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Bimodule categories and monoidal 2-structure

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BIMODULE CATEGORIES AND MONOIDAL 2-STRUCTURE

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ABSTRACT

BIMODULE CATEGORIES AND MONOIDAL 2-STRUCTURE:

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We define a notion of tensor product of bimodule categories and prove that with this product the 2-category of $\mathcal{C}$-bimodule categories for fixed tensor $\mathcal{C}$ is a monoidal 2-category in the sense of Kapranov and Voevodsky ([KV91]). We then provide a monoidal-structure preserving 2-equivalence between the 2-category of $\mathcal{C}$-bimodule categories and $Z(\mathcal{C})$-module categories (module categories over the center of $\mathcal{C}$). The (braided) tensor structure of $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ for (braided) fusion categories *over* braided fusion $\mathcal{D}$ is introduced. For a finite group G we show that de-equivariantization is equivalent to the tensor product over $\text{Rep}(G)$. The fusion rules for the Grothendeick ring of $\text{Rep}(G)$-module categories are derived and it is shown that the group of invertible $\text{Rep}(G)$-module categories is isomorphic to $H^2(G, k^\times)$, extending results in [ENO09].

INTRODUCTION

0.1 Generalities

Over the last century it has become evident that the study of algebraic structures from a module theoretic perspective is effective and powerful. The essential paradigm hinges on the observation that one structure may "act" on another and that in studying such actions one may learn something about the structures involved. The application of this basic notion has led to the development of a vast machinery of techniques and methods. In the theory of group representations, for example, one defines the action of a group on a vector space by specifying an association between elements of a group and linear transformations on a fixed space. Much can be understood about groups by making observations about the sorts of linear transformations which can arise by this process and in particular the traces of these linear maps (character theory). To study Lie algebras one defines an associative algebra as a certain quotient of the tensor algebra and then studies modules over this algebra. This notion also occurs naturally in more physical contexts, such as Boundary Conformal Field Theory (see for example [Car], [JF03], [VP01]). To any CFT is associated a ring-like object which acts on boundary conditions in a higher-dimensional space. Considerations about how similar constructions can be deformed have helped lead to the development of the theory of quantum groups, Hopf algebras and algebraic category theory, and have deep applications in theoretical physics (see [Maj02], [Str07], [FMS99] among many others).

It is beneficial to consider the ways in which modules interact. The collection of modules defined over a given structure will generally form a category with extra algebraic structure allowing the application of extended classical results and constructions from ring and group theory. In precisely this fashion one moves from a "one-dimensional theory" to an enriched categorical theory with analogous but subtler structures yielding analogous but more refined results. Thus the classical picture acts as a cartography for the new theory and provides a narrative over which it develops.

The categories under study are required to satisfy axioms making something akin to linear algebra possible (so called abelian categories). Fusion categories are defined to be abelian categories equipped with a monoidal structure (multiplication) that behaves nicely with respect to other important operations. The notion of "monoidal category" is an abstraction of the notion of a ring and is intended to capture ring-like properties on an axiomatic level. Similarly we may define symmetric r braided tensor categories as abstractions of a commutative ring, and module category as an abstracted module. Module categories, introduced by Bernstein in [Ber95] and studied in [Ost03], [EO04] among many others, form the basic objects of study in this thesis. The definition of a module category involves describing the action of a monoidal category just as classical modules describe actions of rings. Because we are dealing with more abstract structures the new axioms take the form of commuting diagrams whose vertices are objects and whose edges consist of appropriately defined maps which form part of the definition of the action. These maps dictate the appropriate associativity and unit constraints, in the categorical context, that one would see expressed in equations such as $(xy)z = x(yz)$ and $1x = x$ in algebra. As with rings and modules, one would like some meaningful way by which to relate pairs of module categories. There we have functions preserving module structure (linearity) and here we have functors preserving module category structure, so called module functors. One primary difference in the categorical setting is that here we have a way of relating pairs of module functors. Since module functors themselves are required to satisfy certain axioms (again taking the form of commuting diagrams) we may define module transformations as transformations preserving this structure in the appropriate fashion.

Just as modules over a fixed ring form a category, module categories over a fixed monoidal category form an appropriately enriched structure, called a 2-category. Just as in certain circumstances the category of modules over a fixed ring may itself have the structure of a monoidal category (under tensor product of modules) so may the associated 2-category of module categories have under certain conditions a monoidal structure making it a monoidal 2-category. As we move from monoidal category to monoidal 2-category the basic data expressed in diagrams (2-dimensional versions of equations in the lower-dimensional case) are replaced by 3-dimensional diagrams, *polytopes*, which represent restrictions on the ways cells of various levels are allowed to interact. Now instead of just 0-cells (objects) and 1-cells (morphisms) we have 2-cells (morphisms between morphisms). A priori there is no reason why the theory should fail to continue beyond level two yielding 3-cells, 4-cells etc. Although it is possible to define higher level structures leading to n-categoies, and even ∞-categories, we leave this to future endeavor.

The basic "nice" condition allowing us to define a tensor product between module categories occurs when we require that module categories are really *bimodule* categories. One instance in which this happens arises naturally when we stipulate that the underlying monoidal category is *braided*, a notion generalizing the idea of ring commutativity in an appropriately categorical way. In such a case we can define a tensor product of bimodule categories in a way reminiscent of the definition of the tensor product of modules; by stipulating an object universal for certain types of functors. If the bimodule categories in question are taken over a fixed monoidal category $\mathcal{C}$ we denote this new tensor product $\boxtimes_{\mathcal{C}}$. As the notation suggests $\boxtimes_{\mathcal{C}}$ reduces to a well known product for abelian categories developed by Deligne in [Del90]: in the case that $\mathcal{C} = Vec$, the category of vector spaces, we have $\boxtimes_{\mathcal{C}} = \boxtimes$.

A major part of this thesis has focused on asking and answering basic theoretical questions about $\boxtimes_{\mathcal{C}}$ and the associated monoidal 2-category of bimodule categories. It turns out that $\boxtimes_{\mathcal{C}}$ shares, in categorical analogue, many properties of the classical module theoretic tensor product, e.g. weak associativity, Frobenius reciprocity, and unitality with respect to the underlying monoidal category. These results, as in the classical case, provide powerful tools required for difficult calculations and form a basic starting point from which to develop algebraic aspects of the theory.

0.2 Thesis outline

First steps in defining this extended product involve defining *balanced functors* from the Deligne product of a pair of module categories. This approach mimics the classical definition of tensor product of modules as universal object for balanced or *middle linear* morphisms. Tensor product of module categories is then defined in terms of a universal functor factoring balanced functors. In Theorem 2.3.1 we prove that the tensor product exists; explicitly we prove that, for $\mathcal{M}$ a right $\mathcal{C}$-module category and $\mathcal{N}$ a left $\mathcal{C}$-module category there is a canonical equivalence $\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N} \simeq \underline{Fun}_{\mathcal{C}}(\mathcal{M}^{op}, \mathcal{N})$ where the category on the right is the appropriate category of $\mathcal{C}$-module functors.

In order to apply the tensor product of module categories we provide results in §2.3 giving 2-category analogues to classical formulas relating tensor product and hom-functor. In this setting the classical hom functor is replaced by the 2-functor $\underline{Fun}_{\mathcal{C}}$ giving categories of right exact $\mathcal{C}$-module functors.

In §4.1 we prove

Theorem 0.2.1. *For any monoidal category $\mathcal{C}$ the associated 2-category $\mathcal{B}(\mathcal{C})$ of $\mathcal{C}$-bimodule categories equipped with the tensor product $\boxtimes_{\mathcal{C}}$ becomes a (non-semistrict) monoidal 2-category in the sense of [KV91].*

In Chapter 5 we discuss the tensor product for a special class of module categories. Here we assume our module categories are equipped with the structure of fusion categories and that their centers contain a faithful image of some fixed braided fusion category $\mathcal{D}$ (such categories are said to be tensor *over* $\mathcal{D}$, see Definition 5.0.7). Under these circumstances the tensor product itself has the structure of a fusion category.

If the module categories are braided the tensor product is braided. We describe these structures explicitly.

As an immediate application we prove in Chapter 6 that de-equivariantization of a tensor category can be represented as a tensor product over Rep(G), the category of finite dimensional representations of a finite group G. Let A be the regular algebra in Rep(G). For tensor category $\mathcal{C}$ *over* Rep(G) the de-equivariantization $\mathcal{C}_G$ is defined to be the tensor category of A-modules in $\mathcal{C}$. This definition was given in [DGNO10] and studied extensively there. We prove

Theorem 0.2.2. *There is a canonical tensor equivalence $\mathcal{C}_G \simeq \mathcal{C} \boxtimes_{Rep(G)} Vec$ such that the canonical functor $\mathcal{C} \to \mathcal{C} \boxtimes_{Rep(G)} Vec$ is identified with the canonical (free module) functor $\mathcal{C} \to \mathcal{C}_G$.*

In §7 we introduce the notion of the center of a bimodule category generalizing the notion of the center of a monoidal category. We then prove a monoidal-structure preserving 2-equivalence between the monoidal 2-category of $\mathcal{C}$-bimodule categories, denoted $\mathcal{B}(\mathcal{C})$, and $Z(\mathcal{C})$-Mod, module categories over the center $Z(\mathcal{C})$:

Theorem 0.2.3. *There is a canonical monoidal equivalence between 2-categories $\mathcal{B}(\mathcal{C})$ and $Z(\mathcal{C})$-Mod.*

In §8 we give a second application of the monoidal structure in $\mathcal{B}(\mathcal{C})$. To be precise we show that, for arbitrary finite group G, fusion rules for Rep(G)-module categories over $\boxtimes_{\text{Rep}(G)}$ correspond to products in the twisted Burnside ring over G (see e.g. [OY01] and [Ros07]). As a side effect we show that the group of indecomposable invertible Rep(G)-module categories is isomorphic to $H^2(G, k^\times)$ thus generalizing results in [ENO09] given for finite abelian groups.

CHAPTER I

PRELIMINARIES, BACKGROUND

Very little in this section is new. Where it seemed necessary sources have been indicated. In most cases what is included here has become standard and so we omit references (suggested general references: [Mac00], [BK01], [Kas95] along with those already given in the introduction).

1.1 Abelian categories

As mentioned in the introduction we are interested in studying an enriched, *categorified* version of the theory of rings and modules. The proper context in which to do this should provide tools and structures allowing us to do something akin to linear algebra in this extended region of discourse. In this section we will outline the basic sorts of categories with which we will have occasion to work in later sections.

Definition 1.1.1. An *additive category* is a category $\mathcal{C}$ satisfying the following.

i) Every hom set has the structure of an abelian group with respect to which composition of morphisms is a group homorphism.

ii) $\mathcal{C}$ has a *zero object* 0 with the property that $\mathrm{Hom}(0,0) = 0$.

iii) (Existence of direct sums.) for any objects $X_1, X_2 \in \mathcal{C}$ there exists an object $Z := X_1 \oplus X_2 \in \mathcal{C}$ and morphisms $j_i : X_i \to Z$, $p_i : Z \to X_i$ for $i = 1,2$ such that $p_i \circ j_i = id_{X_i}$ and $j_1 \circ p_1 + j_2 \circ p_2 = id_Z$ and Z is unique object up to a unique isomorphism having this property.

The object Z in (*iii*) is called the direct sum of X_1 and X_2 and is denoted $X_1 \oplus X_2$. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is said to be an additive functor if the associated functions $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ are group homomorphisms.

Definition 1.1.2. Let k be any field. An additive category is *k-linear* if each hom set has the structure of a vector space over k with respect to which composition of morphisms is bilinear. A functor between k-linear categories is a k-linear functor if the associated functions between hom sets are linear transformations.

Let $\mathcal{C}$ be an additive category, and $f : X \to Y$ a morphism in $\mathcal{C}$. Then the *kernel* of f (if it exists) is the unique (up to a unique isomorphism) object K together with a morphism $\kappa : K \to X$ such that $f \circ \kappa = 0$ and if $\kappa' : K' \to X$ is any other morphism with this property there is a morphism $j : K' \to K$ with $\kappa \circ j = \kappa'$. Typically we denote the kernel of f by $\ker(f)$. Similarly one defines the *cokernel* of f to be an object $\mathrm{coker}(f)$ and a morphism $c : Y \to \mathrm{coker}(f)$ with the property that $c \circ f = 0$ and which is universal with respect to this property in a way analogous to the universality defining $\ker(f)$. If $\ker(f) = 0$ f is said to be *injective*, and *surjective* if $\mathrm{coker}(f) = 0$. In the case that f is injective we call X a subobject of Y and if f is surjective we call Y a quotient object of X. In an additive category there is no guarantee that kernels and cokernels exist. We will require that they do.

Definition 1.1.3. Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is an *abelian category* if it satisfies the further property that for any morphism $f : X \to Y$ there is a composition

$\ker(f) \xrightarrow{\kappa} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} \mathrm{coker}(f)$ with $j \circ i = f$ and $\mathrm{coker}(\kappa) = I = \ker(c)$. The object I is called the *image* of f. In particular, kernels and cokernels exist in an abelian category.

Definition 1.1.4. In an abelian category $\mathcal{C}$ an object is said to be *simple* if its only subobjects are itself and 0. If there are only finitely many isomorphism classes of simple objects then $\mathcal{C}$ is called *finite.* If an object Y can be written as the direct sum of simple objects Y is called *semisimple*, and $\mathcal{C}$ is called semisimple if all of its objects are semisimple.

Any category for which the class of objects form a set will be called *small.* An important theorem of Mitchell shows that the category of modules over a fixed ring is the typical example of an abelian category. We include it here without proof for the sake of completeness. See [Fre64] and [Mit64] for a more thorough discussion.

Theorem 1.1.5 (Mitchell). *Every small abelian category is equivalent to a full subcategory of the category of left modules over an associative unital ring. If the category is k-linear then the ring is a k-algebra.*

We end this section on abelian categories with a few definitions familiar from topology, the theory of modules, and representation theory which will be of importance to us in the sequel.

Definition 1.1.6. An *exact sequence* in an abelian category is a diagram of the form

$$\cdots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_i} X_i \xrightarrow{f_{i+1}} X_{i+1} \xrightarrow{f_{i+2}} \cdots$$

where $\ker(f_{i+1})$ is the image of f_i for every i. That is $f_{i+1}f_i = 0$. If all but finitely many of the X_i are 0 then this is called a *finite* exact sequence.

Definition 1.1.7. Functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *right exact* if F takes short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ to exact sequences $F(A) \to F(B) \to F(C) \to 0$ in $\mathcal{B}$. Similarly one defines left exact functors. Denote by $\underline{Fun}(\mathcal{A}, \mathcal{B})$ the category of right exact functors $\mathcal{A} \to \mathcal{B}$.

1.2 Monoidal and fusion categories

In the rest of this thesis all categories are assumed to be abelian and k-linear, have finite-dimensional hom spaces, and all functors are assumed to be additive and k-linear. Even though most of what we do here is valid over fields of positive characteristic we assume at the outset that k is a fixed field of characteristic 0.

Definition 1.2.1. A *monoidal category* $\mathcal{C}$ consists of the following: a category $\mathcal{C}$ containing an object 1 called the *unit* of $\mathcal{C}$, an exact-in-both-variables bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, natural isomorphisms $a : \otimes(\otimes \times id) \to \otimes(id \times \otimes)$, $r_X : X \otimes 1 \simeq X$, $\ell_X : 1 \otimes X \simeq X$ whenever $X \in \mathcal{C}$, required to satisfy the following commutative diagrams:

$$\begin{tikzcd}
 & ((WX)Y)Z \arrow[dl, "a_{W,X,Y}\otimes Z"'] \arrow[dr, "a_{WX,Y,Z}"] & \\
(W(XY))Z \arrow[d, "a_{W,XY,Z}"'] & & (WX)(YZ) \arrow[d, "a_{W,X,YZ}"] \\
W((XY)Z) \arrow[rr, "W\otimes a_{X,Y,Z}"'] & & W(X(YZ))
\end{tikzcd}
\qquad
\begin{tikzcd}
(X1)Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X\otimes Y"'] & & X(1Y) \arrow[dl, "X\otimes \ell_Y"] \\
 & XY &
\end{tikzcd}$$

for any objects $W, X, Y, Z \in \mathcal{C}$. Here, as in the sequel, we may abbreviate tensor products as juxtaposition in an effort to save space. The natural isomorphism a is called an *associativity constraint* and ℓ, r *unit constraints* of $\mathcal{C}$. The monoidal category $\mathcal{C}$ is said to be *strict* if all the natural isomorphisms a, r, ℓ are identity.

Remark 1.2.2. Denote by $\boxtimes$ the product of abelian categories introduced in [Del90]. This is an object in the category of abelian categories universal for right-exact in both variables bifunctors from the cartesian product category $\mathcal{C} \times \mathcal{D}$. If $(\mathcal{C}, \otimes, 1, a, \ell, r)$ and $(\mathcal{D}, \otimes, 1', a', \ell', r')$ are monoidal categories then $\mathcal{C} \boxtimes \mathcal{D}$ has the structure of a monoidal category as follows: $(X_1 \boxtimes X_2) \otimes (Y_1 \boxtimes Y_2) = (X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2)$ with associativity constraint $a \boxtimes a'$, unit object $1 \boxtimes 1'$ and unit constraints $\ell \boxtimes \ell'$, $r \boxtimes r'$.

Definition 1.2.3. Let $\mathcal{C} = (\mathcal{C}, \otimes, a, \ell, r, 1), \mathcal{D} = (\mathcal{D}, \otimes, a', \ell', r', 1')$ be monoidal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a *monoidal functor* if it comes with natural isomorphisms $f_{X,Y} : F(X \otimes Y) \simeq F(X) \otimes F(Y)$ and $u : F(1) \simeq 1'$ satisfying the following hexagon and squares for every $X, Y, Z \in \mathcal{C}$.

$$\begin{tikzcd}
F(XY)F(Z) \arrow[d, "f_{X,Y}\otimes id"'] & F((XY)Z) \arrow[l, "f_{XY,Z}"'] \arrow[r, "F(a_{X,Y,Z})"] & F(X(YZ)) \arrow[d, "f_{X,YZ}"] \\
(F(X)F(Y))F(Z) \arrow[dr, "a'_{F(X),F(Y),F(Z)}"'] & & F(X)(F(YZ)) \arrow[dl, "id\otimes f_{Y,Z}"] \\
& F(X)(F(Y)F(Z)) &
\end{tikzcd}$$

$$\begin{tikzcd}
F(X)\otimes F(1) \arrow[r, "id\otimes u"] & F(X)\otimes 1' \arrow[d, "r'_{F(X)}"] \\
F(X\otimes 1) \arrow[u, "f_{X,1}"] \arrow[r, "F(r_X)"'] & F(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(1)\otimes F(X) \arrow[r, "u\otimes id"] & 1'\otimes F(X) \arrow[d, "\ell'_{F(X)}"] \\
F(1\otimes X) \arrow[u, "f_{1,X}"] \arrow[r, "F(\ell_X)"'] & F(X)
\end{tikzcd}$$

In order to emphasize or designate the linearity constraint f for a functor F we may on occasion write (F, f). The functor is said to be a *strict* monoidal functor if the natural isomorphism f is identity.

Following terminology from regular category theory we will say that two monoidal categories are *monoidally equivalent* if there is a monoidal functor between them which is an equivalence. As it turns out, by MacLane's famous "strictness theorem," we may justifiably assume all monoidal categories to be strict. We include the statement here for completeness and because we will use it extensively in what follows.

Theorem 1.2.4 (MacLane strictness theorem). *Any monoidal category is monoidally equivalent to a strict one.*

A nice proof of Theorem 1.2.4 may be found in Joyal and Street's 1993 paper on braided tensor categories [JS93]. In what follows we will assome monoidal categories strict unless stated otherwise. The primary benefit of having such a theorem is that it provides notational convenience simplifying diagrams and calculations. For example it allows us to replace the expressions $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ with the now unambiguous expression $X \otimes Y \otimes Z$, and allows us to dispense with structural constraints.

Definition 1.2.5. Let $(F, f), (G, g) : \mathcal{C} \to \mathcal{D}$ be two monoidal functors. A *monoidal natural transformation* $\eta : F \to G$ is a natural transformation satisfying the rectangle

$$\begin{tikzcd}
F(X \otimes Y) \arrow[r, "\eta_{X \otimes Y}"] \arrow[d, "f_{X,Y}"'] & G(X \otimes Y) \arrow[d, "g_{X,Y}"] \\
F(X) \otimes F(Y) \arrow[r, "\eta_X \otimes \eta_Y"'] & G(X) \otimes G(Y)
\end{tikzcd}$$

for any $X, Y \in \mathcal{C}$.

As in many familiar classical situations (Rep(G), R-Mod for commutative R, pointed toplogical spaces, etc) there is a natural notion of duality. The following definition gives a categorical axiomatization of this concept.

Definition 1.2.6. Let $\mathcal{C}$ be a monoidal category, and let X be an object of $\mathcal{C}$. An object Y is said to be a *right dual* of X if there are morphisms $\mathrm{ev}_X : Y \otimes X \to 1$ and $\mathrm{coev}_X : 1 \to X \otimes Y$, called *evaluation* and *coevaluation*, such that both of the compositions

$$X = 1 \otimes X \xrightarrow{\mathrm{coev}_X} X \otimes Y \otimes X \xrightarrow{\mathrm{ev}_X} X \otimes 1 = X$$
$$Y = Y \otimes 1 \xrightarrow{\mathrm{coev}_X} X \otimes Y \otimes X \xrightarrow{\mathrm{ev}_X} 1 \otimes Y = Y$$

are equal to identity. Similarly one defines a *left dual* of X to be an object V together with morphisms $\mathrm{ev}'_X : X \otimes V \to 1$ and $\mathrm{coev}'_X : 1 \to V \otimes X$ making both of the compositions

$$X = X \otimes 1 \xrightarrow{\mathrm{coev}'_X} X \otimes V \otimes X \xrightarrow{\mathrm{ev}'_X} 1 \otimes X = X$$
$$V = 1 \otimes V \xrightarrow{\mathrm{coev}'_X} V \otimes X \otimes V \xrightarrow{\mathrm{ev}'_X} V \otimes 1 = V$$

identity.

It is well known that if X possesses any left (right) dual then it is unique up to a unique isomorphism. In this case the left (right) dual object of X is denoted ${}^*X$ (resp. X^*). Furthermore this process of associating to an object its duals, should such dual objects exist, extends to morphisms. Explicitly, if $f : X \to Y$ is a morphism between objects X, Y possessing right duals then define the right dual $f^* : Y^* \to X^*$ of f by the composition

$$Y^* = Y^* \otimes 1 \xrightarrow{coev_X} Y^* \otimes X \otimes X^* \xrightarrow{id \otimes f \otimes id} Y^* \otimes Y \otimes X^* \xrightarrow{\mathrm{ev}_Y} 1 \otimes X^* = X^*.$$

Similarly one defines the left dual ${}^*f : {}^*Y \to {}^*X$.

Definition 1.2.7. A monoidal category is said to be *rigid* if every object possesses both a right and a left dual object.

Definition 1.2.8. Let $\mathcal{C}$ be an abelian k-linear monoidal category having finite dimensional hom spaces with respect to which the bifunctor $\otimes$ is bilinear. $\mathcal{C}$ is called a *tensor category* if it is finite, rigid and has a simple unit object 1. $\mathcal{C}$ is called a *fusion category* if it is tensor and semisimple.

Also of interest is the notion of invertible object in a tensor category.

Definition 1.2.9. An object X is *invertible* if there is an object Y such that $X \otimes Y \simeq 1 \simeq Y \otimes X$. If every simple object is invertible the category is said to be *pointed.*

1.2.1 Braiding, center.

The definitions given thus far in §1.2 describe basic categorical analogues to the objects of study in the classical theory of rings. The next definition describes the categorical version of a commutative ring.

Definition 1.2.10. A monoidal category $\mathcal{C}$ is said to be *braided* if it is equipped with a class of natural isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V$$

satisfying the pair of hexagons

$$
\begin{tikzcd}
& U(VW) \arrow[r, "c_{U,VW}"] & (VW)U \arrow[dr, "a_{V,W,U}"] & \\
(UV)W \arrow[ur, "a_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes W"'] & & & V(WU) \\
& (VU)W \arrow[r, "a_{V,U,W}"'] & V(UW) \arrow[ur, "V\otimes c_{U,W}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (UV)W \arrow[r, "c_{UV,W}"] & W(UV) \arrow[dr, "a^{-1}_{W,U,V}"] & \\
U(VW) \arrow[ur, "a^{-1}_{U,V,W}"] \arrow[dr, "U\otimes c_{V,W}"'] & & & (WU)V \\
& U(WV) \arrow[r, "a^{-1}_{U,W,V}"'] & (UW)V \arrow[ur, "c_{U,W}\otimes V"'] &
\end{tikzcd}
$$

for all objects $U, V, W \in \mathcal{C}$.

When $\mathcal{C}$ is strict these reduce to commuting triangles giving equations

$$
\begin{aligned}
c_{U,V\otimes W} &= (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W) \\
c_{U\otimes V,W} &= (c_{U,W} \otimes id_V)(id_U \otimes c_{V,W}).
\end{aligned}
$$

In any braided monoidal category the isomorphisms $c_{X,Y}, c_{Y,X}$ are composable. We adapt the following definition from [Mug00], [Mug03].

Definition 1.2.11. Two objects X, Y in a braided monoidal category are said to *centralize* each other if $c_{X,Y}c_{Y,X} = id_{Y\otimes X}$.

Let $\mathcal{D}$ be a fusion subcategory of a braided fusion category $\mathcal{C}$. Following [DGNO10] we make the following definition.

Definition 1.2.12. The *centralizer* $\mathcal{D}'$ of $\mathcal{D}$ is the full subcategory of objects of $\mathcal{C}$ that centralize each objects of $\mathcal{D}$. The centralizer $\mathcal{C}'$ is sometimes called the *Müger center* of $\mathcal{C}$.

In the next two examples G is a finite group.

Example 1.2.13. Rep(G), the category of finite dimensional representations of G, is a braided tensor category with the usual tensor product.

Example 1.2.14. The category Vec_G^ω of finite dimensional G-graded vector spaces twisted by $\omega \in H^3(G, k^\times)$ is a rigid monoidal category. Simple objects are given by k_g (g^{th} component k, 0 elsewhere) with unit object k_1. Associativity is given by $a_{k_g,k_h,k_m} = \omega(g,h,m)id$ on simple objects, tensor product is defined by

$$(V \otimes W)_g = \bigoplus_{hk=g} V_h \otimes W_k$$

and $(V^*)_g = ({}^*V)_g = V_{g^{-1}}$. In general Vec_G^ω is not braided.

Definition 1.2.15. The *center* $Z(\mathcal{C})$ of a monoidal category $\mathcal{C}$ is the category having as objects pairs (X, c) where $X \in \mathcal{C}$ and for every $Y \in \mathcal{C}$ $c_Y : Y \otimes X \to X \otimes Y$ is a family of natural isomorphisms satisfying the hexagon

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "c_{X Y,Z}"] & Z \otimes (X \otimes Y) \arrow[dr, "a^{-1}_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "a^{-1}_{X,Y,Z}"] \arrow[dr, "id_X \otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \\
 & X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes id_Y"'] &
\end{tikzcd}$$

for all $Y, Z \in \mathcal{C}$. Here a is the associativity constraint for the monoidal structure in $\mathcal{C}$. A morphism $(X, c) \to (X', c')$ is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ satisfying the equation $c'_Y(f \otimes id_Y) = (id_Y \otimes f)c_Y$ for every $Y \in \mathcal{C}$.

The center $Z(\mathcal{C})$ has the structure of a monoidal category as follows. Define the tensor product $(X,c)\otimes(X',c') = (X\otimes X', \tilde{c})$ where $\tilde{c}$ is defined by the composition

$$\begin{array}{ccccc} Y\otimes(X\otimes X') & \xrightarrow{a^{-1}_{Y,X,X'}} & (Y\otimes X)\otimes X' & \xrightarrow{c_Y} & (X\otimes Y)\otimes X' \\ \downarrow{\scriptstyle \tilde{c}_Y} & & & & \downarrow{\scriptstyle a_{X,Y,X'}} \\ (X\otimes X')\otimes Y & \xleftarrow[a^{-1}_{X,X',Y}]{} & X\otimes(X'\otimes Y) & \xleftarrow[c'_Y]{} & X\otimes(Y\otimes X') \end{array}$$

If r and ℓ are the right and left unit constraints for the monoidal structure in $\mathcal{C}$ then the unit object for the monoidal structure in $Z(\mathcal{C})$ is given by $(1, r^{-1}\ell)$ as one may easily check. Suppose now that $\mathcal{C}$ is rigid and $X\in\mathcal{C}$ has right dual X^* (recall Definition 1.2.6). Then $(X,c)\in Z(\mathcal{C})$ has right dual $(X^*,\bar{c})$ where $\bar{c}_Y := (c^{-1}_{{}^*Y})^*$ and ${}^*Y$ is the left dual of Y. One may also check that $Z(\mathcal{C})$ is braided by $c_{(X,c)\otimes(X',c')} := c'_X$.

There is a canonical inclusion of monoidal category $\mathcal{C}$ into its center given by $X\mapsto (X,c_X)$. It is well known that the center $Z(\mathcal{C})$ is in some sense "larger" than $\mathcal{C}$. This differs from the classical analogue in which a ring contains its center. We generalize the notion of center in §7.

1.2.2 Pre-metric groups

Everything in this subsection may be found in [DGNO10]. We refer the reader to [DGNO10], [Kas95] and [BK01] for definitions and other information relating to braided fusion categories.

Recall that a *quadratic form* on an abelian group G having values in an abelian group B is a map $q: G\to B$ such that $q(g^{-1}) = q(g)$ and the symmetric function $b(g,h) := \frac{q(gh)}{q(g)q(h)}$ is bimultiplicative. We call $b: G\times G\to B$ the bimultiplicative form associated to q. If $B = k^\times$ we call b the bicharacter associated to q.

Definition 1.2.16. A *pre-metric group* is a pair (G, q) where G is a finite abelian group and $q : G \to k^\times$ is a quadratic form. A morphism of pre-metric groups $(G_1, q_1) \to (G_2, q_2)$ is a homomorphism $\varphi : G_1 \to G_2$ such that $q_2 \circ \varphi = q_1$.

The set of isomorphism classes of the simple objects of any pointed braided fusion category $\mathcal{C}$ form a group G. For $g \in G$ denote by $q(g) \in k^\times$ the braiding $c_{X,X} \in \mathrm{Aut}(X \otimes X)$ where X is in g. Then $g \mapsto q(g)$ is a quadratic form $G \to k^\times$. In this way $\mathcal{C}$ determines the pre-metric group (G, q).

Conversely every pre-metric group (G, q) determines a pointed braided fusion category $\mathcal{C}(G, q)$ as follows. As a fusion category $\mathcal{C}(G, q)$ is Vec_G, the category of (finite-dimensional) G-graded vector spaces. For X homogeneous object of degree g define the twist $\theta_X = q(g)$. Then the braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfies

$$c_{X,Y} c_{Y,X} = b(g, h) id_{Y \otimes X} \tag{1}$$

where b is the bicharacter determined by q. In the special case that q comes from a bicharacter $\beta : G \times G \to k^\times$ via the equation $q(x) = \beta(x, x)$, the associated braiding is $c_{X,Y} = \beta(g, h)\tau$ for τ the linear twist. These two constructions define reciprocal equivalences between the category of pre-metric groups and the (truncated 2-) category of pointed braided fusion categories.

1.3 Module categories

In §1.2 we described the basic objects of study for a categorical version of classical ring theory. In this section we will define the categorical analogue of the classical theory of modules. The first definition is crucial for this thesis.

Definition 1.3.1. Let $\mathcal{C}$ be a monoidal category. A *left $\mathcal{C}$-module category* (M, μ) is a category $\mathcal{M}$ together with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and a family of natural isomorphisms $\mu_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$, $\ell_M : 1 \otimes M \to M$ for $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$ subject to the coherence diagrams

$$\begin{tikzcd}
 & ((WX)Y)M \arrow[dl, "\mu_{W,X,Y} \otimes M"'] \arrow[dr, "\mu_{WX,Y,M}"] & \\
(W(XY))M \arrow[d, "\mu_{W,XY,M}"'] & & (WX)(YM) \arrow[d, "\mu_{W,X,YM}"] \\
W((XY)M) \arrow[rr, "W \otimes \mu_{X,Y,M}"'] & & W(X(YM))
\end{tikzcd}
\qquad
\begin{tikzcd}
(X1)M \arrow[rr, "\mu_{X,1,M}"] \arrow[dr, "r_X \otimes M"'] & & X(1M) \arrow[dl, "X \otimes \ell_M"] \\
 & XM &
\end{tikzcd}$$

Similarly one defines the structure of *right* module category on $\mathcal{M}$. If the structure maps are identity we say $\mathcal{M}$ is *strict* as a module category over $\mathcal{C}$.

Example 1.3.2. Any monoidal category $\mathcal{C}$ is a module category over itself with action given by monoidal structure. This is referred to as the *regular* module category structure on $\mathcal{C}$.

Example 1.3.3. Let G be a finite group with subgroup H. For 2-cocycle $\mu \in H^2(H, k^\times)$ the category $\mathrm{Rep}_\mu(H)$ of projective representations of H corresponding to Schur multiplier μ constitutes a $\mathrm{Rep}(G)$-module category with module category structure defined by $W \otimes V := \mathrm{res}^G_H(W) \otimes V$ whenever $W \in \mathrm{Rep}(G)$, $V \in \mathrm{Rep}_\mu(H)$ and $\mathrm{res} : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is the restriction functor.

For any $X \in \mathcal{C}$ we get a functor $L_X : \mathcal{M} \to \mathcal{M}$ given by $M \mapsto X \otimes M$ (left multiplication by X). It is natural to ask about the existence of adjoints of L_X. The following definition introduces a convenient technical tool for dealing with module categories. For the next definition assume $\mathcal{M}$ is a $\mathcal{C}$-module category for semisimple $\mathcal{C}$. Denote by Vec the braided tensor category of finite dimensional vector spaces.

Definition 1.3.4. For $M, N \in \mathcal{M}$ the *internal hom* $\underline{\mathrm{Hom}}(M, N)$ is defined to be the object in $\mathcal{C}$ representing the functor $\mathrm{Hom}_{\mathcal{M}}(\_\_ \otimes M, N) : \mathcal{C} \to Vec$. That is, for any object $X \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{M}}(X \otimes M, N) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, N))$$

naturally in Vec. It follows from Yoneda's Lemma that $\underline{\mathrm{Hom}}(M, N)$ is well defined up to a unique isomorphism and $\underline{\mathrm{Hom}}(-,-)$ is a bifunctor.

Definition 1.3.5. For $\mathcal{M}, \mathcal{N}$ left $\mathcal{C}$-module categories a functor $F : \mathcal{M} \to \mathcal{N}$ is said to be a *$\mathcal{C}$-module functor* if F comes equipped with a family of natural isomorphisms $f_{X,M} : F(X \otimes M) \to X \otimes F(M)$ satisfying the coherence diagrams

$$\begin{tikzcd}
 & F((XY)M) \arrow[dl, "F(\mu_{X,Y,M})"'] \arrow[dr, "f_{XY,M}"] & \\
F(X(YM)) \arrow[d, "f_{X,YM}"'] & & (XY)F(M) \arrow[d, "\mu_{X,Y,F(M)}"] \\
XF(YM) \arrow[rr, "X\otimes f_{Y,M}"'] & & X(Y(F(M)))
\end{tikzcd}
\qquad
\begin{tikzcd}
F(1M) \arrow[rr, "F(\ell_M)"] \arrow[dr, "f_{1,M}"'] & & F(M) \\
 & 1F(M) \arrow[ur, "\ell_{F(M)}"'] &
\end{tikzcd}$$

whenever $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$. We may write (F, f) when referring to such a functor. A natural transformation $\tau : F \Rightarrow G$ for bimodule functors $(F, f), (G, g)$:

$\mathcal{M} \to \mathcal{N}$ is said to be a *module natural transformation* whenever the diagram

$$\begin{tikzcd} F(X \otimes M) \arrow[r, "\tau_{X\otimes M}"] \arrow[d, "f_{X,M}"'] & G(X \otimes M) \arrow[d, "g_{X,M}"] \\ X \otimes F(M) \arrow[r, "id_X \otimes \tau_{X \otimes N}"'] & X \otimes G(M) \end{tikzcd}$$

commutes for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

In what follows we will have occasion to deal with categories of module functors. We fix notation now.

Definition 1.3.6. The category of left $\mathcal{C}$-module functors from $\mathcal{M} \to \mathcal{N}$ with morphisms given by module natural transformations will be denoted $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$. The subcategory of *right-exact* $\mathcal{C}$-module functors (recall Definition 1.1.7) will be denoted $\underline{Fun_{\mathcal{C}}}(\mathcal{M}, \mathcal{N})$.

It is known that the category $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is abelian. Furthermore if $\mathcal{M}, \mathcal{N}$ are semisimple then so is $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ (see [ENO05] for details).

In much of this thesis we will be concerned with categories for which there are left *and* right module structures which interact in a consistent and predictable way. In the next subsection we will discuss this in more detail and for now simply give a definition.

Definition 1.3.7. Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. $\mathcal{M}$ is said to be a $(\mathcal{C}, \mathcal{D})$*-bimodule category* if $\mathcal{M}$ is a $\mathcal{C} \boxtimes \mathcal{D}^{op}$-module category. If $\mathcal{M}$ and $\mathcal{N}$ are $(\mathcal{C}, \mathcal{D})$-bimodule categories call $F : \mathcal{M} \to \mathcal{N}$ a $(\mathcal{C}, \mathcal{D})$*-bimodule functor* if it is a $\mathcal{C} \boxtimes \mathcal{D}^{op}$-module functor.

Recall MacLane's strictness theorem for monoidal categories stating that every monoidal category is equivalent to a strict one (Theorem 1.2.4). Next we prove a generalized version for module categories which reduces to the monoidal strictness theorem in the regular module case. Our proof mimics the proof of the monoidal strictness theorem found in [JS93].

