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Kadison -Singer algebras with applications to von Neumann algebras

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KADISON-SINGER ALGEBRAS WITH APPLICATIONS TO VON NEUMANN ALGEBRAS

BY

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ABSTRACT

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I develop the theory of Kadison-Singer algebras, introduced recently by Ge and Yuan. I prove basic structure theorems, construct several new examples and explore connections to other areas of operator algebras. In chapter 1, I survey those aspects of the theory of non-selfadjoint algebras that are relevant to this work. In chapter 2, I define Kadison-Singer algebras and give different proofs of results of Ge-Yuan, which will be further extended in the last chapter. In chapter 3, I analyse in detail a class of elementary Kadison-Singer algebras that contain H^∞ and describe their lattices of projections. In chapter 4, I use ideas from free probability theory to construct Kadison-Singer algebras with core the free group factors $L(F_r)$ for $r < 2$. I then introduce two constructions that yield new Kadison-Singer algebras - The maximal join and the minimal join. In chapter 5, I analyse tensor products of Kadison-Singer algebras, showing that they are never Kadison-Singer. I then show how under certain conditions, one may construct a Kadison-Singer algebra with core the tensor product of the core of two given Kadison-Singer algebras.

CHAPTER 1

NON-SELFADJOINT OPERATOR ALGEBRAS

Given an operator T, a subspace $\mathcal{K}$ is invariant for T if T maps $\mathcal{K}$ into itself. Invariant subspaces determine triangular representations for an operator and if P is the orthogonal projection onto an invariant subspace for T, we have that $(I-P)TP = 0$. For operators on a finite dimensional vector space, we have a maximal nest of invariant subspaces, but it is unknown if every operator on an infinite dimensional Hilbert space has even a single invariant subspace.

The *invariant subspace problem* has been extensively studied, as have more general and more special versions of the problem. There are operators on Banach spaces that do not have invariant subspaces and large classes of operators on Hilbert spaces do have invariant subspaces. A weakening of the invariant subspace problem, the *Transitive algebra problem* asks if there are weak operator closed algebras other than $\mathcal{B}(\mathcal{H})$ that do not have common invariant subspaces. These two problems have inspired much research into the theory of operator algebras that are not closed under the adjoint operation, the so called *non-self adjoint* algebras.

Two fundamental notions in the theory of non-selfadjoint algebras are those of Alg and Lat. Let $\mathcal{L}$ be a set of orthogonal projections in $\mathcal{B}(\mathcal{H})$. Define $\text{Alg}(\mathcal{L}) = \{T \in \mathcal{B}(\mathcal{H}) : TP = PTP, \text{for all } P \in \mathcal{L}\}$. We have that $\text{Alg}(\mathcal{L})$ is a weak-operator closed subalgebra of $\mathcal{B}(\mathcal{H})$. At times, to avoid confusion, we will denote $\text{Alg}(\mathcal{L})$ by $\text{Alg}_{\mathcal{H}}(\mathcal{L})$ Similarly, for a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, define $\text{Lat}(\mathcal{S}) = \{P \in \mathcal{B}(\mathcal{H}) : P \text{ a projection, and } PTP = TP, \text{ for all } T \in \mathcal{S}\}$. Then

Lat($\mathcal{S}$) is a strong-operator closed lattice of projections. An algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is called a reflexive operator algebra if Alg(Lat($\mathcal{A}$)) $= \mathcal{A}$. Similarly, a lattice $\mathcal{L}$ is called a reflexive lattice if Lat(Alg($\mathcal{L}$)) $= \mathcal{L}$.

The study of non-self adjoint algebras, as a separate mathematical theory was initiated by Kadison and Singer in 1960 with their study of *Triangular algebras.* Given a Hilbert space $\mathcal{H}$ and a maximal abelian self-adjoint algebra(masa) $\mathcal{A}$, they called an algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ triangular if $\mathfrak{A} \cap (\mathfrak{A})^* = \mathcal{A}$. The masa $\mathcal{A}$ was denoted the *diagonal* of $\mathfrak{A}$.

Information on the strucutre of the triangular algebra can be gleaned by looking at the lattice of invariant projections of the algebra, all of which lie in the diagonal. The weak operator closed algebra generated by these projections is called the *core* of the algebra $\mathfrak{A}$. The core can be trivial, i.e., with the only invariant projections $\{0, I\}$, in which case the triangular algebra is called irreducible. The core, in certain cases, can also be equal to the diagonal, in which case, the triangular algebra is called hyperreducible. A simple Zorn's lemma argument shows that triangular algebras are contained in maximal triangular algebras and further, a result of Arveson from [1] shows that hyperreducible triangular algebras are contained in maximal hyperreducible triangular algebras. Kadison and Singer argued that parallel to masas giving bases of the Hilbert space $\mathcal{H}$, maximal hyperreducible triangular algebras corresponded to ordered bases for the Hilbert space.

The lattices of invariant projections of maximal triangular algebras are totally ordered (by inclusion) and closed in the strong operator topology. A *nest* is a complete chain of subspaces, which is equivalent to the collection of projections onto elements in the nest being totally ordered and closed in the strong operator toplogy. We will abuse notation and use the term *nest* to refer both to a collection of subspaces as above and a collection of projections as above.

In their paper, Kadison and Singer were able to classify all hyperreducible maximal triangular algebras by the structure of their lattice of invariant subspaces.

Theorem 1.0.1. *(Kadison-Singer,[15]) Let* ν *be a finite regular Borel measure on* $[0,1]$ *with* $\nu(\{0\}) = 0$. *The set* $\{P_t = M_{\chi[0,t)}, P_t^+ = M_{\chi[0,t]}\}$ *is a maximal nest* $\mathfrak{L}_0$ *in* $\mathcal{B}(L^2([0,1], d\nu)$.

Any multiplicity 1 *nest is unitarily equivalent to one such.*

In 1966, Ringrose generalized the class of hyperreducible maximal triangular algebras by introducing the class of nest algebras. As mentioned above, *nest* is a chain of closed subspaces of a Hilbert space $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$, which is closed under (arbitrary)intersection and closed union. For any closed subspace N of $\mathcal{H}$, let $P(N)$ denote the orthogonal projection onto N. Given a nest $\mathcal{N}$, the set $\mathfrak{L} = \{P(N) : N \in \mathcal{N}\}$ is a totally ordered(in the natural ordering in $\mathcal{B}(\mathcal{H})$) lattice of projections containing 0, I and closed in the strong-operator topology. Ringrose[27] showed that $\mathfrak{L}$ is a reflexive lattice. Alg($\mathfrak{L}$) is called the associated *nest algebra*.

A more general class of lattices is the class of the so called commutative subspace lattices. A commutative subspace lattice is a collection of closed subspaces of $\mathcal{H}$, closed under (arbitrary)intersection and closed union, containing $\{0\}$ and $\mathcal{H}$ and such that the projections onto any two elements in the collection commute. If $\mathfrak{L}$ is the collection of projections onto elements of a commutative subspace lattice, then a result of Arveson[1] says that $\mathfrak{L}$ is a reflexive lattice. Alg($\mathfrak{L}$) is called the associated *commutative subspace lattice*(CSL) algebra. Nest and CSL algebra are formally the simplest non-selfadjoint algebras and have been studied very extensively.

In this thesis, one of our key goals is to relate non-selfadjoint and self-adjoint algebras. A very natural way of doing this is the following: Minimal generating sets of projections in von Neumann algebras should correspond to maximal triangular non-selfadjoint algebras. The focus on projections is natural and has an extensive predigree. In the 30's, in a series of famous papers, von Neumann made an extensive study of *continuous geometries*. von Neumann started with the space of all projections in a matrix algebra and sought infinite dimensional analogues, in a manner parallel to the passage from type I factors to type II_1 factors. Continuous geometries in the sense of von Neumann were lattices of projections that satisfied among others, a complementation condition and a modularity condition. von Neumann was able to show that these axioms could be used to extract a dimension function on these continuous geometries. The lattice of all projections in a finite von Neumann algebra are an example of a continuous geometry. Minimal generating sets of projections can be viewed as *generators* of continuous geometries.

In the case of nest and CSL algebras, the generating sets of projections of the *core* are commutative and in particular distributive. Distributive lattices of projections are the simplest possible lattices of projections and their analysis is straightforward. Under mild conditions, these lattices are reflexive and the algebra of invariant operators contain abundantly many finite rank operators. The analysis of the invariant algebras is fundamentally related to the problem of spectral synthesis in classical Harmonic analysis. When one passes to cores that are not abelian, it is very hard to find minimal generating lattices that are non-distributive. The analysis requires very different tools, ideas from the theory of von Neumann algebras such as free probability theory.

CHAPTER 2

KADISON-SINGER ALGEBRAS

Ge and Yuan, in [8] introduced the class of *Kadison-Singer* algebras. These are maximal reflexive algebras containing a prescribed von Neumann algebra as diagonal. There were several motivations for the work. The first motivation was to seek a generalization of triangular algebras as has been mentioned in the introduction. One well known prior generalization was due to Muhly, Saito and Solel. These authors studied triangular algebras *within* a von Neumann algebra in the following way. Given a von Neumann algebra $\mathfrak{M}$ and a maximal abelian subalgebra $\mathcal{A}$, an algebra $\mathfrak{A} \subset \mathfrak{M}$ is said to be triangular if

$$\mathfrak{A} \cap (\mathfrak{A})^* = \mathcal{A}$$

We now give the definition of a Kadison-Singer algebra, note the duality between algebras and lattices and prove a basic proposition that the lattices of invariant projections of Kadison-Singer algebra have some nice properties.

Definition 2.0.1. *Given a von Neumann subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, an algebra $\mathfrak{A}$, $\mathfrak{M} \subseteq \mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ is Kadison-Singer with diagonal $\mathfrak{M}$ if*

1. $\mathfrak{A} \cap \mathfrak{A}^* = \mathfrak{M}$
2. $\mathfrak{A}$ *is reflexive.*
3. $\mathfrak{A}$ *is maximal with respect to* 1 *and* 2.

The lattice of invariant projections of a Kadison-Singer algebra(from now on denoted KS algebra) corresponds to a minimal generating lattice of the commutant of the diagonal algebra, which is a von Neumann algebra. We formalize this notion.

Definition 2.0.2. *Given a von Neumann subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, a lattice $\mathfrak{L}$ in $\mathfrak{M}$ is a Kadison-Singer lattice(from now on denoted KS lattice) for $\mathfrak{M}$ if*

1. $(\mathfrak{L})'' = \mathfrak{M}$
2. *$\mathfrak{L}$ is a reflexive lattice.*
3. *$\mathfrak{L}$ is minimal with respect to* 1 *and* 2*.*

The reflexivity of a lattice of projections in a (von Neumann)algebra is a notion that depends upon the particular representation we choose. The following theorem shows, however, that when $\mathfrak{M}$ is a factor, we may speak of KS lattices without reference to a representation. By the representation of a von Neumann algebra, we will mean a normal $*$ preserving injective homomorphism. Recall the following fundamental fact about representations of von Neumann algebras[30, Theorem IV.5.5].

Theorem 2.0.2. *Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{K})$ be a representation. Then, there is a Hilbert space $\overline{\mathcal{K}}$ and a projection P_π in $\mathcal{B}(\mathcal{H} \otimes \overline{\mathcal{K}})$ such that P_π commutes with $\mathfrak{M} \otimes I \subset \mathcal{B}(\mathcal{H} \otimes \overline{\mathcal{K}})$ and π is unitarily equivalent to the representation $\mathfrak{M} \to P_\pi \mathcal{B}(\mathcal{H} \otimes \overline{\mathcal{K}}) P_\pi$ given by $T \to P_\pi(T \otimes I)$.*

Thus, any representation can be derived from the given imbedding through an amplification and then, a cut down using a projection in the commutant. This allows us to prove the following theorem,

Theorem 2.0.3. *Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\mathfrak{L} \subset \mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a KS lattice for $\mathfrak{M}$. Then, for any other faithful representation $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{K})$, $\pi(\mathfrak{L})$ is a KS lattice in $\mathcal{B}(\mathcal{K})$.*

Proof. Let $\mathcal{H}_1$ be a Hilbert space and consider the representation $\tilde{\pi} : \mathfrak{M} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_1)$ given by $\tilde{\pi}(\cdot) = \pi(\cdot) \otimes I_{\mathcal{H}_1}$. A simple calculation gives us that

$$\mathrm{Lat}(\mathrm{Alg}((\pi \otimes I)(\mathfrak{L}))) = \mathrm{Lat}(\mathrm{Alg}(\pi(\mathfrak{L})) \otimes I_{\mathcal{H}_1}$$

and thus, $(\pi \otimes I)(\mathfrak{L})$ is reflexive if and only if $\pi(\mathfrak{L})$ is. Thus, the reflexivity(or the abscence of reflexivity) of $\mathfrak{L}$ is not changed under dilations of representations.

Next, let P be a projection in $\pi(\mathfrak{M})'$, such that the map from $\pi(\mathfrak{M}) \to P\pi(\mathfrak{M})$ given by $\pi(x) \to P\pi(x)$ for $x \in \mathfrak{M}$ is faithful. Consider the representation $\tilde{\pi} = \pi \mid_{P\mathcal{H}}$. Another simple calculation gives us that

$$P\,\mathrm{Lat}(\mathrm{Alg}_{\mathcal{H}}(\pi(\mathfrak{L}))) = \mathrm{Lat}(\mathrm{Alg}_{P\mathcal{H}}(\pi_1(\mathfrak{L})))$$

Thus, $\pi_1(\mathfrak{L})$ is reflexive if and only if $\pi(\mathfrak{L})$ is. Therefore, the reflexivity(or the abscence of reflexivity) of $\mathfrak{L}$ is not changed under the compression of representations.

Thus, the reflexivity of a lattice in a von Neumann algebra does not depend upon the particular faithful representation chosen. And since the property that a subset generates a von Neumann algebra is representation independent, the theorem follows. □

We note here a couple of basic fact for future reference.

Remark 2.0.1. *Let $\mathcal{N}$ be a nest in $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{L}$ be the set of projections onto elements in $\mathcal{N}$. It is a classical theorem of Ringrose[27], that $\mathfrak{L}$ is a reflexive lattice. If $\mathfrak{K} \subsetneq \mathfrak{L}$ is any SOT closed sublattice, then $(\mathfrak{K})'' \subsetneq (\mathfrak{L})''$. Thus, $\mathfrak{L}$ is a KS lattice and* $\mathrm{Alg}(\mathfrak{L})$ *(the associated nest algebra) is a KS algebra with diagonal $\mathfrak{L}'$.*

Remark 2.0.2. *Let $\mathfrak{A}$ be a CSL algebra that is not a nest algebra. Let $\mathfrak{L} = \mathrm{Lat}(\mathfrak{A})$. Write $\mathfrak{L} = \{P_\lambda : \lambda \in \Lambda\}$. Since $\mathfrak{L}$ is not totally ordered, there are projections that are not order-comparable to every other element in $\mathfrak{L}$. Pick such a projection P_0 in $\mathfrak{L}$ and let $\mathfrak{L}_1$ be the lattice generated by $\{P : P \in \mathfrak{L}, P \geq P_0 \text{ or } P \leq P_0\}$. For any projection $P \in \mathfrak{L}$, $P = P \vee P_0 + PP_0 - P_0$ and since each of the projections on the right hand side is in $(\mathfrak{L})''$, so is P. Thus, $\mathfrak{L}_1$ is also a generating lattice for $\mathfrak{A}$. It is easy to see that every operator of the form $T = P_0T(I - P_0)$ is in* $\mathrm{Alg}(\mathfrak{L}_1)$, *i.e.* $P_0\mathcal{B}(\mathcal{H})(I - P_0) \in \mathrm{Alg}(\mathfrak{L}_1)$.

However, letting P_1 be a projection in $\mathfrak{L}$ that is not order-comparable to P_0, we see that $(I - P_1)P_0\mathcal{B}(\mathcal{H})(I - P_0)P_1 \neq 0$ and thus, $\mathrm{Alg}(\mathfrak{L}) \subsetneq \mathrm{Alg}(\mathfrak{L}_1)$. *i.e. $\mathfrak{A}$ is not a Kadison-Singer algebra.*

More, in fact is true. Not only is a CSL algebra not a Kadison-Singer algebra, it is actually contained in a nest algebra with the same diagonal. One may extract a minimal generating nest from the Commutative Subspace Lattice, using Arveson's representation theorem.

Theorem 2.0.4. *Given a CSL algebra $\mathfrak{A}$, there is a nest algebra $\mathfrak{B}$ containing $\mathfrak{A}$ such that $\mathfrak{B}^* \cap \mathfrak{B} = \mathfrak{A}^* \cap \mathfrak{A}$. i.e., every CSL algebra is contained in a Kadison-Singer algebra with the same diagonal.*

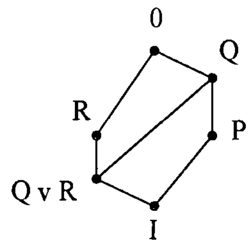
The proof is a standard application of Arveson's representation theorem and follows almost immediately from [5, Lemma 22.3] and the discussion preceding it. We omit it.

2.1 A general contruction of Kadison-Singer algebras due to Ge-Yuan

In their first paper, Ge and Yuan gave a general construction for Kadison-Singer algebras. It exploited the fact that for certain lattices of projections, reflexivity can be determined from the lattice structure. A lattice(of projections) $\mathfrak{L}$ is traditionally called a small lattice if it is finite. A lattice is said to be distributive if for each triple P, Q, R in $\mathfrak{L}$,

$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

The distributivity condition is equivalent to saying that there is a *model* for the lattice as Borel subsets of a measure space. A theorem of Longstaff says that every small, distributive lattice is reflexive in any representation. In particular, the six element lattice $\{0, P, Q, R, Q \vee R, I\}$ with $Q < P$, $P \vee R = I$ and $P \wedge R = 0$ is reflexive in any representation.



Let now $\mathfrak{N}$ be a von Neumann algebra generated by a projection q and a self-adjoint element H that does not have 1 in its point spectrum. Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C})$ and let P be the projection I in $\mathfrak{M} \otimes e_{11}$. Let Q be the projection $q \otimes e_{11}$ and let R be the projection $H \otimes e_{11} + \sqrt{H(I-H)} \otimes e_{12} + \sqrt{H(I-H)} \otimes e_{21} + (I-H) \otimes e_{22}$. Or, in matrix form,

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & (I-H) \end{pmatrix}.$$

We have that $Q < P$, $P \wedge R = 0$, $P \vee R = I$. Thus, the lattice generated by P, Q and R is precisely, $\mathfrak{L} = \{0, P, Q, R, Q \vee R, I\}$ and this lattice is reflexive as noted above. The only strict sublattices are

1. $\{0, R, Q \vee R, I\}$
2. $\{0, Q, P, I\}$
3. $\{0, P, Q \vee R, I\}$
4. $\{0, P, Q, Q \vee R, I\}$
5. Three element lattices $\{0, P, I\}$, etc.

The first lattice has R in the commutant and lattices 2, 3, 4 have Q in the commutant. Since the original lattice has neither in the commutant, the algebra generated by each of the four sublattices is strictly larger. Also, the lattices in 5 generate a two dimensional subalgebra, which clearly cannot be $\mathfrak{M}$.This shows that the lattice $\mathfrak{L}$ is a Kadison-Singer lattice.

