcal{D}$. Then $C_1 \boxtimes_D C_2$ is also a fusion category over $\mathcal{D}$.

We break up the proof of Theorem 5.1.5 into two parts: first we will show that $C_1 \boxtimes_D C_2$ is fusion and then show in Proposition 5.1.8 that it is fusion over $\mathcal{D}$ in the sense of Definition 5.0.7.

**Proposition 5.1.6.** Under the conditions of Theorem 5.1.5 $C_1 \boxtimes_D C_2$ is fusion.

**Proof.** Thanks to Proposition 5.1.1 it remains only to check that $C_1 \boxtimes_D C_2$ is rigid and semisimple. We begin with a general result. Recall that a dominant functor $F$ is one for which the codomain category and the category $\text{Im}(F)$ coincide (Definition 1.3.16).
Lemma 5.1.7. Let $C, D$ be semisimple tensor categories with $C$ fusion, and let $F : C \to D$ be a strict dominant tensor functor. Then $D$ is fusion.

Proof. Let $\hat{F}$ denote the tensor subcategory of $D$ generated by objects in the image of $F$. Since $F$ is a tensor functor $\hat{F}$ is itself fusion with rigidity inherited from that in $C$: duality is given by $F(X)^* = F(X^*)$ and $\text{ev}_{F(X)} := F(\text{ev}_X) : F(X)^* \otimes F(X) \to F(1) = 1$. Similarly $\text{coev}_{F(X)} := F(\text{coev}_X)$.

It is our task to define duality for a general object in $D$. To this end fix $Y \in D$. Let $X \in C$ be an object such that $F(X)$ contains $Y$ as a subobject. Write $F(X) = Y \oplus Z$ for some object $Z \in D$. Now $F(X^*) \otimes Y$ is a subobject of $F(X^*) \otimes F(X)$. Define the object $Z^*$ to be the largest subobject of $F(X^*)$ having the property that

$$F(\text{ev}_X)|_{Z^* \otimes Y} = 0.$$ 

Define $Y^*$ to be the complement of $Z^*$ in $F(X^*)$, i.e. $F(X^*) = Y^* \oplus Z^*$. Thus the object $Y^* \otimes Y$ is a subobject of $F(X^*) \otimes F(X)$, and we may therefore restrict $F(\text{ev}_X)$ to define morphism $e_Y := F(\text{ev}_X)|_{Y^* \otimes Y}$. To be explicit, let $\rho_{Y^*} : Y^* \hookrightarrow F(X^*)$ and $\rho_Y : Y \hookrightarrow F(X)$ be inclusions of the indicated subobjects. Then we have defined $e_Y := F(\text{ev}_X) \circ (\rho_{Y^*} \otimes \rho_Y)$.

Next let $\pi_Y : F(X) \to Y$ and $\pi_{Y^*} : F(X) \to Y^*$ be projections. Then define $\text{coev}_Y := (\pi_Y \otimes \pi_{Y^*}) \circ F(\text{coev}_X) : Y \otimes Y^* \to 1$. Neither $e_Y$ nor $\text{coev}_Y$ is identically zero because of the choice of subobject $Y^*$. We claim that $e_Y, \text{coev}_Y$ together with the identifications made above make $Y^*$ a bona fide left dual for $Y \in D$. It remains to
check the usual identities. On the definitions of \(co_Y\) and \(e_Y\) we have

\[(1_Y \otimes co_Y)(e_Y \otimes 1_Y) = (1_Y \otimes F(ev_X))(\pi_Y \otimes \rho_Y \cdot \pi_Y \otimes \rho_Y)(F(coev_X) \otimes 1_Y).
\]

Using the basic identity \(\rho_Y \cdot \pi_Y = id_F(x) - \rho_Z \cdot \pi_Z\) this becomes a difference of two maps with only the "positive" one non-zero because of the definition of \(Z^\ast\). The remaining non-zero part fits into the following diagram as the lower horizontal composition.

\[
\begin{array}{cccccc}
F(X) & \xrightarrow{F(coev_X) \otimes id} & F(X)F(X^\ast)F(X) & \xrightarrow{id \otimes F(ev_X)} & F(X) \\
\rho_Y & & \pi_Y \otimes id & & \pi_Y \\
Y & \xrightarrow{F(coev_X) \otimes id} & YF(X)Y & \xrightarrow{id \otimes F(ev_X)} & Y
\end{array}
\]

All subdiagrams commute trivially. The horizontal composition across the top of the diagram is \(id\) (this is the equation required of rigidity of \(X\) in \(C\)). Tracing around the perimeter gives \((id_Y \otimes e_Y)(co_Y \otimes id_Y) = \pi_Y \rho_Y\) which is \(id_Y\) (the other basic identity relating \(\rho\) and \(\pi\)). The second equation \(id_Y^\ast = (e_Y \otimes id_Y^\ast)(id_Y^\ast \otimes co_Y)\) follows similarly.

Now we complete the proof of Proposition 5.1.6. By Lemma 2.1.8 the universal balanced functor \(B_{1,2} : C_1 \boxtimes C_2 \to C_1 \boxtimes_D C_2\) is dominant, hence \(C_1 \boxtimes_D C_2\) is rigid. Also since \(C_i\) are both semisimple the category \(C_1 \boxtimes_D C_2\) is semisimple because it is equivalent to the category of functors \(Fun_D(C_1^{op}, C_2)\) (Theorem 2.3.1) which is semisimple by [ENO05, Theorem 2.16].

**Proposition 5.1.8.** Under the conditions of Theorem 5.1.5 \(C_1 \boxtimes_D C_2\) is fusion over
Proof. Let $\varphi_i : \mathcal{D} \rightarrow Z(C_i)$ be the braided inclusions putting fusion categories $C_i$ over $\mathcal{D}$. Note that $\mathcal{D}$ sits inside $Z(C_1 \boxtimes_{\mathcal{D}} C_2)$ by the composition

$$\varphi := \mathcal{D} \hookrightarrow \mathcal{D} \boxtimes \mathcal{D} \xrightarrow{\varphi_1 \boxtimes \varphi_2} Z(C_1) \boxtimes Z(C_2) = Z(C_1 \boxtimes C_2) \xrightarrow{ZB_{1,2}} Z(C_1 \boxtimes_{\mathcal{D}} C_2)$$

(14)

sending $D \mapsto (B_{1,2}(D \boxtimes 1), \tilde{\gamma}_{B_{1,2}(D \boxtimes 1)})$. Here $\tilde{\gamma}$ is the braiding in $Z(C_1 \boxtimes_{\mathcal{D}} C_2)$ as in the last part of the proof of Proposition 5.2.3 and $ZB_{1,2}$ is the functor sending $(A, c_A) \mapsto (B_{1,2}(A), \tilde{\gamma}_{B_{1,2}(A)})$ for any object $(A, c_A)$ in the center of $C_1 \boxtimes C_2$. To complete the proof of Proposition 5.1.8 we must show that the composition 14 is an inclusion and that it is braided.

1. $\varphi$ is an inclusion. Since $\varphi_1 \boxtimes \varphi_2$ is an inclusion on account of $\varphi_i$ being so we must check only that $ZB_{1,2}$ is an inclusion on the tensor subcategory generated by objects of the form $(D \boxtimes 1, c_{D, -} \boxtimes 1)$. But this is obvious.

2. Braiding of $\varphi$. Since both $\varphi_1, \varphi_2$ are braided the functor $\varphi_1 \boxtimes \varphi_2$ is also braided (this is perfectly general and has nothing whatever to do with the other properties of $\varphi_i$). Note also that $ZB_{1,2}$ is braided; this follows from the fact that $B_{1,2}$ is a braided functor (this is shown in Proposition 5.2.3). \qed

Corollary 5.1.9. For $C_1, C_2; \mathcal{D}$ as in the hypothesis of Theorem 5.1.5 the category of $\mathcal{D}$-module functors $\text{Fun}_\mathcal{D}(C_1, C_2)$ has the structure of a fusion category over $\mathcal{D}$. 

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Proof. This follows immediately from the comments at the end of the proof of Proposition 5.1.6. □

Now that we have a multiplication in the category of tensor (fusion) categories over a fixed braided fusion category it is natural to ask if this extends to morphisms of such categories. This is the content of the following proposition. Recall what it means for a functor to be a functor over a braided fusion category (Definition 5.1.2).

Proposition 5.1.10. Let \((F, f) : C_1 \to E_1\) and \((G, g) : C_2 \to E_2\) be tensor functors over \(\mathcal{D}\) for \(C_i, E_i\) tensor categories over braided fusion \(\mathcal{D}\). Then \(F \boxtimes \mathcal{D} G\) has canonical structure of a tensor functor over \(\mathcal{D}\).

Proof. Define functors \(F_1 : C_1 \to E_1 \boxtimes \mathcal{D} E_2\) and \(G_2 : C_2 \to E_1 \boxtimes \mathcal{D} E_2\) by

\[
F_1(X) := B_{1,2}(F(X) \boxtimes 1), \quad G_2(Y) := B_{1,2}(1 \boxtimes G(Y)).
\]

Using the braided inclusions from Proposition 5.1.8 it is easy to check that \(F_1, G_2\) are functors over \(\mathcal{D}\). Observe that \(F_1(X) \otimes G_2(Y) = G_2(Y) \otimes F_1(X)\) so we have a trivial relative braiding between \(F_1, G_2\). Applying Proposition 5.1.4 to \(\otimes(F_1 \boxtimes G_2) = B_{1,2}(F \boxtimes G)\) we get a unique tensor functor \(C_1 \boxtimes \mathcal{D} C_2 \rightarrow E_1 \boxtimes \mathcal{D} E_2\). This is exactly \(F \boxtimes \mathcal{D} G\). □

5.2 Tensor product of braided fusion categories

In this section we discuss when the relative tensor product of a pair of braided fusion categories is itself braided. It turns out that in order for such a braiding to exist we
will need a restricted version of the phenomenon described in Definition 5.0.7.

**Definition 5.2.1.** If \( \mathcal{C}, \mathcal{D} \) are braided then we say \( \mathcal{C} \) is *braided over* \( \mathcal{D} \) if there is a braided inclusion \( \mathcal{D} \hookrightarrow \mathcal{C}' \) where \( \mathcal{C}' \) is the centralizer of \( \mathcal{C} \) (Definition 1.2.12).

**Example 5.2.2.** Let \( G_i \) be finite abelian groups and \( q_i : G_i \to \mathbb{F}^\times \) quadratic forms satisfying \( q_i(g) = \beta_i(g, g) \) for some bicharacters \( \beta_i \) on \( G_i \), \( i = 1, 2 \). One easily checks that \( (G_1 \times G_2, p) \) is a pre-metric group for \( p(g, h) = q_1(g)q_2(h) \). As a quadratic form \( p \) comes from the bicharacter on \( G_1 \times G_2 \) given by \( (g_1, h_1, g_2, h_2) \mapsto \beta_1(g_1, g_2)\beta_2(h_1, h_2) \).

Now suppose we have embeddings \( G \hookrightarrow G_i \) for a finite group \( G \) such that \( q_i(g) = q_2(g) \) for all \( g \in G \). Then the pair \( (G, q) \) is a metric group for \( q := q_i|_G \). Denote by \( \tilde{G} \) the subgroup of \( G_1 \times G_2 \) given by the set \( \{(x, x^{-1})|x \in G\} \) and suppose that \( p \) descends to a quadratic form on \( (G_1 \times G_2)/\tilde{G} \). Since \( p \) is constant on \( \tilde{G} \)-cosets we have \( q_i(x)q(g) = q_i(gx)q(1) \) for \( x \in G_i, g \in G \). As a result \( b_i(g, x) = q(1)^{-1} \) and since \( b_i(1, 1) = 1 \) we may conclude that

\[
b_i(g, x) = 1 \quad (15)
\]

for \( i = 1, 2 \).