Theorem 1.3.8. *Any module category is module equivalent to a strict module category.*

Proof. Let $(\mathcal{M}, \mu, r)$ be a right $\mathcal{C}$-module category for some strict monoidal category $\mathcal{C}$. The strategy is to show that $\mathcal{M}$ is module equivalent to a $\mathcal{C}$-module category $\mathcal{M}'$ which is defined to be a category of functors on which $\mathcal{C}$ acts by functor composition and which is therefore strict. We begin by recalling that $\mathcal{C}$ is monoidally equivalent to the category of $\mathcal{C}$-module endofunctors $Fun_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ with equivalence given by $X \mapsto F^X$. $F^X : \mathcal{C} \to \mathcal{C}$ is the functor sending $1 \mapsto X$ (1 is unit object in $\mathcal{C}$): $F^X(Y) = X \otimes Y$.

Define $\mathcal{M}'$ to have objects given by pairs (F, f) where F is a functor $\mathcal{C} \to \mathcal{M}$ and $f_{X,Y} : F(X) \otimes Y \to F(X \otimes Y)$ is a natural isomorphism in $\mathcal{M}$ satisfying the diagram

$$\begin{tikzcd} F(X) \otimes (Y \otimes Z) \arrow[r, "f_{X,Y\otimes Z}"] & F(X \otimes Y \otimes Z) \\ (F(X) \otimes Y) \otimes Z \arrow[u, "\mu_{F(X),Y,Z}"] \arrow[r, "f_{X,Y} \otimes id"'] & F(X \otimes Y) \otimes Z \arrow[u, "f_{X\otimes Y,Z}"'] \end{tikzcd}$$

for every triple $X, Y, Z \in \mathcal{C}$. In short, F is a *right* $\mathcal{C}$-module functor with module linearity given by f. A morphism $\theta : (F^1, f^1) \to (F^2, f^2)$ in $\mathcal{M}'$ is defined to be a natural transformation $\theta : F^1 \to F^2$ satisfying the diagram in Definition 1.3.5 making it a module natural transformation. Composition in $\mathcal{M}'$ is vertical composition of natural transformations.

Now note that $\mathcal{M}'$ is a right $\mathcal{C}$-module category: for $X \in \mathcal{C}$ and $(F, f) \in \mathcal{M}'$ define $(F, f) \otimes X := (F \circ F^X, f^X)$ where f^X is defined by

$$FF^X(Y) \otimes Z = F(X \otimes Y) \otimes Z \xrightarrow{f_{X\otimes Y, Z}} F(X \otimes Y \otimes Z) = FF^X(Y \otimes Z).$$

Note that the action of $\mathcal{C}$ on $\mathcal{M}'$ is strict since composition of functors is strictly associative and $F^1 = id$. We show that $\mathcal{M}$ is module equivalent to $\mathcal{M}'$.

For $M \in \mathcal{M}$ define functor $L_M : \mathcal{C} \to \mathcal{M}$ by left M multiplication in $\mathcal{C}$, i.e. $X \mapsto M \otimes X$. This allows us to define functor $L : \mathcal{M} \to \mathcal{M}'$ by

$$L(M) := (L_M, \mu_{M,-,-}).$$

It is evident that $L(M)$ is an object in $\mathcal{M}'$: the diagram required of $\mu_{M,-,-}$ is precisely the pentagon in the definition of module category. We show that L is both essentially surjective and fully faithful.

To see essential surjectivity observe that any $(F, f) \in \mathcal{M}'$ is isomorphic to $L_{F(1)}$. Indeed $f_{1,X} : L_{F(1)}(X) = F(1) \otimes X \simeq F(X)$ for any $X \in \mathcal{C}$, and $f_{1,-}$ is natural.

Next let $\theta : L_M \to L_N$ be a morphism in $\mathcal{M}'$ for $M, N \in \mathcal{M}$. Define the morphism $\varphi : M \to N$ in $\mathcal{M}$ by the composition

$$\varphi := M \xrightarrow{r_M^{-1}} M \otimes 1 \xrightarrow{\theta_1} N \otimes 1 \xrightarrow{r_N} N.$$

We claim that for all $Z \in \mathcal{C}$ one has $\theta_Z = \varphi \otimes Z$, whence $\theta = L(\varphi)$ and L is thereby full. To see this consider the following diagram.

$$\begin{tikzcd}
M \otimes Z \arrow[r, "r_M^{-1} \otimes Z"] \arrow[d, "\varphi \otimes Z"'] & (M \otimes 1) \otimes Z \arrow[r, "\mu_{M,1,Z}"] \arrow[d, "\theta_1 \otimes Z"] & M \otimes (1 \otimes Z) \arrow[r, "M \otimes \ell_Z"] \arrow[d, "\theta_{1 \otimes Z}"] & M \otimes Z \arrow[d, "\theta_Z"] \\
N \otimes Z \arrow[r, "r_N^{-1} \otimes Z"'] & (N \otimes 1) \otimes Z \arrow[r, "\mu_{N,1,Z}"'] & N \otimes (1 \otimes Z) \arrow[r, "N \otimes \ell_Z"'] & N \otimes Z
\end{tikzcd}$$

Rectangle on the left is definition of φ, middle rectangle commutes since θ is a morphism in $\mathcal{M}'$, right rectangle commutes on naturality of θ. Top and bottom horizontal compositions are identity (two applications of the triangle axiom which forms a part of the definition of module category). Thus perimeter is identical to the equation $\varphi \otimes Z = \theta_Z$ and L is full. On the other hand if $L(f) = L(g)$ for any morphisms $f, g \in \mathcal{M}$ then the square of naturality for r implies $f = g$, and L is also therefore faithful. This completes the proof that L is an equivalence.

We now show that L is a module functor and finish the proof of the theorem. Define natural isomorphism

$$J_{M,Y} := \mu_{M,Y,-} : (L(M \otimes Y), \mu_{M \otimes Y,-,-}) \to (L(M) \otimes Y, \mu_{M,Y \otimes -,-}).$$

Pentagon in the definition of module category implies that $J_{M,Y}$ is a morphism in $\mathcal{M}'$ *and* that J is a module functor. We are done. $\square$

1.3.1 Bimodule categories

For right $\mathcal{C}$-module category $\mathcal{M}$ having module associativity μ define $\tilde{\mu}_{X,Y,M} = \mu_{M,{}^*Y,{}^*X}$. Then $\mathcal{M}^{op}$ has left $\mathcal{C}$-module category structure given by $(X, M) \mapsto M \otimes {}^*X$ with module associativity $\tilde{\mu}^{-1}$. Similarly, if $\mathcal{M}$ has left $\mathcal{C}$-module structure with associativity σ then $\mathcal{M}^{op}$ has right $\mathcal{C}$-module category structure $(M, Y) \mapsto Y^* \otimes M$ with associativity $\tilde{\sigma}^{-1}$ for $\tilde{\sigma}_{M,X,Y} := \sigma_{Y^*,X^*,M}$.

Proposition 1.3.9. *These actions determine a $(\mathcal{D}, \mathcal{C})$-bimodule structure*

$$(Y \boxtimes X, M) \mapsto X^* \otimes M \otimes {}^*Y$$

on $\mathcal{M}^{op}$ whenever $\mathcal{M}$ has $(\mathcal{C}, \mathcal{D})$-bimodule structure. If γ are the bimodule coherence isomorphisms for the left/right module structures in $\mathcal{M}$ (see Proposition 1.3.10), then $\tilde{\gamma}_{Y,M,X} = \gamma_{X^*,M,{}^*Y}$ are those for $\mathcal{M}^{op}$.

In the sequel whenever $\mathcal{M}$ is a bimodule category $\mathcal{M}^{op}$ will always refer to $\mathcal{M}$ with the bimodule structure described in Proposition 1.3.9.

Proposition 1.3.10. *Let $\mathcal{C}$, $\mathcal{D}$ be strict monoidal catgories. Suppose $\mathcal{M}$ has both left $\mathcal{C}$-module and right $\mathcal{D}$-module category structures μ^l, μ^r and a natural family of isomorphisms $\gamma_{X,M,Y} : (X \otimes M) \otimes Y \to X \otimes (M \otimes Y)$ for X in $\mathcal{C}$, Y in $\mathcal{D}$ making* *the pentagons*

$$\begin{tikzcd}
((XY)M)Z \arrow[r, "\gamma"] \arrow[d, "\mu^l\otimes id"'] & (XY)(MZ) \arrow[dd, "\mu^l"] \\
(X(YM))Z \arrow[d, "\gamma"'] & \\
X((YM)Z) \arrow[r, "id\otimes\gamma"'] & X(Y(MZ))
\end{tikzcd}
\quad
\begin{tikzcd}
(XM)(YZ) \arrow[r, "\gamma"] \arrow[d, "\mu^r"'] & X(M(YZ)) \arrow[dd, "id\otimes\mu^r"] \\
((XM)Y)Z \arrow[d, "\gamma\otimes id"'] & \\
(X(MY))Z \arrow[r, "\gamma"'] & X((MY)Z)
\end{tikzcd}
\quad
\begin{tikzcd}
(1M)1 \arrow[r, "\gamma_{1,M,1}"] \arrow[d, "\ell_M"'] & 1(M1) \arrow[dd, "r_M"] \\
M1 \arrow[d, "r_M"'] & \\
M & 1M \arrow[l, "\ell_M"]
\end{tikzcd}$$

commute. Then $\mathcal{M}$ has canonical $(\mathcal{C},\mathcal{D})$-bimodule category structure.

Proof. Throughout abbreviate $\overline{X} := X_1 \boxtimes X_2$ in $\mathcal{C}\boxtimes\mathcal{D}^{op}$. Suppose given μ^l, μ^r and γ as in the statement of the proposition. Observe that $\mathcal{C}\boxtimes\mathcal{D}^{op}$ acts on $\mathcal{M}$ by $(X\boxtimes Y)\otimes' M := (X\otimes M)\otimes Y$ where the $\otimes$ on the right are the given module structures assumed for $\mathcal{M}$. For $M\in\mathcal{M}$ define natural isomorphism $\mu : \otimes'(id_{\mathcal{C}\boxtimes\mathcal{D}^{op}}\times\otimes')\to \otimes'(\otimes'\times id_{\mathcal{M}})$ by the composition

$$\mu_{\overline{X},\overline{Y},M} = (\gamma_{X_1,Y_1M,Y_2}\otimes id_{Y_2})(\mu^l_{X_1,Y_1,M}\otimes id_{Y_2X_2})(\mu^r_{(X_1Y_1)M,Y_2,X_2}).$$

Thus $\mu_{\overline{X},\overline{Y},M} : ((X_1\otimes Y_1)\otimes M)\otimes(Y_2\otimes Y_1)\to(X_1\otimes((Y_1\otimes M)\otimes Y_2))\otimes X_2$ in the language of left and right module structures .

Consider the partitioned diagram below whose periphery is the appropriate diagram for μ written as the composition which defines it. To save space we elide identity morphisms, morphism subscripts, and objects occurring at internal vertices. Label the subdiagrams Di.

$$\begin{tikzcd}
((X_1Y_1Z_1)M)(Z_2Y_2X_2) \arrow[r, "\mu^r"] \arrow[d, "\mu^r"'] \arrow[dr, phantom, "D1"] & (((X_1Y_1Z_1)M)Z_2)(Y_2X_2) \arrow[rr, "\mu^l"] \arrow[d, "\mu^r"] \arrow[dr, phantom, "D2"] & & (((X_1Y_1)(Z_1M))Z_2)(Y_2X_2) \arrow[dl, "\mu^r"'] \arrow[d, "\gamma"] \\
(((X_1Y_1Z_1)M)(Z_2Y_2))X_2 \arrow[r, "\mu^r"] \arrow[d, "\mu^l"'] \arrow[dr, phantom, "D3"] & {} \arrow[r, "\mu^l"] \arrow[d, "\mu^l"] \arrow[dr, phantom, "D4"] & {} \arrow[d, "\mu^l"'] \arrow[dr, "\gamma"'] \arrow[r, phantom, "D5"] & ((X_1Y_1)((Z_1M)Z_2))(Y_2X_2) \arrow[d, "\mu^l"] \\
(((X_1((Y_1Z_1)M))(Z_2Y_2))X_2 \arrow[r, "\mu^r"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "D6"] & {} \arrow[r, "\mu^l"] \arrow[d, "\gamma"] \arrow[dr, phantom, "D7"] & {} \arrow[d, "\gamma"] \arrow[dr, phantom, "D8"] & (((X_1Y_1)((Z_1M)Z_2))Y_2)X_2 \arrow[d, "\mu^l"] \\
(X_1(((Y_1Z_1)M)(Z_2Y_2))X_2 \arrow[d, "\mu^r"'] & {} \arrow[r, "\mu^l"] \arrow[dl, "\gamma"'] \arrow[dr, phantom, "D9"] & {} \arrow[r, "\gamma"] \arrow[d, "\gamma"] \arrow[dr, phantom, "D10"] & (X_1(Y_1((Z_1M)Z_2)))Y_2)X_2 \arrow[d, "\gamma"] \\
(X_1(((Y_1Z_1)M)Z_2)Y_2)X_2 \arrow[rr, "\mu^l"'] & & X_1(((Y_1(Z_1M))Z_2)Y_2)X_2 \arrow[r, "\gamma"'] & (X_1((Y_1((Z_1M)Z_2))Y_2)X_2
\end{tikzcd}$$

Diagrams $D1$, $D4$ are the associativity diagrams for μ^r, μ^l, diagrams $D2$, $D3$, $D5$, $D7$, $D9$, $D10$ are naturality diagrams for either μ^l or γ, and diagrams $D6$ and $D8$ are the second and first diagrams given at the beginning of this remark. $\square$

Remark 1.3.11. For bimodule structure $(\mathcal{M}, \mu)$, γ is given by $\gamma_{X,M,Y} = \mu_{X\boxtimes 1, 1\boxtimes Y, M}$ over the inherent left and right module category structures. In this way we get the converse of Proposition 1.3.10: every bimodule structure gives separate left and right module category structures and the special constraints described therein in a predictable way.

Remark 1.3.12. We saw in Proposition 1.3.10 that bimodule category structure can be described separately as left and right structures which interact in a predictable fashion. We make an analogous observation for bimodule functors. Let $F : (\mathcal{M}, \gamma) \to (\mathcal{N}, \delta)$ be a functor with left $\mathcal{C}$-module structure f^ℓ and right $\mathcal{D}$-module structure f^r, where $(\mathcal{M}, \gamma)$ and $(\mathcal{N}, \delta)$ are $(\mathcal{C}, \mathcal{D})$-bimodule categories with bimodule consistency isomorphisms γ, δ as above. Then F is a $(\mathcal{C}, \mathcal{D})$-bimodule functor iff the hexagon

$$\begin{tikzcd}
F(X \otimes (M \otimes Y)) \arrow[d, "f^{\ell}_{X,M\otimes Y}"'] & F((X \otimes M) \otimes Y) \arrow[l, "F\gamma_{X,M,Y}"'] \arrow[r, "f^{r}_{X\otimes M,Y}"] & F(X \otimes M) \otimes Y \arrow[d, "f^{\ell}_{X,M}\otimes Y"] \\
X \otimes F(M \otimes Y) \arrow[r, "X\otimes f^{r}_{M,Y}"'] & X \otimes (F(M) \otimes Y) & (X \otimes F(M)) \otimes Y \arrow[l, "\delta_{X,F(M),Y}"]
\end{tikzcd}$$

commutes for all X in $\mathcal{C}$, Y in $\mathcal{D}$, M in $\mathcal{M}$. The proof is straightforward and so we do not include it.

1.3.2 Exact module categories

It is desirable to restrict the general study of module categories in order to render questions of classification tractable. In their beautiful paper [EO04] Etingof and Ostrik suggest the class of *exact* module categories as an appropriate restriction intermediary between the semisimple and general (non-semisimple, possibly non-finite) cases. Let P be an object in any abelian category. We say P is *projective* if the functor $\mathrm{Hom}(P, -)$ is exact.

Definition 1.3.13 ([EO04]). A module category $\mathcal{M}$ over tensor category $\mathcal{C}$ is said to be *exact* if for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$ the object $P \otimes M$ is projective.

It turns out that module category exactness is equivalent to exactness of certain functors. We will not require the general formulation here but give the next lemma for exact module categories because exactness ensures adjoints for module functors.

Lemma 1.3.14. *For $\mathcal{M}, \mathcal{N}$ exact left $\mathcal{C}$-module categories the association*

$$Func_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \xrightarrow{ad} Func_{\mathcal{C}}(\mathcal{N}, \mathcal{M})^{op}$$

sending F to its left adjoint is an equivalence of abelian categories. If $\mathcal{M}, \mathcal{N}$ are bimodule categories then this equivalence is bimodule.

Proof. By Lemma 3.21 in *loc. cit.* such adjoints exist and since adjoints are unique up to isomorphism the association is bijective on ismomorphism classes of objects (functors). For $F : \mathcal{M} \to \mathcal{N}$ linearity of F^{ad} over $\mathcal{C}$ comes from that for F via the composition

$$\begin{aligned}
\alpha_R &:= \operatorname{Hom}_{\mathcal{M}}(F^{ad}(X \otimes N), R) \simeq \operatorname{Hom}_{\mathcal{N}}(X \otimes N, F(R)) \\
&\simeq \operatorname{Hom}_{\mathcal{N}}(N, X^* \otimes F(R)) \simeq \operatorname{Hom}_{\mathcal{N}}(N, F(X^* \otimes R)) \\
&\simeq \operatorname{Hom}_{\mathcal{M}}(F^{ad}(N), X^* \otimes R) \simeq \operatorname{Hom}_{\mathcal{M}}(X \otimes F^{ad}(N), R)
\end{aligned}$$

for $X \in \mathcal{C}$, $N \in \mathcal{N}$, $R \in \mathcal{M}$. The third $\simeq$ is linearity of F. Define

$$\alpha_{F^{ad}(X \otimes N)}(id) : X \otimes F^{ad}(N) \xrightarrow{\sim} F^{ad}(X \otimes N)$$

The diagrams required to show that α gives $\mathcal{C}$-linearity for F^{ad} in $Func_{\mathcal{C}}(\mathcal{N}, \mathcal{M})^{op}$ are not difficult to draw but tedious and non-enlightening and so we omit them. Now assume that the module categories involved are bimodule. Define left and right

$\mathcal{C}$-module action on F^{ad} by the equations

$$X \otimes F^{ad} := (F \otimes {}^*X)^{ad}, \quad F^{ad} \otimes X := ({}^*X \otimes F)^{ad}.$$

This defines bimodule action $(X \boxtimes Y) \otimes F^{ad} := ({}^*Y \otimes (F \otimes {}^*X))^{ad}$ with bimodule coherence the adjoints of those for F: If $\gamma_{X,Y} : X \otimes (F \otimes Y) \to (X \otimes F) \otimes Y)$ are those for F then those for F^{ad} are given by $\gamma'_{X,Y} = (\gamma_{{}^*Y,{}^*X})^{ad}$. $\square$

Note 1.3.15 (Notation). For $\mathcal{C}$ and $\mathcal{D}$ finite tensor categories we define a new category whose objects are exact $(\mathcal{C}, \mathcal{D})$-bimodule categories with morphisms $(\mathcal{C}, \mathcal{D})$-bimodule functors. Denote this category $\mathcal{B}(\mathcal{C}, \mathcal{D})$. When $\mathcal{C} = \mathcal{D}$ this is the category of exact bimodule categories over $\mathcal{C}$, which we denote $\mathcal{B}(\mathcal{C})$. For $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{B}(\mathcal{C}, \mathcal{D})$ denote by $Fun_{\mathcal{C},\mathcal{D}}(\mathcal{M}, \mathcal{N})$ the category of $(\mathcal{C}, \mathcal{D})$-bimodule functors from $\mathcal{M}$ to $\mathcal{N}$. It is evident that for exact $(\mathcal{C}, \mathcal{D})$ bimodule category $\mathcal{M}$ and $(\mathcal{C}, \mathcal{E})$ bimodule category $\mathcal{N}$ the category of module functors $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ has the structure of a $(\mathcal{D}, \mathcal{E})$ bimodule category with action $(X \boxtimes Y) \otimes F := F(- \otimes X) \otimes Y$. For finite exact module categories $\mathcal{M}, \mathcal{N}$ the category of functors $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is known to be an exact module category over the tensor category $Fun_{\mathcal{C}}(\mathcal{N}, \mathcal{N})$ with action given by composition of functors (Lemma 3.30 *loc. cit.*).

1.3.3 Dominant functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and define its image $Im(F)$ to be the full subcategory of $\mathcal{B}$ having objects given by all subquotients of objects of the form $F(X)$ for any $X \in \mathcal{A}$.

Definition 1.3.16. The functor F is said to be *dominant* if $Im(F) = \mathcal{B}$.

It is an easy exercise to show that $Im(F)$ is itself an abelian category. Furthermore if $\mathcal{A}, \mathcal{B}$ are tensor categories and F a tensor functor then $Im(F)$ is a tensor subcategory of $\mathcal{B}$. Indeed if A_1, A_1 are quotients of subobjects Z_1, Z_2 of $F(X_1), F(X_2)$ for X_i objects of $\mathcal{A}$ then exactness of tensor structure $\otimes$ of $\mathcal{B}$ implies that $A_1 \otimes A_2$ is a quotient of $Z_1 \otimes Z_2$ which is a subobject of $F(X_1) \otimes F(X_2) \simeq F(X_1 \otimes X_2)$. Hence $A_1 \otimes A_2$ is a subquotient of $F(X_1 \otimes X_2)$ and is therefore an object of $Im(F)$. The unit object 1 is contained in $Im(F)$ because it is a trivial subobject of $F(1)$, and constraints come from those in $\mathcal{B}$.

It is also evident that if $\mathcal{A}, \mathcal{B}$ are semisimple then dominance of F means that any object of $\mathcal{B}$ is actually a subobject of $F(X)$ for some $X \in \mathcal{A}$.

1.4 2-categories and monoidal 2-categories

Recall that a 2-category is a generalized version of an ordinary category where we have cells of various degrees and rules dictating how cells of different degrees interact. There are two ways to compose 2-cells α, β: *vertical* composition $\beta\alpha$ and *horizontal* composition $\beta * \alpha$ as described by the diagrams below.

$$
\begin{tikzcd}
A \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{description, name=G}] \arrow[r, bend right=60, "h"'{name=H}] \arrow[Rightarrow, from=F, to=G, "\beta"] \arrow[Rightarrow, from=G, to=H, "\alpha"] & B
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow\alpha\beta"] & B
\end{tikzcd},
\qquad
\begin{tikzcd}
A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left=50, "f'"] \arrow[r, bend right=50, "h'"'] \arrow[r, phantom, "\Downarrow\beta"] & C
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
A \arrow[r, bend left=50, "f'f"] \arrow[r, bend right=50, "h'h"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & C
\end{tikzcd}
$$

It is required that $\alpha * \beta = (\beta \bullet h)(f' \bullet \alpha) = (h' \bullet \alpha)(\beta \bullet f)$ where $\bullet$ signifies composition between 1-cells and 2-cells giving 2-cells (see [Lei04] for a thorough treatment of higher category theory and [Ben67], [Kel82] for theory of enriched categories). For fixed monoidal category $\mathcal{C}$ we have an evident 2-category with 0-cells $\mathcal{C}$-module categories, 1-cells $\mathcal{C}$-module functors and 2-cells monoidal natural transformations.

Example 1.4.1. The category of rings defines a 2-category with 0-cells rings, 1-cells bimodules and 2-cells tensor products.

A *monoidal* 2-category is essentially a 2-category equipped with a monoidal structure that acts on pairs of cells of various types. For convenience we reproduce, in part, the definition of monoidal 2-category as it appears in [KV91].

Definition 1.4.2. Let $\mathcal{A}$ be a strict 2-category. A *(lax) monoidal structure* on $\mathcal{A}$ consists of the following data:

M1. An object $1 = 1_{\mathcal{A}}$ called the unit object

M2. For any two objects A, B in $\mathcal{A}$ a new object $A \otimes B$, also denoted AB

M3. For any 1-morphism $u : A \to A'$ and any object B a pair of 1-morphisms $u \otimes B : A \otimes B \to A' \otimes B$ and $B \otimes u : B \otimes A \to B \otimes A'$

M4. For any 2-morphism

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow T"] & A' \end{tikzcd}$$

and object B there exist 2-morphisms

$$\begin{tikzcd} A \otimes B \arrow[r, bend left, "u\otimes B"] \arrow[r, bend right, "u'\otimes B"'] \arrow[r, phantom, "\Downarrow T\otimes B"] & A' \otimes B \end{tikzcd} \qquad \begin{tikzcd} B \otimes A \arrow[r, bend left, "B\otimes u"] \arrow[r, bend right, "B\otimes u'"'] \arrow[r, phantom, "\Downarrow B\otimes T"] & B \otimes A' \end{tikzcd}$$

M5. For any three objects A, B, C an isomorphism $a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$

M6. For any object A isomorphisms $l_A : 1 \otimes A \to A$ and $r_A : A \otimes 1 \to A$

M7. For any two morphisms $u : A \to A'$, $v : B \to B'$ a 2-isomorphism

$$\begin{tikzcd}
A \otimes B \arrow[r, "A \otimes v"] \arrow[d, "u \otimes B"'] \arrow[dr, phantom, "\swarrow \otimes_{u,v}"] & A \otimes B' \arrow[d, "u \otimes B'"] \\
A' \otimes B \arrow[r, "A' \otimes v"] & A' \otimes B'
\end{tikzcd}$$

M8. For any pair of composable morphisms $A \xrightarrow{u} A' \xrightarrow{u'} A''$ and object B 2-isomorphisms

$$\begin{tikzcd}
A \otimes B \arrow[r, "(u'u) \otimes B"] \arrow[d, "u \otimes B"'] & A'' \otimes B \\
A' \otimes B \arrow[ur, "u' \otimes B"'] \arrow[u, phantom, ""] \arrow[ur, phantom, bend left, "\nearrow \otimes_{u,u',B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes A \arrow[r, "B \otimes (u'u)"] \arrow[d, "B \otimes u"'] & B \otimes A'' \\
B \otimes A' \arrow[ur, "B \otimes u'"'] \arrow[ur, phantom, bend left, "\nearrow \otimes_{B,u,u'}"]
\end{tikzcd}$$

M9. For any four objects A, B, C, D a 2-morphism

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "A \otimes a_{B,C,D}"] \arrow[d, "a_{A,B,C \otimes D} \otimes D"'] & A \otimes ((B \otimes C) \otimes D) \arrow[dd, "a_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "a_{A \otimes B,C,D}"'] \arrow[r, phantom, "\swarrow a_{A,B,C,D}"] & \\
((A \otimes B) \otimes C) \otimes D & (A \otimes (B \otimes C)) \otimes D \arrow[l, "a_{A,B,C \otimes D}"]
\end{tikzcd}$$

M10. For any morphism $u : A \to A', v : B \to B', w : C \to C'$ 2-isomorphisms

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "a_{A,B,C}"] \arrow[d, "u \otimes (B \otimes C)"'] \arrow[dr, phantom, "\swarrow a_{u,B,C}"] & (A \otimes B) \otimes C \arrow[d, "(u \otimes B) \otimes C"] \\
A' \otimes (B \otimes C) \arrow[r, "a_{A',B,C}"'] & (A' \otimes B) \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "a_{A,B,C}"] \arrow[d, "A \otimes (v \otimes C)"'] \arrow[dr, phantom, "\swarrow a_{A,v,C}"] & (A \otimes B) \otimes C \arrow[d, "(A \otimes v) \otimes C"] \\
A \otimes (B' \otimes C) \arrow[r, "a_{A,B',C}"'] & (A \otimes B') \otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "a_{A,B,C}"] \arrow[d, "A\otimes(B\otimes w)"'] & (A \otimes B) \otimes C \arrow[d, "(A\otimes B)\otimes w"] \arrow[dl, phantom, "\Swarrow a_{A,B,w}"] \\
A \otimes (B \otimes C') \arrow[r, "a_{A,B,C'}"'] & (A \otimes B) \otimes C'
\end{tikzcd}$$

M11. For any two objects A, B 2-isomorphism

$$\begin{tikzcd}
A \otimes (B \otimes 1) \arrow[r, "A\otimes r_B"] \arrow[d, "a_{A,B,1}"'] & A \otimes B \\
(A \otimes B) \otimes 1 \arrow[ur, "r_{A\otimes B}"'] \arrow[u, phantom, shift left=6, "\Nearrow \rho_{A,B}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes (A \otimes B) \arrow[r, "l_{A\otimes B}"] \arrow[d, "a_{1,A,B}"'] & A \otimes B \\
(1 \otimes A) \otimes B \arrow[ur, "l_A\otimes B"'] \arrow[u, phantom, shift left=6, "\Nearrow \lambda_{A,B}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (1 \otimes B) \arrow[r, "A\otimes l_B"] \arrow[d, "a_{A,1,B}"'] & A \otimes B \\
(A \otimes 1) \otimes B \arrow[ur, "r_A\otimes B"'] \arrow[u, phantom, shift left=6, "\Nearrow \mu_{A,B}"] &
\end{tikzcd}$$

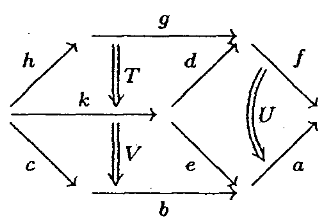
M12. For any morphism $u : A \to A'$ 2-isomorphisms

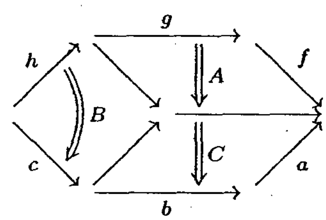
$$\begin{tikzcd}
1 \otimes A \arrow[r, "1\otimes u"] \arrow[d, "l_A"'] & 1 \otimes A' \arrow[d, "l_{A'}"] \arrow[dl, phantom, "\Swarrow l_u"] \\
A \arrow[r, "u"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes 1 \arrow[r, "u\otimes 1"] \arrow[d, "r_A"'] & A' \otimes 1 \arrow[d, "r_{A'}"] \arrow[dl, phantom, "\Swarrow r_u"] \\
A \arrow[r, "r"'] & A'
\end{tikzcd}$$

M13. A 2-isomorphism $\epsilon : r_1 \Rightarrow l_1$.

These data are further required to satisfy a series of axioms given in the form of *commutative polytopes* listed by Kapranov and Voevodsky. As well as describing the sort of naturality we should expect (extending that appearing in the definition of 2-cells for categories of functors) these polytopes provide constraints on the various cells at different levels and dictates how they are to inteact. For the sake of brevity we do not list them here but will refer to the diagrams in the original paper when needed. In [KV91] these polytopes are indicated using hieroglyphic notation. The Stasheff polytope, for example, (which they signify by $(\bullet \otimes \bullet \otimes \bullet \otimes \bullet \otimes \bullet)$, pg. 217) describes how associativity 2-cells and their related morphisms on pentuples of 0-cells interact. In the sequel we will adapt their hieroglyphic notation without explanation.

We digress briefly to explain what is meant by "commuting polytope." This notion will be needed for the proof of Theorem 0.2.1 Our discussion is taken from *loc. cit.*. In a strict 2-category $\mathcal{A}$ algebraic expressions may take the form of 2-dimensional cells subdivided into smaller cells indicating the way in which the larger 2-cells are to be composed. This procedure is referred to as *pasting.* Consider the diagram below left.



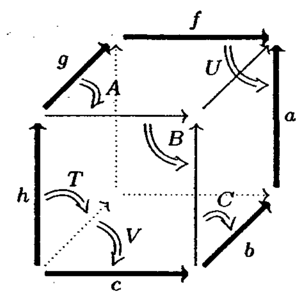


Edges are 1-cells and faces (double arrows) are 2-cells in $\mathcal{A}$; $T : gh \Rightarrow dk$, $V : ek \Rightarrow bc$, $U : fd \Rightarrow ae$. The diagram represents a 2-cell $fgh \Rightarrow abc$ in $\mathcal{A}$ as follows. It is possible to compose 1-cell F and 2-cell α obtaining new 2-cells $F * \alpha, \alpha * F$ whenever these compositions make sense. If $\alpha : G \Rightarrow H$, these are new 2-cells $FG \Rightarrow FH$ and $GF \Rightarrow HF$, respectively. Pasting of diagram above left represents the composition

$$fgh \overset{f*T}{\Longrightarrow} fdk \overset{U*k}{\Longrightarrow} aek \overset{a*V}{\Longrightarrow} abc.$$

For 2-composition abbreviated by juxtaposition the pasting is then $(a*V)(U*k)(f*T)$.

In case the same external diagram is subdivided in different ways a new 3-dimensional polytope may be formed by gluing along the common edges. Thus the two 2-dimension diagrams can be combined along the edges fgh and abc to form the new 3-dimensional polytope



We have labeled only those edges common to the two original figures. As an aid to deciphering polytope commutativity we will denote the boundary with bold arrows as above. To say that the polytope commutes is to say that the results of the pastings of the two sections of its boundary agree. In such a case we say that the pair of diagrams composing the figure are equal: the 2-cells they denote in $\mathcal{A}$ coincide.

CHAPTER II

TENSOR PRODUCT OF BIMODULE CATEGORIES

The next few chapters contain a description of the data giving the 2-category of $\mathcal{C}$-bimodule categories for a fixed monoidal category $\mathcal{C}$ the structure of a monoidal 2-category. In the rest of this thesis all categories are assumed to be abelian and k-linear, have finite-dimensional hom spaces, and all functors are assumed to be additive and k-linear. Even though most of what we do here is valid over fields of positive characteristic, we assume at the outset that k is a fixed field of characteristic 0.

2.1 Preliminary definitions and first properties

Recall definition of right exactness (Definition 1.1.7).

Definition 2.1.1. Suppose $(\mathcal{M}, \mu)$ right, $(\mathcal{N}, \eta)$ left $\mathcal{C}$-module categories. A functor $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ is said to be $\mathcal{C}$*-balanced* if there are natural isomorphisms $b_{M,X,Y}$: $F((M \otimes X) \boxtimes N) \simeq F(M \boxtimes (X \otimes N))$ satisfying the pentagon

$$\begin{tikzcd}
F((M \otimes (X \otimes Y)) \boxtimes N) \arrow[rr, "b_{M,X\otimes Y,N}"] \arrow[d, "\mu_{M,X,Y}"] & & F(M \boxtimes ((X \otimes Y) \otimes N)) \arrow[d, "\eta_{X,Y,N}"] \\
F(((M \otimes X) \otimes Y) \boxtimes N) \arrow[dr, "b_{M\otimes X,Y,N}"'] & & F(M \boxtimes (X \otimes (Y \otimes N))) \\
& F((M \otimes X) \boxtimes (Y \otimes N)) \arrow[ur, "b_{M,X,Y\otimes N}"'] &
\end{tikzcd}$$

whenever X, Y are objects of $\mathcal{C}$ and $M \in \mathcal{M}$.

Of course Definition 2.1.1 can be extended to functors from the Deligne product of more than two categories.

Definition 2.1.2. Let $F : \mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \cdots \boxtimes \mathcal{M}_n \to \mathcal{N}$ be a functor of abelian categories and suppose that, for some i, $1 \leq i \leq n-1$, $\mathcal{M}_i$ is a right $\mathcal{C}$-module category and $\mathcal{M}_{i+1}$ a left $\mathcal{C}$-module category. Then F is said to be *balanced in the i^{th} position* if there are natural isomorphisms $b^i_{X,M_1,\ldots,M_n} : F(M_1 \boxtimes \cdots \boxtimes (M_i \otimes X) \boxtimes M_{i+1} \boxtimes \cdots \boxtimes M_n) \simeq F(M_1 \boxtimes \cdots \boxtimes M_i \boxtimes (X \otimes M_{i+1}) \boxtimes \cdots \boxtimes M_n)$ whenever M_i are in $\mathcal{M}_i$ and X is in $\mathcal{C}$. The b^i are required to satisfy a diagram analogous to that described in Definition 2.1.1.

One may also define *multibalanced* functors F balanced at multiple positions simultaneously. We will need, and so define, only the simplest nontrivial case.

Definition 2.1.3. Let $\mathcal{M}_1$ be right $\mathcal{C}$-module, $\mathcal{M}_2$ $(\mathcal{C}, \mathcal{D})$-bimodule, and $\mathcal{M}_3$ a left $\mathcal{D}$-module category. The functor $F : \mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \mathcal{M}_3 \to \mathcal{N}$ is said to be *completely balanced* (or *2-balanced*) if for $X \in \mathcal{C}$, $Y \in \mathcal{D}$, $N \in \mathcal{M}_2$, $M \in \mathcal{M}_1$ and $P \in \mathcal{M}_3$ there are natural isomorphisms

$$b^1_{M,X,N,P} : F((M \otimes X) \boxtimes N \boxtimes P) \simeq F(M \boxtimes (X \otimes N) \boxtimes P)$$

$$b^2_{M,N,Y,P} : F(M \boxtimes (N \otimes Y) \boxtimes P) \simeq F(M \boxtimes N \boxtimes (Y \otimes P))$$

satisfying the balancing diagrams in Definition 2.1.1 and the consistency pentagon

$$\begin{tikzcd}
F((M \otimes X) \boxtimes (N \otimes Y) \boxtimes P) \arrow[r, "b^2_{M\otimes X,N,Y,P}"'] \arrow[d, "b^1_{M,X,N\otimes Y,P}"'] & F((M \otimes X) \boxtimes N \boxtimes (Y \otimes P)) \arrow[dd, "b^1_{M,X,N,Y\otimes P}"] \\
F(M \boxtimes (X \otimes (N \otimes Y)) \boxtimes P) \arrow[d, "\gamma^{-1}_{X,N,Y}"'] & \\
F(M \boxtimes ((X \otimes N) \otimes Y) \boxtimes P) \arrow[r, "b^2_{M,X,Y\otimes N}"'] & F(M \boxtimes (X \otimes N) \boxtimes (Y \otimes P))
\end{tikzcd}$$

Here γ is the family of natural isomorphisms associated to the bimodule structure in $\mathcal{M}_2$ (see Remark 1.3.10). Whenever F from $\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \cdots \boxtimes \mathcal{M}_n$ is balanced in "all" positions call F $(n-1)$-*balanced* or *completely balanced*. In this case the consistency axioms take the form of commuting polytopes. For example the consistency axiom for 4-balanced functors is equivalent to the commutativity of a polytope having eight faces (four pentagons and four squares) which reduces to a cube on elision of γ-labeled edges. With this labeling scheme the 1-balanced functors are the original ones given in Definition 2.1.1.

