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Kadison -Singer algebras

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Dedication

To my parents, whose love and support sustained me throughout.

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ABSTRACT

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In this dissertation, we defined a new class of non selfadjoint operator algebras–Kadison-Singer algebras or KS-algebras for simplicity. These algebras combine triangularity, reflexivity and von Neumann algebra property into one consideration. Generally speaking, KS-algebras are reflexive, maximal triangular with respect to its "diagonal subalgebra". Many selfadjoint features are preserved in them and concepts can be borrowed directly from the theory of von Neumann algebras. In fact, a more direct connection of KS-algebras and von Neumann algebras is through the lattice of invariant projections of a KS-algebra. The lattice is reflexive and "minimally generating" in the sense that it generates the commutant of the diagonal as a von Neumann algebra.

This dissertation consists of three chapters. In chapter 1, we give some background and the definition of Kadison-Singer algebras(as well as corresponding Kadisalong with some basic properties of KS-algebras. In chapter 2, we construct Kadision-Singer factors with hyperfinite factors as their diagonals, study their commutant and describe the corresponding Kadison-Singer lattices in details. At the end, a lattice invariant is introduced to distinguish these lattices. In chapter 3, we first review the results of reflexive algebras determined by two projections, then describe the reflexive lattice generated by three free projections and show that it is a Kadison-Singer lattice and thus the corresponding algebra is a Kadison-Singer algebra. We also show that this lattice is homeomorphic to two-dimensional sphere $\mathcal{S}^2$(plus two distinct points corresponding to 0 and I). Then we introduce a notation of connectedness of projections in a lattice of projections in a finite von Neumann and show that all connected components form another lattice, called a reduced lattice. Reduced

lattices of most of our examples were computed. We end this dissertation by discussing maximal triangularity in different aspects.

Chapter 1

Background and Preliminary

In [34], Kadison and Singer initiate the study of non-self-adjoint algebras of bounded operators on Hilbert spaces. They introduce a class of algebras they call *triangular operator algebras.* An algebra $\mathcal{T}$ is triangular (relative to a factor $\mathcal{M}$) when $\mathcal{T} \cap \mathcal{T}^*$ is a maximal abelian (self-adjoint) algebra in the factor $\mathcal{M}$. When the factor is the algebra of all $n \times n$ complex matrices, this condition guarantees that there is a unitary matrix U such that the mapping $A \to UAU^*$ transforms $\mathcal{T}$ onto a subalgebra of the upper triangular matrices.

Beginning with [34], the theory of non-self-adjoint operator algebras has undergone a vigorous development parallel to, but not nearly as explosive as, that of the self-adjoint theory, the C*- and von Neumann algebra theories. The self-adjoint theory began with the 1929-30 von Neumann article [42]. In [42, 6, 7, 43, 8], F. J. Murray and J. von Neumann introduced and studied certain algebras of Hilbert space operators. Those algebras are now called von Neumann Algebras. They are strong-operator closed self-adjoint subalgebras of the algebra of all bounded linear transformations on a Hilbert space. Factors are von Neumann algebras whose centers consist of scalar multiples of the identity operator. Every von Neumann algebra is a direct sum (or direct integral) of factors. In [6] Murray and von Neumann classified factors into type I_n , I_∞, II_1 , II_∞, III factors. Since then the theory has been extensively studied, and many important progress has been made.

Over the same period, considerable effort has gone into the study of triangular operator algebras (see, for example, [28]) and [15]) and another class of non-self-adjoint operator algebras, the *reflexive algebras* (see, for example, [30], [13], [16], and [23]). Many definitive and interesting results are obtained during the course of these investigations. For the most part, these more detailed results rely on relations to compact, or even finite-rank, operators. This direction is taken in the seminal article [34], as well. In Section 3.2 of [34], a detailed and complete classification is given for an important class of (maximal) triangular algebras; but much depends on the analysis of those $\mathcal{T}$ for which (the "diagonal") $\mathcal{T} \cap \mathcal{T}^*$ is generated by one-dimensional projections. On the other hand, the emphasis of C*- and von Neumann algebra theory is on those algebras where compact operators are (almost) absent.

The parallel development of self-adjoint and non-self-adjoint operator theories has not produced the synergistic interactions we would have expected from subjects that are so closely and naturally related, and so likely to benefit from cross connections with one another. The purpose of our study is to recapture the synergy that should exist between the powerful techniques that have developed in self-adjoint-operator-algebra theory and those of the non-self-adjoint theory by conjoining the two theories. We do this by embodying those theories in a single class of algebras.

For the rest of this chapter, we will give the definition of Kadison-Singer Algebras along with some easy facts of this new class of algebras.

Suppose $\mathcal{H}$ is a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. A *triangular (operator) algebra* is a subalgebra $\mathcal{T}$ of $\mathcal{M}$ such that $\mathcal{T} \cap \mathcal{T}^* = \mathfrak{A}$, a maximal abelian selfadjoint subalgebra (masa) of $\mathcal{M}$. One of the interesting cases is when $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

Let $\mathcal{P}$ be a set of (orthogonal) projections in $\mathcal{B}(\mathcal{H})$. Define $\mathcal{A}lg(\mathcal{P}) = \{T \in \mathcal{B}(\mathcal{H}) : TP = PTP, \text{ for all } P \in \mathcal{P}\}$. Then $\mathcal{A}lg(\mathcal{P})$ is a weak-operator closed subalgebra of $\mathcal{B}(\mathcal{H})$. Similarly, for a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, define $\mathcal{L}at(\mathcal{S}) = \{P \in \mathcal{B}(\mathcal{H}) : P \text{ a projection, } TP = PTP, \text{ for all } T \in \mathcal{S}\}$. Then $\mathcal{L}at(\mathcal{S})$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$ is called a *reflexive (operator) algebra* if $\mathcal{B} = \mathcal{A}lg(\mathcal{L}at(\mathcal{B}))$. Similarly, a lattice $\mathcal{L}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a *reflexive lattice (of projections)* if $\mathcal{L} = \mathcal{L}at(\mathcal{A}lg(\mathcal{L}))$. A *nest* is a totally ordered reflexive lattice. If $\mathcal{L}$ is a nest, then $\mathcal{A}lg(\mathcal{L})$ is called a *nest al-*

gebra. Nest algebras are generalizations of (hyperreducible) "maximal triangular" algebras introduced by Kadison and Singer in [34]. Kadison and Singer also show that nest algebras are the only maximal triangular reflexive algebras (with a commutative lattice of invariant projections). Motivated by this, we give the following definition:

Definition 1.0.1. *A subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ is called a Kadison-Singer (operator) algebra (or KS-algebra) if $\mathfrak{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathfrak{A}\cap\mathfrak{A}^*$ of $\mathfrak{A}$, in the sense that if there is another reflexive subalgebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathfrak{A}\subseteq\mathfrak{B}$ and $\mathfrak{B}\cap\mathfrak{B}^* = \mathfrak{A}\cap\mathfrak{A}^*$, then $\mathfrak{A}=\mathfrak{B}$. When the diagonal of a KS-algebra is a factor, we call the KS-algebra a KS-factor or a Kadison-Singer factor. A lattice $\mathcal{L}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a Kadison-Singer lattice (or KS-lattice) if $\mathcal{L}$ is a minimal reflexive lattice that generates the von Neumann algebra $\mathcal{L}''$, or equivalently $\mathcal{L}$ is reflexive and $\mathcal{A}lg(\mathcal{L})$ is a Kadison-Singer algebra.*

Clearly nest algebras are KS-algebras. Since a nest generates an abelian von Neumann algebra, we may view nest algebras as "type I" KS-algebras and general KS-algebras as "quantized" nest algebras. The maximality condition for a KS-algebra requires that the associated lattice is "reflexive and minimal" in the sense that there is no smaller reflexive sublattice that generates the commutant of the diagonal algebra.

The following lemma is an immediate consequence of the above definition.

Lemma 1.0.1. *Suppose $\mathfrak{A}$ is a Kadison-Singer algebra in $\mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is the commutant of $\mathfrak{A}\cap\mathfrak{A}^*$ in $\mathcal{B}(\mathcal{H})$. Then $\mathcal{L}at(\mathfrak{A})\subseteq\mathcal{M}$ and generates $\mathcal{M}$ as a von Neumann algebra.*

Proof. Since $\mathfrak{A}\cap\mathfrak{A}^*$ is a von Neumann algebra and $\mathcal{L}at(\mathfrak{A}\cap\mathfrak{A}^*)\subseteq\mathcal{M}$, we have $\mathcal{L}at(\mathfrak{A})\subseteq\mathcal{M}$. Let $\mathcal{N}$ be the von Neumann algebra generated by $\mathcal{L}at(\mathfrak{A})$. Then $\mathcal{N}$ is a subalgebra of $\mathcal{M}$, which implies that $\mathcal{M}'\subseteq\mathcal{N}'$. It is clear that $\mathcal{N}'\subseteq\mathcal{A}lg(\mathcal{L}at(\mathfrak{A}))=\mathfrak{A}$ and is selfadjoint. Thus $\mathcal{N}'\subseteq\mathfrak{A}\cap\mathfrak{A}^*=\mathcal{M}'$. Now $\mathcal{N}'=\mathcal{M}'$, which implies that $\mathcal{N}=\mathcal{M}$ □

When $\mathfrak{A}$ is a KS-algebra and $\mathfrak{A}\cap\mathfrak{A}^*$ is a factor of type I, II or III, then $\mathfrak{A}$ is called a KS-factor of the same type. In the same way, we can further classify KS-factors into type II_1, II_∞, etc., similar to usual factors. A KS-algebra $\mathfrak{A}$ is said to be in a *standard form*, or a *standard* KS-algebra, if the diagonal $\mathfrak{A} \cap \mathfrak{A}^*$ of $\mathfrak{A}$ is in a standard form, i.e., $\mathfrak{A} \cap \mathfrak{A}^*$ has a cyclic and separating vector in $\mathcal{H}$. In this case, the von Neumann algebra generated by $\mathcal{L}at(\mathfrak{A})$ (or the core, see [34]) is also in a standard form.

In the next two chapters, we will give some nontrivial examples of KS-algebras, in particular, KS-factors of type II and III. The following theorem shows that all type II and type III KS-algebras are truly non selfadjoint algebras.

Theorem 1.0.1. *If $\mathfrak{A}$ is a KS-algebra of type II or type III in $\mathcal{B}(\mathcal{H})$, then $\mathfrak{A}$ is not selfadjoint.*

Proof. Assume on the contrary that $\mathfrak{A}$ is selfadjoint. From our assumption we know that $\mathfrak{A}'$ contains a 2×2 matrix subalgebra $\mathcal{M}_2$. Let E_{ij}, $i, j = 1, 2$, be a matrix unit system for $\mathcal{M}_2$. Then one can construct a reflexive lattice $\mathcal{L}$ generated by all projections in the relative commutant of $\mathcal{M}_2$ in $\mathfrak{A}'$ and two non commuting projections E_{11} and $\frac{1}{2}\sum_{i,j} E_{ij}$ in $\mathcal{M}_2$. It is easy to see that $\mathcal{L}$ generates $\mathfrak{A}'$ as a von Neumann algebra. One easily checks that $\mathcal{A}lg(\mathcal{L})$ is non selfadjoint but reflexive. Moreover its diagonal is equal to the commutant of $\mathcal{L}$, which agrees with $\mathfrak{A}$. This contradicts to the assumption that $\mathfrak{A}$ is a KS-algebra. □

Similar argument shows that any nontrivial standard KS-algebra, even in the case of type I, is not selfadjoint. Standard KS-algebras can be viewed as *maximal* upper triangular algebras with a von Neumann algebra as its diagonal.

Two Kadison-Singer algebras are said to be *isomorphic* if there is a norm preserving (algebraic) isomorphism between the two algebras. Two KS-algebras are called *unitarily equivalent* if there is a unitary operator between the underlying Hilbert spaces that induces an isomorphism between the KS-algebras.

It is easy to see that an isomorphism between two Kadison-Singer algebras induces a * isomorphism between the diagonal subalgebras.

For lattices of projections on a Hilbert space, the definition of an isomorphism is subtle. We consider a simple example where a lattice $\mathcal{L}_0$ contains two free projections of trace $\frac{1}{2}$ and $0, I$ in a type II_1 factor. As a lattice (with respect to union, intersection and ordering), it is isomorphic to the lattice generated by two rank-one projections on a two-dimensional euclidean space. We shall call such an isomorphism (which preserves only the lattice structure) an *algebraic (lattice) isomorphism.* An isomorphism between two lattices, in this paper, is an isomorphism that also induces a * isomorphism between the von Neumann algebras they generate. To avoid confusion, sometimes we call such isomorphisms *spatial isomorphisms* between two lattices of projections.

Chapter 2

Hyperfinite Kadison-Singer Factors and its Lattices

2.1 Hyperfinite KS-factor

In this section, we shall construct some hyperfinite Kadison-Singer factors. We begin with a UHF C^*-algebra obtained by taking the completion(with respect to operator norm) of $\otimes_1^\infty M_n(\mathbb{C})$(see [18]), denoted by $\mathfrak{A}_n$ (or equivlently, $\mathfrak{A}_n = \overline{\otimes_1^\infty M_n(\mathbb{C})}$), where $2 \le n \in \mathbb{N}$ is a fixed nature number. We denote by $M_n^{(k)}(\mathbb{C})$ the kth copy of $M_n(\mathbb{C})$ in $\mathfrak{A}_n$ and $E_{ij}^{(k)}$, i, $j = 1, \ldots, n$, the standard matrix unit system for $M_n^{(k)}(\mathbb{C})$, for $k = 1, 2, \ldots$. Then we may write $\mathfrak{A}_n = \overline{M_n^{(1)} \otimes M_n^{(2)} \otimes \cdots}$. Let $\mathcal{N}_m = M_n^{(1)}(\mathbb{C}) \otimes M_n^{(2)}(\mathbb{C}) \otimes \cdots \otimes M_n^{(m)}(\mathbb{C}) (\cong M_{n^m}(C))$. Then $\mathfrak{A}_n = \overline{\cup_{m=1}^\infty \mathcal{N}_m}$. Now, we construct inductively a family of projections in $\mathcal{N}_m$.

When $m = 1$, define $P_{1j} = \sum_{i=1}^j E_{ii}^{(1)}$, $j = 1, \ldots, n-1$, $P_{1n} = \frac{1}{n}\sum_{s,t=1}^n E_{st}^{(1)}$. Suppose for $k \le m-1$, $j = 1, \ldots, n$, $P_{kj} (\in \mathcal{N}_k)$ are defined. Now we define

$$P_{mj} = P_{m-1,n-1} + (I - P_{m-1,n-1}) \sum_{i=1}^{j} E_{ii}^{(m)}, \qquad j = 1, \ldots, n-1, \tag{2.1}$$

$$P_{mn} = P_{m-1,n-1} + (I - P_{m-1,n-1}) \left(\frac{1}{n} \sum_{s,t=1}^{n} E_{st}^{(m)} \right). \tag{2.2}$$

Denote by $\mathcal{L}_m$ the lattice generated by $\{P_{kj} : 1 \le k \le m, 1 \le j \le n\}$ and $\mathcal{L}_\infty = \cup_m \mathcal{L}_m$, the lattice generated by $\{P_{kj} : k \ge 1, 1 \le j \le n\}$.

Let ρ_n be a faithful state on $M_n(\mathbb{C})$. We extend ρ_n to a state on $\mathfrak{A}_n$, denoted by ρ, i.e., $\rho = \rho_n \otimes \rho_n \otimes \cdots$. Let $\mathcal{H}$ be the Hilbert space obtained by GNS construction on $(\mathfrak{A}_n, \rho)$.

It is well known (see [36]) that the weak-operator closure of $\mathfrak{A}_n$ in $\mathcal{B}(\mathcal{H})$ is a hyperfinite factor $\mathcal{R}$ (when ρ is a trace, the factor $\mathcal{R}$ is type II_1). Then $\mathcal{L}_m$ and $\mathcal{L}_\infty$ become lattices of projections in $\mathcal{R}$.

Before state the main result of this section, we first prove the following fact.

Lemma 2.1.1. *$\mathcal{L}_m$ generate $\mathcal{N}_m$, in another word, $\mathcal{L}_m'' = \mathcal{N}_m$.*

Proof. We shall prove this lemma by induction on m.

When $m = 1$, it's easy to see that $E_{ii}^{(1)} \in \mathcal{L}_1''$, then $nE_{ii}^{(1)}P_{1n}E_{jj}^{(1)} = E_{ij}^{(1)} \in \mathcal{L}_1''$. So we proved the statement when $m = 1$.

We shall assume that the statement hold for $m \leq k$, i.e. $\mathcal{L}_m'' = \mathcal{N}_m$. Because $\text{Ł}_k \subset \mathcal{L}_{k+1}$, we have $\mathcal{N}_k \subset \mathcal{L}_{k+1}''$. This implies $\sum_{i=1}^{j} E_{ii}^{(k+1)}$ ($j = 1, \ldots, n-1$) and $\sum_{s,t=1}^{n} E_{st}^{(k+1)}$ are in $\mathcal{L}_{k+1}''$. Applying the same argument in the $m = 1$ case, we have $\mathcal{L}_{k+1}'' = \mathcal{N}_{k+1}$. Hence the induction is completed. □

With the notation above, we state the main result of this section as follows.

Theorem 2.1.1. *If $\mathfrak{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ such that*

1. *$\mathcal{A}lg(\mathcal{L}_\infty) \subset \mathfrak{A}$,*

2. *$\mathfrak{A} \cap \mathfrak{A}^* = \mathcal{A}lg(\mathcal{L}_\infty) \cap \mathcal{A}lg(\mathcal{L}_\infty)^*$,*

then $\mathfrak{A} = \mathcal{A}lg(\mathcal{L}_\infty)$.

This theorem implies that $\mathcal{A}lg(\mathcal{L}_\infty)$ is a KS-algebra.

Corollary 2.1.1. *$\mathcal{A}lg(\mathcal{L}_\infty)$ is a Kadsion-Singer factor containing the hyperfinite factor $\mathcal{R}'$ as its diagonal.*

Our above defined hyperfinite KS-factor depends $n(\geq 2)$ appeared in the UHF algebra construction. We shall see in next section that, when ρ is a trace, for different n, the Kadison-Singer algebras constructed above are not unitarily equivalent.

To prove Theorem 2.1.1, we need some lemmas.

Lemma 2.1.2. *With $\mathcal{L}_1 \subset \mathcal{N}_1$ defined above and $E_{ij}^{(1)}$, $i,j = 1,2,\ldots,n$, the matrix units for $\mathcal{N}_1$, we have*

$$\mathcal{A}lg(\mathcal{L}_1) = \{T \in \mathcal{B}(\mathcal{H}) : \quad E_{ii}^{(1)}TE_{jj}^{(1)} = 0, \qquad 1 \leq j < i \leq n;$$
$$\sum_{j=1}^{n} E_{11}^{(1)}TE_{j1}^{(1)} = \sum_{j=2}^{n} E_{12}^{(1)}TE_{j1}^{(1)} = \cdots = E_{1n}^{(1)}TE_{n1}^{(1)}\}.$$

Proof. Let T be an element in $\mathcal{A}lg(\mathcal{L}_1)$. Since $P_{1j} = \sum_{i=1}^{j} E_{ii}^{(1)} \in \mathcal{L}_1 \in \mathcal{L}_1$ for $j = 1, \ldots,$ $n-1$, we know that $E_{ii}^{(1)}TE_{jj}^{(1)} = 0$, $1 \leq j < i \leq n$. From $TP_{1n}^{(1)} = P_{1n}^{(1)}TP_{1n}^{(1)}$, we have

$$nT\sum_{i,j=1}^{n} E_{ij}^{(1)} = (\sum_{i,j=1}^{n} E_{ij}^{(1)})T(\sum_{i,j=1}^{n} E_{ij}^{(1)}).$$

Multiplying the above equation by $E_{1l}^{(1)}$ on left and $E_{11}^{(1)}$ on right, we have

$$nE_{1l}^{(1)}T\sum_{i=1}^{n} E_{i1}^{(1)} = n\sum_{i=1}^{n} E_{1l}^{(1)}TE_{i1}^{(1)} = (\sum_{j=1}^{n} E_{1j}^{(1)})T(\sum_{i=1}^{n} E_{i1}^{(1)}) = \sum_{i,j=1}^{n} E_{1i}^{(1)}TE_{j1}^{(1)}.$$

The right hand side is independent of l. By letting $l = 1, \ldots, n$ and applying $E_{ii}^{(1)}TE_{jj}^{(1)} = 0$ when $1 \leq j < i \leq n$, we have that

$$\sum_{i=1}^{n} E_{11}^{(1)}TE_{i1}^{(1)} = \sum_{i=2}^{n} E_{12}^{(1)}TE_{i1}^{(1)} = \cdots = E_{1n}^{(1)}TE_{n1}^{(1)}.$$

It is easy to check that when T satisfies those identities in the lemma, T must be an element in $\mathcal{A}lg(\mathcal{L}_1)$. □

In terms of matrix representations of elements in $\mathcal{A}lg(\mathcal{L}_1)$ with respect to matrix units in $\mathbb{N}_1$, we know from then Lemma above that such an element T is upper triangular. In another word, we can write T as

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ 0 & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{nn} \end{pmatrix}.$$

Moreover, one can arbitrarily choose the strictly upper triangular part of T and use equations

$$\sum_{i=1}^{n} T_{1i} = \sum_{i=2}^{n} T_{2i} = \cdots = T_{nn},$$

to determine the diagonal entries of T so that $T \in \mathcal{A}lg(\mathcal{L}_1)$.

Lemma 2.1.3. *For any T in $\mathcal{A}lg(\mathcal{L}_1)$, there are T_1 in $\mathcal{A}lg(\mathcal{L}_1) \cap \mathcal{L}_1'$, T_2 in $\mathcal{A}lg(\mathcal{L}_\infty)$ ($\subset \mathcal{A}lg(\mathcal{L}_1)$) such that $T = T_1 + T_2$. In particular, when $E_{nn}^{(1)} T E_{nn}^{(1)} = 0$, $T = T_2 \in \mathcal{A}lg(\mathcal{L}_\infty)$.*

Proof. Suppose $T \in \mathcal{A}lg(\mathcal{L}_1)$ and let

$$T_1 = \sum_{i=1}^{n} E_{in}^{(1)} T E_{ni}^{(1)}, \qquad T_2 = T - T_1. \tag{2.3}$$

It is easy to check that $E_{ii}^{(1)} T_1 E_{jj}^{(1)} = 0$ when $i \neq j$, by Lemma 2.1.2, $T_1 \in \mathcal{A}lg(\mathcal{L}_1)$. Moreover, for all l, k,

$$E_{lk}^{(1)} T_1 = E_{lk}^{(1)} \sum_{i=1}^{n} E_{in}^{(1)} T E_{ni}^{(1)} = E_{ln}^{(1)} T E_{nk}^{(1)} = T_1 E_{lk}^{(1)}.$$

This implies that $T_1 \in \mathcal{L}_1'$ $(= \mathcal{N}_1')$.