Let us translate this into the language of pointed braided fusion categories à la §1.2.2. Pre-metric inclusions \( G \hookrightarrow G_i \) correspond to braided inclusions \( \mathcal{C}(G, q) \to \mathcal{C}(G_i, q_i) \). Equation 15 becomes \( c_{X,Y}^i = 1 \) (\( c^i \) the braiding in \( \mathcal{C}(G_i, q_i) \)) whenever \( X, Y \) are homogeneous objects of \( Vec_{G_i} \) of degrees \( g \in G, x \) respectively. Thus the images of the braided inclusions are contained in M"uger centers \( \mathcal{C}(G_i, q_i)' \). As braided fusion
categories it is evident that \( \mathcal{C}(G_1 \times G_2/\tilde{G}, p) \simeq \mathcal{C}(G_1, q_1) \boxtimes_{\mathcal{C}(G,q)} \mathcal{C}(G_2, q_2) \).

The next proposition describes braiding for the tensor category \( \mathcal{C}_1 \boxtimes_D \mathcal{C}_2 \) whenever \( \mathcal{C}_i \) are braided over \( D \) in the sense of Definition 5.2.1. Note that the Deligne product of any pair of braided categories has braiding given by Deligne product of the individual braided structures. In what remains of this section we extend this observation to the relative product over a braided fusion category.

**Proposition 5.2.3.** Let \( \mathcal{C}_i, i = 1, 2 \) be braided fusion categories braided over \( D \). Then \( \mathcal{C}_1 \boxtimes_D \mathcal{C}_2 \) has canonical braiding such that \( B_{1,2} \) is a braided functor.

*Proof.* By Propositions 5.2 and 5.3 in [JS93] braided structures on \( \mathcal{C}_1 \boxtimes_D \mathcal{C}_2 \) are in correspondence with tensor structures on the monoidal product \( \otimes : (\mathcal{C}_1 \boxtimes_D \mathcal{C}_2) \otimes \to \to \mathcal{C}_1 \boxtimes_D \mathcal{C}_2 \). Thus to prove the proposition we consider such tensor structures.

Let \( c^i \) be braiding on \( \mathcal{C}_i \) and let \( \Lambda : (\mathcal{C}_1 \boxtimes \mathcal{C}_2) \otimes \to \to \mathcal{C}_1 \boxtimes_D \mathcal{C}_2 \) be as in the proof of Proposition 5.1.1. The category \( (\mathcal{C}_1 \boxtimes \mathcal{C}_2) \otimes \to \to \mathcal{C}_1 \boxtimes_D \mathcal{C}_2 \) has monoidal structure coming from the one in \( \mathcal{C}_1 \otimes \mathcal{C}_2 \) in the obvious way. We will adopt the convention of abbreviating objects of the form \( X_1 \boxtimes X_2 \in \mathcal{C}_1 \boxtimes \mathcal{C}_2 \) by \( \overline{X} \) and on occasion write \( \overline{X}_i = X_i \) for the "coordinates" of \( \overline{X} \). Thus \( (\overline{X} \otimes \overline{Y})_1 = X_1 \otimes Y_1 \). Any tensor structure \( \lambda_{\overline{X} \otimes \overline{Y}, \overline{U} \otimes \overline{V}} : \Lambda((\overline{X} \otimes \overline{Y}) \otimes (\overline{U} \otimes \overline{V})) \simeq \Lambda(\overline{X} \otimes \overline{Y}) \otimes \Lambda(\overline{U} \otimes \overline{V}) \) is of the form

\[
\lambda_{\overline{X} \otimes \overline{Y}, \overline{U} \otimes \overline{V}} : B_{1,2}(X_1 Y_1 V_1 \boxtimes X_2 Y_2 V_2) \sim B_{1,2}(X_1 Y_1 U_1 V_1 \boxtimes X_2 Y_2 U_2 V_2)
\]

where we have used the definition of \( \Lambda \) to write (co)domain of \( \lambda \) in terms of \( B_{1,2} \).
Given braidings in $C_i$ the most natural possibilities are

$$\lambda_{X \otimes Y, U \otimes V} = B_{1,2}(1_{X_1} \otimes C^i(U_1, Y_1) \otimes 1_{Y_1} \otimes 1_{X_2} \otimes C^j(U_2, Y_2) \otimes 1_{Y_2})$$

where $C^i(A, B) \in \{c^{i}, c^{-1}_i\}$. Leaving out tensoring with identity the diagram needed in order for $\Lambda$ to have monoidal structure $\lambda$ is

$$\Lambda(\bar{U} W Y \otimes \bar{V} X Z) \xrightarrow{C^1(WY, V) \otimes C^2(W_2, V_2)} \Lambda(\bar{U} \otimes \bar{V}) \Lambda(W Y \otimes X Z)$$

$$\Lambda(\bar{U} W \otimes \bar{V} X) \Lambda(Y \otimes Z) \xrightarrow{C^1(W_1, V_1) \otimes C^2(W_2, V_2)} \Lambda(\bar{U} \otimes \bar{V}) \Lambda(W \otimes X) \Lambda(Y \otimes Z)$$

This is really two diagrams: one for $C^1$ and another for $C^2$. The diagram corresponding to $C^1$ comes down to showing commutativity of the diagram

$$WYVX \xrightarrow{C^1(WY, V) \otimes \text{id}_X} VWYX$$

$$\xrightarrow{id_W \otimes C^1(Y, VX)} \xrightarrow{\text{id}_V \otimes C^1(Y, X)} VWXY$$

where subscripts have been left off for simplicity. This diagram commutes if we choose $C^1 = c^1$ to be the braiding in $C_1$. Similar considerations lead to choosing $C^2 = c^2$ to be the braiding in $C_2$.

**Lemma 5.2.4.** The natural isomorphism $\lambda$ descends to a canonical tensor structure on $\overline{\Lambda} : (C_1 \boxtimes_D C_2)^{\otimes 2} \to C_1 \boxtimes_D C_2$.

*Proof.* To prove the lemma we must only show that $\lambda : \Lambda \otimes \to \otimes(\Lambda \otimes \Lambda) : (C_1 \boxtimes C_2)^{\otimes 4} \to C_1 \boxtimes_C C_2$ is $D$-balanced in positions 1, 3, 5, 7. The 1-balancing of $\lambda$ is
equivalent to commutativity of the perimeter

\[
B_{1,2}(X_1DU_1V_1 \otimes X_2U_2V_2) \xrightarrow{c_{1,1}^1 \otimes c_{2,2}^2} B_{1,2}(X_1Dy_1U_1V_1 \otimes X_2Y_2U_2V_2)
\]

which commutes trivially. Upper and lower horizontals are \(\lambda_{\overline{X}D} \otimes \overline{U} \otimes \overline{V}\) and \(\lambda_{D \overline{X} \otimes \overline{V} \otimes \overline{V}}\) where \(\overline{XD} := X_1D \otimes X_2\) and verticals are \(\Lambda\) 1-balancing. Balancing in the 7th position is similar. The 3-balancing of \(\lambda\) comes down to the diagrams

\[
\begin{array}{c}
XUYDV & \xrightarrow{c_{U,Y}} & XYDUV \\
\downarrow c_{D,Y} & & \downarrow c_{D,U} \\
XUYVD & \xrightarrow{c_{U,Y}} & XUYVD
\end{array}
\quad
\begin{array}{c}
DXUYV & \xrightarrow{c_{U,Y}} & DXYUV \\
\downarrow c_{D,X} & & \downarrow c_{D,D} \\
XUDYV & \xrightarrow{c_{U,Y}} & XDYUV
\end{array}
\]

where indices and tensor with identity morphisms have been elided. In the diagram on the left \(c = c^1\) and on the right \(c = c^2\). Each subdiagram is either naturality or pentagon for the braiding. The double edges follow from \(D\) injecting into the Müger centers. The 5-balancing requires similar diagrams.

Our discussion implies that \(\lambda\) descends to \(\overline{\lambda} : \overline{\Lambda} \otimes \to \otimes(\Lambda \otimes \Lambda)\), a natural isomor-
phism, as indicated in the diagram.

Using basic properties of $B_{1,2}$ this becomes $\overline{\lambda} : \overline{\Lambda} \otimes \rightarrow \overline{\Lambda}(\overline{\Lambda} \otimes \overline{\Lambda})$ and hence a tensor structure for $\overline{\Lambda}$, proving the lemma. \qed

In the language of [JS93] the functor $\overline{\Lambda}$ gives $C_1 \boxtimes_D C_2$ a multiplication $\Phi : C_1 \boxtimes_D C_2 \times C_1 \boxtimes_D C_2 \rightarrow C_1 \boxtimes_D C_2$ defined by $\Phi(A, B) = \overline{\Lambda}(A \otimes B)$. Part of the data describing a multiplication in $C_1 \boxtimes D C_2$ involves isomorphisms $\Phi(A, 1) \simeq A$, $\Phi(1, B) \simeq B$ which we can assume are identity (on assuming strictness of tensor structure $\overline{\Lambda}$ in $C_1 \boxtimes_D C_2$). Natural isomorphisms $\overline{\lambda}$ give an isomorphism $\Phi(A, A') \otimes \Phi(B, B') \simeq \Phi(A \otimes B, A' \otimes B')$.

Proposition 5.3 of loc. cit. implies that $C_1 \boxtimes_D C_2$ acquires a braided structure $c$ making the diagram commute:

$$\xymatrix@C=20pt{\Lambda(A \boxtimes 1) \otimes \Lambda(1 \boxtimes B) = A \otimes B \ar[r]^{\gamma_{A,B}} & B \otimes A = \Lambda(1 \boxtimes B) \otimes \Lambda(A \boxtimes 1) \\
\Lambda((A \boxtimes 1) \otimes (1 \boxtimes B)) = \Lambda(A \boxtimes B) = \Lambda((1 \boxtimes B) \otimes (A \boxtimes 1))}$$

Denote by $\gamma_{U \boxtimes V}$ the natural isomorphism $B_{1,2}(c_{U_1,Y_1} \boxtimes c_{U_2,Y_2}) : \Lambda \rightarrow \Lambda^{op}$ for any pair
of objects \( \overline{U}, \overline{V} \in C_1 \boxtimes C_2 \). Note that

\[
\lambda_{\overline{X}, \overline{Y}, \overline{Z}} = B_{1,2}(id \otimes c_{1,1} \otimes id \otimes id \otimes c_{1,1} \otimes id) = id_{\Lambda(\overline{X} \boxtimes \overline{Y})}
\]

and

\[
\lambda_{\overline{X}, \overline{Y}, \overline{Z}} = B_{1,2}(c_{X_1,Y_1} \otimes c_{X_2,Y_2}) = \gamma_{\overline{X} \boxtimes \overline{Y}}.
\]

Balancing of \( \lambda \) in positions 3 and 5 implies balancing of \( \gamma \) in positions 1, 3, hence a unique natural isomorphism \( \overline{\gamma} : \overline{\Lambda} \to \overline{\Lambda}^{\text{op}} \) satisfying \( \overline{\gamma} \ast B_{1,2}^{\text{op}} = \gamma \). Uniqueness gives \( \overline{\lambda}_{\overline{X},-,-} = \overline{\gamma} \) and \( \overline{\lambda}_{-,\overline{Y},-} = id \). Thus braiding on the relative tensor product is equal to \( \overline{\gamma} \): for \( A, B \in C_1 \boxtimes_D C_2 \) we have \( c_{A,B} = \overline{\gamma}_{A,B} \). \( \square \)

5.3 Module categories over \( C_1 \boxtimes_D C_2 \)

In this section we are interested in studying module categories over fusion (tensor) categories of the form \( C_1 \boxtimes_D C_2 \) where we retain above notation. We begin with a general lemma relating balancing and module category structure.

**Lemma 5.3.1.** Let \( \mathcal{M} \) be a strict \( \mathcal{C}, \mathcal{D} \)-module category and let \( \mathcal{N} \) be a left \( \mathcal{C} \)-module category. Then any \( \mathcal{D} \)-balanced left \( \mathcal{C} \)-module functor \( \mathcal{M} \boxtimes \mathcal{N} \to A \) descends to a functor \( \mathcal{M} \boxtimes_D \mathcal{N} \to A \) having canonical \( \mathcal{C} \)-module structure.