We summarize this,

Theorem 2.1.1. *Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C})$, be a von Neumann algebra where $\mathfrak{N}$ is a von Neumann algebra generated by a projection and a self-adjoint element. Then, there is a Kadison-Singer algebra with core $\mathfrak{M}$.*

The utility of the construction comes from the fact that a very large number of von Neumann algebras can be realised in the following manner. We have the following proposition,

Proposition 2.1.1. *Any von Neumann algebra can be realised as the diagonal of a Kadison-Singer algebra.*

Proof. Let $\mathfrak{M}$ be a von Neumann algebra. By a theorem of Haagerup, any von Neumann algebra $\mathfrak{M}$ has a *Standard form* representation $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$, which has the property that $\pi(\mathfrak{M})$ is anti-isomorphic to its commutant, $\pi(\mathfrak{M})'$, which we denote by $\mathfrak{N}$. It is a classical result that if $\mathfrak{M}$ is a von Neumann algebra, then $\mathfrak{M}\bar{\otimes}\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional is generated by one self-adjoint element and a projection.

Let θ be the representation $\pi \otimes I_{\mathcal{H}} \otimes I_2$, of $\mathfrak{M}$ on $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^{2n})$. The commutant of $\theta(\mathfrak{M})$ is equal to $\mathfrak{N} \otimes \mathcal{B}(\mathcal{H}) \otimes M_2(\mathbb{C})$. Note that $\mathfrak{N}\bar{\otimes}\mathcal{B}(\mathcal{H})$ is generated by a single self-adjoint element and a projection. Thus, the hypothesis of theorem(2.1.1) are satisfied and we have a Kadison-Singer algebra with core $\mathfrak{N}\bar{\otimes}\mathcal{B}(\mathcal{H}) \otimes M_2(\mathbb{C})$, that is, a Kadison-Singer algebra with diagonal $\mathfrak{M}$. □

2.2 Ge-Yuan's construction for hyperfinite factors

Write the hyperfinite II_1 factor R as the closure of the algebra $\otimes_{n=1}^{\infty} M_m(\mathbb{C})$ under the product trace. Let $\{E_{ij}\}$, $1 \leq i, j \leq m$ denote a standard set of matrix units for $M_n(\mathbb{C})$.

Let, for $1 \leq l \leq n-1$, $P_i = \sum_{l=i}^{i} E_{ll}$ and $Q = \frac{1}{m} \sum_{l,k=1}^{m} E_{lk}$. Let us define recursively,

$$\mathfrak{L}_1^m = \{P_1, \cdots, P_{m-1}, Q\}$$

$$\mathfrak{L}_2^m = \{Q \otimes \mathfrak{L}_1^m, P_1 + Q \otimes \mathfrak{L}_1^m, \cdots, P_{n-1} + Q \otimes \mathfrak{L}_1^m\}$$

$$\mathfrak{L}_3^m = \{Q \otimes \mathfrak{L}_2^m, P_1 + Q \otimes \mathfrak{L}_2^m, \cdots, P_{n-1} + Q \otimes \mathfrak{L}_2^m\}$$

$\vdots$

$$\mathfrak{L}_{k+1}^m = \{Q \otimes \mathfrak{L}_k^m, P_1 + Q \otimes \mathfrak{L}_k^m, \cdots, P_{n-1} + Q \otimes \mathfrak{L}_k^m\}$$

$\vdots$

$$\mathfrak{L}^m = \bigcup_{n=1}^{\infty} \mathfrak{L}_n^m$$

Then, Ge-Yuan proved the following theorem.

Theorem 2.2.1. *$\mathfrak{L}^m$ is a Kadison-Singer lattice for R. Further, $\mathfrak{L}^m$ is not isomorphic to $\mathfrak{L}^n$ for $m \neq n$.*

They were able to describe the reflexive closure of $\mathfrak{L}^m$ explicitly and showed the following.

Theorem 2.2.2. *Every projection P in $\mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}^m))$ has a representation as a sequence, $P \sim (a_1, a_2, \cdots,)$ where $a_n \in \{0, 1, \cdots, m-1\}$ with $\tau(P) = \sum_{n=1}^{\infty} \frac{a_n}{m^n}$. In particular, there are two projections with trace any m'ary fraction in $(0,1)$ and one projection with trace any non-m'ary fraction in $(0,1)$. Writing $a_n = P(n)$, we have that for two projections $P, Q \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}^m))$, $P \leq Q$ iff*

$$P(n) \leq Q(n) \quad \text{for} \quad \forall n$$

Further, $P \vee Q$ and $P \wedge Q$ are the co-ordinatewise max *and* min

We note here a simple consequence that we will have occasion to use later.

Proposition 2.2.1. *1. For any $\epsilon > 0$, there are non-trivial projections(i.e. not equal to 0 or I) $Q, (P_\lambda)_{\lambda \in \Lambda}$, with $\tau(Q) \leq \epsilon$ satisfying the following two conditions*

a) For each $\lambda \in \Lambda$, $(\bigvee_{P \in \mathfrak{L}, P \ngeq P_\lambda} P) < I$.

b) For any projection P in $\mathfrak{L}^m$, either $P \leq Q$ or $P \geq P_\lambda$ for some $\lambda \in \Lambda$.

2. For any $\epsilon > 0$, there are non-trivial projections(i.e. not equal to 0 or I) $Q, (P_\lambda)_{\lambda \in \Lambda}$, with $\tau(Q) \geq 1 - \epsilon$ satisfying the following two conditions

a) For each $\lambda \in \Lambda$, $(\bigwedge_{P \in \mathfrak{L}, P \nleq P_\lambda} P) > 0$.

b) For any projection P in $\mathfrak{L}^m$, either $P \geq Q$ or $P \leq P_\lambda$ for some $\lambda \in \Lambda$.

Proof. Choose k such that $\frac{1}{m^k} < \epsilon$ and let $Q \sim (0, \cdots, 0, 0, \overset{k+1}{m-1}, m-1, m-1, \cdots)$ and $P_i \sim (0, \cdots, 0, 0, \overset{i}{1}, 0, 0, 0, \cdots)$ for $1 \leq i \leq k-1$. Then, we claim that $\{P_i\}_{i=1}^{k-1}$ and Q will do the job for us.

Any projection R with associated sequence zero in the first k entries will satisfy $R \leq Q$ and any other projection S must have at least one of its first k entries non-zero, say, the i'th one and thus, $S \geq P_i$. Thus, we conclude that for any $R \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}^m))$,

$$\text{Either} \quad R \leq Q \quad \text{or} \quad R \geq P_i \quad \text{for some } 1 \leq i < k$$

Further, $\bigvee_{P \gtrsim P_i} P \sim (1, 1, \cdots, \overset{i}{0}, 1, \cdots)$ and thus, $\bigvee_{P \gtrsim P_i} P < I$. We see that the conditions in (1) are satisfied. The verification of (2) is similar. □

In section (4), we will introduce two constructions - The tensor product and the join of two Kadison-Singer lattices, one of them satisfying a certain technical condition, which we call property F. The previous proposition verifies that the Kadison-Singer lattices for the hyperfinite II_1 factor satisfies property F.

2.3 A new proof of the previous result

Ge-Yuan proved the previous theorem by carefully analysing the algebra of invariant operators and showing that any larger algebra would have a larger diagonal. The same result can also be obtained by looking at the lattice $\mathfrak{L}_m$ and determining what algebras sublattices of the lattice generate. The proofs are very different in nature and we include it because it was the inspiration for the major theorems in this thesis, as proved in section 4. Another key point is the following: It is an insight of Liming Ge that the study of Kadison-Singer algebras is intimately related to generator problems in von Neumann algebras and that this point of view should form a useful link between non-selfadjoint and selfadjoint theory.

We only prove this theorem in the case when $m = 2$. The proof when $m > 2$ is more intricate, but the basic idea is similar.

Proposition 2.3.1. *Let $\mathfrak{L}_2$ be defined as above. Then the lattice is reflexive.*

Proof. We only outline the proof. The proof has the following steps:

1. Any projection R in the reflexive closure satisfies: Either $R \leq Q$ or $R \geq P$. We may thus write $R = P + (I - P) \otimes R_1$ in the first case or $R = Q \otimes R_1$ in the second.

2. R_1 satisfies: Either $R_1 \leq I \otimes Q$ or $R \geq I \otimes P$. We may thus write $R_1 = 1 \otimes P + 1 \otimes (I - P) \otimes R_2$ in the first case or $R_1 = I \otimes Q \otimes R_2$ in the second.

3. Suppose the trace of the projection R satisfies $\frac{k}{2^n} \leq \tau(R) \leq \frac{k+1}{2^n}$ for some $0 \leq k \leq 2^n - 1$. Let P_1, P_2 be the two projections in $\mathfrak{L}_2$ of trace $\frac{k}{2^n}$. Then, $P_i \leq R$ for some $i \in \{1, 2\}$ and thus, $||R - P_i||_2 \leq \frac{1}{2^n}$.

4. The lattice $\mathfrak{L}_2$ is SOT closed and thus, we can conclude from all this that

$$\mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}_2)) = \mathfrak{L}_2$$

.

The proposition follows. □

Theorem 2.3.1. *$\mathfrak{L}_2$ is a Kadison-Singer lattice.*

Proof. Let $\mathfrak{K}$ be a reflexive sublattice that generates R. We claim that $P \otimes I$ and $Q \otimes I$ both belong to $\mathfrak{K}$. First, we note that any projection in $\mathfrak{K}$ is either less than $Q \otimes I$ or is greater than $P \otimes I$. Suppose $Q \otimes I$ does not belong to $\mathfrak{K}$, let Q_1 be the supremum of all the elements less than $Q \otimes I$. Since $\mathfrak{K}$ is SOT closed, $Q_1 \lneq Q \otimes I$. Then, we see that $(I - Q_1) \wedge (P \otimes I)$ is a nontrivial projection in R that commutes with $\mathfrak{K}$. Thus, $\mathfrak{K}$ is not minimal generating. A similar argument shows that $P \otimes I$ must belong to any minimal generating lattice as well.

An inductive argument shows that the twin projections corresponding to each finite binary sequence must belong to any minimal generating lattice and since reflexive lattices are SOT closed, we have that the conclusion extends to all binary sequences, i.e $\mathfrak{K} \supset \mathfrak{L}_2$,. The conclusion follows. □

2.4 Ge-Yuan's construction with three projections

In [7], Ge and Yuan gave a concrete construction for non-hyperfinite type II_1 factors. The most interesting case deals with three free trace $\frac{1}{2}$ projections that generate $L(F_{1\frac{1}{2}})$. Write these three projections as

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & I-H \end{pmatrix}$$

$$Q = \begin{pmatrix} K & \sqrt{K(1-K)}U \\ U^*\sqrt{K(1-K)} & U^*(I-K)U \end{pmatrix}$$

Theorem 2.4.1. *(Ge-Yuan)* $\mathrm{Alg}(\{P,Q,R\})$ *is a Kadison-Singer algebra. Further,*

$$\mathrm{Lat}(\mathrm{Alg}(\{P,Q,R\})) = \{P_\lambda \cup P : \lambda \in \mathbb{C}\}$$

, where $P_\lambda = \begin{pmatrix} L & \sqrt{L(1-L)}V \\ V^*\sqrt{L(1-L)} & V^*(I-L)V \end{pmatrix}$, *with* L, V *determined by*

$$\sqrt{L}\sqrt{(I-L)^{-1}}V = \lambda\sqrt{H}\sqrt{(I-H)^{-1}} + (1-\lambda)\sqrt{K}\sqrt{(I-K)^{-1}}U$$

The lattice $\mathrm{Lat}(\mathrm{Alg}(\{P,Q,R\}))$*(endowed with the SOT) is topologically homeomorphic to* S^2*. Also, for any two elements in the lattice,* P_1, P_2*, we have that* $P_1 \vee P_2 = I$ *and* $P_1 \wedge P_2 = 0$ *and in particular, each element in the lattice has trace* $\frac{1}{2}$.

The Kadison-Singer lattice, above, of Ge-Yuan, is extremely non-distributive and is further formalized as an example of a *KS manifold*, about which we will have to say later. There is nothing special about the freeness in the theorem and one can perform the same construction, after a couple of adjustments for any von Neumann algebra generated by three projections $\{P_1, P_2, P_3\}$ satisfying $P_i \vee P_j = I$ and $P_i \wedge P_j = 0$ for $1 \le i \ne j \le 3$.

One consequence of this is that we are assured of the existence of Kadison-Singer lattices for $L(F_r)$, with $r \le \frac{3}{2}$. In a subsequent section, we will show how we can in fact construct Kadison-Singer lattices for $r < 2$. It turns out that $r = 2$ is a critical value, where simple methods of constructing Kadison-Singer lattices break down.

CHAPTER 3

A FAMILY OF ELEMENTARY KADISON-SINGER ALGEBRAS

From this chapter onwards, given our interest in von Neumann algebras, we will focus upon lattices rather than algebras. A natural attempt to get minimal lattices is the following: Look at the lattices generated by nests and a single projection and analyse the relefexivity and generating properties of these. To get things started, we assume that that the extending projection is in fact rank 1. We creatively call such lattices one point extensions of nests. In this chapter, we will prove a series of more and more general theorems, culminating in a theorem about the extension of a general nest by a rank one projection is Kadison-Singer. Later, we will study more general extensions in the setting of the Hardy space.

3.1 Lattices generated by a maximal multiplicity 1 nest and a one dimensional projection

Recall the following theorem of Kadison and Singer on a representation theorem for uniform multiplicity 1 nests(i.e. those whose associated projection lattice generates a masa in $\mathcal{B}(\mathcal{H})$).

Theorem 3.1.1. *(Kadison-Singer,[15]) Let ν be a finite regular Borel measure on $[0,1]$ with $\nu(\{0\}) = 0$. The set $\{P_t = M_{\chi[0,t)}, P_t^+ = M_{\chi[0,t]}\}$ is a maximal nest $\mathfrak{L}_0$ in $\mathcal{B}(L^2([0,1], d\nu)$. Any maximal multiplicity 1 nest is unitarily equivalent to one such.*

Note that the projections P_t^+ appear in the theorem to account for point masses in the measure μ. Now, fix a finite regular Borel measure ν on $[0,1]$ as above and let $\mathfrak{L}_0$ be as above. Note that this nest may contain atoms, i.e. when $P_t < P_t^+$. Let f be a function in $\mathcal{H} = L^2([0,1], d\nu)$ such that $f \neq 0$ a.e. with respect to the measure ν. Let P_f be the projection onto the one dimensional subspace spanned by f. We will prove the following theorem:

Theorem 3.1.2. *The lattice $\mathfrak{L}$ generated by $\{\mathfrak{L}_0, P_f\}$ i.e. by $\{P_t, P_t^+, P_f : t \in [0,1]\}$ is a Kadison-Singer lattice with trivial diagonal.*

Any element in the commutant of $\mathfrak{L}_0$ is of the form M_g for an L^∞ function g and if M_g commutes with P_f, there is a scalar α, so that $M_g f = \alpha f$. Then, $(g - \alpha)f = 0$ a.e. and thus $g = \alpha$ a.e. Therefore the commutant of $\mathfrak{L}$ is trivial and hence $\mathrm{Alg}(\mathfrak{L})$ is an algebra with trivial diagonal.

We will use the following standard notation for rank one operators. Given vectors ξ and η, the symbol $\eta \otimes \xi$ will denote the operator

$$(\eta \otimes \xi)(\zeta) = <\zeta, \eta> \xi$$

Lemma 3.1.1. $\mathfrak{L} = \{P, P_f, P \vee P_f : P \in \mathfrak{L}_0\}$

Proof. We need to check that the above lattice is closed under the lattice operations. We note that since $f \neq 0$ a.e., for any $P \in \mathfrak{L}_0$, we have that $P \wedge P_f = 0$. To prove our lemma, the only thing we need to check is that if $P, Q \in \mathfrak{L}$ with $P < Q < I$ than $(P \vee P_f) \wedge Q = P$.

Assume the contrary. If $(P \vee P_f) \wedge Q \neq P$, there is a vector h in $\mathrm{Ran}(Q) \setminus \mathrm{Ran}(P)$ such that $h = \alpha f + \beta g$ for some $g \in \mathrm{Ran}(P)$ where $\alpha \neq 0, \beta$ are scalars. Thus, $f = \dfrac{h - \beta g}{\alpha}$ and since $(I - Q)\dfrac{h - \beta g}{\alpha} = 0$, we have that $(I - Q)f = 0$, which is impossible by our assumption that $f \neq 0$ a.e. Hence, $(P \vee P_f) \wedge Q = P$ and we have that

$$\mathfrak{L} = \{P, P_f, P \vee P_f : P \in \mathfrak{L}_0\}$$

□

Lemma 3.1.2. *The lattice $\mathfrak{L}$ is minimal generating, i.e, if $\mathfrak{K}$ is a strict sublattice, $(\mathfrak{K})'' \subsetneq (\mathfrak{L})'' = \mathcal{B}(\mathcal{H})$*

Proof. Let $\mathfrak{K} \subsetneq \mathfrak{L}$ be a strict sublattice. The proof splits into two cases.

Case 1: Suppose first that $\mathfrak{K}$ does not contain P_f. Let $P_0 = \wedge\{P \in \mathfrak{L}_0 : P \vee P_f \in \mathfrak{K}\}$. Since $P_f \notin \mathfrak{K}$, $P_0 > 0$. Then the operator P_0 commutes with $\mathfrak{K}$ and hence, $(\mathfrak{K})'' \subsetneq (\mathfrak{L})'' = \mathcal{B}(\mathcal{H})$.

Case 2: Suppose that $P_f \in \mathfrak{K}$. Let $P_0 = \wedge\{P \mid P \in \mathfrak{L}_0 \text{ and } P \in \mathfrak{K}\}$. Assume first that $P_0 \neq 0$. If $Q \in \mathfrak{L}_0$ such that $Q \vee P_f \in \mathfrak{K}$, then $P_0 \wedge (Q \vee P_f) = P_0 \wedge Q$ is in $\mathfrak{K}$ as well. Hence $Q \geq P_0$. Let $\mathfrak{L}_1 = \{P \in \mathfrak{L}_0 : P \vee P_f \in \mathfrak{K}\}$. $\mathfrak{L}_1$ is a strict sublattice of $\mathfrak{L}_0$. We see that $\mathfrak{K} = \{P, P_f, P \vee P_f : P \in \mathfrak{L}_1\}$. The same conclusion holds if $P_0 = 0$, by taking a decreasing net of projections P_α in $\mathfrak{L}_0$, decreasing to 0.

Now, It is easy to see that there are projections $P_1 < P_2 \in \mathfrak{L}_0 \setminus \mathfrak{L}_1$, such that if $P_1 < P < P_2$, then $P \notin \mathfrak{L}_1$. Pick an operator $T = (P_2 - P_1)T(P_2 - P_1)$ with $Tf = 0$ and we see that T commutes with $\mathfrak{K}$. Thus, $(\mathfrak{K})'' \subsetneq (\mathfrak{L})'' = \mathcal{B}(\mathcal{H})$

Thus, $\mathfrak{L}$ is minimal generating. □

We note the following elementary fact concerning reflexive lattices. Let $\mathfrak{L}$ be a lattice of projections. Then, the lattice is reflexive iff for every vector g, the projection onto the subspace $\overline{\mathrm{Alg}(\mathfrak{L})g}$ lies in $\mathfrak{L}$.