Clearly $T_2 \in \mathcal{A}lg(\mathcal{L}_1)$. Thus $T_2 P_{1k} = P_{1k} T_2 P_{1k}$, for $k = 1, \ldots, n$. We need to show that $T_2 P_{jk} = P_{jk} T_2 P_{jk}$, for $j \geq 2$ and $k = 1, \ldots, n$. By the definition of P_{jk}, we know that $I - P_{jk} \leq E_{nn}^{(1)}$ for $j \geq 2$. Now, from $T_2 \in \mathcal{A}lg(\mathcal{L}_1)$, we have

$$E_{nn}^{(1)} T_2 = E_{nn}^{(1)} \sum_{1 \leq l \leq k \leq n} E_{ll}^{(1)} T_2 E_{kk}^{(1)} = E_{nn}^{(1)} T_2 E_{nn}^{(1)} = E_{nn}^{(1)} (T - T_1) E_{nn}^{(1)} = 0.$$

This implies that $0 = (I - P_{jk}) T_2 = (I - P_{jk}) T_2 P_{jk}$. Thus we have $T_2 \in \mathcal{A}lg(\mathcal{L}_\infty)$. □

Lemma 2.1.4. *For any $m \geq 1$, if $T \in \mathcal{A}lg(\mathcal{L}_m)$ and $(I - P_{m,n-1})T = 0$, then $T \in \mathcal{A}lg(\mathcal{L}_\infty)$.*

When $m = 1$, the proof is given above. For a general m, the argument is similar. We omit its details here. From the construction of P_{mk}'s, we know that the differences between elements in $\mathcal{A}lg(\mathcal{L}_m)$ and those in $\mathcal{A}lg(\mathcal{L}_{m+1})$ only occur within $I - P_{m,n-1}$ ($= E_{nn}^{(1)} \otimes \cdots \otimes E_{nn}^{(m)}$). Thus we have the following lemma.

Lemma 2.1.5. *If $T \in \mathcal{A}lg(\mathcal{L}_m)$, then $T \in \mathcal{A}lg(\mathcal{L}_{m+1})$ if and only if, for $j = 1, \ldots, n$, the projections $(I - P_{m,n-1}) P_{m+1,j} (I - P_{m,n-1})$ are invariant under $(I - P_{m,n-1}) T (I - P_{m,n-1})$.*

Proof. Since T is in $\mathcal{A}lg(\mathcal{L}_m)$, we have $T \in \mathcal{A}lg(\mathcal{L}_{m+1})$ if and only if $(I - P_{m+1,j}) T P_{m+1,j} = 0$, for $j = 1, \ldots, n$. By the definition, we know $P_{m+1,j} = P_{m,n-1} + (I - P_{m,n-1}) Q_j$, where

$Q_j = \sum_{i=1}^{j} E_{ii}^{(m+1)}$ for $j = 1, \ldots, n-1$, $Q_n = \frac{1}{n}\sum_{s,t=1}^{n} E_{st}^{(m+1)}$. Thus

$$\begin{aligned} 0 &= (I - P_{m+1,j})TP_{m+1,j} \\ &= (I - P_{m,n-1})(I - Q_j)T[P_{m,n-1} + (I - P_{m,n-1})Q_j] \\ &= (I - Q_j)(I - P_{m,n-1})T(I - P_{m,n-1})Q_j. \end{aligned}$$

This implies that $(I - P_{m,n-1})P_{m+1,j}(I - P_{m,n-1})$ are invariant under $(I - P_{m,n-1})T(I - P_{m,n-1})$. □

Inductively, we can easily prove the following lemma which generalizes Lemma 2.1.3.

Lemma 2.1.6. *If $T \in \mathcal{A}lg(\mathcal{L}_m)$, then there are $T_1, \ldots, T_{m+1}$ in $\mathcal{A}lg(\mathcal{L}_m)$ such that $T = T_1 + \cdots + T_{m+1}$, where $T_i \in \mathcal{N}_{i-1}' \cap \mathcal{A}lg(\mathcal{L}_\infty)$, $(I - P_{i,n-1})T_i = 0$ for $i = 1, \ldots, m$ (here we let $\mathcal{N}_0 = \mathbb{C}I$), and $T_{m+1} \in \mathcal{N}_m' \cap \mathcal{A}lg(\mathcal{L}_m)$.*

Proof. The case when $m = 1$ is proved in Lemma 2.1.3. For the case when $m = k+1$, we assume that the statement is hold for $m = 1, \ldots, k$.

If $T \in \mathcal{A}lg(\mathcal{L}_{k+1})$, because $\mathcal{L}_k \subset \mathcal{L}_{k+1}$, we have $\mathcal{A}lg(\mathcal{L}_{k+1}) \subset \mathcal{A}lg(\mathcal{L}_k)$. Thus by our assumption, $T = \sum_{i=1}^{k+1} T_i$, where $T_{k+1} \in \mathcal{N}_k' \cap \mathcal{A}lg(\mathcal{L}_{k+1})$. By Lemma 2.1.5, it is not hard to show that $T_{k+1} = \widetilde{T_{k+1}} + T_{k+2}$, where $\widetilde{T_{k+1}} \in \mathcal{N}_k' \cap \mathcal{A}lg(\mathcal{L}_\infty) T_{k+2} \in \mathcal{N}_{k+1}' \cap \mathcal{A}lg(\mathcal{L}_{k+1})$. This completes the proof of the lemma. □

For any $T \in \mathcal{A}lg(\mathcal{L}_m)$, by Lemma 2.1.6, $T = (\sum_{i=1}^{m} T_i) + T_{m+1}$. Moreover, since $T_i \in \mathcal{A}lg(\mathcal{L}_\infty)$ ($i = 1, \ldots, m$), we have $\sum_{i=1}^{m} T_i$ is in $\mathcal{A}lg(\mathcal{L}_\infty)$. So we get the following corollary.

Corollary 2.1.2. *If $T \in \mathcal{A}lg(\mathcal{L}_m)$, there exist T_1, T_2 such that $T = T_1 + T_2$, and $T_1 \in \mathcal{N}_m' T_2 \in \mathcal{A}lg(\mathcal{L}_\infty)$.*

The following lemma is the key to prove the maximality of $\mathcal{A}lg(\mathcal{L}_\infty)$.

Lemma 2.1.7. *Suppose T is an element in $\mathcal{B}(\mathcal{H})$ and $\mathfrak{A}$ is the algebra generated by T and $\mathcal{A}lg(\mathcal{L}_\infty)$. If $\mathfrak{A} \cap \mathfrak{A}^* = \mathcal{A}lg(\mathcal{L}_\infty) \cap \mathcal{A}lg(\mathcal{L}_\infty)^* = \mathcal{R}'$, then $T \in \mathcal{A}lg(\mathcal{L}_1)$.*

Proof. Suppose $T \in A$ is given. From the comments preceding Lemma 2.1.3 and Lemma 2.1.4, by taking a difference from an element in $\mathcal{A}lg(\mathcal{L}_1)$, we may assume that, with respect to matrix units $E_{ij}^{(1)}$ in $\mathcal{N}_1$, T is lower triangular, i.e., $E_{ii}^{(1)}TE_{jj}^{(1)} = 0$ for $i < j$.

Now we want to show that T is diagonal. If the strictly lower triangular entries of T are not all zero, then let i_0 be the largest integer such that $E_{i_0i_0}^{(1)}TE_{jj}^{(1)} \neq 0$ for some $j < i_0$. Among all such j, let j_0 be the largest. Then we have that $E_{ii}^{(1)}TE_{jj}^{(1)} = 0$ if $i > j$ and $i > i_0$; or $i = i_0 > j > j_0$. It is easy to check (from Lemma 2.1.3, 2.1.4) that $E_{j_0,i_0-1}^{(1)} - E_{j_0i_0}^{(1)} \in \mathcal{A}lg(\mathcal{L}_\infty)$. Then $T(E_{j_0,i_0-1}^{(1)} - E_{j_0i_0}^{(1)}) \in \mathfrak{A}$. Define $T_1 = T(E_{j_0,i_0-1}^{(1)} - E_{j_0i_0}^{(1)})$. Then

$$\begin{aligned} T_1 &= \sum_{n \geq k \geq l \geq 1} E_{kk}^{(1)}TE_{ll}^{(1)}(E_{j_0,i_0-1}^{(1)} - E_{j_0i_0}^{(1)}) \\ &= \sum_{i_0 \geq k \geq j_0} E_{kk}^{(1)}T(E_{j_0,i_0-1}^{(1)} - E_{j_0i_0}^{(1)}). \end{aligned}$$

Let

$$\begin{aligned} T_2 &= E_{i_0i_0}^{(1)}T(E_{j_0,i_0-1}^{(1)} - E_{j_0i_0}^{(1)}), \\ T_3 &= \sum_{i_0 > k \geq j_0} E_{kk}^{(1)}T(E_{j_0,i_0-1}^{(1)} - E_{j_0i_0}^{(1)}). \end{aligned}$$

Then $T_1 = T_2 + T_3$. From Lemma 2.1.3 again, $T_3 \in \mathcal{A}lg(\mathcal{L}_\infty)$. This implies that $T_2 \in \mathfrak{A}$.

Let $E_{i_0i_0}^{(1)}TE_{j_0i_0}^{(1)} = HV$ be the polar decomposition (in $\mathcal{B}(\mathcal{H})$), where H is positive and V a partial isometry. From our assumption that $E_{i_0i_0}^{(1)}TE_{j_0j_0}^{(1)} \neq 0$, we have $H \neq 0$, $E_{i_0i_0}^{(1)}H = HE_{i_0i_0}^{(1)} = H$ and $E_{i_0i_0}^{(1)}V = VE_{i_0i_0}^{(1)} = V$. Then $T_2 = HVE_{i_0,i_0-1}^{(1)} - HV$. Define

$$\begin{aligned} T_4 &= -E_{i_0-1,i_0}^{(1)}V^*E_{i_0,i_0-1}^{(1)} + E_{i_0-1,i_0}^{(1)}V^*E_{i_0i_0}^{(1)}, \\ T_5 &= E_{i_0-1,i_0}^{(1)}HE_{i_0,i_0-1}^{(1)} - E_{i_0-1,i_0}^{(1)}HE_{i_0i_0}^{(1)}. \end{aligned}$$

It is easy to check, from Lemma 2.1.3, that $T_4, T_5 \in \mathcal{A}lg(\mathcal{L}_\infty)$. Let

$$\begin{aligned} T_6 = T_2T_4 + T_5 &= (HVE_{i_0,i_0-1}^{(1)} - HV)(-E_{i_0-1,i_0}^{(1)}V^*E_{i_0,i_0-1}^{(1)} + E_{i_0-1,i_0}^{(1)}V^*E_{i_0i_0}^{(1)}) \\ &\quad + E_{i_0-1,i_0}^{(1)}HE_{i_0,i_0-1}^{(1)} - E_{i_0-1,i_0}^{(1)}HE_{i_0i_0}^{(1)} \\ &= -HE_{i_0,i_0-1}^{(1)} + HE_{i_0i_0}^{(1)} + E_{i_0-1,i_0}^{(1)}HE_{i_0,i_0-1}^{(1)} - E_{i_0-1,i_0}^{(1)}HE_{i_0i_0}^{(1)} \\ &= -HE_{i_0,i_0-1}^{(1)} + H + E_{i_0-1,i_0}^{(1)}HE_{i_0,i_0-1}^{(1)} - E_{i_0-1,i_0}^{(1)}H. \end{aligned}$$

Clearly $T_6 \in \mathfrak{A}$ and $T_6^* = T_6$. But T_6 is not upper triangular. Thus $T_6 \notin \mathcal{A}lg(\mathcal{L}_\infty)$. This implies that $\mathfrak{A} \cap \mathfrak{A}^* \neq \mathcal{A}lg(\mathcal{L}_\infty) \cap \mathcal{A}lg(\mathcal{L}_\infty)^*$. This contradiction shows that T must be diagonal. Thus we have that $T = \sum_{j=1}^n E_{jj}^{(1)} T E_{jj}^{(1)}$. Now we show that $E_{11}^{(1)} T E_{11}^{(1)} = E_{1j}^{(1)} T E_{j1}^{(1)}$ for $j = 1, \ldots, n$.

Assume that there is an i such that $E_{11}^{(1)} T E_{11}^{(1)} \neq E_{1i}^{(1)} T E_{i1}^{(1)}$. Define

$$T_7 = (E_{11}^{(1)} - E_{1i}^{(1)})T = E_{11}^{(1)} T E_{11}^{(1)} - E_{1i}^{(1)} T E_{ii}^{(1)}.$$

Because $E_{11}^{(1)} - E_{1i}^{(1)}$ is in $\mathcal{A}lg(\mathcal{L}_\infty)$, we see that $T_7 \in \mathfrak{A}$. Again write $T_8 = -E_{1i}^{(1)} T E_{i1}^{(1)} + E_{1i}^{(1)} T E_{ii}^{(1)}$. One checks (by Lemma 2.1.3) that $T_8 \in \mathcal{A}lg(\mathcal{L}_\infty)$. Then

$$0 \neq T_7 + T_8 = E_{11}^{(1)} T E_{11}^{(1)} - E_{1i}^{(1)} T E_{i1}^{(1)} \in \mathfrak{A}.$$

Set $T_7 + T_8 = V'H'$, the polar decomposition with V' a partial isometry. One easily checks that $V'^* - V'^* E_{12}^{(1)} \in \mathcal{A}lg(\mathcal{L}_\infty)$. Then

$$(V'^* - V'^* E_{12}^{(1)})(T_7 + T_8) = H' \in \mathfrak{A}.$$

Since H' is selfadjoint, $H' \in \mathfrak{A} \cap \mathfrak{A}^*$. But $E_{11}^{(1)} H' E_{11}^{(1)} \neq 0$ (with $E_{22}^{(1)} H' E_{22}^{(1)} = \cdots = E_{nn}^{(1)} H' E_{nn}^{(1)} = 0$). Thus $H' \notin \mathcal{N}_1'(\supseteq \mathcal{A}lg(\mathcal{L}_\infty) \cap \mathcal{A}lg(\mathcal{L}_\infty)^*)$. This implies that $\mathfrak{A} \cap \mathfrak{A}^* \neq \mathcal{A}lg(\mathcal{L}_\infty) \cap \mathcal{A}lg(\mathcal{L}_\infty)^*$. This contradiction shows that $E_{11}^{(1)} T E_{11}^{(1)} = \cdots = E_{1n}^{(1)} T E_{n1}^{(1)}$. Therefore $T \in \mathcal{L}_1' \subseteq \mathcal{A}lg(\mathcal{L}_1)$. □

Now we are ready to prove Theorem 2.1.1.

The proof of Theorem 2.1.1. Without the loss of generality, we may assume that $\mathfrak{A}$ is generated by T and $\mathcal{A}lg(\mathcal{L}_\infty)$. From the above lemma, we have that $T \in \mathcal{A}lg(\mathcal{L}_1)$. Suppose $T \in \mathcal{A}lg(\mathcal{L}_m)$ but $T \notin \mathcal{A}lg(\mathcal{L}_{m+1})$. From Lemma 3.6, we write $T = S + T'$ such that $S \in \mathcal{A}lg(\mathcal{L}_\infty)$ and $T' \in \mathcal{N}_m' \cap \mathcal{A}lg(\mathcal{L}_m)$. When we restrict all operators to the commutant of $\mathcal{N}_m$ and working with matrix units $E_{ij}^{(m+1)}$, similar computation as in the proof of Lemma 2.1.7 will show that $T' \in \mathcal{A}lg(\mathcal{L}_{m+1})$. This contradiction shows that $T \in \cap_{m=1}^\infty \mathcal{A}lg(\mathcal{L}_m) = \mathcal{A}lg(\mathcal{L}_\infty)$. □

In the above theorem, we did not assume the closedness of $\mathfrak{A}$ under any topology. Thus $\mathcal{A}lg(\mathcal{L}_\infty)$ has an algebraic maximality property. Next section, we will show that $\mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$ is the strong-operator closure of $\mathcal{L}_\infty$.

2.2 Kadison-Singer lattices $\mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$

It is hard to determine whether a given lattice is a Kadison-Singer lattice. The only known class is the family of nests [34]. Some finite distributive lattices (see [30] and [20]) are Kadison-Singer lattices if they have a minimal generating property (We will give more examples in next chapter). In this section, we will show that the strong-operator closure of $\mathcal{L}_\infty$ defined in Section 2.1 is a Kadison-Singer lattice. From now, we will denote ρ by τ, if ρ is a trace state on $\mathfrak{A}_n$. Let $\mathcal{R}$ be the hyperfinite factor generated by $\mathcal{L}_\infty$ (or $\mathfrak{A}_n$), then the commutant $\mathcal{R}'$ of $\mathcal{R}$ is the diagonal subalgebra of $\mathcal{A}lg(\mathcal{L}_\infty)$. Moreover, the state ρ can be extended to be a state on $\mathcal{R}$, still denote by ρ (when ρ is a trace, re-denote as τ).

Next theorem is the main result of this section.

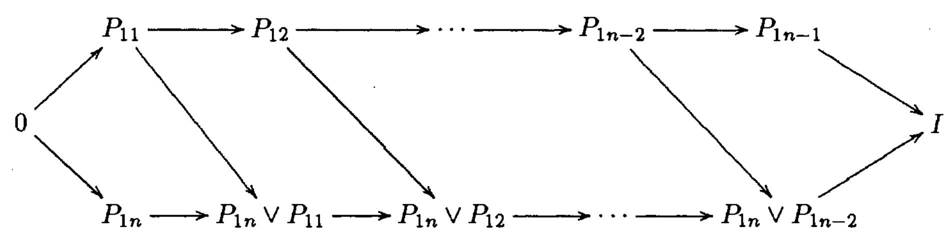
Theorem 2.2.1. *Let $\mathcal{L}^{(n)}$ be the strong-operator closure of $\mathcal{L}_\infty$, $\mathcal{H}$ the Hilbert space obtained by GNS construction on $(\mathfrak{A}_n, \rho)$. Then we have that $\mathcal{L}^{(n)} = \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$.*

To understand the lattice structure of $\mathcal{L}^{(n)}$, we first analyze the lattice properties of $\mathcal{L}_m$. From the definition of P_{1j}, $j = 1, \ldots, n$, the generators of $\mathcal{L}_1$, we know that $\mathcal{L}_1$ consists of a nest $\{0, P_{11}, \ldots, P_{1,n-1}, I\}$ in $\mathcal{N}_1$ ($\cong M_n(\mathbb{C})$) on the diagonal and a minimal projection P_{1n}. It is easy to see that $P_{1n} \wedge P_{1j} = 0$ for $1 \le j \le n-1$, and their unions give rise to another nest $\{0, P_{1n}, P_{1n} \vee P_{11}, \ldots, P_{1n} \vee P_{1,n-1} = I\}$ in $\mathcal{N}_1$. The lattice $\mathcal{L}_1$ is the union of these two nests. For any $1 \le k \le n-1$, there are two distinct projections in $\mathcal{L}_1$ such that they have the same trace $\frac{k}{n}$.With respect to matrix units, $E_{ij}^{(1)}$, i, $j = 1, \ldots, n$, in $\mathcal{N}_1(\cong M_n(\mathbb{C}))$, we can write the projections in $\mathcal{L}_1$ as following matrices. For $0 \le k \le n-1$

(Let $P_{1n} \vee P_{10} = P_{1n}$),

$$P_{1n} \vee P_{1k} = \begin{pmatrix} \overbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{k} & \\ & \underbrace{\begin{pmatrix} \frac{1}{n-k} & \cdots & \frac{1}{n-k} \\ \vdots & \ddots & \vdots \\ \frac{1}{n-k} & \cdots & \frac{1}{n-k} \end{pmatrix}}_{n-k} \end{pmatrix}.$$

The Hasse graph of $\mathcal{L}_1$ is



Easy matrix computation will give the projections in $\mathcal{L}_2$. The pattern of double nests appears in $\mathcal{L}_2$ between any two trace values $\frac{k}{n}$ and $\frac{k+1}{n}$, $0 \leq k \leq n-1$. To describe all these projections, we need more notation. For $k = 1, 2, \ldots$, define

$$E_i^{(k)} = \sum_{l=1}^{i} E_{ll}^{(k)} \quad i = 1, \ldots, n;$$

$$F_i^{(k)} = \frac{1}{i} \sum_{n \geq l, m > n-i} E_{lm}^{(k)}, \quad i = 1, \ldots, n.$$

Since $E_{ij}^{(k)}$ and $E_{i'j'}^{(k')}$ are tensorial relations for $k \neq k'$, we have $E_i^{(k)}$ and $F_i^{(k)}$ are projections in $\mathcal{N}'_{k-1} \cap \mathcal{N}_k$ ($\mathcal{N}_0 = \mathbb{C}I$). Also $E_i^{(k)} F_j^{(k)} = 0$ when $j \leq n-i$. If for ρ, we assume that

$$\rho(E_i^{(k)}) = x_i \quad (x_n = 1); \quad \rho(F_i^{(k)}) = y_i,$$

and let $c = max\{y_1, \ldots, y_n\}$ (since ρ is faithful, we have $0 < c < 1$). Notice for trace τ, $\tau(E_i^{(k)}) = \frac{i}{n}$, $\tau(F_i^{(k)}) = \frac{1}{n}$.

With the above notation, we can state our result about the structure of $\mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$ as the following lemma.

Lemma 2.2.1. *Suppose $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$ and $P \neq 0, I$. Then $P = E_i^{(1)} + F_{n-i}^{(1)}Q$, for some $i \in \{0, 1, \ldots, n-1\}$, $Q \in \mathcal{N}_1'$, and also $Q \in \mathcal{L}at(\mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}_\infty))$. When P is given in this form, $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$.*

Proof. First we show that if $P = E_i^{(1)} + F_{n-i}^{(1)}Q$ with Q described in the lemma, then $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$.