**Proof.** Denote by \( f \) the balancing isomorphisms for \( F \), and for \( X \in \mathcal{C} \) write \( \varphi_{X,\mathcal{M}} : F(X \otimes M) \to X \otimes F(M) \) for \( \mathcal{C} \)-module structure of \( F \). If \( L_X \) is the functor associated to left \( X \)-multiplication then we can view \( \varphi \) as a natural isomorphism \( \varphi_X : F(L_X \boxtimes 1) \to L_X F \) (here \( 1 = id_{\mathcal{N}} \)). Recall that left \( X \)-multiplication in \( \mathcal{M} \boxtimes_D \mathcal{N} \) is given by \( \overline{L_X} \), the
unique endofunctor on \( \mathcal{M} \boxtimes \mathcal{D} \mathcal{N} \) determined by \( B_{\mathcal{M}, \mathcal{N}} \circ (L_X \boxtimes 1) : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes \mathcal{D} \mathcal{N} \).

It is trivial to check that both \( F(L_X \boxtimes 1) \), \( L_X F \) are balanced functors \( \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A} \) with balancing coming from \( f \). Also one checks that \( \varphi_X \) is balanced natural transformation. Thus we have unique \( \varphi_X : F(L_X \boxtimes 1) \to L_X F : \mathcal{M} \boxtimes \mathcal{D} \mathcal{N} \to \mathcal{A} \). Using basic properties of \( B_{\mathcal{M}, \mathcal{N}} \) we see that \( F(L_X \boxtimes 1) = F(L_X) \) and \( L_X F = L_X \bar{F} \). Thus components of \( \varphi_X \) are given by \( \varphi_{X A} : F(X \otimes A) \to X \otimes \bar{F}(A) \) for a typical object \( A \in \mathcal{M} \boxtimes \mathcal{D} \mathcal{N} \).

Extending this construction to all objects in \( \mathcal{C} \) we get \( \mathcal{C} \)-linearity for \( \bar{F} \). \( \square \)

Let us now return to pre Lemma 5.3.1 notation. In what follows assume all module categories to be strict, an assumption we can justify thanks to Theorem 1.3.8. The next proposition relates module structure over the tensor product to module structure over braided fusion category \( \mathcal{D} \).

**Proposition 5.3.2.** Any module category over \( \mathcal{C}_1 \boxtimes \mathcal{D} \mathcal{C}_2 \) admits a canonical \( \mathcal{D} \)-bimodule category structure with respect to which the left and right module structures agree.

**Proof.** Suppose braided inclusions \( \varphi_i : \mathcal{D} \hookrightarrow Z(C_i) \) put \( C_i \) over braided \( \mathcal{C} \) as above. Let \( \mathcal{M} \) be a left \( \mathcal{C}_1 \boxtimes \mathcal{D} \mathcal{C}_2 \)-module category. Then define left and right \( \mathcal{D} \)-module category structures on \( \mathcal{M} \) in the following way. For \( M \in \mathcal{M} \) set

\[
D \otimes M := B_{1,2}(\varphi_1(D) \boxtimes 1) \otimes M, \quad M \otimes D := B_{1,2}(1 \boxtimes \varphi_2(D)) \otimes M.
\]

Note that left module associativity of the action comes from tensor structure of \( \varphi_1 \) and module associativity on the right comes from tensor structure of \( \varphi_2 \). Note also
that since

\[ B_{1,2}(\varphi_1(D) \boxtimes 1) \otimes B_{1,2}(1 \boxtimes \varphi_2(D)) = B_{1,2}(\varphi_1(D) \boxtimes \varphi_2(E)) = B_{1,2}(1 \boxtimes \varphi_2(E)) \otimes B_{1,2}(\varphi_1(D) \boxtimes 1) \]

left \( C_1 \) and right \( C_2 \) module structures are strictly consistent: \( (D \otimes M) \otimes E = D \otimes (M \otimes E) \). It is evident that \( b_{1,D,1} \otimes id_M : D \otimes M \rightarrow M \otimes D \).

**Theorem 5.3.3.** Let \( C_i \) be tensor categories over braided fusion category \( D \). Then \( \boxtimes_D \) is a functor \( C_1 \text{-Mod} \boxtimes_{C_2} \text{-Mod} \rightarrow C_1 \boxtimes_D C_2 \text{-Mod} \). Furthermore \( \boxtimes_D \) is bilinear with respect to composition of functors.

**Proof.** Let \( M \in C_1 \text{-Mod}, N \in C_2 \text{-Mod} \) and for convenience assume that the braided inclusions \( \varphi_i : D \rightarrow Z(C_i) \) are both strict as tensor functors. Centrality of \( D \) in \( C_i \) allows us to define \( D \)-bimodule structure on both \( M \) and \( N \) by stipulating that left, right actions agree. We break up the proof of Theorem 5.3.3 into two parts. First we show that \( \boxtimes_D \) has the proper codomain category and then show that it respects the relevant structures.

**Proposition 5.3.4.** \( M \boxtimes_D N \) has canonical structure of a \( C_1 \boxtimes_D C_2 \)-module category.
Proof. Define $C_1 \boxtimes_D C_2$-module structure on $\mathcal{M} \boxtimes_D \mathcal{N}$ using the diagram

$$
\begin{array}{ccc}
C_1 \boxtimes C_2 \boxtimes \mathcal{M} \boxtimes \mathcal{N} & \xrightarrow{\tau(23)} & C_1 \boxtimes \mathcal{M} \boxtimes C_2 \boxtimes \mathcal{N} \\
B_{1,2} \otimes B_{\mathcal{M},\mathcal{N}} & & \mathcal{M} \boxtimes \mathcal{N} \\
\downarrow & & \downarrow B_{\mathcal{M},\mathcal{N}} \\
C_1 \boxtimes_D C_2 \boxtimes \mathcal{M} \boxtimes_D \mathcal{N} & \rightarrow & \mathcal{M} \boxtimes_D \mathcal{N}
\end{array}
$$

For convenience abbreviate $T := B_{\mathcal{M},\mathcal{N}} \circ \otimes^2 \circ \tau(23)$. We wish to descend $T$ to the functor indicated in the diagram by the unadorned horizontal arrow.

We first check that the composition $T$ is $\mathcal{D}$-balanced in positions 1 and 3 (Definition 2.1.3). Let $X,Y,D,M,N$ be objects in $C_1,C_2,D,\mathcal{M},\mathcal{N}$. Then

$$
B_{\mathcal{M},\mathcal{N}}(XDM \otimes YN) \xrightarrow{b} B_{\mathcal{M},\mathcal{N}}(XM \otimes DY N)
$$

gives balancing $b_{X,M,D,Y,N} : T(XD \otimes Y \otimes M \otimes N) \rightarrow T(X \otimes DY \otimes M \otimes N)$ in position 1 for $T$. Balancing in position 3 can be written in terms of both balancing for $B_{\mathcal{M},\mathcal{N}}$ and the central structure in $C_2$. Explicitly

$$
B_{\mathcal{M},\mathcal{N}}(XMD \otimes YN) \xrightarrow{b} B_{\mathcal{M},\mathcal{N}}(XM \otimes DY N) \xrightarrow{c^2} B_{\mathcal{M},\mathcal{N}}(XM \otimes YDN)
$$

where as usual $c^2$ is the braiding in $Z(C_i)$. It is evident that these candidate balancings in positions 1, 3 satisfy the balancing diagrams for those positions, hence $T$ is so balanced. We therefore get a unique right exact functor (the unlabeled horizontal arrow in the diagram) which we will also call $\otimes$.  

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Next we check module associativity. It is easy to check that the compositions

$$(C_1 \boxtimes C_2) \boxtimes (M \boxtimes N) \xrightarrow{(\Gamma, \text{id})} (C_1 \boxtimes C_2) \boxtimes (M \boxtimes N) \xrightarrow{T} M \boxtimes_D N$$

and

$$(C_1 \boxtimes C_2) \boxtimes (M \boxtimes N) \xrightarrow{(B_{1,2}, T)} (C_1 \boxtimes_D C_2) \boxtimes (M \boxtimes_D N) \xrightarrow{T} M \boxtimes_D N$$

are equal (here we use $\Gamma$ for monoidal structure in $C_1 \boxtimes C_2$ as in Proposition 5.1.1). Also it's easy (but tedious) to show that they are each balanced in positions 1, 3, 5; all balancings may be written in terms of $c'$ and balancings for $B_{1,2}$ and $B_{M, N}$. Thus they descend to functors $\overline{T}(\Gamma \boxtimes \text{id})$ and $\overline{T}(B_{1,2} \boxtimes T) : (C_1 \boxtimes_D C_2) \boxtimes (M \boxtimes_D N) \to M \boxtimes_D N$ which are equal. Using basic properties of universal balanced functors one therefore has

$$\overline{T}(\Lambda \boxtimes 1) = \overline{T}(1 \boxtimes \overline{T})$$

where again we use $\Lambda$ to denote monoidal structure in $C_1 \boxtimes_D C_2$ as in Proposition 5.1.1. This is precisely the statement that $M \boxtimes_D N$ has (strict) $C_1 \boxtimes_D C_2$-module category structure. Unit object of the action is clearly $1 = B_{1,2}(1 \boxtimes 1)$ and the required unit constraints obtain. \( \square \)

The next result is the module counterpart to the corresponding result for tensor functors proved above (Proposition 5.1.10) showing that the class of module functors over $D$ is closed under the relative product $\boxtimes_D$.

**Proposition 5.3.5.** Let $F_i : M_i \to N_i$, $i = 1, 2$, be a pair of functors where for each
$i$ is $C_i$-module. Then $F_1 \boxtimes_D F_2 : M_1 \boxtimes_D M_2 \to \mathcal{N}_1 \boxtimes_D \mathcal{N}_2$ has canonical structure of a $C_1 \boxtimes_D C_2$-module functor.

Proof. Denote by $B_M$ the universal $D$-balanced functor $B_{M_1, M_2} : M_1 \boxtimes D M_2 \to M_1 \boxtimes D M_2$. Similarly define $B_N$. Let $F_1$ have $C_1$-module structure $t$ and $F_2$ have $C_2$-module structure $t'$. Define $F := B_N(F_1 \boxtimes F_2) : M_1 \boxtimes M_2 \to \mathcal{N}_1 \boxtimes D \mathcal{N}_2$. One easily checks that $F$ is balanced by $t'^{-1}_D, b_{F_1(M), D, F_2(N)} t_{M, D} : F(MD \boxtimes N) \to F(M \boxtimes DN)$ for $M, N \in M_1, M_2$ and $D \in D$ and where $b$ is balancing for $B_N$. We therefore get functor $F_1 \boxtimes_D F_2 : M_1 \boxtimes D M_2 \to \mathcal{N}_1 \boxtimes D \mathcal{N}_2$ with the usual uniqueness property.

We now show that it respects $C_1 \boxtimes_D C_2$-module structure. Then note that we have a natural isomorphism

$$B_N \ast (t \boxtimes t') \ast \tau : B_N(F_1 \boxtimes F_2) \tau \to B_N(\boxtimes(F_1 \boxtimes F_1), \boxtimes(F_2 \boxtimes F_2)) \tau.$$  (16)

Using the definition of $C_1 \boxtimes_D D_2$-module structure $\boxtimes_M, \boxtimes_N$ for $M_1 \boxtimes_D M_2$ and $\mathcal{N}_1 \boxtimes_D \mathcal{N}_2$ as described in the proof of Proposition 5.3.4 and the definition of the tensor product $F_1 \boxtimes_D F_2$ we see that this is a natural isomorphism $(F_1 \boxtimes_D F_2) \boxtimes_M (B_{1, 2} \boxtimes B_M) \to \boxtimes_N(1 \boxtimes (F_1 \boxtimes_D F_2))B_{1, 2} \boxtimes B_M$. All structures are easily seen to be balanced therefore equation 16 descends to a unique natural isomorphism $t \boxtimes t' : (F_1 \boxtimes_D F_2) \boxtimes_M \to \boxtimes_N(1 \boxtimes (F_1 \boxtimes_D F_2))$. This is precisely to say that $F_1 \boxtimes_D F_2$ has the structure of a module functor.

With Proposition 5.3.5 we have shown that $\boxtimes_D$ as described in the statement of Theorem 5.3.3 is a functor. This completes the proof of the theorem. \qed
The next result examines functoriality of relative tensor product.