Lemma 3.1.3. *The lattice $\mathfrak{L}$ is reflexive.*

Proof. We now show that the lattice is reflexive. Let $g \in L^2([0,1], d\nu)$. Let P_0 be the smallest projection in the nest $\mathfrak{L}_0$ so that $P_0 g = g$. If $P_0 = I$, let Q_0 be the smallest projection in the nest so that $(I - Q_0)g = \alpha(I - Q_0)f$ for some scalar α. We will show that

Case 1: If $P_0 = I$ and $Q_0 = I$, $\overline{\mathrm{Alg}(\mathfrak{L})g} = \mathcal{H}$

Case 2: If $P_0 < I$, $\overline{\mathrm{Alg}(\mathfrak{L})g} = \mathrm{Ran}(P_0)$.

Case 3: If $P_0 = I$ and $Q_0 < I$, $\overline{\mathrm{Alg}(\mathfrak{L})g} = \mathrm{Ran}(Q_0 \vee P_f)$

By the note before the statement of this theorem, this will prove that $\mathfrak{L}$ is reflexive. We handle the three cases below.

Case 1: Suppose $P_0 = Q_0 = I$. Let $P \in \mathfrak{L}_0$. Any operator of the form $k \otimes h$, where $h \in \mathrm{Ran}(P)$, $k \in \mathrm{Ran}(I-P)$ and $< k, f >= 0$, is invariant under $\mathfrak{L}$. For any $P < I$, we may find $k = (I-P)k$ such that $< k, f >= 0$, but $< k, g >\neq 0$.(Otherwise $< k, f >= 0 \Rightarrow < k, g >= 0 \Rightarrow (I-P)f = (I-P)g \Rightarrow Q_0 < I)$. Therefore,

$$\overline{\mathrm{Alg}(\mathfrak{L})g} \supset \overline{\mathrm{Span}(\{h \mid h \in \cup_{P<I}\mathrm{Ran}(P)\} \cup \{g\})}$$

Thus,

$$\overline{\mathrm{Alg}(\mathfrak{L})g} = \mathcal{H}$$

Case 2: Suppose $P_0 < I$. Then, $\overline{\mathrm{Alg}(\mathfrak{L})g} \subset \overline{\mathrm{Alg}(\mathfrak{L}_0)g} \subset \mathrm{Ran}(P_0)$. For the reverse inequality, note that we may choose for $P < P_0$, $h \in \mathrm{Ran}(P), k \in \mathrm{Ran}(I-P)$, $< k, f >= 0$, but $< k, g >\neq 0$. Then $k \otimes h$ is in $\mathrm{Alg}(\mathfrak{K})$ and $(k \otimes h)g =< g, k > h$. Thus, $\overline{\mathrm{Alg}(\mathfrak{L})g} \supset \overline{\mathrm{Span}\{(h \mid h \in \cup_{\mathrm{Ran}(P):P<P_0}\} \cup \{g\})}$. Thus,

$$\overline{\mathrm{Alg}(\mathfrak{L})g} = \mathrm{Ran}(P_0)$$

Case 3: Suppose $P_0 = I, Q_0 < I$. Then, by the definition of $(I - Q_0)g = \alpha f$ for some scalar α. Let $g_1 = g - \alpha f$. Noting the argument in the second case, we see that $\overline{\mathrm{Alg}(\mathfrak{L})g} \subset \mathrm{Ran}(Q_0 \vee P_f)$. The reverse inequality is very similar to the computation in Case 1 and we omit it. We conclude that

$$\overline{\mathrm{Alg}(\mathfrak{L})g} = \mathrm{Ran}(Q_0 \vee P_f)$$

$\square$

3.2 Extending nests by a rank 1 projection

Let $\mathfrak{L}_0$ be a nest of projections in $\mathcal{B}(\mathcal{H})$. Since the algebra $(\mathfrak{L}_0)''$ is abelian, we may find a separating unit vector Ω for this algebra. The map from $\mathfrak{L}_0$ to $[0,1]$ given by $P \to < P\Omega, \Omega >$ for P in $\mathfrak{L}_0$ is a topological homeomorphism from $\mathfrak{L}_0$ endowed with the SOT to $[0,1]$ with the natural metric topology. The image of $\mathfrak{L}_0$ under this map is a compact subset of $[0,1]$, which we denote by $\mathcal{S}$. We will use $\mathcal{S}$ to index elements in $\mathfrak{L}_0$ in the natural way, with P_t representing the projection in $\mathfrak{L}_0$ such that $< P_t\Omega, \Omega >= t$. We thus write $\mathfrak{L}_0 = \{P_t : t \in \mathcal{S} \subset [0,1]$. In this identification, P_0 is 0 and P_1 is $I_{\mathcal{H}}$. Also, given a vector ξ in $\mathcal{H}$, we will denote the rank one projection onto the span of ξ by P_ξ.

Consider then, a nest $\mathfrak{L}_0 = \{P_t : t \in \mathcal{S} \subset [0,1]\}$(with $P_0 = 0_{\mathcal{H}}$ and $P_1 = I_{\mathcal{H}}$) and indexed as above, where $\mathcal{S}$ is a compact subset of $[0,1]$ and a rank one projection P_ξ. Let $\mathfrak{L}$ be the lattice generated by the elements in $\{\mathfrak{L}_0, P_\xi\}$. We will show that under mild conditions $\mathfrak{L}$ is a Kadison-Singer lattice. We note two necessary conditions needed for this consequence. Recall that a Kadison-Singer lattice is a minimal generating reflexive lattice for the von Neumann algebra it generates.

We assume first of all that P_ξ is not order comparable to every element in $\mathfrak{L}_0$, i.e $P_\xi \lneq 0_+$, to avoid te trivial case of the rank one projection extending the nest. Now, note that if P_ξ belongs to the abelian algebra $(\mathfrak{L}_0)''$ and P_ξ is not itself in $\mathfrak{L}_0$, the lattice $\mathfrak{L}$ clearly cannot be Kadison-Singer. Another situation to be avoided is the following. Suppose there is a non-trivial projection P_{t_0} (with t_0 in $\mathcal{S}$) in $\mathfrak{L}_0$ such that $P_\xi P_{t_0} = 0$. Then the lattice $\mathfrak{K}$ generated by

$$\{P_t \vee P_\xi, P_s : t \geq t_0 \text{ and } s, t \in \mathcal{S} \subset [0,1]\},$$

which is a sublattice of $\mathfrak{L}$, generates the same algebra and $\mathrm{Alg}(\mathfrak{K}) \supsetneq \mathrm{Alg}(\mathfrak{L})$. For the first assertion, note that $P_\xi = (P_{t_0} \vee P_\xi)(I - P_{t_0})$ and thus P_ξ belongs to $(\mathfrak{K})''$. Consequently, $(\mathfrak{K})'' \supset (\mathfrak{L})''$. For the second assertion, note that for any pair of vectors η in $\mathrm{Ran}(P_{t_0})$ and ξ in $\mathrm{Ran}(I - P_{t_0})$, we have that $\xi \otimes \eta$ is in $\mathrm{Alg}(\mathfrak{K}) \setminus \mathrm{Alg}(\mathfrak{L})$.

The two conditions essentially capture all the cases where a one point extension of a nest is not a Kadison-Singer lattice. We have the following theorem,

Theorem 3.2.1. *Let $\mathfrak{L}_0 = \{P_t : t \in \mathcal{S} \subset [0,1]\}$, where $\mathcal{S}$ is a compact subset of $[0,1]$ be a nest and let P_ξ be a rank one projection not in $\mathfrak{L}_0$. Let $\mathfrak{L}$ be the lattice generated by the elements in $\{\mathfrak{L}_0, P_\xi\}$. Suppose the following three conditions are satisfied,*

1. *P_ξ is not in $(\mathfrak{L}_0)''$*

2. *For every $t \neq 0$, $P_t P_\xi \neq 0$.*

3. *Let P_{t_0} be the smallest projection so that $P_{t_0}\xi = \xi$. Then, $(P_{t_0})_-\xi \neq P_{s_0}\xi$ for any $P_{s_0} < (P_{t_0})_-$.*

Then, $\mathfrak{L}$ is a Kadison-Singer lattice. Conversely, if $\mathfrak{L}$ is a Kadison-Singer lattice, then the above conditions are satisfied.

Proof. For the converse part, note that we have shown in the paragraph preceding the proof that if conditions (1) and (2) fail, then the lattice cannot be Kadison-Singer. Suppose condition (3) fails. Let $\mathfrak{K}$ be the lattice generated by

$$\{P_t \vee P_\xi, P_s : s \leq s_0 \text{ and } s,t \in \mathcal{S} \subset [0,1]\}.$$

We have that $P_{t_0} = P_{t_0} \vee P_f$, $(P_{t_0})_- - P_{s_0} = (P_{t_0})_- \vee P_f - P_{s_0} \vee P_f$ and for $s \geq s_0$, $(P_{t_0})_- - P_s = (P_{t_0})_- \vee P_f - P_s \vee P_f$ and thus, $(\mathfrak{K})'' = (\mathfrak{L})''$. And, if $\eta \in \text{Ran}(P_t) \ominus \text{Ran}((P_t)_-)$, then, $\eta \otimes \xi$ is in $\text{Alg}(\mathfrak{K}) \setminus \text{Alg}(\mathfrak{L})$.

Now for the forward inclusion. We first show that the lattice $\mathfrak{L}$ is reflexive. Pick any vector g. We will show that $\overline{\text{Alg}(\mathfrak{L})g}$ is the range of some projection in $\mathfrak{L}$. This will show reflexivity. Let P_0 be the smallest projection in the nest $\mathfrak{L}_0$ so that $P_0 g = g$. If $P_0 = I$, let Q_0 be the smallest projection in the nest so that $(I - Q_0)g = \alpha(I - Q_0)f$ for some scalar α. We will show that

Case 1: If $P_0 = I$ and $Q_0 = I$, $\overline{\text{Alg}(\mathfrak{L})g} = \mathcal{H}$

Case 2: If $P_0 < I$, $\overline{\text{Alg}(\mathfrak{L})g} = \text{Ran}(P_0)$.

Case 3: If $P_0 = I$ and $Q_0 < I$, $\overline{\text{Alg}(\mathfrak{L})g} = \text{Ran}(Q_0 \vee P_f)$

We handle each of the cases below.

Case 1: Any rank one operator $\eta \otimes \zeta$ with $\eta \in \text{Ran}(I - P_t)$ and $\zeta \in \text{Ran}((P_t)_+)$ for some P_t in $\mathfrak{L}_0$ and with $\eta \perp \xi$ belongs to $\text{Alg}(\mathfrak{L})$. Thus,

$$\overline{\text{Alg}(\mathfrak{L})g} \supset \bigcup_{t|P_t<I} \text{Ran}((P_t)_+) = \mathcal{H}$$

Case 2: Any rank one operator $\eta \otimes \zeta$ with $\eta \in \text{Ran}(I - P_t)$ and $\zeta \in \text{Ran}((P_t)_+)$ for some P_t in $\mathfrak{L}_0$ with $P_t < P_0$ and with $\eta \perp \xi$ belongs to $\text{Alg}(\mathfrak{L})$. Thus,

$$\overline{\text{Alg}(\mathfrak{L})g} \supset \bigcup_{t|P_t<P_0} \text{Ran}((P_t)_+) = \text{Ran}(P_0)$$

Case 3: Suppose $P_t < Q_0$, we can always find a η in $\text{Ran}(I - P_t)$, with $\eta \perp \xi$ and $< \eta, g > \neq 0$. Otherwise, $(I - P_t)\xi$ is a scalar multiple of $(I - P_t)g$, which is impossible as $P_t < Q_0$. Thus, any rank one operator $\eta \otimes \zeta$ with $\eta \in \text{Ran}(I - P_t)$ and $\zeta \in \text{Ran}((P_t)_+)$ for some P_t in $\mathfrak{L}_0$ with $P_t < Q_0$ and with $\eta \perp \xi$ belongs to $\text{Alg}(\mathfrak{L})$. Thus,

$$\overline{\text{Alg}(\mathfrak{L})g} \supset \bigcup_{t|P_t<Q_0} \text{Ran}((P_t)_+) = \text{Ran}(Q_0)$$

Writing $g = Q_0(g) + Q_0 g$ and noting that $P_t(g)$ is in $\overline{\text{Alg}(\mathfrak{L})g}$, we note that $(I - Q_0)g$ and thus $(I - Q_0)\xi$ is in $\overline{\text{Alg}(\mathfrak{L})g}$. It follows that ξ is in $\overline{\text{Alg}(\mathfrak{L})g}$ and thus, $\overline{\text{Alg}(\mathfrak{L})g}$ contains $\text{Ran}(Q_0 \vee P_\xi)$. The reverse direction is trivial.

We have thus shown that

Proposition 3.2.1. *Any lattice $\mathfrak{L}$ generated(as a lattice) by a (general) nest $\mathfrak{L}_0$ and a rank one projection P_ξ,*

$$\mathfrak{L} = \{\mathfrak{L}_0, P_\xi\}$$

is reflexive.

Let now $\mathfrak{K}$ be a sublattice of $\mathfrak{L}$. Suppose that P_ξ does not belong to $\mathfrak{K}$. Let s_0 be the smallest number in $\mathcal{S}$ so that $P_{s_0} \vee P_\xi$ is in $\mathfrak{K}$. We see that $P_{s_0} > 0$ and that $\mathfrak{K} \subset \{P_t, P_s \vee P_\xi : t \in \mathcal{S}, s \geq s_0, s \in \mathcal{S}\}$. Choose a projection P_{t_0} with $P_{t_0} \leq P_{s_0}$ and $P_{t_0}\xi \neq \xi$(which is avaliable due to the hypothesis 1 in the theorem). Then, we have that P_{t_0} does not commute with P_ξ, but commutes with $\mathfrak{K}$. Thus, $(\mathfrak{K})'' \subsetneq (\mathfrak{L})''$.

Since P_ξ is a rank one projection, we have that $(P_t \vee P_\xi) \wedge P_s = P_{\min(t,s)}$ unless $P_t \vee P_\xi \leq P_s$. Let t_0 be the minimal element in $\mathcal{T}$ such that $P_{t_0}\xi = \xi$. Therefore, we have that $\mathfrak{L} = \{P_t, P_\xi, P_s \vee P_\xi : t \in \mathcal{T}, s \in \mathcal{T} \cap [0, t_0]\}$. Thus, $\mathfrak{L}$ is the lattice sum of a smaller lattice $\mathfrak{L}_1 = \{P_t, P_\xi, P_s \vee P_\xi : t \in \mathcal{T} \cap [0, t_0]\}$ and a nest $\mathfrak{L}_2 = \{P_t : t \geq t_0\}$. It is clear that we may assume that $P_t\xi \neq \xi$ for any element P_t in $\mathfrak{L}_0$ to prove our theorem that $\mathfrak{L}$ is minimal generating. Let us make this assumption.

With this assumption and noting condition (3), any sublattice $\mathfrak{K}$ of $\mathfrak{L}$ is of the form $\mathfrak{K} = \{P_t, P_f : t \in \mathcal{S} \subset \mathcal{T}\}$. Such lattices are reflexive by proposition(3.2.1) and we will show that if $\mathcal{S} \subsetneq \mathcal{T}$, then $(\mathfrak{K})'' \subsetneq (\mathfrak{L})''$.

If $\mathcal{S} \subsetneq \mathcal{T}$, there exist s, t such that if $s < r < t$, then $P_r \in \mathfrak{L} \setminus \mathfrak{K}$ and further, $(s,t) \cap \mathcal{T} \neq \{\phi\}$. Now, a moment's thought shows that we may pick an operator T, with $T = (P_t - P_s)T(P_t - P_s)$, $T\xi = 0$ and $TP_r \neq P_rT$. This shows that $(\mathfrak{K})'' \subsetneq (\mathfrak{L})''$ and we are done. □

Remark 3.2.1. *We will call the lattices generated by a nest and a rank one projection one point extensions of nests. Lance's proof that the cohomology of nest algebras is trivial can be adapted with the slightest modifications to show that* Alg($\mathfrak{L}$) *has trivial Hochschild cohomology for every one point extension* $\mathfrak{L}$*, KS or otherwise.*

3.3 Extensions by finite rank projections

The analysis of the last section does not extend nicely to cover the case of lattices $\mathfrak{L}$ generated by $\{\mathfrak{L}_0, P\}$ where $\mathfrak{L}_0$ is a nest and P is a finite rank projection of rank greater than 1. For instance, if $\mathfrak{L}_0$ is the Volterra nest in $\mathcal{B}(L^2([0,\infty))$, i.e. $\mathfrak{L}_0 = \{P_t : t \in [0,\infty)\}$, P_t the orthogonal projection onto $\{f : f(x) = 0$ a.e. on $[0,t)$ w.r.t dm$\}$ and P is the projection onto the subspace spanned by e^{-x} and e^{-2x}, then $\mathfrak{L}$ is not reflexive. Indeed the projection onto the one dimensional subspace e^{-x} lies in Lat(Alg($\mathfrak{L}$)). Necessary and sufficient conditions for reflexivity are easy to obtain, but very messy to write down. The prescence of atoms in the nest $\mathfrak{L}_0$ queer the pitch further.

Rather than study finite rank extensions in full generality, in this section, we will focus upon certain natural extensions of a specific nest. In the Hardy space $H^2(S^1)$(which we will abbreviate by H^2), let P_n be the projection onto $z^n H^2$ for $n \geq 1$ and let $Q_n = (P_n)^\perp$, i.e. Q_n is the projection onto the span of $\{z^m : 0 \leq m \leq n-1\}$. Let $\mathfrak{L}_0$ be the lattice $\mathfrak{L}_0 = \{P_n : n \geq 1\}$. $\mathfrak{L}_0$ is a maximal nest(and atomic), which we will refer to hitherto as the basis nest in H^2.

For any $\lambda \in \mathbb{D}$, the function $\frac{1}{1-\lambda z}$ is in H^2. Let $\lambda_1, \cdots, \lambda_k$ be n points, none of them zero, in $\mathbb{D}$ and denote the set $\{\lambda_1, \cdots, \lambda_k\}$ by Λ. Denote by Q_Λ the projection onto the span of $\{\frac{1}{1-\lambda_i z} \mid 1 \leq i \leq k\}$. We will also denote this projection by $Q_{\{\lambda_1, \cdots, \lambda_k\}}$. It is easy to see that if we let B_Λ be the Blaschke product

$$B_\Lambda = \prod_{i=1}^{n} \frac{\overline{\lambda_i}}{|\lambda_i|} \frac{\lambda_i - z}{1 - \overline{\lambda}_i z}$$

and P_Λ the projection onto the subspace of H^2 spanned by functions that vanish at the k points $\{\lambda_1, \cdots, \lambda_k\}$, then P_Λ is the projection onto $B_\Lambda H^2$ and $Q_\Lambda = P_\Lambda^\perp$.