For any $T \in \mathcal{A}lg(\mathcal{L}_\infty)$, by Lemma 2.1.3, there is $T_1 \in \mathcal{A}lg(\mathcal{L}_\infty)$, $(I - P_{1,n-1})T_1 = 0$ $T_2 \in \mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}_\infty)$ such that $T = T_1 + T_2$. Since $E_i^{(1)} = P_{1i} \in \mathcal{L}_1 \subseteq \mathcal{L}_\infty$, we have $(I - E_i^{(1)})T_jE_i^{(1)} = 0$, for $j \in \{1, 2\}$. One can check directly that $F_{n-i}^{(1)}(I - E_i^{(1)}) = F_{n-i}^{(1)} = (I - E_i^{(1)})F_{n-i}^{(1)}$. Thus $F_{n-i}^{(1)}TE_i^{(1)} = 0$. Since Q commutes with $\mathcal{N}_1$, we have

$$\begin{aligned}(I-P)TP &= (I - E_i^{(1)} - F_{n-i}^{(1)}Q)T(E_i^{(1)} + F_{n-i}^{(1)}Q)\\ &= (I - E_i^{(1)})TF_{n-i}^{(1)}Q - F_{n-i}^{(1)}QTE_i^{(1)} - F_{n-i}^{(1)}QTF_{n-i}^{(1)}Q\\ &= (I - E_i^{(1)})TF_{n-i}^{(1)}Q - QF_{n-i}^{(1)}(I - E_i^{(1)})TF_{n-i}^{(1)}Q.\end{aligned}$$

The above equations hold when T is replaced by T_1 or T_2. From our assumptions that $Q \in \mathcal{L}at(\mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}_\infty))$, $E_i^{(1)}, F_{n-i}^{(1)} \in \mathcal{N}_1$ and $T_2 \in \mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}_\infty)$, we have

$$(I-P)T_2P = ((I - E_i^{(1)})F_{n-i}^{(1)} - QF_{n-i}^{(1)}(I - E_i^{(1)}))T_2Q = F_{n-1}^{(1)}(I-Q)T_2Q = 0.$$

Next we show that $(I - E_i^{(1)})T_1F_{n-i}^{(1)} = 0$, which implies that $(I-P)T_1P = 0$. Note that

$$(n-i)(I - E_i^{(1)})T_1F_{n-i}^{(1)} = (\sum_{j=i+1}^{n} E_{jj}^{(1)})T_1(\sum_{l,m=i+1}^{n} E_{lm}^{(1)}) = \sum_{j,m=i+1}^{n}\sum_{l=j}^{n} E_{jj}^{(1)}T_1E_{lm}^{(1)}.$$

By Lemma 2.1.3 and $(I - P_{n-1}^{(1)})T_1 = E_{nn}^{(1)}T_1 = 0$, we have $\sum_{l=j}^{n} E_{jj}^{(1)}T_1E_{lm}^{(1)} = 0$. So $(n-i)(I - E_i^{(1)})T_1F_{n-i}^{(1)} = 0$. Thus $(I-P)TP = 0$ which implies that $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$.

Now for any $P \in \mathcal{L}at\mathcal{A}lg(\mathcal{L}_\infty))$, $P \neq 0, I$, let i_0, $1 \leq i_0 \leq n$, be the smallest integer such that $E_{i_0i_0}^{(1)}PE_{i_0i_0}^{(1)} \neq E_{i_0i_0}^{(1)}$. Then $E_{ii}^{(1)}PE_{ii}^{(1)} = E_{ii}^{(1)}$ for $1 \leq i \leq i_0 - 1$ and $P = E_{i_0-1}^{(1)} + P_1$, where P_1 is a projection and $E_i^{(1)}P_1 = 0$ for $i \leq i_0 - 1$. First we assume that $i_0 \leq n-1$. For any $A \in \mathcal{B}(\mathcal{H})$ and $i_1 \geq i_0 + 1$, define $A_{i_1} = E_{i_0i_0}^{(1)}A(E_{i_0i_0}^{(1)} - E_{i_0i_1}^{(1)})$. Then $A_{i_1} \in \mathcal{A}lg(\mathcal{L}_\infty)$. Since $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$, we have

$$0 = (I - E_{i_0-1}^{(1)} - P_1)A_{i_1}(E_{i_0-1}^{(1)} + P_1) = (I - P_1)E_{i_0i_0}^{(1)}A(E_{i_0i_0}^{(1)} - E_{i_0i_1}^{(1)})P_1.$$

From $E^{(1)}_{i_0i_0}(I-P_1)E^{(1)}_{i_0i_0} \neq 0$, the above equation implies that $E^{(1)}_{i_0i_0}P_1 = E^{(1)}_{i_0i_1}P_1$, for all $i_1 \geq i_0$. So, multiplying by $P_1E^{(1)}_{i_0i_0} = P_1E^{(1)}_{ji_0}$ (the adjoint of the above equation) on the right hand side, we have $E^{(1)}_{i_0i_0}P_1E^{(1)}_{i_0i_0} = E^{(1)}_{i_0i_1}P_1E^{(1)}_{ji_0}$ for all $i_1, j \geq i_0$. This implies that $P_1 = F^{(1)}_{n-i_0+1}Q$, where Q is a projection in $\mathcal{N}'_1$. If $i_0 = n$, then P_1 can be written as $F^{(1)}_1 Q$ for $Q \in \mathcal{N}'_1$. From $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$, it is easy to see that $Q \in \mathcal{L}at(\mathcal{N}'_1 \cap \mathcal{A}lg(\mathcal{L}_\infty))$.

□

Lemma 2.2.2. *Suppose $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$. Then there exist $Q \in \mathcal{N}'_k \cap \mathcal{L}at(\mathcal{N}'_k \cap \mathcal{A}lg(\mathcal{L}_\infty))$ and integers a_k such that*

$$P = E^{(1)}_{a_1} + F^{(1)}_{n-a_1}E^{(2)}_{a_2} + \cdots + (\prod_{i=1}^{k-1} F^{(i)}_{n-a_i})E^{(k)}_{a_k} + (\prod_{i=1}^{k} F^{(i)}_{n-a_i})Q, \tag{2.4}$$

where $0 \leq a_i \leq n-1$. If $Q = 0$ or I, then $P \in \mathcal{L}_\infty$. Let $x_0 = 0$, we have

$$\rho(P) = x_{a_1} + y_{n-a_1}x_{a_2} + \cdots + (\prod_{i=1}^{k-1} y_{n-a_i})x_{a_k} + (\prod_{i=1}^{k} y_{n-a_i})\rho(Q).$$

Specially, if ρ is a trace (denote by τ), then

$$\tau(P) = \sum_{i=1}^{k}\frac{a_i}{n^i} + \frac{\tau(Q)}{n^k} \in \left[\sum_{i=1}^{k}\frac{a_i}{n^i},\ \sum_{i=1}^{k}\frac{a_i}{n^i} + \frac{1}{n^k}\right] \subseteq [0,1].$$

The above lemma follows easily from induction. The details are similar to the proof of Lemma 2.2.1.

Proof of Theorem 2.1.1. Recall that the Hilbert space $\mathcal{H}$ is obtained by GNS construction on $(\mathfrak{A}_n, \rho)$. We denote the unit vector in $\mathcal{H}$ corresponding to I by ξ. Thus for any $A \in \mathfrak{A}_n$, $\rho(A) = \langle T\xi, \xi\rangle$. It is routine to check that ξ is a cyclic, separating vector of $\mathcal{R}$ (the SOT-closure of $\mathfrak{A}_n$).

In order to prove the strong-operator closure of $\mathcal{L}_\infty$ is $\mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$, we need to show that for any $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$ and $\varepsilon > 0$, there is a projection P_ε in $\mathcal{L}_\infty$ such that $\|(P_\varepsilon - P)\xi\| \leq \varepsilon$. If $P \notin \mathcal{L}_\infty$, let $k \in \mathbb{N}$ be the number such that $c^{\frac{k}{2}} < \varepsilon$ ($c = max\{y_1, \ldots, y_n\}$), by Lemma 2.2.2 we have

$$P = E^{(1)}_{a_1} + F^{(1)}_{n-a_1}E^{(2)}_{a_2} + \cdots + (\prod_{i=1}^{k-1} F^{(i)}_{n-a_i})E^{(k)}_{a_k} + (\prod_{i=1}^{k} F^{(i)}_{n-a_i})Q.$$

Clearly, if let

$$P_\varepsilon = E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)}) E_{a_k}^{(k)} \in \mathcal{L}_\infty,$$

we have

$$\|(P_\varepsilon - P)\xi\|^2 = < (\prod_{i=1}^{k} F_{n-a_i}^{(i)}) Q\xi, \xi > = \rho((\prod_{i=1}^{k} F_{n-a_i}^{(i)}) Q)$$
$$= (\prod_{i=1}^{k} y_{n-a_i}) \rho(Q) < c^k,$$

the last equation hold because ρ is product state, this implies that $\|(P_\varepsilon - P)\xi\| \leq \varepsilon$. □

If ρ is a trace state, we will able to describe the trace value of the projections in $\mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$ completely.

Theorem 2.2.2. *If ρ(denote by τ) is a trace state. Let $\mathcal{L}^{(n)}$ be the strong-operator closure of $\mathcal{L}_\infty$, $\mathcal{H}$ the Hilbert space obtained by GNS construction on $(\mathfrak{A}_n, \tau)$. Then we have that $\mathcal{L}^{(n)} = \mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$. For any $r \in (0,1)$, if there are $a, l \in \mathbb{N}$ such that $r = \frac{a}{n^l}$, then there are two distinct projections in $\mathcal{L}^{(n)}$ with trace value r; otherwise there is only one projection in $\mathcal{L}^{(n)}$ with trace r.*

Proof. We only need to prove the statement about trace value. For $P \in \mathcal{L}^{(n)}$, if $\tau(P) = \sum_{i=1}^{k} \frac{a_i}{n^i}$, where $0 \leq a_i \leq n-1$ and $a_k \neq 0$. By lemma 2.2.2 , there are only two cases, either

$$P = \sum_{j=1}^{k} (\prod_{i=1}^{j-1} F_{n-a_i}^{(i)}) E_{a_j}^{(j)}, \qquad (\text{let } \prod_{i=1}^{0} F_{n-a_i}^{(i)} = I) \quad \text{or}$$
$$P = \sum_{j=1}^{k-1} (\prod_{i=1}^{j-1} F_{n-a_i}^{(i)}) E_{a_j}^{(j)} + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)}) E_{a_k - 1}^{(k)} + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)}) F_{n-a_k+1}^{(k)}$$

Note that the above two projections correspond to the case when $Q = 0$ for the decomposition $\tau(P) = \sum_{i=1}^{k} \frac{a_i}{n^i}$, or respectively $Q = I$ for $\tau(P) = \sum_{i=1}^{k-1} \frac{a_i}{n^i} + \frac{a_k - 1}{n^k} + \frac{1}{n^k}$ in Lemma 2.2.2. Thus for any $r = \frac{a}{n^l}$ for some integer $l > 0$ and any integer a such that $0 < a < n^l$, there are exactly two projections in $\mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$ with trace r.

When $r \in (0,1)$ and $r \neq \frac{a}{n^l}$ for any positive integer l and any integer a with $0 < a < n^l$, we shall show that there is a unique P in $\mathcal{L}at(\mathcal{A}lg(\mathcal{L}_\infty))$ with trace r. For the given r, there is a unique expansion $r = \sum_{k=1}^{\infty} \frac{a_k}{n^k}$, where a_k is an integer with $0 \le a_k \le n-1$, there are infinitely many non zero a_k's and infinitely many $a_k \ne n-1$. (This is because repeating $n-1$ as coefficients from certain place on will result r being $\frac{a}{n^l}$, e.g., $0.09999\cdots = 0.1$ when $n = 10$.) In fact, Lemma 2.2.2 gives the existence and uniqueness of such a projection:

$$P = E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + F_{n-a_1}^{(1)} F_{n-a_2}^{(2)} E_{a_3}^{(3)} + \cdots .$$

It is not hard to see that P is the strong-operator limit of finite sums. The finite sums

$$Q_k = \sum_{j=1}^{k} (\prod_{i=1}^{j-1} F_{n-a_i}^{(i)}) E_{a_j}^{(j)} \in \mathcal{L}_\infty, \quad k = 1, 2, \ldots,$$

$Q_1 < Q_2 < \cdots < Q_k < \cdots < P$ and $\lim_{k\to\infty} \tau(Q_k) = r = \tau(P)$. □

For the rest of this section, we will show that for $n \ne k$, $\mathcal{L}^{(n)}$ and $\mathcal{L}^{(k)}$ are not algebraically isomorphic as lattices. Thus we give infinitely many non isomorphic Kadison-Singer lattices.

Theorem 2.2.3. *For $n \ne k$, $\mathcal{L}^{(n)}$ and $\mathcal{L}^{(k)}$ are not algebraically isomorphic as lattices.*

To prove this theorem, we need several lemmas. First we give some more notations.

Definition 2.2.1. *For any subset $\mathcal{S}$ of $\mathcal{L}^{(n)}$, let*

$$\mathcal{Z}(\mathcal{S}) = \{P \in \mathcal{L}^{(n)} : \ P \wedge Q = 0, \text{ for all } Q \in \mathcal{S}\}.$$

Lemma 2.2.3. *A projection P is a minimum in $\mathcal{L}^{(n)}$ if and only if $P = (\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)} (m = 0, 1, \ldots)$. When $m = 0$, $P = E_1^{(1)}$.*

The lemma is easy to check by Lemma 2.2.2, we omit the proof here.

Lemma 2.2.4. *Suppose $Q \in \mathcal{N}_{m+1}' \cap \mathcal{L}at(\mathcal{N}_{m+1}' \cap \mathcal{A}lg(\mathcal{L}^{(n)}))$, $m \ge 0$ $0 \le a_i \le n-1$, and*

$$P = E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{m-1} F_{n-a_i}^{(i)}) E_{a_m}^{(m)} + (\prod_{i=1}^{m} F_{n-a_i}^{(i)}) E_{a_{m+1}}^{(m+1)}$$
$$+ (\prod_{i=1}^{m} F_{n-a_i}^{(i)}) F_{n-a_{m+1}}^{(m+1)} Q \in \mathcal{L}^{(n)} \prod_{i=1}^{k} F_{n-a_i}^{(i)} = 0, k < 0).$$

then $P \in \mathcal{Z}((\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)})$ if and only if $a_{m+1} = 0$.

Proof. For $a_{m+1} > 0$, let

$$P_1 = E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{m-1} F_{n-a_i}^{(i)}) E_{a_m}^{(m)} + (\prod_{i=1}^{m} F_{n-a_i}^{(i)}) E_{a_{m+1}}^{(m+1)} (\leq P),$$

we will show that $(\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)} \leq P_1$, which implies $a_{m+1} = 0$. If $m = 0$, $P_1 = E_{a_1}^{(1)}$, it is obvious that $E_1^{(1)} \leq E_{a_1}^{(1)}$. From now on, assume $m > 0$. We have

$$\begin{aligned} P_1(\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}) &= [E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{m-2} F_{n-a_i}^{(i)}) E_{a_{m-1}}^{(m-1)}](\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}) \\ &+ [(\prod_{i=1}^{m-1} F_{n-a_i}^{(i)}) E_{a_m}^{(m)} + (\prod_{i=1}^{m} F_{n-a_i}^{(i)}) E_{a_{m+1}}^{(m+1)}](\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}). \end{aligned}$$

Note $[E_{a_m}^{(m)} + F_{n-a_m}^{(m)}] F_n^{(m)} = F_n^{(m)}$, thus

$$\begin{aligned} &[(\prod_{i=1}^{m-1} F_{n-a_i}^{(i)}) E_{a_m}^{(m)} + (\prod_{i=1}^{m} F_{n-a_i}^{(i)}) E_{a_{m+1}}^{(m+1)}](\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}) \\ &= (\prod_{i=1}^{m-1} F_{n-a_i}^{(i)})(\prod_{i=1}^{m-1} F_n^{(i)})[E_{a_m}^{(m)} + F_{n-a_m}^{(m)}] F_n^{(m)} E_1^{(m+1)} \\ &= (\prod_{i=1}^{m-1} F_{n-a_i}^{(i)})(\prod_{i=1}^{m-1} F_n^{(i)}) F_n^{(m)} E_1^{(m+1)}, \end{aligned}$$

this shows that

$$\begin{aligned} P_1(\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}) &= [E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{m-2} F_{n-a_i}^{(i)}) E_{a_{m-1}}^{(m-1)} + (\prod_{i=1}^{m-1} F_{n-a_i}^{(i)})] \\ &\times (\prod_{i=1}^{m-1} F_n^{(i)}) F_n^{(m)} E_1^{(m+1)}. \end{aligned}$$

Similar computation shows $P_1(\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}) = (\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}$, which implies $(\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)} \leq P_1$.

Conversely, if $a_{m+1} = 0$,

$$P = E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{m-1} F_{n-a_i}^{(i)}) E_{a_m}^{(m)} + (\prod_{i=1}^{m} F_{n-a_i}^{(i)}) F_n^{(m+1)} Q.$$

Let $\xi \in P(\mathcal{H}) \wedge (\prod_{i=1}^{m} F_n^{(i)}) E_1^{(m+1)}(\mathcal{H})$, and $E = (I - E_{n-1}^{(1)})(I - E_{n-1}^{(2)}) \cdots (I - E_{n-1}^{(m+1)})$. We have $E\xi = E(\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}\xi = 0$. But

$$0 = PEP\xi = (\prod_{i=1}^m F_{n-a_i}^{(i)})F_n^{(m+1)}E(\prod_{i=1}^m F_{n-a_i}^{(i)})F_n^{(m+1)}Q\xi$$
$$= (\prod_{i=1}^m F_{n-a_i}^{(i)}(I - E_{n-1}^{(i)})F_{n-a_i}^{(i)})F_n^{(m+1)}(I - E_{n-1}^{m+1})F_n^{(m+1)}Q\xi$$
$$= \frac{1}{n}(\prod_{i=1}^m \frac{1}{n-a_i})(\prod_{i=1}^m F_{n-a_i}^{(i)})F_n^{(m+1)}Q\xi.$$

This shows that $\xi \in (\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}(\mathcal{H}) \wedge P_1(\mathcal{H})$, here $P_1 = P - (\prod_{i=1}^m F_{n-a_i}^{(i)})F_n^{(m+1)}Q$. Let $\widetilde{E} = (I - E_{n-1}^{(1)})(I - E_{n-1}^{(2)}) \cdots (I - E_{n-1}^{(m)})$. Then $\widetilde{E}\xi = \widetilde{E}P_1\xi = 0$ and

$$0 = (\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}\widetilde{E}(\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}\xi = (\prod_{i=1}^m F_n^{(i)}(I - E_n^{(i)})F_n^{(i)})E_1^{(m+1)}\xi$$
$$= \frac{1}{n^m}(\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}\xi = \frac{1}{n^m}\xi.$$

This shows that $\xi = 0$, thus $P \wedge (\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)} = 0$. □

Lemma 2.2.5. *For any minimum projection* $(\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)} (m = 0, 1, \ldots)$ *in* $\mathcal{L}^{(n)}$, *we have*

$$\mathcal{Z}(\mathcal{Z}(\{(\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}\})) = \{(\prod_{i=1}^m F_n^{(i)})E_k^{(m+1)} | k = 0, 1, \ldots n-1\}.$$

Proof. By the lemma above, we know

$$\mathcal{Z}(\{(\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}\})$$
$$= \{E_{a_1}^{(1)} + F_{n-a_1}^{(1)}E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{m-1} F_{n-a_i}^{(i)})E_{a_m}^{(m)} + (\prod_{i=1}^m F_{n-a_i}^{(i)})F_n^{(m+1)}Q :$$
$$0 \le a_i \le n-1, Q \in \mathcal{N}_{m+1}' \cap Lat(\mathcal{N}_{m+1}' \cap \mathcal{A}lg(\mathcal{L}^{(n)}))\}.$$

Suppose

$$E = E_{b_1}^{(1)} + F_{n-b_1}^{(1)}E_{b_2}^{(2)} + \cdots + (\prod_{i=1}^{m-1} F_{n-b_i}^{(i)})E_{b_m}^{(m)} + (\prod_{i=1}^m F_{n-b_i}^{(i)})E_{b_{m+1}}^{(m+1)}$$
$$+ (\prod_{i=1}^m F_{n-b_i}^{(i)})F_{n-b_{m+1}}^{(m+1)}Q \in \mathcal{Z}(\mathcal{Z}(\{(\prod_{i=1}^m F_n^{(i)})E_1^{(m+1)}\})),$$

because $E_{b_1}^{(1)} + F_{n-b_1}^{(1)}E_{b_2}^{(2)} + \cdots + (\prod_{i=1}^{m-1} F_{n-b_i}^{(i)})E_{b_m}^{(m)} \in \mathcal{Z}(\{(\prod_{i=1}^{m} F_n^{(i)})E_1^{(m+1)}\})$, we have $E = (\prod_{i=1}^{m} F_n^{(i)})[E_{b_{m+1}}^{(m+1)} + F_{n-b_{m+1}}^{(m+1)}Q]$. Also $E_{b_{m+1}}^{(m+1)} + F_{n-b_{m+1}}^{(m+1)}Q = E_{b_{m+1}}^{(m+1)} \vee F_n^{(m+1)}Q$ and $(\prod_{i=1}^{m} F_n^{(i)})F_n^{(m+1)}Q \le E$ implies that $Q = 0$, $E = (\prod_{i=1}^{m} F_n^{(i)})E_{b_{m+1}}^{(m+1)} (0 \le b_{m+1} \le n-1)$. Now it is not hard to show the result as in the proof of Lemma 2.2.4. □

Proof of Theorem 2.2.3 . By Lemma 2.2.5, for any minimum projection P in $\mathcal{L}^{(n)}$, we have $\#\mathcal{Z}(\mathcal{Z}(P)) = n$, this is an invariant of $\mathcal{L}^{(n)}$. Thus we have the theorem. □

2.3 The commutant of $\mathcal{A}lg(\mathcal{L}^{(n)})$

In this section, we shall prove as a subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space obtained by GNS construction on $(\mathfrak{A}_n, \rho)$, the center of $\mathcal{A}lg(\mathcal{L}^{(n)})$ is the commutant of itself. In order to state our result, we introduce the following notation.