**Proposition 5.3.6.** Let \( C_i, i \in \{1, 2\} \) be categories fusion over braided fusion \( D \), and let \( \mathcal{M}_i \) be right \( C_i \)-module categories, \( \mathcal{N}_i \) left \( C_i \)-module categories. Then there is a canonical equivalence \((\mathcal{M}_1 \boxtimes_D \mathcal{M}_2) \boxtimes_{C_1 \boxtimes_D C_2} (\mathcal{N}_1 \boxtimes_D \mathcal{N}_2) \cong (\mathcal{M}_1 \boxtimes_{C_1} \mathcal{N}_1) \boxtimes_D (\mathcal{M}_2 \boxtimes_{C_2} \mathcal{N}_2)\).

**Proof.** The proposition is proved by showing the existence of unique balanced functors \( L, L', R, R' \) making the diagram below commute. We present the diagram here in full prematurely and explain its various attributes in the following paragraph as we work through the proof. To save space we haven't been completely explicit in indexing universal balanced functors \( B \), and rely on context to alleviate confusion.

The functor \( \tau_{(23)} \) permutes the second and third tensorands. It is easy to see that the composition \( B_{\mathcal{M}_1 \boxtimes \mathcal{M}_2} \mathcal{N}_1 \boxtimes \mathcal{N}_2 \circ \tau_{(23)} \) is \( D \)-balanced in positions 1 and 3 (Definition 2.1.3), hence the existence of unique balanced functor \( L \) making the subdiagram in the upper left commute. Similarly \( B_{1,2} \boxtimes_{C_1 \boxtimes_D C_2} B_{1,2} \circ L \) is \( D \)-balanced giving unique balanced \( L' \) making lower left subdiagram commute.

Moving to the right side of the diagram one checks that composition \( B_{1,1} \boxtimes B_{2,2} \circ \tau_{(23)} \) is \( C_1 \boxtimes_D C_2 \)-balanced giving unique balanced \( R \) making upper right subdiagram commute. Existence of the functor \( R' \) is slightly trickier. Observe that composition
of functors down the vertical center forms half of the diagram defining the functor $B_{1,2} \boxtimes C_1 \boxtimes D_2 B_{1,2}$ (the rectangular subdiagram in the left of the diagram below).

Denote $\Gamma := B \circ B_{1,1} \boxtimes B_{2,2} \circ \tau_{(23)}$, the composition of right-most vertical and top right functors. Using the $D$-balancing of $B_{1,1}$ and $B_{2,2}$ as well as the bimodule-linearity of functors $B$ one shows that $\Gamma$ is $D$-balanced in positions 1, 3 and thus we have a unique balanced functor $\Gamma' : (M_1 \boxtimes_D M_2) \boxtimes (N_1 \boxtimes_D N_2) \to (M_1 \boxtimes_{C_1} N_1) \boxtimes_D (M_2 \boxtimes_{C_2} N_2)$ such that $\Gamma' \circ B_{1,2} \boxtimes B_{1,2} = \Gamma$. In fact $\Gamma'$ is $C_1 \boxtimes_D C_2$-balanced giving unique balanced functor $R'$. This is precisely $R'$ in the first diagram.

Every cell commutes and therefore the exterior contour also commutes. Retaining notation above this means

$$B \circ B_{1,1} \boxtimes B_{2,2} = R' L' \circ B \circ B_{1,1} \boxtimes B_{2,2}$$

since $\tau_{1,3}^2 = id$. Universality of functors $B$ and $B_{1,1} \boxtimes B_{2,2}$ implies that $R' L' = id$. Reasoning similar to the above yields $L' R' = id$, hence $L', R'$ are quasi-inverse and the proposition is proved.

Note 5.3.7. In the case that categories $\mathcal{M}_i, \mathcal{N}_i$ have bimodule category structure the equivalence in Proposition 5.3.6 is an equivalence of bimodule categories.
Corollary 5.3.8. Suppose that $\mathcal{M}$ is a $C_1$-bimodule category, $\mathcal{N}$ is a $C_2$-bimodule category, and assume that $\mathcal{M}$ and $\mathcal{N}$ are invertible. Then $\mathcal{M} \boxtimes_D \mathcal{N}$ is invertible as a $C_1 \boxtimes_D C_2$-bimodule category and has inverse $\mathcal{M}^{-1} \boxtimes_D \mathcal{N}^{-1}$.

Proof. Theorem 5.3.3 implies

$$(\mathcal{M} \boxtimes_{C_1} \mathcal{N}) \boxtimes_{C_1 \boxtimes_D C_2} (\mathcal{M}^{-1} \boxtimes_{C_2} \mathcal{N}^{-1}) \simeq (\mathcal{M} \boxtimes_{C_1} \mathcal{M}^{-1}) \boxtimes_D (\mathcal{N} \boxtimes_{C_2} \mathcal{N}^{-1}) \simeq C_1 \boxtimes_D C_2$$

giving the result. \qed
6.1 (De)-equivariantization: background

Most of the background information in this section is taken from [DGNO10]. Let \( \mathcal{M} \) be a monoidal category. Recall that an action of \( \mathcal{M} \) on \( \mathcal{C} \) is a monoidal functor \( F : \mathcal{M} \to \text{End}(\mathcal{C}) \) where \( \text{End}(\mathcal{C}) \) denotes the category of \( k \)-linear endofunctors on \( \mathcal{C} \).

For finite group \( G \) denote by \( G \) the finite monoidal category having objects elements of \( G \), only trivial morphisms, and with tensor product given by multiplication in \( G \). Then an action of \( G \) on \( \mathcal{C} \) is the same as an action of \( G \) on \( \mathcal{C} \). The tensor category \( \text{Vec}_G \) of finite dimensional \( G \)-graded vector spaces identifies with the \( k \)-linear hull of \( G \) and hence an action of \( G \) on \( \mathcal{C} \) is the same thing as a \( k \)-linear action of \( \text{Vec}_G \) on \( \mathcal{C} \).

Let \( \text{Rep}(G) \) denote the braided category of finite dimensional representations of \( G \). Then we have an equivalence of 2-categories

\[
\{ \text{k-linear categories with } G \text{-action} \} \cong \{ \text{k-linear categories with } \text{Rep}(G) \text{-action} \},
\]

called equivariantization and de-equivariantization.
6.1.1 Equivariantization

In this section we describe how a category with $G$-action has canonical $\text{Rep}(G)$-module structure. Let $F : G \to \text{End}(C)$, $g \mapsto F_g$ be an action of $G$ on $C$ and let $\gamma_{g,h} : F_g F_h \simeq F_{gh}$ be the isomorphism giving $F$ the structure of a monoidal functor.

A pair $(X, u)$ for object $X \in C$ is said to be $G$-equivariant if there is a natural family $u_g : F_g(X) \simeq X$ of natural isomorphisms making the diagram

\[
\begin{array}{ccc}
F_g(F_h(X)) & \xrightarrow{F_g(u_h)} & F_g(X) \\
\gamma_{g,h}(X) & & u_g \\
F_{gh}(X) & \xrightarrow{u_{gh}} & X
\end{array}
\]

commute for $g, h \in G$. Morphisms of equivariant objects are defined to be morphisms in $C$ commuting with $u_g$ for all $g \in G$. Evidently we have a category of $G$-equivariant objects in $C$, denoted $C^G$.

The category $C^G$ has $\text{Rep}(G)$-module category structure as follows. For representation $(V, \rho)$ and $(X, u) \in C^G$ we define $(V \otimes X, u^V)$ by the composition

$u_g^V := F_g(X \otimes V) \simeq F_g(X) \otimes V \xrightarrow{u_g \otimes \rho(g)} X \otimes V$

6.1.2 De-equivariantization

Here we describe how a $\text{Rep}(G)$-module category carries a natural structure of a category on which $G$ acts. Recall that the regular object $A$ in $\text{Rep}(G)$ can be viewed as the algebra $\text{Fun}(G, k)$ of $k$-valued functions on $G$. As a representation $G$ acts on $A$ by right translation. Any $\text{Rep}(G)$-module category $D$ thus contains a subcategory
of $A$-modules, which we denote by $D_G$ and call the de-equivariantization of $D$.

### 6.2 Monoidal 2-structure and (de)-equivariantization

We keep notation as above. Braiding of $\text{Rep}(G)$ implies an embedding of 2-categories $\text{Rep}(G)\text{-Mod} \hookrightarrow B(\text{Rep}(G))$ into the monoidal 2-category consisting of $\text{Rep}(G)$-bimodule categories. This is symptomatic of the observation that every module category over a braided monoidal category is really a bimodule category. Denote by $\boxtimes_{\text{Rep}(G)}$ the monoidal 2-structure in $B(\text{Rep}(G))$.

Denote by $G\text{-Mod}$ the 2-category consisting of categories with $G$-action. $G\text{-Mod}$ has monoidal structure as follows. Let $F, E$ be objects in $G\text{-Mod}$ with $G$-actions given by monoidal functors $F, E$ respectively. Then $G$ acts on $F \boxtimes E$ via $F \boxtimes E$. We write $F \boxtimes E$ to indicate the category $F \boxtimes E$ with this action.

**Proposition 6.2.1.** The correspondence $C \leftrightarrow C^G$ between the 2-categories $G\text{-Mod}$ and $B(\text{Rep}(G))$ respects monoidal structure.

**Proof.** Denote by $\text{Fun}_G(\text{Vec}, C)$ the category of functors which commute with the action of $G$ where we view $\text{Vec}$ as having trivial $G$-action. $\text{Fun}_G(\text{Vec}, C)$ carries a natural $\text{Rep}(G)$-module category structure as follows: for $(V, \rho) \in \text{Rep}(G)$ define $(H, h) \otimes (V, \rho) := (H^V, h^V)$ where $H^V(W) := H(W) \otimes V$ and where $h^V$ is given by the composition

\[ h^V_g := H^V(F_g k) = H(F_g k) \otimes V^{h_g \otimes \rho(g)}F_g H(k) \otimes V = F_g(H^V(k)). \]

One easily checks that the relevant module coherence diagram for $h^V$ follows from
those satisfied by $h$ and $\rho$.

**Lemma 6.2.2.** For $\mathcal{C} \in G$-Mod there is a canonical equivalence of $\text{Rep}(G)$-module categories $\mathcal{C}^G \simeq \text{Fun}_G(\text{Vec}, \mathcal{C})$.

**Proof.** For $(X, u) \in \mathcal{C}^G$ denote by $H_X : \text{Vec} \to \mathcal{C}$ the unique functor having $H_X(k) = X$. Then $u_g : H_X(k) \simeq F_g H_X(k)$ gives $H_X$ the structure of a $G$-module functor. In the opposite direction any $G$-module functor $(H, h) : \text{Vec} \to \mathcal{C}$ determines a $G$-equivariant object of $\mathcal{C}$: $G$-module structure $h$ on $H$ corresponds to a natural isomorphism $h_g : H(k) \simeq F_g(H(k))$ where $F : G \to \text{End}(\mathcal{C})$ is the action of $G$ on $\mathcal{C}$.

Let $v_g := h_g^{-1} : F_g(H(k)) \simeq H(k)$ and observe that the $G$-module diagram satisfied by $h$ translates into the diagram making $(H(k), v_g)$ an object in the equivariantization $\mathcal{C}^G$. Clearly these two constructions are inverse.

It remains to check that this correspondence respects $\text{Rep}(G)$-module category structures. Let $(X, u) \in \mathcal{C}^G$ and $(V, \rho) \in \text{Rep}(G)$. Then the functor associated to $(X \otimes V, u^V)$ is $(H_{X \otimes V}, u^V)$ and this is trivially naturally isomorphic to the functor $(H_X, u) \otimes (V, \rho)$.  

Remark 4.3 in [DGNO10] implies that, as abelian categories, $\text{Fun}_G(\text{Vec}, \mathcal{C}) \simeq \text{Fun}_{\text{Vec}G}(\text{Vec}, \mathcal{C})$. Write $\text{Fun}_{\text{Vec}G}(\text{Vec}, \mathcal{C}) := \overline{\mathcal{C}}$. It is trivial that this equivalence respects $\text{Rep}(G)$-module structure, and hence as $\text{Rep}(G)$-module categories $\mathcal{C}^G \simeq \overline{\mathcal{C}}$.