Definition 2.1.4. The *tensor product* of right $\mathcal{C}$-module category $\mathcal{M}$ and left $\mathcal{C}$-module category $\mathcal{N}$ consists of an abelian category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ and a right exact $\mathcal{C}$-balanced functor $B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ universal for *right exact* $\mathcal{C}$-balanced functors from $\mathcal{M} \boxtimes \mathcal{N}$.

Remark 2.1.5. In [Tam01] constructions similar to these were defined for k-linear categories as part of a program to study the representation categories of Hopf algebras and their duals. Balanced functors appeared under the name *bilinear functors*, and the tensor product there is given in terms of generators and relations instead of the universal properties used here. The tensor product was defined and applied extensively by [ENO09] in the study of semisimple module categories over fusion $\mathcal{C}$.

Remark 2.1.6. Universality here means that for any right exact $\mathcal{C}$-balanced functor $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ there exists a unique right exact functor $\overline{F}$ such that the diagram on the left commutes.

$$\begin{tikzcd}
\mathcal{M} \boxtimes \mathcal{N} \arrow[r, "F"] \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{A} \\
\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \arrow[ur, "\overline{F}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M} \boxtimes \mathcal{N} \arrow[r, "U"] \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] \arrow[dr, "F"] & \mathcal{U} \arrow[d, "F'"] \arrow[dl, dotted, "\alpha"'] \\
\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \arrow[r, "\overline{F}"'] & \mathcal{A}
\end{tikzcd}$$

The category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ and the functor $B_{\mathcal{M},\mathcal{N}}$ are defined up to a unique equivalence. This means that if $U : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{U}$ is a second right exact balanced functor with $F = F'U$ for unique right exact functor F' there is a unique equivalence of abelian categories $\alpha : \mathcal{U} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ making the diagram on the right commute.

Remark 2.1.7. The definition of balanced functor may be easily adapted to bifunctors from $\mathcal{M} \times \mathcal{N}$ instead of $\mathcal{M} \boxtimes \mathcal{N}$. In this case the definition of tensor product becomes object universal for balanced functors *right exact in both variables* from $\mathcal{M} \times \mathcal{N}$ (Remark 1.2.2). This is the approach taken by Deligne in [Del90]. One easily checks that our definition reduces to Deligne's for $\mathcal{C} = Vec$. This provides some justification for defining the relative tensor product in terms of right-exact functors as opposed to functors of some other sort.

Lemma 2.1.8. *Let $\mathcal{M}, \mathcal{N}$ be right, left $\mathcal{C}$-module categories for $\mathcal{C}$ a monoidal category. Then the universal balanced functor $B_{\mathcal{M},\mathcal{N}}$ from Definition 2.1.4 is dominant*

(Definition 1.3.16).

Proof. Let F be any balanced functor from $\mathcal{M} \boxtimes \mathcal{N}$, and let $\overline{F}$ be the unique functor from $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ with $\overline{F} B_{\mathcal{M},\mathcal{N}} = F$. For the inclusion $i : Im(B_{\mathcal{M},\mathcal{N}}) \hookrightarrow \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ define $F' := \overline{F} i$. Then it is obvious that $F' B_{\mathcal{M},\mathcal{N}} = F$, hence F factors through $Im(B_{\mathcal{M},\mathcal{N}})$ uniquely. As a consequence of the universality of the relative tensor product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} = Im(B_{\mathcal{M},\mathcal{N}})$. $\square$

The following lemma is a straightforward application of the tensor product universality from Definition 2.1.4. We list it here for reference in the sequel.

Lemma 2.1.9. *Let F, G be right exact functors $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathcal{A}$ such that $F B_{\mathcal{M},\mathcal{N}} = G B_{\mathcal{M},\mathcal{N}}$. Then $F = G$.*

Proof. In the diagram

$$\begin{tikzcd}
\mathcal{M} \boxtimes \mathcal{N} \arrow[r, "B_{\mathcal{M},\mathcal{N}}"] \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] \arrow[dr, "T"] & \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \arrow[d, "F"] \arrow[dl, dotted, "\alpha"'] \\
\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \arrow[r, "G"'] & \mathcal{A}
\end{tikzcd}$$

for $T = F B_{\mathcal{M},\mathcal{N}} = G B_{\mathcal{M},\mathcal{N}}$ the unique equivalence α is $id_{\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}}$. $\square$

Definition 2.1.10. For $\mathcal{M}$ a right $\mathcal{C}$-module category and $\mathcal{N}$ a left $\mathcal{C}$-module category denote by $\underline{Fun}^{bal}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{A})$ the category of right exact $\mathcal{C}$-balanced functors. Morphisms are natural transformations $\tau : (F, f) \to (G, g)$ where f and g are balancing isomorphisms for F and G satisfying, whenever $M \in \mathcal{M}$ and $N \in \mathcal{N}$,

$$\begin{tikzcd}
F((M \otimes X) \boxtimes N) \arrow[r, "\tau_{M\otimes X,N}"] \arrow[d, "f_{M,X,N}"'] & G((M \otimes X) \boxtimes N) \arrow[d, "g_{M,X,N}"] \\
F((M \boxtimes (X \otimes N)) \arrow[r, "\tau_{M,X\otimes N}"'] & G(M \boxtimes (X \otimes N))
\end{tikzcd}$$

for X in $\mathcal{C}$. Call morphisms in a category of balanced functors *balanced natural transformations.* Similarly we can define $\underline{Fun}_i^{bal}(\mathcal{M}_1 \boxtimes \cdots \boxtimes \mathcal{M}_n, \mathcal{A})$ to be the category of right exact functors "balanced in the i^{th} position" requiring of morphisms a diagram similar to that above.

2.2 Module category theoretic structure of tensor product

In this section we examine functoriality of $\boxtimes_{\mathcal{C}}$ and discuss module structure of the tensor product.

For $\mathcal{M}$ a right $\mathcal{C}$-module category, $\mathcal{N}$ a left $\mathcal{C}$-module category, universality of $B_{\mathcal{M},\mathcal{N}}$ implies an equivalence between categories of functors

$$\mathcal{Y} : \underline{Fun}^{bal}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{A}) \xrightarrow{\sim} \underline{Fun}(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \mathcal{A}) \tag{2}$$

sending $F \mapsto \overline{F}$ (here overline is as in Definition 2.1.4). Quasi-inverse $\mathcal{W}$ sends $G \mapsto GB_{\mathcal{M},\mathcal{N}}$ with balancing $G * b$, b the balancing of $B_{\mathcal{M},\mathcal{N}}$. On natural transformations τ, $\mathcal{W}$ is defined by $\mathcal{W}(\tau) = \tau * B_{\mathcal{M},\mathcal{N}}$ where $*$ is the product of 2-morphism and 1-morphism: components are given by $\mathcal{W}(\tau)_{M\boxtimes N} = \tau_{B_{\mathcal{M},\mathcal{N}}(M\boxtimes N)}$. One easily checks that $\mathcal{YW} = id$ so that $\mathcal{W}$ is a strict right quasi-inverse for $\mathcal{Y}$. Let $J : \mathcal{WY} \to id$

be any natural isomorphism. Then components of J are balanced isomorphisms $J_{(F,f)} : (F, \overline{F} * b) \to (F, f)$ where f is balancing for functor F. Being balanced means commutativity of the diagram

$$\begin{tikzcd}
F(M \otimes X \boxtimes N) \arrow[r, "\overline{F}(b_{M,X,N})"] \arrow[d, "J_{MX\boxtimes N}"'] & F(M \boxtimes X \otimes N) \arrow[d, "J_{M\boxtimes XN}"] \\
F(M \otimes X \boxtimes N) \arrow[r, "f_{M,X,N}"'] & F(M \boxtimes X \otimes N)
\end{tikzcd}$$

for any $M \in \mathcal{M}, X \in \mathcal{C}, N \in \mathcal{N}$. Hence any balancing structure f on the functor F is conjugate to $\overline{F} * b$ in the sense that

$$f_{M,X,N} = J_{M\boxtimes XN} \circ \overline{F}(b_{M,X,N}) \circ J^{-1}_{MX\boxtimes N}. \tag{3}$$

Remark 2.2.1. Let $F, G : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ be right exact $\mathcal{C}$-balanced functors. To understand how $\mathcal{Y}$ acts on balanced natural transformation $\tau : F \to G$ recall that to any functor $E : \mathcal{S} \to \mathcal{T}$ we associate the *comma category*, denoted $(E, \mathcal{T})$, having objects triples $(X, Y, q) \in \mathcal{S} \times \mathcal{T} \times \mathrm{Hom}_{\mathcal{T}}(E(X), Y)$. A morphism $(X, Y, q) \to (X', Y', q')$ is a pair of morphisms (h, k) with the property that $k \circ q = q' \circ E(h)$. For E right exact and $\mathcal{S}, \mathcal{T}$ abelian, $(E, \mathcal{T})$ is abelian ([FGR75]).

Let $\overline{F}$ be the unique right exact functor having $\overline{F}B_{\mathcal{M},\mathcal{N}} = F$ and consider the comma category $(\overline{F}, \mathcal{A})$. Natural balanced transformation τ determines a functor $S_\tau : \mathcal{M} \boxtimes \mathcal{N} \to (\overline{F}, \mathcal{A})$, $X \mapsto (B_{\mathcal{M},\mathcal{N}}(X), G(X), \tau_X)$ and $f \mapsto (F(f), G(f))$. It is evident that S_τ is right exact and inherits $\mathcal{C}$-balancing from that in $B_{\mathcal{M},\mathcal{N}}$, G and τ. Thus we have a unique functor $\overline{S_\tau} : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to (\overline{F}, \mathcal{A})$ with $\overline{S_\tau}B_{\mathcal{M},\mathcal{N}} = S_\tau$. Write

$\overline{S_\tau} = (S_1, S_2, \sigma)$. Using Lemma 2.1.9 one shows that $S_1 = id_{\mathcal{M}\boxtimes_\mathcal{C}\mathcal{N}}$ and $S_2 = \overline{G}$. Then $\sigma(Y) : \overline{F}(Y) \to \overline{G}(Y)$ for $Y \in \mathcal{M} \boxtimes_\mathcal{C} \mathcal{N}$. This is precisely $\overline{\tau} : \overline{F} \to \overline{G}$.

Given right exact right $\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{M}'$ and right exact left $\mathcal{C}$-module functor $G : \mathcal{N} \to \mathcal{N}'$ note that $B_{\mathcal{M}',\mathcal{N}'}(F \boxtimes G) : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M}' \boxtimes_\mathcal{C} \mathcal{N}'$ is $\mathcal{C}$-balanced. Thus the universality of B implies the existence of a unique right exact functor $F \boxtimes_\mathcal{C} G := \overline{B_{\mathcal{M}',\mathcal{N}'}(F \boxtimes G)}$ making the diagram

$$\begin{tikzcd}
\mathcal{M} \boxtimes \mathcal{N} \arrow[r, "F\boxtimes G"] \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{M}' \boxtimes \mathcal{N}' \arrow[d, "B_{\mathcal{M}',\mathcal{N}'}"] \\
\mathcal{M} \boxtimes_\mathcal{C} \mathcal{N} \arrow[r, "F\boxtimes_\mathcal{C} G"'] & \mathcal{M}' \boxtimes_\mathcal{C} \mathcal{N}'
\end{tikzcd}$$

commute. One uses Lemma 2.1.9 (see the next diagram) to show that $\boxtimes_\mathcal{C}$ is functorial on 1-cells: $(F' \boxtimes_\mathcal{C} E')(F \boxtimes_\mathcal{C} E) = F'F \boxtimes_\mathcal{C} E'E$.

$$\begin{tikzcd}
\mathcal{M} \boxtimes \mathcal{N} \arrow[r, "F\boxtimes E"] \arrow[dd, "B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{M}' \boxtimes \mathcal{N}' \arrow[r, "F'\boxtimes E'"] \arrow[d, "B_{\mathcal{M}',\mathcal{N}'}"] & \mathcal{M}'' \boxtimes \mathcal{N}'' \arrow[dd, "B_{\mathcal{M}'',\mathcal{N}''}"] \\
& \mathcal{M}' \boxtimes_\mathcal{C} \mathcal{N}' \arrow[dr, "F'\boxtimes_\mathcal{C} E'"] & \\
\mathcal{M} \boxtimes_\mathcal{C} \mathcal{N} \arrow[ur, "F\boxtimes_\mathcal{C} E"] \arrow[rr, "F'F\boxtimes_\mathcal{C} E'E"'] & & \mathcal{M}'' \boxtimes_\mathcal{C} \mathcal{N}''
\end{tikzcd}$$

Thus the 2-cells in M7. of Definition 1.4.2 are identity. If we define $F \otimes \mathcal{N} := F \boxtimes_\mathcal{C} id_\mathcal{N}$ (Definition 3.1.5) then the 2-cells in M8. are identity as well.

Remark 2.2.2. Next we consider how $\boxtimes_\mathcal{C}$ can be applied to pairs of module natural transformations. Apply $B_{\mathcal{N},\mathcal{N}'}$ to the right of the diagram for the Deligne product of

τ and σ

$$\begin{tikzcd} \mathcal{M} \boxtimes \mathcal{M}' \arrow[r, bend left, "F \boxtimes E"] \arrow[r, bend right, "G \boxtimes H"'] \arrow[r, phantom, "\Downarrow \tau\boxtimes\sigma"] & \mathcal{N} \boxtimes \mathcal{N}' \arrow[r, "B_{\mathcal{N},\mathcal{N}'}"] & \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{N}' \end{tikzcd}$$

giving natural transformation

$$(\tau \boxtimes \sigma)' := B_{\mathcal{N},\mathcal{N}'} * (\tau \boxtimes \sigma) : B_{\mathcal{N},\mathcal{N}'}(F \boxtimes E) \Rightarrow B_{\mathcal{N},\mathcal{N}'}(G \boxtimes H) \tag{4}$$

having components $B_{\mathcal{N},\mathcal{N}'} * (\tau \boxtimes \sigma)_{A \boxtimes B} = B_{\mathcal{N},\mathcal{N}'}(\tau_A \boxtimes \sigma_B)$. Here $*$ indicates composition between cells of different index (in this case a 1-cell and a 2-cell with the usual 2-category structure in **Cat**).

It is easy to see that this is a balanced natural transformation, i.e. a morphism in the category of balanced right exact functors $\underline{Fun}^{bal}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{M}' \boxtimes_{\mathcal{C}} \mathcal{N}')$. Using comma category $(F \boxtimes_{\mathcal{C}} F', \mathcal{M}' \boxtimes_{\mathcal{C}} \mathcal{N}')$ we get

$$\tau \boxtimes_{\mathcal{C}} \sigma := \overline{(\tau \boxtimes \sigma)'} : F \boxtimes_{\mathcal{C}} F' \Rightarrow G \boxtimes_{\mathcal{C}} G'. \tag{5}$$

Note also that $\boxtimes_{\mathcal{C}}$ is functorial over vertical composition of 2-cells: $(\tau' \boxtimes_{\mathcal{C}} \sigma')(\tau \boxtimes_{\mathcal{C}} \sigma) = \tau'\tau \boxtimes_{\mathcal{C}} \sigma'\sigma$ whenever the compositions make sense. Though we do not prove it here observe also that $\boxtimes_{\mathcal{C}}$ preserves horizontal composition $\bullet$ of 2-cells:

$$(\tau' \bullet \tau) \boxtimes_{\mathcal{C}} (\sigma' \bullet \sigma) = (\tau' \boxtimes_{\mathcal{C}} \sigma') \bullet (\tau \boxtimes_{\mathcal{C}} \sigma).$$

For the following proposition recall that, for left $\mathcal{C}$-module category $\mathcal{M}$, the functor $L_X : \mathcal{M} \to \mathcal{M}$ sending $M \mapsto X \otimes M$ for $X \in \mathcal{C}$ fixed is right exact. This follows from the fact that $\mathrm{Hom}(X^* \otimes N, \_\_)$ is left exact for any $N \in \mathcal{M}$.

Proposition 2.2.3. *Let $\mathcal{M}$ be a $(\mathcal{C}, \mathcal{E})$-bimodule category and $\mathcal{N}$ an $(\mathcal{E}, \mathcal{D})$-bimodule category. Then $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category and $B_{\mathcal{M},\mathcal{N}}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule functor.*

Proof. For X in $\mathcal{C}$ define functor $L_X : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes \mathcal{N} : M \boxtimes N \mapsto (X \otimes M) \boxtimes N$. Then there is a unique right exact $\overline{L_X}$ making the diagram on the left commute; bimodule consistency isomorphisms in $\mathcal{M}$ make L_X balanced.

$$\begin{tikzcd} \mathcal{M} \boxtimes \mathcal{N} \arrow[r, "B_{\mathcal{M},\mathcal{N}} L_X"] \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \\ \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[ur, "\overline{L_X}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} \boxtimes \mathcal{N} \arrow[r, "B_{\mathcal{M},\mathcal{N}} R_Y"] \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \\ \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[ur, "\overline{R_Y}"'] & \end{tikzcd}$$

Similarly, for Y in $\mathcal{D}$ define endofunctor $R_Y : M \boxtimes N \mapsto M \boxtimes (N \otimes Y)$. Then there is unique right exact $\overline{R_Y}$ making the diagram on the right commute; bimodule consistency isomorphisms in $\mathcal{N}$ make R_Y balanced. $\overline{L_X}$ and $\overline{R_Y}$ define left/right module category structures on $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$. Indeed for μ the left module associativity in $\mathcal{M}$ note that $B_{\mathcal{M},\mathcal{N}}(\mu_{X,Y,M} \boxtimes id_N) : \overline{L_X}\,\overline{L_Y} B_{\mathcal{M},\mathcal{N}} \simeq \overline{L_{X \otimes Y}} B_{\mathcal{M},\mathcal{N}}$ is an isomorphism in $\underline{Fun}^{bal}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N})$ so corresponds to an isomorphism $\overline{L_X}\,\overline{L_Y} \simeq \overline{L_{X \otimes Y}}$ in $\underline{\mathrm{End}}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N})$ which therefore satisfies the diagram for left module associativity in $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$. Composing diagonal arrows we obtain the following commutative diagram.

$$\begin{tikzcd} \mathcal{M} \boxtimes \mathcal{N} \arrow[r, "B_{\mathcal{M},\mathcal{N}} L_X"] \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\overline{R_Y}"] & \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \\ \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[ur, "\overline{L_X}"] \arrow[urr, "\overline{R_Y}\,\overline{L_X}"'] & & \end{tikzcd}$$

Note then that

$$\overline{L_X}\,\overline{R_Y}B_{\mathcal{M},\mathcal{N}} = \overline{R_Y}\,\overline{L_X}B_{\mathcal{M},\mathcal{N}}$$

and since $\overline{R_Y}\,\overline{L_X}B_{\mathcal{M},\mathcal{N}}$ is balanced Lemma 2.1.9 implies $\overline{R_Y}\,\overline{L_X} = \overline{L_X}\,\overline{R_Y}$. Suppose $Q \in \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$. Then $(X \boxtimes Y) \otimes Q := \overline{L_X}\,\overline{R_Y}Q = \overline{R_Y}\,\overline{L_X}Q$ defines $(\mathcal{C}, \mathcal{D})$-bimodule category structure on $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$. Note also that since the bimodule consistency isomorphisms in $\mathcal{M} \boxtimes \mathcal{N}$ are trivial the same holds in $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$. As a result $B_{\mathcal{M},\mathcal{N}}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule functor. □

In the sequel we may use L_X to denote left action of $X \in \mathcal{C}$ in $\mathcal{M} \boxtimes \mathcal{N}$ and for the induced action on $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$. Similarly for R_X.

Remark 2.2.4. The above construction is equivalent to defining left and right module category structures as follows. For the right module structure

$$\otimes : (\mathcal{M}\mathcal{N}) \boxtimes \mathcal{C} \xrightarrow{\alpha^1_{\mathcal{M},\mathcal{N},\mathcal{C}}} \mathcal{M}(\mathcal{N} \boxtimes \mathcal{C}) \xrightarrow{id \otimes} \mathcal{M}\mathcal{N}$$

where α^1 is defined in Lemma 3.1.1 and where tensor product of module categories has been written as juxtaposition. The left action is similarly defined using α^2 and left module structure of $\mathcal{M}$ in second arrow.

Proposition 2.2.5. *Let $\mathcal{M}$ be a $(\mathcal{C}, \mathcal{D})$-bimodule category. Then there are canonical $(\mathcal{C}, \mathcal{D})$-bimodule equivalences $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D} \simeq \mathcal{M} \simeq \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M}$.*

Proof. Observing that the $\mathcal{D}$-module action $\otimes$ in $\mathcal{M}$ is balanced let $l_{\mathcal{M}} : \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D} \to \mathcal{M}$ denote the unique exact functor factoring $\otimes$ through $B_{\mathcal{M},\mathcal{D}}$. Define $U : \mathcal{M} \to$

$\mathcal{M} \boxtimes \mathcal{D}$ by $M \mapsto M \boxtimes 1$ and write $U' = B_{\mathcal{M},\mathcal{D}}U$. We wish to show that $l_{\mathcal{M}}$ and U' are inverses.

Note first that $l_{\mathcal{M}}U' = id_{\mathcal{M}}$. Now define natural isomorphism $\tau : B_{\mathcal{M},\mathcal{D}} \Rightarrow U'\otimes$ by $\tau_{M,X} = b^{-1}_{M,X,1}$ where b is balancing isomorphism for $B_{\mathcal{M},\mathcal{D}}$. As a balanced natural isomorphism τ corresponds to an isomorphism $\overline{\tau} : \overline{B_{\mathcal{M},\mathcal{D}}} = id_{\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{D}} \Rightarrow \overline{U'\otimes}$ in the category $\underline{\mathrm{End}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D})$. Commutativity of the diagram

$$\begin{tikzcd}
\mathcal{M} \boxtimes \mathcal{D} \arrow[r, "\otimes"] \arrow[d, "B_{\mathcal{M},\mathcal{D}}"'] & \mathcal{M} \arrow[d, "U'"] \\
\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D} \arrow[r, "\overline{U'\otimes}"'] \arrow[ur, "l_{\mathcal{M}}"] & \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D}
\end{tikzcd}$$

implies $U'l_{\mathcal{M}} = \overline{U'\otimes}$ so that $id_{\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{D}} \simeq U'l_{\mathcal{M}}$ via $\overline{\tau}$. In proving $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{M}$ one lifts the left action of $\mathcal{C}$ for an equivalence $r_{\mathcal{M}} : \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \xrightarrow{\sim} \mathcal{M}$. Strict associativity of the module action on $\mathcal{M}$ implies that both $r_{\mathcal{M}}$ and $l_{\mathcal{M}}$ are trivially balanced. □

Corollary 2.2.6. *Let $(F, f) : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathcal{B}(\mathcal{C})$ where f is left $\mathcal{C}$-module linearity for F. Then there is a natural isomorphism $Fr_{\mathcal{M}} \xrightarrow{\sim} r_{\mathcal{N}}(id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F)$ satisfying a polytope version of the diagram for module functors in Definition 1.3.5. A similar result holds for the equivalence l.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{C} \boxtimes \mathcal{M} \arrow[rrr, "id_{\mathcal{C}} \boxtimes F"] \arrow[dr, "B_{\mathcal{C},\mathcal{M}}"] \arrow[dd, "\otimes"'] & & & \mathcal{C} \boxtimes \mathcal{N} \arrow[dl, "B_{\mathcal{C},\mathcal{N}}"'] \arrow[dd, "\otimes"] \\
& \mathcal{C}\mathcal{M} \arrow[r, "id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F"] \arrow[dl, "r_{\mathcal{M}}"] & \mathcal{C}\mathcal{N} \arrow[dr, "r_{\mathcal{N}}"'] & \\
\mathcal{M} \arrow[rrr, "F"'] & & & \mathcal{N}
\end{tikzcd}$$

The top rectangle is definition of $id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F$, right triangle definition of functor $r_{\mathcal{N}}$, and left triangle definition of $r_{\mathcal{M}}$. The outer edge commutes up to f. We therefore have natural isomorphism $f : Fr_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}} \to r_{\mathcal{N}}(id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F)B_{\mathcal{C},\mathcal{N}}$. Now observe that, using the regular module structure in $\mathcal{C}$ we have the following isomorphisms.

$$\begin{aligned}
Fr_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}}(XY \boxtimes M) &= F((XY)M) \\
&= F(X(YM)) = Fr_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}}(X \boxtimes YM), \\
r_{\mathcal{N}}(id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F)B_{\mathcal{C},\mathcal{N}}(XY \boxtimes M) &= (XY)F(M) \xrightarrow{\sim} XF(YM) \\
&= r_{\mathcal{N}}(id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F)B_{\mathcal{C},\mathcal{N}}(X \boxtimes YM)
\end{aligned}$$

Here $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$ and $\sim$ is $id_X \otimes f^{-1}_{Y,M}$. Using the relations required of the module structure f described in Definition 1.3.5 one sees that the second isomorphism constitutes a $\mathcal{C}$-balancing for the functor $r_{\mathcal{N}}(id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F)B_{\mathcal{C},\mathcal{N}}$. Thus both functors are balanced. Using the relations for f from Definition 1.3.5 a second time shows that f is actually a balanced natural isomorphism $Fr_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}} \to r_{\mathcal{N}}(id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F)B_{\mathcal{C},\mathcal{N}}$. Hence we may descend to a natural isomorphism $r_F := \overline{f} : Fr_{\mathcal{M}} \to r_{\mathcal{N}}(id_{\mathcal{C}} \boxtimes_{\mathcal{C}} F)$. The associated polytopes are given in Polytope 4.1.2, Chapter 4. The result for l is similar. $\square$

Corollary 2.2.6 shows, predictably, that functoriality of l, r depends on module linearity of the underlying functors. In particular, if F is a strict module functor l_F and r_F are both identity. As an example note that the associativity is strict as a module functor (this follows from Proposition 3.1.6) and so $r_{a_{\mathcal{M},\mathcal{N},\mathcal{P}}} = id$ for the relevant module categories. Similarly for l. Thus polytopes of the form $(1 \otimes \bullet \otimes \bullet \otimes \bullet)$ (pg. 222 in [KV91]) describing interaction between a, l and r commute trivially.

Remark 2.2.7. $r_{\mathcal{M}} : \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \to \mathcal{M}$ is itself a strict left $\mathcal{C}$-module functor as follows. Let $X \in \mathcal{C}$ and let L_X be left $\mathcal{C}$-module action in $\mathcal{C} \boxtimes \mathcal{M}$. Replacing L_X with $id \boxtimes_{\mathcal{C}} F$ in the diagram given in the proof of Corollary 2.2.6 and chasing around the resulting diagram allows us to write the equation

$$L'_X r_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}} = r_{\mathcal{M}} \overline{L_X} B_{\mathcal{C},\mathcal{M}}$$

where L'_X is left X-multiplication in $\mathcal{M}$ and $\overline{L_X}$ the induced left X-multiplication in $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M}$. Thus $L'_X r_{\mathcal{M}} = r_{\mathcal{M}} \overline{L_X}$, which is precisely the statement that $r_{\mathcal{M}}$ is strict as a $\mathcal{C}$-module functor. Thus Corollary 2.2.6 implies that $r_{r_{\mathcal{M}}} = id$ for any $\mathcal{C}$-module category $\mathcal{M}$. If $\mathcal{M}$ is a bimodule category it is evident that $r_{\mathcal{M}}$ is also a strict right module functor and hence strict as a bimodule functor.

Proposition 2.2.8. *For $(\mathcal{C},\mathcal{D})$-bimodule category $\mathcal{M}$ and $(\mathcal{C},\mathcal{E})$-bimodule category $\mathcal{N}$ the category of right exact $\mathcal{C}$-module functors $\underline{Fun_{\mathcal{C}}}(\mathcal{M},\mathcal{N})$ has canonical structure of a $(\mathcal{D},\mathcal{E})$-bimodule category.*

Proof. $((X \boxtimes Y) \otimes F)(M) = F(M \otimes X) \otimes Y$ defines $\mathcal{D} \boxtimes \mathcal{E}^{rev}$-action on $\underline{Fun_{\mathcal{C}}}(\mathcal{M},\mathcal{N})$. Right exactness of $(X \boxtimes Y) \otimes F$ comes from right exactness of F and of module action in $\mathcal{M},\mathcal{N}$. $\mathcal{D} \boxtimes \mathcal{E}^{rev}$ acts on the module part f of F by

$$((X \boxtimes Y) \otimes f)_{Z,M} = \gamma_{Z,F(M \otimes X),Y} f_{Z,M \otimes X} F(\gamma_{Z,M,X})$$

where γ is the bimodule consistency for the left and right module structures in $\mathcal{M}$, $\mathcal{N}$ (Proposition 1.3.10). The required diagrams commute since they do for f.

Next let $\tau : F \Rightarrow G$ be a natural left $\mathcal{C}$-module transformation for right exact left $\mathcal{C}$-module functors $(F, f), (G, g) : \mathcal{M} \to \mathcal{N}$. Define action of $X \boxtimes Y$ on τ by $((X \boxtimes Y) \otimes \tau)_M = \tau_{M \otimes X} \otimes id_Y : ((X \boxtimes Y) \otimes F)(M) \to ((X \boxtimes Y) \otimes G)(M)$. Then $(X \boxtimes Y) \otimes \tau$ is a natural left $\mathcal{C}$-module transformation. Indeed the diagram

$$\begin{tikzcd}
F((Z \otimes M) \otimes X) \otimes Y \arrow[r, "\tau_{(Z \otimes M) \otimes X} \otimes id_Y"] \arrow[d, "F\gamma"'] & G((Z \otimes M) \otimes X) \otimes Y \arrow[d, "G\gamma"] \\
F(Z \otimes (M \otimes X)) \otimes Y \arrow[r, "\tau_{Z \otimes (M \otimes X)} \otimes id_Y"] \arrow[d, "f_{Z, M \otimes X}"'] & G(Z \otimes (M \otimes X)) \otimes Y \arrow[d, "g_{Z, M \otimes X}"] \\
(Z \otimes F(M \otimes X)) \otimes Y \arrow[r, "id_Z \otimes \tau_{M \otimes X} \otimes id_Y"] \arrow[d, "\gamma"'] & (Z \otimes G(M \otimes X)) \otimes Y \arrow[d, "\gamma"] \\
Z \otimes (F(M \otimes X) \otimes Y) \arrow[r, "\tau_{M \otimes X} \otimes id_Y"] & Z \otimes (G(M \otimes X) \otimes Y)
\end{tikzcd}$$

commutes. The top rectangle is the rectangle of naturality for τ. The middle rectangle expresses the fact that τ is a natural left $\mathcal{C}$-module transformation. The bottom rectangle is the rectangle of naturality for γ. Perimeter is the diagram expressing that $(X \boxtimes Y) \otimes \tau$ is a module natural transformation. $\square$

Remark 2.2.9. $\mathcal{Y}$ in equation (2) at the beginning of this section is an equivalence of $(\mathcal{D}, \mathcal{F})$-bimodule categories

$$\underline{Fun}_{\mathcal{C}}^{bal}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{S}) \to \underline{Fun}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}, \mathcal{S}) \tag{6}$$

whenever $\mathcal{M} \in \mathcal{B}(\mathcal{C}, \mathcal{E})$, $\mathcal{N} \in \mathcal{B}(\mathcal{E}, \mathcal{D})$, $\mathcal{S} \in \mathcal{B}(\mathcal{C}, \mathcal{F})$. If balanced right exact bimodule functor $u : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{U}$ is universal for such functors from $\mathcal{M} \boxtimes \mathcal{N}$ then $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N} \simeq \mathcal{U}$ as bimodule categories.

To see the first claim let F be $\mathcal{E}$-balanced left $\mathcal{C}$-module functor $\mathcal{M} \boxtimes \mathcal{N} \to \mathcal{S}$ with module linearity f and balancing t. For $X \in \mathcal{C}$ denote by $L_X : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes \mathcal{N}$ left action of X, and define natural isomorphisms $f_X : FL_X \simeq L_X F$ by $(f_X)_A = f_{X,A}$ whenever $A \in \mathcal{M} \boxtimes \mathcal{N}$. Note that $L_X F$ has balancing $id_X \boxtimes t$ and that FL_X is balanced by

$$t_{X \otimes M,Y,N} F(\gamma^{-1}_{X,M,Y} \boxtimes id_{\mathcal{N}}) : (FL_X)((M \otimes Y) \boxtimes N) \simeq (FL_X)(M \boxtimes (Y \otimes N))$$

whenever $M \in \mathcal{M}$, $Y \in \mathcal{E}$ and $N \in \mathcal{N}$. Using Lemma 2.1.9 one verifies that $\overline{FL_X} = \overline{F} \circ \overline{BL_X}$ and $\overline{L_X F} = L_X \overline{F}$. Note that $\overline{BL_X}$ is the induced left action of X in $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$ which we will also denote L_X. Naturality of f implies that f_X is balanced hence and application of $\mathcal{Y}$ gives $\overline{f_X} : \overline{F} L_X \simeq L_X \overline{F}$ naturally in $Fun(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}, \mathcal{S})$. One checks that $\overline{F}$ is bimodule functor with module linearity $\overline{f}_{X,Q} = (\overline{f_X})_Q$ whenever $Q \in \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$ ($\overline{f}$ satisfies required diagrams because f does).

We may therefore write $\overline{(F, f)} = (\overline{F}, \overline{f})$ for the functor in $Fun_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}, \mathcal{S})$. We now show that $\mathcal{Y}$ respects the bimodule structure in the functor categories. For $Y \in \mathcal{D}$, $Z \in \mathcal{F}$ and $Q \in \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$ one checks easily that

$$\mathcal{Y}(Y \otimes F)(Q) = \overline{FR_Y}(Q) = \overline{F} \circ \overline{BR_Y}(Q) = \overline{F}(Q \otimes Y) = (Y \otimes \mathcal{Y}(F))(Q)$$

and similarly that $\mathcal{Y}(F \otimes Z)(Q) = (\mathcal{Y}(F) \otimes Z)(Q)$ making $\mathcal{Y}$ a bimodule functor.

For the second claim, universality of both $B_{\mathcal{M},\mathcal{N}}$ and u gives unique equivalence α of abelian categories making the diagram

$$\begin{tikzcd} \mathcal{M}\boxtimes\mathcal{N} \arrow[d, "u"'] \arrow[dr, "B_{\mathcal{M},\mathcal{N}}"] & \\ \mathcal{U} \arrow[r, "\alpha"'] & \mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N} \end{tikzcd}$$

commute. Thus α is the unique exact functor factoring $B_{\mathcal{M},\mathcal{N}}$, and since the latter is a balanced bimodule functor α inherits this property by the first part of the proposition.

2.3 Relative tensor product as category of functors

The purpose of this section is to prove an existence theorem for the relative tensor product by providing a canonical equivalence with a certain category of module functors. Let $\mathcal{M},\mathcal{N}$ be exact right, left module categories over tensor category $\mathcal{C}$, and define $I:\mathcal{M}\boxtimes\mathcal{N}\to\underline{Fun_{\mathcal{C}}}(\mathcal{M}^{op},\mathcal{N})$ by

$$I: M\boxtimes N\mapsto \underline{\mathrm{Hom}}_{\mathcal{M}}(-,M)\otimes N$$

where $\underline{\mathrm{Hom}}_{\mathcal{M}}$ means internal hom for right $\mathcal{C}$-module structure in $\mathcal{M}$ (Definition 1.3.4). Using the formulas satisfied by internal hom for right module category structure we see that images under I are indeed $\mathcal{C}$-module functors:

$$\begin{aligned} I(M\boxtimes N)(X\otimes M') &= \underline{\mathrm{Hom}}_{\mathcal{M}}(X\otimes M',M)\otimes N=\underline{\mathrm{Hom}}_{\mathcal{M}}(M',{}^*X\otimes M)\otimes N\\ &= X\otimes\underline{\mathrm{Hom}}_{\mathcal{M}}(M',M)\otimes N=X\otimes I(M\boxtimes N)(M'). \end{aligned}$$

Using similar relations one easily shows that I is $\mathcal{C}$-balanced. Hence I descends to a unique right-exact functor $\overline{I} : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \underline{Fun_{\mathcal{C}}}(\mathcal{M}^{op}, \mathcal{N})$ satisfying $\overline{I}B_{\mathcal{M},\mathcal{N}} = I$.

In the opposite direction define $J : \underline{Fun_{\mathcal{C}}}(\mathcal{M}^{op}, \mathcal{N}) \to \mathcal{M} \boxtimes \mathcal{N}$ as follows. For F a $\mathcal{C}$-module functor $\mathcal{M}^{op} \to \mathcal{N}$ let $J(F)$ be the object representing the functor $M \boxtimes N \mapsto \mathrm{Hom}(N, F(M))$, that is $\mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, J(F)) = \mathrm{Hom}_{\mathcal{N}}(N, F(M))$. Now denote by $J' : \underline{Fun_{\mathcal{C}}}(\mathcal{M}^{op}, \mathcal{N}) \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ the composition $B_{\mathcal{M},\mathcal{N}}J$.

Theorem 2.3.1. *Let $\mathcal{C}$ be a rigid monoidal category. For $\mathcal{M}$ a right $\mathcal{C}$-module category and $\mathcal{N}$ a left $\mathcal{C}$-module category there is a canonical equivalence*

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \simeq \underline{Fun_{\mathcal{C}}}(\mathcal{M}^{op}, \mathcal{N}).$$

If $\mathcal{M}, \mathcal{N}$ are bimodule categories this equivalence is bimodule.

Proof. In order to prove the theorem we simply show that $\overline{I}$ and J' defined above are quasi-inverses. This will follow easily if we can first show that I, J are quasi-inverses, and so we dedicate a separate lemma to proving this.