Let

$$T^{(k)} = \sum_{i=1}^{n} E_{in}^{(k)} \quad k = 1, 2, \ldots.$$

Lemma 2.3.1. *A operator $A(\in \mathcal{B}(\mathcal{H}))$ is in $\mathcal{A}lg(\mathcal{L}^{(n)})'$ if and only if $A = a_0 I + A_1 T^{(1)}$, here $a_0 \in \mathbb{C}$, $A_1 \in \mathcal{N}_1' \cap (\mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}^{(n)}))'$.*

Proof. First note for any $B \in \mathcal{B}(\mathcal{H})$, $0 < l < n$, $k > l$, $E_{ll}^{(1)}B(E_{ll}^{(1)} - E_{lk}^{(1)}) \in \mathcal{A}lg(\mathcal{L}^{(n)})$. If $A \in \mathcal{A}lg(\mathcal{L}^{(n)})'$, we have

$$\begin{aligned} E_{ll}^{(1)}B(E_{ll}^{(1)} - E_{lk}^{(1)})[\sum_{i,j} E_{ii}^{(1)}AE_{jj}^{(1)}] &= \sum_{j=1}^{n} E_{ll}^{(1)}B(E_{ll}^{(1)} - E_{lk}^{(1)})AE_{jj}^{(1)} \\ = \sum_{i=1}^{n} E_{ii}^{(1)}AE_{ll}^{(1)}BE_{ll}^{(1)} - \sum_{i=1}^{n} E_{ii}^{(1)}AE_{ll}^{(1)}BE_{lk}^{(1)} &= [\sum_{i,j} E_{ii}^{(1)}AE_{jj}^{(1)}]E_{ll}^{(1)}B(E_{ll}^{(1)} - E_{lk}^{(1)}). \end{aligned}$$

By the above equation, it is not hard to see that when $i \ne l$, $E_{ii}^{(1)}AE_{ll}^{(1)}BE_{ll}^{(1)} = 0$, which implies $E_{ii}^{(1)}AE_{ll}^{(1)} = 0$. If $l = 1$, multiply $E_{11}^{(1)}$ on both side of the above equation, we have $E_{11}^{(1)}BE_{11}^{(1)}E_{11}^{(1)}AE_{11}^{(1)} = E_{11}^{(1)}AE_{11}^{(1)}BE_{11}^{(1)}$, so there must exists $a_0 \in \mathbb{C}$ such that $E_{11}^{(1)}AE_{11}^{(1)} = a_0 E_{11}^{(1)}$. For $k < n$, multiplying the above equation by $E_{11}^{(1)}$ on left and $E_{kk}^{(1)}$ on right, we have

$$-E_{11}^{(1)}BE_{1k}^{(1)}AE_{kk}^{(1)} = -E_{11}^{(1)}AE_{11}^{(1)}BE_{1k}^{(1)} = -a_0 E_{11}^{(1)}BE_{1k}^{(1)},$$

which implies $E_{kk}^{(1)}AE_{kk}^{(1)} = a_0 E_{kk}^{(1)}$, $(k < n)$.

Similarly for $l = 1$ and $k < n$, Multiplying the equation by $E_{11}^{(1)}$ on left and $E_{nn}^{(1)}$ on right, we have $E_{11}^{(1)}BE_{11}^{(1)}(E_{11}^{(1)} - E_{1k}^{(1)})AE_{nn}^{(1)} = 0$, thus $E_{11}^{(1)}AE_{nn}^{(1)} = E_{1k}^{(1)}AE_{nn}^{(1)}$. Also since for $k = n$,

$$E_{11}^{(1)}BE_{11}^{(1)}(E_{11}^{(1)} - E_{1n}^{(1)})AE_{nn}^{(1)} = -E_{11}^{(1)}AE_{11}^{(1)}BE_{1n}^{(1)} = -a_0 E_{11}^{(1)}BE_{1n}^{(1)},$$

we have $E_{11}^{(1)}AE_{nn}^{(1)} = E_{1n}^{(1)}AE_{nn}^{(1)} - a_0 E_{1n}^{(1)}$. This implies there exists $A_1 \in \mathcal{N}_1'$, such that $A = a_0 I + A_1 T^{(1)}$. Next we need to show that $A_1 \in (\mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}^{(n)}))'$. Without lose of generality, we may assume that $a_0 = 0$, i.e. $A = A_1 T^{(1)}$.

First it is not hard to check that for any $B \in \mathcal{A}lg(\mathcal{L}^{(n)})$ such that $E_{nn}^{(1)}BE_{nn}^{(1)} = 0$, we have $AB = BA$. Indeed,

$$AB = A_1 \sum_{i=1}^{n} E_{in}^{(1)}(I - E_{nn}^{(1)})B = 0,$$

also note $\sum_{j=1}^{n} E_{11}^{(1)}TE_{jn}^{(1)} = \sum_{j=2}^{n} E_{12}^{(1)}TE_{jn}^{(1)} = \cdots = 0$,

$$\begin{aligned} BA &= (\sum_{i \le j}^{n} E_{ii}^{(1)}BE_{jj}^{(1)})(\sum_{k=1}^{n} E_{kn}^{(1)})A_1 \\ &= \sum_{i=1}^{n}(\sum_{j=i}^{n} E_{ii}^{(1)}BE_{jn}^{(1)}) = 0. \end{aligned}$$

By Lemma 2.2.1, any operator in $\mathcal{A}lg(\mathcal{L}^{(n)})$ can be written as the sum of B_1, B_2, where $E_{nn}^{(1)}B_1E_{nn}^{(1)} = 0$, $B_2 \in \mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}^{(n)})$. So $A = A_1T^{(1)}$ is in $\mathcal{A}lg(\mathcal{L}^{(n)})'$, if and only if $A_1 \in \mathcal{N}_1' \cap (\mathcal{N}_1' \cap \mathcal{A}lg(\mathcal{L}^{(n)}))'$ □

Remark 2.3.1. *If we write $A \in \mathcal{A}lg(\mathcal{L}^{(n)})'$ as operator matrix with respect to the matrix units in $\mathcal{N}_1$, we have*

$$A = \begin{pmatrix} a_0 & & & A_1 \\ & \ddots & & \vdots \\ & & a_0 & A_1 \\ & & & a_0 + A_1 \end{pmatrix}.$$

And it is easy to see that $\mathcal{A}lg(\mathcal{L}^{(n)})' \subset \mathcal{A}lg(\mathcal{L}_1)$.

Inductively, we can easily prove the following theorem.

Theorem 2.3.1. *$A(\in \mathcal{B}(\mathcal{H}))$ is in $\mathcal{A}lg(\mathcal{L}^{(n)})'$, if and only if for any $k \geq 0$, there exist $\{a_i\}_{i=0}^{k} \subset \mathbb{C}$, and $A_{k+1} \in \mathcal{N}'_{k+1} \cap (\mathcal{N}'_{k+1} \cap \mathcal{A}lg(\mathcal{L}^{(n)}))'$, such that*

$$A = a_0 I + a_1 T^{(1)} + \cdots + a_k \prod_{i=1}^{k} T^{(i)} + (\prod_{i=1}^{k+1} T^{(i)}) A_{k+1}.$$

Specially, $A \in \mathcal{A}lg(\mathcal{L}_k)$.

Since for any $k > 0$, $\mathcal{A}lg(\mathcal{L}^{(n)})' \subset \mathcal{A}lg(\mathcal{L}_k)$, we have the following corollary.

Corollary 2.3.1. *The commutant of $\mathcal{A}lg(\mathcal{L}^{(n)})$ is its center.*

Generally speaking, for non selfadjoint algebra $\mathfrak{A}$, $\mathfrak{A}'' \neq \mathfrak{A}$. We claim that for $n \geq 3$, $\mathcal{A}lg(\mathcal{L}^{(n)})'' \neq \mathcal{A}lg(\mathcal{L}^{(n)})$. Here we only show this fact for $n = 3$; for $n > 3$, the proof is similar. We need to find a $T \in \mathcal{A}lg(\mathcal{L}^{(3)})'' \setminus \mathcal{A}lg(\mathcal{L}^{(3)})$. Let

$$T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{(with respect to the matrix units in } \mathcal{N}_1).$$

We will check that $T \in \mathcal{A}lg(\mathcal{L}^{(3)})'' \setminus \mathcal{A}lg(\mathcal{L}^{(3)})$. By Lemma 2.3.1 and the remark after it, the following equation

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} a_0 & 0 & A_1 \\ 0 & a_0 & A_1 \\ 0 & 0 & a_0 + A_1 \end{pmatrix} = \begin{pmatrix} a_1 & -a_1 & 0 \\ a_1 & -a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a_0 & 0 & A_1 \\ 0 & a_0 & A_1 \\ 0 & 0 & a_0 + A_1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

implies that $T \in \mathcal{A}lg(\mathcal{L}^{(3)})''$, but obviously T is not upper-triangular, so T is not in $\mathcal{A}lg(\mathcal{L}^{(3)})$.

Although for $n \geq 3$, $\mathcal{A}lg(\mathcal{L}^{(n)})'' \neq \mathcal{A}lg(\mathcal{L}^{(n)})$, when $n = 2$, we have $\mathcal{A}lg(\mathcal{L}^{(2)})'' = \mathcal{A}lg(\mathcal{L}^{(2)})$. For the rest of this section, we will prove this fact.

Lemma 2.3.2. *For any $k \in \mathbb{N}$, $T^{(k)}$ is nilpotent.*

Proof.

$$(T^{(k)})^2 = \sum_{i,j}^{n} E_{in}^{(k)} E_{jn}^{(k)} = T^{(k)}.$$

□

Lemma 2.3.3. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is nilpotent and denote the range projection of $Ran(T)$, $Ran(I-T)$ by P_1, P_2. Then $A \in \mathcal{B}(\mathcal{H})$ commute with T if and only if $(I-P_i)BP_i = 0 (i = 1,2)$, i.e. $A \in \mathcal{A}lg(P_1, P_2)$.*

Proof. First by $AT = TA = TAT$, we have $(I-T)AT = 0$ and $TA(I-T) = 0$, which implies $A \in \mathcal{A}lg(P_1, P_2)$. Conversely, note $P_1T = T$, $TP_2 = 0$. If $(I-P_2)AP_2 = 0$, we have $0{=}T(I-P_2)AP_2(I-T) = TA(I-T)$. Similarly, $(I-T)AT = 0$, i.e. $TA = AT$. □

Note $\prod_{i=1}^{k} T^{(i)} (\in \mathcal{A}lg(\mathcal{L}^{(n)})')$ $(k = 1, 2 \ldots)$ are nilpotents. Next we will give the range projections of these nilpotents.

Lemma 2.3.4. *The range projections of $\prod_{i=1}^{k} T^{(i)}$ and $I - \prod_{i=1}^{k} T^{(i)}$ are*

$$P_1 = Ran(\prod_{i=1}^{k} T^{(i)}) = \prod_{i=1}^{k} F_n^{(i)},$$

$$P_2 = Ran(I - \prod_{i=1}^{k} T^{(i)}) = E_{n-1}^{(1)} + F_1^{(1)} E_{n-1}^{(2)} + \cdots + (\prod_{i=1}^{k-1} F_1^{(i)}) E_{n-1}^{(k)}.$$

Proof. Since $\prod_{i=1}^{k} T^{(i)}$ and $I - \prod_{i=1}^{k} T^{(i)}$ are nilpotents, we only need to check

$$\prod_{i=1}^{k} T^{(i)} = P_1 (\prod_{i=1}^{k} T^{(i)}), \quad P_1 = (\prod_{i=1}^{k} T^{(i)}) P_1 \text{ and}$$

$$I - \prod_{i=1}^{k} T^{(i)} = P_2 (I - \prod_{i=1}^{k} T^{(i)}), \quad P_2 = (I - \prod_{i=1}^{k} T^{(i)}) P_2.$$

The first two equation is easy, here we only prove the last two.

First note $I - \prod_{i=1}^{k} T^{(i)} = P_2(I - \prod_{i=1}^{k} T^{(i)})$ if and only if $I - P_2 = (I - P_2) \prod_{i=1}^{k} T^{(i)}$.

By

$$I - P_2 = I - E_{n-1}^{(1)} - F_1^{(1)} E_{n-1}^{(2)} - \cdots - (\prod_{i=1}^{k-1} F_1^{(i)}) E_{n-1}^{(k)}$$

$$= F_1^{(1)} - F_1^{(1)} E_{n-1}^{(2)} - \cdots - (\prod_{i=1}^{k-1} F_1^{(i)}) E_{n-1}^{(k)} \quad (I - E_{n-1}^{(k)} = F_1^{(k)} = E_{nn}^{(k)})$$

$$= \ldots = \prod_{i=1}^{k} F_1^{(i)} = \prod_{i=1}^{k} E_{nn}^{(i)},$$

we have

$$(I - P_2) \prod_{i=1}^{k} T^{(i)} = (\prod_{i=1}^{k} E_{nn}^{(i)}) \prod_{i=1}^{k} (\sum_{j=1}^{n} E_{jn}^{(i)}) = \prod_{i=1}^{k} E_{nn}^{(i)} = (I - P_2)$$

The last equation hold if and only if $\prod_{i=1}^{k} T^{(i)} P_2 = 0$. By the fact

$$T^{(i)} E_{n-1}^{(i)} = (\sum_{j=1}^{n} E_{jn}^{(i)})(\sum_{l=1}^{n-1} E_{ll}^{(i)}) = 0,$$

it is easy to check $\prod_{i=1}^{k} T^{(i)} P_2 = 0$. □

Recall that for $n = 2$, $\mathcal{L}_\infty$ is generated by

$$P_{m1} = P_{m-1,1} + (I - P_{m-1,1}) E_{11}^{(m)},$$

$$P_{m2} = P_{m-1,1} + (I - P_{m-1,1}) \left(\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(m)} \right),$$

$m = 1, 2 \ldots$, and $P_{11} = E_{11}^{(1)}$, $P_{12} = \frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(1)}$. So we have

$$P_{m1} = E_{11}^{(1)} + E_{22}^{(1)} E_{11}^{(2)} + \cdots + (\prod_{i=1}^{m-1} E_{22}^{(i)}) E_{11}^{(m)},$$

$$P_{m2} = E_{11}^{(1)} + E_{22}^{(1)} E_{11}^{(2)} + \cdots + (\prod_{i=1}^{m-2} E_{22}^{(i)}) E_{11}^{(m-1)} + (\prod_{i=1}^{m-1} E_{22}^{(i)})(\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(m)}).$$

Theorem 2.3.2. $\mathcal{A}lg(\mathcal{L}^{(2)})'' = \mathcal{A}lg(\mathcal{L}^{(2)})$.

Proof. We will show the range projections of $\prod_{i=1}^{k} T^{(i)}$ and $I - \prod_{i=1}^{k} T^{(i)}$ generate $\mathcal{L}_\infty$.

By Lemma 2.3.4, when $n = 2$,

$$Ran(\prod_{i=1}^{k} T^{(i)}) = \prod_{i=1}^{k} F_n^{(i)} = \prod_{i=1}^{k} (\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(i)}),$$

$$Ran(I - \prod_{i=1}^{k} T^{(i)}) = E_{11}^{(1)} + E_{22}^{(1)} E_{11}^{(2)} + \cdots + (\prod_{i=1}^{k-1} E_{22}^{(i)}) E_{11}^{(k)}.$$

By definition $Ran(I - \prod_{i=1}^{k} T^{(i)}) = P_{k1}(k = 1, 2 \ldots)$, $Ran(T^{(1)}) = P_{12}$. Next we show that for $k = 1, 2 \ldots$,

$$Ran(\prod_{i=1}^{k+1} T^{(i)}) \vee Ran(I - \prod_{i=1}^{k} T^{(i)}) = \prod_{i=1}^{k+1} (\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(i)}) \vee P_{k,1}$$
$$= E_{11}^{(1)} + E_{22}^{(1)} E_{11}^{(2)} + \cdots + (\prod_{i=1}^{k-1} E_{22}^{(i)}) E_{11}^{(k)} + (\prod_{i=1}^{k} E_{22}^{(i)})(\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(k+1)}) = P_{k+1,2}.$$

Since $I - P_{k+1,2} = (\prod_{i=1}^{k} E_{22}^{(i)})(I - \frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(k+1)})$, it is not hard to check

$$(I - P_{k+1,2}) Ran(\prod_{i=1}^{k+1} T^{(i)}) = 0, \quad (I - P_{k+1,2}) P_{k,1} = 0.$$

The above implies $Ran(\prod_{i=1}^{k+1} T^{(i)}) \vee Ran(I - \prod_{i=1}^{k} T^{(i)}) \leq P_{k+1,2}$. To prove $Ran(\prod_{i=1}^{k+1} T^{(i)}) \vee Ran(I - \prod_{i=1}^{k} T^{(i)}) \geq P_{k+1,2}$, we only need to show for any $\xi \in (\prod_{i=1}^{k} E_{22}^{(i)})(\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(k+1)})\mathcal{H}$, $\xi = \xi_1 + \xi_2$, where $\xi_1 \in P_{k,1}\mathcal{H}$, $\xi_2 \in \prod_{i=1}^{k+1} (\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(i)})\mathcal{H}$. In fact, let

$$\xi_2 = 2^k \prod_{i=1}^{k+1} (\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(i)}) \xi = \prod_{i=1}^{k} [(\sum_{s,t=1}^{2} E_{st}^{(i)}) E_{22}^{(i)}] (\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(k+1)}) \xi$$
$$= \prod_{i=1}^{k} (E_{12}^{(i)} + E_{22}^{(i)})(\frac{1}{2} \sum_{s,t=1}^{2} E_{st}^{(k+1)}) \xi = \xi - \xi_1.$$

It is clear that $\xi_1 \in P_{k,1}\mathcal{H}$.

First note $\mathcal{A}lg(\mathcal{L}^{(2)}) \subset \mathcal{A}lg(\mathcal{L}^{(2)})''$. Nows suppose $A \in \mathcal{A}lg(\mathcal{L}^{(2)})''$, by Theorem 2.3.1 we have $A(\prod_{i=1}^{k} T^{(i)}) = (\prod_{i=1}^{k} T^{(i)})A(k = 1, 2 \ldots)$, then Lemma 2.3.3 and the above result implies $A \in \mathcal{A}lg(\mathcal{L}_\infty)$, i.e. $\mathcal{A}lg(\mathcal{L}^{(2)})'' \subset \mathcal{A}lg(\mathcal{L}^{(2)})$. This complete the proof $\square$

Chapter 3

Reflexive lattices generated by three projections and the corresponding Kadison-Singer algebra

In this chapter we will prove that the reflexive lattice generated by a double triangle (a special lattice with only three nontrivial projections) in finite von Neumann algebra is, in general, isomorphic to the two-dimensional sphere S^2 (plus two distinct points corresponding to 0 and I), and the corresponding reflexive algebra is a Kadison-Singer algebra. In particular, we show that the algebra leave three free projections invariant is a Kadison-Singer algebra. This shows that many factors are (minimally) generated by a reflexive lattice of projections which is topologically homeomorphic to S^2.

First we recall some basic facts about reflexive lattices generated by two projections.

3.1 Reflexive lattices generated by two projections

In [29] P.R.Halmos studied the reflexive lattices generated by two projections. Here we list some results without proof.

Definition 3.1.1. *Suppose P and Q are two projections in $\mathcal{B}(\mathcal{H})$. We say these two projections are in "general position" if $P \wedge Q = 0$, $P \vee Q = I$, $(I-P) \wedge Q$ and $(I-P) \vee Q = I$.*

Remark 3.1.1. *If $\mathcal{H}_1$, $\mathcal{H}_2$ are the ranges of P, Q. Then P, Q are in general position if and only if $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$, $\mathcal{H}_1^\perp \cap \mathcal{H}_2 = \{0\}$, $\mathcal{H}_1 \cap \mathcal{H}_2^\perp = \{0\}$, $\mathcal{H}_1^\perp \cap \mathcal{H}_2^\perp = \{0\}$.*

Lemma 3.1.1. *Suppose two projections P and Q are in general position, let $\mathcal{M} = \{P, Q\}''$ be the von Neumann algebra generated by P, Q. Then $P\mathcal{M}P$ is an abelian von Neumann algebra generated by PQP.*

Lemma 3.1.2. *With the notation in the above lemma, we have in $\mathcal{M}$, there is a system of 2×2 matrix units $\{E_{ij}\}_{i,j}^2$ with $P = E_{11}$, and $\mathcal{M} \cong P\mathcal{M}P \otimes M_2(\mathbb{C})$, $P \sim Q$.*

Remark 3.1.2. *By the above lemma, $\mathcal{M}$ is spatially isomorphic to $\mathfrak{A} \otimes M_2(\mathbb{C})$ for some abelian von Neumann algebra $\mathfrak{A}$ generated by a positive operator H $(0 \leq H \leq I)$. Also with respect to the canonical matrix units in $M_2(\mathbb{C})$, we can write P, Q as*

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix}.$$

Since P, Q are in general position, we have $Ker(H) = \{0\}$, $Ker(I - H) = \{0\}$. In fact, if two projections P, Q have the matrix form described above, then they are in general position if and only if $Ker(H) = \{0\}$, $Ker(I - H) = \{0\}$.

Generally, if P and Q are two projections onto closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of the Hilbert space $\mathcal{H}$. We could decompose $\mathcal{H}$ as $\mathcal{H}_1 \cap \mathcal{H}_2 \oplus \mathcal{H}_1 \cap \mathcal{H}_2^\perp \oplus \mathcal{H}_1^\perp \cap \mathcal{H}_2 \oplus \mathcal{H}_1^\perp \cap \mathcal{H}_2^\perp \oplus \widetilde{\mathcal{H}}$. It is easy to see that this decomposition is invariant under P and Q. Moreover on $\widetilde{\mathcal{H}}$, P and Q

are in general position. Thus we may write P, Q as

$$P = \begin{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$Q = \begin{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix} \end{pmatrix}.$$

By the discussion above, we have the following fact.