Let $\mathfrak{L}$ be the lattice generated by $\{Q_n, Q_{\lambda_1, \cdots, \lambda_k} : n \geq 1\}$. It will also be conveninent to consider another representation for Q_Λ and Q_n.

We note that $H^2(S^1)$ is isomorphic to the Hilbert space $l^2(\mathbb{N} \cup \{0\})$ under the map $V : H^2(S^1) \to l^2(\mathbb{N} \cup \{0\})$ given by $Vz^n = e_n$. We have

$$V(\sum_n a_n z^n) = (a_0, a_1, \cdots)$$
$$V(\frac{1}{1-\lambda z}) = (1, \lambda, \lambda^2, \cdots)$$

We will abuse notation and identify subspaces of $H^2(S^1)$ and their images under V. The function $\frac{1}{1-\lambda z}(\lambda \in \mathbb{D})$ is the reproducing kernel(also called the Szego Kernel) for $\overline{\lambda}$, i.e, for any $f \in H^2$,

$$< f, \frac{1}{1-\lambda z} >= f(\overline{\lambda})$$

We will denote $V(\frac{1}{1-\lambda z})$ by $\tilde{\lambda}$. We note that the one-dimensional subspaces Span($\tilde{\lambda}$) as well as the n dimensional subspaces Q_n are invariant for U^*, the adjoint of the unilateral shift. Thus, Alg($\mathfrak{L}$) contains $(H^\infty)^*$.

We investigate the question of determining KS sublattices of the lattice $\mathfrak{L}$ generated by $\{\mathfrak{L}_0, Q_{\{\lambda_1,\cdots,\lambda_k\}}\}$.

Proposition 3.3.1. $\mathfrak{L} = \{Q_n, Q_\Lambda, Q_n \vee Q_\Lambda : n \in \mathbb{N}\}$

Proof. Let, as above, B_Λ be the Blaschke product

$$B_\Lambda = \prod_{i=1}^{n} \frac{\overline{\lambda_i}}{|\lambda_i|} \frac{\lambda_i - z}{1 - \overline{\lambda}_i z}$$

We first claim that $Q_n \wedge Q_\Lambda = 0$. To see this, note that since $(Q_n \wedge Q)^\perp$ is invariant for U, $(Q_n \wedge Q)^\perp = \phi H^2$ for some inner function ϕ. And since $(Q_n \wedge Q)^\perp$ contains $z^n H^2$ and $B_\Lambda H^2$, ϕ divides both z^n and B_Λ. It is easy to see that this implies that ϕ must be constant. Thus, $(Q_n \wedge Q)^\perp = H^2$.

To complete the proof, we need to check that $(Q_n \vee Q_\Lambda) \wedge Q_m = Q_{\min(m,n)}$. For $n \geq m$, this is clear. Let $n < m$

$$\begin{aligned} ((Q_n \vee Q_\Lambda) \wedge Q_m)^\perp &= (Q_n^\perp \wedge Q_\lambda^\perp) \vee P_m^\perp \\ &= (z^n H^2 \wedge B_\Lambda H^2) \vee z^m H^2 \\ &= (z^n B_\Lambda H^2) \vee z^m H^2 \end{aligned}$$

The greatest common divisor of $z^n B_\Lambda$ and z^m is z^n and thus, $((Q_n \vee Q_\Lambda) \wedge Q_m)^\perp = z^n H^2$, i.e. $(Q_n \vee Q_\Lambda) \wedge Q_m = Q_n$. The proposition follows.

□

Proposition 3.3.2. *Let $\mathfrak{L} = \{Q_n, Q_\Lambda, Q_n \vee Q_\Lambda : n \in \mathbb{N}\}$, where Q_Λ is the k dimensional projection onto span $\{\frac{1}{1-\lambda_i z} : 1 \leq i \leq k,\ |\lambda_i| < 1\ \lambda_i \neq 0\}$. Then, the lattice is reflexive iff $|\lambda_1|^k = \cdots = |\lambda_k|^k$. If $|\lambda_i| = \min\{|\lambda_j| : 1 \leq j \leq k\}$ for $i = 1, \cdots, k_0$ and $|\lambda_i| >$ min$\{|\lambda_j| : 1 \leq j \leq k\}$ for $i > k_0$, then $Q_{\lambda_1,\cdots\lambda_{k_0}} \in$ Lat(Alg($\mathfrak{L}$)).*

Proof. Any operator that leaves $\mathfrak{L}$ invariant must be upper triangular with respect to the nest $\mathfrak{L}_0$. Suppose first that $|\lambda_i| = \min\{|\lambda_j| : 1 \leq j \leq k\}$ for $1 \leq i \leq k_0$ and $|\lambda_i| > \min\{|\lambda_j| : 1 \leq j \leq k\}$ for $k_0 < i \leq k$. We will show that $Q_{\lambda_1,\cdots,\lambda_{k_0}}$ is in Lat(Alg($\mathfrak{L}$)).

Let $T \in$ Alg($\mathfrak{L}$) and let $T\tilde{\lambda}_j = \sum_{i=1}^{k} \alpha_i \tilde{\lambda}_i$, for $j \leq k_0$. We will show that $\alpha_i = 0$ ($i > k_0$). Once we prove this, we will have that $Q_{\lambda_1,\cdots,\lambda_{k_0}}$ is in Lat(Alg($\mathfrak{L}$)). Fix $j = 1$, the computation is identical for $1 < j \leq k_0$.

Since $TQn = Q_nTQ_n$, we have that $(I - Q_n)T(I - Q_n) = (I - Q_n)T$. Then,

$$(T(I - Q_n)\tilde{\lambda}_1)(n) = (T\tilde{\lambda}_1)(n) = \sum_{i=1}^{k} \alpha_i \lambda_i^n$$

$||(I - Q_n)\tilde{\lambda}_1|| = |\frac{\lambda_1^n}{1 - \lambda_1}|$. We have that

$$||\sum_{i=1}^{k} \alpha_i \lambda_i^n|| \leq C|\lambda_1^n| \quad \forall n$$

and this implies that $\alpha_i = 0 \; \forall i : |\lambda_i| > |\lambda_1|$. This proves one half of (2). For the other half, assume that $|\lambda_1| = \cdots = |\lambda_k|$. For each n, choose a vector x_n such that

$$< x_n, (\lambda_1^n, \lambda_1^{n+1}, \cdots, \lambda_1^{n+k-1}) = \lambda_1^n$$

$$< x_n, (\lambda_2^n, \lambda_2^{n+1}, \cdots, \lambda_2^{n+k-1}) = 0$$

$$\cdots$$

$$< x_n, (\lambda_k^n, \lambda_k^{n+1}, \cdots, \lambda_k^{n+k-1}) = 0$$

This can always be done as the vectors $(\lambda_i^n, \lambda_i^{n+1}, \cdots, \lambda_i^{n+k-1})$ are linearly independent. This is because the Vandermonde matrix

$$T = \begin{pmatrix} \lambda_1^n & \lambda_2^n & \cdots & \lambda_k^n \\ \lambda_1^{n+1} & \lambda_2^{n+1} & \cdots & \lambda_k^{n+1} \\ \lambda_1^{n+2} & \lambda_2^{n+2} & \cdots & \lambda_k^{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n+k-1} & \lambda_2^{n+k-1} & \cdots & \lambda_k^{n+k-1} \end{pmatrix}$$

is invertible. It is also clear that x_n can be chosen with $||x_n|| < C||T^{-1}||$ for some scalar C. The matrix A with

$$A(n, n+i) = \left\{ \begin{array}{cc} x_n(i+1) & 0 \leq i \leq k \\ 0 & \text{otherwise} \end{array} \right\}$$

is a band matrix with k rows(and bounded), is in $\text{Alg}(\{Q_n : n \geq 1\})$(as it is upper triangular) and satisfies

$$A\tilde{\lambda}_1 = \tilde{\lambda}_1$$

$$A\tilde{\lambda}_i = 0 \quad (1 < i \leq k)$$

and hence, leaves Q invariant. Thus, $A \in \text{Alg}(\mathfrak{L})$. Using this, we see that given any vector ξ in the range of Q, $\text{Alg}(\mathfrak{L})\xi = Q\mathcal{H}$. There is thus no strict subprojection of Q in $\text{Lat}(\text{Alg}(\mathfrak{L}))$.

Suppose $P \in \text{Lat}(\text{Alg}(\mathfrak{L}))$, let n be the smallest number, $n \geq 1$ so that $P \leq P_n$. It is easy to see that $P = P_n$ in this case. Suppose there is no such n, let m be the smallest number, $m \geq 0$, so that $P \leq P_m \vee Q$. By using the above construction, it is easy to see that $P = P_n \vee Q$. Suppose there is no such m either, it is easy to see that $P = I$. This shows that $\mathfrak{L}$ is reflexive if $|\lambda_1| = \cdots = |\lambda_k|$.

Also, if $\Lambda = \{\lambda_i \mid |\lambda_i| = \min\{|\lambda_i| : 1 \leq i \leq k\}\}$, we see that $\{Q_n, Q_\Lambda, Q_n \vee Q_\Lambda,\ n \geq 1\}$ is a reflexive lattice with trivial diagonal. □

Proposition 3.3.3. *Let $\mathfrak{L} = \{Q_n, Q_\Lambda, Q_n \vee Q_\Lambda : n \in \mathbb{N}\}$, where Q_Λ is the k dimensional projection onto span $\{\frac{1}{1-\lambda_i z} : 1 \leq i \leq k,\ |\lambda_i| < 1\ \lambda_i \neq 0\}$. Let $p = \sum_{n=0}^{k} a_n \lambda^n$ be the minimal polynomial with zeros at each of the $\{\lambda_i\}_{i=1}^{k}$. Then, the lattice $\mathfrak{L}$ does not generate $B(H^2)$ iff for some divisor d of k, $a_i \neq 0$ if $d \mid i$ and $a_i = 0$ otherwise .*

Proof. Any operator D that commutes with $\{Q_n : n \geq 1\}$ must be diagonal and we may write it as $D = \text{diag}(d_0, \cdots, d_n, \cdots)$. Let $x_n = (\lambda_1^{n-1}, \cdots, \lambda_k^{n-1})$ and let T be the Vandermonde matrix,

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \end{pmatrix}$$

which is invertible. Let $S = T^{-1}$. We have that

$$x_n S = (0, \cdots, 0, \overbrace{1}^{n}, 0, \cdots, 0), \text{ for } 1 \leq n \leq k.$$

For $1 \leq i \leq k$, let L_i be the infinite column vector given by

$$L_i(n) = (x_n S)(i), \quad 1 \leq n < \infty.$$

Note that for $1 \leq n \neq m \leq k$, $L_n(n) = 1$ and $L_n(m) = 0$. We see that the vectors $\{L_i : 1 \leq i \leq k\}$ have the same span as $\{\tilde{\lambda_i} : 1 \leq i \leq k\}$ as S is non-singular. Now, suppose the diagonal operator $D = \text{diag}(d_0, \cdots, d_n, \cdots)$ leaves $\text{span}\{L_i : 1 \leq i \leq k\}$ invariant, i.e. $DL_i = \sum_{j=1}^{k} \alpha_j^i L_j$. Then, for $j \neq i$,

$$DL_i(j) = d_j L_i(j) = 0 = \alpha_j^i L_j(j) = \alpha_j^i \qquad \text{And thus, } \alpha_j^i = 0$$

Thus, for some scalars, $c_1, \cdots, c_k$,

$$DL_i = c_i L_i \quad 1 \leq i \leq k$$

For any n and $1 \leq i \leq k$,

$$d_n(x_n S)(i) = d_n L_i(n) = (DL_i)(n) = c_i L_i(n) = c_i(x_n S)(i)$$

If $(x_n S)(i) \neq 0$, $d_n = c_i$. For any n, since S is non-singular, not all of the k numbers $((x_n S)(i))_{1 \leq i \leq k}$ can be zero. Thus, each entry d_n of the diagonal operator can only take on finitely many values, $c_1, \cdots, c_k$. If $(x_n S)(i) \neq 0$ and $(x_n S)(j) \neq 0$, then $c_i = c_j = d_n$.

Define an equivalence relation E on $X = \{1, \cdots, k\}$ by $i \sim_E j$ if there is a n such that $(x_n S)(i) \neq 0$ and $(x_n S)(j) \neq 0$. If $i \sim_E j$, by the previous paragraph, $c_i = c_j$. Assume now that $E \subsetneq X \times X$. Then, there is a partition of X into sets $A_1, A_2 \cdots A_l \subset \{1, \cdots, k\}$ such that if $i \sim_E j$, then $\{i, j\}$ belongs to exactly one of $\{A_m\}_{m=1}^l$.

Thus, any x_n is in the linear span of exactly one of the $\{x_j \mid j \in A_i\}_{i=1}^l$. Let $B_i = \{n \mid x_n \in \text{Span}(\{x_j : j \in A_i\})\}$. For any m, the k vectors $x_{m+1}, \cdots, x_{m+k}$ are linearly independent. There can be at most $|A_i|$ of these that lie in Span($\{x_j : j \in A_i\}$) for each i and thus exactly $|A_i|$ respectively, of each.

Thus, $|B_i \cap [m+1, m+k]| = |A_i|$ for every m and i and we conclude that B_i is closed under translation by k, i.e, $B_i = \{A_i + nk\}$ $(n \geq 1)$. Let b_i be the smallest element in A_i. The $|A_i + 1|$ vectors $\{x_j : j \in A_i \cup x_{b_i+k}\}$ lie in a space of dimension $|A_i|$ and are thus linearly dependent. Thus, there is a polynomial $\sum_{j \in A_i} c_j^i \lambda^j + c_i \lambda^{b_i+k}$ that is zero at each of the λ_i. Therefore, the polynomial $p_i = \sum_{j \in A_i} c_j^i \lambda^{j-b_i} + c_i \lambda^k$ is zero at each of the λ_i and must thus be the minimal polynomial.

This means that vectors $\{x_k \cup x_{j-b_i} : j \in A_i\}$ are linearly dependent. Since we have assumed that $1 \in A_1$, we have that the set $\{j - b_i : j \in A_i\}$ is contained in A_1. Thus, $A_i = A_1 + b_i$ for each i and given that $\cup_i A_i = \{1, \cdots, k\}$, we conclude that there is a divisor d of k such that $A_i = \{i + dj : 0 \leq j \leq \frac{k}{d} - 1\}$. The theorem is proved. □

Suppose we are given k points $\Lambda = \{\lambda_1, \cdots, \lambda_k\}$ in $\mathbb{D}$. We now investigate the Kadison-Singer sublattices of the lattice generated by $\{Q_n, Q : n \in \mathbb{Z}\}$.

Theorem 3.3.1. *Let λ_i, $1 \leq i \leq k$, where $k > 1$, Q_n and Q be as above. Then, there are uncountably many non-isomorphic Kadison-Singer sublattices of $\{Q_n, Q : n \in \mathbb{Z}\}$.*

Proof. Assume that $\mathfrak{L}$ generates $\mathcal{B}(H^2)$. The proof in the other case is similar. For each $m \geq k$, we can find a unique polynomial p_m of degree less than equal to $k-1$ such that $(\lambda_i)^m = p_m(\lambda_i)$ for $1 \leq i \leq k$. Let A_m be the subset of $\{0, 1, \cdots, k-1\}$ consisting of j such that the coefficient of λ^j in p_m is non-zero. Let $n(\Lambda)$ be the smallest number such that $\bigcup_{m=k}^{n(\Lambda)} A_m = \{0, 1, \cdots, k-1\}$.

Now, let S be any infinite sequence of numbers $\{1, 2, \cdots, n(\Lambda), n_1, n_2, \cdots\}$ with $n(\Lambda) < n_1$ such that

1. $n_1 - n(\Lambda) \leq k$

2. $n_i < n_{i+1}$ for $i \geq 1$

3. $n_{i+1} - n_i \leq k$ for $i \geq 1$

4. Either $n_2 - n(\Lambda) > k$ or $n_{i+2} - n_i > k$ for some $i \geq 1$

Then, we claim that the lattice generated by $\mathfrak{K} = \{Q_l, Q : l \in S\}$ is Kadison-Singer with the same core as $\mathfrak{L}$, i.e. $\mathcal{B}(H^2)$. There are clearly uncountably infinitely many non-isomorphic such lattices.

The reflexivity of the lattice $\mathfrak{K}$ follows as in the previous theorem. Conditions 1, 2, 3 assure that the core is $\mathcal{B}(H^2)$ and condition 4 ensures that any sublattice has core strictly smaller than $\mathcal{B}(H^2)$.

□

3.4 Extensions by infinite codimension projections

Suppose B_Λ, where $\Lambda = \{\lambda_1, \cdots, \lambda_k\}$ is a Blaschke product and f is orthogonal to $B_\Lambda H^2$. Then, f belongs to the span of $\{\tilde{\lambda}_i\}_{1 \leq i \leq k}$ and thus, extends to an analytic function on $B(0, \min(\frac{1}{|\lambda_i|}))$. On the other hand, suppose ϕ is a singular inner function and f is a function orthogonal to ϕH^2. Suppose further that f extends to an analytic function on $B(0, r)$, where $r > 1$. Then, $f(rz)$ is an element in H^2 that is orthogonal to $\phi(\frac{z}{r})H^2$(in H^2). A classical result of Smirnov says that any singular inner function has a representation

$$S(z) = K \exp(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta))$$

where $d\mu$ is a finite positive regular Borel measure, singular with respect to the lebesgue measure and K is a constant of modulus 1. Thus,

$$S(\frac{z}{r}) = K \exp(-\frac{1}{2\pi} \int_0^{2\pi} \frac{re^{i\theta} + z}{re^{i\theta} - z} d\mu(\theta))$$

and it is easy to see that $\log(S(\frac{e^{i\theta}}{r}))$ is in L^1, with L^1 norm less than $\frac{1}{(1-r)^2}$. This implies that $S(\frac{z}{r})$ is outer and thus, $\phi(\frac{z}{r})H^2 = H^2$. This gives us in turn that $f = 0$. We have the following theorem.

Theorem 3.4.1. *Let $\psi = B\phi$ be an inner function, with B a Blaschke product not equal to z^n for any $n \geq 0$ and ϕ a singular inner function, not equal to 1. Then the lattice $\mathfrak{L}$ generated by $\{z^n H^2, \psi H^2\}$ is not reflexive. Indeed, BH^2 is in* $\mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}))$.