Lemma 2.3.2. *I, J are quasi-inverses.*

Proof. Let us first discuss internal homs for the $\mathcal{C}$-module structure in $\mathcal{M}\boxtimes\mathcal{N}$ induced by $X \otimes (M \boxtimes N) := (X \otimes M) \boxtimes N$. Let X be any simple object in $\mathcal{C}$. Then one shows, using the relations for internal hom in $\mathcal{M}$ and $\mathcal{N}$ separately, that the internal hom in $\mathcal{M} \boxtimes \mathcal{N}$ is given by

$$\underline{\mathrm{Hom}}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, S \boxtimes T) = \underline{\mathrm{Hom}}_{\mathcal{M}}(M, S) \otimes \underline{\mathrm{Hom}}_{\mathcal{N}}(N, T) \tag{7}$$

where the $\otimes$ is of course that in $\mathcal{C}$. Using this and the definitions of I and J we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, JI(S \boxtimes T)) &= \mathrm{Hom}_{\mathcal{N}}(N, \underline{\mathrm{Hom}}_{\mathcal{M}}(M, S) \otimes T) \\
&= \mathrm{Hom}_{\mathcal{C}}(1, \underline{\mathrm{Hom}}_{\mathcal{N}}(N, \underline{\mathrm{Hom}}_{\mathcal{M}}(M, S) \otimes T)) \\
&= \mathrm{Hom}_{\mathcal{C}}(1, \underline{\mathrm{Hom}}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, S \boxtimes T)) \\
&= \mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, S \boxtimes T).
\end{aligned}$$

The third line is an application of (7). The first and the last line imply that the functor $M \boxtimes N \mapsto \mathrm{Hom}_{\mathcal{N}}(N, \underline{\mathrm{Hom}}_{\mathcal{M}}(M, S) \otimes T)$ is represented by both $S \boxtimes T$ and $JI(S \boxtimes T)$, and these objects must therefore be equal up to a unique isomorphism, hence $JI \simeq id$.

Next we show that $IJ \simeq id$. Let F be any functor $\mathcal{M}^{op} \to \mathcal{N}$. From the first part of this proof we may write the following equation (up to unique linear isomorphism):

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{N}}(N, IJ(F)(M)) &= \mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, JIJ(F)) \\
&= \mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, J(F)) = \mathrm{Hom}_{\mathcal{N}}(N, F(M)).
\end{aligned}$$

Thus both $IJ(F)(M)$ and $F(M)$ are representing objects for the functor $N \mapsto \mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(M \boxtimes N, J(F))$ for each fixed $M \in \mathcal{M}$. Thus $IJ(F)(M) = F(M)$ up to a unique isomorphism. The collection of all such isomorphisms gives a natural isomorphism $IJ(F) \simeq F$, and therefore $IJ \simeq id$. This, with the first part of this proof, is equivalent to the statement that J is a quasi-inverse for I, proving the lemma. □

Now we are ready to complete the proof of Theorem 2.3.1. Using the definition of J' and $\overline{I}$ write $J'\overline{I}B_{\mathcal{M},\mathcal{N}} = B_{\mathcal{M},\mathcal{N}}JI \simeq B_{\mathcal{M},\mathcal{N}}$. By uniqueness (Lemma 2.1.9) it therefore follows that $J'\overline{I} \simeq id$. Also $\overline{I}J' = \overline{I}B_{\mathcal{M},\mathcal{N}}J = IJ \simeq id$, and we are done. □

As an immediate corollary to Theorem 2.3.1 and associativity of relative tensor product (equation 9, given below) we are able to prove a module category theoretic version of a theorem which appears in many places, notably as Frobenius reciprocity for induced representations of finite groups ([Ser77, §3.3]) and generally as a classical adjunction in the theory of modules.

Corollary 2.3.3 (Frobenius Reciprocity). *Let $\mathcal{M}$ be a $(\mathcal{C},\mathcal{D})$-bimodule category, $\mathcal{N}$ a $(\mathcal{D},\mathcal{F})$-module category, and $\mathcal{A}$ a $(\mathcal{C},\mathcal{F})$-module category. Then there is a canonical equivalence*

$$\underline{Fun}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}, \mathcal{A}) \simeq \underline{Fun}_{\mathcal{D}}(\mathcal{N}, \underline{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{A})) \tag{8}$$

as $(\mathcal{E},\mathcal{F})$-bimodule categories.

Proof. To see this we will first use Lemma 1.3.14 to describe the behaviour of the tensor product under *op*. Observe that

$$(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})^{op} \simeq \underline{Fun}_{\mathcal{D}}(\mathcal{M}^{op}, \mathcal{N})^{op} \simeq \underline{Fun}_{\mathcal{D}}(\mathcal{N}, \mathcal{M}^{op}) \simeq \mathcal{N}^{op} \boxtimes_{\mathcal{D}} \mathcal{M}^{op}$$

applying Theorem 2.3.1 twice (first and third) and Lemma 1.3.14 for the second step.

Now we may write

$$\begin{aligned}\underline{Fun}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}, \mathcal{A}) \simeq (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})^{op} \boxtimes_{\mathcal{C}} \mathcal{A} &\simeq (\mathcal{N}^{op} \boxtimes_{\mathcal{D}} \mathcal{M}^{op}) \boxtimes_{\mathcal{C}} \mathcal{A} \\ &\simeq \mathcal{N}^{op} \boxtimes_{\mathcal{D}} (\mathcal{M}^{op} \boxtimes_{\mathcal{C}} \mathcal{A}) \\ &\simeq \underline{Fun}_{\mathcal{C}}(\mathcal{N}, \underline{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{A})).\end{aligned}$$

□

Theorem 2.3.3 states that functor $\mathcal{M} \boxtimes_{\mathcal{D}} - : \mathcal{B}(\mathcal{D}, \mathcal{E}) \to \mathcal{B}(\mathcal{C}, \mathcal{E})$ is left adjoint to functor $\underline{Fun}_{\mathcal{C}}(\mathcal{M}, -) : \mathcal{B}(\mathcal{C}, \mathcal{E}) \to \mathcal{B}(\mathcal{D}, \mathcal{E})$.

CHAPTER III

ASSOCIATIVITY AND UNIT CONSTRAINTS FOR $\mathcal{B}(\mathcal{C})$

3.1 Tensor product associativity

In this section we discuss associativity of tensor product. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be tensor categories. Let $\mathcal{A}$ be a right $\mathcal{C}$-module category, $\mathcal{M}$ a $\mathcal{C}$-$\mathcal{D}$-bimodule category, $\mathcal{N}$ a $\mathcal{D}$-$\mathcal{E}$-bimodule category and $\mathcal{P}$ a left $\mathcal{E}$-module category. In an effort to save space we will at times abbreviate tensor product by juxtaposition.

Lemma 3.1.1. *$\mathcal{A}\boxtimes(\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}) \simeq (\mathcal{A}\boxtimes\mathcal{M})\boxtimes_{\mathcal{D}}\mathcal{N}$ and $(\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N})\boxtimes\mathcal{A} \simeq \mathcal{M}\boxtimes_{\mathcal{D}}(\mathcal{N}\boxtimes\mathcal{A})$ as abelian categories.*

Proof. Let $F : \mathcal{A} \boxtimes \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{S}$ be totally balanced (Definition 2.1.3). For A in $\mathcal{A}$ define functor $F_A : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{S}$ by $M \boxtimes N \mapsto F(A \boxtimes M \boxtimes N)$ on simple tensors and $f \mapsto F(id_A \boxtimes f)$ on morphisms. Note that functors F_A are balanced since F is totally balanced. Thus for any object A there is a unique functor $\overline{F_A} : \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{S}$ satisfying the diagram below left. The $\overline{F_A}$ allow us to define functor $F' : \mathcal{A} \boxtimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \to \mathcal{S} : A \boxtimes Q \mapsto \overline{F_A}(Q)$ whenever Q is an object of $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$

giving the commutative upper right triangle in the diagram on the right.

$$\begin{tikzcd}
\mathcal{M} \boxtimes \mathcal{N} \arrow[d, "B_{\mathcal{M},\mathcal{N}}"'] \arrow[dr, "F_A"] & \\
\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\overline{F_A}"'] & \mathcal{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \boxtimes \mathcal{M} \boxtimes \mathcal{N} \arrow[r, "B_{\mathcal{M},\mathcal{N}}"] \arrow[d, "B_{\mathcal{A}\boxtimes\mathcal{M},\mathcal{N}}"'] \arrow[dr, "F"] & \mathcal{A} \boxtimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \arrow[d, "F'"] \arrow[dl, dotted] \\
(\mathcal{A} \boxtimes \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\overline{F}"'] & \mathcal{S}
\end{tikzcd}$$

Since the functors $B_{\mathcal{A}\boxtimes\mathcal{M},\mathcal{N}}$, $B_{\mathcal{M},\mathcal{N}}$, $\overline{F}$ and F' are unique by the various universal properties by which they are defined, both $\mathcal{A} \boxtimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$ and $(\mathcal{A} \boxtimes \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N}$ are universal factorizations of F and must therefore be connected by a unique equivalence

$$\alpha^2_{\mathcal{A},\mathcal{M},\mathcal{N}} : \mathcal{A} \boxtimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \xrightarrow{\sim} (\mathcal{A} \boxtimes \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N}$$

(perforated arrow in diagram). One obtains natural equivalence $\alpha^1_{\mathcal{M},\mathcal{N},\mathcal{A}} : (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \boxtimes \mathcal{A} \xrightarrow{\sim} \mathcal{M} \boxtimes_{\mathcal{D}} (\mathcal{N} \boxtimes \mathcal{A})$ by giving the same argument "on the other side," i.e. by first defining $F_N : \mathcal{A} \boxtimes \mathcal{M} \to \mathcal{S}$ for fixed $N \in \mathcal{N}$ and proceeding analogously. □

Remark 3.1.2. For bimodule category $\mathcal{A}$ Remark 2.2.9 implies that α^i are bimodule equivalences.

Lemma 3.1.3. *For α^1 in Lemma 3.1.1 $(\mathcal{A} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}})\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}} : (\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}) \boxtimes \mathcal{N} \to \mathcal{A} \boxtimes_{\mathcal{C}} (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$ is balanced.*

Proof. Treat $\mathcal{M}$ as having right $\mathcal{C}$-module structure coming from its bimodule structure, and similarly give $\mathcal{N}$ its left $\mathcal{C}$-module structure. Recall, as above, we define $R_X : \mathcal{M} \to \mathcal{M}$ and $L_X : \mathcal{N} \to \mathcal{N}$ right and left action of $X \in \mathcal{C}$ on $\mathcal{M}, \mathcal{N}$ respectively. We will use superscripts to keep track of where $\mathcal{C}$-action is taking place, e.g. $R^{\mathcal{M}}_Y$ means right action of Y in $\mathcal{M}$. Recall also that we have right $\mathcal{D}$-action

$id_{\mathcal{A}} \boxtimes_{\mathcal{C}} R_X : \mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M} \to \mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}$, for $X \in \mathcal{D}$, which we denote also by R_X. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{A}\boxtimes\mathcal{M}\boxtimes\mathcal{N} \arrow[rrrr, "B\boxtimes id"] \arrow[drrrr, "L_X^{\mathcal{N}}"] \arrow[ddrrr, bend left, "B(id\boxtimes L_X)"] \arrow[ddrrr, bend right, "B(R_X\boxtimes id)"'] \arrow[ddrrr, phantom, "\overset{\simeq}{\Uparrow} b"] \arrow[dddr, "R_X^{\mathcal{M}}"] \arrow[ddd, "B\boxtimes id"'] & & & & \mathcal{A}\mathcal{M}\boxtimes\mathcal{N} \arrow[r, "id\boxtimes L_X"] & \mathcal{A}\mathcal{M}\boxtimes\mathcal{N} \arrow[dd, "\alpha^1"] \\
& & & & \mathcal{A}\boxtimes\mathcal{M}\boxtimes\mathcal{N} \arrow[ur, "B\boxtimes id"] \arrow[dl, "id\boxtimes B"'] \arrow[dr, "B"] & \\
& & & \mathcal{A}\boxtimes\mathcal{M}\mathcal{N} \arrow[dr, "B"] & & \mathcal{A}(\mathcal{M}\boxtimes\mathcal{N}) \arrow[dl, "id\boxtimes_{\mathcal{C}} B"] \\
\mathcal{A}\mathcal{M}\boxtimes\mathcal{N} \arrow[d, "R_X\boxtimes id"'] & \mathcal{A}\boxtimes\mathcal{M}\boxtimes\mathcal{N} \arrow[urr, "id\boxtimes B"] \arrow[dl, "B\boxtimes id"] \arrow[dr, "B"] & & & \mathcal{A}(\mathcal{M}\mathcal{N}) & \\
\mathcal{A}\mathcal{M}\boxtimes\mathcal{N} \arrow[rr, "\alpha^1"'] & & \mathcal{A}(\mathcal{M}\boxtimes\mathcal{N}) \arrow[urr, "id\boxtimes_{\mathcal{C}} B"] & & &
\end{tikzcd}$$

Leftmost rectangle is (definition of R_X) $\boxtimes\, id_{\mathcal{N}}$, top rectangle is tautologically $B \boxtimes L_X$, upper right and lower left triangles are definition of α^1, lower right rectangles definition of $id_{\mathcal{A}} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}}$ and b is $id_{\mathcal{A}} \boxtimes$ (balancing isomorphism for $B_{\mathcal{M},\mathcal{N}}$). An application of Lemma 2.1.9 then gives

$$(id_{\mathcal{A}} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}})\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}}(R_X \boxtimes id_{\mathcal{N}}) \overset{b}{\simeq} (id_{\mathcal{A}} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}})\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}}(id_{\mathcal{A}\boxtimes_{\mathcal{C}}\mathcal{M}} \boxtimes L_X)$$

Since b satisfies the balancing axiom (Definition 2.1.1) for $B_{\mathcal{M},\mathcal{N}}$ it satisfies it here. This is precisely the statement that $(\mathcal{A} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}})\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}}$ is balanced. □

Proposition 3.1.4. *If $\mathcal{A}$ and $\mathcal{N}$ are bimodules we have $(\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N} \simeq \mathcal{A} \boxtimes_{\mathcal{C}} (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$ as bimodule categories.*

Proof. We plan to define the stated equivalence as the image of the functor $(\mathcal{A} \boxtimes_{\mathcal{C}}$

$B_{\mathcal{M},\mathcal{N}})\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}} : (\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}) \boxtimes N \to \mathcal{A} \boxtimes_{\mathcal{C}} (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$ under $\mathcal{Y}$ (equation (2)). Lemma 3.1.3 implies that indeed $\mathcal{Y}$ is defined there. With notation as above define a^1 and a^2 using the universality of B by the following diagrams.

$$
\begin{tikzcd}
(\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}) \boxtimes \mathcal{N} \arrow[r, "\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}}"] \arrow[d, "B_{\mathcal{A}\mathcal{M},\mathcal{N}}"'] & \mathcal{A} \boxtimes_{\mathcal{C}} (\mathcal{M} \boxtimes \mathcal{N}) \arrow[d, "id_{\mathcal{A}} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}}"] \\
(\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "a^1_{\mathcal{A},\mathcal{M},\mathcal{N}}"'] & \mathcal{A} \boxtimes_{\mathcal{C}} (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \boxtimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \arrow[r, "\alpha^2_{\mathcal{A},\mathcal{M},\mathcal{N}}"] \arrow[d, "B_{\mathcal{A},\mathcal{M}\mathcal{N}}"'] & (\mathcal{A} \boxtimes \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[d, "B_{\mathcal{A},\mathcal{M}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}}"] \\
\mathcal{A} \boxtimes_{\mathcal{C}} (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \arrow[r, "a^2_{\mathcal{A},\mathcal{M},\mathcal{N}}"'] & (\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N}
\end{tikzcd}
$$

α^i are defined in Lemma 3.1.1.To see that a^1 and a^2 are quasi-inverses consider the diagram

$$
\begin{tikzcd}
\mathcal{A} \boxtimes (\mathcal{M}\mathcal{N}) \arrow[dr, "\alpha^2"'] \arrow[dd, "B_{\mathcal{A},\mathcal{M}\mathcal{N}}"'] & & \mathcal{A} \boxtimes \mathcal{M} \boxtimes \mathcal{N} \arrow[ll, "id_{\mathcal{A}} \boxtimes B_{\mathcal{M},\mathcal{N}}"'] \arrow[rr, "B_{\mathcal{A},\mathcal{M}\boxtimes\mathcal{N}}"] \arrow[dl, "B_{\mathcal{A}\boxtimes\mathcal{M},\mathcal{N}}"'] \arrow[dr, "B_{\mathcal{A},\mathcal{M}} \boxtimes id_{\mathcal{N}}"] & & \mathcal{A}(\mathcal{M} \boxtimes \mathcal{N}) \arrow[dd, "id_{\mathcal{A}} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}}"] \\
& (\mathcal{A} \boxtimes \mathcal{M})\mathcal{N} \arrow[dr, "B_{\mathcal{A},\mathcal{M}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}}"] & & (\mathcal{A}\mathcal{M}) \boxtimes \mathcal{N} \arrow[ur, "\alpha^1"'] \arrow[dl, "B_{\mathcal{A}\mathcal{M},\mathcal{N}}"'] & \\
\mathcal{A}(\mathcal{M}\mathcal{N}) \arrow[rr, "a^2"'] & & (\mathcal{A}\mathcal{M})\mathcal{N} \arrow[rr, "a^1"'] & & \mathcal{A}(\mathcal{M}\mathcal{N})
\end{tikzcd}
$$

The triangles in upper left and right are those defining α^2, α^1 respectively. The central square is the definition of $B_{\mathcal{A},\mathcal{M}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}}$, and the left and right squares those defining a^2 and a^1. Thus the perimeter commutes, giving

$$
\begin{aligned}
& a^1 a^2 B_{\mathcal{A},\mathcal{M}\mathcal{N}}(id_{\mathcal{A}} \boxtimes B_{\mathcal{M},\mathcal{N}}) = (id_{\mathcal{A}} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}}) B_{\mathcal{A},\mathcal{M}\boxtimes\mathcal{N}} \\
\Rightarrow \quad & a^1 a^2 B_{\mathcal{A},\mathcal{M}\mathcal{N}}(id_{\mathcal{A}} \boxtimes B_{\mathcal{M},\mathcal{N}}) = B_{\mathcal{A},\mathcal{M}\mathcal{N}}(id_{\mathcal{A}} \boxtimes B_{\mathcal{M},\mathcal{N}}) \\
\Rightarrow \quad & a^1 a^2 B_{\mathcal{A},\mathcal{M}\mathcal{N}}(\alpha^2)^{-1} B_{\mathcal{A}\boxtimes\mathcal{M},\mathcal{N}} = B_{\mathcal{A},\mathcal{M}\mathcal{N}}(\alpha^2)^{-1} B_{\mathcal{A}\boxtimes\mathcal{M},\mathcal{N}} \\
\Rightarrow \quad & a^1 a^2 B_{\mathcal{A},\mathcal{M}\mathcal{N}} = B_{\mathcal{A},\mathcal{M}\mathcal{N}} \\
\Rightarrow \quad & a^1 a^2 = id_{\mathcal{A}(\mathcal{M}\mathcal{N})}
\end{aligned}
$$

where the first implication follows from the square defining $id_{\mathcal{A}} \boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}}$, the second by the definition of α^2, the third by Lemma 2.1.9 (for $B_{\mathcal{A}\boxtimes\mathcal{M},\mathcal{N}}$, $B_{\mathcal{A},\mathcal{M}\mathcal{N}}$, resp.). Using a similar diagram one derives $a^2 a^1 = id_{(\mathcal{A}\mathcal{M})\mathcal{N}}$ hence the a^i are equivalences and by Remark 2.2.9 they are bimodule equivalences. $\square$

In what follows denote

$$a_{\mathcal{A},\mathcal{M},\mathcal{N}} := a^1_{\mathcal{A},\mathcal{M},\mathcal{N}} : (\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M}) \boxtimes_{\mathcal{D}} \mathcal{N} \simeq \mathcal{A} \boxtimes_{\mathcal{C}} (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}). \tag{9}$$

In order to prove coherence for a (Proposition 3.1.8) we will need a couple of simple technical lemmas together with results about the naturality of a. In the monoidal category setting associativity of monoidal product is required to be natural in each of its indices, which are taken as objects in the underlying category. In describing monoidal structure in the 2-category setting we also require associativity though stipulate that it be natural in its indices *up to* 2-isomorphism (see M.10 in Definition 1.4.2). For us this means, in the first index,

$$a_{F,\mathcal{M},\mathcal{N}} : a_{\mathcal{B},\mathcal{M},\mathcal{N}}(F\mathcal{M})\mathcal{N} \overset{\sim}{\Rrightarrow} F(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})a_{\mathcal{A},\mathcal{M},\mathcal{N}}$$

for bimodule functor $F : \mathcal{A} \to \mathcal{B}$. Similarly we need 2-isomorphisms for F in the remaining positions. The content of Proposition 3.1.6 is that all such 2-isomorphims are actually identity. Before stating it we give a definition to introduce a notational convenience.

Definition 3.1.5. For right exact right $\mathcal{C}$-module functor $F : \mathcal{A} \to \mathcal{B}$ define 1-cell $F\mathcal{M} := F \boxtimes_{\mathcal{C}} id_{\mathcal{M}} : \mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{M} \to \mathcal{B} \boxtimes_{\mathcal{C}} \mathcal{M}$ and note that $F\mathcal{M}$ is right exact. Similarly we can act on such functors from the right.

Proposition 3.1.6 (Associativity "2-naturality"). *We have*

$$a_{\mathcal{B},\mathcal{M},\mathcal{N}}(F\mathcal{M})\mathcal{N} = F(\mathcal{M}\mathcal{N})a_{\mathcal{A},\mathcal{M},\mathcal{N}}.$$

Analogous relations hold for the remaining indexing valencies of a.

Proof. We will prove the stated naturality of a for 1-cells appearing in the first index. A similar proof with analogous diagrams gives the others. Recall α^1 defined in Lemma

3.1.1. Consider the diagram:

$$
\begin{tikzcd}
(\mathcal{AM})\mathcal{N} \arrow[rrr, "(F\mathcal{M})\mathcal{N}"] \arrow[dddd, "a"'] & & & (\mathcal{BM})\mathcal{N} \arrow[dddd, "a"] \\
& (\mathcal{AM}) \boxtimes \mathcal{N} \arrow[r, "(F\mathcal{M})"] \arrow[ul, "B_{\mathcal{AM},\mathcal{N}}"'] \arrow[dd, bend right=60, "\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}}"'] & (\mathcal{BM}) \boxtimes \mathcal{N} \arrow[ur, "B_{\mathcal{BM},\mathcal{N}}"] \arrow[dd, bend left=60, "\alpha^1_{\mathcal{B},\mathcal{M},\mathcal{N}}"] & \\
& \mathcal{A} \boxtimes \mathcal{M} \boxtimes \mathcal{N} \arrow[r, "F"'] \arrow[u, "B_{\mathcal{A},\mathcal{M}}"'] \arrow[d, "B_{\mathcal{A},\mathcal{M}\boxtimes\mathcal{N}}"] & \mathcal{B} \boxtimes \mathcal{M} \boxtimes \mathcal{N} \arrow[u, "B_{\mathcal{B},\mathcal{M}}"] \arrow[d, "B_{\mathcal{B},\mathcal{M}\boxtimes\mathcal{N}}"'] & \\
& \mathcal{A}(\mathcal{M} \boxtimes \mathcal{N}) \arrow[r, "F(\mathcal{M}\boxtimes\mathcal{N})"'] \arrow[dl, "\mathcal{A}B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{B}(\mathcal{M} \boxtimes \mathcal{N}) \arrow[dr, "\mathcal{B}B_{\mathcal{M},\mathcal{N}}"] & \\
\mathcal{A}(\mathcal{MN}) \arrow[rrr, "F(\mathcal{MN})"'] & & & \mathcal{B}(\mathcal{MN})
\end{tikzcd}
$$

The top, bottom and center rectangles follow from Definition 3.1.5 and definition of tensor product of functors. Commutativity of all other subdiagrams is given in proof of Proposition 3.1.4. External contour is the stated relation. □

Remark 3.1.7. Observe that the proof of Proposition 3.1.6 also gives 2-naturality of α^1: the center square with attached arches gives the equation

$$\alpha^1_{\mathcal{B},\mathcal{M},\mathcal{N}}((F\mathcal{M}) \boxtimes id_{\mathcal{N}}) = F(\mathcal{M} \boxtimes \mathcal{N})\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}}. \tag{10}$$

Lemma 3.1.8. *The hexagon*

$$
\begin{tikzcd}
\mathcal{A}(\mathcal{MN}) \boxtimes \mathcal{P} \arrow[d, "\alpha^1_{\mathcal{A},\mathcal{MN},\mathcal{P}}"'] & (\mathcal{AM})\mathcal{N} \boxtimes \mathcal{P} \arrow[l, "a_{\mathcal{A},\mathcal{M},\mathcal{N}}"'] \arrow[d, "B_{(\mathcal{AM})\mathcal{N},\mathcal{P}}"] \\
\mathcal{A}(\mathcal{MN} \boxtimes \mathcal{P}) \arrow[d, "B_{\mathcal{MN},\mathcal{P}}"'] & ((\mathcal{AM})\mathcal{N})\mathcal{P} \arrow[d, "a_{\mathcal{AMN}}"] \\
\mathcal{A}((\mathcal{MN})\mathcal{P}) & (\mathcal{A}(\mathcal{MN}))\mathcal{P} \arrow[l, "a_{\mathcal{A},\mathcal{MN},\mathcal{P}}"]
\end{tikzcd}
$$

commutes.

Proof. The arrow $B_{\mathcal{A}(\mathcal{M}\mathcal{N}),\mathcal{P}}$ drawn from the upper-left most entry in the hexagon to the lower-right most entry divides the diagram into a pair of rectangles. The upper right rectangle is the definition of $a_{\mathcal{A},\mathcal{M},\mathcal{N}} \boxtimes_{\mathcal{E}} id_{\mathcal{P}}$ and the lower left rectangle is the definition of $a_{\mathcal{A},\mathcal{M}\mathcal{N},\mathcal{P}}$. ☐

In the case of monoidal categories the relevant structure isomorphisms are required to satisfy axioms which take the form of commuting diagrams. In the 2-monoidal case we make similar requirements of the structure morphisms but here, because of the presence of higher dimensional structures, it is necessary to weaken these axioms by requiring only that their diagrams commute up to some 2-morphisms. Above we have defined a 2-associativity isomorphism $a_{\mathcal{M},\mathcal{N},\mathcal{P}} : (\mathcal{M}\mathcal{N})\mathcal{P} \to \mathcal{M}(\mathcal{N}\mathcal{P})$. In the definition of monoidal 2-category a is required to satisfy the pentagon which appears in the lower dimensional monoidal case, but only up to 2-isomorphism. The content of Proposition 3.1.9 is that, in the 2-category of bimodule categories, the monoidal structure $\boxtimes_{\mathcal{C}}$ *strictly* satisfies the associated hexagon just as in the monoidal category setting. For us this means that the 2-isomorphism $a_{\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{P}}$ (see M9. Definition 1.4.2) is actually identity for any bimodule categories $\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{P}$ for which the relevant tensor products make sense.

Proposition 3.1.9 (2-associativity hexagon). *The diagram of functors commutes.*

$$\begin{tikzcd}
((\mathcal{A}\mathcal{M})\mathcal{N})\mathcal{P} \arrow[r, "a_{\mathcal{A}\mathcal{M},\mathcal{N},\mathcal{P}}"'] \arrow[d, "a_{\mathcal{A},\mathcal{M},\mathcal{N}}\boxtimes_{\mathcal{E}}\mathcal{P}"'] & (\mathcal{A}\mathcal{M})(\mathcal{N}\mathcal{P}) \arrow[dd, "a_{\mathcal{A},\mathcal{M},\mathcal{N}\mathcal{P}}"'] \\
(\mathcal{A}(\mathcal{M}\mathcal{N}))\mathcal{P} \arrow[d, "a_{\mathcal{A},\mathcal{M}\mathcal{N},\mathcal{P}}"'] & \\
\mathcal{A}((\mathcal{M}\mathcal{N})\mathcal{P}) \arrow[r, "\mathcal{A}\boxtimes_{\mathcal{C}} a_{\mathcal{M},\mathcal{N},\mathcal{P}}"'] & \mathcal{A}(\mathcal{M}(\mathcal{N}\mathcal{P}))
\end{tikzcd}$$

Proof. Consider the diagram below. We first show that the faces peripheral to the embedded hexagon commute and then show that the extended perimeter commutes.

$$\begin{tikzcd}
(\mathcal{A}\mathcal{M})\mathcal{N}\boxtimes\mathcal{P} \arrow[rrr, "\alpha^1_{\mathcal{A}\mathcal{M},\mathcal{N},\mathcal{P}}"] \arrow[dd, "a_{\mathcal{A},\mathcal{M},\mathcal{N}}"'] \arrow[dr, "B_{(\mathcal{A}\mathcal{M})\mathcal{N},\mathcal{P}}"] & & & (\mathcal{A}\mathcal{M})(\mathcal{N}\boxtimes\mathcal{P}) \arrow[dl, "id_{\mathcal{A}\mathcal{M}}\boxtimes_{\mathcal{D}} B_{\mathcal{N},\mathcal{A}}"'] \arrow[dddd, "a_{\mathcal{A},\mathcal{M},\mathcal{N}\boxtimes\mathcal{P}}"] \\
& ((\mathcal{A}\mathcal{M})\mathcal{N})\mathcal{P} \arrow[r, "a_{\mathcal{A}\mathcal{M},\mathcal{N},\mathcal{P}}"'] \arrow[d, "a_{\mathcal{A},\mathcal{M},\mathcal{N}}\boxtimes_{\mathcal{E}} id_{\mathcal{P}}"] & (\mathcal{A}\mathcal{M})(\mathcal{N}\mathcal{P}) \arrow[dd, "a_{\mathcal{A},\mathcal{M},\mathcal{N}\mathcal{P}}"] & \\
\mathcal{A}(\mathcal{M}\mathcal{N})\boxtimes\mathcal{P} \arrow[dd, "\alpha^1_{\mathcal{A},\mathcal{M}\mathcal{N},\mathcal{P}}"'] & (\mathcal{A}(\mathcal{M}\mathcal{N}))\mathcal{P} \arrow[d, "a_{\mathcal{A},\mathcal{M}\mathcal{N},\mathcal{P}}"] & & \\
& \mathcal{A}((\mathcal{M}\mathcal{N})\mathcal{P}) \arrow[r, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}} a_{\mathcal{M},\mathcal{N},\mathcal{P}}"'] & \mathcal{A}(\mathcal{M}(\mathcal{N}\mathcal{P})) & \\
\mathcal{A}(\mathcal{M}\mathcal{N}\boxtimes\mathcal{P}) \arrow[rrr, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}}\alpha^1_{\mathcal{M}\mathcal{N}\mathcal{P}}"'] \arrow[ur, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}} B_{\mathcal{M}\mathcal{N},\mathcal{P}}"] & & & \mathcal{A}(\mathcal{M}(\mathcal{N}\boxtimes\mathcal{P})) \arrow[ul, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}}(id_{\mathcal{M}}\boxtimes_{\mathcal{D}} B_{\mathcal{N},\mathcal{P}})"']
\end{tikzcd}$$

The top rectangle is the definition of $a_{\mathcal{A}\mathcal{M},\mathcal{N},\mathcal{P}}$, the rectangle on the right is naturality of a as in Proposition 3.1.6, the bottom rectangle the definition of a tensored on the left by $\mathcal{A}$, and the hexagon is Lemma 3.1.8. To prove commutativity of the extended perimeter subdivide it as indicated below.

$$
\begin{tikzcd}
(\mathcal{AM})\mathcal{N}\boxtimes\mathcal{P} \arrow[rrrr, "\alpha^1_{\mathcal{AM},\mathcal{N},\mathcal{P}}"] \arrow[dd, "a_{\mathcal{A},\mathcal{M},\mathcal{N}\boxtimes\mathcal{P}}"'] & & & & (\mathcal{AM})(\mathcal{N}\boxtimes\mathcal{P}) \arrow[dddd, "a_{\mathcal{A},\mathcal{M},\mathcal{N}}"] \\
& & & (\mathcal{AM})\boxtimes\mathcal{N}\boxtimes\mathcal{P} \arrow[ulll, "B_{\mathcal{AM},\mathcal{N}}"] \arrow[ur, "B_{\mathcal{AM},\mathcal{N}\boxtimes\mathcal{P}}"'] \arrow[dl, "\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}}"'] \arrow[dd, "\alpha^1_{\mathcal{A},\mathcal{M},\mathcal{N}\boxtimes\mathcal{P}}"] & \\
\mathcal{A}(\mathcal{MN})\boxtimes\mathcal{P} \arrow[dd, "\alpha^1_{\mathcal{A},\mathcal{MN},\mathcal{P}}"'] & & \mathcal{A}(\mathcal{M}\boxtimes\mathcal{N})\boxtimes\mathcal{P} \arrow[ll, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}}"'] \arrow[dr, "\alpha^1_{\mathcal{A},\mathcal{M}\boxtimes\mathcal{N},\mathcal{P}}"'] & & \\
& & & \mathcal{A}(\mathcal{M}\boxtimes\mathcal{N}\boxtimes\mathcal{P}) \arrow[dlll, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}}(B_{\mathcal{M},\mathcal{N}}\boxtimes\mathcal{P})"'] \arrow[dr, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}} B_{\mathcal{M},\mathcal{N}\boxtimes\mathcal{P}}"] & \\
\mathcal{A}(\mathcal{MN}\boxtimes\mathcal{P}) \arrow[rrrr, "id_{\mathcal{A}}\boxtimes_{\mathcal{C}}\alpha^1_{\mathcal{MNP}}"'] & & & & \mathcal{A}(\mathcal{M}(\mathcal{N}\boxtimes\mathcal{P}))
\end{tikzcd}
$$

The upper and lower triangles are the definitions of $\alpha^1_{\mathcal{AM},\mathcal{N},\mathcal{P}}$ and $\mathcal{A}\boxtimes_*$ (definition of $\alpha^1_{\mathcal{M},\mathcal{N},\mathcal{P}}$), respectively (Lemma 3.1.1). Right rectangle is definition of $a_{\mathcal{A},\mathcal{M},\mathcal{N}\boxtimes\mathcal{P}}$. Upper left rectangle is (definition of $a_{\mathcal{A},\mathcal{M},\mathcal{N}}$) $\boxtimes\,\mathcal{P}$, and the lower left rectangle is explained in Remark 3.1.7. The central triangle commutes as follows. Using the definition of α^1 given in the proof of Proposition 3.1.4 we can draw the diagram

$$
\begin{tikzcd}
& & \mathcal{AM}\boxtimes\mathcal{N}\boxtimes\mathcal{P} \arrow[dl, "\alpha_2"'] \arrow[dd, "\alpha_1"] \\
\mathcal{A}\boxtimes\mathcal{M}\boxtimes\mathcal{N}\boxtimes\mathcal{P} \arrow[r, "B_2"] \arrow[urr, bend left, "B_1"] \arrow[drr, bend right, "B_3"'] & \mathcal{A}(\mathcal{M}\boxtimes\mathcal{N})\boxtimes\mathcal{P} \arrow[dr, "\alpha_3"] & \\
& & \mathcal{A}(\mathcal{M}\boxtimes\mathcal{N}\boxtimes\mathcal{P})
\end{tikzcd}
$$

where we have abbreviated the various α^1 appearing in the statement of the lemma by α_i and associated functors B occurring in their definitions B_i in such a way that

$$\alpha_1 B_1 = B_3, \quad \alpha_3 B_2 = B_3, \quad \alpha_2 B_1 = B_2.$$

These equations imply $B_3 = \alpha_3 \alpha_2 \alpha_1^{-1} B_3$ where $B_3 = B_{\mathcal{A},\mathcal{M}\boxtimes\mathcal{N}\boxtimes\mathcal{P}}$. Apply Lemma 2.1.9 to write $id = \alpha_3 \alpha_2 \alpha_1^{-1}$. Now equating paths in the large diagram allows us to write

$$a_{\mathcal{A},\mathcal{M},\mathcal{N}\boxtimes\mathcal{P}}(\alpha^1_{\mathcal{A}\mathcal{M},\mathcal{N},\mathcal{P}})(B_{\mathcal{A}\mathcal{M},\mathcal{N}} \boxtimes \mathcal{P}) = (\mathcal{A} \boxtimes_* \alpha^1_{\mathcal{M}\mathcal{N}\mathcal{P}})(\alpha^1_{\mathcal{A},\mathcal{M}\mathcal{N},\mathcal{P}})(a_{\mathcal{A},\mathcal{M},\mathcal{N}} \boxtimes \mathcal{P}) B_{\mathcal{A}\mathcal{M},\mathcal{N}} \boxtimes \mathcal{P}$$

and a final application of Lemma 2.1.9 gives the relation expressing commutativity of outer pentagon. $\square$

Let $\mathcal{M}_i$ be a $(\mathcal{C}_{i-1}, \mathcal{C}_i)$-bimodule category tensor categories $\mathcal{C}_i$ $0 \leq i \leq n+1$. Then one extends the arguments above to completely balanced functors (Definition 2.1.2) of larger index to show that any meaningful arrangement of parentheses in the expression $\mathcal{M}_1 \boxtimes_{\mathcal{C}_1} \mathcal{M}_2 \cdots \boxtimes_{\mathcal{C}_{n-1}} \mathcal{M}_n$ results in an equivalent bimodule category.

Remark 3.1.10. Proposition 3.1.9 implies that the 2-morphism described in M9 of Definition 1.4.2 is actually identity. The primary polytope associated to associativity in the monoidal 2-category setting is the Stasheff polytope which commutes in this case. It is obvious that the modified tensor product $\hat{\otimes}$ with associativity ([KV91] §4) is identity and that nearly every face commutes strictly. The two non-trivial remaining faces (one on each hemisphere) agree trivially. We refer the reader to the original paper for details and notation.

3.2 Unit constraints

Recall from Proposition 2.2.5 the equivalences $l_{\mathcal{M}} : \mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{D} \simeq \mathcal{M}$ and $r_{\mathcal{M}} : \mathcal{C}\boxtimes_{\mathcal{C}}\mathcal{M} \simeq \mathcal{M}$. This section's first proposition explains how l, r interact with 2-associativity.

Proposition 3.2.1. $(id_{\mathcal{M}} \boxtimes_{\mathcal{D}} l_{\mathcal{N}})a_{\mathcal{M},\mathcal{N},\mathcal{E}} = l_{\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}}$, $r_{\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}}(a_{\mathcal{C},\mathcal{M},\mathcal{N}}) = r_{\mathcal{M}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}}$. *Also the triangle*

$$\begin{tikzcd}
(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D}) \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[d, "a_{\mathcal{M},\mathcal{D},\mathcal{N}}"'] \arrow[dr, "l_{\mathcal{M}}\boxtimes_{\mathcal{D}} id_{\mathcal{N}}"] & \\
\mathcal{M} \boxtimes_{\mathcal{D}} (\mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N}) \arrow[r, "id_{\mathcal{M}}\boxtimes_{\mathcal{D}} r_{\mathcal{N}}"'] & \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}
\end{tikzcd}$$

commutes up to a natural isomorphism.