Lemma 3.1.3. *For any projections P, Q in $\mathcal{B}(\mathcal{H})$, $P \vee Q - P \sim Q - P \wedge Q$ in the von Neumann algebra generated by P and Q. Specially, if P, Q are in some von Neumann algebra $\mathcal{M}$, and τ is a trace state on $\mathcal{M}$, we have*

$$\tau(P \vee Q) = \tau(P) + \tau(Q) - \tau(P \wedge Q).$$

In [30] Halmos proved that lattices generated by two projections are always reflexive. For the rest of this section, we will provide another proof of this fact for the lattice generated by two projections which are in general position. And the method used here will be applied to study reflexive lattices generated by three projections in the next section.

By remark 3.1.2, we may assume that P, Q are in $M_2(\mathcal{B}(\mathcal{H}))$(acting on $\mathcal{H} \oplus \mathcal{H}$), and

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix}.$$

Lemma 3.1.4. *With above notations,* $\mathcal{A}lg(P,Q) = \{\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in M_2(\mathcal{B}(\mathcal{H})) : \sqrt{H}T_3\sqrt{I-H} - \sqrt{I-H}T_1\sqrt{H} = \sqrt{I-H}T_2\sqrt{I-H}\}$.

Proof. First note that a operator $T = \begin{pmatrix} T_1 & T_2 \\ T_4 & T_3 \end{pmatrix}$ in $M_2(\mathcal{B}(\mathcal{H}))$ has P as its invariant subspace if and only if $T_4 = 0$. Let $U = \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix}$, since

$$Q = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix} = \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix}.$$

we have $(I-Q)TQ = 0$ if and only if $(I-P)UTUP = 0$. Which implies that UTU is also upper-triangular with respect to P, by

$$\begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ \sqrt{I-H} & -\sqrt{H} \end{pmatrix} = \begin{pmatrix} * & * \\ \sqrt{I-H}T_1\sqrt{H} + \sqrt{I-H}T_2\sqrt{I-H} - \sqrt{H}T_3\sqrt{I-H} & * \end{pmatrix},$$

we have $\sqrt{I-H}T_1\sqrt{H} + \sqrt{I-H}T_2\sqrt{I-H} - \sqrt{H}T_3\sqrt{I-H} = 0$. □

The proof of next corollary is easy computation, we omit the details here.

Corollary 3.1.1. *Suppose P, Q are two projections in general position, we have* $\mathcal{L}at(\mathcal{A}lg(P,Q)) = \{0, P, Q, I\}$. *And $\mathcal{A}lg(P,Q)$ is Kadison-Singer algebra.*

Lemma 3.1.5. *Suppose* $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix}$ *are two projections in $M_2(\mathcal{B}(\mathcal{H}))$, H and $I-H$ are invertible. If the algebra $\mathfrak{A}$ generated by* $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ *and $\mathcal{A}lg(P,Q)$ satisfies $\mathfrak{A}^* \cap \mathfrak{A} = \mathcal{A}lg(P,Q)^* \cap \mathcal{A}lg(P,Q)$, then T must be in $\mathcal{A}lg(P,Q)$.*

Proof. First note

$$\mathcal{A}lg(P,Q)^* \cap \mathcal{A}lg(P,Q) = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \{H\}''\}. \tag{3.1}$$

By Lemma 3.1.4, we have

$$T_1 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} - \begin{pmatrix} -T_{12}\sqrt{\frac{I-H}{H}} & T_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{11} + T_{12}\sqrt{\frac{I-H}{H}} & 0 \\ T_{21} & T_{22} \end{pmatrix} \in \mathfrak{A}.$$

Thus

$$T_2 = T_1 \begin{pmatrix} I & -\sqrt{\frac{H}{I-H}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{11} + T_{12}\sqrt{\frac{I-H}{H}} & -(T_{11} + T_{12}\sqrt{\frac{I-H}{H}})\sqrt{\frac{H}{I-H}} \\ T_{21} & -T_{21}\sqrt{\frac{H}{I-H}} \end{pmatrix} \in \mathfrak{A},$$

which implies that $T_3 = \begin{pmatrix} 0 & 0 \\ T_{21} & -T_{21}\sqrt{\frac{H}{I-H}} \end{pmatrix}$ is in $\mathfrak{A}$. Let H_1U_1 be the polar decomposition of T_{21} in $\mathcal{B}(\mathcal{H})$. We obtain

$$T_4 = T_3 \begin{pmatrix} U_1^*\sqrt{\frac{I-H}{H}} & -U_1^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ H_1\sqrt{\frac{I-H}{H}} & -H_1 \end{pmatrix} \in \mathfrak{A}.$$

So we have

$$T_4 + \begin{pmatrix} -\sqrt{\frac{I-H}{H}}H_1\sqrt{\frac{I-H}{H}} & \sqrt{\frac{I-H}{H}}H_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{I-H}{H}}H_1\sqrt{\frac{I-H}{H}} & \sqrt{\frac{I-H}{H}}H_1 \\ H_1\sqrt{\frac{I-H}{H}} & -H_1 \end{pmatrix} \in \mathfrak{A}^* \cap \mathfrak{A},$$

by (3.1), the above is true if and only if $H_1 = 0$.

From now on, we assume that $T_{21} = 0$. Also because $\begin{pmatrix} T_{11} & \sqrt{\frac{H}{I-H}}A_{22} - A_{11}\sqrt{\frac{H}{I-H}} \\ 0 & T_{22} \end{pmatrix} \in \mathcal{A}lg(P,Q)$, without lose of generality, we may assume that $T = \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}$. So we only need to show that T must be 0. If $T \neq 0$, let

$$T_5 = T - \begin{pmatrix} -T_{12}\sqrt{\frac{I-H}{H}} & T_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{12}\sqrt{\frac{I-H}{H}} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{A},$$

and U_2H_2 be the polar decomposition of $T_{12}\sqrt{\frac{I-H}{H}}$ in $\mathcal{B}(\mathcal{H})$, we have

$$\begin{pmatrix} U_2^* & -U_2^*\sqrt{\frac{H}{I-H}} \\ 0 & 0 \end{pmatrix} T_5 = \begin{pmatrix} H_2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{A}^* \cap \mathfrak{A}.$$

This implies that $H_2 = 0$, thus $T = 0$. □

Remark 3.1.3. *Generally, if $\mathcal{H}$ is infinite dimensional Hilbert space, P and Q are in general position does not implies H and $I-H$ are invertible. But we still have $ker(H)=0$, $ker(I-H)=0$, by spectrum theorem, we know that there exist a family of projections $\{P_n\}_{n=1}^{\infty}$ in the Abelian von Neumann algebra $\{H\}''$, such that $P_n \leq P_m$ $(n \leq m)$, P_n converge to I with respect to the strong operator topology, and $H|_{P_n\mathcal{H}}$, $(I-H)|_{P_n\mathcal{H}}$ are invertible. Moreover $\begin{pmatrix} P_n & 0 \\ 0 & P_n \end{pmatrix}$ is in $\mathcal{A}lg(P,Q)$.*

Theorem 3.1.1. *In $M_2(\mathcal{B}(\mathcal{H}))$, suppose $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{pmatrix}$ are two projections in general position. If the algebra $\mathfrak{A}$ generated by $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and $\mathcal{A}lg(P,Q)$ satisfies $\mathfrak{A}^* \cap \mathfrak{A} = \mathcal{A}lg(P,Q)^* \cap \mathcal{A}lg(P,Q)$, we have $T \in \mathcal{A}lg(P,Q)$.*

Proof. Let P_n be the projections in remark 3.1.3, denote $\begin{pmatrix} P_n & 0 \\ 0 & P_n \end{pmatrix}$ by $\widetilde{P_n}$. Because $\widetilde{P_n}T\widetilde{P_n}$ converge to T with respect to SOT, and $\mathcal{A}lg(P,Q)$ is SOT-closed. To prove this theorem, we only need to show that for any n, $\widetilde{P_n}T\widetilde{P_n} \in \mathcal{A}lg(P,Q)$.

Because $\widetilde{P_n} \in \mathcal{A}lg(P,Q)$, we have $\widetilde{\mathfrak{A}} \subset \widetilde{P_n}\mathfrak{A}\widetilde{P_n}$, where $\widetilde{\mathfrak{A}}$ is the algebra generated by $\widetilde{P_n}T\widetilde{P_n}$ and $\widetilde{P_n}\mathcal{A}lg(P,Q)\widetilde{P_n}$. Furthermore, $\widetilde{\mathfrak{A}}^* \cap \widetilde{\mathfrak{A}} \subset \widetilde{P_n}\mathfrak{A}\widetilde{P_n}^* \cap \widetilde{P_n}\mathfrak{A}\widetilde{P_n} = \widetilde{P_n}(\mathfrak{A}^* \cap \mathfrak{A})\widetilde{P_n} = \widetilde{P_n}(\mathcal{A}lg(P,Q)^* \cap \mathcal{A}lg(P,Q))\widetilde{P_n} = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \{H\}'', P_nA = A\}$. By Lemma 3.1.4, it is not hard to see that restricted on $\widetilde{P_n}(\mathcal{H} \oplus \mathcal{H})$, $P\widetilde{P_n}$, $Q\widetilde{P_n}$ are in general position, and $\widetilde{P_n}\mathcal{A}lg(P,Q)\widetilde{P_n}|_{\widetilde{P_n}\mathcal{H}\oplus\mathcal{H}} = \mathcal{A}lg(P|_{\widetilde{P_n}\mathcal{H}\oplus\mathcal{H}}Q|_{\widetilde{P_n}\mathcal{H}\oplus\mathcal{H})})$. Now apply Lemma 3.1.5 to $\widetilde{P_n}T\widetilde{P_n}|_{\widetilde{P_n}\mathcal{H}\oplus\mathcal{H}}$, and $P|_{\widetilde{P_n}\mathcal{H}\oplus\mathcal{H}}$, $Q|_{\widetilde{P_n}\mathcal{H}\oplus\mathcal{H})}$ will give the result that we want. □

3.2 Reflexive lattices generated by three projections

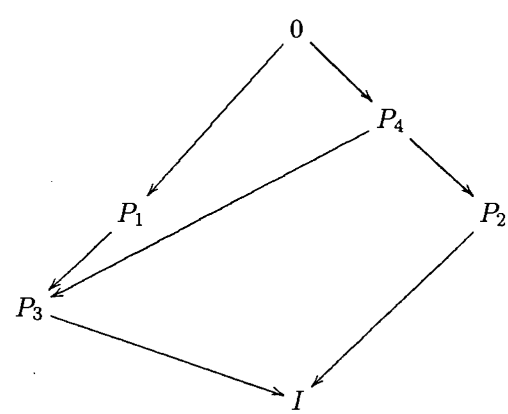
As we have seen in last section, lattices generated by two projections are always reflexive. But lattices generated by three projections are complicated. Most of factors acting on a separable Hilbert space are known to be generated by three projections or a projection and a positive operator ([24]).

Example 3.2.1. *Suppose $\mathcal{M}$ is a factor acting on $\mathcal{H}$ and write $\mathcal{M} = M_2(\mathcal{N})$, where $\mathcal{N}$ is a subfactor of $\mathcal{M}$. We assume that $\mathcal{N}$ is generated by a projection P and a positive element H with $0 \leq H \leq I$ and $ker(H) = \{0\}$, $ker(I - H) = \{0\}$. Here we view $M_2(\mathbb{C})$ as a subalgebra of $\mathcal{M}$ and $\mathcal{N}$ as the relative commutant of $M_2(\mathbb{C})$ in $\mathcal{M}$. Let E_{11}, E_{12}, E_{21} and E_{22} be the standard matrix unit system for $M_2(\mathbb{C})$. Define*

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I - H \end{pmatrix},$$

$$P_3 = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix}, P_4 = P_2 \wedge P_3$$

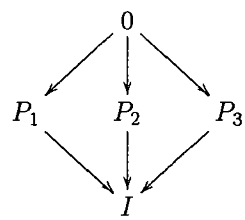
Note that $P_1 \wedge P_2 = 0$, $P_1 \vee P_2 = I$. Then $\mathcal{L} = \{0, I, P_1, P_2, P_3, P_4\}$ generated by $\{P_1, P_2, P_3\}$ is a distributive lattice, its Hasse graph is



thus by the result in [20], this lattice is reflexive. From our construction, we know that $\mathcal{M}$ is generated by $\mathcal{L}$ as a von Neumann algebra. One also easily checks that any proper sublattice of $\mathcal{L}$ does not generate $\mathcal{M}$. Thus $\mathcal{L}$ is a Kadison-Singer lattice and $Alg(\mathcal{L})$ is a Kadison-Singer factor. From this construction, we can realize most of the factors as diagonals of Kadison-Singer algebras. For example, one may choose $\mathcal{N}$ as a factor of type II_1 generated by a projection P of trace $\frac{1}{2}$ and a positive operator H such that P and H are free, and H has the same distribution (with respect to the trace on $\mathcal{N}$) as the function $\cos^2 \frac{\pi}{2}\theta$ on $[0, 1]$ (with respect to Lebesgue measure). Let τ be the trace on $\mathcal{M}$. In this case,

$\tau(P_1) = \tau(P_2) = \frac{1}{2}$, $\tau(P_3) = \frac{3}{4}$ *and* $\tau(P_4) = \frac{1}{4}$. *Then* $\mathcal{A}lg(\mathcal{L})$ *is a Kadison-Singer factor of type* II_1.

It is hard to determine when a lattice is reflexive even for a finite lattice. Finite distributive lattices are reflexive [20]. But most of the lattice are not distributive. The simplest non distributive lattice is a *double triangle*, which has the following Hasse graph:



It contains $0, I$ and three projections P_1, P_2 and P_3 so that $P_i \vee P_j = I$ and $P_i \wedge P_j = 0$ for any $i \neq j$ and $i, j = 1, 2, 3$. Any lattice that contains a double triangle sublattice is not distributive. Three free projections with trace $\frac{1}{2}$ in a factor of type II_1 (together with $0, I$) form a double triangle lattice. In the following, we first describe factors generated by free projections. For basic theory on freeness and distributions, we refer to [4].

Let G_n be the free product of $\mathbb{Z}_2$ with itself n times, for $n \geq 2$, or $= \infty$. When $n \geq 3$, G_n is an i.c.c. group so its associated group von Neumann algebra $\mathcal{L}_{G_n}$ is a factor of type II_1 acting on $l^2(G_n)$ ([35]). If $U_1, \ldots, U_n$ are canonical generators for $\mathcal{L}_{G_n}$ corresponding to the generators of G_n with $U_j^2 = I$. Then $\frac{I-U_j}{2}$, $j = 1, \ldots, n$, are projections of trace $\frac{1}{2}$. Let $\mathcal{F}_n$ be the lattice consisting of these n free projections and $0, I$.

Clearly $\mathcal{F}_n$ is a minimal lattice which generates $\mathcal{L}_{G_n}$ as a von Neumann algebra. Is $\mathcal{A}lg(\mathcal{F}_n)$, $n \geq 3$, a Kadison-Singer algebra? What is $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_n))$? When $n = 2$, we have seen in the last section that $\mathcal{F}_2$ is reflexive and thus $\mathcal{A}lg(\mathcal{F}_2)$ is maximal and hence a Kadison-Singer algebra. We shall answer the above questions for the case when $n = 3$ and show that $\mathcal{A}lg(\mathcal{F}_3)$ is a KS-algebra and $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\}$ is homeomorphic to S^2, the two-dimensional sphere.

We shall realize $\mathcal{L}_{G_3}$ as the von Neumann algebra generated by $M_2(\mathbb{C})$ and its relative commutant $\mathcal{M}$ in $\mathcal{L}_{G_3}$ and write projection generators of $\mathcal{L}_{G_3}$ in terms of 2×2 matrices (with respect to the standard matrix units in $M_2(\mathbb{C})$) given by the following equations:

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & I - H_1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix}.$$

The freeness among P_1, P_2, P_3 require that H_1, H_2 and V be free, H_1 and H_2 have the same distribution as $\cos^2 \frac{\pi}{2}\theta$ on $[0,1]$ with respect to Lebesgue measure and V a Haar unitary element. Then the subalgebra $\mathcal{M}$ of $\mathcal{L}_{G_3}$ is the von Neumann algebra generated by H_1, H_2 and V. Now $\mathcal{F}_3 = \{0, I, P_1, P_2, P_3\}$, $\mathcal{H} = l^2(G_3)$.

When $M_2(\mathbb{C})$ is a subalgebra of $\mathcal{L}_{G_3}$, we may also view $\mathcal{B}(\mathcal{H}) = M_2(\mathbb{C}) \otimes \mathcal{B}$, where $\mathcal{B}$ is the commutant of $M_2(\mathbb{C})$ in $\mathcal{B}(\mathcal{H})$. Thus all operators can be written as 2×2 matrices with entries from $\mathcal{B}$. In fact, when $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ for some Hilbert space $\mathcal{H}_1$, then $\mathcal{B} \cong \mathcal{B}(\mathcal{H}_1)$.

Since $P_1 \in \mathcal{F}_3$, any operator T belonging to $\mathcal{A}lg(\mathcal{F}_3)$ must be upper triangular. The following lemma follows from the invariance of P_2 and P_3 under T. The computation is straight forward (exactly as in the proof of Lemma 3.1.4).

Lemma 3.2.1. *With notation given above,* $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{A}lg(\mathcal{F}_3)$, *where* $T_1, T_2, T_3 \in \mathcal{B}$, *if and only if*

$$\sqrt{I-H_1}T_2\sqrt{I-H_1} = \sqrt{H_1}T_3\sqrt{I-H_1} - \sqrt{I-H_1}T_1\sqrt{H_1};$$
$$\sqrt{I-H_2}T_2V^*\sqrt{I-H_2} = \sqrt{H_2}VT_3V^*\sqrt{I-H_2} - \sqrt{I-H_2}T_1\sqrt{H_2}.$$

Using unbounded operators affiliated with $\mathcal{L}_{G_3}$, one can construct many finite rank operators in $\mathcal{A}lg(\mathcal{F}_3)$. Unbounded operators affiliated with a finite von Neumann form an algebra [35]. Any finitely many unbounded operators have a common dense domain.

Let ξ and η be vectors in the common domain of $\sqrt{H_1(I-H_1)^{-1}}$, $\sqrt{H_2(I-H_2)^{-1}}V$ and the adjoint $(\sqrt{H_2(I-H_2)^{-1}}V)^*$. We shall use $x \otimes y$ to denote the rank one operator defined by:

$$x \otimes y\colon\ z \longmapsto <z, x> y, \quad \forall z \in \mathcal{H},$$

for any $z \in \mathcal{H}$ with x and y arbitrarily given. Now let

$$T_1 = \xi \otimes (\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V)\eta$$
$$T_3 = ((\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V)^*\xi) \otimes \eta,$$

by Lemma 3.2.1, we can determine T_2:

$$T_2 = \sqrt{H_1(I-H_1)^{-1}}T_3 - T_1\sqrt{H_1(I-H_1)^{-1}}.$$

Then $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{A}lg(\mathcal{F}_3)$ is a rank two operator. This shows that $\mathcal{A}lg(\mathcal{F}_3)$ contains many finite rank (and thus compact) operators. In fact, later we will see that $\mathcal{A}lg(\mathcal{F}_3)$ contains "almost" a copy of $\mathcal{B}(\mathcal{H})$.

The following is a technical result that will be used frequently. The result might be well known. Here we provide a proof for the sake of completeness.

Lemma 3.2.2. *Suppose U is a Haar unitary element in a factor $\mathcal{M}$ of type II_1 and A is an element in (or an unbounded operator affiliated with) $\mathcal{M}$ such that A and U are free with each other. Then any nonzero scalar λ can not be a point spectrum of AU.*

Proof. First note that $0 \neq \lambda \in \mathbb{C}$ is a point spectrum of AU if and only if 1 is a point spectrum of $\frac{1}{\lambda}AU$. Since $\frac{1}{\lambda}A$, U still satisfy the conditions of the lemma, we only need to show that 1 is not a point spectrum of AU. If A is an unbounded operator affiliated with $\mathcal{M}$, and there exists $\beta \in \mathcal{H}$ such that $AU\beta = \beta$. We have $P = \overline{\{\mathcal{M}'\beta\}} \in \mathcal{M}$, and for any $\xi \in P\mathcal{H}$, $AU\xi = \xi$. Since $\mathcal{M}$ is a type II_1 factor, we could find a projection $Q \in \{A\}''$, such that AQ is bounded, and $Q \wedge UPU^* \neq 0$. Since we can find $\xi \in P\mathcal{H}$ such that $U\xi \in Q\mathcal{H}$, we obtain $AQU\xi = AU\xi = \xi$. This implies that AQ and U also satisfy the conditions of the lemma. So we only need to prove the lemma for the case when $\|A\| < \infty$, $\lambda = 1$.

Assume that A is bounded and 1 is a point spectrum of AU. For any $|\lambda| = 1$, by symmetry and freeness of A and λU, we know that 1 must be a point spectrum for $A(\lambda U)$. This implies that λ^{-1} is a point spectrum of AU. Suppose P_β is the spectral projection of AU supported at $\beta \in \mathbb{C}$. Then P_{λ_1} is equivalent to P_{λ_2} for any $|\lambda_1| = |\lambda_2| = 1$. Now we claim that if $n \geq 2$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n different complex number with norm 1, then $P_{\lambda_n} \wedge (\vee_{i=1}^{n-1} P_{\lambda_i}) = 0$. If this claim is true, let τ be the faithful normal trace state on $\mathcal{M}$, we have $\tau(P_{\lambda_1}) = \tau(P_{\lambda_2})$, because $P_{\lambda_1} \sim P_{\lambda_2}$. Thus by the equation in Lemma 3.1.3, we obtain

$$1 \geq \tau(\vee_{i=1}^{n} P_{\lambda_i}) = n\tau(P_{\lambda_1}), \quad \forall n \in \mathbb{N},$$

it is clear the above inequality is true only if $\tau(P_{\lambda_1}) = 0$, which implies that 1 is not a point spectrum of AU. Next we will prove the claim by induction on n.