Proof. Working with the dual lattice $\mathfrak{K}$, let T be an upper triangular contraction preserving $(\psi H^2)^\perp$. Suppose T maps an element f in the span of $\{\tilde{\lambda}_i\}_{1\leq i\leq k}$ to g, where $g \perp \phi H^2$. By the previous paragraph, we may assume that $|\tilde{f}(n)| \leq s^n$ for some $s < 1$. Then, $|\tilde{Tf}(n)| \leq s^n$ and $\tilde{Tf}$ extends to an analytic function on $B(0, \frac{1}{s})$. Then, again by the previous paragraph, $g = \tilde{Tf}$ cannot be orthogonal to ϕH^2 and thus, $(BH^2)^\perp$ is indeed in the reflexive closure of $\mathfrak{K}$. ☐

The lattice generated by the basis nest and the projection corresponding to a singular inner function is even more interesting. For simplicity, let us fix attention upon the point mass singular inner function $\phi = \exp\left(\frac{z+1}{z-1}\right)$. Let $\mathfrak{L}$ be the lattice generated by $\{P_n, Q = P_{\phi H^2} \mid n \geq 1\}$. It is easy to see that $\mathfrak{L} = \{P_n, Q, P_n \wedge Q \mid n \geq 1\}$, that $\mathfrak{L}$ generates $\mathcal{B}(H^2)$ and that

$$\mathrm{Lat}(\mathrm{Alg}(\mathfrak{L})) \subset \{P_n \wedge Q_a : n \geq 0, 0 \leq a \leq 1\}$$

where Q_a is the projection onto $\phi_a H^2 = \exp\left(a\frac{z+1}{z-1}\right)$.

We are unable to determine exactly what the reflexive closure of $\mathfrak{L}$ is, but we suspect it is $\{P_n \wedge Q_a : n \geq 0, 0 \leq a \leq 1\}$. It is easy to show that any generating sublattice $\mathfrak{K}$ of $\mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}))$ contains a sublattice $\mathfrak{K}_1$ of the form $\{P_{n_i}, Q_a, P_{n_i} \wedge Q_a\}$ for some $0 < a \leq 1$ and some infinite sequence $0 < n_1 < n_2 < \cdots$. We further suspect all such lattices $\mathfrak{K}_1$ generate $\mathcal{B}(H^2)$, which, if confirmed, would give us an example of a reflexive algebra(with trivial diagonal) not contained in any Kadison-Singer algebra with the same(trivial) diagonal.

CHAPTER 4

LATTICES IN FINITE VON NEUMANN ALGEBRAS

Lattices of projections in finite von Neumann algebras are much more tractable than lattices of projections in $\mathcal{B}(\mathcal{H})$. It is easy to construct two maximal nests in $\mathcal{B}(\mathcal{H})$, such that the intersections and unions of any two elements, one from each lattice are trivial. For instance, we could take the first nest to be the Volterra nest in $L^2(S^1, dm)$, where dm denotes Lebesgue measure and the second nest to be the Fourier nest, $\{P_n : n \in \mathbb{Z}\}$ in $\mathcal{B}(L^2(S^1, dm))$, P_n the projection onto $\overline{\text{span}\{z^m : m \geq n\}}^{\|\cdot\|}$. That any two elements, one in each, have intersection 0 and union I follows from the F. and M. Riesz theorem. In finite von Neumann algebras, the prescence of the trace ensures the "modularity" condition of Birkhoff-von Neumann and makes computations much easier.

We prove two elementary theorems that we will use in the next section to construct new KS lattices out of old.

Theorem 4.0.2. *Let $\mathfrak{M}$ be a finite von Neumann algebra and let τ be a (faithful, normal) trace on $\mathfrak{M}$. Let $\mathfrak{L}_1 = \{P_\lambda : \lambda \in \Lambda\}$, where Λ is a lattice, be a lattice in $\mathfrak{M}$, with $\mathfrak{L}_1$ closed in the strong operator topology,. Note that Λ is then necessarily a complete lattice. Further suppose $\bigwedge_{\lambda\in\Lambda} P_\lambda = 0$ and if $P = \bigvee_{\lambda\in\Lambda} P_\lambda$, then $\tau(P) = \frac{1}{2}$. Let Q be another projection in $\mathfrak{M}$, of trace half and satisfying $P \wedge Q = 0$. Let $Q_\lambda = P_\lambda \vee Q$. Let $\mathfrak{L}$ be the lattice generated* *by $\mathfrak{L}_0$ and Q(i.e the smallest lattice containing $\mathfrak{L}_0$ and Q). Then,*

$$\mathfrak{L} = \{P_\lambda, Q_\lambda : \lambda \in \Lambda\}.$$

Proof. Note that $Q_0 = Q$ and thus, $\mathfrak{L}$ and $\{P_\lambda, Q_\lambda : \lambda \in \Lambda\}$ generate the same lattice. To prove the theorem, we need to show that

$$\{P_\lambda, Q_\lambda : \lambda \in \Lambda\}$$

is closed under the lattice operations.

For any $\lambda_1\lambda_2 \in \Lambda$, $\max\{\lambda_1, \lambda_2\}$ and $\min\{\lambda_1, \lambda_2\}$ will denote the join and meet in the lattice Λ. With this notation, $P_{\lambda_1} \vee P_{\lambda_2} = P_{\max\{\lambda_1,\lambda_2\}}$ and $P_{\lambda_1} \wedge P_{\lambda_2} = P_{\min\{\lambda_1,\lambda_2\}}$.

We begin by noting that given $\lambda_1, \lambda_2 \in \Lambda$,

- $P_{\lambda_1} \vee Q_{\lambda_2} = P_{\lambda_1} \vee (P_{\lambda_2} \vee Q) = P_{\max\{\lambda_1,\lambda_2\}} \vee Q = Q_{\max\{\lambda_1,\lambda_2\}}$
- $Q_{\lambda_1} \vee Q_{\lambda_2} = (P_{\lambda_1} \vee Q) \vee (P_{\lambda_2} \vee Q) = P_{\max\{\lambda_1,\lambda_2\}} \vee Q = Q_{\max\{\lambda_1,\lambda_2\}}$

Fix $\lambda_1, \lambda_2 \in \Lambda$. We now show that $Q_{\lambda_1} \wedge Q_{\lambda_2} = Q_{\min\{\lambda_1,\lambda_2\}}$ and that $P_{\lambda_1} \wedge Q_{\lambda_2} = P_{\min\{\lambda_1,\lambda_2\}}$, which will give us the desired conclusion.

Since τ is a faithful trace, for any pair of projections, $P_1, P_2 \in \mathfrak{M}$, $\tau(P_1 \vee P_2) + \tau(P_1 \wedge P_2) = \tau(P_1) + \tau(P_2)$. In particular, for $\lambda_1, \lambda_2 \in \Lambda$, $\tau(P_{max\{\lambda_1,\lambda_2\}}) + \tau(P_{min\{\lambda_1,\lambda_2\}}) = \tau(P_{\lambda_1}) + \tau(P_{\lambda_2})$

$$\begin{aligned}
\tau(Q_{\lambda_1}) &= \tau(P_{\lambda_1} \vee Q) = \tau(P_{\lambda_1}) + \tau(Q) - \tau(P_{\lambda_1} \wedge Q) \\
&= \tau(P_{\lambda_1}) + \frac{1}{2} \qquad\qquad (\text{As} \quad P_{\lambda_1} \wedge Q = 0) \\
\tau(Q_{\lambda_1} \wedge Q_{\lambda_2}) &= \tau(Q_{\lambda_1}) + \tau(Q_{\lambda_2}) - \tau(Q_{\lambda_1} \vee Q_{\lambda_2}) \\
&= \tau(P_{\lambda_1}) + \frac{1}{2} + \tau(P_{\lambda_2}) + \frac{1}{2} - \tau(P_{max\{\lambda_1,\lambda_2\}}) - \frac{1}{2} \\
&= \tau(P_{min\{\lambda_1,\lambda_2\}}) + \frac{1}{2}
\end{aligned}$$

We have that $Q_{\min\{\lambda_1,\lambda_2\}} \leq Q_{\lambda_1}$ and $Q_{\min\{\lambda_1,\lambda_2\}} \leq Q_{\lambda_2}$. Thus, $Q_{\min\{\lambda_1,\lambda_2\}} \leq Q_{\lambda_1} \wedge Q_{\lambda_2}$. And since $\tau(Q_{\min\{\lambda_1,\lambda_2\}}) = \frac{1}{2} + \tau(P_{\min\{\lambda_1,\lambda_2\}})$, we have

$$Q_{\lambda_1} \wedge Q_{\lambda_2} = Q_{\min\{\lambda_1,\lambda_2\}} \tag{4.1}$$

We have that $P_{\min\{\lambda_1,\lambda_2\}} \leq P_{\lambda_1} \wedge Q_{\lambda_2}$. For equality, it is enough to check they have the same trace. Note that

$$\tau(P_{\lambda_1} \wedge Q_{\lambda_2}) = \tau(P_{\lambda_1}) + \frac{1}{2} + \tau(P_{\lambda_2}) - \frac{1}{2} - \tau(P_{\max\{\lambda_1,\lambda_2\}}) = \tau(P_{\min\{\lambda_1,\lambda_2\}}).$$

Thus,

$$\tau(P_{\min\{\lambda_1,\lambda_2\}}) = \tau(P_{\lambda_1} \wedge Q_{\lambda_2}) = \tau(P_{\min\{\lambda_1,\lambda_2\}}).$$

Hence,

$$P_{\lambda_1} \wedge Q_{\lambda_2} = P_{\min\{\lambda_1,\lambda_2\}} \tag{4.2}$$

We have thus shown that $\{P_\lambda, Q_\lambda : \lambda \in \Lambda\}$ is closed under the lattice operations and hence, is the lattice closure of $\mathfrak{L}$. □

Theorem 4.0.3. *Let $\mathfrak{M}$ be a finite von Neumann algebra and let τ be a (faithful, normal) trace on $\mathfrak{M}$. Let P and Q be projections of trace half in $\mathfrak{M}$, satisfying*

$$P \vee Q = I \qquad \textit{and hence,} \qquad P \wedge Q = 0.$$

Let $\mathfrak{L}_1 = \{P_\lambda : \lambda \in \Lambda\}$, $\mathfrak{L}_2 = \{P_\mu : \mu \in \Delta\}$ where Λ, Δ are lattices, be lattices in $\mathfrak{M}$, both lattices closed in the strong operator topology, $\bigwedge_{\lambda\in\Lambda} P_\lambda = 0$, $\bigvee_{\lambda\in\Lambda} P_\lambda = P$, $\bigwedge_{\mu\in\Delta} P_\mu = P$, $\bigvee_{\mu\in\Delta} P_\mu = I$. Note that Λ, Δ are necessarily complete lattices. Let $\mathfrak{L}$ be the lattice generated by $\mathfrak{L}_1, \mathfrak{L}_2$ and Q. Then,

$$\mathfrak{L} = \{(P_\lambda \vee Q) \wedge P_\mu : \lambda \in \Lambda, \mu \in \Delta\}.$$

We first state and prove a lemma.

Lemma 4.0.1. *Assume the conditions of theorem 4.0.3. Given* $\lambda_1, \lambda_2 \in \Lambda, \mu_1, \mu_2 \in \Delta$,

1. $\tau(P_{\mu_1} \vee Q) = \tau(P_{\mu_1}) - \frac{1}{2}$
2. $\tau((P_{\lambda_1} \vee Q) \wedge P_{\mu_1}) = \tau(P_{\lambda_1}) + \tau(P_{\mu_1}) - \frac{1}{2}$
3. $(P_{\lambda_1} \vee Q) \wedge P_{\mu_1} = P_{\lambda_1} \vee (Q \wedge P_{\mu_1})$
4. $((P_{\lambda_1} \vee Q) \wedge P_{\mu_1}) \wedge ((P_{\lambda_2} \vee Q) \wedge P_{\mu_2}) = (P_{min\{\lambda_1,\lambda_2\}} \vee Q) \wedge P_{min\{\mu_1,\mu_2\}}$
5. $(P_{\lambda_1} \vee Q) \wedge P_{\mu_1}) \vee ((P_{\lambda_2} \vee Q) \wedge P_{\mu_2}) = (P_{max\{\lambda_1,\lambda_2\}} \vee Q) \wedge P_{max\{\mu_1,\mu_2\}}$

Proof. For (1), note that

$$\tau(P_{\mu_1} \vee Q) = \tau(P_{\mu_1}) + \tau(Q) - \tau(P_{\mu_1} \vee Q) = \tau(P_{\mu_1}) + \frac{1}{2} - 1 = \tau(P_{\mu_1}) - \frac{1}{2}.$$

For (2),

$$\begin{aligned} \tau((P_{\lambda_1} \vee Q) \wedge P_{\mu_1}) &= \tau(P_{\lambda_1} \vee Q) + \tau(P_{\mu_1}) - \tau((P_{\lambda_1} \vee Q) \vee P_{\mu_1}) \\ &= \tau(P_{\lambda_1}) + \frac{1}{2} + \tau(P_{\mu_1}) - 1 \\ &= \tau(P_{\lambda_1}) + \tau(P_{\mu_1}) - \frac{1}{2} \end{aligned}$$

We also have,

$$\begin{aligned} \tau(P_{\lambda_1} \vee (Q \wedge P_{\mu_1}) &= \tau(P_{\lambda_1}) + \tau(Q \wedge P_{\mu_1}) - \tau((P_{\lambda_1} \wedge Q \wedge P_{\mu_1}) \\ &= \tau(P_{\lambda_1}) + \tau(P_{\mu_1}) - \frac{1}{2} \end{aligned}$$

For (3), $(P_{\lambda_1} \vee Q) \wedge P_{\mu_1} \geq P_{\lambda_1} \vee (Q \wedge P_{\mu_1})$. And the equality follows from equality of trace.

For (4), we first recall from note (4.1) in the proof of 4.0.2 that $(P_{\lambda_1} \vee Q) \wedge (P_{\lambda_2} \vee Q) = (P_{\min\{\lambda_1,\lambda_2\}} \vee Q)$.

$$\begin{aligned} ((P_{\lambda_1} \vee Q) \wedge P_{\mu_1}) \wedge ((P_{\lambda_2} \vee Q) \wedge P_{\mu_2}) &= (P_{\lambda_1} \vee Q) \wedge (P_{\lambda_2} \vee Q) \wedge P_{\mu_1} \wedge P_{\mu_2} \\ &= (P_{\min\{\lambda_1,\lambda_2\}} \vee Q) \wedge P_{\min\{\mu_1,\mu_2\}} \end{aligned}$$

and the claim is proved.

For (5),

$$\begin{aligned}((P_{\lambda_1} \vee Q) \wedge P_{\mu_1}) \vee ((P_{\lambda_2} \vee Q) \wedge P_{\mu_2}) &= (P_{\lambda_1} \vee (Q \wedge P_{\mu_1}) \vee (P_{\lambda_2} \vee (Q \wedge P_{\mu_2}) \\ &= P_{\lambda_1} \vee P_{\lambda_2} \vee (Q \wedge P_{\mu_1}) \vee (Q \wedge P_{\mu_2}) \\ &= (P_{\max\{\lambda_1,\lambda_2\}} \vee Q) \wedge P_{\max\{\mu_1,\mu_2\}}\end{aligned}$$

The lemma is proved. □

Proof. (of **Theorem** 4.0.3): We first show that $\mathfrak{L}$ as in the lemma contains $\{P_\lambda : \lambda \in \Lambda, Q, P_\mu : \mu \in \Delta\}$. If $P_\lambda = P$, noting that $P \vee Q = I$,

$$(P \vee Q) \wedge P_\mu = P_\mu.$$

And if $P_\mu = P$, noting that that $P_\lambda \leq (P_\lambda \vee Q) \wedge P$ and from 2, that, $\tau(P_\lambda \vee Q) \wedge P) = \tau(P_\lambda)$,

$$(P_\lambda \vee Q) \wedge P = P_\lambda.$$

Finally, if $P_\lambda = 0, P_\mu = I$,

$$(0 \vee Q) \wedge I = Q.$$

That $\mathfrak{L}$ is closed under the lattice operations follows from 4.0.1.

□

4.1 The lattice generated by a half nest and a projection

Let $\mathfrak{N} \subset \mathcal{B}(\mathcal{K})$ be a finite von Neumann algebra containing an SOT closed nest of projections, $\mathfrak{L}_0 = \{p_t : t \in [0,1]\}$, where for $t \in [0,1]$, $\tau(p_t) = t$. Let $H \in \mathfrak{N}$ be a positive contraction whose point spectrum does not contain 1. Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C}) \subset \mathcal{B}(\mathcal{K}) \otimes M_2(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ and let (e_{ij}), $1 \leq i,j \leq 2$ be a family of matrix units for $M_2(\mathbb{C})$. We will represent elements in $\mathfrak{M}$ with 2×2 matrices over $\mathfrak{N}$, using this system of matrix units.

Let, for $0 \leq t \leq \frac{1}{2}$, $P_t = p_{2t} \otimes e_{11}$. Denote the lattice $\{P_t : 0 \leq t \leq \frac{1}{2}\}$ in $\mathfrak{M}$ by $\mathfrak{L}_1$. The operator

$$Q = \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & 1-H \end{pmatrix}$$

is a projection. Let $\mathfrak{L}$ be the lattice generated by the elements in $\{\mathfrak{L}_1, Q\} \subset \mathfrak{M}$. We claim that the lattice closure of $\mathfrak{L}$ is a KS lattice for the von Neumann algebra $\mathfrak{L}''$.