As monoidal categories $G$-Mod and $\text{Vec}G$-Mod are equivalent. We will use $\otimes$ to denote monoidal structure in both places. For $G$-module categories $\mathcal{C}, \mathcal{D}$ we have the
following Rep(G)-module equivalences:

\[
(C \otimes D)^G \simeq \overline{C} \otimes \overline{D} \simeq \overline{C} \boxtimes_{\text{Rep}(G)} \overline{D} \simeq C^G \boxtimes_{\text{Rep}(G)} D^G.
\]  

(17)

First and last equivalences are Lemma 6.2.2 and the second is Theorem 8.3.2. \(\square\)

6.3 On de-equivariantization and relative tensor product

The main result of this section is the proof of Theorem 0.2.2. We begin with the following lemma.

Lemma 6.3.1. Let \(\mathcal{C}, \mathcal{D}\) be fusion categories and let \(F: \mathcal{C} \to \mathcal{D}\) be a surjective tensor functor. Let \(I\) be its right adjoint. Then

1. \(I(1)\) is an algebra in \(Z(\mathcal{C})\).

2. \(\mathcal{D}\) is tensor equivalent to the category \(\text{Mod}_\mathcal{C}(I(1))\) of right \(I(1)\)-modules in \(\mathcal{C}\).

3. The equivalence in (2) identifies \(F\) with the free module functor \(X \mapsto X \otimes I(1)\).

Proof. To prove (1) observe that \(\mathcal{D}\) is a \(Z(\mathcal{C})\)-module category with action \(X \otimes Y := F'(X) \otimes Y\) where \(F': Z(\mathcal{C}) \to \mathcal{D}\) is \(F\) composed with functor forgetting central structure. Under this action \(\text{Hom}(1, 1) = I(1)\) (see Definition 1.3.4) so by Lemma 5 in [Ost03] \(I(1)\) is an algebra in \(Z(\mathcal{C})\). Note that since \(I(1)\) is an algebra in \(Z(\mathcal{C})\) we have tensor structure on \(\text{Mod}_\mathcal{C}(I(1))\): \(X \otimes I(1) = I(1) \otimes X\) so for \(I(1)\)-modules \(X, Y\) \(X \otimes_{I(1)} Y\) makes sense. Theorem 1 in the same paper says that \(\text{Mod}_\mathcal{C}(I(1)) \cong \mathcal{D}\) as
module categories over \( C \) via \( F \) in (3). Observe that

\[
F(X) \otimes_{I(1)} F(Y) = (X \otimes I(1)) \otimes_{I(1)} (Y \otimes I(1)) = (X \otimes Y) \otimes I(1) = F(X \otimes Y).
\]

Hence \( F : X \mapsto X \otimes I(1) \) respects tensor structure. This completes the proof of the lemma.

In what follows \( G \) is a finite group and we write \( \mathcal{E} := \text{Rep}(G) \), the symmetric fusion category of finite dimensional representations of \( G \) in \( \text{Vec} \). Let \( \mathcal{C} \) be tensor category over \( \mathcal{E} \) (Definition 5.0.7) which we thereby view as a right \( \mathcal{E} \)-module category. Let \( A \) be the regular representation of \( G \). \( A \) has the structure of an algebra in \( \mathcal{E} \) and we therefore have the notion of \( A \)-module in \( \mathcal{C} \). Denote by \( C_G \) the category \( \text{Mod}_C(A) \) of \( A \)-modules in \( \mathcal{C} \). There is functor \( \text{Free} : \mathcal{C} \to C_G \), \( X \mapsto X \otimes A \) left adjoint to the functor \( \text{Forg} : C_G \to \mathcal{C} \) which forgets \( A \)-module structure ([DGNO10, §4.1.9]). We are now ready to prove the theorem.

*Proof of Theorem 0.2.2.* Let \( F := B_{C, \text{vec}} : \mathcal{C} \boxtimes \text{Vec} \to \mathcal{C} \boxtimes \text{Vec} \) be the canonical surjective right exact functor described in Definition 2.1.4 which is tensor by Proposition 5.1.1, and let \( I \) be its right adjoint. Lemma 6.3.1 gives us tensor equivalence \( \text{Mod}_C(I(1)) \cong \mathcal{C} \boxtimes \text{Vec} \). Denote by \( A' \) the image of the regular algebra \( A \) in \( \mathcal{E} \) under the composition

\[
\mathcal{E} \to Z(C) \to \mathcal{C}.
\]

We claim that \( I(1) \) is \( A' \)

Let \( X, Y \in \mathcal{C} \) be in distinct indecomposable \( \mathcal{E} \)-module subcategories of \( \mathcal{C} \). Since
the indecomposable $\mathcal{E}$-module subcategories of $\mathcal{C}$ are respected by $F$ the images of $X,Y$ under $F$ are in distinct $\mathcal{E}$-module components of $\mathcal{C} \boxtimes \mathcal{E} \text{Vec}$. Not only does this imply that $F(X)$ and $F(Y)$ are not isomorphic but in fact $\text{Hom}(F(X), F(Y)) = 0$. Thus if $F(X)$ contains a copy of the unit object $1 \in \mathcal{C} \boxtimes \mathcal{E} \text{Vec}$ then $X$ and $1 \in \mathcal{C}$ must belong to the same indecomposable $\mathcal{E}$-module subcategory of $\mathcal{C}$. Thus any object whose $F$-image contains the unit object must be contained in the image of $\mathcal{E}$ in $\mathcal{C}$ under the composition (18).

Note that the restriction of $F$ to the image of $\mathcal{E}$ in $\mathcal{C}$ gives a fiber functor $\mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E} \text{Vec} = \text{Vec}$. By [DGNO10, §2.13] the choice of a fiber functor from $\mathcal{E}$ determines a group $G_F \simeq G$ having the property that $\text{Fun}(G_F)$ is regular algebra $A$ in $\text{Rep}(G)$ and as such is canonically isomorphic to $I(1)$. Thus we have tensor equivalence $\text{Mod}_\mathcal{C}(A) = \mathcal{C}_G \simeq \mathcal{C} \boxtimes \mathcal{E} \text{Vec}$ and the proof is complete. □
CHAPTER VII

MODULE CATEGORIES OVER

BRAIDED MONOIDAL CATEGORIES

In what follows \( \mathcal{C} \) is a fixed tensor category (Definition 1.2.8) and all module categories are assumed to be exact. Recall (Definition 1.2.10) that \( \mathcal{C} \) is said to be braided if \( \mathcal{C} \) is equipped with a class of natural isomorphisms

\[
c_{V,W} : V \otimes W \to W \otimes V
\]

for objects \( V, W \in \mathcal{C} \) satisfying a pair of hexagons describing how they interact with tensor associativity. When \( \mathcal{C} \) is strict these reduce to the equations

\[
c_{U,V \otimes W} = (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W)
\]

(19)

\[
c_{U \otimes V,W} = (c_{U,W} \otimes id_V)(id_U \otimes c_{V,W}).
\]

(20)

7.1 The center of a bimodule category

In this section we describe a construction which associates to a strict \( \mathcal{C} \)-bimodule category \( \mathcal{M} \) a new category having the structure of a \( Z(\mathcal{C}) \)-bimodule category. Note that as monoidal categories \( \mathcal{C}^{op} \), which we have been using to denote the opposite
category, is canonically monoidal equivalent to the category $C^{\text{rev}}$, the category $C$ with monoidal product reversed. We will therefore not distinguish between them and use the single notation $C^{\text{op}}$.

For the first proposition assume $C$ to be braided by $c_{X,Y} : X \otimes Y \to Y \otimes X$. Our first proposition is well known and we provide a proof only for completeness.

**Proposition 7.1.1.** Let $\mathcal{M}$ be a left $C$-module category. Then $\mathcal{M}$ has canonical structure of $C$-bimodule category.

**Proof.** We begin with the following lemma.

**Lemma 7.1.2.** $\mathcal{M}$ is right $C$-module category via $(M, X) \mapsto X \otimes M$ where $\otimes$ is left $C$-module structure.

**Proof.** For left module associativity $a$ define natural isomorphism

$$a'_{M,X,Y} = a_{Y,X,M}(id_M \otimes c_{X,Y}) : M \otimes (X \otimes Y) \to (M \otimes X) \otimes Y$$

for $X, Y \in C$ and $M \in \mathcal{M}$. In terms of the left module structure by which $M \otimes X$ is defined $a'_{M,X,Y} = a_{Y,X,M}(c_{X,Y} \otimes id_M) = (X \otimes Y) \otimes M \to Y \otimes (X \otimes M)$. We show that $a'$ is module associativity for right module structure. Consider diagram

```
(XYZ)M <-a_{X,Y,Z}-(XZY)M <-a_{X,Y,Z}-(ZYX)M <-a_{X,Y,Z}-(ZXY)M
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
|                     |                     |                     |
```
The upper left rectangle is naturality of \( c \), upper right triangle naturality of \( a \), leftmost triangle is equation (20), triangle in lower half of diagram is equation (19), central bottom rectangle is naturality of \( a \) and rightmost rectangle is \( a \)-pentagon in \( C \). The two directed components of the external contour are precisely \( a'_{M,X,Y,Z}a'_{M,X,Y,Z} \) and \( (a'_{M,X,Y} \otimes Z)a'_{M,X,Y,Z} \). The diagrams for action of unit in \( C \) are even easier. \( \square \)

Define action of \( X \otimes Y \in C \otimes C^{rev} \) using left and right actions, i.e. \( (X \otimes Y) \otimes M = Y \otimes (X \otimes M) \). Define

\[
\gamma_{X,M,Y} = a_{X,Y,M}(c_{Y,X} \otimes id_M)a^{-1}_{Y,X,M} : Y \otimes (X \otimes M) \to X \otimes (Y \otimes M).
\]

In order to verify that the candidate action is indeed bimodule we must show that \( \gamma \) satisfies the necessary pentagons (Remark 1.3.10). Commutativity of the first pentagon follows from an examination of the diagram below.

Every peripheral rectangle is either the definition of \( \gamma \) or the module associativity satisfied by \( a \). Note that top left vertex can be connected to the lower center vertex by the map \( c_{Z,X} \otimes id_{Y \otimes M} \) making commutative rectangle expressing naturality of \( a \) in first
index. Lower center vertex can be connected to uppermost right vertex by the map $id_X \otimes c_{Z,Y} \otimes id_M$ making commutative rectangle expressing naturality of $a$ in the second index. Commutativity of this new external triangle is (equation (19))$\otimes M$. Thus the internal pentagon commutes, and this is precisely the first diagram in Remark 1.3.10. Commutativity of second pentagon is similar. This completes the proof of Proposition 7.1.1. □

Next we generalize the notion of center to module categories.

**Definition 7.1.3.** Let $M$ be a $C$-bimodule category. A *central structure on $M$* is a family of natural isomorphisms $\varphi_{X,M} : X \otimes M \simeq M \otimes X$, $X \in C$, one for each object $M \in M$, satisfying the condition

\[
\begin{array}{ccc}
(XY)M & \xrightarrow{\varphi_{XY,M}} & M(XY) \\
\downarrow a^X_{X,Y,M} & & \downarrow a^M_{M,XY} \\
X(YM) & \xrightarrow{\varphi_{X,M \otimes Y}} & (MX)Y \\
\downarrow X \otimes \varphi_{Y,M} & & \downarrow \varphi_{X,M \otimes Y} \\
X(MY) & \xrightarrow{\gamma_{X,Y,M}} & (XM)Y \\
\end{array}
\]

whenever $Y \in C$ where $a^X, a^M$ are left and right module associativity in $M$ and $\gamma$ bimodule consistency (Proposition 1.3.10). $\varphi_M$ is called the *centralizing isomorphism* associated to $M$. If such a central structure exists $M$ is said to be *central* over $C$.