It is clear that for $n = 2$, the statement is true. Assume that the claim if true for $n \leq k \in \mathbb{N}$. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are $k+1$ norm 1 complex numbers, and $\xi \in (P_{\lambda_{k+1}} \wedge (\vee_{i=1}^{k} P_{\lambda_i}))\mathcal{H}$. Then for any $\varepsilon > 0$, there are $\xi_i \in P_{\lambda_i}\mathcal{H} (i = 1, 2, \ldots k)$ such that

$$\|\xi - \xi_1 - \xi_2 - \ldots - \xi_k\| < \varepsilon,$$

we have

$$\|\lambda_{k+1}\xi - \lambda_1\xi_1 - \lambda_2\xi_2 - \ldots - \lambda_k\xi_k\| < \|A\|\varepsilon.$$

Combining the above two inequalities, we obtain

$$\|(\lambda_{k+1} - \lambda_k)\xi - (\lambda_1 - \lambda_k)\xi_1 - (\lambda_2 - \lambda_k)\xi_2 - \ldots - (\lambda_{k-1} - \lambda_k)\xi_{k-1}\| < (\|A\| + 1)\varepsilon,$$

which equivalent to

$$\|\xi - \frac{(\lambda_1 - \lambda_k)}{(\lambda_{k+1} - \lambda_k)}\xi_1 - \frac{(\lambda_2 - \lambda_k)}{(\lambda_{k+1} - \lambda_k)}\xi_2 - \ldots - \frac{(\lambda_{k-1} - \lambda_k)}{(\lambda_{k+1} - \lambda_k)}\xi_{k-1}\| < \frac{(\|A\| + 1)}{\|\lambda_{k+1} - \lambda_k\|}\varepsilon.$$

This implies that ξ is in $(P_{\lambda_{k+1}} \wedge (\vee_{i=1}^{k-1} P_{\lambda_i}))\mathcal{H}$. Since $P_{\lambda_{k+1}} \wedge (\vee_{i=1}^{k-1} P_{\lambda_i}) = 0$, we have $\xi = 0$, thus complete the proof. □

In the following, we shall describe all elements in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$. Unbounded operators will be used in our computation. All unbounded operators affiliated with the factor $\mathcal{L}_{G_3} = M_2(\mathcal{M})$ form an algebra. From function calculus, many unbounded operators we encounter in this paper can be viewed as (positive) functions defined on $(0, 1)$ with respect to Lebesgue measure. When H $(= H_1$ or $H_2)$ is identified with $\cos^2 \frac{\pi}{2}\theta$. Then $I - H$, $\sqrt{H(I-H)}$, $\sqrt{H^{-1}}$,

$\sqrt{(I-H)^{-1}}$, etc., can be viewed as trigonometric functions and they are all determined by any one of them. Lemma 3.3 also tells that many linear combinations of free (non selfadjoint) operators such as $\sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V = \sqrt{H_1(I-H_1)^{-1}}(I - \sqrt{H_1^{-1}(I-H_1)}\sqrt{H_2(I-H_2)^{-1}}V)$ are invertible (with unbounded inverses).

When S, T are unbounded operators affiliated with $\mathcal{L}_{G_3}$ or a finite von Neumann algebra and $X \in \mathcal{B}(\mathcal{H})$, then SXT is an unbounded operator that can be viewed as the weak-operator limit of bounded operators of the form $SE_\epsilon XF_\epsilon T$ for projections E_ϵ and F_ϵ in $\mathcal{L}_{G_3}$ (or the finite von Neumann algebra) so that SE_ϵ and $F_\epsilon T$ are bounded and E_ϵ, F_ϵ have strong operator limit I (as $\epsilon \to 0$). Thus, for any operator X in a weak-operator dense subalgebra $\cup_\epsilon E_\epsilon \mathcal{B}(\mathcal{H}) F_\epsilon$ of $\mathcal{B}(\mathcal{H})$, the operator SXT is a bounded operator.

Using unbounded operators, we may restate the Lemma 3.2.1 in the following form.

Lemma 3.2.3. *With $H_1, H_2, V \in \mathcal{M} \subset \mathcal{L}_{G_3}$ given above, let $S = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V$ be an unbounded operator affiliated with $\mathcal{M}$. If $T \in \mathcal{A}lg(\mathcal{F}_3)$, then there is an $A \in \mathcal{B}$ such that*

$$T = \begin{pmatrix} A & \sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}} \\ 0 & S^{-1}AS \end{pmatrix}.$$

Conversely, if $A \in \mathcal{B}$ such that $S^{-1}AS$ and $\sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}}$ are bounded operators, then the above T belongs to $\mathcal{A}lg(\mathcal{F}_3)$.

The above lemma shows that $\mathcal{A}lg(\mathcal{F}_3)$ is quite large, in particular, $\mathcal{A}lg(\mathcal{F}_3) \cap \mathcal{L}_{G_3}$ is infinite dimensional. The following result follows easily from the above lemma and shows that all nontrivial projections in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ have trace $\frac{1}{2}$.

Corollary 3.2.1. *For any $Q \in \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I, P_1\}$, we have that $Q \wedge P_1 = 0$, $Q \vee P_1 = I$, and $\tau(Q) = \frac{1}{2}$.*

Proof. For any $Q \in \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$, $Q \wedge P_1 \in \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$. Thus $Q \wedge P_1$ is invariant under all T given in Lemma 3.4 and thus the A in $(1,1)$ entry of T. This implies that $Q \wedge P_1 = P_1$ or 0. Similarly we can show that $Q \vee P_1 = I$ or P_1. For $Q \in \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I, P_1\}$, by the equation in Lemma 3.1.3, we have $\tau(Q) = \tau(Q \vee P_1) - \tau(P_1) + \tau(Q \wedge P_1) = 1 - \frac{1}{2} + 0 = \frac{1}{2}$. $\square$

This corollary actually shows that for any distinct projections Q_1, Q_2 in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\}$, $Q_1 \wedge Q_2 = 0$, and $Q_1 \vee Q_2 = I$.

For any $X \in \mathcal{B}(\mathcal{H})$ or an unbounded operator X affiliated with a (finite) von Neumann algebra, we shall use $\text{supp}(X)$ to denote the support of X, i.e., the range projection of X^*X. When $\text{supp}(X) = \text{supp}(X^*) = I$, X has an (unbounded) inverse.

Theorem 3.2.1. *For any projection Q in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I, P_1\}$, there are K and U in $\mathcal{M}$, such that*

$$Q = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & U^*(I-K)U \end{pmatrix},$$

where $\sqrt{K(I-K)^{-1}}$ (or K) and U are determined by the polar decomposition of $(1+a)\sqrt{H_1(I-H_1)^{-1}} - a\sqrt{H_2(I-H_2)^{-1}}V = aS + \sqrt{H_1(I-H_1)^{-1}} = \sqrt{K(I-K)^{-1}}U$ for some $a \in \mathbb{C}$. Moreover for any given a in $\mathbb{C}$, the polar decomposition determines U and K uniquely which give rise to a projection Q (in the above form) in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$.

Proof. Suppose Q is given in the theorem. From Corollary 3.2.1, we know that $\text{supp}(I - K) = I$. From Lemma 3.2.3, for any $A \in \mathcal{B}$ (the commutant of $M_2(\mathbb{C})$ in $\mathcal{B}(l^2(G_3))$, $\mathcal{B} \cong \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$) such that $S^{-1}AS$ and $\sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}}$ are bounded, then

$$T = \begin{pmatrix} A & \sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}} \\ 0 & S^{-1}AS \end{pmatrix} \in \mathcal{A}lg(\mathcal{F}_3),$$

here $S = \sqrt{H_1(I-H_1)^{-1}} - \sqrt{H_2(I-H_2)^{-1}}V$. Thus $(I-Q)TQ = 0$. This implies that

$$\begin{aligned}(I-K)A\sqrt{K(I-K)}U &+ (I-K)\left[\sqrt{H_1(I-H_1)^{-1}}S^{-1}AS\right.\\ &\left.- A\sqrt{H_1(I-H_1)^{-1}}\right]U^*(I-K)U - \sqrt{K(I-K)}US^{-1}ASU^*(I-K)U = 0,\end{aligned}$$

Since $\text{supp}(I-K) = I$, $I - K$ is invertible. We have

$$\begin{aligned}&\sqrt{I-K}A\left[\sqrt{K} - \sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}\right]\\ &= \left[\sqrt{K}U - \sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}\right]S^{-1}ASU^*\sqrt{I-K}.\end{aligned}$$

This gives us

$$A[\sqrt{K} - \sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}](SU^*\sqrt{I-K})^{-1}$$
$$= \sqrt{(I-K)^{-1}}[\sqrt{K}U - \sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}]S^{-1}A.$$

The above equation holds for all A in a weak-operator dense subalgebra of $\mathcal{B}$ ($\cong \mathcal{B}(\mathcal{H}_1)$). Thus there is an $a \in \mathbb{C}$ such that

$$a\sqrt{I-K} = [\sqrt{K}U - \sqrt{I-K}\sqrt{H_1(I-H_1)^{-1}}]S^{-1},$$
$$aSU^*\sqrt{I-K} = \sqrt{K} - \sqrt{H_1(I-H_1)^{-1}}U^*\sqrt{I-K}.$$

This implies that

$$\sqrt{K(I-K)^{-1}}U = aS + \sqrt{H_1(I-H_1)^{-1}}.$$

Conversely, when K and U are given by this equation, all above equations hold. From Lemma 3.2.3 one checks easily that Q given in the theorem lies in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$. □

3.3 $\mathcal{A}lg(\mathcal{F}_3)$ is a Kadison-Singer algebra

In this section, we shall prove that $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ is a Kadison-Singer lattice which implies that $\mathcal{A}lg(\mathcal{F}_3)$ is a Kadison-Singer algebra.

Lemma 3.3.1. *For any two distinct projections Q_1, Q_2 in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I, P_1\}$, we have that $\mathcal{A}lg(\{P_1, Q_1, Q_2\}) = \mathcal{A}lg(\mathcal{F}_3)$.*

Proof. By Theorem 3.2.1, we may assume that, for $i = 1, 2$,

$$Q_i = \begin{pmatrix} K_i & \sqrt{K_i(I-K_i)}U_i \\ U_i^*\sqrt{K_i(I-K_i)} & U_i^*(I-K_i)U_i \end{pmatrix}$$
$$\sqrt{K_i(I-K_i)^{-1}}U_i = (1+a_i)\sqrt{H_1(I-H_1)^{-1}} - a_i\sqrt{H_2(I-H_2)^{-1}}V$$
$$= a_iS + \sqrt{H_1(I-H_1)^{-1}}$$

where $a_1, a_2 \in \mathbb{C}$ and $a_1 \neq a_2$. Then we have

$$\sqrt{K_1(I-K_1)^{-1}}U_1 - \sqrt{K_2(I-K_2)^{-1}}U_2 = (a_1 - a_2)S.$$

Replacing S by $(a_1 - a_2)S$ in Lemma 3.2.3, we know that $\mathcal{A}lg(\{P_1, Q_1, Q_2\}) = \mathcal{A}lg(\mathcal{F}_3)$. $\square$

Lemma 3.3.2. *For any three distinct projections Q_1, Q_2 and Q_3 in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\}$, we always have that $P_1 \in \mathcal{L}at(\mathcal{A}lg(\{Q_1, Q_2, Q_3\}))$.*

Proof. By Theorem 3.3.1, we could assume that

$$Q_i = \begin{pmatrix} K_i & \sqrt{K_i(I-K_i)}U_i \\ U_i^*\sqrt{K_i(I-K_i)} & U_i^*(I-K_i)U_i \end{pmatrix} \text{ and}$$

$$\begin{aligned}\sqrt{K_i(I-K_i)^{-1}}U_i &= (1+a_i)\sqrt{H_1(I-H_1)^{-1}} - a_i\sqrt{H_2(I-H_2)^{-1}}V \\ &= a_iS + \sqrt{H_1(I-H_1)^{-1}},\end{aligned}$$

where $a_i \in \mathbb{C}(i = 1, 2, 3)$.

To porve this lemma, we only need to show that if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{A}lg(\{Q_1, Q_2, Q_3\})$$

then $A_{21} = 0$. Now assume that the above A belongs to $\mathcal{A}lg(\{Q_1, Q_2, Q_3\})$. Then $(I - Q_i)AQ_i = 0$, for $i = 1, 2, 3$. Thus we have

$$\begin{aligned}&\sqrt{I-K_i}(A_{11}\sqrt{K_i(I-K_i)}U_i + A_{12}U_i^*(I-K_i)U_i) \\ &\qquad = \sqrt{K_i}U_i(A_{21}\sqrt{K_i(I-K_i)}U_i + A_{22}U_i^*(I-K_i)U_i).\end{aligned}$$

Because $(\cap_{i=1}^3 \mathcal{D}(U_i^*(I-K_i)^{-1}U_i)) \cap (\mathcal{D}(S)) \cap (\mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}))$ is dense in $P_1\mathcal{H}$ (here for closed operator T, $\mathcal{D}(T)$ is T's domain), then the above equation is true if and only if for any $\xi \in (\cap_{i=1}^3 \mathcal{D}(U_i^*(I-K_i)^{-1}U_i)) \cap (\mathcal{D}(S)) \cap (\mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}))$,

$$\begin{aligned}&A_{11}\sqrt{K_i(I-K_i)^{-1}}U_i\xi + A_{12}\xi = \\ &\qquad \sqrt{K_i(I-K_i)^{-1}}U_i[A_{21}\sqrt{K_i(I-K_i)^{-1}}U_i\xi + A_{22}\xi].\end{aligned}$$

Now we choose a family of projections E_ϵ, E_ϵ has strong operator limit I as $\epsilon \to 0$, and $E_\epsilon S$, $E_\epsilon\sqrt{H_1(I-H_1)^{-1}}$ are bounded operators, thus for $i = 1, 2, 3$,

$$E_\epsilon\sqrt{K_i(I-K_i)^{-1}}U_i = a_iE_\epsilon S + E_\epsilon\sqrt{H_1(I-H_1)^{-1}},$$

are bounded operators. For i, $j = 1, 2, 3$, we have

$$E_\epsilon\sqrt{K_i(I-K_i)^{-1}}U_iA_{21}\sqrt{K_i(I-K_i)^{-1}}U_i\xi - E_\epsilon\sqrt{K_j(I-K_j)^{-1}}U_jA_{21}\sqrt{K_j(I-K_j)^{-1}}U_j\xi$$
$$= (a_i - a_j)[E_\varepsilon A_{11}S\xi - E_\varepsilon SA_{22}\xi].$$

Form this we conclude that

$$(a_1-a_2)E_\epsilon\sqrt{K_3(I-K_3)^{-1}}U_3A_{21}\sqrt{K_3(I-K_3)^{-1}}U_3\xi +$$
$$(a_2-a_3)E_\epsilon\sqrt{K_1(I-K_1)^{-1}}U_1A_{21}\sqrt{K_1(I-K_1)^{-1}}U_1\xi +$$
$$(a_3-a_1)E_\epsilon\sqrt{K_2(I-K_2)^{-1}}U_2A_{21}\sqrt{K_2(I-K_2)^{-1}}U_2\xi = 0.$$

Using the relation $E_\epsilon\sqrt{K_i(I-K_i)^{-1}}U_i = a_iE_\epsilon S + E_\epsilon\sqrt{H_1(I-H_1)^{-1}}$ again, we easily get

$$[a_3^2(a_1-a_2) + a_1^2(a_2-a_3) + a_2^2(a_3-a_1)]E_\epsilon SA_{21}S\xi$$
$$+ [a_3(a_1-a_2) + a_1(a_2-a_3) + a_2(a_3-a_1)][E_\epsilon SA_{21}\sqrt{H_1(I-H_1)^{-1}}\xi$$
$$+ E_\epsilon\sqrt{H_1(I-H_1)^{-1}}A_{21}S\xi]$$
$$+ [(a_1-a_2) + (a_2-a_3) + (a_3-a_1)]E_\epsilon\sqrt{H_1(I-H_1)^{-1}}A_{21}\sqrt{H_1(I-H_1)^{-1}}\xi = 0,$$

which implies that

$$(a_1-a_2)(a_1-a_3)(a_2-a_3)E_\epsilon SA_{21}S\xi = 0.$$

Since $ker(S) = \{0\}$, we have $F_\epsilon = \mathcal{R}(S^*E_\epsilon)$(the range projection of S^*E_ϵ) also has strong operator limit I as $\epsilon \to 0$, thus the equation above implies that $F_\epsilon A_{21}S\xi = 0$. We also know $\{S\xi|\xi \in (\cap_{i=1}^3\mathcal{D}(U_i^*(I-K_i)^{-1}U_i)) \cap (\mathcal{D}(S)) \cap (\mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}))\}$ is dense in $P_1\mathcal{H}$, this implies $F_\epsilon A_{21} = 0$. Let $\epsilon \to 0$, we have $A_{21} = 0$ and the lemma follows. □

Lemma 3.3.3. *We define a one to one map from* $\mathbb{C}$ *into* $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ *as following:*

$$\mathbb{C} \to \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \quad : a \to Q_a = \begin{pmatrix} K_a & \sqrt{K_a(I-K_a)}U_a \\ U_a^*\sqrt{K_a(I-K_a)} & U_a^*(I-K_a)U_a \end{pmatrix}$$
$$\sqrt{K_a(I-K_a)^{-1}}U_a = aS + \sqrt{H_1(I-H_1)^{-1}}.$$

This map is continuous if $\mathbb{C}$ *is endowed with its canonical topology and* $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ *has the* $\|.\|_2$ *norm topology.*

Proof. We will show that when $a \to a_0$, $\|Q_a - Q_{a_0}\|_2 \to 0$. From the equation:

$$\begin{aligned}\|Q_a - Q_{a_0}\|_2^2 &= 1 - 2\tau(Q_a Q_{a_0}) \\ &= 1 - tr(K_a K_{a_0}) - tr(\sqrt{K_a(I-K_a)}U_a U_{a_0}^* \sqrt{K_{a_0}(I-K_{a_0})}) \\ &- tr(U_a^* \sqrt{K_a(I-K_a)}\sqrt{K_{a_0}(I-K_{a_0})}U_{a_0}) \\ &- tr(U_a^*(I-K_a)U_a U_{a_0}^*(I-K_{a_0})U_{a_0}),\end{aligned}$$

it is easy to see that we only need to show when $a \to a_0$,

$$|tr((K_a - K_{a_0})K_{a_0})| \to 0, \tag{3.2}$$

$$|tr((\sqrt{K_a(I-K_a)}U_a - \sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})U_{a_0}^*\sqrt{K_{a_0}(I-K_{a_0})})| \to 0, \tag{3.3}$$

$$|tr((U_a^*\sqrt{K_a(I-K_a)} - U_{a_0}^*\sqrt{K_{a_0}(I-K_{a_0})})\sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})| \to 0, \tag{3.4}$$

$$|tr(K_a - K_{a_0})| \to 0, \quad |tr((U_a^* K_a U_a - U_{a_0}^* K_{a_0} U_{a_0})U_{a_0}^* K_{a_0} U_{a_0})| \to 0. \tag{3.5}$$

Here we only provide the proof for (3.2) and (3.3), the rest can be proved similarly.

First note that for any $a \in \mathbb{C}$

$$\begin{aligned}&K_a(I-K_a)^{-1} \\ &\quad = |a|^2 SS^* + aS\sqrt{H_1(I-H_1)^{-1}} + \overline{a}\sqrt{H_1(I-H_1)^{-1}}S^* + \sqrt{H_1(I-H_1)^{-1}}.\end{aligned}$$

Let $F(a) = |a|^2 SS^* + aS\sqrt{H_1(I-H_1)^{-1}} + \overline{a}\sqrt{H_1(I-H_1)^{-1}}S^* + \sqrt{H_1(I-H_1)^{-1}}$. Thus we have

$$I - K_a = (I + F(a))^{-1}.$$

For any $\epsilon > 0$, we can find a projection $F_\epsilon \in \mathcal{M}$, such that SS^*E_ϵ, $S\sqrt{H_1(I-H_1)^{-1}}E_\epsilon$, $\sqrt{H_1(I-H_1)^{-1}}S^*E_\epsilon$, $\sqrt{H_1(I-H_1)^{-1}}E_\epsilon$ are bounded operators, and $tr(I - F_\epsilon) = tr(I - E_\epsilon) < \epsilon^2$, where $E_\epsilon = \mathcal{R}((I+F(a))^{-1}F_\epsilon)(\in \mathcal{M})$ is the range projection of $(I+F(a))^{-1}F_\epsilon$. So there exists a $\beta > 0$, if $|a - a_0| < \beta$, we have $\|F(a)E_\epsilon - F(a_0)E_\epsilon\| < \epsilon$. Thus

$$\begin{aligned}|tr((K_a - K_{a_0})K_{a_0})| &\le |tr(K_{a_0}(K_a - K_{a_0})F_\varepsilon)| + |tr(K_{a_0}(K_a - K_{a_0})(I - F_\varepsilon))| \\ &\le |tr(K_{a_0}(I+F(a))^{-1}(F(a) - F(a_0))(I+F(a_0))^{-1}F_\varepsilon)| + 2\varepsilon \le 3\varepsilon.\end{aligned}$$

This implies (3.2).