Proof. The first two statements follow easily from definitions of α^1 (Lemma 3.1.1), module structure in $\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}$ and those of l and r. This means that the 2-isomorphisms ρ and λ in M11 of Defintition 1.4.2 are both trivial.

The diagram below relating l and r commutes only up to balancing isomorphism b for $B_{\mathcal{M},\mathcal{N}}$ where we write $b : B_{\mathcal{M},\mathcal{N}}(\otimes \boxtimes id_{\mathcal{N}}) \Rightarrow B_{\mathcal{M},\mathcal{N}}(id_{\mathcal{M}} \boxtimes \otimes)$.

$$\begin{tikzcd}
(\mathcal{M}\boxtimes\mathcal{D})\mathcal{N} \arrow[rrrr, "\alpha_{\mathcal{M},\mathcal{D},\mathcal{N}}"] \arrow[dddd, "B_{\mathcal{M},\mathcal{D}}"'] \arrow[dddrr, "\otimes"'] & & & & \mathcal{M}\boxtimes(\mathcal{D}\mathcal{N}) \arrow[dddd, "B_{\mathcal{M},\mathcal{C}\mathcal{N}}"] \\
& & \mathcal{M}\boxtimes\mathcal{D}\boxtimes\mathcal{N} \arrow[ull, "B_{\mathcal{M}\boxtimes\mathcal{D},\mathcal{N}}"'] \arrow[urr, "B_{\mathcal{D},\mathcal{N}}"] \arrow[dr, "B_{\mathcal{M},\mathcal{D}\boxtimes\mathcal{N}}"] & & \\
& {} \arrow[r, phantom, "\overset{b}{\Rightarrow}"] & & \mathcal{M}(\mathcal{D}\boxtimes\mathcal{N}) \arrow[dl, "\otimes"] \arrow[ddr, "B_{\mathcal{D},\mathcal{N}}"] & \\
& & \mathcal{M}\mathcal{N} & & \\
(\mathcal{M}\mathcal{D})\mathcal{N} \arrow[urr, "l_{\mathcal{M}}"'] \arrow[rrrr, "a_{\mathcal{M},\mathcal{D},\mathcal{N}}"'] & & & & \mathcal{M}(\mathcal{D}\mathcal{N}) \arrow[ull, "r_{\mathcal{N}}"']
\end{tikzcd}$$

Top triangle is definition of α^1, rectangle is definition of $id_{\mathcal{M}} \boxtimes_{\mathcal{D}} B_{\mathcal{D},\mathcal{N}}$, lower right triangle is $\mathcal{M} \boxtimes_{\mathcal{D}}$(definition of $r_{\mathcal{N}}$), triangle on left is (definition of $l_{\mathcal{M}}$)$\boxtimes_{\mathcal{D}} \mathcal{N}$, and central weakly commuting rectangle is definition of balancing b for $B_{\mathcal{M},\mathcal{N}}$. The perimeter is a diagram occuring in the proof of Proposition 3.1.4 (we have been sloppy with the labeling of the arrow across the top). Since all other non-labeled faces commute we may write, after chasing paths around the diagram,

$$l_{\mathcal{M}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}}(B_{\mathcal{M},\mathcal{D}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}})B_{\mathcal{M}\boxtimes\mathcal{D},\mathcal{N}} \overset{b}{\simeq} (id_{\mathcal{M}} \boxtimes_{\mathcal{D}} r_{\mathcal{N}})a_{\mathcal{M},\mathcal{D},\mathcal{N}}(B_{\mathcal{M},\mathcal{D}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}})B_{\mathcal{M}\boxtimes\mathcal{D},\mathcal{N}}.$$

Applying Lemma 2.1.9 twice we obtain a unique natural isomorphism

$$\mu_{\mathcal{M},\mathcal{N}} : l_{\mathcal{M}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}} \xrightarrow{\sim} (id_{\mathcal{M}} \boxtimes_{\mathcal{D}} r_{\mathcal{N}})a_{\mathcal{M},\mathcal{D},\mathcal{N}} \tag{11}$$

having the property that $\mu_{\mathcal{M},\mathcal{N}} * ((B_{\mathcal{M},\mathcal{D}} \boxtimes_{\mathcal{D}} id_{\mathcal{N}})B_{\mathcal{M}\boxtimes\mathcal{D},\mathcal{N}}) = b$, the balancing in $B_{\mathcal{M},\mathcal{N}}$. □

CHAPTER IV

PROOF OF THEOREM 0.2.1

4.1 The commuting polytopes

In this section we finish verifying that the list of requirements given in the definition of monoidal 2-category ([KV91]), Definition 1.4.2 of this thesis, are substantiated by the scenario where we take as underlying 2-category $\mathcal{B}(\mathcal{C})$. Recall that for a fixed monoidal category $\mathcal{C}$ the 2-category $\mathcal{B}(\mathcal{C})$ is defined as having 0-cells $\mathcal{C}$-bimodule categories, 1-cells $\mathcal{C}$-bimodule functors and 2-cells monoidal natural transformations. M1-M11 are evident given what we have discussed so far; explicitly, and in order, these are given in Proposition 2.2.5, Proposition 2.2.3, Definition 3.1.5, Remark 2.2.2 (take one of the 2-cells to be identity transformation on identity functor), Equation 9, Proof of Proposition 2.2.5, Definition 3.1.5 (trivial, composition with id commutes), Polytope 4.1.3, Proposition 3.1.9 (trivial), Proposition 3.1.6 ($a_{F,\mathcal{M},\mathcal{N}} = id$ for bimodule functor F), Proof of Proposition 3.2.1. Commutativity of the Stasheff polytope follows from Proposition 3.1.9 (see Remark 3.1.10).

The data introduced throughout are required to satisfy several commuting polytopes describing how they are to interact. Fortunately for us only a few of these require checking since many of the structural morphisms above are identity. Because of this we prove below only those verifications which are not immediately evident. Recall (Definition 3.1.5) that we define action $\mathcal{M}F$ of bimodule category $\mathcal{M}$ on module functor F.

Polytope 4.1.1. *For $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \mathcal{B}(\mathcal{C})$, the pastings*

$$\begin{tikzcd}
((\mathcal{C}\mathcal{M})\mathcal{N})\mathcal{P} \arrow[rr, "(r_{\mathcal{M}}\mathcal{N})\mathcal{P}"] \arrow[d, "a_{\mathcal{C},\mathcal{M},\mathcal{N}}\mathcal{P}"'] \arrow[dr, "a_{\mathcal{C}\mathcal{M},\mathcal{N},\mathcal{P}}"] & \arrow[d, phantom, "\Uparrow a_{r_{\mathcal{M}},\mathcal{N},\mathcal{P}}"] & (\mathcal{M}\mathcal{N})\mathcal{P} \arrow[d, "a_{\mathcal{M},\mathcal{N},\mathcal{P}}"] \\
(\mathcal{C}(\mathcal{M}\mathcal{N}))\mathcal{P} \arrow[d, "a_{\mathcal{C},\mathcal{M}\mathcal{N},\mathcal{P}}"'] \arrow[r, phantom, "\Downarrow a_{\mathcal{C},\mathcal{M},\mathcal{N},\mathcal{P}}"] & (\mathcal{C}\mathcal{M})(\mathcal{N}\mathcal{P}) \arrow[r, "r_{\mathcal{M}}\mathcal{N}\mathcal{P}"] \arrow[dr, "a_{\mathcal{C},\mathcal{M},\mathcal{N}\mathcal{P}}"'] & \mathcal{M}(\mathcal{N}\mathcal{P}) \\
\mathcal{C}((\mathcal{M}\mathcal{N})\mathcal{P}) \arrow[rr, "\mathcal{C}a_{\mathcal{M},\mathcal{N},\mathcal{P}}"'] & & \mathcal{C}(\mathcal{M}(\mathcal{N}\mathcal{P})) \arrow[u, "r_{\mathcal{M}(\mathcal{N}\mathcal{P})}"'] \arrow[u, phantom, shift left=6, "\Uparrow \rho_{\mathcal{M},\mathcal{N}\mathcal{P}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
((\mathcal{C}\mathcal{M})\mathcal{N})\mathcal{P} \arrow[rr, "(r_{\mathcal{M}}\mathcal{N})\mathcal{P}"] \arrow[d, "a_{\mathcal{C},\mathcal{M},\mathcal{N}}\mathcal{P}"'] \arrow[dr, phantom, "\Uparrow \rho_{\mathcal{M},\mathcal{N}}\mathcal{P}"] & & (\mathcal{M}\mathcal{N})\mathcal{P} \arrow[d, "a_{\mathcal{M},\mathcal{N},\mathcal{P}}"] \\
(\mathcal{C}(\mathcal{M}\mathcal{N}))\mathcal{P} \arrow[urr, "r_{\mathcal{M}\mathcal{N}}\mathcal{P}"] \arrow[d, "a_{\mathcal{C},\mathcal{M}\mathcal{N},\mathcal{P}}"'] \arrow[r, phantom, "\Uparrow \rho_{\mathcal{M}\mathcal{N},\mathcal{P}}"] & \arrow[d, phantom, "\Uparrow r_{a_{\mathcal{M},\mathcal{N},\mathcal{P}}}"] & \mathcal{M}(\mathcal{N}\mathcal{P}) \\
\mathcal{C}((\mathcal{M}\mathcal{N})\mathcal{P}) \arrow[uurr, "r_{(\mathcal{M}\mathcal{N})\mathcal{P}}"] \arrow[rr, "\mathcal{C}a_{\mathcal{M},\mathcal{N},\mathcal{P}}"'] & & \mathcal{C}(\mathcal{M}(\mathcal{N}\mathcal{P})) \arrow[u, "r_{\mathcal{M}(\mathcal{N}\mathcal{P})}"']
\end{tikzcd}$$

correspond to the same 2-cell.

Proof. Note that every face commutes (all labeling 2-cells are identity) except for $r_{a_{\mathcal{M},\mathcal{N},\mathcal{P}}}$ in the second diagram. Thus the pastings give the same 2-cell if we can show $r_{a_{\mathcal{M},\mathcal{N},\mathcal{P}}}$ is also identity. By comments following the proof of Corollary 2.2.6 this is equivalent to showing that $a_{\mathcal{M},\mathcal{N},\mathcal{P}}$ is a strict module functor, i.e. that the module linearity w associated to a is identity. For $X \in \mathcal{C}$ note that for simple tensor $(MN)P = B_{\mathcal{M}\mathcal{N},\mathcal{P}}(B_{\mathcal{M},\mathcal{N}} \boxtimes \mathcal{P})(M \boxtimes N \boxtimes P)$

$$a_{\mathcal{M},\mathcal{N},\mathcal{P}}(X \otimes (MN)P) = (X \otimes M)(NP) = X \otimes a_{\mathcal{M},\mathcal{N},\mathcal{P}}((MN)P)$$

so by two applications of Lemma 2.1.9 $w = id$. $\square$

The remaining four polytopes describe 2-naturality of the action of the unit object in our monoidal 2-category (recall that the unit object in $\mathcal{B}(\mathcal{C})$ is $\mathcal{C}$ itself). The first concerns 2-naturality of μ, λ and ρ.

Polytope 4.1.2. *For $F : \mathcal{M} \to \mathcal{M}'$ a morphism in $\mathcal{B}(\mathcal{C})$ and any $\mathcal{C}$-bimodule category $\mathcal{N}$ the polytopes*

$$
\begin{tikzcd}
(\mathcal{M}\mathcal{C})\mathcal{N} \arrow[rr, "a"] \arrow[dr, "l_{\mathcal{M}}"'] \arrow[dd, "(F\mathcal{C})\mathcal{N}"'] \arrow[d, phantom, "\Rightarrow l_{F\mathcal{N}}"] & \arrow[d, phantom, "\Uparrow \mu_{\mathcal{M},\mathcal{N}}"] & \mathcal{M}(\mathcal{C}\mathcal{N}) \arrow[dl, "r_{\mathcal{N}}"] \arrow[dd, "F(\mathcal{C}\mathcal{N})"] \\
{} & \mathcal{M}\mathcal{N} \arrow[dd, "F\mathcal{N}"] & \\
(\mathcal{M}'\mathcal{C})\mathcal{N} \arrow[rr, "a"] \arrow[dr, "l_{\mathcal{M}'}"'] & \arrow[d, phantom, "\Uparrow \mu_{\mathcal{M}',\mathcal{N}}"] & \mathcal{M}'(\mathcal{C}\mathcal{N}) \arrow[dl, "r_{\mathcal{N}}"] \\
& \mathcal{M}'\mathcal{N} &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{N}\mathcal{C})\mathcal{M} \arrow[rr, "a"] \arrow[dr, "l_{\mathcal{N}}"'] \arrow[dd, "(\mathcal{N}\mathcal{C})F"'] & \arrow[d, phantom, "\Uparrow \mu_{\mathcal{N},\mathcal{M}}"] & \mathcal{N}(\mathcal{C}\mathcal{M}) \arrow[dl, "r_{\mathcal{M}}"] \arrow[dd, "\mathcal{N}(\mathcal{C}F)"] \arrow[d, phantom, "\Rightarrow \mathcal{N}r_F"] \\
{} & \mathcal{N}\mathcal{M} \arrow[dd, "\mathcal{N}F"] & {} \\
(\mathcal{N}\mathcal{C})\mathcal{M}' \arrow[rr, "a"] \arrow[dr, "l_{\mathcal{N}}"'] & \arrow[d, phantom, "\Uparrow \mu_{\mathcal{N},\mathcal{M}}"] & \mathcal{N}(\mathcal{C}\mathcal{M}') \arrow[dl, "r_{\mathcal{M}'}"] \\
& \mathcal{N}\mathcal{M}' &
\end{tikzcd}
$$

commute. Similarly there are commuting prisms for upper left vertex corresponding to the remaining four permutations of $\mathcal{M}, \mathcal{C}, \mathcal{N}$ with upper and lower faces commuting up to either λ or ρ.

In [KV91] these triangular prisms are labeled $(\to \otimes 1 \otimes \bullet), (1 \otimes \to \otimes \bullet)$, etc.

Proof. We verify commutativity of the second polytope. Commutativity of the other prisms is proved similarly. Denote by $*$ mixed composition of cells. Commutativity of polytope on the right is equivalent to the equation

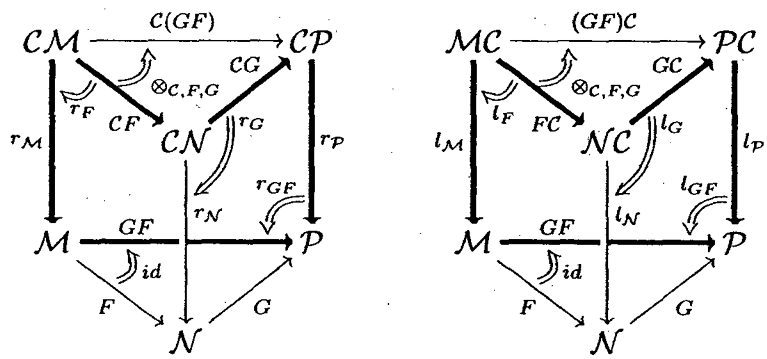
$$(id_{\mathcal{N}} \boxtimes_{\mathcal{C}} \overline{f})((id_{\mathcal{N}} \boxtimes_{\mathcal{C}} F) * \mu_{\mathcal{N},\mathcal{M}}) = \mu_{\mathcal{N},\mathcal{M}'} * (id_{\mathcal{N}\boxtimes_{\mathcal{C}}\mathcal{C}} \boxtimes_{\mathcal{C}} F) \tag{12}$$

where f is module structure of F and $\overline{f} = r_F$ (recall Corollary 2.2.6). Let LHS and RHS denote the left and right sides of (12). Then one easily shows that both

LHS$*((B_{\mathcal{N},\mathcal{C}\boxtimes_\mathcal{C}}id_\mathcal{M})B_{\boxtimes_\mathcal{N}\mathcal{C},\mathcal{M}})_{M\boxtimes X\boxtimes N}$ and RHS$*((B_{\mathcal{N},\mathcal{C}\boxtimes_\mathcal{C}}id_\mathcal{M})B_{\boxtimes_\mathcal{N}\mathcal{C},\mathcal{M}})_{M\boxtimes X\boxtimes N}$, for $N \in \mathcal{N}, X \in \mathcal{C}, M \in \mathcal{M}$, are equal to $b'_{N,X,F(M)}$ where b' is the balancing for $B_{\mathcal{N},\mathcal{M}'}$. Two applications of Lemma 2.1.9 now imply that LHS=RHS. $\square$

The next polytope concerns functoriality of the 2-cells l_F, r_F.

Polytope 4.1.3. *Let $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{G} \mathcal{P}$ be composible 1-morphisms in $\mathcal{B}(\mathcal{C})$. Then the prisms*



commute.

Proof. We prove commutativity of the first prism. Commutativity of the second follows similarly. It is obvious that $\otimes_{\mathcal{C},F,G}$ is trivial (it is just composition of functors). First polytope is the condition $r_{GF} = (G * r_F)(r_G * \mathcal{C}F)$. Let f be left $\mathcal{C}$-linearity for F, g that for G. Then $(G, g)(F, f) := (GF, g \bullet f)$ where $(g \bullet f)_{X,M} = g_{X,F(M)}G(f_{X,M})$ is left $\mathcal{C}$-linearity for GF. One checks directly that

$$(G * r_F)(r_G * \mathcal{C}F) * B_{\mathcal{C},\mathcal{M}} = (g \bullet f)^{-1}.$$

r_{GF} is defined as the unique 2-isomorphism for which $r_{GF} * B_{\mathcal{C},\mathcal{M}} = (g \bullet f)^{-1}$ so Lemma 2.1.9 gives the result. $\square$

Polytope 4.1.4. *For any 2-cell $\alpha : F \Rightarrow G$ in $\mathcal{B}(\mathcal{C})$ the cylinders*

$$
\begin{tikzcd}
\mathcal{C}\mathcal{M} \arrow[rr, bend left, "\mathcal{C}F"] \arrow[rr, "\mathcal{C}G"'] \arrow[rr, phantom, bend left=15, "\Downarrow \mathcal{C}\alpha"] \arrow[dd, "r_{\mathcal{M}}"'] \arrow[dr, phantom, "\Leftarrow r_G"] & & \mathcal{C}\mathcal{N} \arrow[dd, "r_{\mathcal{N}}"] \arrow[dl, phantom, "r_F \Leftarrow"] \\
& \; & \\
\mathcal{M} \arrow[rr, dotted, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & \mathcal{N}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}\mathcal{C} \arrow[rr, bend left, "F\mathcal{C}"] \arrow[rr, "G\mathcal{C}"'] \arrow[rr, phantom, bend left=15, "\Downarrow \alpha\mathcal{C}"] \arrow[dd, "l_{\mathcal{M}}"'] \arrow[dr, phantom, "\Leftarrow l_G"] & & \mathcal{N}\mathcal{C} \arrow[dd, "l_{\mathcal{N}}"] \arrow[dl, phantom, "l_F \Leftarrow"] \\
& \; & \\
\mathcal{M} \arrow[rr, dotted, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & \mathcal{N}
\end{tikzcd}
$$

commute.

Proof. Again we check commutativity of the first polytope. The first cylinder is the condition $(\alpha * r_{\mathcal{M}})r_F = r_G(r_{\mathcal{N}} * \mathcal{C}\alpha)$ where $\mathcal{C}\alpha$ is the 2-cell defined by $\mathbf{id}_{\mathcal{C}} \boxtimes_{\mathcal{C}} \alpha$ and $\mathbf{id}_{\mathcal{C}}$ means natural isomorphism $id : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$. One verifies this directly using the bimodule condition on α. Again one checks first that components after right $*$-composing with the appropriately indexed universal functor B agree. Thus for $X \in \mathcal{C}$ and $M \in \mathcal{M}$ we have

$$
\begin{aligned}
((\alpha * r_{\mathcal{M}})r_F * B_{\mathcal{C},\mathcal{M}})_{X \boxtimes M} &= \alpha_{X \otimes M} f_{X,M}^{-1} \\
(r_G(r_{\mathcal{N}} * \mathcal{C}\alpha) * B_{\mathcal{C},\mathcal{M}})_{X \boxtimes M} &= g_{X,M}^{-1}(id_X \otimes \alpha_M)
\end{aligned}
$$

and since α is a natural module transformation the compositions on the right agree. Applying Lemma 2.1.9 for $B_{\mathcal{C},\mathcal{M}}$ gives the result. □

Polytope 4.1.5. *For $\mathcal{M}$ in $\mathcal{B}(\mathcal{C})$, the pastings*

$$
\begin{tikzcd}
(\mathcal{C}\mathcal{C})\mathcal{M} \arrow[r, "a_{\mathcal{C},\mathcal{C},\mathcal{M}}"] \arrow[dd, "r_{\mathcal{C}}\boxtimes_{\mathcal{C}} id_{\mathcal{M}}"'] \arrow[dr, phantom, "\Leftarrow \rho_{\mathcal{C},\mathcal{M}}"] & \mathcal{C}(\mathcal{C}\mathcal{M}) \arrow[d, "r_{\mathcal{C}\mathcal{M}}"] \arrow[ddl, "id_{\mathcal{C}}\boxtimes_{\mathcal{C}} r_{\mathcal{M}}"'] \\
 & \mathcal{C}\mathcal{M} \arrow[d, "r_{\mathcal{M}}"] \arrow[d, phantom, shift right=4ex, "\Downarrow r_{r_{\mathcal{M}}}"] \\
\mathcal{C}\mathcal{M} \arrow[r, "r_{\mathcal{M}}"'] & \mathcal{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{C}\mathcal{C})\mathcal{M} \arrow[r, "a_{\mathcal{C},\mathcal{C},\mathcal{M}}"] \arrow[dd, "r_{\mathcal{C}}\boxtimes_{\mathcal{C}} id_{\mathcal{M}}"'] \arrow[dr, "l_{\mathcal{C}}\boxtimes_{\mathcal{C}} id_{\mathcal{M}}"'] \arrow[dr, phantom, shift left=3ex, "\Downarrow \mu_{\mathcal{C},\mathcal{M}}"] & \mathcal{C}(\mathcal{C}\mathcal{M}) \arrow[d, "id_{\mathcal{C}}\boxtimes_{\mathcal{C}} r_{\mathcal{M}}"] \\
 & \mathcal{C}\mathcal{M} \arrow[d, "r_{\mathcal{M}}"] \\
\mathcal{C}\mathcal{M} \arrow[ur, "id"'] \arrow[r, "r_{\mathcal{M}}"'] & \mathcal{M}
\end{tikzcd}
$$

give the same 2-isomorphism. Each of the remaining two orderings of the multiset $\{\mathcal{C}, \mathcal{C}, \mathcal{M}\}$ determines an analogous pair of pastings, and hence a unique 2-isomorphism.

Remark 4.1.6. Note that the pair of diagrams is determined by the order of the objects in the upper left vertex. Keeping parentheses fixed, there are related pairs of diagrams for the remaining two orderings of the multiset $\{\mathcal{C}, \mathcal{C}, \mathcal{M}\}$. Each pair determines a pair of pastings, and each such pair of pastings similarly determines a unique 2-isomorphism.

Proof. We give proof in the diagrammed case. The other two are similar. Bimodule linearity for $r_{\mathcal{M}}$ is trivial (since $r_{\mathcal{M}}$ is strict à la Remark 2.2.7). The equation

$$r_{\mathcal{M}} * \mu_{\mathcal{C},\mathcal{M}} = id \tag{13}$$

is therefore the content of Polytope 4.1.5. To see this choose natural isomorphism $J : r_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}} = \otimes \to \otimes = r_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}}$ having components $J_{X \boxtimes M} := r_{\mathcal{M}}(b_{1,X,M})$ where b is the balancing for $B_{\mathcal{C},\mathcal{M}}$. According to the definition of $\mu_{\mathcal{C},\mathcal{M}}$ in 11 we see that

$(r_{\mathcal{M}} * \mu_{\mathcal{C},\mathcal{M}}) * (B_{\mathcal{C},\mathcal{C}} \boxtimes_{\mathcal{C}} id_{\mathcal{M}}) B_{\mathcal{C}\boxtimes\mathcal{C},\mathcal{M}} = r_{\mathcal{M}} * b$. Now using the fact that $r_{\mathcal{M}}$ is trivially balanced (proof of Proposition 2.2.5) the natural isomorphism J is balanced: that is, we have commutativity of the diagram

$$\begin{tikzcd}
r_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}}(X \otimes Y \boxtimes M) = (XY)M \arrow[r, "(r_{\mathcal{M}}*b)_{X,Y,M}"] \arrow[d, "J_{XY\boxtimes M}=(r_{\mathcal{M}}*b)_{1,XY,M}"'] & r_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}}(X \boxtimes Y \otimes M) = X(YM) \arrow[d, "J_{X\boxtimes YM}=(r_{\mathcal{M}}*b)_{1,X,YM}"] \\
r_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}}(X \otimes Y \boxtimes M) = (XY)M \arrow[r, equal] & r_{\mathcal{M}} B_{\mathcal{C},\mathcal{M}}(X \boxtimes Y \otimes M) = X(YM)
\end{tikzcd}$$

This follows from the balancing diagram satisfied by b. Using the relations given in the balancing diagram for b we derive the relations $b_{1,XY,M} = b_{1,X,YM} b_{X,Y,M}$ and $b_{X,1,M} = id$ which together imply $b_{1,XY,M} = id$ for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$. Thus the vertical arrows in the diagram above are identity hence $r_{\mathcal{M}} * b = id$. On an application of the uniqueness of the descended 2-cells (Lemma 2.1.9) we must have $r_{\mathcal{M}} * \mu_{\mathcal{C},\mathcal{M}} = id$, which is (13). □

This completes verification of the polytopes required for monoidal 2-category structure, and therefore completes the proof of Theorem 0.2.1.

CHAPTER V

TENSOR PRODUCT OF FUSION CATEGORIES OVER BRAIDED FUSION CATEGORIES

In this chapter we are interested in examining the relative tensor product of monoidal categories. That is, if monoidal categories $\mathcal{C}_1, \mathcal{C}_2$ also happen to be module categories over a fusion category $\mathcal{D}$ when is $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ monoidal? When can we give $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ a braided structure? What is its center? Clearly it is possible to formulate many interesting questions. We hope to answer some of them here. We will need the following definition.

Definition 5.0.7 ([DGNO10]). Let $\mathcal{C}$ be a monoidal category. Then $\mathcal{C}$ is said to be tensor *over* braided fusion category $\mathcal{D}$ if there is a braided tensor functor $\varphi : \mathcal{D} \to Z(\mathcal{C})$.

Typically we will identify $\mathcal{D}$ with its image in the center $Z(\mathcal{C})$. Evidently this gives $\mathcal{C}$ the structure of a $\mathcal{D}$-bimodule category: if $X \in \mathcal{C}$ and $D \in \mathcal{D}$ define $D \otimes X := \varphi(D) \otimes X$ where $\otimes$ on the right is in $Z(\mathcal{C})$ and where we identify $\varphi(D) \otimes X$ with its image under the canonical surjection $Z(\mathcal{C}) \to \mathcal{C}$. Right $\mathcal{D}$-module category structure is given by $X \otimes \varphi(D)$.

5.1 Tensor product of monoidal categories

Unless otherwise noted assume all tensor categories are semisimple. Let $\mathcal{D}$ be a braided fusion category and let $\varphi_i : \mathcal{D} \to Z(\mathcal{C}_i)$, $i = 1, 2$, be braided inclusions so that $\mathcal{C}_i$ are tensor over $\mathcal{D}$. Further assume that the compositions $\pi_i\varphi_i$ are fully faithful functors ($\pi_i : Z(\mathcal{C}_i) \to \mathcal{C}_i$ are the canonical surjections). We may thus consider $\mathcal{D}$ as a braided fusion subcategory of both $\mathcal{C}_i$.

5.1.1 Monoidal structure of $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$

Let $\mathcal{C}_i$ be monoidal categories over braided fusion category $\mathcal{D}$. Let $\tau : \mathcal{C}_1 \boxtimes \mathcal{C}_2 \to \mathcal{C}_2 \boxtimes \mathcal{C}_1$ be the functor $X \boxtimes Y \mapsto Y \boxtimes X$, and denote by $B_{1,2} : \mathcal{C}_1 \boxtimes \mathcal{C}_2 \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ the universal balanced functor described in Definition 2.1.4.

Proposition 5.1.1. *$\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ has canonical structure of a monoidal category with respect to which $B_{1,2}$ is a strict monoidal functor.*

Proof. Denote by Γ the composition of functors

$$\Gamma := \mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \mathcal{C}_1 \boxtimes \mathcal{C}_2 \xrightarrow{\tau(23)} \mathcal{C}_1 \boxtimes \mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \mathcal{C}_2 \xrightarrow{(\otimes_1, \otimes_2)} \mathcal{C}_1 \boxtimes \mathcal{C}_2$$

and define $\Lambda = B_{1,2} \circ \Gamma$. It is evident that Λ is balanced. Thus we get unique functor $\overline{\Lambda}$ making the diagram

$$\begin{tikzcd}
\mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \mathcal{C}_1 \boxtimes \mathcal{C}_2 \arrow[d, "B^{1,3}_{\mathcal{C}_1,\mathcal{C}_2}"'] \arrow[dr, "\Lambda"] & \\
(\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2) \boxtimes (\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2) \arrow[r, "\overline{\Lambda}"'] & \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2
\end{tikzcd}$$

commute. Here $B^{1,3}_{\mathcal{C}_1,\mathcal{C}_2} = B_{1,2} \boxtimes B_{1,2}$ is the universal functor for right exact functors balanced in positions 1, 3 (Definition 2.1.3) from the abelian category at the apex.

Associativity for $\overline{\Lambda}$ is verified as follows. Abbreviate functors

$$\begin{aligned} \Lambda_\ell &:= \Lambda(\Gamma \boxtimes id_{\mathcal{C}_1 \boxtimes \mathcal{C}_2}) : (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 3} \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \\ \Lambda_r &:= \Lambda(id_{\mathcal{C}_1 \boxtimes \mathcal{C}_2} \boxtimes \Gamma) : (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 3} \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2. \end{aligned}$$

We leave verification that Λ_ℓ, Λ_r are balanced in positions 1, 3 and 5 to the motivated reader. One checks easily that $\Lambda_\ell = \overline{\Lambda}(\overline{\Lambda} \boxtimes id_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2}) B^{\boxtimes 3}_{1,2}$ and $\Lambda_r = \overline{\Lambda}(id_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} \boxtimes \overline{\Lambda}) B^{\boxtimes 3}_{1,2}$. Thus by uniqueness of $\overline{\Lambda_\ell}$, $\overline{\Lambda_r}$ we must have

$$\begin{aligned} \overline{\Lambda_\ell} &= \overline{\Lambda}(\overline{\Lambda} \boxtimes id_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2}) \\ \overline{\Lambda_r} &= \overline{\Lambda}(id_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} \boxtimes \overline{\Lambda}). \end{aligned}$$

Next let a^i be associativity constraints in $\mathcal{C}_i$. Then $B_{1,2} * a^1 \boxtimes a^2 : \Lambda_\ell \to \Lambda_r$ is a balanced natural transformation and we thus get a unique natural isomorphism

$$B_{1,2} * \overline{a^1 \boxtimes a^2} : \overline{\Lambda}(\overline{\Lambda} \boxtimes id_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2}) \xrightarrow{\sim} \overline{\Lambda}(id_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} \boxtimes \overline{\Lambda}).$$

This is precisely the associativity diagram required of $\overline{\Lambda}$ evincing it a bona fide tensor structure on $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$. Observe that unit object for $\overline{\Lambda}$ comes from identity objects of $\mathcal{C}_i$ in the obvious way: $\mathbf{1} = B_{1,2}(\mathbf{1} \boxtimes \mathbf{1})$.

Tensor strictness of $B_{1,2}$ follows from the fact that monoidal structure in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$

is defined by the functor Γ. Indeed if $U := X \boxtimes Y$ and $V := X' \boxtimes Y'$ are objects in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ we have, from the definition of Λ and Γ,

$$B_{1,2}\Gamma(U \boxtimes V) = \overline{\Lambda}(B_{1,2} \boxtimes B_{1,2})(U \boxtimes V).$$

The LHS is $B_{1,2}$ evaluated on the tensor product $U \otimes V$ in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ and the RHS is the tensor product $B_{1,2}(U) \otimes B_{1,2}(V)$ in $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$. It is clear that both sides equal $B_{1,2}((X \otimes X') \boxtimes (Y \otimes Y'))$. □

5.1.2 Functors over $\mathcal{D}$

In this subsection we are interested in studying the (the as yet undefined) monoidal 2-category of tensor categories over a fixed braided fusion category. The next definition is an essential step in this direction.

Definition 5.1.2. Suppose $\mathcal{C}_1, \mathcal{C}_2$ are tensor categories over braided fusion category $\mathcal{D}$, and denote by ψ_i the the compositions $\mathcal{D} \hookrightarrow Z(\mathcal{C}_i) \to \mathcal{C}_i$. A tensor functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be a *functor over* $\mathcal{D}$ if $F\psi_1 = \psi_2$.

Definition 5.1.2 stipulates that functors over $\mathcal{D}$ are precisely those respecting the relevant braided injections. We require one further definition to form the functorial counterpart to Proposition 5.1.1.

Definition 5.1.3. Suppose $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are tensor categories and let $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{D} \to \mathcal{B}$ be tensor functors with tensor structures f, g respectively. A *relative braiding* for the pair F, G is a family of natural isomorphisms $c_{X,Y} : F(X) \otimes G(Y) \to$

$G(Y) \otimes F(X)$ satisfying the pentagons

$$\begin{tikzcd} F(XY)G(Z) \arrow[r, "f_{X,Y}"] \arrow[dd, "c_{XY,Z}"'] & F(X)F(Y)G(Z) \arrow[d, "c_{Y,Z}"] \\ & F(X)G(Z)F(Y) \arrow[d, "c_{X,Z}"] \\ G(Z)F(XY) \arrow[r, "f_{X,Y}"'] & G(Z)F(X)F(Y) \end{tikzcd} \qquad \begin{tikzcd} F(X)G(VZ) \arrow[r, "g_{V,Z}"] \arrow[dd, "c_{X,VZ}"'] & F(X)G(V)G(Z) \arrow[d, "c_{X,V}"] \\ & G(V)F(X)G(Z) \arrow[d, "c_{X,Z}"] \\ G(VZ)F(X) \arrow[r, "g_{V,Z}"'] & G(V)G(Z)F(X) \end{tikzcd}$$

for all $X, Y \in \mathcal{C}$ and $V, Z \in \mathcal{D}$.

Assume the category $\mathcal{B}$ in Definition 5.1.3 is braided. Then any pair of tensor functors into $\mathcal{B}$ are related by a relative braiding having components given by the braiding indexed by objects in the images of F and G. This follows from naturality of the tensor structures f, g and the braiding hexagon.

Proposition 5.1.4. *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{A}$ be tensor categories over braided fusion category $\mathcal{D}$ and let $F_i : \mathcal{C}_i \to \mathcal{A}$ be tensor functors over $\mathcal{D}$ related by a relative braiding. Then F_1, F_2 determine a unique tensor functor $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \to \mathcal{A}$.*

Proof. Let t, t' be tensor structures for F_1, F_2 respectively and let c denote the relative braiding as in Definition 5.1.3. Denote by c^i the braided structure in $Z(\mathcal{C}_i)$. The functors F_i determine a tensor functor $F : \mathcal{C}_1 \boxtimes \mathcal{C}_2 \to \mathcal{A}$ defined by sending $X \boxtimes Y \mapsto F_1(X) \otimes F_2(Y)$. We show that F is a tensor functor below.

The proof of Proposition 5.1.4 divides into three parts. First we show that F is $\mathcal{D}$ balanced. Then we show that the functors $F\otimes$ and $\otimes(F \boxtimes F)$ are both tensor and balanced, and finally that tensor structure $f : F\otimes \xrightarrow{\sim} \otimes(F \boxtimes F)$ is a balanced natural isomorphism. This will imply that all these structures descend to the relative product.

1. F balancing. Denote by $b_{X,D,Y}$ the composition

$$F_1(X \otimes D) \otimes F_2(Y) \xrightarrow{t_{X,D}} F_1(X) \otimes F_1(D) \otimes F_2(Y) \xrightarrow{t'^{-1}_{D,Y}} F_1(X) \otimes F_2(D \otimes Y).$$

for $X \in \mathcal{C}_1$, $Y \in \mathcal{C}_2$ and $D \in \mathcal{D}$. This composition makes sense because $F_1(D) = F_2(D)$ thanks to Definition 5.1.3. We show that b satisfies the balancing diagram, thus balancing for F. This is a straightfoward calculation: simply observe that the diagram commutes for $D, E \in \mathcal{D}$:

$$\begin{tikzcd}
F_1(XDE)F_2(Y) \arrow[rr, "t_{XD,E}"] \arrow[d, "t_{X,DE}"'] & & F_1(XD)F_1(E)F_2(Y) \arrow[dl, "t_{X,D}"'] \arrow[d, "t'^{-1}_{E,Y}"] \\
F_1(X)F_1(DE)F_2(Y) \arrow[d, "t'^{-1}_{DE,Y}"'] & F_1(X)F_1(D)F_1(E)F_2(Y) \arrow[l, "t'^{-1}_{D,E}"] & F_1(XD)F_2(EY) \arrow[d, "t_{X,D}"] \\
F_1(X)F_2(DEY) & & F_1(X)F_1(D)F_2(EY) \arrow[ll, "t'^{-1}_{D,EY}"] \arrow[ul, "t'^{-1}_{E,Y}"]
\end{tikzcd}$$

Note that $t_{D,E} = t'_{D,E}$ via Definition 5.1.3. The rectangles are therefore diagrams required of tensor structures for F_1, F_2 and triangle commutes trivially. Perimeter is the balancing diagram for b.