Pictorially,

$$P_t = \begin{pmatrix} p_{2t} & 0 \\ 0 & 0 \end{pmatrix} P_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q = \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & 1-H \end{pmatrix}$$

As in the preface to lemma(4.0.2), let $Q_t = P_t \vee Q$. lemma(4.0.2) gives us that

$$\mathfrak{L} = \{P_t, Q_t : 0 \leq t \leq \frac{1}{2}\}.$$

Lemma 4.1.1. $\mathrm{Alg}(\mathfrak{L}) = \{ \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} : A_i \in \mathcal{B}(\mathcal{K})(i = 1, 2, 3)$ *and*

$$\sqrt{1-H}A_1\sqrt{H} + \sqrt{1-H}A_3\sqrt{1-H} = \sqrt{H}A_2\sqrt{1-H}$$

and

$$(1 - p_t)A_1 p_t = 0, \quad 0 \leq t \leq 1 \ \}.$$

Or, as an equation involving unbounded operators affiliated with a finite von Neumann algebra,

$$A_1\sqrt{H(I-H)^{-1}} + A_3 = \sqrt{H(I-H)^{-1}}A_2.$$

Proof. If T is an operator that leaves every element in $\mathfrak{L}$ invariant, T in particular leaves $P_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ invariant. Thus, T can be written in the form $T = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}$. Because of the condition that it leaves $\begin{pmatrix} p_t & 0 \\ 0 & 0 \end{pmatrix}$, where $0 \leq t \leq 1$ invariant, we have

$$(1 - p_t)A_1 p_t = 0, \quad 0 \leq t \leq 1.$$

. Since T leaves Q invariant, $(I-Q)TQ=0$,

$$\begin{pmatrix} I-H & -\sqrt{H(1-H)} \\ -\sqrt{H(1-H)} & H \end{pmatrix}\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & 1-H \end{pmatrix}=0$$

Since we have assumed that the $1 \notin \mathrm{sp}_p(H)$, it is easy to see that this yields us precisely,

$$\sqrt{1-H}A_1\sqrt{H}+\sqrt{1-H}A_3\sqrt{1-H}=\sqrt{H}A_2\sqrt{1-H}.$$

Or equivalently,

$$A_1\sqrt{H(I-H)^{-1}}+A_3=\sqrt{H(I-H)^{-1}}A_2.$$

These two conditions characterise $\mathrm{Alg}(\mathfrak{L})$ □

Remark 4.1.1. *Note that the set* $\mathcal{S}=\{A_2 : \exists A_1, A_3$ *such that* $\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L})\}$, *regarded as a subset of* $\mathcal{B}((I-P_{\frac{1}{2}})\mathcal{H})$ *is dense in* $\mathcal{B}((I-P_{\frac{1}{2}})\mathcal{H})$ *in the SOT. To see this, let* P_ϵ *be the spectral projection of* H *corresponding to* $[0,1-\epsilon]$. $P_\epsilon(I-H)\mid_{P_\epsilon\mathcal{H}}$ *is invertible as an operator in* $\mathcal{B}(P_\epsilon\mathcal{H})$ *and* $P_\epsilon\sqrt{H(I-H)^{-1}}\mid_{P_\epsilon\mathcal{H}}$ *is bounded in* $\mathcal{B}(P_\epsilon\mathcal{H})$. *Choose* $A_2 = P_\epsilon A_2 P_\epsilon$. *Let* $A_1=0$ *and* $A_3=P_\epsilon\sqrt{H(I-H)^{-1}}A_2$, *which is a bounded operator satisfying* $A_3=P_\epsilon A_3 P_\epsilon$. *We see that* $\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}$ *is in* $\mathrm{Alg}(\mathfrak{L})$. *Letting* $\epsilon \to 0$, *we see that* $\mathcal{S}$ *contains all bounded operators in* $\mathcal{B}((I-P_{\frac{1}{2}})\mathcal{H})$ *so that* $\exists\epsilon>0, A=P_\epsilon A P_\epsilon$. *Thus,* $\mathcal{S}$ *is SOT dense in* $\mathcal{B}((I-P_{\frac{1}{2}})\mathcal{H})$. *Similarly,* $\mathcal{T}=\{A_1 : \exists A_2, A_3$ *such that* $\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L})\}$, *regarded as a subset of* $\mathcal{B}(P_{\frac{1}{2}}\mathcal{H})$ *is dense in the nest algebra* $\mathrm{Alg}_{P_{\frac{1}{2}}\mathcal{H}}(\{p_t : t \in [0,1]\})$ *in the SOT.*

Lemma 4.1.2. $\mathfrak{L}$ *is reflexive.*

Proof. Let $P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}))$. We claim that $P=P_t$ or $P=Q_t$ for some $t\in[0,1]$. We make two simple obervations to begin the proof. Suppose $P \geq P_{\frac{1}{2}}$. Then $P=\begin{pmatrix} I & 0 \\ 0 & \tilde{P} \end{pmatrix}$ for some projection $\tilde{P} \in \mathcal{B}((I-P_{\frac{1}{2}})\mathcal{H})$. If $T=\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L})$, then since $\tilde{P}$ is invariant under T, a simple computation gives us that

$$\tilde{P}A_2\tilde{P} = A_2\tilde{P}.$$

Since $\mathcal{S} = \{A_2 : \exists A_1, A_3$ such that $\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \text{Alg}(\mathfrak{L})\}$ is SOT dense in $\mathcal{B}((I - P_{\frac{1}{2}})\mathcal{H})$, by remark(4.1.1), we have that

$$P \in \text{Lat}(\text{Alg}(\mathfrak{L})), P \geq P_{\frac{1}{2}} \Rightarrow P = P_{\frac{1}{2}} \text{ or } P = I. \tag{4.3}$$

Suppose $P \in \text{Lat}(\text{Alg}(\mathfrak{L}))$, $P \leq P_{\frac{1}{2}}$. P can be written in the form $P = \begin{pmatrix} \tilde{P} & 0 \\ 0 & 0 \end{pmatrix}$ for some projection $\tilde{P} \in \mathcal{B}(P_{\frac{1}{2}}\mathcal{H})$. If $T = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \text{Alg}(\mathfrak{L})$, then since $\tilde{P}$ is invariant under T, another simple computation shows us,

$$\tilde{P}A_1\tilde{P} = A_1\tilde{P}.$$

By remark(4.1.1), $\mathcal{T} = \{A_1 : \exists A_2, A_3$ such that $\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \text{Alg}(\mathfrak{L})\}$, regarded as a subset of $\mathcal{B}(P_{\frac{1}{2}}\mathcal{H})$ is dense in the nest algebra $\text{Alg}_{P_{\frac{1}{2}}\mathcal{H}}(\{p_t : t \in [0,1]\})$ in the SOT. Since $\{p_t : t \in [0,1]\}$ is reflexive, $\tilde{P} = p_t$ for some $t \in [0,1]$. Thus,

$$P \in \text{Lat}(\text{Alg}(\mathfrak{L})), P \leq P_{\frac{1}{2}} \Rightarrow \exists t \in [0, \frac{1}{2}] \mid P = P_t. \tag{4.4}$$

Now, if $\tau(P) < \frac{1}{2}$, $P_{\frac{1}{2}} \leq P \vee P_{\frac{1}{2}} \lneq I$ and by (4.3) and (4.4),

$$P \in \text{Lat}(\text{Alg}(\mathfrak{L})), \tau(P) < \frac{1}{2} \Rightarrow \exists t \in [0, \frac{1}{2}], P = P_t. \tag{4.5}$$

We next note that $\mathfrak{L}$ is also generated (as a lattice) by $\{P_{\frac{1}{2}}, Q \vee P_t : t \in [0, \frac{1}{2}]\}$(as $P_t = Q_t \wedge P_{\frac{1}{2}}$). The operator $U = \begin{pmatrix} \sqrt{H} & \sqrt{1-H} \\ -\sqrt{1-H} & \sqrt{H} \end{pmatrix}$ is a unitary and $U^*QU = P_{\frac{1}{2}}$, while $U^*P_{\frac{1}{2}}U = Q = \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & 1-H \end{pmatrix}$. $U^*(Q \vee P_t)U \geq U^*QU = P_{\frac{1}{2}}$ and if we let

$$\begin{pmatrix} I & 0 \\ 0 & r_t \end{pmatrix} = U^*(Q \vee P_t)U,$$

then $\{r_t : t \in [0, \frac{1}{2}]\}$ is a nest in $\mathcal{B}((I - P_{\frac{1}{2}})\mathcal{H})$.

An identical analysis shows that

$$P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L})), P \geq Q \Rightarrow \exists t \in [0, \frac{1}{2}] \text{ such that } P = Q \vee P_t, \tag{4.6}$$

and that

$$P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L})), P \leq Q \Rightarrow P = 0 \text{ or } P = 0. \tag{4.7}$$

If $P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L})), \tau(P) > \frac{1}{2}$, then by (4.6), (4.7), $0 < \tau(P \wedge Q) \leq \frac{1}{2}$

$$P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L})), \tau(P) > \frac{1}{2} \Rightarrow \exists t \in [0, \frac{1}{2}], P = Q_t \tag{4.8}$$

Suppose $\tau(P) = \frac{1}{2}$. Let, for $t \in [0, \frac{1}{2}]\}$, $R_t = (P \vee P_t) \wedge Q$. $R_t \leq Q$ and by (4.6), for any $t \in [0, \frac{1}{2}]$, $R_t = 0$ or $R_t = Q$. The map $t \to R_t$ is continuous(with respect to the toplogy given by the $|| \cdot ||_2$ on $\mathfrak{M}$) and has image in the set $\{0, Q\}$ and hence, is constant.

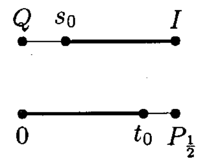
If $R_t = 0 \; \forall t \in [0, \frac{1}{2}] \Rightarrow \tau(P \vee P_t) \leq \frac{1}{2}$ and in particular, $\tau(P \vee P_{\frac{1}{2}}) \leq \frac{1}{2}$. $\frac{1}{2} \leq \tau(P_{\frac{1}{2}}) \leq \tau(P \vee P_{\frac{1}{2}}) \leq \frac{1}{2}$ and by the faithfulness of the trace, $P = P_{\frac{1}{2}}$.

And if $R_t = Q \; \forall t \in [0, \frac{1}{2}]$, setting $t = 0$, $P \wedge Q = Q$. $\tau(P \wedge Q) = \tau(P) = \frac{1}{2}$ and by the faithfulness of the trace, $P = Q$.

Combining this with (4.5),(4.8), we have that the lattice $\mathfrak{L} = \{P_t, Q_t : t \in [0, \frac{1}{2}]\}$ is reflexive.

Theorem 4.1.1. *$\mathfrak{L}$ is minimal reflexive. More in fact is true. If $\mathfrak{K}$ is a complete lattice in $\mathfrak{M}$, $\mathfrak{K} \subsetneq \mathfrak{L}$, then $\mathfrak{K}'' \subsetneq \mathfrak{L}'' = \mathfrak{M}$.*

Proof. Given $\mathfrak{K}$ as above, let $\mathcal{S} = \{s \in [0, \frac{1}{2}] | Q_s \in \mathfrak{K}\}$ and $\mathcal{T} = \{t \in [0, \frac{1}{2}] | P_t \in \mathfrak{K}\}$. Since $\mathfrak{K}$ is complete(which is equivalent to SOT closedness) $\mathcal{S}, \mathcal{T}$ are closed subsets of $[0, \frac{1}{2}]$. Let t_0 be the largest element in $\mathcal{T}$ and let s_0 be the smallest element in $\mathcal{S}$. The figure below is a pictorial representation of the sublattice $\mathfrak{K}$.



The lattice $\mathfrak{K}$

The proof splits into three cases.

Case 1: $s_0 \gneq 0$.

Let $R = (P_{s_0} \vee Q) \wedge (1 - P_{\frac{1}{2}})$. Note that $R \leq (P_{s_0} \vee Q) = Q_{s_0}$ and hence, $R \leq Q_t \; \forall t \in \mathcal{S}$. Further, $R \leq (I - P_{\frac{1}{2}})$ and hence, $RP_t = 0 \; \forall t \in [0, \frac{1}{2}]$.

$$\tau(R) = \tau(P_{s_0} \vee Q) + \tau(1 - P_{\frac{1}{2}}) - \tau(P_{s_0} \vee Q \vee (1 - P_{\frac{1}{2}})) = s_0 + \frac{1}{2} + \frac{1}{2} - 1 = s_0 \neq 0, 1.$$

Thus, R is a non-trivial projection in $\mathfrak{M}$ that commutes with $\{P_t : t \in [0, \frac{1}{2}],\; Q_s : s \in \mathcal{S}\}$, and therefore with $\mathfrak{K}''$. Thus, $\mathfrak{K}$ cannot generate $\mathfrak{M}$.

Case 2: $t_0 \lneq \frac{1}{2}$.

$R = (1 - P_{t_0}) \wedge Q$ is a non-trivial projection in $\mathfrak{M}$ that commutes with $\{P_t : t \in \mathcal{T},\; Q_s : s \in [0, \frac{1}{2}]\}$ and hence with $\mathfrak{K}$. Again, $\mathfrak{K}$ cannot generate $\mathfrak{M}$.

Case 3: $s_0 = 0$ and $t_0 = \frac{1}{2}$.

Since $P_t \wedge Q_s = P_{\min(s,t)}$, setting $t = t_0 = \frac{1}{2}$, we see that if $P_t \wedge Q_s \in \mathfrak{K}$ then $P_{\frac{1}{2}} \wedge Q_s = P_s \in \mathfrak{K}$. Thus, $\mathcal{T} \supseteq \mathcal{S}$. And since $P_t \vee Q_s = Q_{\max(s,t)}$, setting $s = s_0 = 0$, if $P_t \in \mathfrak{K}$, $P_t \vee Q_0 = P_t \vee Q = Q_t \in \mathfrak{K}$. Thus, $\mathcal{S} \supseteq \mathcal{T}$. We therefore have that $\mathcal{S} = \mathcal{T}$.

Thus,

$$\mathfrak{K} = \{P_t \mid t \in \mathcal{T},\ Q,\ P_t \vee Q \mid t \in \mathcal{T}\}.$$

Note that $\frac{1}{2} \in \mathcal{T}$ and hence, $P_{\frac{1}{2}} \in \mathfrak{K}$. Recall that,

$$P_t = \begin{pmatrix} p_{2t} & 0 \\ 0 & 0 \end{pmatrix} P_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q = \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & 1-H \end{pmatrix}$$

We have thus,

$$\begin{aligned}
\mathfrak{K}'' &= \{P_t | t \in \mathcal{T}, Q\}'' \\
&= \{p_{2t} | t \in \mathcal{T}, H\}'' \otimes \mathcal{M}_2(\mathbb{C}) \\
&\subsetneq \{p_t | t \in [0,1], H\}'' \otimes \mathcal{M}_2(\mathbb{C}) \qquad \text{(as } H, \{p_t\} \text{ are free)} \\
&= \mathfrak{M}''.
\end{aligned}$$

In step 3 we use the fact that if $\mathfrak{A} \subsetneq \mathfrak{B}$, $\mathfrak{C}$ are von Neumann subalgebras of a finite von Neumann algebra, and $\mathfrak{C}$ is free from $\mathfrak{B}$, then $\mathfrak{A} * \mathfrak{C} \subsetneq \mathfrak{B} * \mathfrak{C}$, where we take the reduced free product with respect to traces on $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$.

The theorem is proved. □

An entirely similar computation gives us a little more.

Remark 4.1.2. *Let $\mathfrak{N} \subset \mathcal{B}(\mathcal{K})$ be a finite von Neumann algebra. Let $\mathfrak{L}_0 = \{p_\lambda : \lambda \in \Lambda\}$ be an arbitrary KS lattice, with $(\mathfrak{L}_0)'' \in \mathfrak{N}$. Let $H \in \mathfrak{N}$ be a positive contraction whose point spectrum does not contain 1 and which is free from $(\mathfrak{L}_0)''$. Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C}) \subset \mathcal{B}(\mathcal{K}) \otimes M_2(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ and let $(e_{ij})_{1 \le i,j \le 2}$ be a system of matrix units for $M_2(\mathbb{C})$.*

Let, for $\lambda \in \Lambda$, $P_\lambda = p_\lambda \otimes e_{11}$. Denote the lattice $\{P_\lambda : \lambda \in \Lambda\}$ in $\mathfrak{M}$ by $\mathfrak{L}_1$. The operator

$$Q = \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & 1-H \end{pmatrix}$$

is a projection. Let $\mathfrak{L}$ be the lattice in $\mathfrak{M}$ generated by the elements in $\{\mathfrak{L}_1, Q\}$. Then $\mathfrak{L} = \{P_\lambda, P_\lambda \vee Q, \lambda \in \Lambda\}$ and $\mathfrak{L}$ is a KS lattice(for $(\mathfrak{L})''$) isomorphic(as lattices) to the union of two disjoint copies of the lattice $\mathfrak{L}_0$ with itself.

4.2 Full nests with a trace half projection

We begin by recalling some standard facts in free probability theory. Given a von Neumann probability space $(\mathfrak{M}, \tau)$, the distribution of an element X in $\mathfrak{m}$ is the linear map ϕ : $\mathbb{C}[x] \to \mathbb{C}$, given by $\phi(p(x)) = \tau(p(X))$ for any polynomial p. The following definitions were introduced by Voiculescu, see for instance [32].

Definition 4.2.1. *An element A in a von Neumann probability space $(\mathfrak{M}, \tau)$ is* quarter-circular *if it is self-adjoint and has distribution given by*

$$\tau(A^n) = \int_0^1 t^n \sqrt{1-t^2} dt \qquad \textit{for } n = 0, 1, 2, \cdots .$$

An element $S \in (\mathfrak{M}, \tau)$ is $((0,1))$ semi-circular *if S is self-adjoint and has distribution given by*

$$\tau(A^n) = \int_{-1}^1 t^n \sqrt{1-t^2} dt \qquad \textit{for } n = 0, 1, 2, \cdots .$$

And an element $C \in (\mathfrak{M}, \tau)$ is $((0,1))$ circular *if $C = S_1 + iS_2$, where S_1, S_2 are $((0,1))$ free semi-circular elements. Finally, a unitary U in a von Neumann probability space is said to be a Haar unitary if all its moments save the zeroth one are 0, i.e $\tau(U^n) = 0$ for $n = \pm 1, \pm 2, \cdots$.*

Remark 4.2.1. *It was shown by Voiculescu in [32] that any circular element C in $(\mathfrak{M}, \tau)$ has polar decomposition (inside the von Neumann algebra $\mathfrak{M}$) $C = UA$, where U is a Haar unitary, A is quarter circular and U, A are free with respect to τ.*

We extend the analysis to cover the case of a full nest and a trace half projection that satisfies a freeness relation. Let $\mathfrak{N} \in \mathcal{B}(\mathcal{K})$ be a finite von Neumann algebra with a faithful normal trace τ, with $\mathfrak{N}$ satisfying the following several conditions.

1. $\mathfrak{N}$ contains $\mathfrak{L}_0 = \{p_t \mid 0 \leq t \leq 1\}$, a complete lattice(i.e SOT closed) in $\mathfrak{N}$, with $\tau(p_t) = t$.

2. $\mathfrak{N}$ contains H, an operator with the same distribution as $A^2(1+A^2)^{-1}$ where A is a quarter circular element. We see that $||H|| = \frac{1}{2}$. Note that A and hence, H, has no point spectrum. In particular, H is a contraction without 1 in its point spectrum.

3. $\mathfrak{N}$ contains U, a Haar unitary.

4. The three algebras $(\{p_t : t \in [0,1]\})'', (\{H\})'', (\{U\})''$ are free.

Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C}) \subset \mathcal{B}(\mathcal{K}) \otimes M_2(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ and let $(e_{ij})_{1\leq,j\leq 2}$ be a system of matrix units for $M_2(\mathbb{C})$. Let Q be the operator $Q = H \otimes e_{11} + \sqrt{H(I-H)}U \otimes e_{12} + U^*\sqrt{H(I-H)} \otimes e_{21} + U^*(I-H)U \otimes e_{22}$. Or, in matrix form,

$$Q = \begin{pmatrix} H & \sqrt{H(1-H)}U \\ U^*\sqrt{H(1-H)} & U^*(I-H)U \end{pmatrix}.$$

It is easy to see that Q is a projection. Let, for $t \in [0, \frac{1}{2}]$, $P_t = p_{2t} \otimes e_{11}$ and for $s \in [0, \frac{1}{2}]$, $P_{\frac{1}{2}+s} = e_{11} + p_{2t} \otimes e_{22}$. Let $t, s \in [0, \frac{1}{2}]$. Then, in matrix form, we have,

$$P_t = \begin{pmatrix} p_{2t} & 0 \\ 0 & 0 \end{pmatrix} \qquad P_{\frac{1}{2}} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad P_{\frac{1}{2}+s} = \begin{pmatrix} I & 0 \\ 0 & p_{2s} \end{pmatrix}$$

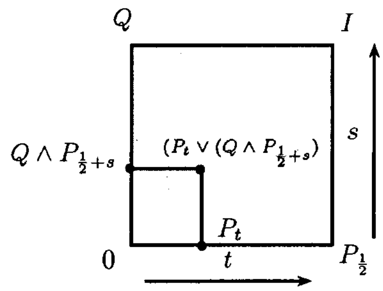
Let $\mathfrak{L}_1 = \{P_t : t \in [0, \frac{1}{2}]\}$, $\mathfrak{L}_2 = \{P_{\frac{1}{2}+s} : s \in [0, \frac{1}{2}]\}$. Let $\mathfrak{L}$ be the lattice in $\mathfrak{M}$ generated(as a lattice) by $\mathfrak{L}_1, \mathfrak{L}_2$ and Q. It is easy to see that $(\mathfrak{L})''$ is a II_1 factor.