Note that when $M$ is strict as a bimodule category the hexagon reduces to

\[
\begin{array}{ccc}
XMY & \xrightarrow{\varphi_{X,M \otimes id_Y}} & MXY \\
\downarrow id_X \otimes \varphi_{Y,M} & & \downarrow \varphi_{XY,M} \\
XYM & & \\
\end{array}
\]
In what follows assume $C$ is a strict monoidal category.

**Definition 7.1.4.** The center $Z_C(M)$ of $M$ over $C$ consists of objects given by pairs $(M, \varphi_M)$ where $M \in M$ and where $\varphi_M$ is a family of natural isomorphisms such that the isomorphisms $\varphi_{X,M} : X \otimes M \simeq M \otimes X$ satisfy Definition 7.1.3 for $X \in C$.

A morphism from $(M, \varphi_M)$ to $(N, \varphi_N)$ in $Z_C(M)$ is a morphism $t : M \to N$ in $M$ satisfying $\varphi_{X,N}(id_X \otimes t) = (t \otimes id_X) \varphi_{X,M}$.

**Note 7.1.5.** Definition 7.1.4 appeared in [GNN09] in connection with centers of braided fusion categories.

**Example 7.1.6.** For $C$ viewed as having a regular bimodule category structure $Z_C(C) = Z(C)$, the center of $C$.

**Definition 7.1.7.** Let $M, N$ be bimodule categories central over $C$. Then $C$-bimodule functor $T : M \to N$ is called *central* if the diagram

\[
\begin{array}{ccc}
T(X \otimes M) & \xrightarrow{f_{X,M}} & X \otimes T(M) \\
\downarrow_{T(\varphi_{X,M})} & & \downarrow_{\varphi_{X,T(M)}} \\
T(M \otimes X) & \xrightarrow{f_{M,X}} & T(M) \otimes X
\end{array}
\]

commutes for all $X \in C$, $M \in M$, where $\varphi$ denotes centralizing natural isomorphisms in $M$ and $N$. $f$ is linearity isomorphism for $T$. A *central natural transformation* $\tau : F \Rightarrow G$ for central functors $F, G : M \to N$ is a bimodule natural transformation.
\( F \Rightarrow G \) with the additional requirement that, for \( X \in \mathcal{C}, M \in \mathcal{M} \) the diagram

\[
\begin{array}{ccc}
X \otimes F(M) & \xrightarrow{\phi_{X,F(M)}} & F(M) \otimes X \\
\downarrow \tau_M & & \downarrow \tau_M \otimes X \\
X \otimes G(M) & \xrightarrow{\phi_{X,G(M)}} & G(M) \otimes X
\end{array}
\]

commutes.

It is evident that centrality of natural transformations is preserved by vertical (and horizontal) composition, and we thus have a category (indeed a bicategory) \( Z(\mathcal{M}, \mathcal{N}) \) for central bimodule categories \( \mathcal{M}, \mathcal{N} \) consisting of central functors \( \mathcal{M} \to \mathcal{N} \) where morphisms are central natural transformations.

**Lemma 7.1.8.** \( Z_C(\mathcal{M}) \) is a \( Z(C) \)-bimodule category.

**Proof.** Assume \( \mathcal{M} \) is strict bimodule category. We have left action of \( Z(C) \) on \( Z_C(\mathcal{M}) \) given as follows: for \( (X, c_X) \in Z(C) \) and \( (M, \phi_M) \in Z_C(\mathcal{M}) \) define \( (X, c_X) \otimes (M, \phi_M) = (X \otimes M, \phi_{X \otimes M}) \) where for \( Y \in \mathcal{C} \)

\[
\phi_{Y,X \otimes M} := Y \otimes X \otimes M \xrightarrow{c_{X,Y} \otimes M} X \otimes Y \otimes M \xrightarrow{X \otimes \phi_{Y,M}} X \otimes M \otimes Y
\]

so that \( X \otimes M \in Z_C(\mathcal{M}) \). Define right action of \( Z(C) \) by \( (M, \phi_M) \otimes (X, c_X) = (M \otimes X, \phi_{M \otimes X}) \) where

\[
\phi_{Y,M \otimes X} := Y \otimes M \otimes X \xrightarrow{\phi_{Y,M} \otimes X} M \otimes Y \otimes X \xrightarrow{M \otimes c_{Y,X}} M \otimes X \otimes Y
\]

putting \( M \otimes X \in Z_C(\mathcal{M}) \). It is easy to check that these actions are consistent in the
Proposition 7.1.9. $Z_C(\mathcal{M})$ has a canonical central structure over $Z(\mathcal{C})$.

Proof. $\varphi_{X,M} : (X \otimes M, \varphi_{X \otimes M}) \to (M \otimes X, \varphi_{M \otimes X})$ is a morphism in $Z_C(\mathcal{M})$ as can be seen by the diagram

\[
\begin{array}{ccc}
YXM & \xrightarrow{Y \otimes \varphi_{X,M}} & YM X \\
\downarrow \varphi_{Y,X,M} & & \downarrow \varphi_{Y,M \otimes X} \\
XYM & \xrightarrow{\varphi_{XY,M}} & MYX \\
\downarrow X \otimes \varphi_{Y,M} & & \downarrow M \otimes \varphi_{Y,X} \\
XMY & \xrightarrow{\varphi_{X,M \otimes Y}} & MXY
\end{array}
\]

Triangles are Definition 7.1.3 for $\varphi$ and the square is $\mathcal{C}$-naturality of $\varphi$. 

Proposition 7.1.10. For $\mathcal{C}$-bimodule category $\mathcal{M}$ we have canonical $Z(\mathcal{C})$-bimodule equivalence $\text{Fun}_{\mathcal{C}^{\text{op}}} (\mathcal{C}, \mathcal{M}) \simeq Z_C(\mathcal{M})$.

Proof. For simplicity assume $\mathcal{M}$ is strict as a $\mathcal{C}$-bimodule category. Define functor $\Delta : \text{Fun}_{\mathcal{C}^{\text{op}}} (\mathcal{C}, \mathcal{M}) \simeq Z_C(\mathcal{M})$ by sending $F \mapsto (F(1), f^r \circ f^{-1})$ where $f^r_X : F(X) \simeq X \otimes F(1)$ and $f^r_X : F(X) \simeq F(1) \otimes X$ are left/right module linearity isomorphisms for $F$. The diagram below implies $(F(1), f^r \circ f^{-1}) \in Z_C(\mathcal{M})$:

\[
\begin{array}{ccc}
F(1)XY & \xleftarrow{f_{XY}} & F(XY) & \xrightarrow{f_{XY}} & XYF(1) \\
\downarrow f_X \otimes Y & & \downarrow f_X \otimes Y & & \downarrow x \otimes f_Y \\
F(X)Y & \xrightarrow{f_X \otimes Y} & XF(1)Y & \xleftarrow{x \otimes f_Y} & XF(Y)
\end{array}
\]

Left and right triangles are diagrams expressing module linearity of $F$ and square is bimodularity of $F$ (Remark 1.3.12). Inverting all $\ell$ superscripted isomorphisms gives
the diagram required for centrality of \( f^r \circ f^{t-1} \).

To complete definition of functor \( \text{Fun}_{\mathcal{C} \otimes \mathcal{C}^{\text{op}}}(\mathcal{C}, \mathcal{M}) \to Z\mathcal{C}(\mathcal{M}) \) we must define action on natural bimodule transformations. For \( \tau : F \Rightarrow G \) a morphism in the category of functors \( \text{Fun}_{\mathcal{C} \otimes \mathcal{C}^{\text{op}}}(\mathcal{C}, \mathcal{M}) \) note that \( \tau_1 : (F(1), f^r \circ f^{t-1}) \to (G(1), g^r \circ g^{t-1}) \) is a morphism in \( Z\mathcal{C}(\mathcal{M}) \): indeed, diagram required of \( \tau_1 \) as central morphism is given by pasting together left/right module diagrams for \( \tau \) along the edge \( \tau_X : F(X) \to G(X) \).

We now define quasi-inverse \( \Gamma \) for functor \( \Delta \). For \( M \in \mathcal{M} \) denote by \( F_M \) the functor \( \mathcal{C} \to \mathcal{M} \) defined by \( F_M(X) := X \otimes M \). Right exactness of \( F_M \) follows from (contravariant) left exactness of \( \text{Hom}(\_ , \text{Hom}(M, M)) \). Since \( \mathcal{M} \) is a strict \( \mathcal{C} \)-bimodule category \( F_M \) is strict as a left \( \mathcal{C} \)-module functor. For \( (M, \varphi_M) \in Z\mathcal{C}(\mathcal{M}) \) we give \( F_M \) the structure of a right \( \mathcal{C} \)-module functor via

\[
F_M(X) = X \otimes M \xrightarrow{\varphi_X M} M \otimes X = F_M(1) \otimes X
\]  

and with this \( F_M \) is \( \mathcal{C} \)-bimodule. Define \( \Gamma(M, \varphi_M) := F_M \) with the bimodule structure given in (21). It is now trivial to verify that \( \Delta \Gamma = \text{id} \) and that \( \Gamma \Delta \) is naturally equivalent to \( \text{id} \) via \( f^t \). Finally, it is easy to see that \( \Gamma \) is a strict \( Z(\mathcal{C}) \)-bimodule functor.

As a corollary we get a well known result which appears for example in [EO04].

**Corollary 7.1.11.** \((\mathcal{C} \otimes \mathcal{C}^{\text{op}})^\mathcal{C} \simeq Z(\mathcal{C}) \) canonically as monoidal categories.

Here, as elsewhere, we have used \( \mathcal{C}^\mathcal{M} \) to denote the category of \( \mathcal{C} \)-module endo-functors \( \text{End}_\mathcal{C}(\mathcal{M}) \) for \( \mathcal{C} \)-module category \( \mathcal{M} \).
7.2 The 2-categories $\mathcal{B}(C)$ and $Z(C)$-Mod

Recall that $\mathcal{B}(C)$ denotes the category of exact $C$-bimodule categories. The main result of this section is Theorem 7.2.3 giving an equivalence $\mathcal{B}(C) \simeq Z(C)$-Mod which is suitably monoidal. Before we give the first proposition of this subsection recall that $C$ has a trivial $Z(C)$-module category structure given by the forgetful functor.

**Proposition 7.2.1.** The 2-functor $\mathcal{B}(C) \to Z(C)$-Mod given by $\mathcal{M} \mapsto Z_C(\mathcal{M}) = \text{Fun}_{C,\text{op}}(C, \mathcal{M})$ is an equivalence with inverse given by $\mathcal{N} \mapsto \text{Fun}_{Z(C)}(C_{\text{op}}, \mathcal{N})$.

**Proof.** In Proposition 7.1.10 we saw that $Z_C(\mathcal{M})$ is a $Z(C)$-module category whenever $\mathcal{M}$ is a $C$-bimodule category (here module structure is just composition of functors). The category of $Z(C)$-module functors $\text{Fun}_{Z(C)}(C_{\text{op}}, \mathcal{N})$ for $Z(C)$-module category $\mathcal{N}$ has the structure of a $C$-bimodule category with actions

$$(F \otimes X)(Z) := F(X \otimes Z), \quad (Y \otimes F)(Z) := F(Z \otimes Y).$$

To see that $\text{Fun}_{C,\text{op}}(C, -)$ and $\text{Fun}_{Z(C)}(C_{\text{op}}, -)$ are quasi-inverses first note that

$$\text{Fun}_{Z(C)}(C_{\text{op}}, \text{Fun}_{C,\text{op}}(C, \mathcal{N})) \simeq \text{Fun}_{C,\text{op}}(C \boxtimes_{Z(C)} C_{\text{op}}, \mathcal{N}) \simeq \text{Fun}_{C,\text{op}}(Z(C)_{c}^{\ast}, \mathcal{N})$$

as $C$-bimodule categories for any bimodule category $\mathcal{N}$ where we have used equation (22) freely. Theorem 3.27 in loc. cit. gives a canonical equivalence $(C_{\mathcal{M}})_{\mathcal{M}}^{\ast} \simeq C$ for any (exact) $C$-module category $\mathcal{M}$. In the case that $\mathcal{M} = C$ this and Corollary 7.1.11 imply $Z(C)_{c}^{\ast} \simeq ((C \boxtimes C_{\text{op}})_{c})_{c}^{\ast} \simeq C \boxtimes C_{\text{op}}$. Thus the last category of functors in (22) is
canonically equivalent to \( \text{Fun}_{Z(C)_{\text{op}}} (C \otimes C_{\text{op}}, N) \simeq N \).