2. Tensor structure of F. In what follows we will be required to draw diagrams having vertices labeled by sextuples of objects. In order to simplify and condense notation let us adopt the following convention: write $F_1(X)F_2(Y) := (X)(Y)$ for the tensor product of the images of $X \in \mathcal{C}_1, Y \in \mathcal{C}_2$ in $\mathcal{A}$. Since all monoidal categories are assumed strict we lose nothing by so doing. Denote objects of $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ using overline and subscripts: $\overline{X} = X_1 \boxtimes X_2$ etc. Define natural isomorphisms $f_{\overline{X},\overline{Y}} : F(\overline{X} \otimes \overline{Y}) \to F(\overline{X}) \otimes F(\overline{Y})$ by the composition

$$\begin{aligned} F(\overline{X}\,\overline{Y}) = (X_1Y_1)(X_2Y_2) \xrightarrow{t\otimes t'} (X_1)(Y_1)(X_2)(Y_2) &\xrightarrow{c_{Y_1,X_2}} (X_1)(X_2)(Y_1)(Y_2) \\ &= F(\overline{X})F(\overline{Y}). \end{aligned}$$

We now show that f provides F with the structure of a tensor functor. This will require the defining diagrams for the relative braiding.

$$\begin{tikzcd}
F(\overline{X}\,\overline{Y}\,\overline{Z}) \arrow[rr, "t\otimes t'"] \arrow[d, "t\otimes t'"'] & & (X_1)(Y_1Z_1)(X_2)(Y_2Z_2) \arrow[dl, "t\otimes t'"'] \arrow[d, "c_{Y_1Z_1,X_2}"] \\
(X_1Y_1)(Z_1)(X_2Y_2)(Z_2) \arrow[r, "t\otimes t'"] \arrow[d, "c_{Z_1,X_2Y_2}"'] & (X_1)(Y_1)(Z_1)(X_2)(Y_2)(Z_2) \arrow[d, "c_{Z_1,X_2}"] & F(\overline{X})F(\overline{Y}\,\overline{Z}) \arrow[d, "t\otimes t'"] \\
F(\overline{X}\,\overline{Y})F(\overline{Z}) \arrow[d, "t\otimes t'"'] & (X_1)(Y_1)(X_2)(Z_1)(Y_2)(Z_2) \arrow[r, "c_{Y_1,X_2}"'] \arrow[dl, "c_{Z_1,Y_2}"'] & F(\overline{X})(Y_1)(Z_1)(Y_2)(Z_2) \arrow[d, "c_{Z_1,Y_2}"] \\
(X_1)(Y_1)(X_2)(Y_2)F(\overline{Z}) \arrow[rr, "c_{Y_1,X_2}"'] & & F(\overline{X})F(\overline{Y})F(\overline{Z})
\end{tikzcd}$$

All subdiagrams are either relative braiding diagram or diagrams for tensor structure in the F_i. Perimeter is tensor diagram for (F, f).

3. Balancing of $F\otimes$. In this part of the proof we show that the composition $F\otimes : (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 2} \to \mathcal{A}$ of F with the monoidal structure in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is $\mathcal{D}$ balanced in positions 1 and 3. This is necessary for F to descend to a functor from the relative product in a way which respects monoidal structure. For $D \in \mathcal{D}$ define natural isomorphism

$b^1_{X_1,D,X_2,\overline{Y}} : F \otimes (X_1 D \boxtimes X_2 \boxtimes \overline{Y}) \simeq F \otimes (X_1 \boxtimes DX_2 \boxtimes \overline{Y})$ by the composition

$$\begin{aligned}(X_1DY_1)(X_2Y_2) \quad &\xrightarrow{c^1_{D,Y_1}} \quad (X_1Y_1D)(X_2Y_2) \\ &\xrightarrow{t_{X_1Y_1,D}} \quad (X_1Y_1)(D)(X_2Y_2) \xrightarrow{t'^{-1}_{D,X_2Y_2}} (X_1Y_1)(DX_2Y_2)\end{aligned}$$

where we continue to use notation introduced in part 2 of this proof. Note that the third expression is not ambiguous because F_1, F_2 agree on $\mathcal{D}$. Let $D, E \in \mathcal{D}$. The following diagram shows that b^1 provides a balancing of $F\otimes$ in the first position.

$$\begin{tikzcd}
(X_1DEY_1)(X_2Y_2) \arrow[r, "c^1_{E,Y_1}"] \arrow[d, "c^1_{DE,Y_1}"'] & (X_1DY_1E)(X_2Y_2) \arrow[r, "t_{X_1DY_1,E}"] \arrow[dl, "c^1_{D,Y_1}"'] & (X_1DY_1)(E)(X_2Y_2) \arrow[d, "t'^{-1}_{E,X_2Y_2}"] \arrow[dl, "c^1_{D,Y_1}"] \\
(X_1Y_1DE)(X_2Y_2) \arrow[r, "t_{X_1Y_1D,E}"'] \arrow[d, "t_{X_1Y_1,DE}"'] & (X_1Y_1D)(E)(X_2Y_2) \arrow[d, "t_{X_1Y_1,D}"'] \arrow[dr, "t'^{-1}_{E,X_2Y_2}"] & (X_1DY_1)(EX_2Y_2) \arrow[d, "c^1_{D,Y_1}"] \\
(X_1Y_1)(DE)(X_2Y_2) \arrow[r, "t_{D,E}", "t'_{D,E}"'] \arrow[d, "t'^{-1}_{DE,X_2,Y_2}"'] & (X_1Y_1)(D)(E)(X_2Y_2) \arrow[dr, "t'^{-1}_{E,X_2Y_2}"] & (X_1Y_1D)(EX_2Y_2) \arrow[d, "t_{X_1Y_1,D}"] \\
(X_1Y_1)(DEX_2Y_2) & & (X_1Y_1)(D)(EX_2Y_2) \arrow[ll, "t'^{-1}_{D,EX_2Y_2}"]
\end{tikzcd}$$

Every subdiagram is either braiding hexagon, naturality of t or tensor structure for t, t'. One similarly defines $b^2_{\overline{X},Y_1,D,X_2} : F \otimes (\overline{X} \boxtimes Y_1 D \boxtimes Y_2) \simeq F \otimes (\overline{X} \boxtimes Y_1 \boxtimes DY_2)$ giving balancing in position 2 over $\mathcal{D}$.

4. Balancing of $\otimes(F \boxtimes F)$. In this part of the proof we show that the composition $\otimes(F \boxtimes F) : (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 2} \to \mathcal{A}$ of the monoidal structure in $\mathcal{A}$ with $F \boxtimes F$ is $\mathcal{D}$ balanced in positions 1 and 3. Begin by defining natural isomorphism $(id \otimes t'^{-1}_{D,X_2})(t_{X_1,D} \otimes id) : F_1(X_1D)F_2(X_2)F(\overline{Y}) \to F_1(X_1)F_2(DX_2)F(\overline{Y})$. Using the diagrams required of t, t'

it is easy to show this satisfies the diagram required of a balancing for $\otimes(F \boxtimes F)$ in position 1. Doing so for position 3 is just as easy.

5. *Balancing of $f : F\otimes \to \otimes(F \boxtimes F)$.* Recall the definition of the tensor structure f for F from Part 1 of this proof. We show that satisfies the diagram required of a balanced natural transformation in both positions 1 and 3. In position 1 this means showing that $(b_{X_1,D,X_2} \otimes id) f_{X_1 D \boxtimes X_2, \overline{Y}} = f_{X_1 \boxtimes D X_2, \overline{Y}} b^1_{X_1,D,X_2,\overline{Y}}$ where b^1 is balancing of $F\otimes$ in position 1 as in Part 3 of this proof and b is balancing of F. To show this consider the diagram

$$
\begin{tikzcd}
(X_1DX_2)F(\overline{Y}) \arrow[r, "t_{X_1,DX_2}"] \arrow[d, "c^1_{D,X_2}"'] & (X_1)(DX_2)F(\overline{Y}) \arrow[r, "t\otimes t'"] \arrow[d, "c^1_{D,X_2}"] & (X_1)(D)(X_2)(Y_1)(Y_2) \arrow[r, "t_{X_1,D}"] \arrow[ddr, "c_{X_2,Y_1}"'] & (X_1D)(X_2)(Y_1)(Y_2) \arrow[d, "c_{X_2,Y_1}"] \\
(X_1X_2D)F(\overline{Y}) \arrow[r, "t_{X_1,X_2D}"] \arrow[d, "t_{X_1X_2,D}"'] & (X_1)(X_2D)F(\overline{Y}) \arrow[dr, "t_{X_2,D}\otimes t'_{X_2,Y_2}"] & & (X_1D)(Y_1)(X_2)(Y_2) \arrow[d, "t_{X_1,D}"] \\
F(\overline{X})(D)F(\overline{Y}) \arrow[rr, "t_{X_1,X_2}\otimes t'_{Y_1,Y_2}"] \arrow[d, "t'^{-1}_{D,Y_1Y_2}"'] & & (X_1)(X_2)(D)(Y_1)(Y_2) \arrow[uu, "t_{X_1,D}"] & (X_1)(D)(Y_1)(X_2)(Y_2) \arrow[d, "t'^{-1}_{D,Y_1}"] \\
F(\overline{X})(DY_1Y_2) \arrow[rr, "t_{X_1,X_2}\otimes t'_{DY_1,Y_2}"] & & (X_1)(X_2)(DY_1)(Y_2) \arrow[r, "c_{X_2,DY_1}"] \arrow[u, "t'_{D,Y_1}"] & (X_1)(DY_1)(X_2)(Y_2)
\end{tikzcd}
$$

Every subdiagram is either naturality or tensor diagrams for t, t' or relative braiding of F_1, F_2. Since the composition of morphisms across the top is $t_{X_1D,X_2} \otimes t'_{Y_1,Y_2}$ the perimeter is the balancing diagram for f in position 1. Showing that f is balanced in position 3 requires a similar diagram and is just as easy.

Parts 1-5 above imply that there are unique functors and natural isomorphism $\overline{f} : \overline{F\otimes} \to \overline{\otimes(F \boxtimes F)} : (\mathcal{C}_1 \boxtimes_D \mathcal{C}_2)^{\boxtimes 2} \to \mathcal{A}$ satisfying $\overline{f} * B^{\boxtimes 2}_{1,2} = f$. Using basic properties of the functor $B_{1,2}$ and the definition of Λ from Proposition 5.1.1 one shows that $\overline{F\otimes} = \overline{F}\,\overline{\Lambda}$ and $\overline{\otimes(F \boxtimes F)} = \otimes(\overline{F} \boxtimes \overline{F})$ where $\overline{F} : \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \to \mathcal{A}$ is the unique functor with $\overline{F}B_{1,2} = F$. Thus $\overline{f}$ provides $\overline{F}$ with the structure of a tensor functor in a canonical way. The proposition is proved. □

Proposition 5.1.4, though perhaps interesting in its own right, is of immediate value in that it implies closure over relative product $\boxtimes_{\mathcal{D}}$ of the class of functors over $\mathcal{D}$. This we prove in Proposition 5.1.10.

5.1.3 The fusion category $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$

In this section we show that the relative product of two fusion categories over braided $\mathcal{D}$ inherits the structure of a fusion category over $\mathcal{D}$. Notation is retained from the previous section.

Theorem 5.1.5. *Let $\mathcal{C}_i$, $i = 1, 2$ be fusion categories over $\mathcal{D}$. Then $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is also a fusion category over $\mathcal{D}$.*

We break up the proof of Theorem 5.1.5 into two parts: first we will show that $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is fusion and then show in Proposition 5.1.8 that it is fusion *over* $\mathcal{D}$ in the sense of Definition 5.0.7.

Proposition 5.1.6. *Under the conditions of Theorem 5.1.5 $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is fusion.*

Proof. Thanks to Proposition 5.1.1 it remains only to check that $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is rigid and semisimple. We begin with a general result. Recall that a dominant functor F is one for which the codomain category and the category $Im(F)$ coincide (Definition 1.3.16).

Lemma 5.1.7. *Let $\mathcal{C}, \mathcal{D}$ be semisimple tensor categories with $\mathcal{C}$ fusion, and let $F : \mathcal{C} \to \mathcal{D}$ be a strict dominant tensor functor. Then $\mathcal{D}$ is fusion.*

Proof. Let $\hat{F}$ denote the tensor subcategory of $\mathcal{D}$ generated by objects in the image of F. Since F is a tensor functor $\hat{F}$ is itself fusion with rigidity inherited from that in $\mathcal{C}$: duality is given by $F(X)^* = F(X^*)$ and $\mathrm{ev}_{F(X)} := F(\mathrm{ev}_X) : F(X)^* \otimes F(X) \to F(1) = 1$. Similarly $\mathrm{coev}_{F(X)} := F(\mathrm{coev}_X)$.

It is our task to define duality for a general object in $\mathcal{D}$. To this end fix $Y \in \mathcal{D}$. Let $X \in \mathcal{C}$ be an object such that $F(X)$ contains Y as a subobject. Write $F(X) = Y \oplus Z$ for some object $Z \in \mathcal{D}$. Now $F(X^*) \otimes Y$ is a subobject of $F(X^*) \otimes F(X)$. Define the object Z^* to be the *largest* subobject of $F(X^*)$ having the property that

$$F(\mathrm{ev}_X)|_{Z^* \otimes Y} = 0.$$

Define Y^* to be the complement of Z^* in $F(X^*)$, i.e. $F(X^*) = Y^* \oplus Z^*$. Thus the object $Y^* \otimes Y$ is a subobject of $F(X^*) \otimes F(X)$, and we may therefore restrict $F(\mathrm{ev}_X)$ to define morphism $e_Y := F(\mathrm{ev}_X)|_{Y^* \otimes Y}$. To be explicit, let $\rho_{Y^*} : Y^* \hookrightarrow F(X^*)$ and $\rho_Y : Y \hookrightarrow F(X)$ be inclusions of the indicated subobjects. Then we have defined $e_Y := F(\mathrm{ev}_X) \circ (\rho_{Y^*} \otimes \rho_Y)$.

Next let $\pi_Y : F(X) \to Y$ and $\pi_{Y^*} : F(X) \to Y^*$ be projections. Then define $co_Y := (\pi_Y \otimes \pi_{Y^*}) \circ F(\mathrm{coev}_X) : Y \otimes Y^* \to 1$. Neither e_Y nor co_Y is identically zero because of the choice of subobject Y^*. We claim that e_Y, co_Y together with the identifications made above make Y^* a bona fide left dual for $Y \in \mathcal{D}$. It remains to check the usual identities. On the definitions of co_Y and e_Y we have

$$(1_Y \otimes co_Y)(e_Y \otimes 1_Y) = (1_Y \otimes F(\mathrm{ev}_X))(\pi_Y \otimes \rho_{Y^*}\pi_{Y^*} \otimes \rho_Y)(F(\mathrm{coev}_X) \otimes 1_Y).$$

Using the basic identity $\rho_{Y^*}\pi_{Y^*} = id_{F(X^*)} - \rho_{Z^*}\pi_{Z^*}$ this becomes a difference of two maps with only the "positive" one non-zero because of the definition of Z^*. The remaining non-zero part fits into the following diagram as the lower horizontal composition.

$$\begin{tikzcd}
F(X) \arrow[rr, "F(\mathrm{coev}_X)\otimes id"] & & F(X)F(X^*)F(X) \arrow[rr, "id\otimes F(\mathrm{ev}_X)"] \arrow[dr, "\pi_Y\otimes id"] & & F(X) \arrow[d, "\pi_Y"] \\
Y \arrow[u, "\rho_Y"] \arrow[r, "F(\mathrm{coev}_X)\otimes id"'] & F(X)F(X^*)Y \arrow[ur, "id\otimes\rho_Y"] \arrow[r, "\pi_Y\otimes id"'] & YF(X)Y \arrow[r, "id\otimes\rho_Y"'] & YF(X^*)F(X) \arrow[r, "id\otimes F(\mathrm{ev}_X)"'] & Y
\end{tikzcd}$$

All subdiagrams commute trivially. The horizontal composition across the top of the diagram is id (this is the equation required of rigidity of X in $\mathcal{C}$). Tracing around the perimeter gives $(id_Y \otimes e_Y)(co_Y \otimes id_Y) = \pi_Y\rho_Y$ which is id_Y (the other basic identity relating ρ and π). The second equation $id_{Y^*} = (e_Y \otimes id_{Y^*})(id_{Y^*} \otimes co_Y)$ follows similarly. □

Now we complete the proof of Proposition 5.1.6. By Lemma 2.1.8 the universal balanced functor $B_{1,2} : \mathcal{C}_1 \boxtimes \mathcal{C}_2 \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is dominant, hence $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is rigid. Also since $\mathcal{C}_i$ are both semisimple the category $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is semisimple because it is equivalent to the category of functors $Fun_{\mathcal{D}}(\mathcal{C}_1^{op}, \mathcal{C}_2)$ (Theorem 2.3.1) which is semisimple by [ENO05, Theorem 2.16]. □

Proposition 5.1.8. *Under the conditions of Theorem 5.1.5 $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is fusion over*

$\mathcal{D}$.

Proof. Let $\varphi_i : \mathcal{D} \to Z(\mathcal{C}_i)$ be the braided inclusions putting fusion categories $\mathcal{C}_i$ over $\mathcal{D}$. Note that $\mathcal{D}$ sits inside $Z(\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2)$ by the composition

$$\varphi := \mathcal{D} \hookrightarrow \mathcal{D} \boxtimes \mathcal{D} \xrightarrow{\varphi_1 \boxtimes \varphi_2} Z(\mathcal{C}_1) \boxtimes Z(\mathcal{C}_2) = Z(\mathcal{C}_1 \boxtimes \mathcal{C}_2) \xrightarrow{ZB_{1,2}} Z(\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2) \tag{14}$$

sending $D \mapsto (B_{1,2}(D \boxtimes 1), \overline{\gamma}_{B_{1,2}(D\boxtimes 1)})$. Here $\overline{\gamma}$ is the braiding in $Z(\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2)$ as in the last part of the proof of Proposition 5.2.3 and $ZB_{1,2}$ is the functor sending $(A, c_A) \mapsto (B_{1,2}(A), \overline{\gamma}_{B_{1,2}(A)})$ for any object (A, c_A) in the center of $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. To complete the proof of Proposition 5.1.8 we must show that the composition 14 is an inclusion and that it is braided.

1. φ is an inclusion. Since $\varphi_1 \boxtimes \varphi_2$ is an inclusion on account of φ_i being so we must check only that $ZB_{1,2}$ is an inclusion on the tensor subcategory generated by objects of the form $(D \boxtimes 1, c^1_{D,-} \boxtimes 1)$. But this is obvious.

2. Braiding of φ. Since both φ_1, φ_2 are braided the functor $\varphi_1 \boxtimes \varphi_2$ is also braided (this is perfectly general and has nothing whatever to do with the other properties of φ_i). Note also that $ZB_{1,2}$ is braided; this follows from the fact that $B_{1,2}$ is a braided functor (this is shown in Proposition 5.2.3). □

Corollary 5.1.9. *For $\mathcal{C}_1, \mathcal{C}_2; \mathcal{D}$ as in the hypothesis of Theorem 5.1.5 the category of $\mathcal{D}$-module functors $Fun_{\mathcal{D}}(\mathcal{C}_1, \mathcal{C}_2)$ has the structure of a fusion category over $\mathcal{D}$.*

Proof. This follows immediately from the comments at the end of the proof of Proposition 5.1.6. □

Now that we have a multiplication in the category of tensor (fusion) categories over a fixed braided fusion category it is natural to ask if this extends to morphisms of such categories. This is the content of the following proposition. Recall what it means for a functor to be a functor *over* a braided fusion category (Definition 5.1.2).

Proposition 5.1.10. *Let $(F, f) : \mathcal{C}_1 \to \mathcal{E}_1$ and $(G, g) : \mathcal{C}_2 \to \mathcal{E}_2$ be tensor functors over $\mathcal{D}$ for $\mathcal{C}_i, \mathcal{E}_i$ tensor categories over braided fusion $\mathcal{D}$. Then $F \boxtimes_{\mathcal{D}} G$ has canonical structure of a tensor functor over $\mathcal{D}$.*

Proof. Define functors $F_1 : \mathcal{C}_1 \to \mathcal{E}_1 \boxtimes_{\mathcal{D}} \mathcal{E}_2$ and $G_2 : \mathcal{C}_2 \to \mathcal{E}_1 \boxtimes_{\mathcal{D}} \mathcal{E}_2$ by

$$F_1(X) := B_{1,2}(F(X) \boxtimes 1), \qquad G_2(Y) := B_{1,2}(1 \boxtimes G(Y)).$$

Using the braided inclusions from Proposition 5.1.8 it is easy to check that F_1, G_2 are functors over $\mathcal{D}$. Observe that $F_1(X) \otimes G_2(Y) = G_2(Y) \otimes F_1(X)$ so we have a trivial relative braiding between F_1, G_2. Applying Proposition 5.1.4 to $\otimes(F_1 \boxtimes G_2) = B_{1,2}(F \boxtimes G)$ we get a unique tensor functor $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \to \mathcal{E}_1 \boxtimes_{\mathcal{D}} \mathcal{E}_2$. This is exactly $F \boxtimes_{\mathcal{D}} G$. □

5.2 Tensor product of braided fusion categories

In this section we discuss when the relative tensor product of a pair of braided fusion categories is itself braided. It turns out that in order for such a braiding to exist we will need a restricted version of the phenomenon described in Definition 5.0.7.

Definition 5.2.1. If $\mathcal{C}, \mathcal{D}$ are braided then we say $\mathcal{C}$ is *braided over* $\mathcal{D}$ if there is a braided inclusion $\mathcal{D} \hookrightarrow \mathcal{C}'$ where $\mathcal{C}'$ is the centralizer of $\mathcal{C}$ (Definition 1.2.12).

Example 5.2.2. Let G_i be finite abelian groups and $q_i : G_i \to k^\times$ quadratic forms satisfying $q_i(g) = \beta_i(g,g)$ for some bicharacters β_i on G_i, $i = 1, 2$. One easily checks that $(G_1 \times G_2, p)$ is a pre-metric group for $p(g,h) = q_1(g)q_2(h)$. As a quadratic form p comes from the bicharacter on $G_1 \times G_2 \times G_1 \times G_2$ given by $(g_1, h_1, g_2, h_2) \mapsto \beta_1(g_1, g_2)\beta_2(h_1, h_2)$.

Now suppose we have embeddings $G \hookrightarrow G_i$ for a finite group G such that $q_1(g) = q_2(g)$ for all $g \in G$. Then the pair (G, q) is a metric group for $q := q_i|_G$. Denote by $\tilde{G}$ the subgroup of $G_1 \times G_2$ given by the set $\{(x, x^{-1}) | x \in G\}$ and suppose that p descends to a quadratic form on $(G_1 \times G_2)/\tilde{G}$. Since p is constant on $\tilde{G}$-cosets we have $q_i(x)q(g) = q_i(gx)q(1)$ for $x \in G_i, g \in G$. As a result $b_i(g,x) = q(1)^{-1}$ and since $b_i(1,1) = 1$ we may conclude that

$$b_i(g,x) = 1 \tag{15}$$

for $i = 1, 2$.

Let us translate this into the language of pointed braided fusion categories à la §1.2.2. Pre-metric inclusions $G \hookrightarrow G_i$ correspond to braided inclusions $\mathcal{C}(G,q) \to \mathcal{C}(G_i, q_i)$. Equation 15 becomes $c^i_{X,Y} = 1$ (c^i the braiding in $\mathcal{C}(G_i, q_i)$) whenever X, Y are homogeneous objects of Vec_{G_i} of degrees $g \in G$, x respectively. Thus the images of the braided inclusions are contained in Müger centers $\mathcal{C}(G_i, q_i)'$. As braided fusion categories it is evident that $\mathcal{C}(G_1 \times G_2/\tilde{G}, p) \simeq \mathcal{C}(G_1, q_1) \boxtimes_{\mathcal{C}(G,q)} \mathcal{C}(G_2, q_2)$.

The next proposition describes braiding for the tensor category $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ whenever $\mathcal{C}_i$ are braided over $\mathcal{D}$ in the sense of Definition 5.2.1. Note that the Deligne product of any pair of braided categories has braiding given by Deligne product of the individual braided structures. In what remains of this section we extend this observation to the relative product over a braided fusion category.

Proposition 5.2.3. *Let $\mathcal{C}_i$, $i = 1, 2$ be braided fusion categories braided over $\mathcal{D}$. Then $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ has canonical braiding such that $B_{1,2}$ is a braided functor.*

Proof. By Propositions 5.2 and 5.3 in [JS93] braided structures on $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ are in correspondence with tensor structures on the monoidal product $\otimes : (\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2)^{\boxtimes 2} \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$. Thus to prove the proposition we consider such tensor structures.

Let c^i be braiding on $\mathcal{C}_i$ and let $\Lambda : (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 2} \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ be as in the proof of Proposition 5.1.1. The category $(\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 2}$ has monoidal structure coming from the one in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ in the obvious way. We will adopt the convention of abbreviating objects of the form $X_1 \boxtimes X_2 \in \mathcal{C}_1 \boxtimes \mathcal{C}_2$ by $\overline{X}$ and on occasion write $\overline{X}_i = X_i$ for the "coordinates" of $\overline{X}$. Thus $(\overline{X} \otimes \overline{Y})_1 = X_1 \otimes Y_1$. Any tensor structure $\lambda_{\overline{X} \boxtimes \overline{Y}, \overline{U} \boxtimes \overline{V}}$: $\Lambda((\overline{X} \boxtimes \overline{Y}) \otimes (\overline{U} \boxtimes \overline{V})) \simeq \Lambda(\overline{X} \boxtimes \overline{Y}) \otimes \Lambda(\overline{U} \boxtimes \overline{V})$ is of the form

$$\lambda_{\overline{X} \boxtimes \overline{Y}, \overline{U} \boxtimes \overline{V}} : B_{1,2}(X_1 U_1 Y_1 V_1 \boxtimes X_2 U_2 Y_2 V_2) \xrightarrow{\sim} B_{1,2}(X_1 Y_1 U_1 V_1 \boxtimes X_2 Y_2 U_2 V_2)$$

where we have used the definition of Λ to write (co)domain of λ in terms of $B_{1,2}$.

Given braidings in $\mathcal{C}_i$ the most natural possibilities are

$$\lambda_{\overline{X\boxtimes Y},\overline{U\boxtimes V}} = B_{1,2}(1_{X_1} \otimes C^1(U_1, Y_1) \otimes 1_{V_1} \boxtimes 1_{X_2} \otimes C^2(U_2, Y_2) \otimes 1_{V_2})$$

where $C^i(A, B) \in \{c^i_{A,B},\ c^{i\,-1}_{B,A}\}$. Leaving out tensoring with identity the diagram needed in order for Λ to have monoidal structure λ is

$$\begin{array}{ccc}
\Lambda(\overline{U}\,\overline{W}\,\overline{Y} \boxtimes \overline{V}\,\overline{X}\,\overline{Z}) & \xrightarrow{C^1(W_1Y_1,V_1)\boxtimes C^2(W_2Y_2,V_2)} & \Lambda(\overline{U} \boxtimes \overline{V})\Lambda(\overline{W}\,\overline{Y} \boxtimes \overline{X}\,\overline{Z}) \\
\downarrow{\scriptstyle C^1(Y_1,V_1X_1)\boxtimes C^2(Y_2,V_2X_2)} & & \downarrow{\scriptstyle C^1(Y_1,X_1)\boxtimes C^2(Y_2,X_2)} \\
\Lambda(\overline{U}\,\overline{W} \boxtimes \overline{V}\,\overline{X})\Lambda(\overline{Y} \boxtimes \overline{Z}) & \xrightarrow[C^1(W_1,V_1)\boxtimes C^2(W_2,V_2)]{} & \Lambda(\overline{U} \boxtimes \overline{V})\Lambda(\overline{W} \boxtimes \overline{X})\Lambda(\overline{Y} \boxtimes \overline{Z})
\end{array}$$

This is really two diagrams: one for C^1 and another for C^2. The diagram corresponding to C^1 comes down to showing commutativity of the diagram

$$\begin{array}{ccc}
WYVX & \xrightarrow{C^1(WY,V)\otimes id_X} & VWYX \\
{\scriptstyle id_W\otimes C^1(Y,VX)}\downarrow & & \downarrow{\scriptstyle id_{VW}\otimes C^1(Y,X)} \\
WVXY & \xrightarrow[C^1(W,V)\otimes id_{XY}]{} & VWXY
\end{array}$$

where subscripts have been left off for simplicity. This diagram commutes if we choose $C^1 = c^1$ to be the braiding in $\mathcal{C}_1$. Similar considerations lead to choosing $C^2 = c^2$ to be the braiding in $\mathcal{C}_2$.

Lemma 5.2.4. *The natural isomorpism λ descends to a canonical tensor structure on $\overline{\Lambda} : (\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2)^{\boxtimes 2} \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$.*

Proof. To prove the lemma we must only show that $\lambda : \Lambda\otimes \to \otimes(\Lambda \boxtimes \Lambda) : (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 4} \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ is $\mathcal{D}$-balanced in positions 1, 3, 5, 7. The 1-balancing of λ is equivalent to commutativity of the perimeter

$$
\begin{tikzcd}
B_{1,2}(X_1DU_1Y_1V_1 \boxtimes X_2U_2Y_2V_2) \arrow[r, "c^1_{U_1,Y_1}\boxtimes c^2_{U_2,Y_2}"] \arrow[d, "c^1_{D,U_1Y_1V_1}"'] & B_{1,2}(X_1DY_1U_1V_1 \boxtimes X_2Y_2U_2V_2) \arrow[d, "c^1_{D,Y_1U_1V_1}"] \\
B_{1,2}(X_1U_1Y_1V_1D \boxtimes X_2U_2Y_2V_2) \arrow[r, "c^1_{U_1,Y_1}\boxtimes c^2_{U_2,Y_2}"] \arrow[d, "b"'] & B_{1,2}(X_1Y_1U_1V_1D \boxtimes X_2Y_2U_2V_2) \arrow[d, "b"] \\
B_{1,2}(X_1U_1Y_1V_1 \boxtimes DX_2U_2Y_2V_2) \arrow[r, "c^1_{U_1,Y_1}\boxtimes c^2_{U_2,Y_2}"] & B_{1,2}(X_1Y_1U_1V_1 \boxtimes DX_2Y_2U_2V_2)
\end{tikzcd}
$$

which commutes trivially. Upper and lower horizontals are $\lambda_{\overline{XD}\boxtimes\overline{Y},\overline{U}\boxtimes\overline{V}}$ and $\lambda_{D\overline{X}\boxtimes\overline{Y},\overline{U}\boxtimes\overline{V}}$ where $\overline{XD} := X_1D\boxtimes X_2$ and verticals are Λ 1-balancing. Balancing in the 7th position is similar. The 3-balancing of λ comes down to the diagrams

$$
\begin{tikzcd}
XUYDV \arrow[rr, "c_{U,YD}"] \arrow[dr, "c_{U,Y}"'] \arrow[dd, "c_{D,V}"'] & & XYDUV \arrow[dl, shift left, "c_{D,U}"] \arrow[dd, "c_{D,UV}"] \\
& XYUDV \arrow[ur, shift left, "c_{U,D}"] \arrow[dr, "c_{D,V}"'] & \\
XUYVD \arrow[rr, "c_{U,Y}"'] & & XYUVD
\end{tikzcd}
\qquad
\begin{tikzcd}
DXUYV \arrow[rr, "c_{U,Y}"] \arrow[dr, "c_{D,X}"] \arrow[dd, "c_{D,XU}"'] & & DXYUV \arrow[dd, "c_{D,X}"] \\
& XDUYV \arrow[dl, shift right, "c_{D,U}"'] \arrow[dr, "c_{U,Y}"] & \\
XUDYV \arrow[ur, shift right, "c_{U,D}"'] \arrow[rr, "c_{U,DY}"'] & & XDYUV
\end{tikzcd}
$$

where indices and tensor with identity morphisms have been elided. In the diagram on the left $c = c^1$ and on the right $c = c^2$. Each subdiagram is either naturality or pentagon for the braiding. The double edges follow from $\mathcal{D}$ injecting into the Müger centers. The 5-balancing requires similar diagrams.

Our discussion implies that λ descends to $\overline{\lambda} : \overline{\Lambda\otimes} \to \overline{\otimes(\Lambda\boxtimes\Lambda)}$, a natural isomorphism, as indicated in the diagram.

$$\begin{tikzcd}
(\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 4} \arrow[dd, "B_{1,2}^{\boxtimes 4}"'] \arrow[dr, squiggly, bend left=40, "\Lambda\otimes"{name=A}] \arrow[dr, "\otimes(\Lambda\boxtimes\Lambda)"'{name=B}] & \\
& \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \\
(\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2)^{\boxtimes 4} \arrow[ur, squiggly, "\overline{\Lambda\otimes}"{name=C}] \arrow[ur, bend right=40, "\overline{\otimes(\Lambda\boxtimes\Lambda)}"'{name=D}] &
\arrow[from=A, to=B, Rightarrow, "\lambda"]
\arrow[from=C, to=D, Rightarrow, "\overline{\lambda}"]
\end{tikzcd}$$

Using basic properties of $B_{1,2}$ this becomes $\overline{\lambda} : \overline{\Lambda}\,\overline{\otimes} \to \overline{\Lambda}(\overline{\Lambda} \boxtimes \overline{\Lambda})$ and hence a tensor structure for $\overline{\Lambda}$, proving the lemma. $\square$

In the language of [JS93] the functor $\overline{\Lambda}$ gives $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ a multiplication $\Phi : \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \times \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ defined by $\Phi(A, B) = \overline{\Lambda}(A \boxtimes B)$. Part of the data describing a multiplication in $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ involves isomorphisms $\Phi(A, 1) \simeq A$, $\Phi(1, B) \simeq B$ which we can assume are identity (on assuming strictness of tensor structure $\overline{\Lambda}$ in $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$). Natural isomorphisms $\overline{\lambda}$ give an isomorphism $\Phi(A, A') \otimes \Phi(B, B') \simeq \Phi(A \otimes B, A' \otimes B')$. Proposition 5.3 of *loc. cit.* implies that $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ acquires a braided structure c making the diagram commute:

$$\begin{tikzcd}
\overline{\Lambda}(A \boxtimes 1) \otimes \overline{\Lambda}(1 \boxtimes B) = A \otimes B \arrow[rr, "c_{A,B}"] \arrow[d, "\overline{\lambda}^{-1}_{A\boxtimes 1, 1\boxtimes B}"'] & & B \otimes A = \overline{\Lambda}(1 \boxtimes B) \otimes \overline{\Lambda}(A \boxtimes 1) \\
\overline{\Lambda}((A \boxtimes 1) \otimes (1 \boxtimes B)) \arrow[r, equal] & \overline{\Lambda}(A \boxtimes B) \arrow[r, equal] & \overline{\Lambda}((1 \boxtimes B) \otimes (A \boxtimes 1)) \arrow[u, "\overline{\lambda}_{1\boxtimes B, A\boxtimes 1}"']
\end{tikzcd}$$

Denote by $\gamma_{\overline{U}\boxtimes\overline{Y}}$ the natural isomorphism $B_{1,2}(c^1_{U_1,Y_1} \boxtimes c^2_{U_2,Y_2}) : \Lambda \to \Lambda^{op}$ for any pair of objects $\overline{U}, \overline{Y} \in \mathcal{C}_1 \boxtimes \mathcal{C}_2$. Note that

$$\lambda_{\overline{X}\boxtimes 1, 1\boxtimes\overline{Y}} = B_{1,2}(id \otimes c^1_{1,1} \otimes id \boxtimes id \otimes c^2_{1,1} \otimes id) = id_{\Lambda(\overline{X}\boxtimes\overline{Y})}$$

and

$$\lambda_{1\boxtimes\overline{X}, \overline{Y}\boxtimes 1} = B_{1,2}(c^1_{X_1,Y_1} \boxtimes c^2_{X_2,Y_2}) = \gamma_{\overline{X}\boxtimes\overline{Y}}.$$

Balancing of λ in positions 3 and 5 implies balancing of γ in positions 1, 3, hence a unique natural isomorphism $\overline{\gamma} : \overline{\Lambda} \to \overline{\Lambda^{op}}$ satisfying $\overline{\gamma} * B^{\boxtimes 2}_{1,2} = \gamma$. Uniqueness gives $\overline{\lambda}_{1\boxtimes -, -\boxtimes 1} = \overline{\gamma}$ and $\overline{\lambda}_{-\boxtimes 1, 1\boxtimes -} = id$. Thus braiding on the relative tensor product is equal to $\overline{\gamma}$: for $A, B \in \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ we have $c_{A,B} = \overline{\gamma}_{A,B}$. □

5.3 Module categories over $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$

In this section we are interested in studying module categories over fusion (tensor) categories of the form $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ where we retain above notation. We begin with a general lemma relating balancing and module category structure.

Lemma 5.3.1. *Let $\mathcal{M}$ be a strict $\mathcal{C}, \mathcal{D}$-module category and let $\mathcal{N}$ be a left $\mathcal{C}$-module category. Then any $\mathcal{D}$-balanced left $\mathcal{C}$-module functor $\mathcal{M} \boxtimes \mathcal{N} \to \mathcal{A}$ descends to a functor $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{A}$ having canonical $\mathcal{C}$-module structure.*

Proof. Denote by f the balancing ismorphisms for F, and for $X \in \mathcal{C}$ write $\varphi_{X,M} : F(X \otimes M) \to X \otimes F(M)$ for $\mathcal{C}$-module structure of F. If L_X is the functor associated to left X-multiplication then we can view φ as a natural isomorphism $\varphi_X : F(L_X \boxtimes 1) \to L_X F$ (here $1 = id_{\mathcal{N}}$). Recall that left X-multiplication in $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is given by $\overline{L_X}$, the unique endofunctor on $\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}$ determined by $B_{\mathcal{M},\mathcal{N}}\circ(L_X\boxtimes 1):\mathcal{M}\boxtimes\mathcal{N}\to\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}$. It is trivial to check that both $F(L_X\boxtimes 1)$, $L_X F$ are balanced functors $\mathcal{M}\boxtimes\mathcal{N}\to\mathcal{A}$ with balancing coming from f. Also one checks that φ_X is balanced natural transformation. Thus we have unique $\overline{\varphi_X}:\overline{F(L_X\boxtimes 1)}\to\overline{L_X F}:\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}\to\mathcal{A}$. Using basic properties of $B_{\mathcal{M},\mathcal{N}}$ we see that $\overline{F(L_X\boxtimes 1)}=\overline{F}(\overline{L_X})$ and $\overline{L_X F}=L_X\overline{F}$. Thus components of $\overline{\varphi_X}$ are given by $\overline{\varphi_X}_A:\overline{F}(X\otimes A)\to X\otimes\overline{F}(A)$ for a typical object $A\in\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{N}$. Extending this construction to all objects in $\mathcal{C}$ we get $\mathcal{C}$-linearity for $\overline{F}$. □

Let us now return to pre Lemma 5.3.1 notation. In what follows assume all module categories to be strict, an assumption we can justify thanks to Theorem 1.3.8. The next proposition relates module structure over the tensor product to module structure over braided fusion category $\mathcal{D}$.