Theorem 4.2.1. *With the assumptions as above, $\mathfrak{L} \subset \mathfrak{M}$ is a KS lattice for the von Neumann algebra $\mathfrak{L}''$.*

Before we give the proof of the theorem, we make some preliminary computations and prove a lemma and a proposition.

By theorem(4.0.3)(apply it with projections $P_{\frac{1}{2}}, Q$),

$$\mathfrak{L} = \{(P_t \vee Q) \wedge P_{\frac{1}{2}+s} : t, s \in [0, \frac{1}{2}]\}$$



The lattice $\mathfrak{L}$

Lemma 4.2.1. $\mathrm{Alg}(\mathfrak{L}) = \{ \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} : p_t A_1 p_t = A_1 p_t$ *for* $t \in [0,1], p_t A_2 p_t = A_2 p_t$ *for* $t \in [0,1]$ *and*

$$\sqrt{1-H} A_1 \sqrt{H} + \sqrt{1-H} A_3 \sqrt{1-H} = \sqrt{H} A_2 \sqrt{1-H} \ \}.$$

Also, the sets

$$\mathcal{S} = \{ A_1 : \exists A_s, A_3 \text{ such that } \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L}) \},$$

$$\mathcal{T} = \{ A_2 : \exists A_1, A_3 \text{ such that } \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L}) \},$$

regarded as subsets of $\mathcal{B}(P_{\frac{1}{2}})\mathcal{H})$ *(respectively,* $\mathcal{B}((I - P_{\frac{1}{2}})\mathcal{H}))$ *are dense in* $\mathrm{Alg}_{P_{\frac{1}{2}}\mathcal{H}}(\{p_t : t \in [0,1]\})$ *(respectively,* $\mathrm{Alg}_{(I-P_{\frac{1}{2}})\mathcal{H}}(\{p_t : t \in [0,1]\})$*) in the SOT.*

Proof. This is the same computation as in lemma(4.1.1). ☐

Proposition 4.2.1. *The projection* $Q = \begin{pmatrix} H & \sqrt{H(1-H)}U \\ U^* \sqrt{H(1-H)} & U^*(1-H)U \end{pmatrix}$ *where* U, H *as in the statement of theorem(4.2.1), is the projection onto the subspace*

$$\mathcal{H}_1 = \begin{pmatrix} \tilde{H} U \eta \\ \eta \end{pmatrix}, \text{ where } \eta \in (I - P_{\frac{1}{2}})\mathcal{H},$$

where $\tilde{H} = \sqrt{H(1-H)^{-1}}$ *(bounded and of norm 1). And the projection* $(P_t \vee Q) \wedge P_{\frac{1}{2}+s}$, *where* $t, s \in [0, \frac{1}{2}]$, *is the projection onto the subspace*

$$\mathcal{H}_1 = \begin{pmatrix} \xi + \tilde{H}U\eta \\ \eta \end{pmatrix}, \text{ where } \xi \in \mathrm{Ran}(p_{2t}) \subset P_{\frac{1}{2}}\mathcal{H}, \eta \in \mathrm{Ran}(p_{2s}) \subset (I - P_{\frac{1}{2}})\mathcal{H}$$

Proof. If $Q\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, we have that

$$\begin{pmatrix} H\xi + \sqrt{H(1-H)}U\eta \\ U^*\sqrt{H(1-H)}\xi + U^*(I-H)U\eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

i.e.

$$\begin{pmatrix} (I-H)\xi - \sqrt{H(1-H)}U\eta \\ U^*\sqrt{H(1-H)}\xi - U^*HU\eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Comparing the first co-ordinates,

$$\sqrt{H(1-H)}U\eta = (I-H)\xi$$

Thus,

$$\xi = \tilde{H}U\eta.$$

Also, comparing the second co-ordinates,

$$U^*\sqrt{H(1-H)}\xi + U^*(1-H)U\eta = \eta$$

Thus, we get the identical equation

$$\xi = \tilde{H}U\eta.$$

The proof of the second assertion is similar. □

Proposition 4.2.2. $\mathfrak{L}$ *is reflexive.*

Proof. First. let $P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}))$, with $P \leq P_{\frac{1}{2}}$. P can be written in the form $P = \begin{pmatrix} \tilde{P} & 0 \\ 0 & 0 \end{pmatrix}$ for some projection $\tilde{P} \in \mathcal{B}(P_{\frac{1}{2}}\mathcal{H})$. If $T = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L})$, then since P

is invariant under T, a simple computation shows us,

$$\tilde{P}A_1\tilde{P} = A_1\tilde{P}.$$

By 4.2.1,

$$\mathcal{S} = \{A_1 : \exists A_s, A_3 \text{ such that } \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L})\},$$

is dense in $\mathrm{Alg}_{P_{\frac{1}{2}}\mathcal{H}}(\{p_t : t \in [0,1]\})$. Since nests are reflexive, we have that there exists a t in $[0,1]$ such that $\tilde{P} = p_t$, i.e

$$P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L})) \text{ with } P \leq P_{\frac{1}{2}} \Rightarrow \exists t \in [0, \frac{1}{2}] \text{ such that } P = P_t$$

Similarly, we have that

$$P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L})) \text{ with } P \geq P_{\frac{1}{2}} \Rightarrow \exists s \in [0, \frac{1}{2}] \text{ such that } P = P_{\frac{1}{2}+s}$$

Let $P \in \mathrm{Lat}(\mathrm{Alg}(\mathfrak{L}))$. $P \wedge P_{\frac{1}{2}} = P_t$ for some $t \in [0, \frac{1}{2}]$ and $P \vee P_{\frac{1}{2}} = P_{s+\frac{1}{2}}$ for some $s \in [0, \frac{1}{2}]$. We claim that $P = (P_t \vee Q) \wedge P_{s+\frac{1}{2}}$.

Let $\tilde{P} = (P_t \vee Q) \wedge P_{s+\frac{1}{2}}$. Then, $\tilde{P} \wedge P_{\frac{1}{2}} = P_t$ and $\tilde{P} \vee P_{\frac{1}{2}} = P_{s+\frac{1}{2}}$.

Decompose the Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = P_t\mathcal{H} \oplus (P_{\frac{1}{2}} - P_t)\mathcal{H} \oplus (P_{\frac{1}{2}+s} - P_{\frac{1}{2}})\mathcal{H} \oplus (I - P_{\frac{1}{2}+s})\mathcal{H}.$$

Choose a set of partial isometries between the components, which will allow us to write each operator in $\mathcal{B}(\mathcal{H})$ as a 4×4 matrix. In particular,

$$\tilde{P} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & K & \sqrt{K(I-K)}U & 0 \\ 0 & U^*\sqrt{K(I-K)} & C & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & L & \sqrt{L(I-L)}V & 0 \\ 0 & V^*\sqrt{L(I-L)} & D & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ where K, L do not have 1 in their point spectra, C, D do not have 0 in their point spectra, $\sqrt{K(I-K)}U, \sqrt{L(I-L)}V$ are polar decompositions and

$$\begin{aligned} C - C^2 &= U^*K(I-K)U \\ D - D^2 &= V^*L(I-L)V \end{aligned} \tag{4.9}$$

Any element $T \in \mathrm{Alg}(\mathfrak{L})$, can be written in the form, $T = \begin{pmatrix} A_1^1 & A_1^3 & A_3^1 & A_3^3 \\ 0 & A_1^2 & A_3^4 & A_3^2 \\ 0 & 0 & A_2^1 & A_2^3 \\ 0 & 0 & 0 & A_2^2 \end{pmatrix}$.

Since T leaves P and $\tilde{P}$ invariant, we get

$$\begin{aligned} (I-K)A_1^2\sqrt{K(I-K)}U + (I-K)A_3^4C &= \sqrt{K(I-K)}UA_2^1C \\ (I-L)A_1^2\sqrt{L(I-L)}V + (I-L)A_3^4D &= \sqrt{L(I-L)}VA_2^1C \end{aligned}$$

Now, from remark(4.1.1), it follows that we may choose a net $(T_\alpha)_{\alpha\in\mathcal{I}}$(with T_α bounded) with $(P_{\frac{1}{2}} - P_t)T_\alpha \mid_{(P_{\frac{1}{2}}-P_t)\mathcal{H}} \to_{\mathrm{SOT}} 0$ and $(P_{\frac{1}{2}+s} - P_{\frac{1}{2}})T_\alpha \mid_{(P_{\frac{1}{2}+s}-P_{\frac{1}{2}})\mathcal{H}} \to_{\mathrm{SOT}} I$. In the limit, we get,

$$\begin{aligned} \left(A_3^4 - \sqrt{K(I-K)^{-1}}U\right) C &= 0 \\ \left(A_3^4 - \sqrt{L(I-L)^{-1}}V\right) D &= 0 \end{aligned}$$

Since K, L do not have 1 in their point spectra and C, D do not have 0 in their point spectra,

$$A_3^4 = \sqrt{K(I-K)^{-1}}U = \sqrt{L(I-L)^{-1}}V.$$

Hence, $\sqrt{K(I-K)^{-1}}U = \sqrt{L(I-L)^{-1}}V$ and therefore, $K = L$ and $U = V$.

C, D may have 1 in their point spectra, but not 0. From (4.9), $C - C^2 = D - D^2$. Thus, $C = D$. We conclude that $P = \tilde{P}$ and $\mathfrak{L}$ is reflexive. □

Proof. (of theorem (4.2.1)) The proof will split into three cases.

Case 1: $\{P_{\frac{1}{2}}, Q\} \in \mathfrak{K}$

If $P = (P_t \vee Q) \wedge P_{\frac{1}{2}+s} \in \mathfrak{K}$, then $P \wedge P_{\frac{1}{2}} = P_t \in \mathfrak{K}$ and $P \vee P_{\frac{1}{2}} = P_{\frac{1}{2}+s} \in \mathfrak{K}$. Let $\mathcal{T} = \{t : P_t \in \mathfrak{K}\} \subset [0, \frac{1}{2}]$ and $\mathcal{S} = \{s : P_{\frac{1}{2}+s} \in \mathfrak{K}\} \subset [0, \frac{1}{2}]$. $\{0, \frac{1}{2}\} \subset \mathcal{T}, \mathcal{S} \subset [0, \frac{1}{2}]$. If $\mathcal{T} = \mathcal{S} = [0, \frac{1}{2}]$, then $\mathfrak{K} = \mathfrak{L}$ and $\mathfrak{K}$ is not a strict sublattice. Assume without loss of generality that $\mathcal{T} \neq [0, \frac{1}{2}]$.

$$\mathfrak{K} \subseteq \{P_t : t \in \mathcal{T}, P_{\frac{1}{2}+s} : s \in [0, \frac{1}{2}], Q\}.$$

Denote $\mathcal{A}_1 = (\{P_t : t \in \mathcal{T}\})''$ and $\mathcal{A}_2 = (\{P_t : t \in [0, \frac{1}{2}]\})''$. ($\mathcal{A}_1 \subsetneq \mathcal{A}_2$). Denote $\mathcal{B} = (\{P_{\frac{1}{2}+t} : t \in [\frac{1}{2}, 1]\})''$. We see that $(\mathfrak{K})''$ is generated by

$$\begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{B} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} H & \sqrt{H(1-H)}U \\ U^*\sqrt{H(1-H)} & U^*(1-H)U \end{pmatrix}$$

Conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$, we see that it is unitarily conjugate to the algebra generated by

$$\begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & U\mathcal{B}U^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} H & \sqrt{H(1-H)} \\ \sqrt{H(1-H)} & (1-H) \end{pmatrix}$$

where $\mathcal{A}_1$ and $U\mathcal{B}U^*$ are now free. It is easy to see that this algebra is equal to $M_2(\mathbb{C}) \otimes \{\mathcal{A}_1, U\mathcal{B}U^*, H\}$.

Thus, if $\{P_{\frac{1}{2}}, Q\} \in \mathfrak{K}$,

$$(\mathfrak{K})'' = M_2(\mathbb{C}) \otimes \{\mathcal{A}_1 * \mathcal{B} * \{H\}''\} \subsetneq M_2(\mathbb{C}) \otimes \{\mathcal{A}_2 * \mathcal{B} * \{H\}''\} = (\mathfrak{L})'', \qquad (4.10)$$

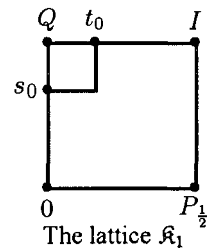
as $\mathcal{A}_1 \subsetneq \mathcal{A}_2$.

Case 2: $Q \notin \mathfrak{K}$

Since we assume that $\mathfrak{K}$ is strongly closed, there are s_0, t_0, with $\{s_0, t_0\} \in [0, \frac{1}{2}]$ such that $\mathfrak{K} \subseteq \{(P_t \vee Q) \wedge P_{\frac{1}{2}+s} \mid s \leq s_0 \text{ or } t \geq t_0\}$. Denote the latter lattice by $\mathfrak{K}_1$.

It is easy to see that

$$\mathfrak{K}'' \subseteq \{(P_t \vee Q) \wedge P_{\frac{1}{2}+s} \mid s \leq s_0 \text{ or } t \geq t_0\}'' = \{P_t \mid_{t=0}^{t=1}, Q \vee P_{t_0}, Q \wedge P_{\frac{1}{2}+s_0}\}''$$



The lattice $\mathfrak{K}_1$

By (4.2.1), the projection $Q \vee P_{t_0}$ is the projection onto the subspace

$$\mathcal{H}_1 = \begin{pmatrix} \xi + \tilde{H}U\eta \\ \eta \end{pmatrix} \qquad \xi \in \operatorname{Ran}(p_{t_0})$$

And the projection $Q \wedge P_{\frac{1}{2}+s_0}$ is the projection onto the subspace

$$\mathcal{H}_1 = \begin{pmatrix} \tilde{H}U\eta \\ \eta \end{pmatrix} \qquad \eta \in \operatorname{Ran}(p_{s_0})$$

An operator T that commutes with $\mathfrak{K}$ may be written in the form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \qquad \text{where } T_1, T_2 \text{ commute with } \{p_t : t \in [0, \frac{1}{2}]\}$$

$$T_1(1-p_{t_0})\tilde{H}U - (1-p_{t_0})\tilde{H}UT_2 = 0$$

$$T_1\tilde{H}Up_{s_0} - \tilde{H}Up_{s_0}T_2 = 0 \tag{4.11}$$

Let us denote $C = \tilde{H}U$. By remark(4.2.1), C is a circular element. Let $C' = C - p_{t_0}C(1-p_{s_0})$, ie C with a corner snipped off. We have that

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ commutes with } D' = \begin{pmatrix} 0 & C' \\ (C')^* & 0 \end{pmatrix}$$

If T further commutes with $D = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$, then $(\mathfrak{K})'' = (\mathfrak{L})''$. We will show that there is an operator T that commutes with D' but not D.

Let $\mathcal{A} = \left(\begin{pmatrix} p_t & 0 \\ 0 & p_s \end{pmatrix} : t, s \in [0, \frac{1}{2}]\right)''$, which is abelian. Let S_1, S_2 be two semicircular elements free from C as well as $\{p_t : t \in [0, \frac{1}{2}]\}$. Let $S = \begin{pmatrix} S_1 & C \\ C^* & S_2 \end{pmatrix}$. Then, S is a semicircular element free from $\mathcal{A}$ [33]. Let $S' = \begin{pmatrix} S_1 & C' \\ (C')^* & S_2 \end{pmatrix}$. We will show that $(S, \mathcal{A})'' \supsetneq (S', \mathcal{A})''$, which will prove our theorem.

By enlarging $\mathfrak{K}_1$ if necessary, assume that $t_0 = 1 - s_0 = \frac{1}{n}$ for some $n \in \mathbb{N}$. Write the II_1 factor $(\mathfrak{L})'' \cong \mathfrak{M}_1 \otimes M_{2n}(\mathbb{C})$ for a II_1 factor $\mathfrak{M}_1$. Choose a system of matrix units $(e_{ij})_{1\le i,j\le 2n}$ for $M_{2n}(\mathbb{C})$, with $\sum_i^{2n} e_{ii} = I$ and $e_{ii} \in \mathcal{A}$ for $1 \le i \le 2n$. We write elements in $(\mathfrak{L})''$ as $2n \times 2n$ matrices over $\mathfrak{M}_1$. Choose a self-adjoint generator A for $\mathcal{A}$. With this identification, $A = \text{diag}(d_i)_{1\le i\le 2n}$ and $S = (s_{ij})_{1\le i,j\le 2n}$. with $\{s_{ii}, 1 \le i \le 2n\}$ semicircular, $\{s_{ij}, 1 \le i < j \le 2n\}$ circular, $s_{ij} = s_{ji}^*, 1 \le i,j \le 2n$ and the elements $\{d_i : 1 \le i \le 2n, s_{ij} : 1 \le i < j \le 2n\}$ free.

Let $s_{i2} = u_{i2}t_{i2}$, for $1 \le i \le 2n$, be the polar decompositions. $u_{22} = I$ and the elements $u_{i2}, i \ne 2$ are Haar unitaries. Let $U = \text{diag}(u_{i2})_{1\le i\le 2n}$. Then,

$$U^*AU = \text{diag}(u_{i2}^* d_i u_{i2})_{1\le i\le 2n}$$

$$U^*SU = (u_{i2}^* s_{ij} u_{j2}^*)_{1\le i,j\le 2n}$$

$$U^*SU = \begin{pmatrix} u_{12}^* s_{11} u_{12}^* & t_{12} & u_{12}^* s_{13} u_{32}^* & \cdots & u_{12}^* s_{1,2n} u_{2n,2}^* \\ u_{22}^* s_{21} u_{12}^* & t_{22} & u_{22}^* s_{23} u_{32}^* & \cdots & u_{22}^* s_{2,2n} u_{2n,2}^* \\ u_{32}^* s_{31} u_{12}^* & t_{32} & u_{32}^* s_{33} u_{32}^* & \cdots & u_{3,2}^* s_{3,2n} u_{2n,2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{2n,2}^* s_{2n,1} u_{12}^* & t_{2n,2} & u_{2n,2}^* s_{2n,3} u_{32}^* & \cdots & u_{2n,2}^* s_{2n,2n} u_{2n,2}^* \end{pmatrix}$$

Then we have $\{(U^*AU)(i,i),(U^*SU)(i,i),(U^*SU)(i,j)$ where $1 \le i < j \le 2n\}$ are a family of $4n^2+1$ free elements (with $(U^*SU)(i,i)$ semicircular and $(U^*SU)(i,j)$ circular).