In order to prove (3.3), for any $\epsilon > 0$, we choose a projection $P_\epsilon \in \mathcal{M}$ such that SP_ε, $\sqrt{H_1(I-H_1)^{-1}}P_\varepsilon$ are bounded operators, and $tr(I-P_\varepsilon) < \varepsilon^2$. Then there exists $\beta_1 > 0$, if $|a-a_0| < \beta_1$, we have

$$\|\sqrt{K_a(I-K_a)^{-1}}U_aP_\epsilon - \sqrt{K_{a_0}(I-K_{a_0})^{-1}}U_{a_0}P_\epsilon\| \leq \epsilon.$$

Then we obtain

$$\begin{aligned}
&|tr((\sqrt{K_a(I-K_a)}U_a - \sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})U_{a0}^*\sqrt{K_{a_0}(I-K_{a_0})})| \\
&\leq |tr(U_{a0}^*\sqrt{K_{a_0}(I-K_{a_0})}(\sqrt{K_a(I-K_a)}U_a - \sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})P_\varepsilon)| \\
&+ |tr(U_{a0}^*\sqrt{K_{a_0}(I-K_{a_0})}(\sqrt{K_a(I-K_a)}U_a - \sqrt{K_{a_0}(I-K_{a_0})}U_{a_0})(I-P_\varepsilon))| \\
&\leq |tr(U_{a0}^*\sqrt{K_{a_0}(I-K_{a_0})}(I-K_a)[\sqrt{K_a(I-K_a)^{-1}}U_aP_\varepsilon - \sqrt{K_{a_0}(I-K_{a_0})^{-1}}U_{a_0}P_\varepsilon])| \\
&+ |tr(\sqrt{K_{a_0}(I-K_{a_0})^{-1}}U_{a_0}P_\varepsilon U_{a0}^*\sqrt{K_{a_0}(I-K_{a_0})}(K_{a_0}-K_a))| + 2\varepsilon \\
&\leq |tr(\sqrt{K_{a_0}(I-K_{a_0})^{-1}}U_{a_0}P_\varepsilon U_{a0}^*\sqrt{K_{a_0}(I-K_{a_0})}(K_{a_0}-K_a))| + 3\varepsilon.
\end{aligned}$$

Now similar argument as in the proof of (3.2) will give us (3.3). □

Lemma 3.3.4. *With the notations in the above lemma, we have when $a \to \infty$, $\|Q_a - P_1\|_2 \to 0$.*

Proof. Since $\|Q_a - P_1\|_2^2 = 1 - tr(K_a)$, we only need to show that when $a \to \infty$, $tr(I-K_a) \to 0$. Again let

$$F(a) = |a|^2(SS^* + \frac{1}{\bar{a}}S\sqrt{H_1(I-H_1)^{-1}} + \frac{1}{a}\sqrt{H_1(I-H_1)^{-1}}S^* + \frac{1}{|a|^2}\sqrt{H_1(I-H_1)^{-1}}).$$

Since SS^* is invertible, for any $\epsilon > 0$, we can choose a projection $E \in \mathcal{M}$, such that $\tau(E) > 1-\epsilon$, ESS^*E, $ES\sqrt{H_1(I-H_1)^{-1}}E$, $E\sqrt{H_1(I-H_1)^{-1}}E$, are all bounded, and $ESS^*E \geq \beta E (\beta > 0)$. Thus there exist $c > 0$, s.t. if $|a| > c$, we have

$$\begin{aligned}
&ESS^*E + \frac{1}{\bar{a}}ES\sqrt{H_1(I-H_1)^{-1}}E \\
&\quad + \frac{1}{a}E\sqrt{H_1(I-H_1)^{-1}}S^*E + \frac{1}{|a|^2}E\sqrt{H_1(I-H_1)^{-1}}E > \frac{\beta}{2}E.
\end{aligned}$$

By [14] Lemma 3.2, we know $tr(e_{F(a)}([|a|^2\frac{\beta}{2},+\infty])) \geq 1-\epsilon$, where $e_{F(a)}([\frac{|a|^2\beta}{2},+\infty]) = e$ is the spectral projection of $F(a)$, such that $eF(a)e \geq \frac{|a|^2\beta}{2}e$. Choose $|a|$ large enough, we will have

$$\begin{aligned} tr(I-K_a) &= tr((I+F(a))^{-1}e) + tr((I+F(a))^{-1}(1-e)) \\ &\leq \frac{2}{|a|^2\beta} + \epsilon \leq 2\epsilon. \end{aligned}$$

This implies when $a \to \infty$, $tr(I-K_a) \to 0$. □

Now the follow theorem follows easily from our lemmas.

Theorem 3.3.1. *With the above notation, we have that $\mathcal{A}lg(\mathcal{F}_3)$ is a KS-algebra and $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ is determined by the following: $P \in \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ and $P \neq 0, I, P_1$, if and only if*

$$P = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & U^*(I-K)U \end{pmatrix},$$

where K and U are uniquely determined by the following polar decomposition with any $a \in \mathbb{C}$: $(a+1)\sqrt{H_1(I-H_1)^{-1}} - a\sqrt{H_2(I-H_2)^{-1}}V = \sqrt{K(I-K)^{-1}}U$. As a consequence, we have $\tau(P) = \frac{1}{2}$ and as a tends to ∞, the projection P converges strongly to P_1. Thus $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\}$ is homeomorphic to S^2.

Proof. By Lemma 3.3.3 and 3.3.4, the following one to one map:

$$S^2 \to \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\} : \quad a \to \begin{cases} Q_a, & a \in \mathbb{C} \\ P_1, & a = \infty \end{cases}$$

is continuous form S^2 onto $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\}$. Since S^2 is compact, and $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\}$ is Hausdorff, we have $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3)) \setminus \{0, I\}$ is homeomorphic to S^2, i.e. the one point compacification of $\mathbb{C}$.

We still need to show the minimality of $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$. Clearly for any sublattice $\mathcal{L}_1$ containing only two projections Q_1, Q_2 in $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$, $\mathcal{L}_1$ can not generate the type II_1 factor $\mathcal{L}_{G_3}$. Lemma 3.3.1 and 3.3.2 shows that any reflexive sublattice of $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ containing more than two nontrivial projections must agree with $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$. Thus $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$ is a Kadison-Singer lattice. □

Although the above theorem is stated for $\mathcal{F}_3$, similar results hold for any double triangle lattice in finite von Neumann algebras. We shall provide more details in the appendix.

When a lattice contains four or more projections in a von Neumann algebra, the situation is not clear. Even for $\mathcal{F}_4$ (the lattice generated by four free projections), we know from our above result that $\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_4))$ contains several copies of S^2. But we do not have a complete characterization of this lattice. The following theorem shows that $\mathcal{A}lg(\mathcal{F}_\infty)$ contains no nonzero compact operators.

Theorem 3.3.2. *Let $\mathcal{F}_\infty$ be the lattice generated by countably infinitely many free projections of trace $\frac{1}{2}$ in $\mathcal{L}_{G_\infty}$, where G_∞ is the free product of countably infinitely many copies of $\mathbb{Z}_2$. Then $\mathcal{A}lg(\mathcal{F}_\infty)$ does not contain any nonzero compact operators.*

In order to prove this theorem, we need following facts.

Lemma 3.3.5. *Suppose $\{\xi_i\}_{i=1}^n$ are unit vectors in $l^2(G_\infty)$. For any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for $k > m$, we have $| < U_k\xi_1, \xi_i > | < \varepsilon (i = 1, 2, \ldots, n)$, i.e. $\{U_n\}_{n=1}^\infty$ has weak-operator limit zero as $n \to 0$, where U_j, $j = 1, 2, \ldots$ are the standard generators of $\mathcal{L}_{G_n}$.*

Proof. In convention, we will denote the element of the orthonormal basis of $l^2(G_\infty)$ by $g(\in G_\infty)$. Thus $\xi_i = \sum_{g\in G_\infty}^\infty a_g^i g$. For any finite unit vectors, there exists a subset $\{g_j\}_{j=1}^{n_0}$ of G_∞ such that $\|P\xi_i\| > 1 - \varepsilon^2 (i = 1, 2 \ldots, n)$, where P is the orthogonal projection onto the closed subspace spanned by $\{g_j\}_{j=1}^{n_0}$. Choosing $m \in \mathcal{N}$ such that for any $k > m$, the group element corresponding to U_k does not appear in $\{g_j\}_{j=1}^{n_0}$. Let $\xi_i = \xi_i^1 + \xi_i^2$, where $\xi_i^1 = P\xi_i$, by the property of free group, we have $< U_k\xi_1^1, \xi_i^1 >= 0$. Thus

$$\begin{aligned}
| < U_k\xi_1, \xi_i > | &= | < U_k\xi_1^1 + U_k\xi_1^2, \xi_i^1 + \xi_i^2 > | \\
&= | < U_k\xi_1^1, \xi_i^2 > + < U_k\xi_1^2, \xi_i > | \\
&\leq \|U_k\xi_1^1\|\|\xi_i^2\| + \|U_k\xi_1^2\| \leq 2\varepsilon.
\end{aligned}$$

□

Lemma 3.3.6. *Suppose $T \in \mathcal{B}(l^2(G_\infty))$ is compact, then $\lim_k \|TU_k\| = 0$ (here we treat U_k as the element of the orthonormal basis of $l^2(G_\infty)$).*

Proof. Without lose of generality, we may assume that $\|T\| = 1$. If $\lim_k \|TU_k\| \neq 0$, by the compactness of T, we could also assume that $\lim_k TU_k = \xi \neq 0$. Then for any $0 < \varepsilon < \frac{\|\xi\|}{3}$, there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, we have $\|TU_{k_0} - TU_k\| < \varepsilon$. This implies $\|TU_{k_0}\| \geq \frac{2\|\xi\|}{3}$. Thus for any $N \in \mathbb{N}$, we obtain

$$\begin{aligned} 1 \geq \|T^*TU_{k_0}\|^2 &\geq \sum_{k=k_0}^{k_0+N} |< T^*TU_{k_0}U_k >|^2 \\ &\geq N\|TU_{k_0}\|^2(\|TU_{k_0}\| - \varepsilon)^2 \\ &\geq \frac{N\|TU_{k_0}\|^2\|\xi\|^2}{9}, \end{aligned}$$

This is a contradiction, and therefore our assumption is wrong. □

Proof of Theorem 3.3.2. Let U_j, $j = 1, 2, \ldots$, be the standard generators of $\mathcal{L}_{G_n}$ and $P_j = \frac{I - U_j}{2}$ the free projections in $\mathcal{F}_\infty$. Suppose $T \in \mathcal{A}lg(\mathcal{F}_\infty)$ is a non-zero compact operator and $\|T\| = 1$. Without lose of generality, we may assume that $T1 = \xi \neq 0$. Indeed, since $T \neq 0$, there must exists $g \in G$, $Tg \neq 0$, we can replace T with $TR_{g^{-1}}$ (Note $TR_{g^{-1}} \in \mathcal{A}lg(\mathcal{F}_\infty)$, where $(R_{g^{-1}}\xi)(g_1) = \xi(g_1 g^{-1})$, for any $\xi \in l^2(G_\infty)$).

From $(I - P_j)TP_j = 0$, we know that $(I + U_j)T(I - U_j) = 0$ for $j = 1, 2, \ldots$. By the above lemmas, we have

$$\lim_n \|(1 - U_n)\xi\| = \lim_n \|(1 - U_n)TU_n\| \leq 2 \lim_n \|TU_n\| = 0.$$

Since U_j has weak-operator limit zero, as $j \to \infty$, there is $n_0 \in \mathbb{N}$ such that for any $m > n_0$, $|< U_m\xi, \xi >| < \frac{1}{2}\|\xi\|^2$. Therefore, we have

$$\begin{aligned} \|(1 - U_m)\xi\|^2 &= < (1 - U_m)\xi, (1 - U_m)\xi > \\ &\geq \|\xi\|^2. \end{aligned}$$

It is clear that the above two equations contradict each other, thus our assumption must be wrong. □

3.4 Reduced lattices

Definition 3.4.1. *Suppose $\mathcal{L}$ is a lattice of projections in a finite von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ. Two projections P and Q in $\mathcal{L}$ are said to be connected if, for any $\epsilon > 0$, there are elements $P_1, P_2, \ldots, P_n$ in $\mathcal{L}$ such that $P_1 = P$, $P_n = Q$, $|\tau(P_j - P_{j+1})| < \epsilon$, and either $P_j \leq P_{j+1}$ or $P_j \geq P_{j+1}$, for $j = 1, \ldots, n-1$. Define the connected component $O(P)$ of P to be the set of all projections in $\mathcal{L}$ that are connected with P. Let $\Gamma_0(\mathcal{L})$ be the set of all connected components in $\mathcal{L}$.*

We shall see that $\Gamma_0(\mathcal{L})$ carries an induced lattice structure from $\mathcal{L}$ and we call $\Gamma_0(\mathcal{L})$ the *reduced lattice* of $\mathcal{L}$. It is clear that if $\mathcal{L}$ is a continuous nest, then $\Gamma_0(\mathcal{L})$ contains only one point. A basic fact on connected components is given in the following.

Proposition 3.4.1. *Suppose $\mathcal{L}$ is a lattice of projections in a finite von Neumann algebra $\mathcal{M}$, $P, Q \in \mathcal{L}$. If $O(P) \neq O(Q)$, then $O(P) \cap O(Q) = \emptyset$*

The proof of this proposition follows easily from the definition. If $O(P)$ and $O(Q)$ are two elements in $\Gamma_0(\mathcal{L})$, then, for any $Q_1 \in O(Q)$, it is easy to see that $O(P \vee Q) = O(P \vee Q_1)$. Indeed, by the definition, for any ε, there exist P_1, P_2, …, P_n, $P_1 = Q$, $P_n = Q_1$, $|\tau(P_j - P_{j+1})| < \epsilon$, and for $j = 1$, …, $n-1$, $P_j \leq P_{j+1}$ or $P_j \geq P_{j+1}$. Without lose of generality, we may assume $P_j \leq P_{j+1}$, thus we have $P_{j+1} = P_j \vee P'$, $\tau(P') \leq \varepsilon$,

$$\begin{aligned} &|\tau(P \vee P_j) - \tau(P \vee P_{j+1})| = |\tau(P \vee P_j) - \tau(P \vee P_j \vee P')| \\ =&|\tau(P') - \tau((P \vee P_j) \wedge P')| \leq \varepsilon. \end{aligned}$$

This implies $P \vee Q$ and $P \vee Q_1$ are connected. Thus $O(P \vee Q)$ depends on the components $O(P)$ and $O(Q)$, not the choices of P and Q in the components. We define $O(P) \vee O(Q) = O(P \vee Q)$. Similarly we define that $O(P) \wedge O(Q) = O(P \wedge Q)$. It is easy to shows that $\Gamma_0(\mathcal{L})$ is a lattice. The following theorem is immediate.

Theorem 3.4.1. *Suppose $\mathcal{L}$ is a lattice of projections in a finite von Neumann algebra $\mathcal{M}$ with a (faithful normal) trace τ. If $\tau(\mathcal{L})$ contains only finitely many trace values, then $\Gamma_0(\mathcal{L})$ is the same as $\mathcal{L}$ (i.e., every connected component contains only one element in $\mathcal{L}$)*

From this theorem, we know that $\Gamma_0(\mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))) = \mathcal{L}at(\mathcal{A}lg(\mathcal{F}_3))$. Suppose $\mathcal{L}$ is a Kadison-Singer lattice and $\mathcal{L}$ generates a finite von Neumann algebra. If $\Gamma_0(\mathcal{L})$ contains only one point, then we call $\mathcal{L}$ *contractible.* Thus continuous nests are contractible.

Reduced lattices can not contain continuous nests. The following theorem shows that all possible trace values can appear in a reduced lattice.

Theorem 3.4.2. *Suppose $\mathcal{L}^{(n)}$ is the lattice given in Section 2.1. Then $\Gamma_0(\mathcal{L}^{(n)}) = \mathcal{L}^{(n)}$*

To prove the preceding theorem, we need the following facts (we will use the notations introduced in the last chapter).

Lemma 3.4.1. *Suppose $P = E_i^{(1)} + F_{n-i}^{(1)}Q$ is a projection in $L^{(n)}$, then $\widetilde{P} = E_j^{(1)} + F_{n-j}^{(1)}\widetilde{Q} \leq P$ if and only if $j \leq i$, and $\widetilde{Q} \leq Q$. Specially, we have $|\tau(P) - \tau(Q)| = \frac{i-j}{n} + \frac{\tau(Q-\widetilde{Q})}{n}$.*

It is not hard to prove the above lemma by considering the trace value of projections, we omit the proof here. Note the lemma implies that if

$$E_i^{(1)} + F_{n-i}^{(1)}Q = P \leq \widetilde{P} = E_j^{(1)} + F_{n-j}^{(1)}\widetilde{Q},$$

and $|\tau(P - \widetilde{P})| < \frac{1}{n}$, we must have $i = j$.

Lemma 3.4.2. *Suppose $P_1 = E_{i_1}^{(1)} + F_{n-i_1}^{(1)}Q_1$, $P_2 = E_{i_2}^{(1)} + F_{n-i_2}^{(1)}Q_2$ are two projections in $\mathcal{L}^{(n)}$, if $i_1 \neq i_2$, then P_1, P_2 are not connected.*

Proof. Assuming P_1 and P_2 are connected. By definition, for any $\varepsilon < \frac{1}{n}$, there are projections $\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_m$ in $\mathcal{L}^{(n)}$ such that $P_1 = \widetilde{P}_1$, $\widetilde{P}_m = P_2$, $|\tau(\widetilde{P}_j - \widetilde{P}_{j+1})| < \varepsilon$ and $\widetilde{P}_j \leq \widetilde{P}_{j+1}$ or $\widetilde{P}_j \geq \widetilde{P}_{j+1}$. By the discussion above, we have $\widetilde{P}_2 = E_{i_1}^{(1)} + F_{n-i_1}^{(1)}\widetilde{Q_2}$. Similarly, $\widetilde{P}_j = E_{i_1}^{(1)} + F_{n-i_1}^{(1)}\widetilde{Q_j}(j = 1, 2, \ldots m)$. Thus

$$E_{i_1}^{(1)} + F_{n-i_1}^{(1)}\widetilde{Q_m} = \widetilde{P}_m = P_2 = E_{i_2}^{(1)} + F_{n-i_2}^{(1)}Q_2,$$

this contradicts the fact that $i_1 \neq i_2$, and completes the proof. □

Now it is easy to prove Theorem 3.4.2 by induction.

Proof. Note if $P_1 \neq P_2$, there is $k \in \mathbb{N}$ such that

$$P_1 = E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{k-2} F_{n-a_i}^{(i)}) E_{a_{k-1}}^{(k-1)} + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)}) E_{a_k}^{(k)} + (\prod_{i=1}^{k} F_{n-a_i}^{(i)}) Q_1,$$

$$P_2 = E_{a_1}^{(1)} + F_{n-a_1}^{(1)} E_{a_2}^{(2)} + \cdots + (\prod_{i=1}^{k-2} F_{n-a_i}^{(i)}) E_{a_{k-1}}^{(k-1)} + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)}) E_{b_k}^{(k)} + (\prod_{i=1}^{k-1} F_{n-a_i}^{(i)}) F_{n-b_k}^{(k)} Q_2,$$

and $a_k \neq b_k$, then by induction, and apply lemma 3.4.2 , it's easy to show that P_1 and P_2 are not connected. □

3.5 Maximality conditions

In the definition of Kadison-Singer algebras, we require that the algebra be maximal in the class of reflexive algebras with the same diagonal. Our examples of KS-algebras given in chapter 2 are "maximal triangular" in the class of all algebras with the same diagonal, i.e., an algebraic maximality without reflexiveness or closedness assumptions. In general, the algebraic maximality assumption is a much stronger requirement. We call a subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ *maximal triangular with respect to its diagonal C^*- (or von Neumann) algebra* $\mathfrak{A} \cap \mathfrak{A}^*$ if, for any subalgebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$, $\mathfrak{B}$ contains $\mathfrak{A}$ and has the same diagonal as $\mathfrak{A}$, then $\mathfrak{B}$ is equal to $\mathfrak{A}$. This may lead to new interesting classes of non selfadjoint algebras. Many similar questions as those in [34], e.g., the closedness of $\mathfrak{A}$, can be asked accordingly.

In the following, we give a canonical method to construct a maximal triangular algebra in the class of weak-operator closed algebras with respect to a given von Neumann algebra as its diagonal.

Suppose $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and $H_1, \ldots, H_n$ are positive elements in $\mathcal{M}$ such that $H_1^2, H_2^2, \ldots, H_n^2$ generate $\mathcal{M}$ as a von Neumann algebra. From [24], we know that many von Neumann algebras can be generated by such positive elements, especially all type III and properly infinite von Neumann algebras. Let $\mathcal{H}$ be the direct sum of $n+1$ copies of $\mathcal{H}_0$. Then $\mathcal{B}(\mathcal{H}) \cong M_{n+1}(\mathcal{B}(\mathcal{H}_0))$. With this identification, we shall view both $M_{n+1}(\mathbb{C})$ and $\mathcal{B}(\mathcal{H}_0)$ as subalgebras of $\mathcal{B}(\mathcal{H})$. Let E_{ij}, $i, j = 1, \ldots, n+1$ be a matrix unit system for $M_{n+1}(\mathbb{C})$. We shall write elements in $\mathcal{B}(\mathcal{H})$ in a matrix form with respect to this unit system (with entries from $\mathcal{B}(\mathcal{H}_0)$).

Theorem 3.5.1. *Define*

$$\mathfrak{A} = \left\{ \begin{pmatrix} A & * & \dots & * \\ 0 & H_1^{-1}AH_1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_n^{-1}AH_n \end{pmatrix} : \quad A \in \mathcal{B}(\mathcal{H}_0) \right\},$$

where $*$ denotes all possible elements in $\mathcal{B}(\mathcal{H}_0)$. Then $\mathfrak{A}$ is maximal upper triangular with respect to the diagonal $M_{n+1}(\mathcal{M})'$ ($\cong \mathcal{M}' \cap \mathcal{B}(\mathcal{H}_0)$).

For the rest of this section, we will prove this theorem.

Lemma 3.5.1. *$\mathfrak{A}$ is weak operator closed, and $\mathfrak{A} \cap \mathfrak{A}^* = M_{n+1}(\mathcal{M})'$.*

Proof. By the definition of $\mathfrak{A}$, $A \in \mathfrak{A}$ if and only if

$$E_{i,i}AE_{j,j} = 0 (i > j), \tag{3.6}$$

$$E_{1,1}AE_{1,1} = H_i E_{1,i+1} A E_{i+1,1} {H_i}^{-1}. \tag{3.7}$$

It is clear if $T_\alpha \xrightarrow{\text{WOT}} T$, $T_\alpha \in \mathfrak{A}$, $E_{i,i}TE_{j,j} = \lim_\alpha E_{i,i}T_\alpha E_{j,j} = 0 (i > j)$, and

$$\begin{aligned} E_{1,1}TE_{1,1} &= \lim_\alpha E_{1,1}T_\alpha E_{1,1} \\ &= \lim_\alpha H_i E_{1,i+1} T_\alpha E_{i+1,1} {H_i}^{-1} \\ &= H_i E_{1,i+1} T E_{i+1,1} {H_i}^{-1}, \end{aligned}$$

this means $T \in \mathfrak{A}$, thus $\mathfrak{A}$ is weak operator closed.