Proposition 5.3.2. *Any module category over $\mathcal{C}_1\boxtimes_{\mathcal{D}}\mathcal{C}_2$ admits a canonical $\mathcal{D}$-bimodule category structure with respect to which the left and right module structures agree.*

Proof. Suppose braided inclusions $\varphi_i:\mathcal{D}\hookrightarrow Z(\mathcal{C}_i)$ put $\mathcal{C}_i$ over braided $\mathcal{C}$ as above. Let $\mathcal{M}$ be a left $\mathcal{C}_1\boxtimes_{\mathcal{D}}\mathcal{C}_2$-module category. Then define left and right $\mathcal{D}$-module category structures on $\mathcal{M}$ in the following way. For $M\in\mathcal{M}$ set

$$D\otimes M:=B_{1,2}(\varphi_1(D)\boxtimes 1)\otimes M,\qquad M\otimes D:=B_{1,2}(1\boxtimes\varphi_2(D))\otimes M.$$

Note that left module associativity of the action comes from tensor structure of φ_1 and module associativity on the right comes from tensor structure of φ_2. Note also that since

$$\begin{aligned} B_{1,2}(\varphi_1(D) \boxtimes 1) \otimes B_{1,2}(1 \boxtimes \varphi_2(D)) &= B_{1,2}(\varphi_1(D) \boxtimes \varphi_2(E)) \\ &= B_{1,2}(1 \boxtimes \varphi_2(E)) \otimes B_{1,2}(\varphi_1(D) \boxtimes 1) \end{aligned}$$

left $\mathcal{C}_1$ and right $\mathcal{C}_2$ module structures are strictly consistent: $(D \otimes M) \otimes E = D \otimes (M \otimes E)$. It is evident that $b_{1,D,1} \otimes id_M : D \otimes M \xrightarrow{\sim} M \otimes D$. $\square$

Theorem 5.3.3. *Let $\mathcal{C}_i$ be tensor categories over braided fusion category $\mathcal{D}$. Then $\boxtimes_{\mathcal{D}}$ is a functor $\mathcal{C}_1$-Mod $\boxtimes \mathcal{C}_2$-Mod $\to \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-Mod. Furthermore $\boxtimes_{\mathcal{D}}$ is bilinear with respect to composition of functors.*

Proof. Let $\mathcal{M} \in \mathcal{C}_1$-Mod, $\mathcal{N} \in \mathcal{C}_2$-Mod and for convenience assume that the braided inclusions $\varphi_i : \mathcal{D} \to Z(\mathcal{C}_i)$ are both strict as tensor functors. Centrality of $\mathcal{D}$ in $\mathcal{C}_i$ allows us to define $\mathcal{D}$-bimodule structure on both $\mathcal{M}$ and $\mathcal{N}$ by stipulating that left, right actions agree. We break up the proof of Theorem 5.3.3 into two parts. First we show that $\boxtimes_{\mathcal{D}}$ has the proper codomain category and then show that it respects the relevant structures.

Proposition 5.3.4. *$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ has canonical structure of a $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-module category.*

Proof. Define $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-module structure on $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ using the diagram

$$\begin{tikzcd}
\mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \mathcal{M} \boxtimes \mathcal{N} \arrow[r, "\tau(23)"] \arrow[dd, "B_{1,2} \boxtimes B_{\mathcal{M},\mathcal{N}}"'] & \mathcal{C}_1 \boxtimes \mathcal{M} \boxtimes \mathcal{C}_2 \boxtimes \mathcal{N} \arrow[d, "\otimes^2 := (\otimes, \otimes)"] \\
 & \mathcal{M} \boxtimes \mathcal{N} \arrow[d, "B_{\mathcal{M},\mathcal{N}}"] \\
\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2 \boxtimes \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r] & \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}
\end{tikzcd}$$

For convenience abbreviate $T := B_{\mathcal{M},\mathcal{N}} \circ \otimes^2 \circ \tau_{(23)}$. We wish to descend T to the functor indicated in the diagram by the unadorned horizontal arrow.

We first check that the composition T is $\mathcal{D}$-balanced in positions 1 and 3 (Definition 2.1.3). Let X, Y, D, M, N be objects in $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}, \mathcal{M}, \mathcal{N}$. Then

$$B_{\mathcal{M},\mathcal{N}}(XDM \boxtimes YN) \xrightarrow{b} B_{\mathcal{M},\mathcal{N}}(XM \boxtimes DYN)$$

gives balancing $b_{XM,D,YN} : T(XD \boxtimes Y \boxtimes M \boxtimes N) \to T(X \boxtimes DY \boxtimes M \boxtimes N)$ in position 1 for T. Balacing in position 3 can be written in terms of both balancing for $B_{\mathcal{M},\mathcal{N}}$ and the central structure in $\mathcal{C}_2$. Explicitly

$$B_{\mathcal{M},\mathcal{N}}(XMD \boxtimes YN) \xrightarrow{b} B_{\mathcal{M},\mathcal{N}}(XM \boxtimes DYN) \xrightarrow{c^2} B_{\mathcal{M},\mathcal{N}}(XM \boxtimes YDN)$$

where as usual c^i is the braiding in $Z(\mathcal{C}_i)$. It is evident that these candidate balancings in positions 1, 3 satisfy the balancing diagrams for those positions, hence T is so balanced. We therefore get a unique right exact functor (the unlabeled horizontal arrow in the diagram) which we will also call $\otimes$.

Next we check module associativity. It is easy to check that the compositions

$$(\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 2} \boxtimes (\mathcal{M} \boxtimes \mathcal{N}) \xrightarrow{(\Gamma, id)} (\mathcal{C}_1 \boxtimes \mathcal{C}_2) \boxtimes (\mathcal{M} \boxtimes \mathcal{N}) \xrightarrow{T} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$$

and

$$(\mathcal{C}_1 \boxtimes \mathcal{C}_2)^{\boxtimes 2} \boxtimes (\mathcal{M} \boxtimes \mathcal{N}) \xrightarrow{(B_{1,2}, T)} (\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2) \boxtimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \xrightarrow{\overline{T}} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$$

are equal (here we use Γ for monoidal structure in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ as in Proposition 5.1.1). Also it's easy (but tedious) to show that they are each balanced in positions 1, 3, 5; all balancings may be written in terms of c^i and balancings for $B_{1,2}$ and $B_{\mathcal{M},\mathcal{N}}$. Thus they descend to functors $\overline{T(\Gamma \boxtimes id)}$ and $\overline{T(B_{1,2} \boxtimes T)} : (\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2)^{\boxtimes 2} \boxtimes (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}) \to \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ which are equal. Using basic properties of universal balanced functors one therefore has

$$\overline{T}(\overline{\Lambda} \boxtimes 1) = \overline{T}(1 \boxtimes \overline{T})$$

where again we use $\overline{\Lambda}$ to denote monoidal structure in $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$ as in Proposition 5.1.1. This is precisely the statement that $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ has (strict) $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-module category structure. Unit object of the action is clearly $1 = B_{1,2}(1 \boxtimes 1)$ and the required unit constraints obtain. $\square$

The next result is the module counterpart to the corresponding result for tensor functors proved above (Proposition 5.1.10) showing that the class of module functors *over* $\mathcal{D}$ is closed under the relative product $\boxtimes_{\mathcal{D}}$.

Proposition 5.3.5. *Let $F_i : \mathcal{M}_i \to \mathcal{N}_i$, $i = 1, 2$, be a pair of functors where for each* *i F_i is $\mathcal{C}_i$-module. Then $F_1 \boxtimes_{\mathcal{D}} F_2 : \mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2 \to \mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2$ has canonical structure of a $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-module functor.*

Proof. Denote by $B_{\mathcal{M}}$ the universal $\mathcal{D}$-balanced functor $B_{\mathcal{M}_1,\mathcal{M}_2} : \mathcal{M}_1 \boxtimes \mathcal{M}_2 \to \mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2$. Similarly define $B_{\mathcal{N}}$. Let F_1 have $\mathcal{C}_1$-module structure t and F_2 have $\mathcal{C}_2$-module structure t'. Define $F := B_{\mathcal{N}}(F_1 \boxtimes F_2) : \mathcal{M}_1 \boxtimes \mathcal{M}_2 \to \mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2$. One easily checks that F is balanced by $t'^{-1}_{D,N} b_{F_1(M),D,F_2(N)} t_{M,D} : F(MD \boxtimes N) \to F(M \boxtimes DN)$ for $M, N \in \mathcal{M}_1, \mathcal{M}_2$ and $D \in \mathcal{D}$ and where b is balancing for $B_{\mathcal{N}}$. We therefore get functor $F_1 \boxtimes_{\mathcal{D}} F_2 : \mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2 \to \mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2$ with the usual uniqueness property.

We now show that it respects $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-module structure. Then note that we have a natural isomorphism

$$B_{\mathcal{N}} * (t \boxtimes t') * \tau : B_{\mathcal{N}}(F_1 \otimes, F_2 \otimes)\tau \to B_{\mathcal{N}}(\otimes(F_1 \boxtimes F_1), \otimes(F_2 \boxtimes F_2))\tau. \tag{16}$$

Using the definition of $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{D}_2$-module structure $\otimes_{\mathcal{M}}, \otimes_{\mathcal{N}}$ for $\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2$ and $\mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2$ as described in the proof of Proposition 5.3.4 and the definition of the tensor product $F_1 \boxtimes_{\mathcal{D}} F_2$ we see that this is a natural isomorphism $(F_1 \boxtimes_{\mathcal{D}} F_2) \otimes_{\mathcal{M}} (B_{1,2} \boxtimes B_{\mathcal{M}}) \to \otimes_{\mathcal{N}}(1 \boxtimes (F_1 \boxtimes_{\mathcal{D}} F_2))B_{1,2} \boxtimes B_{\mathcal{M}}$. All structures are easily seen to be balanced therefore equation 16 descends to a unique natural isomorphism $\overline{t \boxtimes t'} : (F_1 \boxtimes_{\mathcal{D}} F_2) \otimes_{\mathcal{M}} \to \otimes_{\mathcal{N}}(1 \boxtimes (F_1 \boxtimes_{\mathcal{D}} F_2))$. This is precisely to say that $F_1 \boxtimes_{\mathcal{D}} F_2$ has the structure of a module functor. □

With Proposition 5.3.5 we have shown that $\boxtimes_{\mathcal{D}}$ as described in the statement of Theorem 5.3.3 is a functor. This completes the proof of the theorem. □

The next result examines functoriality of relative tensor product.

Proposition 5.3.6. *Let $\mathcal{C}_i, i \in \{1,2\}$ be categories fusion over braided fusion $\mathcal{D}$, and let $\mathcal{M}_i$ be right $\mathcal{C}_i$-module categories, $\mathcal{N}_i$ left $\mathcal{C}_i$-module categories. Then there is a canonical equivalence $(\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2) \boxtimes_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} (\mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2) \simeq (\mathcal{M}_1 \boxtimes_{\mathcal{C}_1} \mathcal{N}_1) \boxtimes_{\mathcal{D}} (\mathcal{M}_2 \boxtimes_{\mathcal{C}_2} \mathcal{N}_2)$.*

Proof. The proposition is proved by showing the existence of unique balanced functors L, L', R, R' making the diagram below commute. We present the diagram here in full prematurely and explain its various attributes in the following paragraph as we work through the proof. To save space we haven't been completely explicit in indexing universal balanced functors B, and rely on context to alleviate confusion.

$$\begin{tikzcd}
\mathcal{M}_1 \boxtimes \mathcal{N}_1 \boxtimes \mathcal{M}_2 \boxtimes \mathcal{N}_2 \arrow[r, "\tau_{(23)}"] \arrow[d, "B_{1,1} \boxtimes B_{2,2}"'] & \mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \mathcal{N}_1 \boxtimes \mathcal{N}_2 \arrow[r, "\tau_{(23)}"] \arrow[d, "B_{\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{N}_1 \boxtimes \mathcal{N}_2}"'] & \mathcal{M}_1 \boxtimes \mathcal{N}_1 \boxtimes \mathcal{M}_2 \boxtimes \mathcal{N}_2 \arrow[d, "B_{1,1} \boxtimes B_{2,2}"] \\
(\mathcal{M}_1 \boxtimes_{\mathcal{C}_1} \mathcal{N}_1) \boxtimes (\mathcal{M}_2 \boxtimes_{\mathcal{C}_2} \mathcal{N}_2) \arrow[r, "L"] \arrow[d, "B"'] & (\mathcal{M}_1 \boxtimes \mathcal{M}_2) \boxtimes_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} (\mathcal{N}_1 \boxtimes \mathcal{N}_2) \arrow[r, "R"] \arrow[d, "B_{1,2} \boxtimes_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} B_{1,2}"] & (\mathcal{M}_1 \boxtimes_{\mathcal{C}_1} \mathcal{N}_1) \boxtimes (\mathcal{M}_2 \boxtimes_{\mathcal{C}_2} \mathcal{N}_2) \arrow[d, "B"] \\
(\mathcal{M}_1 \boxtimes_{\mathcal{C}_1} \mathcal{N}_1) \boxtimes_{\mathcal{D}} (\mathcal{M}_2 \boxtimes_{\mathcal{C}_2} \mathcal{N}_2) \arrow[r, "L'"'] & (\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2) \boxtimes_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} (\mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2) \arrow[r, "R'"'] & (\mathcal{M}_1 \boxtimes_{\mathcal{C}_1} \mathcal{N}_1) \boxtimes_{\mathcal{D}} (\mathcal{M}_2 \boxtimes_{\mathcal{C}_2} \mathcal{N}_2)
\end{tikzcd}$$

The functor $\tau_{(23)}$ permutes the second and third tensorands. It is easy to see that the composition $B_{\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{N}_1 \boxtimes \mathcal{N}_2} \circ \tau_{(23)}$ is $\mathcal{D}$-balanced in positions 1 and 3 (Definition 2.1.3), hence the existence of unique balanced functor L making the subdiagram in the upper left commute. Similarly $B_{1,2} \boxtimes_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} B_{1,2} \circ L$ is $\mathcal{D}$-balanced giving unique balanced L' making lower left subdiagram commute.

Moving to the right side of the diagram one checks that composition $B_{1,1} \boxtimes B_{2,2} \circ \tau_{(23)}$ is $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-balanced giving unique balanced R making upper right subdiagram commute. Existence of the functor R' is slightly trickier. Observe that composition of functors down the vertical center forms half of the diagram defining the functor $B_{1,2} \boxtimes_{C_1 \boxtimes_{\mathcal{D}} C_2} B_{1,2}$ (the rectangular subdiagram in the left of the diagram below).

$$\begin{tikzcd}
\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \mathcal{N}_1 \boxtimes \mathcal{N}_2 \arrow[rr, "\Gamma"] \arrow[dr, "B_{1,2} \boxtimes B_{1,2}"] \arrow[dd, "B_{\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{N}_1 \boxtimes \mathcal{N}_2}"'] & & (\mathcal{M}_1 \boxtimes_{C_1} \mathcal{N}_1) \boxtimes_{\mathcal{D}} (\mathcal{M}_2 \boxtimes_{C_2} \mathcal{N}_2) \\
& (\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2) \boxtimes (\mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2) \arrow[ur, "\Gamma'"] \arrow[dr, "B_{\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2, \mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2}"] & \\
(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \boxtimes_{C_1 \boxtimes_{\mathcal{D}} C_2} (\mathcal{N}_1 \boxtimes \mathcal{N}_2) \arrow[rr, "B_{1,2} \boxtimes_{C_1 \boxtimes_{\mathcal{D}} C_2} B_{1,2}"'] & & (\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2) \boxtimes_{C_1 \boxtimes_{\mathcal{D}} C_2} (\mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2) \arrow[uu, "R'"']
\end{tikzcd}$$

Denote $\Gamma := B \circ B_{1,1} \boxtimes B_{2,2} \circ \tau_{(23)}$, the composition of right-most vertical and top right functors. Using the $\mathcal{D}$-balancing of $B_{1,1}$ and $B_{2,2}$ as well as the bimodule-linearity of functors B one shows that Γ is $\mathcal{D}$-balanced in positions $1, 3$ and thus we have a unique balanced functor $\Gamma' : (\mathcal{M}_1 \boxtimes_{\mathcal{D}} \mathcal{M}_2) \boxtimes (\mathcal{N}_1 \boxtimes_{\mathcal{D}} \mathcal{N}_2) \to (\mathcal{M}_1 \boxtimes_{C_1} \mathcal{N}_1) \boxtimes_{\mathcal{D}} (\mathcal{M}_2 \boxtimes_{C_2} \mathcal{N}_2)$ such that $\Gamma' \circ B_{1,2} \boxtimes B_{1,2} = \Gamma$. In fact Γ' is $C_1 \boxtimes_{\mathcal{D}} C_2$-balanced giving unique balanced functor R'. This is precisely R' in the first diagram.

Every cell commutes and therefore the exterior contour also commutes. Retaining notation above this means

$$B \circ B_{1,1} \boxtimes B_{2,2} = R'L' \circ B \circ B_{1,1} \boxtimes B_{2,2}$$

since $\tau_{1,3}^2 = id$. Universality of functors B and $B_{1,1} \boxtimes B_{2,2}$ implies that $R'L' = id$. Reasoning similar to the above yields $L'R' = id$, hence L', R' are quasi-inverse and the proposition is proved. $\square$

Note 5.3.7. In the case that categories $\mathcal{M}_i, \mathcal{N}_i$ have bimodule category structure the equivalence in Proposition 5.3.6 is an equivalence of bimodule categories.

Corollary 5.3.8. *Suppose that $\mathcal{M}$ is a $\mathcal{C}_1$-bimodule category, $\mathcal{N}$ is a $\mathcal{C}_2$-bimodule category, and assume that $\mathcal{M}$ and $\mathcal{N}$ are invertible. Then $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is invertible as a $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$-bimodule category and has inverse $\mathcal{M}^{-1} \boxtimes_{\mathcal{D}} \mathcal{N}^{-1}$.*

Proof. Theorem 5.3.3 implies

$$(\mathcal{M} \boxtimes_{\mathcal{C}_1} \mathcal{N}) \boxtimes_{\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2} (\mathcal{M}^{-1} \boxtimes_{\mathcal{C}_2} \mathcal{N}^{-1}) \simeq (\mathcal{M} \boxtimes_{\mathcal{C}_1} \mathcal{M}^{-1}) \boxtimes_{\mathcal{D}} (\mathcal{N} \boxtimes_{\mathcal{C}_2} \mathcal{N}^{-1}) \simeq \mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$$

giving the result. $\square$

CHAPTER VI

EQUIVARIANTIZATION AND TENSOR PRODUCT

6.1 (De)-equivariantization: background

Most of the background information in this section is taken from [DGNO10]. Let $\mathcal{M}$ be a monoidal category. Recall that an *action* of $\mathcal{M}$ on $\mathcal{C}$ is a monoidal functor $F : \mathcal{M} \to \underline{\mathrm{End}}(\mathcal{C})$ where $\underline{\mathrm{End}}(\mathcal{C})$ denotes the category of k-linear endofunctors on $\mathcal{C}$.

For finite group G denote by $\underline{G}$ the finite monoidal category having objects elements of G, only trivial morphisms, and with tensor product given by multiplication in G. Then an action of G on $\mathcal{C}$ is the same as an action of $\underline{G}$ on $\mathcal{C}$. The tensor category Vec_G of finite dimensional G-graded vector spaces identifies with the k-linear hull of $\underline{G}$ and hence an action of G on $\mathcal{C}$ is the same thing as a k-linear action of Vec_G on $\mathcal{C}$.

Let $\mathrm{Rep}(G)$ denote the braided category of finite dimensional representations of G. Then we have an equivalence of 2-categories

$$\{k\text{-linear categories with } G\text{-action}\} \rightleftarrows \{k\text{-linear categories with } \mathrm{Rep}(G)\text{-action}\},$$

called *equivariantization* and *de-equivariantization*.

6.1.1 Equivariantization

In this section we describe how a category with G-action has canonical Rep(G)-module structure. Let $F : \underline{G} \to \underline{\mathrm{End}}(\mathcal{C})$, $g \mapsto F_g$ be an action of G on $\mathcal{C}$ and let $\gamma_{g,h} : F_g F_h \simeq F_{gh}$ be the isomorphism giving F the structure of a monoidal functor. A pair (X, u) for object $X \in \mathcal{C}$ is said to be *G-equivariant* if there is a natural family $u_g : F_g(X) \simeq X$ of natural isomorphisms making the diagram

$$\begin{tikzcd} F_g(F_h(X)) \arrow[r, "F_g(u_h)"] \arrow[d, "\gamma_{g,h}(X)"'] & F_g(X) \arrow[d, "u_g"] \\ F_{gh}(X) \arrow[r, "u_{gh}"'] & X \end{tikzcd}$$

commute for $g, h \in G$. Morphisms of equivariant objects are defined to be morphisms in $\mathcal{C}$ commuting with u_g for all $g \in G$. Evidently we have a category of G-equivariant objects in $\mathcal{C}$, denoted $\mathcal{C}^G$.

The category $\mathcal{C}^G$ has Rep(G)-module category structure as follows. For representation (V, ρ) and $(X, u) \in \mathcal{C}^G$ we define $(V \otimes X, u^V)$ by the composition

$$u_g^V := F_g(X \otimes V) \simeq F_g(X) \otimes V \xrightarrow{u_g \otimes \rho(g)} X \otimes V$$

6.1.2 De-equivariantization

Here we describe how a Rep(G)-module category carries a natural structure of a category on which G acts. Recall that the regular object A in Rep(G) can be viewed as the algebra $Fun(G, k)$ of k-valued functions on G. As a representation G acts on A by right translation. Any Rep(G)-module category $\mathcal{D}$ thus contains a subcategory of A-modules, which we denote by $\mathcal{D}_G$ and call the *de-equivariantization* of $\mathcal{D}$.

6.2 Monoidal 2-structure and (de)-equivariantization

We keep notation as above. Braiding of Rep(G) implies an embedding of 2-categories Rep(G)-Mod $\hookrightarrow \mathcal{B}(\text{Rep}(G))$ into the monoidal 2-category consisting of Rep(G)-bimodule categories. This is symptomatic of the observation that every module category over a braided monoidal category is really a bimodule category. Denote by $\boxtimes_{\text{Rep}(G)}$ the monoidal 2-structure in $\mathcal{B}(\text{Rep}(G))$.

Denote by $\underline{G}$-Mod the 2-category consisting of categories with G-action. $\underline{G}$-Mod has monoidal structure as follows. Let $\mathcal{F}$, $\mathcal{E}$ be objects in $\underline{G}$-Mod with G-actions given by monoidal functors F, E respectively. Then G acts on $\mathcal{F} \boxtimes \mathcal{E}$ via $F \boxtimes E$. We write $\mathcal{F} \odot \mathcal{E}$ to indicate the category $\mathcal{F} \boxtimes \mathcal{E}$ with this action.

Proposition 6.2.1. *The correspondence $\mathcal{C} \mapsto \mathcal{C}^G$ between the 2-categories $\underline{G}$-Mod and $\mathcal{B}(Rep(G))$ respects monoidal structure.*

Proof. Denote by $Fun_{\underline{G}}(Vec, \mathcal{C})$ the category of functors which commute with the action of G where we view Vec as having trivial G-action. $Fun_{\underline{G}}(Vec, \mathcal{C})$ carries a natural Rep(G)-module category structure as follows: for $(V, \rho) \in$ Rep(G) define $(H, h) \otimes (V, \rho) := (H^V, h^V)$ where $H^V(W) := H(W) \otimes V$ and where h^V is given by the composition

$$h_g^V := H^V(F_g k) = H(F_g k) \otimes V \xrightarrow{h_g \otimes \rho(g)} F_g H(k) \otimes V = F_g(H^V(k)).$$

One easily checks that the relevant module coherence diagram for h^V follows from those satisfied by h and ρ.

Lemma 6.2.2. *For $\mathcal{C} \in \underline{G}$-Mod there is a canonical equivalence of Rep(G)-module categories $\mathcal{C}^G \simeq Fun_{\underline{G}}(Vec, \mathcal{C})$.*

Proof. For $(X, u) \in \mathcal{C}^G$ denote by $H_X : Vec \to \mathcal{C}$ the unique functor having $H_X(k) = X$. Then $u_g : H_X(k) \simeq F_g H_X(k)$ gives H_X the structure of a $\underline{G}$-module functor. In the opposite direction any $\underline{G}$-module functor $(H, h) : Vec \to \mathcal{C}$ determines a G-equivariant object of $\mathcal{C}$: $\underline{G}$-module structure h on H corresponds to a natural isomorphism $h_g : H(k) \simeq F_g(H(k))$ where $F : \underline{G} \to \underline{\mathrm{End}}(\mathcal{C})$ is the action of G on $\mathcal{C}$. Let $v_g := h_g^{-1} : F_g(H(k)) \simeq H(k)$ and observe that the $\underline{G}$-module diagram satisfied by h translates into the diagram making $(H(k), v_g)$ an object in the equivariantization $\mathcal{C}^G$. Clearly these two constructions are inverse.

It remains to check that this correspondence respects Rep(G)-module category structures. Let $(X, u) \in \mathcal{C}^G$ and $(V, \rho) \in$ Rep(G). Then the functor associated to $(X \otimes V, u^V)$ is $(H_{X \otimes V}, u^V)$ and this is trivially naturally isomorphic to the functor $(H_X, u) \otimes (V, \rho)$. □

Remark 4.3 in [DGNO10] implies that, as abelian categories, $Fun_{\underline{G}}(Vec, \mathcal{C}) \simeq Fun_{Vec_G}(Vec, \mathcal{C})$. Write $Fun_{Vec_G}(Vec, \mathcal{C}) := \overline{\mathcal{C}}$. It is trivial that this equivalence respects Rep(G)-module structure, and hence as Rep(G)-module categories $\mathcal{C}^G \simeq \overline{\mathcal{C}}$. As monoidal categories $\underline{G}$-Mod and Vec_G-Mod are equivalent. We will use $\odot$ to denote monoidal structure in both places. For $\underline{G}$-module categories $\mathcal{C}, \mathcal{D}$ we have the following $\mathrm{Rep}(G)$-module equivalences:

$$(\mathcal{C} \odot \mathcal{D})^G \simeq \overline{\mathcal{C} \odot \mathcal{D}} \simeq \overline{\mathcal{C}} \boxtimes_{\mathrm{Rep}(G)} \overline{\mathcal{D}} \simeq \mathcal{C}^G \boxtimes_{\mathrm{Rep}(G)} \mathcal{D}^G. \tag{17}$$

First and last equivalences are Lemma 6.2.2 and the second is Theorem 8.3.2. $\square$

6.3 On de-equivariantization and relative tensor product

The main result of this section is the proof of Theorem 0.2.2. We begin with the following lemma.

Lemma 6.3.1. *Let $\mathcal{C}$, $\mathcal{D}$ be fusion categories and let $F : \mathcal{C} \to \mathcal{D}$ be a surjective tensor functor. Let I be its right adjoint. Then*

1. *$I(1)$ is an algebra in $Z(\mathcal{C})$.*

2. *$\mathcal{D}$ is tensor equivalent to the category $Mod_{\mathcal{C}}(I(1))$ of right $I(1)$-modules in $\mathcal{C}$.*

3. *The equivalence in (2) identifies F with the free module functor $X \mapsto X \otimes I(1)$.*

Proof. To prove (1) observe that $\mathcal{D}$ is a $Z(\mathcal{C})$-module category with action $X \otimes Y := F'(X) \otimes Y$ where $F' : Z(\mathcal{C}) \to \mathcal{D}$ is F composed with functor forgetting central structure. Under this action $\underline{\mathrm{Hom}}(1, 1) = I(1)$ (see Definition 1.3.4) so by Lemma 5 in [Ost03] $I(1)$ is an algebra in $Z(\mathcal{C})$. Note that since $I(1)$ is an algebra in $Z(\mathcal{C})$ we have tensor structure on $\mathrm{Mod}_{\mathcal{C}}(I(1))$: $X \otimes I(1) \doteq I(1) \otimes X$ so for $I(1)$-modules X, Y $X \otimes_{I(1)} Y$ makes sense. Theorem 1 in the same paper says that $\mathrm{Mod}_{\mathcal{C}}(I(1)) \simeq \mathcal{D}$ as module categories over $\mathcal{C}$ via F in (3). Observe that

$$F(X) \otimes_{I(1)} F(Y) = (X \otimes I(1)) \otimes_{I(1)} (Y \otimes I(1)) = (X \otimes Y) \otimes I(1) = F(X \otimes Y).$$

Hence $F : X \mapsto X \otimes I(1)$ respects tensor structure. This completes the proof of the lemma. $\square$

In what follows G is a finite group and we write $\mathcal{E} := \mathrm{Rep}(G)$, the symmetric fusion category of finite dimensional representations of G in Vec. Let $\mathcal{C}$ be tensor category over $\mathcal{E}$ (Definition 5.0.7) which we thereby view as a right $\mathcal{E}$-module category. Let A be the regular representation of G. A has the structure of an algebra in $\mathcal{E}$ and we therefore have the notion of A-module in $\mathcal{C}$. Denote by $\mathcal{C}_G$ the category $\mathrm{Mod}_{\mathcal{C}}(A)$ of A-modules in $\mathcal{C}$. There is functor $\mathrm{Free} : \mathcal{C} \to \mathcal{C}_G$, $X \mapsto X \otimes A$ left adjoint to the functor $\mathrm{Forg} : \mathcal{C}_G \to \mathcal{C}$ which forgets A-module structure ([DGNO10, §4.1.9]). We are now ready to prove the theorem.

Proof of Theorem 0.2.2. Let $F := B_{\mathcal{C},Vec} : \mathcal{C} \boxtimes Vec \to \mathcal{C} \boxtimes_{\mathcal{E}} Vec$ be the canonical surjective right exact functor described in Definition 2.1.4 which is tensor by Proposition 5.1.1, and let I be its right adjoint. Lemma 6.3.1 gives us tensor equivalence $\mathrm{Mod}_{\mathcal{C}}(I(1)) \simeq \mathcal{C} \boxtimes_{\mathcal{E}} Vec$. Denote by A' the image of the regular algebra A in $\mathcal{E}$ under the composition

$$\mathcal{E} \to Z(\mathcal{C}) \to \mathcal{C}. \tag{18}$$

We claim that $I(1)$ is A'

Let $X, Y \in \mathcal{C}$ be in distinct indecomposible $\mathcal{E}$-module subcategories of $\mathcal{C}$. Since the indecomposible $\mathcal{E}$-module subcategories of $\mathcal{C}$ are respected by F the images of X, Y under F are in distinct $\mathcal{E}$-module components of $\mathcal{C} \boxtimes_{\mathcal{E}} Vec$. Not only does this imply that $F(X)$ and $F(Y)$ are not isomorphic but in fact $\mathrm{Hom}(F(X), F(Y)) = 0$. Thus if $F(X)$ contains a copy of the unit object $1 \in \mathcal{C} \boxtimes_{\mathcal{E}} Vec$ then X and $1 \in \mathcal{C}$ must belong to the same indecomposible $\mathcal{E}$-module subcategory of $\mathcal{C}$. Thus any object whose F-image contains the unit object must be contained in the image of $\mathcal{E}$ in $\mathcal{C}$ under the composition (18).

Note that the restriction of F to the image of $\mathcal{E}$ in $\mathcal{C}$ gives a fiber functor $\mathcal{E} \to \mathcal{E} \boxtimes_{\mathcal{E}} Vec = Vec$. By [DGNO10, §2.13] the choice of a fiber functor from $\mathcal{E}$ determines a group $G_F \simeq G$ having the property that $Fun(G_F)$ is regular algebra A in $\mathrm{Rep}(G)$ and as such is canonically isomorpic to $I(1)$. Thus we have tensor equivalence $\mathrm{Mod}_{\mathcal{C}}(A) = \mathcal{C}_G \simeq \mathcal{C} \boxtimes_{\mathcal{E}} Vec$ and the proof is complete. □

CHAPTER VII

MODULE CATEGORIES OVER BRAIDED MONOIDAL CATEGORIES

In what follows $\mathcal{C}$ is a fixed tensor category (Definition 1.2.8) and all module categories are assumed to be exact. Recall (Definition 1.2.10) that $\mathcal{C}$ is said to be *braided* if $\mathcal{C}$ is equipped with a class of natural isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V$$

for objects $V, W \in \mathcal{C}$ satisfying a pair of hexagons describing how they interact with tensor associativity. When $\mathcal{C}$ is strict these reduce to the equations

$$c_{U,V\otimes W} = (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W) \tag{19}$$

$$c_{U\otimes V,W} = (c_{U,W} \otimes id_V)(id_U \otimes c_{V,W}). \tag{20}$$

7.1 The center of a bimodule category

In this section we describe a construction which associates to a strict $\mathcal{C}$-bimodule category $\mathcal{M}$ a new category having the structure of a $Z(\mathcal{C})$-bimodule category. Note that as monoidal categories $\mathcal{C}^{op}$, which we have been using to denote the opposite category, is canonically monoidal equivalent to the category $\mathcal{C}^{rev}$, the category $\mathcal{C}$ with monoidal product reversed. We will therefore not distinguish between them and use the single notation $\mathcal{C}^{op}$.

For the first proposition assume $\mathcal{C}$ to be braided by $c_{X,Y} : X \otimes Y \to Y \otimes X$. Our first proposition is well known and we provide a proof only for completeness.

Proposition 7.1.1. *Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. Then $\mathcal{M}$ has canonical structure of $\mathcal{C}$-bimodule category.*

Proof. We begin with the following lemma.

Lemma 7.1.2. *$\mathcal{M}$ is right $\mathcal{C}$-module category via $(M, X) \mapsto X \otimes M$ where $\otimes$ is left $\mathcal{C}$-module structure.*

Proof. For left module associativity a define natural isomorphism

$$a'_{M,X,Y} = a_{Y,X,M}(id_M \otimes c_{X,Y}) : M \otimes (X \otimes Y) \to (M \otimes X) \otimes Y$$

for $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$. In terms of the left module structure by which $M \otimes X$ is defined $a'_{M,X,Y} = a_{Y,X,M}(c_{X,Y} \otimes id_M) = (X \otimes Y) \otimes M \to Y \otimes (X \otimes M)$. We show that a' is module associativity for right module structure. Consider diagram

$$\begin{tikzcd}
(XYZ)M \arrow[rr, "c_{X,YZ}"] \arrow[dr, "c_{Y,Z}"] \arrow[dd, "c_{XY,Z}"'] & & (YZX)M \arrow[r, "a_{YZ,X,M}"] \arrow[dr, "c_{Y,Z}"] & (YZ)(XM) \arrow[r, "c_{Y,Z}"] & (ZY)(XM) \arrow[dd, "a_{Z,Y,XM}"] \\
& (XZY)M \arrow[rr, "c_{X,ZY}"] \arrow[dl, "c_{X,Z}"'] & & (ZYX)M \arrow[ur, "a_{ZY,X,M}"'] \arrow[d, "a_{Z,YX,M}"] & \\
(ZXY)M \arrow[rr, "a_{Z,XY,M}"'] \arrow[urrr, "c_{X,Y}"] & & Z((XY)M) \arrow[r, "c_{X,Y}"'] & Z((YX)M) \arrow[r, "a_{Y,X,M}"'] & Z(Y(XM))
\end{tikzcd}$$

The upper left rectangle is naturality of c, upper right triangle naturality of a, leftmost triangle is equation (20), triangle in lower half of diagram is equation (19), central bottom rectangle is naturality of a and rightmost rectangle is a-pentagon in $\mathcal{C}$. The two directed components of the external contour are precisely $a'_{MX,Y,Z}a'_{M,X,YZ}$ and $(a'_{M,X,Y} \otimes Z)a'_{M,XY,Z}$. The diagrams for action of unit in $\mathcal{C}$ are even easier. $\square$

Define action of $X \boxtimes Y \in \mathcal{C} \boxtimes \mathcal{C}^{rev}$ using left and right actions, i.e. $(X \boxtimes Y) \otimes M = Y \otimes (X \otimes M)$. Define

$$\gamma_{X,M,Y} = a_{X,Y,M}(c_{Y,X} \otimes id_M)a^{-1}_{Y,X,M} : Y \otimes (X \otimes M) \to X \otimes (Y \otimes M).$$

In order to verify that the candidate action is indeed bimodule we must show that γ satisfies the necessary pentagons (Remark 1.3.10). Commutativity of the first pentagon follows from an examination of the diagram below.

$$\begin{tikzcd}
(ZXY)M \arrow[rrrr, "c_{Z,XY}"] \arrow[dr, "a_{Z,XY,M}"] \arrow[ddd, "a_{ZX,Y,M}"'] & & & & (XYZ)M \arrow[dl, "a_{XY,Z,M}"'] \arrow[ddd, "a_{X,YZ,M}"] \\
& Z((XY)M) \arrow[rr, "\gamma_{XY,M,Z}"] \arrow[d, "a_{X,Y,M}"'] & & (XY)(ZM) \arrow[d, "a_{X,Y,ZM}"] & \\
& Z(X(YM)) \arrow[r, "\gamma_{X,YM,Z}"] & X(Z(YM)) \arrow[r, "\gamma_{Y,M,Z}"] & X(Y(ZM)) & \\
(ZX)(YM) \arrow[r, "c_{Z,X}"'] \arrow[ur, "a_{Z,X,YM}"] & (XZ)(YM) \arrow[ur, "a_{X,Z,YM}"] & (XZY)M \arrow[l, "a_{XZ,Y,M}"] \arrow[r, "a_{X,ZY,M}"'] & X((ZY)M) \arrow[r, "c_{Z,Y}"'] \arrow[ul, "a_{Z,Y,M}"] & X((YZ)M) \arrow[ul, "a_{Y,Z,M}"']
\end{tikzcd}$$

Every peripheral rectangle is either the definition of γ or the module associativity satisfied by a. Note that top left vertex can be connected to the lower center vertex by the map $c_{Z,X} \otimes id_{Y \otimes M}$ making commutative rectangle expressing naturality of a in first index. Lower center vertex can be connected to uppermost right vertex by the map $id_X \otimes c_{Z,Y} \otimes id_M$ making commutative rectangle expressing naturality of a in the second index. Commutativity of this new external triangle is (equation (19))$\otimes M$. Thus the internal pentagon commutes, and this is precisely the first diagram in Remark 1.3.10. Commutativity of second pentagon is similar. This completes the proof of Proposition 7.1.1. □

Next we generalize the notion of center to module categories.