In the opposite direction we have, for \( Z(C)\)-module category \( \mathcal{M} \),

\[
\text{Fun}_{Z(C)_{\text{op}}} (C, \text{Fun}_{Z(C)} (C_{\text{op}}, \mathcal{M})) \simeq \text{Fun}_{Z(C)} (C_{\text{op}} \otimes Z(C), C_{\text{op}}, \mathcal{M}).
\]  

(23)

Note that \( C_{\text{op}} \otimes Z(C) \simeq (C \otimes C_{\text{op}})^{op} \simeq Z(C) \) (Corollary 7.1.11) and thus the last category of functors in (23) is canonically equivalent to \( \text{Fun}_{Z(C)} (Z(C), \mathcal{M}) \simeq \mathcal{M} \). \( \square \)

**Lemma 7.2.2.** As \( Z(C)\)-bimodule categories \( Z_C (\mathcal{M}) \simeq Z_C (\mathcal{M})^{op} \).

**Proof.** For \( \mathcal{M}, C \) as above we have the bimodule equivalences

\[
\text{Fun}_{Z(C)_{\text{op}}} (C, \mathcal{M}) \simeq \text{Fun}_{Z(C)_{\text{op}}} (\mathcal{M}, C)^{op} \simeq \text{Fun}_{Z(C)_{\text{op}}} (C, \mathcal{M})^{op}.
\]

The first equivalence is Lemma 1.3.14 and the second uses Corollary 2.3.3. By Proposition 7.1.10 the first term is equivalent to \( Z_C (\mathcal{M})^{op} \) and the last to \( Z_C (\mathcal{M})^{op} \). \( \square \)

**Theorem 7.2.3.** The 2-equivalence \( Z_C : B(C) \simeq Z(C)\text{-Mod} \) is monoidal in that

\[ Z_C (\mathcal{M} \otimes_C N) \simeq Z_C (\mathcal{M}) \otimes_{Z(C)} Z_C (N) \]

whenever \( \mathcal{M}, \mathcal{N} \) are \( C\)-bimodule categories.

**Proof.** We have canonical \( Z(C)\)-bimodule equivalences

\[
Z_C (\mathcal{M} \otimes_C N) \simeq \text{Fun}_{Z(C)_{\text{op}}} (C, \mathcal{M} \otimes_C N) \simeq \text{Fun}_{Z(C)_{\text{op}}} (\mathcal{M}, \mathcal{N})
\]

\[
\simeq \text{Fun}_{Z(C)} (Z_C (\mathcal{M}), Z_C (N)) \simeq \text{Fun}_{Z(C)} (Z_C (\mathcal{M})^{op}, Z_C (N))
\]

\[
\simeq \text{Fun}_{Z(C)} (Z(C), Z_C (\mathcal{M}) \otimes_{Z(C)} Z_C (N)) \simeq Z_C (\mathcal{M}) \otimes_{Z(C)} Z_C (N)
\]

The first equivalence is Proposition 7.1.10, the second and fifth are Corollary 2.3.3, the
third follows from the fact that the equivalence of 2-categories $\mathbb{Z}(\mathcal{C})$-$\text{Mod} \simeq (\mathcal{C} \otimes \mathcal{C}^{\text{op}})^{\mathbb{C}}_{\mathcal{C}}$-\text{Mod} (Corollary 7.1.11) preserves categories of 1-cells, and the fourth follows from Lemma 7.2.2. Example 7.1.6 shows that $\mathbb{Z}_c$ preserves units.

\[ \square \]

**Corollary 7.2.4.** Let $\mathcal{M}$ be a $\mathcal{C}$-module category for finite tensor $\mathcal{C}$. There is a canonical 2-equivalence $\mathbb{Z}(\mathcal{C}) \simeq \mathbb{Z}(\mathcal{C}_\mathcal{M}^{\mathbb{C}})$ respecting monoidal structure.

**Proof.** Corollary 3.35 in [EO04] says that $\mathbb{Z}(\mathcal{C}) \simeq \mathbb{Z}(\mathcal{C}_\mathcal{M}^{\mathbb{C}})$. The result follows from Theorem 7.2.3.

\[ \square \]
8.1 Burnside rings

Much in the beginning of this section is basic and can be found for example in [CR87]. Let $G$ be a finite group. Recall that the Burnside Ring $\Omega(G)$ is defined to be the commutative ring generated by isomorphism classes of $G$-sets with addition and multiplication given by disjoint union and cartesian product:

$$\langle H \rangle + \langle K \rangle = G/H \cup G/K$$

$$\langle H \rangle \langle K \rangle = G/H \times G/K$$

Here $\langle H \rangle$ denotes the isomorphism class of the $G$-set $G/H$ for $H < G$ and $G$ acts diagonally over $\times$. Evidently we have

$$\langle H \rangle \langle G \rangle = \langle H \rangle, \quad \langle H \rangle (1) = [G : H] \langle 1 \rangle$$
so $\Omega(G)$ is unital with $1 = \langle G \rangle$. It is a basic exercise to check that multiplication in $\Omega(G)$ satisfies the equation\(^1\)

\[ \langle H \rangle \langle K \rangle = \sum_{HaK \in H \backslash G/K} \langle H \cap aK \rangle. \]

We are interested in a twisted variant of the Burnside ring. Here we take as basis elements $\langle H, \sigma \rangle$ where $G/H$ is a $G$-set and $\sigma$ is a $k^\times$-valued 2-cocycle on $H$. Multiplication of basic elements takes the form

\[ \langle H, \mu \rangle \langle K, \sigma \rangle = \sum_{HaK \in H \backslash G/K} \langle H \cap aK, \mu \sigma^a \rangle \]

where on the right $\mu, \sigma^a$ refer to restriction to the subgroup $H \cap aK$ from $H, aK$, respectively. The cocycle $\sigma^a : aK \times aK \to k^\times$ is defined by $\sigma^a(x, y) = \sigma(x^a, y^a)$.

**Note 8.1.1.** The decomposition for twisted Burnside products described above occurred in [OY01] in order to study crossed Burnside rings, and in [Ros07] in connexion with the extended Burnside ring of semisimple Rep($G$)-module categories $\mathcal{M}$ having exact faithful module functor $\mathcal{M} \to \text{Rep}(G)$.

Recall that indecomposable $Vec_G$-module categories are parametrized by pairs $(H, \mu)$ where $H < G$ and $\mu \in H^2(H, k^\times)$. Denote module category associated to such a pair by $\mathcal{M}(H, \mu)$. Explicitly simple objects of $\mathcal{M}(H, \mu)$ form a $G$-set with stabilizer $H$ and are thus in bijection with cosets in $G/H$. Module associativity

\[ \text{One uses the fact that there is a bijection between the } G\text{-orbits of } (xH, yK) \in G/H \times G/K \text{ and double cosets } H \backslash G/K \text{ given by } (sH, tK) \mapsto Hs^{-1}tK. \text{ The orbit corresponding to the coset } HaK \text{ contains } (H, aK) \text{ with stabilizer } H \cap aK, \text{ thus orbit } O_G(H, aK) \text{ of } (H, aK) \text{ is } G/(H \cap aK) \text{ as } G\text{-sets giving the formula.} \]
is given by scalars $\mu(g_1, g_2)(X)$, for $\mu \in Z^2(G, \text{Fun}(G/H, k^x))$, associated to the
natural isomorphisms $(g_1g_2) \otimes X \to g_1 \otimes (g_2 \otimes X)$ whenever $g_i \in G$ and $X \in G/H$.
Module structures are classified by non-comologous cocycles so we take as module
associativity constraint any representative of the cohomology class $[\mu]$. Identifying $\mu \in
H^2(G, \text{Fun}(G/H, k^x)) = H^2(G, \text{Ind}_H^G k^x)$ with its image in $H^2(H, k^x)$ by Shapiro's
Lemma we may classify such constraints by $H^2(H, k^x)$.

8.2 Vec$_G$-Mod fusion rules

The categories Vec$_G$ and Rep($G$) are Morita equivalent via Vec: $(\text{Vec}_G)_{\text{Vec}} \simeq \text{Rep}(G)$
(send representation $(V, \rho)$ to the functor Vec $\to$ Vec having $F(k) = V$ with Vec$_G$-
linearity given by $\rho$). Since Rep($G$) is braided the category Rep($G$)-Mod has monoidal
structure $\boxtimes_{\text{Rep}(G)}$ (see Proposition 7.1.1). Although Vec$_G$ is not braided the category
Vec$_G$-Mod has monoidal structure as follows. For $\mathcal{M}, \mathcal{N} \in \text{Vec}_G$-Mod define Vec$_G$-
module category structure on $\mathcal{M} \boxtimes \mathcal{N}$ by $g \otimes (m \boxtimes n) := (g \otimes m) \boxtimes (g \otimes n)$ for simple
object $k_g := g$ in Vec$_G$, and linearly extend to all of Vec$_G$. Let $\mathcal{M} \otimes \mathcal{N}$ denote $\mathcal{M} \boxtimes \mathcal{N}$
with this module category structure.

Proposition 8.2.1 (Vec$_G$-Mod fusion rules). With notation as above

$$
\mathcal{M}(H, \mu) \otimes \mathcal{M}(K, \sigma) \simeq \bigoplus_{HaK \in H \setminus G/K} \mathcal{M}(H \cap ^aK, \mu \sigma^a).
$$

Proof. Send $(H, \sigma)$ to module category $\mathcal{M}(H, \sigma)$. This association is clearly well
defined and respects the action of $G$. Applying the proof above for decomposition of
basic elements in $\Omega(G)$ to simple objects in $\mathcal{M}(H, \mu) \otimes \mathcal{N}(K, \sigma)$ verifies the stated
decomposition on the level of objects. We must check only the module associativity constraints for the summand categories. To do this we simply evaluate associativity for a simple object in the summand category having set of objects $G/H \cap ^a K$. We may choose representative $H \otimes a K$. For $g, h \in G$ we have

$$gh \otimes (H \otimes a K) \simeq g \otimes (h \otimes H) \otimes g \otimes (h \otimes a K)$$

via $\mu(g, h)(H) \otimes \sigma(g, h)(a K)$. Noting that $G/K \simeq G/^{a} K$ as $G$-sets, restricting $\varphi : H^2(G, \text{Fun}(G/K, k^\times)) \simeq H^2(^a K, k^\times)$ to coset $a K$ on the right gives $\varphi(\sigma)(k_1, k_2) = \sigma(k_1, k_2)(a K)$ for $k_1, k_2 \in ^a K$. Thus $\varphi(\sigma)^a(k_1, k_2) = \varphi(\sigma)(k_1^a, k_2^a) \in H^2(^a K, k^\times)$, and this we simply denote by $\sigma^a$; module associativity is $\mu \otimes \sigma^a$ which is identical to $\mu \sigma^a$ since each is a scalar on simple objects.

\[\square\]

**Corollary 8.2.2.** The group of invertible irreducible $\text{Vec}_G$-module categories is isomorphic to $H^2(G, k^\times)$.