To prove $\mathfrak{A} \cap \mathfrak{A}^* = M_{n+1}(\mathcal{M})'$, we only need to show that for any self-adjoint operator H in $\mathfrak{A}$, $H \in M_{n+1}(\mathcal{M})'$. By (3.5), (3.6), we have

$$\begin{aligned} E_{i,i}HE_{j,j} &= 0 (i \neq j), \\ E_{1,1}HE_{1,1} &= H_i E_{1,i+1} H E_{i+1,1} {H_i}^{-1}, \\ E_{1,1}HE_{1,1} &= {H_i}^{-1} E_{1,i+1} H E_{i+1,1} H_i. \end{aligned}$$

Since $\{{H_1}^2, {H_2}^2, \dots, {H_n}^2\}'' = \mathcal{M}$, the equations above imply $H \in M_{n+1}(\mathcal{M})'$. □

Lemma 3.5.2. *Suppose the algebra $\mathcal{W}$ generated by T and $\mathfrak{A}$ satisfies $\mathcal{W}^* \cap \mathcal{W} = M_{n+1}(\mathcal{M})'$, we have $E_{i,i}TE_{j,j} = 0$, $i > j$, $i, j = 1, \dots n+1$, where T is an operator in $M_{n+1}(\mathcal{B}(\mathcal{H}_0))$.*

Proof. If there exists $i > 1$ such that $E_{i,i}TE_{1,1} \neq 0$, let

$$E_{1,1}UHE_{1,1} = E_{1,i}TE_{1,1} = E_{1,i}T - \sum_{j=2}^{n+1}, E_{1,i}TE_{j,j}$$

the polar decomposition of $E_{1,i}TE_{1,1} (\in \mathcal{W})$, we have

$$E_{1,1}HE_{1,1} = (E_{1,1}U^*E_{1,1} + \sum_{j=2}^{n+1} H_{j-1}^{-1}E_{j,1}U^*E_{1,j}H_{j-1})E_{1,1}UHE_{1,1} \in \mathcal{W},$$

it is clear that the self adjoint operator $E_{1,1}HE_{1,1}$ is not in $M_{n+1}(\mathcal{M})'$, this contradicts the fact that $\mathcal{W}^* \cap \mathcal{W} = M_{n+1}(\mathcal{M})'$. Other cases can be proved similarly. □

Proof of Theorem 3.5.1. To prove the theorem, we will show that for any $T \in \mathcal{B}(\mathcal{H})$, if the weak operator closed algebra $\mathcal{W}$, generated by T and $\mathfrak{A}$, satisfies $\mathcal{W}^* \cap \mathcal{W} = M_{n+1}(\mathcal{M})'$, then T must be in $\mathfrak{A}$. By Lemma 3.5.2, we may assume that

$$T = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & H_1^{-1}A_2H_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_n^{-1}A_{n+1}H_n \end{pmatrix} \in \mathcal{W}.$$

Next we will prove $A_1 = A_2 = \dots = A_{n+1}$, equivalently $A_1 - A_{n+1} = A_2 - A_{n+1} = \dots = 0$. If this is not true, without lose of generality, we could assume that $A_1 - A_{n+1} \neq 0$ and

$A_1 - A_{n+1} \geq 0$. Actually, if let UK be the polar decomposition of $A_1 - A_{n+1}$, we have

$$\begin{pmatrix} U^* & 0 & \dots & 0 \\ 0 & H_1^{-1}U^*H_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_n^{-1}U^*H_n \end{pmatrix}$$

$$\times \begin{pmatrix} A_1 - A_{n+1} & 0 & \dots & 0 & 0 \\ 0 & H_1^{-1}(A_2 - A_{n+1})H_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & H_{n-1}^{-1}(A_n - A_{n+1})H_{n-1} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} K & 0 & \dots & 0 \\ 0 & H_1^{-1}U^*(A_2 - A_{n+1})H_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{W}.$$

Based on the discussion above, we assume $A_1 \geq 0$, $A_{n+1} = 0$. Choosing a one dimensional projection P in $\mathcal{B}(\mathcal{H})$, such that $PA_1P = c_1P \neq 0$, we have

$$\frac{1}{c_1}\begin{pmatrix} P & 0 & \dots & 0 \\ 0 & H_1^{-1}PH_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_n^{-1}PH_n \end{pmatrix} \times \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & H_1^{-1}A_2H_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} P & 0 & \dots & 0 \\ 0 & H_1^{-1}PH_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_n^{-1}PH_n \end{pmatrix} = \begin{pmatrix} P & 0 & \dots & 0 \\ 0 & c_2H_1^{-1}PH_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{W},$$

where $c_iP = \frac{1}{c_1}PA_iP$ $(i = 2, \dots, n)$. By the function calculous, we may assume $c_i = 1$ $(i = 2, \dots, n)$. Let $\{e_\alpha\}_\alpha$ be an orthonormal basis of $\mathcal{H}$, such that $Pe_{\alpha_0} = e_{\alpha_0}$. If V_α is a partial isometry satisfies $Ve_\alpha = e_{\alpha_0}$, we have $\sum_\alpha V_\alpha P V_\alpha^* = I$. Thus

$$\begin{pmatrix} I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{W}^* \cap \mathcal{W},$$

this contradicts the fact that $\mathcal{W}^* \cap \mathcal{W} = M_{n+1}(\mathcal{M})'$. □

With $\mathfrak{A}$ given in Theorem 3.5.1, suppose P is a projection in $\mathcal{L}at(\mathfrak{A})$ with $P = (P_{ij})_{i,j=1}^{n+1}$, $P_{ij} \in \mathcal{B}(\mathcal{H}_0)$. It is easy to see that P must be diagonal and $(I - P_{ii})TP_{jj} = 0$ for all $i < j$ and any T in $\mathcal{B}(\mathcal{H}_0)$. Thus $P = \sum_{j=1}^{k} E_{jj}$ for some k. We know that such a P lies in $\mathcal{L}at(\mathfrak{A})$. This shows that $\mathcal{L}at(\mathfrak{A}) = \{0, E_{11}, \dots, \sum_{j=1}^{n} E_{jj}, I\}$. Clearly this implies that $\mathfrak{A}$ is not reflexive.

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Double Triangle Lattices in Finite von Neumann algebras

Suppose $\mathfrak{A} \cong \mathcal{M} \otimes M_2(\mathbb{C})$ is a finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, where $\mathcal{M} \otimes I$ is the relative commutant of $I \otimes M_2(\mathbb{C})$ in $\mathfrak{A}$. Let $\{E_{i,j}\}_{i,j=1}^2$ be the canonical system of 2×2 matrix units in $M_2(\mathbb{C})$. With respect to these matrix units, we can write any operator T in $\mathfrak{A}$ as

$$T = T_{1,1} \otimes E_{1,1} + T_{1,2} \otimes E_{1,2} + T_{2,1} \otimes E_{2,1} + T_{2,2} \otimes E_{2,2} = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}.$$

Specially let

$$P_1 = I \otimes E_{1,1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

note if τ is a normalized ($\tau(I) = 1$) normal faithful trace on $\mathfrak{A}$, $\tau(T) = \frac{1}{2}(tr(T_{1,1}) + tr(T_{2,2}))$, where tr is a normalized normal faithful trace on $\mathcal{M}$, then $\tau(P_1) = \frac{1}{2}$.

If $Q \in \mathfrak{A}$ is a projection s.t. $Q \wedge P_1 = 0$, and $Q \vee P_1 = I$, then $\tau(Q) = \frac{1}{2}$ $(\tau(Q \vee P_1) = \tau(P) + \tau(Q) - \tau(Q \wedge P_1) = 1)$. If we write

$$Q = \begin{pmatrix} H_1 & H_2 V \\ V^* H_2 & H_3 \end{pmatrix},$$

where V is an unitary in $\mathcal{M}$ (because $\mathcal{M}$ is finite), H_1, H_2, $H_3 \in \mathcal{M}$ are positive operators. By $Q^2 = Q$, we have

$$H_1 = H_1^2 + H_2^2, \tag{8}$$

$$H_2 V = H_1 H_2 V + H_2 V H_3. \tag{9}$$

(A.1) implies that $H_2 = \sqrt{H_1(I - H_1)}$, so we have

$$(I - H_1)\sqrt{H_1(I - H_1)}V = \sqrt{H_1(I - H_1)}V H_3. \tag{10}$$

Because $Q \vee P_1 = I$, $ker(H_3) = \{0\}$, (A.3) implies that $H_3 = V^*(I - H_1)V$. Also note that $ker(I - H_1) = \{0\}$, since $Q \wedge P_1 = 0$.

Murray and von Neumann proved that the set of operators (generally unbounded) affiliated with $\mathcal{M}$ is an algebra, we denote this set by $\tilde{\mathcal{M}}$. If X, $Y \in \tilde{\mathcal{M}}$, then $X+Y$ and XY are densely defined, closable, and their closures are in $\tilde{\mathcal{M}}$. For any $T \in \tilde{\mathcal{M}}$, since T is a closed operator,

$$\{\begin{pmatrix} T\xi \\ \xi \end{pmatrix} | \xi \in \mathcal{D}(T)\},$$

is a closed subspace of $\mathcal{H}$, where $\mathcal{D}(T)$ is the domain of T. Denote the projection onto this subspace by $\mathcal{G}(T)$.

Lemma .0.3. *If X and Y are two operators (generally unbounded) affiliated with a finite von Neumann algebra $\mathfrak{A}$, then the closure of $X+Y$ (denoted by $X\hat{+}Y$) is also affiliated with $\mathfrak{A}$. If $\xi \in \mathcal{D}(X\hat{+}Y) \cap \mathcal{D}(X)$, we have $\xi \in \mathcal{D}(Y)$, and $(X\hat{+}Y)\xi = X\xi + Y\xi$.*

Proof. Since $\xi \in \mathcal{D}(X\hat{+}Y) \cap \mathcal{D}(X)$, we have $\xi \in \mathcal{D}((X\hat{+}Y) - X)$. Because $\widetilde{\mathcal{M}}$ is a algebra, we have $(X\hat{+}Y)\hat{-}X = Y$, thus $\xi \in \mathcal{D}(Y)$, and $(X\hat{+}Y)\xi = X\xi + Y\xi$. □

Lemma .0.4. *With the notation above, for any projection $Q \in \mathfrak{A}$, $Q \wedge P_1 = 0$ and $Q \vee P_1 = I$, there exist a positive operator H in $\mathcal{M}$ s.t. $ker(I-H) = 0$, and an unitary operator V in $\mathcal{M}$ such that*

$$Q = \begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & V^*(I-H)V \end{pmatrix}.$$

And $Q = \mathcal{G}(\sqrt{H(I-H)^{-1}}V)$.

Proof. We only need to show the last statement. Let

$$U = \begin{pmatrix} \sqrt{H} & \sqrt{I-H}V \\ \sqrt{I-H} & -\sqrt{H}V \end{pmatrix},$$

U is an unitary, and $Q = U^*P_1U$. It is clear

$$\begin{pmatrix} \sqrt{H} & \sqrt{I-H} \\ V^*\sqrt{I-H} & -V^*\sqrt{H} \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{H}\xi \\ V^*\sqrt{I-H}\xi \end{pmatrix} \in \mathcal{G}(\sqrt{H(I-H)^{-1}}V)\mathcal{H}.$$

So $Q \leq \mathcal{G}(\sqrt{H(I-H)^{-1}}V)$.

For any $\xi \in \mathcal{D}(\sqrt{H(I-H)^{-1}}V)$,

$$\begin{pmatrix} \sqrt{H} & \sqrt{I-H}V \\ \sqrt{I-H} & -\sqrt{H}V \end{pmatrix} \begin{pmatrix} \sqrt{H(I-H)^{-1}}V\xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} [H\sqrt{(I-H)^{-1}} + \sqrt{I-H}]V\xi \\ 0 \end{pmatrix} \in P_1\mathcal{H},$$

thus $\mathcal{G}(\sqrt{H(I-H)^{-1}}V) = Q$. □

Lemma .0.5. *Let P_2, P_3 be two projections in $\mathfrak{A}$ such that $P_i \wedge P_1 = 0$, $P_i \vee P_1 = I$, $i = 2$,, 3, by the above lemma, we have*

$$P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)}V_1 \\ V_1^* \sqrt{H_1(I-H_1)} & V_1^*(I-H_1)V_1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V_2 \\ V_2^* \sqrt{H_2(I-H_2)} & V_2^*(I-H_2)V_2 \end{pmatrix}.$$

And $P_2 \wedge P_3 = 0$ if and only if

$$ker(\sqrt{H_1(I-H_1)^{-1}}V_1 - \sqrt{H_2(I-H_2)^{-1}}V_2) = \{0\}.$$

Proof. If $ker(\sqrt{H_1(I-H_1)^{-1}}V_1 - \sqrt{H_2(I-H_2)^{-1}}V_2) \neq \{0\}$, there exist a vector $\xi \in \mathcal{D}(\sqrt{H_1(I-H_1)^{-1}}V_1) \cap \mathcal{D}(\sqrt{H_2(I-H_2)^{-1}}V_2)$, and

$$\sqrt{H_1(I-H_1)^{-1}}V_1\xi = \sqrt{H_2(I-H_2)^{-1}}V_2\xi,$$

this contradicts the fact that $P_2 \wedge P_3 = 0$. □

Remark .0.1. *In the above lemma, we could assume that $V_1 = I$. Indeed, we could replace P_i, $i = 1, 2, 3$, with U^*P_iU, $i = 1, 2, 3$, where $U = \begin{pmatrix} I & 0 \\ 0 & V_1^* \end{pmatrix}$.*

Remark .0.2. *For any two operators (generally unbounded) $X, Y \in \widetilde{\mathcal{M}}$, we denote the closure of $X+Y$ by $X\hat{+}Y$, then $ker(X\hat{+}Y) = \{0\}$ iff $ker(X+Y) = \{0\}$. Indeed, if $ker(X+Y) = \{0\}$, but $ker(X\hat{+}Y) \neq \{0\}$, there exists a projection $Q \in \mathcal{M}$ such that $tr(Q) = a > 0$, and for any $\xi \in Q\mathcal{H}$, $(X\hat{+}Y)\xi = 0$. Then we can choose $\epsilon > 0$ and two projections $E, F \in \mathcal{M}$ such that $tr(E) = tr(F) > 1-\epsilon$, $E\mathcal{H} \subset \mathcal{D}(X)$, $F\mathcal{H} \subset \mathcal{D}(Y)$, and $tr(E \wedge F \wedge Q) > 0$. This means we can find $\xi \in (E \wedge F \wedge Q)\mathcal{H}$, this contradicts the fact that $ker(X+Y) = \{0\}$.*

In order to show that the results in chapter 3 are hold for any double triangle lattices in finite von Neumann algebras, we need the following lemma.

Lemma .0.6. *Suppose P_1, P_2, P_3 are three projections in $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ is a sparable Hilbert space. If $P_i \wedge P_j = 0$, and $P_i \vee P_j = I$, $i \neq j$, and the von Neumann algebra $\mathfrak{A}$ generated by these three projections is finite, then $P_1 \sim I - P_1$ in $\mathfrak{A}$, and $\mathfrak{A}$ is *-isomorphic to $\mathcal{M} \otimes M_2(\mathbb{C})$, where $\mathcal{M} \cong P_1\mathfrak{A}P_1$ is the relative commutant of $M_2(\mathbb{C})$ in $\mathfrak{A}$. Moreover, for any faithful normalized normal trace τ on $\mathfrak{A}$, we have $\tau(P_i) = \frac{1}{2}$.*

Proof. We only need to show that $P_1 \sim I - P_1$ in $\mathfrak{A}$. Let $\mathcal{W} = \mathfrak{A} * L_{\mathbb{Z}_2}$, the reduced free product of $\mathfrak{A}$ with group algebra $L_{\mathbb{Z}_2}$. Denote the group generator of $\mathbb{Z}_2$ by a. Then $\mathcal{W}$ is a finite von Neumann algebra, and P_1, $\frac{L_a+I}{2} = E$ are two trace $\frac{1}{2}$ free projections. Since the von Neumann algebra $\mathbb{N}$ generated by P_1 and E is *-isomorphic to $A \otimes M_2(\mathbb{C})$, where A is *-isomorphic to $L^\infty_{[0,1]}$, and $P_1 \sim I - P_1$ in $\mathcal{N}$. Thus we can choose matrix units $\{E_{i,j}\}_{i,j=1}^2$ in $\mathcal{W}$ such that $E_{1,1} = P_1$ and $\mathcal{W} \cong \mathcal{M} \otimes M_2(\mathbb{C})$, where $\mathcal{M} \cong P_1\mathcal{W}P_1$.

By Lemma A.0.5,

$$P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)}V_1 \\ V_1^*\sqrt{H_1(I-H_1)} & V_1^*(I-H_1)V_1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V_2 \\ V_2^*\sqrt{H_2(I-H_2)} & V_2^*(I-H_2)V_2 \end{pmatrix},$$

where V_1, V_2 are unitary operator in $\mathcal{W}$, H_1, H_2 are two positive operators, and $ker(I-H_i) = \{0\}(i = 1,2)$. By Lemma A.0.5, we have

$$ker(\sqrt{H_1(I-H_1)^{-1}}V_1 - \sqrt{H_2(I-H_2)^{-1}}V_2) = \{0\}.$$

Since

$$T = \begin{pmatrix} 0 & \sqrt{H_1(I-H_1)^{-1}}V_1 \hat{-} \sqrt{H_2(I-H_2)^{-1}}V_2) \\ 0 & 0 \end{pmatrix}$$

is affiliated with $\mathfrak{A}$, let $T = HU$ be the polar decomposition of T in $\mathfrak{A}$, then $U^*U = I - P_1$, $UU^* = P_1$, thus $P_1 \sim I - P_1$ in $\mathfrak{A}$. □

For any double triangle lattice $\mathcal{L} = \{0, P_1, P_2, P_3, I\}$ in a finite von Neumann algebra $\mathfrak{A}$, by the above lemma, we could assume that $\mathfrak{A} = \mathcal{M} \otimes M_2(\mathbb{C})$, and

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} H_1 & \sqrt{H_1(I-H_1)} \\ \sqrt{H_1(I-H_1)} & (I-H_1) \end{pmatrix},$$

$$P_3 = \begin{pmatrix} H_2 & \sqrt{H_2(I-H_2)}V \\ V^*\sqrt{H_2(I-H_2)} & V^*(I-H_2)V \end{pmatrix}.$$

Lemma .0.7. *Suppose $\mathcal{L} = \{0, P_1, P_2, P_3, I\}$ is a double triangle lattice in a finite von Neumann algebra. Without lose of generality, we could assume the von Neumann algebra generated by $\mathcal{L}$ is $\mathfrak{A} = \mathcal{M} \otimes M_2(\mathbb{C})$. With H_1, H_2 and V given above, let*

$$S = \sqrt{H_1(I-H_1)^{-1}} \hat{-} \sqrt{H_2(I-H_2)^{-1}}V$$

be an invertible (unbounded) operator affiliated with $\mathcal{M}$. If $T \in Alg(\mathcal{L})$, then there is an $A \in \mathcal{B}(P_1\mathcal{H})$ such that

$$T = \begin{pmatrix} A & \sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}} \\ 0 & S^{-1}AS \end{pmatrix}.$$

Conversely, if $A \in \mathcal{B}(P_1\mathcal{H})$ such that $S^{-1}AS$ and $\sqrt{H_1(I-H_1)^{-1}}S^{-1}AS - A\sqrt{H_1(I-H_1)^{-1}}$ are bounded operators, then T belongs to $Alg(\mathcal{L})$.

Proof. $T \in \mathcal{A}lg(\mathcal{L})$ if and only if $(I-P_i)TP_i = 0$, $i = 1,2,3$. So

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

and

$$\sqrt{I-H_1}T_2\sqrt{I-H_1} = \sqrt{H_1}T_3\sqrt{I-H_1} - \sqrt{I-H_1}T_1\sqrt{H_1}; \tag{11}$$

$$\sqrt{I-H_2}T_2V^*\sqrt{I-H_2} = \sqrt{H_2}VT_3V^*\sqrt{I-H_2} - \sqrt{I-H_2}T_1\sqrt{H_2}. \tag{12}$$

Since $\mathcal{D}(\sqrt{(I-H_1)^{-1}}) \cap \mathcal{D}(\sqrt{(I-H_2)^{-1}}V)$(and $\mathcal{R}(\sqrt{(I-H_1)^{-1}}) \cap \mathcal{R}(\sqrt{(I-H_2)^{-1}}V)$, $\mathcal{R}(A)$ is the range of operator A) is dense in $P_1\mathcal{H}$, (A.4), (A.5) are true if and only if $\forall \xi \in \mathcal{D}(\sqrt{(I-H_1)^{-1}}) \cap \mathcal{D}(\sqrt{(I-H_2)^{-1}}V)$,

$$T_2\xi = \sqrt{H_1(I-H_1)^{-1}}T_3\xi - T_1\sqrt{H_1(I-H_1)^{-1}}\xi, \tag{13}$$

$$T_2\xi = \sqrt{H_2(I-H_2)^{-1}}VT_3\xi - T_1\sqrt{H_2(I-H_2)^{-1}}V\xi. \tag{14}$$

This equivalent to (by lemma A.0.3)

$$T_2\xi = \sqrt{H_1(I-H_1)^{-1}}T_3\xi - T_1\sqrt{H_1(I-H_1)^{-1}}\xi, \tag{15}$$

$$T_3\xi = S^{-1}T_1S\xi. \tag{16}$$

□

Using the the same method in chapter 3, we have the following theorem.

Theorem .0.2. *Suppose $\mathcal{L} = \{0, P_1, P_2, P_3, I\}$ is a double triangle lattice in a finite von Neumann algebra. Without lose of generality, we could assume the von Neumann algebra generated by $\mathcal{L}$ is $\mathfrak{A} = \mathcal{M} \otimes M_2(\mathbb{C})$. Let $\{E_{i,j}\}_{i,j=1}^2$ be the canonical system of 2×2 matrix units in $M_2(\mathbb{C})$. Then $\mathcal{L}at(\mathcal{A}lg(\{P_1, P_2, P_3\}))$ is determined by the following: $P \in \mathcal{L}at(\mathcal{A}lg(\{P_1, P_2, P_3\}))$ and $P \neq 0, I, P_1$, if and only if*

$$P = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & U^*(I-K)U \end{pmatrix},$$

where K and U are uniquely determined by the following polar decomposition with any $a \in \mathbb{C}$: $(a+1)\sqrt{H_1(I-H_1)^{-1}} - a\sqrt{H_2(I-H_2)^{-1}}V = \sqrt{K(I-K)^{-1}}U$. And $\mathcal{L}at(\mathcal{A}lg(\{P_1, P_2, P_3\})) \setminus \{0, I\}$ is homeomorphic to S^2. Furthermore if the von Neumann algebra $\mathcal{N}$ generated by P_1, P_2, P_3 can not be generated by two projections then $\mathcal{A}lg(\{P_1, P_2, P_3\})$ is a KS-algebra.