Definition 7.1.3. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category. A *central structure on* $\mathcal{M}$ is a family of natural isomorphisms $\varphi_{X,M} : X \otimes M \simeq M \otimes X$, $X \in \mathcal{C}$, one for each object $M \in \mathcal{M}$, satisfying the condition

$$\begin{tikzcd}
(XY)M \arrow[r, "\varphi_{XY,M}"] \arrow[d, "a^{\ell}_{X,Y,M}"'] & M(XY) \arrow[d, "a^{r}_{M,XY}"] \\
X(YM) \arrow[d, "X\otimes\varphi_{Y,M}"'] & (MX)Y \\
X(MY) \arrow[r, "\gamma_{X,M,Y}"'] & (XM)Y \arrow[u, "\varphi_{X,M}\otimes Y"']
\end{tikzcd}$$

whenever $Y \in \mathcal{C}$ where a^{ℓ}, a^{r} are left and right module associativity in $\mathcal{M}$ and γ bimodule consistency (Proposition 1.3.10). φ_M is called the *centralizing isomorphism* associated to M. If such a central structure exists $\mathcal{M}$ is said to be *central* over $\mathcal{C}$.

Note that when $\mathcal{M}$ is strict as a bimodule category the hexagon reduces to

$$\begin{tikzcd}
XMY \arrow[rr, "\varphi_{X,M}\otimes id_Y"] & & MXY \\
& XYM \arrow[ul, "id_X\otimes\varphi_{Y,M}"] \arrow[ur, "\varphi_{XY,M}"'] &
\end{tikzcd}$$

In what follows assume $\mathcal{C}$ is a strict monoidal category.

Definition 7.1.4. The *center* $Z_{\mathcal{C}}(\mathcal{M})$ of $\mathcal{M}$ over $\mathcal{C}$ consists of objects given by pairs (M, φ_M) where $M \in \mathcal{M}$ and where φ_M is a family of natural isomorphisms such that the isomorphisms $\varphi_{X,M} : X \otimes M \simeq M \otimes X$ satisfy Definition 7.1.3 for $X \in \mathcal{C}$. A morphism from (M, φ_M) to (N, φ_N) in $Z_{\mathcal{C}}(\mathcal{M})$ is a morphism $t : M \to N$ in $\mathcal{M}$ satisfying $\varphi_{X,N}(id_X \otimes t) = (t \otimes id_X)\varphi_{X,M}$.

Note 7.1.5. Definition 7.1.4 appeared in [GNN09] in connection with centers of braided fusion categories.

Example 7.1.6. For $\mathcal{C}$ viewed as having a regular bimodule category structure $Z_{\mathcal{C}}(\mathcal{C}) = Z(\mathcal{C})$, the center of $\mathcal{C}$.

Definition 7.1.7. Let $\mathcal{M}, \mathcal{N}$ be bimodule categories central over $\mathcal{C}$. Then $\mathcal{C}$-bimodule functor $T : \mathcal{M} \to \mathcal{N}$ is called *central* if the diagram

$$\begin{tikzcd} T(X \otimes M) \arrow[r, "f_{X,M}"] \arrow[d, "T(\varphi_{X,M})"'] & X \otimes T(M) \arrow[d, "\varphi_{X,T(M)}"] \\ T(M \otimes X) \arrow[r, "f_{M,X}"'] & T(M) \otimes X \end{tikzcd}$$

commutes for all $X \in \mathcal{C}$, $M \in \mathcal{M}$, where φ denotes centralizing natural isomorphisms in $\mathcal{M}$ and $\mathcal{N}$. f is linearity isomorphism for T. A *central natural transformation* $\tau : F \Rightarrow G$ for central functors $F, G : \mathcal{M} \to \mathcal{N}$ is a bimodule natural transformation

$F \Rightarrow G$ with the additional requirement that, for $X \in \mathcal{C}, M \in \mathcal{M}$ the diagram

$$\begin{tikzcd}
X \otimes F(M) \arrow[r, "\varphi_{X,F(M)}"] \arrow[d, "X \otimes \tau_M"'] & F(M) \otimes X \arrow[d, "\tau_M \otimes X"] \\
X \otimes G(M) \arrow[r, "\varphi_{X,G(M)}"'] & G(M) \otimes X
\end{tikzcd}$$

commutes.

It is evident that centrality of natural transformations is preserved by vertical (and horizontal) composition, and we thus have a category (indeed a bicategory) $Z(\mathcal{M}, \mathcal{N})$ for central bimodule categories $\mathcal{M}, \mathcal{N}$ consisting of central functors $\mathcal{M} \to \mathcal{N}$ where morphisms are central natural transformations.

Lemma 7.1.8. *$Z_{\mathcal{C}}(\mathcal{M})$ is a $Z(\mathcal{C})$-bimodule category.*

Proof. Assume $\mathcal{M}$ is strict bimodule category. We have left action of $Z(\mathcal{C})$ on $Z_{\mathcal{C}}(\mathcal{M})$ given as follows: for $(X, c_X) \in Z(\mathcal{C})$ and $(M, \varphi_M) \in Z_{\mathcal{C}}(\mathcal{M})$ define $(X, c_X) \otimes (M, \varphi_M) = (X \otimes M, \varphi_{X \otimes M})$ where for $Y \in \mathcal{C}$

$$\varphi_{Y,X \otimes M} := Y \otimes X \otimes M \xrightarrow{c_{X,Y}^{-1} \otimes M} X \otimes Y \otimes M \xrightarrow{X \otimes \varphi_{Y,M}} X \otimes M \otimes Y$$

so that $X \otimes M \in Z_{\mathcal{C}}(\mathcal{M})$. Define right action of $Z(\mathcal{C})$ by $(M, \varphi_M) \otimes (X, c_X) = (M \otimes X, \varphi_{M \otimes X})$ where

$$\varphi_{Y,M \otimes X} := Y \otimes M \otimes X \xrightarrow{\varphi_{Y,M} \otimes X} M \otimes Y \otimes X \xrightarrow{M \otimes c_{Y,X}} M \otimes X \otimes Y$$

putting $M \otimes X \in Z_{\mathcal{C}}(\mathcal{M})$. It is easy to check that these actions are consistent in the way required of bimodule action. $\square$

Proposition 7.1.9. *$Z_{\mathcal{C}}(\mathcal{M})$ has a canonical central structure over $Z(\mathcal{C})$.*

Proof. $\varphi_{X,M} : (X \otimes M, \varphi_{X\otimes M}) \to (M \otimes X, \varphi_{M\otimes X})$ is a morphism in $Z_{\mathcal{C}}(\mathcal{M})$ as can be seen by the diagram

$$\begin{tikzcd}
YXM \arrow[rr, "Y\otimes\varphi_{X,M}"] \arrow[d, "c_{Y,X}\otimes M"'] \arrow[drr, "\varphi_{YX,M}"'] & & YMX \arrow[d, "\varphi_{Y,M}\otimes X"] \\
XYM \arrow[d, "X\otimes\varphi_{Y,M}"'] \arrow[drr, "\varphi_{XY,M}"] & & MYX \arrow[d, "M\otimes c_{Y,X}"] \\
XMY \arrow[rr, "\varphi_{X,M}\otimes Y"'] & & MXY
\end{tikzcd}$$

Triangles are Definition 7.1.3 for φ and the square is $\mathcal{C}$-naturality of φ. $\square$

Proposition 7.1.10. *For $\mathcal{C}$-bimodule category $\mathcal{M}$ we have canonical $Z(\mathcal{C})$-bimodule equivalence $\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C}, \mathcal{M}) \simeq Z_{\mathcal{C}}(\mathcal{M})$.*

Proof. For simplicity assume $\mathcal{M}$ is strict as a $\mathcal{C}$-bimodule category. Define functor $\Delta : \underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C}, \mathcal{M}) \simeq Z_{\mathcal{C}}(\mathcal{M})$ by sending $F \mapsto (F(1), f^r \circ f^{\ell^{-1}})$ where $f^\ell_X : F(X) \simeq X \otimes F(1)$ and $f^r_X : F(X) \simeq F(1) \otimes X$ are left/right module linearity isomorphisms for F. The diagram below implies $(F(1), f^r \circ f^{\ell^{-1}}) \in Z_{\mathcal{C}}(\mathcal{M})$:

$$\begin{tikzcd}
F(1)XY & & F(XY) \arrow[ll, "f^r_{XY}"'] \arrow[rr, "f^\ell_{XY}"] \arrow[dll, "f^r_{XY}"] \arrow[drr, "f^\ell_{XY}"'] & & XYF(1) \\
F(X)Y \arrow[u, "f^r_X\otimes Y"] \arrow[rr, "f^\ell_X\otimes Y"'] & & XF(1)Y & & XF(Y) \arrow[u, "X\otimes f^\ell_Y"'] \arrow[ll, "X\otimes f^r_Y"]
\end{tikzcd}$$

Left and right triangles are diagrams expressing module linearlity of F and square is bimodularity of F (Remark 1.3.12). Inverting all ℓ superscripted isomorphisms gives the diagram required for centrality of $f^r \circ f^{\ell^{-1}}$.

To complete definition of functor $\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C},\mathcal{M}) \to Z_{\mathcal{C}}(\mathcal{M})$ we must define action on natural bimodule transformations. For $\tau : F \Rightarrow G$ a morphism in the category of functors $\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C},\mathcal{M})$ note that $\tau_1 : (F(1), f^r \circ f^{\ell^{-1}}) \to (G(1), g^r \circ g^{\ell^{-1}})$ is a morphism in $Z_{\mathcal{C}}(\mathcal{M})$: indeed, diagram required of τ_1 as central morphism is given by pasting together left/right module diagrams for τ along the edge $\tau_X : F(X) \to G(X)$.

We now define quasi-inverse Γ for functor Δ. For $M \in \mathcal{M}$ denote by F_M the functor $\mathcal{C} \to \mathcal{M}$ defined by $F_M(X) := X \otimes M$. Right exactness of F_M follows from (contravariant) left exactness of $\mathrm{Hom}(\_\_, \underline{\mathrm{Hom}}(M,M))$. Since $\mathcal{M}$ is a strict $\mathcal{C}$-bimodule category F_M is strict as a left $\mathcal{C}$-module functor. For $(M,\varphi_M) \in Z_{\mathcal{C}}(\mathcal{M})$ we give F_M the structure of a right $\mathcal{C}$-module functor via

$$F_M(X) = X \otimes M \xrightarrow{\varphi_{X,M}} M \otimes X = F_M(1) \otimes X \tag{21}$$

and with this F_M is $\mathcal{C}$-bimodule. Define $\Gamma(M,\varphi_M) := F_M$ with the bimodule structure given in (21). It is now trivial to verify that $\Delta\Gamma = id$ and that $\Gamma\Delta$ is naturally equivalent to id via f^ℓ. Finally, it is easy to see that Γ is a strict $Z(\mathcal{C})$-bimodule functor. □

As a corollary we get a well known result which appears for example in [EO04].

Corollary 7.1.11. $(\mathcal{C} \boxtimes \mathcal{C}^{op})^*_{\mathcal{C}} \simeq Z(\mathcal{C})$ *canonically as monoidal categories.*

Here, as elsewhere, we have used $\mathcal{C}^*_{\mathcal{M}}$ to denote the category of $\mathcal{C}$-module endofunctors $\underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{M})$ for $\mathcal{C}$-module category $\mathcal{M}$.

7.2 The 2-categories $\mathcal{B}(\mathcal{C})$ and $Z(\mathcal{C})$-Mod

Recall that $\mathcal{B}(\mathcal{C})$ denotes the category of exact $\mathcal{C}$-bimodule categories. The main result of this section is Theorem 7.2.3 giving an equivalence $\mathcal{B}(\mathcal{C}) \simeq Z(\mathcal{C})$-Mod which is suitably monoidal. Before we give the first proposition of this subsection recall that $\mathcal{C}$ has a trivial $Z(\mathcal{C})$-module category structure given by the forgetful functor.

Proposition 7.2.1. *The 2-functor $\mathcal{B}(\mathcal{C}) \to Z(\mathcal{C})$-Mod given by $\mathcal{M} \mapsto Z_{\mathcal{C}}(\mathcal{M}) = \underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C}, \mathcal{M})$ is an equivalence with inverse given by $\mathcal{N} \mapsto \underline{Fun}_{Z(\mathcal{C})}(\mathcal{C}^{op}, \mathcal{N})$.*

Proof. In Proposition 7.1.10 we saw that $Z_{\mathcal{C}}(\mathcal{M})$ is a $Z(\mathcal{C})$-module category whenever $\mathcal{M}$ is a $\mathcal{C}$-bimodule category (here module structure is just composition of functors). The category of $Z(\mathcal{C})$-module functors $\underline{Fun}_{Z(\mathcal{C})}(\mathcal{C}^{op}, \mathcal{N})$ for $Z(\mathcal{C})$-module category $\mathcal{N}$ has the structure of a $\mathcal{C}$-bimodule category with actions

$$(F \otimes X)(Z) := F(X \otimes Z), \qquad (Y \otimes F)(Z) := F(Z \otimes Y).$$

To see that $\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C}, -)$ and $\underline{Fun}_{Z(\mathcal{C})}(\mathcal{C}^{op}, -)$ are quasi-inverses first note that

$$\underline{Fun}_{Z(\mathcal{C})}(\mathcal{C}^{op}, \underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C}, \mathcal{N})) \simeq \underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C} \boxtimes_{Z(\mathcal{C})} \mathcal{C}^{op}, \mathcal{N}) \simeq \underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(Z(\mathcal{C})^*_{\mathcal{C}}, \mathcal{N}) \tag{22}$$

as $\mathcal{C}$-bimodule categories for any bimodule category $\mathcal{N}$ where we have used equation 8 freely. Theorem 3.27 in *loc. cit.* gives a canonical equivalence $(\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} \simeq \mathcal{C}$ for any (exact) $\mathcal{C}$-module category $\mathcal{M}$. In the case that $\mathcal{M} = \mathcal{C}$ this and Corollary 7.1.11 imply $Z(\mathcal{C})^*_{\mathcal{C}} \simeq ((\mathcal{C} \boxtimes \mathcal{C}^{op})^*_{\mathcal{C}})^*_{\mathcal{C}} \simeq \mathcal{C} \boxtimes \mathcal{C}^{op}$. Thus the last category of functors in (22) is canonically equivalent to $\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C}\boxtimes\mathcal{C}^{op},\mathcal{N})\simeq\mathcal{N}$.

In the opposite direction we have, for $Z(\mathcal{C})$-module category $\mathcal{M}$,

$$\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C},\underline{Fun}_{Z(\mathcal{C})}(\mathcal{C}^{op},\mathcal{M}))\simeq\underline{Fun}_{Z(\mathcal{C})}(\mathcal{C}^{op}\boxtimes_{\mathcal{C}\boxtimes\mathcal{C}^{op}}\mathcal{C},\mathcal{M}). \tag{23}$$

Note that $\mathcal{C}^{op}\boxtimes_{\mathcal{C}\boxtimes\mathcal{C}^{op}}\mathcal{C}\simeq(\mathcal{C}\boxtimes\mathcal{C}^{op})^*_{\mathcal{C}}\simeq Z(\mathcal{C})$ (Corollary 7.1.11) and thus the last category of functors in (23) is canonically equivalent to $\underline{Fun}_{Z(\mathcal{C})}(Z(\mathcal{C}),\mathcal{M})\simeq\mathcal{M}$. $\square$

Lemma 7.2.2. *As $Z(\mathcal{C})$-bimodule categories $Z_{\mathcal{C}}(\mathcal{M}^{op})\simeq Z_{\mathcal{C}}(\mathcal{M})^{op}$.*

Proof. For $\mathcal{M}$, $\mathcal{C}$ as above we have the bimodule equivalences

$$\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C},\mathcal{M}^{op})\simeq\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{M}^{op},\mathcal{C})^{op}\simeq\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C},\mathcal{M})^{op}.$$

The first equivalence is Lemma 1.3.14 and the second uses Corollary 2.3.3. By Proposition 7.1.10 the first term is equivalent to $Z_{\mathcal{C}}(\mathcal{M}^{op})$ and the last to $Z_{\mathcal{C}}(\mathcal{M})^{op}$. $\square$

Theorem 7.2.3. *The 2-equivalence $Z_{\mathcal{C}}:\mathcal{B}(\mathcal{C})\simeq Z(\mathcal{C})$-Mod is monoidal in that $Z_{\mathcal{C}}(\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N})\simeq Z_{\mathcal{C}}(\mathcal{M})\boxtimes_{Z(\mathcal{C})}Z_{\mathcal{C}}(\mathcal{N})$ whenever $\mathcal{M},\mathcal{N}$ are $\mathcal{C}$-bimodule categories.*

Proof. We have canonical $Z(\mathcal{C})$-bimodule equivalences

$$\begin{aligned}
Z_{\mathcal{C}}(\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N}) &\simeq \underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{C},\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N})\simeq\underline{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{op}}(\mathcal{M}^{op},\mathcal{N})\\
&\simeq \underline{Fun}_{Z(\mathcal{C})}(Z_{\mathcal{C}}(\mathcal{M}^{op}),Z_{\mathcal{C}}(\mathcal{N}))\simeq\underline{Fun}_{Z(\mathcal{C})}(Z_{\mathcal{C}}(\mathcal{M})^{op},Z_{\mathcal{C}}(\mathcal{N}))\\
&\simeq \underline{Fun}_{Z(\mathcal{C})}(Z(\mathcal{C}),Z_{\mathcal{C}}(\mathcal{M})\boxtimes_{Z(\mathcal{C})}Z_{\mathcal{C}}(\mathcal{N}))\simeq Z_{\mathcal{C}}(\mathcal{M})\boxtimes_{Z(\mathcal{C})}Z_{\mathcal{C}}(\mathcal{N})
\end{aligned}$$

The first equivalence is Proposition 7.1.10, the second and fifth are Corollary 2.3.3, the third follows from the fact that the equivalence of 2-categories $Z(\mathcal{C})$-Mod $\simeq (\mathcal{C}\boxtimes\mathcal{C}^{op})^*_{\mathcal{C}}$-Mod (Corollary 7.1.11) preserves categories of 1-cells, and the fourth follows from Lemma 7.2.2. Example 7.1.6 shows that $Z_{\mathcal{C}}$ preserves units. □

Corollary 7.2.4. *Let $\mathcal{M}$ be a $\mathcal{C}$-module category for finite tensor $\mathcal{C}$. There is a canonical 2-equivalence $\mathcal{B}(\mathcal{C}) \simeq \mathcal{B}(\mathcal{C}^*_{\mathcal{M}})$ respecting monoidal structure.*

Proof. Corollary 3.35 in [EO04] says that $Z(\mathcal{C}) \simeq Z(\mathcal{C}^*_{\mathcal{M}})$. The result follows from Theorem 7.2.3. □

CHAPTER VIII

Fusion rules for Rep(G)-module categories

8.1 Burnside rings

Much in the beginning of this section is basic and can be found for example in [CR87]. Let G be a finite group. Recall that the Burnside Ring $\Omega(G)$ is defined to be the commutative ring generated by isomorphism classes of G-sets with addition and multiplication given by disjoint union and cartesian product:

$$\langle H\rangle + \langle K\rangle = G/H \cup G/K$$

$$\langle H\rangle\langle K\rangle = G/H \times G/K$$

Here $\langle H\rangle$ denotes the isomorphism class of the G-set G/H for $H < G$ and G acts diagonally over $\times$. Evidently we have

$$\langle H\rangle\langle G\rangle = \langle H\rangle, \quad \langle H\rangle\langle 1\rangle = [G:H]\langle 1\rangle$$

so $\Omega(G)$ is unital with $1 = \langle G \rangle$. It is a basic exercise to check that multiplication in $\Omega(G)$ satisfies the equation[1]

$$\langle H \rangle \langle K \rangle = \sum_{HaK \in H \backslash G/K} \langle H \cap {}^a K \rangle.$$

We are interested in a twisted variant of the Burnside ring. Here we take as basis elements $\langle H, \sigma \rangle$ where G/H is a G-set and σ is a $k^\times$-valued 2-cocycle on H. Multiplication of basic elements takes the form

$$\langle H, \mu \rangle \langle K, \sigma \rangle = \sum_{HaK \in H \backslash G/K} \langle H \cap {}^a K, \mu \sigma^a \rangle$$

where on the right μ, σ^a refer to restriction to the subgroup $H \cap {}^a K$ from $H, {}^a K$, respectively. The cocycle $\sigma^a : {}^a K \times {}^a K \to k^\times$ is defined by $\sigma^a(x, y) = \sigma(x^a, y^a)$.

Note 8.1.1. The decomposition for twisted Burnside products described above occurred in [OY01] in order to study crossed Burnside rings, and in [Ros07] in connexion with the extended Burnside ring of semisimple Rep(G)-module categories $\mathcal{M}$ having exact faithful module functor $\mathcal{M} \to \text{Rep}(G)$.

Recall that indecomposable Vec_G-module categories are parametrized by pairs (H, μ) where $H < G$ and $\mu \in H^2(H, k^\times)$. Denote module category associated to such a pair by $\mathcal{M}(H, \mu)$. Explicitly simple objects of $\mathcal{M}(H, \mu)$ form a G-set with stabilizer H and are thus in bijection with cosets in G/H. Module associativity

[1] One uses the fact that there is a bijection between the G-orbits of $(xH, yK) \in G/H \times G/K$ and double cosets $H \backslash G/K$ given by $(sH, tK) \mapsto Hs^{-1}tK$. The orbit corresponding to the coset HaK contains (H, aK) with stabilizer $H \cap {}^a K$, thus orbit $\mathcal{O}_G(H, aK)$ of (H, aK) is $G/(H \cap {}^a K)$ as G-sets giving the formula.

is given by scalars $\mu(g_1, g_2)(X)$, for $\mu \in Z^2(G, Fun(G/H, k^\times))$, associated to the natural isomorphisms $(g_1g_2) \otimes X \to g_1 \otimes (g_2 \otimes X)$ whenever $g_i \in G$ and $X \in G/H$. Module structures are classified by non-comologous cocycles so we take as module associativity constraint any representative of the cohomology class $[\mu]$. Identifying $\mu \in H^2(G, Fun(G/H, k^\times)) = H^2(G, Ind_H^G k^\times)$ with its image in $H^2(H, k^\times)$ by Shapiro's Lemma we may classify such constraints by $H^2(H, k^\times)$.

8.2 Vec_G-Mod fusion rules

The categories Vec_G and Rep(G) are Morita equivalent via Vec: $(Vec_G)^*_{Vec} \simeq \text{Rep}(G)$ (send representation (V, ρ) to the functor $Vec \to Vec$ having $F(k) = V$ with Vec_G-linearity given by ρ). Since Rep(G) is braided the category Rep(G)-Mod has monoidal structure $\boxtimes_{\text{Rep}(G)}$ (see Proposition 7.1.1). Although Vec_G is not braided the category Vec_G-Mod has monoidal structure as follows. For $\mathcal{M}, \mathcal{N} \in Vec_G$-Mod define Vec_G-module category structure on $\mathcal{M} \boxtimes \mathcal{N}$ by $g \otimes (m \boxtimes n) := (g \otimes m) \boxtimes (g \otimes n)$ for simple object $k_g := g$ in Vec_G, and linearly extend to all of Vec_G. Let $\mathcal{M} \odot \mathcal{N}$ denote $\mathcal{M} \boxtimes \mathcal{N}$ with this module category structure.

Proposition 8.2.1 (Vec_G-Mod fusion rules). *With notation as above*

$$\mathcal{M}(H, \mu) \odot \mathcal{M}(K, \sigma) \simeq \bigoplus_{HaK \in H \backslash G/K} \mathcal{M}(H \cap {}^aK, \mu\sigma^a).$$

Proof. Send $\langle H, \sigma \rangle$ to module category $\mathcal{M}(H, \sigma)$. This association is clearly well defined and respects the action of G. Applying the proof above for decomposition of basic elements in $\Omega(G)$ to simple objects in $\mathcal{M}(H, \mu) \odot \mathcal{N}(K, \sigma)$ verifies the stated decomposition on the level of objects. We must check only the module associativity constraints for the summand categories. To do this we simply evaluate associativity for a simple object in the summand category having set of objects $G/H \cap {}^aK$. We may choose representative $H \boxtimes aK$. For $g, h \in G$ we have

$$gh \otimes (H \boxtimes aK) \simeq g \otimes (h \otimes H) \boxtimes g \otimes (h \otimes aK)$$

via $\mu(g,h)(H) \boxtimes \sigma(g,h)(aK)$. Noting that $G/K \simeq G/{}^aK$ as G-sets, restricting φ : $H^2(G, Fun(G/K, k^\times)) \simeq H^2({}^aK, k^\times)$ to coset aK on the right gives $\varphi(\sigma)(k_1, k_2) = \sigma(k_1, k_2)(aK)$ for $k_1, k_2 \in {}^aK$. Thus $\varphi(\sigma)^a(k_1, k_2) = \varphi(\sigma)(k_1^a, k_2^a) \in H^2({}^aK, k^\times)$, and this we simply denote by σ^a; module associativity is $\mu \boxtimes \sigma^a$ which is idential to $\mu\sigma^a$ since each is a scalar on simple objects. $\square$

Corollary 8.2.2. *The group of invertible irreducible Vec_G-module categories is isomorphic to $H^2(G, k^\times)$.*

Proof. Without taking twisting into consideration invertible irreducible Vec_G-module categories correspond to invertible basis elements of the Burnside ring $\Omega(G)$. Suppose $\langle H \rangle \langle H' \rangle = \langle G \rangle$ in $\Omega(G)$. Then $\sum \langle H \cap {}^aH' \rangle = \langle G \rangle$ which can happen only if there is a single double coset HH' and if $H \cap {}^aH' = G$, and this occurs only if $H = H' = G$. It follows from Proposition 8.2.1 that

$$\mathcal{M}(G, \mu) \odot \mathcal{M}(G, \mu') = \mathcal{M}(G, \mu\mu')$$

Sending $\mathcal{M}(G, \mu)$ to μ gives the desired isomorphism. $\square$

8.3 Rep(G)-Mod fusion rules

In this section we use the results of the last section together with the equivalence of 2-categories Vec_G-Mod $\longrightarrow$ Rep(G)-Mod to derive fusion rules for the free $\mathbb{Z}_+$-ring generated by simple Rep(G)-module categories. The equivalence is defined by sending $\mathcal{M} \mapsto \overline{\mathcal{M}}$ where

$$\overline{\mathcal{M}} := Fun_{Vec_G}(Vec, \mathcal{M}). \tag{24}$$

Observe that $Fun_{Vec_G}(Vec, Vec)$ acts on $Fun_{Vec_G}(\mathcal{M}, \mathcal{N})$ on the right by the formula $(F \otimes S)(M) = F(M) \odot S(k)$ whenever $M \in \mathcal{M}$ and $S : Vec \longrightarrow Vec$ is a Vec_G-module functor. $F \otimes S$ is trivially a Vec_G-module functor:

$$\begin{aligned}
(F \otimes S)(g \otimes M) &\simeq (g \otimes F(M)) \odot S(k) \\
&= (g \otimes F(M)) \odot (g \otimes S(k)) \\
&= g \otimes (F(M) \odot S(k)) \\
&= g \otimes (F \otimes S)(M).
\end{aligned}$$

The isomorphism is Vec_G-linearity of F and the second line follows from the fact that simple objects of Vec_G (one dimensional vector spaces) act trivially on Vec. Let

$T : Vec \to Vec$ over Vec_G. Associativity of the action is also trivial:

$$\begin{aligned}(F \otimes ST)(M) &= F(M) \odot ST(k) \\ &= F(M) \odot S(k \otimes T(k)) \\ &= F(M) \odot (S(k) \otimes T(k)) \\ &= (F(M) \odot S(k)) \odot T(k) \\ &= (F \otimes S)(M) \odot T(k) = ((F \otimes S) \otimes T)(M)\end{aligned}$$

The second line is tensor product (composition) in $Fun_{Vec_G}(Vec, Vec)$ and the isomorphism is due to the canonical action of Vec on $\mathcal{N}$ given by internal hom:

$$\mathrm{Hom}_{\mathcal{N}}(V \otimes N, N) := \mathrm{Hom}_{Vec}(V, \mathrm{Hom}_{\mathcal{N}}(N, N)). \tag{25}$$

Proposition 8.3.1. *For $H < G$ and $\mu \in H^2(H, k^\times)$ denote by $Rep_\mu(H)$ the category of projective representations of H with Schur multiplier μ. Then $Rep_\mu(H) \simeq \overline{\mathcal{M}(H, \mu)}$ as $Rep(G)$-module categories.*

Proof. Send functor $F : Vec \to \mathcal{M}(H, \mu)$ to $F(k)$. Rep(G)-module structure on Rep$_\mu(H)$ is given by res$\otimes id$: for $V \in$ Rep(G) and $W \in$ Rep$_\mu(H)$ the action is defined by $V \otimes W := \mathrm{res}_H^G(V) \otimes W$ where $\otimes$ on the right is tensor product in Rep$_\mu(G)$. ☐

One of the main results of this section is the following theorem.

Theorem 8.3.2. *The 2-equivalence $\mathcal{M} \mapsto \overline{\mathcal{M}}$ between $(Vec_G\text{-}Mod, \odot)$ and $(Rep(G)$-*

Mod, $\boxtimes_{Rep(G)}$) *is monoidal in the sense that*

$$\overline{\mathcal{M} \odot \mathcal{N}} \simeq \overline{\mathcal{M}} \boxtimes_{Rep(G)} \overline{\mathcal{N}}$$

as Rep(G)*-module categories.*

The action of $\mathrm{Rep}(G) \simeq Fun_{Vec_G}(Vec, Vec)$ is given by composition of functors. Since the correspondence is an equivalence of 2-categories we may identify abelian categories of 1-cells:

$$Fun_{Vec_G}(\mathcal{M},\mathcal{N}) \simeq Fun_{\mathrm{Rep}(G)}(\overline{\mathcal{M}},\overline{\mathcal{N}}). \tag{26}$$

In what follows we provide a few lemmas which show that useful formulas provided earlier for monoidal 2-categories hold also over the category of Vec_G-modules.

Lemma 8.3.3. *The 2-equivalence $\mathcal{M} \mapsto \overline{\mathcal{M}}$ from Vec_G-Mod to Rep(G)-Mod when restricted to 1-cells is an equivalence of right Rep(G)-module categories.*

Proof. The equivalence of 1-cells $\zeta : Fun_{Vec_G}(\mathcal{M},\mathcal{N}) \simeq Fun_{\mathrm{Rep}(G)}(\overline{\mathcal{M}},\overline{\mathcal{N}})$ takes functor $F : \mathcal{M} \to \mathcal{N}$ over Vec_G to the functor defined by $Q \mapsto FQ$ for Rep(G)-module functor $Q : Vec \to \mathcal{M}$. We must check that this correspondence respects Rep(G) action.

Any functor $E : Vec \to Vec$ over Vec_G determines representation $E(k)$, and any representation V determines functor $E^V(k) = V$. $V \in \mathrm{Rep}(G) \simeq \overline{Vec}$ right-acts on $F \in Fun_{\mathrm{Rep}(G)}(\overline{\mathcal{M}},\overline{\mathcal{N}})$ by $(F \otimes V)(Q) = F(Q) \circ E^V$. Writing $< \zeta(F), Q >$ for the functor in $\overline{\mathcal{N}}$ determined by F, Q we have, for $W \in Vec$,

$$\begin{aligned} <\zeta(F \otimes E^V), Q>(W) &= (F \otimes E^V)(Q)(W) \\ &= FQE^V(W) \\ &= <\zeta(F) \otimes E^V, Q>(W). \end{aligned}$$

□

Lemma 8.3.4. *Let $\mathcal{M}, \mathcal{N}$ be left Vec_G-module categories. Then $\mathcal{M} \odot \mathcal{N} \simeq Fun(\mathcal{M}^{op}, \mathcal{N})$ as left Vec_G-module categories.*

Proof. Let $\mathcal{M} := \mathcal{M}(H, \mu)$ and $\mathcal{N} := \mathcal{M}(K, \sigma)$ as above. Define

$$\Phi : \mathcal{M} \odot \mathcal{N} \to Fun(\mathcal{M}^{op}, \mathcal{N}), \quad \Phi(M \odot N)(M') := \mathrm{Hom}(M', M) \otimes N. \tag{27}$$

Clearly Φ is an equivalence of abelian categories (see Lemma 2.3.2 for example) and it remains to show that it respects Vec_G-module structure. The category $Fun(\mathcal{M}^{op}, \mathcal{N})$ carries Vec_G-module structure $(g \otimes F)(M) := g \otimes F(g^{-1} \otimes M)$ for simple objects g in Vec_G. Left action on $\mathcal{M}^{op}$ is given by $X \otimes^{op} M = X \otimes M$ with inverse module associativity. We have

$$\begin{aligned} (gh \otimes F)(M) &= gh \otimes F(h^{-1}g^{-1} \otimes M) \\ &\simeq g \otimes (h \otimes F(h^{-1} \otimes (g^{-1} \otimes M))) \\ &= g \otimes (h \otimes F)(g^{-1} \otimes M) = (g \otimes (h \otimes F))(M) \end{aligned}$$

where $\simeq$ is $\sigma(g,h)\mu^{-1}(h^{-1},g^{-1})$ which is cohomologous to $\sigma(g,h)\mu(g,h)$, i.e. module associativity on functors is given by $\mu\sigma$. For simple objects M, M' in $\mathcal{M}$, $N \in \mathcal{N}$

$$\begin{aligned}(g \otimes \Phi(M \odot N))(M') &= g \otimes (\mathrm{Hom}(g^{-1} \otimes M', M) \otimes N) \\ &\simeq \mathrm{Hom}(M', g \otimes M) \otimes (g \otimes N) \\ &= \Phi(g \otimes (M \odot N))(M')\end{aligned}$$

where $\simeq$ is canonical. Φ respects Vec_G-module structure. □

Lemma 8.3.5. *$Fun_{Vec_G}(\mathcal{M}, \mathcal{N}) \simeq \overline{\mathcal{M}^{op} \odot \mathcal{N}}$ as right Rep(G)-module categories.*

Proof. We have an equivalence $\psi : Fun_{Vec_G}(\mathcal{M}, \mathcal{N}) \to \overline{\mathcal{M}^{op} \odot \mathcal{N}}$, $F \mapsto \psi F$ where $\psi F(V)(M) := F(M) \odot V$ whenever $V \in Vec, M \in \mathcal{M}$ and where we have used Lemma 8.3.4 to express $\mathcal{M}^{op} \odot \mathcal{N}$ as category of functors is an equivalence. ψ has quasi-inverse $F \mapsto F(k)$:

$$\begin{aligned}< \psi(F \otimes V), W > (M) &= (F(M) \odot V) \odot W \\ &\simeq F(M) \odot (V \otimes W) \\ &= \psi F(V \otimes W)(M) \\ &= \psi F(E^V(W))(M) = < \psi F \circ E^V, W > (M).\end{aligned}$$

□

Lemma 8.3.6. *$\overline{\mathcal{M}^{op}} \simeq \overline{\mathcal{M}}^{op}$ as Rep(G)-module categories.*

Proof. $Fun_{Vec_G}(Vec, \mathcal{M}^{op}) \simeq Fun_{Vec_G}(\mathcal{M}, Vec) \simeq Fun_{\mathrm{Rep}(G)}(\overline{\mathcal{M}}, \mathrm{Rep}(G))$ where first $\simeq$ is Lemma 8.3.4 and the second comes from the 2-equivalence. The first term is $\overline{\mathcal{M}^{op}}$ and the last is $\overline{\mathcal{M}}^{op}$. □

Proof of Theorem 8.3.2. With notation as above,

$$\begin{aligned}\overline{\mathcal{M} \odot \mathcal{N}} &\simeq Fun_{Vec_G}(\mathcal{M}^{op}, \mathcal{N}) \\ &\simeq Fun_{\mathrm{Rep}(G)}(\overline{\mathcal{M}^{op}}, \overline{\mathcal{N}}) \\ &\simeq Fun_{\mathrm{Rep}(G)}(\overline{\mathcal{M}}^{op}, \overline{\mathcal{N}}) \simeq \overline{\mathcal{M}} \boxtimes_{\mathrm{Rep}(G)} \overline{\mathcal{N}}.\end{aligned}$$

First line is Lemma 8.3.5, second is Lemma 8.3.3 and third is Lemma 8.3.6. □

Theorem 8.3.2, together with the observation in Remark 8.3.1, immediately gives a formula for $\mathrm{Rep}(G)$-module fusion rules.

Corollary 8.3.7 ($\mathrm{Rep}(G)$-Mod fusion rules). *The twisted Burnside ring $\Omega(G)$ is isomorphic to the ring $K_0(Rep(G)\text{-}Mod)$ of equivalence classes of $Rep(G)$-module categories with multiplication induced by $\boxtimes_{Rep(G)}$. That is, for irreducible $Rep(G)$-module categories $Rep_\mu(H), Rep_\sigma(K)$ we have, as $Rep(G)$-module categories*

$$Rep_\mu(H) \boxtimes_{Rep(G)} Rep_\sigma(K) \simeq \bigoplus_{HaK \in H\backslash G/K} Rep_{\mu\sigma^a}(H \cap {}^aK). \tag{28}$$

Corollary 8.3.8. *The group of invertible irreducible $Rep(G)$-module categories is isomorphic to $H^2(G, k^\times)$.*

Proof. The proof is equivalent to that of Corollary 8.2.2. □

Note 8.3.9. Corollary 8.3.8 generalizes Corollary 3.17(ii) in [ENO09] where it was given for finite abelian groups. Indeed when A is abelian $Vec_A = \mathrm{Rep}(A^*)$ for A^* group homomorphisms $\mathrm{Hom}(A, k^\times)$.

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