**Proof.** Without taking twisting into consideration invertible irreducible $\text{Vec}_G$-module categories correspond to invertible basis elements of the Burnside ring $\Omega(G)$. Suppose $\langle H \rangle \langle H' \rangle = \langle G \rangle$ in $\Omega(G)$. Then $\sum \langle H \cap ^a H' \rangle = \langle G \rangle$ which can happen only if there is a single double coset $HH'$ and if $H \cap ^a H' = G$, and this occurs only if $H = H' = G$. It follows from Proposition 8.2.1 that

$$\mathcal{M}(G, \mu) \otimes \mathcal{M}(G, \mu') = \mathcal{M}(G, \mu \mu')$$

Sending $\mathcal{M}(G, \mu)$ to $\mu$ gives the desired isomorphism.  

\[\square\]
8.3 Rep(G)-Mod fusion rules

In this section we use the results of the last section together with the equivalence of 2-categories \( Vec_G\text{-Mod} \to \text{Rep}(G)\text{-Mod} \) to derive fusion rules for the free \( \mathbb{Z}_+\)-ring generated by simple \text{Rep}(G)-module categories. The equivalence is defined by sending \( \mathcal{M} \mapsto \overline{\mathcal{M}} \) where

\[
\overline{\mathcal{M}} := \text{Fun}_{Vec}(Vec, \mathcal{M}).
\]

(24)

Observe that \( \text{Fun}_{Vec}(Vec, Vec) \) acts on \( \text{Fun}_{Vec}(\mathcal{M}, \mathcal{N}) \) on the right by the formula

\[
(F \otimes S)(M) = F(M) \otimes S(k)
\]

whenever \( M \in \mathcal{M} \) and \( S : Vec \to Vec \) is a \( Vec_G \)-module functor. \( F \otimes S \) is trivially a \( Vec_G \)-module functor:

\[
(F \otimes S)(g \otimes M) = (g \otimes F(M)) \otimes S(k)
= (g \otimes F(M)) \otimes (g \otimes S(k))
= g \otimes (F(M) \otimes S(k))
= g \otimes (F \otimes S)(M).
\]

The isomorphism is \( Vec_G \)-linearity of \( F \) and the second line follows from the fact that simple objects of \( Vec_G \) (one dimensional vector spaces) act trivially on \( Vec \). Let
$T : Vec \to Vec$ over $Vec_G$. Associativity of the action is also trivial:

$$(F \otimes ST)(M) = F(M) \otimes ST(k)$$

$$= F(M) \otimes S(k \otimes T(k))$$

$$= F(M) \otimes (S(k) \otimes T(k))$$

$$= (F(M) \otimes S(k)) \otimes T(k)$$

$$= (F \otimes S)(M) \otimes T(k) = ((F \otimes S) \otimes T)(M)$$

The second line is tensor product (composition) in $Fun_{Vec_G}(Vec, Vec)$ and the isomorphism is due to the canonical action of $Vec$ on $\mathcal{N}$ given by internal hom:

$$\text{Hom}_{\mathcal{N}}(V \otimes N, N) := \text{Hom}_{Vec}(V, \text{Hom}_{\mathcal{N}}(N, N)). \quad (25)$$

**Proposition 8.3.1.** For $H < G$ and $\mu \in H^2(H, k^\times)$ denote by $\text{Rep}_\mu(H)$ the category of projective representations of $H$ with Schur multiplier $\mu$. Then $\text{Rep}_\mu(H) \simeq \overline{\mathcal{M}}(H, \mu)$ as $\text{Rep}(G)$-module categories.

**Proof.** Send functor $F : Vec \to \mathcal{M}(H, \mu)$ to $F(k)$. $\text{Rep}(G)$-module structure on $\text{Rep}_\mu(H)$ is given by $\text{res} \otimes \text{id}$: for $V \in \text{Rep}(G)$ and $W \in \text{Rep}_\mu(H)$ the action is defined by $V \otimes W := \text{res}_H^G(V) \otimes W$ where $\otimes$ on the right is tensor product in $\text{Rep}_\mu(G)$. \qed

One of the main results of this section is the following theorem.

**Theorem 8.3.2.** The 2-equivalence $\mathcal{M} \leftrightarrow \overline{\mathcal{M}}$ between $(Vec_G$-Mod, $\otimes)$ and $(\text{Rep}(G)$-
Mod, $\otimes_{\text{Rep}(G)}$ is monoidal in the sense that

$$
\overline{\mathcal{M}} \otimes \overline{\mathcal{N}} \simeq \overline{\mathcal{M}} \otimes_{\text{Rep}(G)} \overline{\mathcal{N}}
$$

as Rep(G)-module categories.

The action of Rep(G) $\simeq \text{Fun}_{\text{Vec}_G}(\text{Vec}, \text{Vec})$ is given by composition of functors. Since the correspondence is an equivalence of 2-categories we may identify abelian categories of 1-cells:

$$
\text{Fun}_{\text{Vec}_G}(\mathcal{M}, \mathcal{N}) \simeq \text{Fun}_{\text{Rep}(G)}(\overline{\mathcal{M}}, \overline{\mathcal{N}}).
$$

(26)

In what follows we provide a few lemmas which show that useful formulas provided earlier for monoidal 2-categories hold also over the category of Vec$_G$-modules.

**Lemma 8.3.3.** The 2-equivalence $\mathcal{M} \mapsto \overline{\mathcal{M}}$ from Vec$_G$-Mod to Rep(G)-Mod when restricted to 1-cells is an equivalence of right Rep(G)-module categories.

**Proof.** The equivalence of 1-cells $\zeta : \text{Fun}_{\text{Vec}_G}(\mathcal{M}, \mathcal{N}) \simeq \text{Fun}_{\text{Rep}(G)}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ takes functor $F : \mathcal{M} \to \mathcal{N}$ over Vec$_G$ to the functor defined by $Q \mapsto FQ$ for Rep(G)-module functor $Q : \text{Vec} \to \mathcal{M}$. We must check that this correspondence respects Rep(G) action.

Any functor $E : \text{Vec} \to \text{Vec}$ over Vec$_G$ determines representation $E(k)$, and any representation $V$ determines functor $E^V(k) = V$. $V \in \text{Rep}(G) \simeq \overline{\text{Vec}}$ right-acts on $F \in \text{Fun}_{\text{Rep}(G)}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ by $(F \otimes V)(Q) = F(Q) \circ E^V$. Writing $\langle \zeta(F), Q \rangle$ for the
functor in \( N \) determined by \( F, Q \) we have, for \( W \in Vec, \)

\[
< \zeta(F \otimes E^V), Q > (W) = (F \otimes E^V)(Q)(W)
= FQE^V(W)
= < \zeta(F) \otimes E^V, Q > (W).
\]

\[\square\]

**Lemma 8.3.4.** Let \( M, N \) be left \( Vec_G \)-module categories. Then \( M \otimes N \simeq Fun(M^{op}, N) \) as left \( Vec_G \)-module categories.

**Proof.** Let \( M := M(H, \mu) \) and \( N := M(K, \sigma) \) as above. Define

\[
\Phi : M \otimes N \rightarrow Fun(M^{op}, N), \quad \Phi(M \otimes N)(M') := \text{Hom}(M', M) \otimes N.
\]

(27)

Clearly \( \Phi \) is an equivalence of abelian categories (see Lemma 2.3.2 for example) and it remains to show that it respects \( Vec_G \)-module structure. The category \( Fun(M^{op}, N) \) carries \( Vec_G \)-module structure \((g \otimes F)(M) := g \otimes F(g^{-1} \otimes M)\) for simple objects \( g \) in \( Vec_G \). Left action on \( M^{op} \) is given by \( X \otimes^{op} M = X \otimes M \) with inverse module associativity. We have

\[
(gh \otimes F)(M) = gh \otimes F(h^{-1}g^{-1} \otimes M)
\approx g \otimes (h \otimes F(h^{-1} \otimes (g^{-1} \otimes M)))
= g \otimes (h \otimes F)(g^{-1} \otimes M) = (g \otimes (h \otimes F))(M)
\]
where \( \simeq \) is \( \sigma(g, h)\mu^{-1}(h^{-1}, g^{-1}) \) which is cohomologous to \( \sigma(g, h)\mu(g, h) \), i.e. module associativity on functors is given by \( \mu\sigma \). For simple objects \( M, M' \) in \( \mathcal{M}, N \in \mathcal{N} \)

\[
(g \otimes \Phi(M \odot N))(M') = g \otimes (\text{Hom}(g^{-1} \otimes M', M) \otimes N)
\]

\[
\simeq \text{Hom}(M', g \otimes M) \otimes (g \otimes N)
\]

\[
= \Phi(g \otimes (M \odot N))(M')
\]

where \( \simeq \) is canonical. \( \Phi \) respects \( \text{Vec}_G \)-module structure.

\[\square\]

**Lemma 8.3.5.** \( \text{Fun}_{\text{Vec}_G}(\mathcal{M}, \mathcal{N}) \simeq \mathcal{M}^{\text{op}} \odot \mathcal{N} \) as right \( \text{Rep}(G) \)-module categories.

**Proof.** We have an equivalence \( \psi : \text{Fun}_{\text{Vec}_G}(\mathcal{M}, \mathcal{N}) \to \mathcal{M}^{\text{op}} \odot \mathcal{N}, F \mapsto \psi F \) where \( \psi F(V)(M) := F(M) \odot V \) whenever \( V \in \text{Vec}, M \in \mathcal{M} \) and where we have used Lemma 8.3.4 to express \( \mathcal{M}^{\text{op}} \odot \mathcal{N} \) as category of functors is an equivalence. \( \psi \) has quasi-inverse \( F \mapsto \psi(k) \): 

\[
<\psi(F \otimes V), W>(M) = (F(M) \otimes V) \odot W
\]

\[
\simeq F(M) \otimes (V \otimes W)
\]

\[
= \psi F(V \otimes W)(M)
\]

\[
= \psi F(E^V(W))(M) = <\psi F \circ E^V, W>(M).
\]

\[\square\]

**Lemma 8.3.6.** \( \mathcal{M}^{\text{op}} \simeq \mathcal{M}^{\text{op}} \) as \( \text{Rep}(G) \)-module categories.

**Proof.** \( \text{Fun}_{\text{Vec}_G}(\text{Vec}, \mathcal{M}^{\text{op}}) \simeq \text{Fun}_{\text{Vec}_G}(\mathcal{M}, \text{Vec}) \simeq \text{Fun}_{\text{Rep}(G)}(\mathcal{M}, \text{Rep}(G)) \) where
first \simeq is Lemma 8.3.4 and the second comes from the 2-equivalence. The first term is \( \mathcal{M}^{\text{op}} \) and the last is \( \mathcal{M}^{\text{op}} \).

Proof of Theorem 8.3.2. With notation as above,

\[
\mathcal{M} \otimes \mathcal{N} \simeq \text{Fun}_{\text{vcc}}(\mathcal{M}^{\text{op}}, \mathcal{N}) \\
\simeq \text{Fun}_{\text{Rep}(G)}(\mathcal{M}^{\text{op}}, \mathcal{N}) \\
\simeq \text{Fun}_{\text{Rep}(G)}(\mathcal{M}^{\text{op}}, \mathcal{N}) \simeq \mathcal{M} \otimes_{\text{Rep}(G)} \mathcal{N}.
\]

First line is Lemma 8.3.5, second is Lemma 8.3.3 and third is Lemma 8.3.6. □

Theorem 8.3.2, together with the observation in Remark 8.3.1, immediately gives a formula for \( \text{Rep}(G) \)-module fusion rules.

Corollary 8.3.7 (\( \text{Rep}(G) \)-Mod fusion rules). The twisted Burnside ring \( \Omega(G) \) is isomorphic to the ring \( K_0(\text{Rep}(G)\text{-Mod}) \) of equivalence classes of \( \text{Rep}(G) \)-module categories with multiplication induced by \( \otimes_{\text{Rep}(G)} \). That is, for irreducible \( \text{Rep}(G) \)-module categories \( \text{Rep}_{\mu}(H), \text{Rep}_{\sigma}(K) \) we have, as \( \text{Rep}(G) \)-module categories

\[
\text{Rep}_{\mu}(H) \otimes_{\text{Rep}(G)} \text{Rep}_{\sigma}(K) \simeq \bigoplus_{H \triangleleft K \subseteq G/K} \text{Rep}_{\mu \sigma a}(H \cap aK).
\]

Corollary 8.3.8. The group of invertible irreducible \( \text{Rep}(G) \)-module categories is isomorphic to \( H^2(G, k^\times) \).

Proof. The proof is equivalent to that of Corollary 8.2.2. □
Note 8.3.9. Corollary 8.3.8 generalizes Corollary 3.17(ii) in [ENO09] where it was given for finite abelian groups. Indeed when $A$ is abelian $\text{Vec}_A = \text{Rep}(A^*)$ for $A^*$ group homomorphisms $\text{Hom}(A, k^*)$. 
References