It is easy to see that $U^*\mathcal{A}U$ contains the diagonal matrix units $(e_{ii}(1 \le i \le 2n)$. Also, the polar part of $e_{ii}U^*SUe_{22}$ is e_{i2}. Since $e_{ij} = e_{i2}e_{j2}^*$, we see that $(\{U^*\mathcal{A}U, U^*SU\})''$ contains all the matrix units. Therefore,

$$(\{U^*\mathcal{A}U, U^*SU\})'' = M_{2n}((\{(U^*AU)(i,i),(U^*SU)(i,j)\})'').$$

Let $\mathcal{I} = \{\{i,j\} : 1 \le i,j \le 2n, \{i,j\} \ne \{1,2n\} \text{ or } \{2n,1\}\}$. In identical fashion, we see that,

$$(\{U^*\mathcal{A}U, U^*S'U\})'' = M_{2n}((\{(U^*AU)(i,i),(U^*SU)(i,j) : \{i,j\} \in \mathcal{I}\})'').$$

Thus,

$$(S,\mathcal{A})'' \supsetneq (S',\mathcal{A})''$$

We have shown, thus,

$$\text{If } Q \notin \mathfrak{K}, \text{ then } \mathfrak{K}'' \subsetneq \mathfrak{L}'' \tag{4.12}$$

Case 3: $P_{\frac{1}{2}} \notin \mathfrak{K}$

In identical fashion to Case 2, we can prove that,

$$\text{If } P_{\frac{1}{2}} \notin \mathfrak{K}, \text{ then } \mathfrak{K}'' \subsetneq \mathfrak{L}'' \tag{4.13}$$

By (4.10)), (4.12), (4.13), the proof is complete. □

Theorem 4.2.2. *Let $\mathfrak{L}_0 = \{p_\lambda : \lambda \in \Lambda\}$ be a KS lattice and $\mathfrak{L}_1 = \{p_t : t \in \mathcal{S} \subset [0,1]\}$ an SOT closed nest, both in a finite von Neumann algebra $\mathfrak{N}$. Let $\mathfrak{N}_1 = \mathfrak{N} * LF_2$ and let U, H be respectively, a haar unitary and an element with the same distribution as $A^2(1+A^2)^{-1}$,* *where A is quarter-circular and so that U, H are free and free from $\mathfrak{N}_1$. Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C})$ and let $(e_{ij})_{1\leq i,j\leq 2}$. Let Q be the projection,* $Q = \begin{pmatrix} H & \sqrt{H(1-H)}U \\ U^*\sqrt{H(1-H)} & U^*(1-H)U \end{pmatrix}$. *Let $\mathfrak{L} \subseteq \mathfrak{M}$ be the lattice generated by $\{p_\lambda \otimes e_{11}, e_{11} \oplus p_t \otimes e_{22}, Q : \lambda \in \Lambda, t \in \mathcal{S}\}$. Then, $\mathfrak{L}$ is a KS lattice for $\mathfrak{L}''$, isomorphic (as lattices) to $\mathfrak{L}_0 \times \mathfrak{L}_1$.*

Proof. Denote $p_\lambda \otimes e_{11}$ by P_λ and $e_{11} \oplus p_t \otimes e_{22}$ by P_t. Denote e_{11} by P. By (4.0.3),

$$\mathfrak{L} = \{P_\lambda \vee Q) \wedge P_t : \lambda \in \Lambda, t \in \mathcal{S}\}$$

By an identical computation as in (4.2.2), $\mathfrak{L}$ is reflexive. Suppose $\mathfrak{K}$ is a strict sublattice, then we have three cases.

Case 1: $P, Q \in \mathfrak{K}$. Then $\mathfrak{K} = \{(P_\lambda \vee Q) \wedge P_t : \lambda \in \Delta, t \in \mathcal{T}\}$ for subsets $\Delta, \mathcal{T}$ of $\Lambda, \mathcal{S}$ and one of them is a strict subset. The proof is now exactly as in case 1 of the proof of (4.2.1).

Case 2: $Q \notin \mathfrak{K}$. We claim that there are elements $\lambda_0 \in \Lambda, s_0 \in \mathcal{S}$ so that $\mathfrak{K} \subsetneq \{(P_\lambda \vee Q) \wedge P_t : \lambda \in \Lambda, t \in \mathcal{T}$ with $\lambda \geq \lambda_0$ or $s \leq s_0\}$.

For each $s \in \mathcal{S}$, let

$$B_s = \min(\lambda : (P_\lambda \vee Q) \wedge P_t) \in \mathfrak{K}).$$

We note that the map $t \to B_t$ is increasing(as a map between directed sets $\mathcal{S}, \Lambda$). Since $Q \notin \mathfrak{K}$ and $\mathfrak{K}$ is SOT closed, there is a s_0 such that for $s \geq s_0$, $Q \wedge P_{\frac{1}{2}+s} \notin \mathfrak{K}$. $(P_{B_{s_0}} \vee Q) \wedge P_{s_0}$ is an element in $\mathfrak{K}$ and for any element $(P_\lambda \vee Q) \wedge P_s$ in $\mathfrak{K}$, with $s \geq s_0$, $((P_{B_{s_0}} \vee Q) \wedge P_{s_0}) \wedge ((P_\lambda \vee Q) \wedge P_s) = (P_{\min(B_{s_0},\lambda)} \vee Q) \wedge P_{s_0}$. By the definition of B_{s_0}, we must have that $B_{s_0} \leq \lambda$. Therefore, $B_s \geq B_{s_0}$ for $s > s_0$. i.e,

$$\text{If } (P_\lambda \vee Q) \wedge P_s \in \mathfrak{K} \qquad \text{then} \qquad s \leq s_0 \text{ or } \lambda \geq B_{s_0}$$

Therefore, $\mathfrak{K}$ is a sublattice of $\{Q \wedge P_{\frac{1}{2}+s_0}, Q \vee B_{s_0}, P_\lambda, \lambda \in \Lambda, P_{\frac{1}{2}+t}, s \in \mathcal{S}\}$.

The proof now is exactly the same as in Case 2 in the proof of (4.2.1).

Case 3: The proof is exactly as in the preceding case.

□

Definition 4.2.2. ***Maximal join:*** *Let $\mathfrak{L}_0 \subset \mathfrak{N}_0$, $\mathfrak{L}_1 \subset \mathfrak{N}_1$ be two KS lattices, with $(\mathfrak{N}_i, \tau_i)(i = 0, 1)$ finite von Neumann algebras. Let $\mathfrak{N} = \mathfrak{N}_0 * \mathfrak{N}_1 * LF_2$, choose U, H, respectively, a haar unitary and an element with the same distribution as $A^2(1 + A^2)^{-1}$, where A is quarter-circular and so that $(\mathfrak{L}_0)'', (\mathfrak{L}_0)'', U, H\}$ are free with respect to the trace on $\mathfrak{N}$. Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C})$ and let $(e_{ij})_{1 \leq i,j \leq 2}$. Let Q be the projection, $Q = \begin{pmatrix} H & \sqrt{H(1-H)}U \\ U^*\sqrt{H(1-H)} & U^*(1-H)U \end{pmatrix}$. Let $\mathfrak{L} \subseteq \mathfrak{M}$ be the lattice generated by $\{p_\lambda \otimes e_{11}, e_{11} \oplus p_t \otimes e_{22}, Q : \lambda \in \Lambda, t \in \mathcal{S}\}$. We call $\mathfrak{L}$ the maximal join of $\mathfrak{L}_0$ and $\mathfrak{L}_1$ or* $\mathrm{MaxJoin}(\mathfrak{L}_0, \mathfrak{L}_1)$. *It is easy to see that $\mathfrak{L} \cong \mathfrak{L}_0 \times \mathfrak{L}_1$(as lattices).*

Definition 4.2.3. *Let $\mathfrak{S}_1$ denote the Volterra nest. Inductively, define*

$$\mathfrak{S}_n = \mathrm{MaxJoin}(\mathfrak{S}_{n-1}, \mathfrak{S}_{n-1}), \quad \mathfrak{S}_n \subset \mathfrak{M}_n.$$

When $n = 2$, this is just the lattice of theorem(4.2.1).

Theorem 4.2.3. *For every $n \geq 1$, $\mathfrak{S}_n \subset \mathfrak{M}_n$ is a KS lattice and $(\mathfrak{S}_n)'' = LF_{r_n}$, where $r_1 = 1$ and $r_n = 1 + \frac{r_{n-1}}{2}$.*

Proof. Theorem(4.2.1) shows that $\mathfrak{S}_2 \subset \mathfrak{M}_2$ is a KS lattice. We shall give the proof for $\mathfrak{S}_3 \subset \mathfrak{M}_3$ and it will be clear that the same arguments will work for $\mathfrak{S}_n : n \geq 4$.

For the rest of the proof, fix $n = 3$. As in the proof of the previous theorem, let denote e_{11} by P. Any element in $\mathfrak{L}_0$ is indexed by $t, s \in [0, \frac{1}{2}]$. Let us write

$$\mathfrak{L}_0 = \{P(t, s) : t, s \in [0, \frac{1}{2}]$$

The lattice operations are

$$P(t, s) \vee P(t', s') = P(\max(t, t'), \max(s, s'))$$

$$P(t, s) \wedge P(t', s') = P(\min(t, t'), \min(s, s'))$$

Similarly, we may represent

$$\mathfrak{L}_1 = \{R(t, s) : t, s \in [0, \frac{1}{2}],$$

with

$$R(t,s) \vee R(t',s') = R(\max(t,t'), \max(s,s'))$$

$$R(t,s) \wedge R(t',s') = R(\min(t,t'), \min(s,s'))$$

Also,

$$\tau(P(t,s)) = \tau(R(t,s) = t + s - \frac{1}{2}$$

By theorem(4.0.3), and reparamterizing,

$$\mathfrak{L} = \{(P(t,s) \vee Q) \wedge R(t_1,s_1) : s,t,t_1,s_1 \in [0,\frac{1}{4}]\}$$

so that, if $(P(t,s) \vee Q) \wedge R(t_1,s_1) \in \mathfrak{L}$,

$$\tau((P(t,s) \vee Q) \wedge R(t_1,s_1) = t + s + t_1 + s_1$$

Denote the element $(P(t,s) \vee Q) \wedge R(t_1,s_1)$ by $P(t,s,t',s')$. With this parameterization,

$$Q = P(0,0,\frac{1}{4},\frac{1}{4}) \qquad P = P(\frac{1}{4},\frac{1}{4},0,0)$$

And the lattice operations become,

$$P(t,s,t_1,s_1) \vee P(t',s',t_1',s_1') = P(\max(t,t'), \max(s,s'), \max(t_1,t_1'), \max(s_1,s_1'))$$

$$P(t,s,t_1,s_1) \wedge P(t',s',t_1',s_1') = P(\min(t,t'), \min(s,s'), \min(t_1,t_1'), \min(s_1,s_1'))$$

The proof of the reflexivity of $\mathfrak{L}$ is exactly the same as that in proposition(4.2.2). For minimality, let $\mathfrak{K}$ be a strict sublattice. The proof splits into three cases,

Case 1: $P, Q \in \mathfrak{K}$.

The proof that $(\mathfrak{K})'' \subsetneq (\mathfrak{L})''$ is exactly as in Case 1 in the proof of theorem(4.2.1). Also see proof of Case 1 in theorem(4.2.2).

Case 2: $Q \notin \mathfrak{K}$. We will use the symbol $<_P$ to denote the partial order on $[0,\frac{1}{4}] \times [0,\frac{1}{4}]$, given by $(a,b) <_P (c,d)$ if $a \leq b$ and $c \leq d$, where $a,b,c,d, \in [0,\frac{1}{4}]$.

As above, $Q = P(0,0,\frac{1}{4},\frac{1}{4})$. Since $Q \notin \mathfrak{K}$, we may find an $\epsilon > 0$ so that for any $P(a,b,c,d) \in \mathfrak{K}$, if $(a,b,) <_P (\epsilon,\epsilon)$, then $(1-\epsilon, 1-\epsilon) \not<_P (c,d)$. Let

$$c_0 = \max\{c : \exists a,b,d \text{ with } 0 \leq a,b \leq \epsilon, d \in [0,\frac{1}{4}] \text{ such that } P(a,b,c,d) \in \mathfrak{K}\}.$$

And similarly, let

$$d_0 = \max\{d : \exists a,b,c \text{ with } 0 \leq a,b \leq \epsilon, c \in [0,\frac{1}{4}] \text{ such that } P(a,b,c,d) \in \mathfrak{K}\}.$$

It is easy to see that one c_0, d_0 is less than $1-\epsilon$. Assume without loss of generality that $c_0 < 1-\epsilon$. Therefore, for any $P(a,b,c,d) \in \mathfrak{K}$,

$$\text{If } (a,b) <_P (\epsilon,\epsilon), \text{ then } c < 1-\epsilon \tag{4.14}$$

Let $\mathcal{S} = \{P(a,b,c,d) : P(a,b,c,d) \in \mathfrak{K} \text{ and } c \geq 1-\epsilon\}$.

$$\text{If } \mathcal{S} = \{\phi\}, \text{ then } \mathfrak{K} \subset \{P(a,b,c,d) : c \leq 1-\epsilon\} \tag{4.15}$$

Assume next that $\mathcal{S}$ is non-empty. Let $\mathcal{T}_1 = \{P(a,b,c,d) : c \geq 1-\epsilon, a < \epsilon\}$ and $\mathcal{T}_2 = \{P(a,b,c,d) : c \geq 1-\epsilon, b < \epsilon\}$. If both of $\mathcal{T}_1$ and $\mathcal{T}_2$ are non-empty, picking $P(a_1,b_1,c_1,d_1)$ in $\mathcal{T}_1$ and $P(a_2,b_2,c_2,d_2)$ in $\mathcal{T}_2$, we see that

$$P(a_1,b_1,c_1,d_1) \wedge P(a_2,b_2,c_2,d_2) = P(\min(a_1,a_2), \min(b_1,b_2), \min(c_1,c_2), \min(d_1,d_2)).$$

Denoting the latter by $P(a,b,c,d)$, we see that $a < \epsilon, b < \epsilon, c \geq 1-\epsilon$, which contradicts (4.14).

Therefore, one of $\mathcal{T}_1, \mathcal{T}_2$, say $\mathcal{T}_1$ must be empty. Therefore, for any $P(a,b,c,d) \in \mathfrak{K}$, if $a < \epsilon$, then $c < 1-\epsilon$. We have,

$$\text{If } \mathcal{S} \neq \{\phi\}, \text{ then } \mathfrak{K} \subset \{P(a,b,c,d) : a \geq \epsilon \text{ or } c \leq 1-\epsilon\} \tag{4.16}$$

From now on, the proof is exactly as in Case 2 of the proof of theorem(4.2.1).

Case 3: The proof is exactly as in the preceding case.

For the second assertion, note that $(\mathfrak{S}_n)'' = M_2(\mathfrak{S}_{n-1} * LF_2)$ and by Voiculescu's matrix formula,

$$r_n = 1 + \frac{2r_{n-1} + 1 - 1}{4} = 1 + \frac{r_{n-1}}{2}$$

□

Remark 4.2.2. *The lattice $\mathfrak{S}_n$ is isomorphic to $[0,1]^{2^{n-1}}$, for $n \geq 2$, where the join and meet on $[0,1]^{2^{n-1}}$ is the co-ordinate wise max and min.*

Remark 4.2.3. ***A technical condition on lattices:*** *Let $\mathfrak{L}$ be a lattice of projections in a finite vna $(\mathfrak{M}, \tau)$. We say that the lattice satisfies condition F if, for any $\epsilon > 0$, there are projections $Q, (P_\lambda)_{\lambda \in \Lambda}$, with $\tau(Q) \leq \epsilon$ so that that for any projection P in $\mathfrak{L}$, either $P \leq Q$ or $P \geq P_\lambda$ for some $\lambda \in \Lambda$.*

Notice that nests as well as each of the lattices $\mathfrak{S}_n$ constructed above satisfy property F. It is easy to see that if $\mathfrak{L}$ has property F and $\mathfrak{K}$ is any other lattice, $\mathfrak{L} \times \mathfrak{K}$ has property F as well.

The utility of property F comes from the following observation: Let P_0 and P_1 denote the infima and suprema of the elements in $\mathfrak{L}$, and Q_0 and Q_1 respectively the infima and suprema of all the elements in $\mathfrak{K}$. Suppose $\mathfrak{L}_1$ is a sublattice of $\mathfrak{L} \times \mathfrak{K}$, then one of the following holds

Case 1: *Both (P_0, Q_1) and (P_1, Q_0) belong to $\mathfrak{L}_1$*

Case 2: *(P_0, Q_1) does not belong to $\mathfrak{L}_1$. There there exist $\tilde{P} \in \mathfrak{L}$ and $\tilde{Q} \in \mathfrak{K}$ such that for any $(P, Q) \in \mathfrak{L} \times \mathfrak{K}$, either $P \geq \tilde{P}$ or $Q \leq \tilde{Q}$.*

Case 3: *(P_1, Q_0) does not belong to $\mathfrak{L}_1$. There there exist $\tilde{P} \in \mathfrak{L}$ and $\tilde{Q} \in \mathfrak{K}$ such that for any $(P, Q) \in \mathfrak{L} \times \mathfrak{K}$, either $P \leq \tilde{P}$ or $Q \geq \tilde{Q}$.*

A consequence of this is that the maximal join of two KS lattices is again KS as long as one of the lattices satisfies property F.

4.3 Minimal Joins

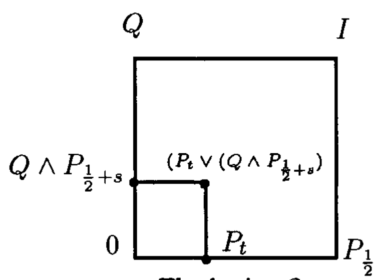
Let $\{p_t\}_{t\in[0,1]}$ be a lattice of projections onto a nest and U a free self-adjoint unitary of trace 0 in a finite von Neumann algebra $(\mathfrak{N}, \tau) \subset \mathcal{B}(\mathcal{K})$, satisfying $\tau(p_t) = t(t \in [0,1])$. Let $q_t = Up_tU^*(t \in [0,1])$. Then, it is easy to see that $(\{p_t\})'', (\{q_t\})''$ are free with respect to the trace τ. Let $\mathfrak{M} = \mathfrak{N} \otimes M_2(\mathbb{C}) \subset \mathcal{B}(\mathcal{K}) \otimes M_2(\mathbb{C}) = \mathcal{B}(\mathcal{H})$ and let $(e_{ij})_{1\le 1,j\le 2}$ be the matrix units for the above copy of $M_2(\mathbb{C})$. We now write elements of $\mathfrak{M}$ as 2×2 matrices over $\mathfrak{N}$.

Let, for $s, t \in [0, \frac{1}{2}]$

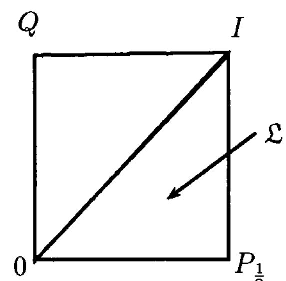
$$P_t = \begin{pmatrix} p_{2t} & 0 \\ 0 & 0 \end{pmatrix}, P_{\frac{1}{2}+t} = \begin{pmatrix} I & 0 \\ 0 & q_{2t} \end{pmatrix}, P_{\frac{1}{2}} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

Let $\mathfrak{L}_0 = \{(P_t \vee Q) \wedge P_{\frac{1}{2}+s}, \quad 0 \le t, s \le \frac{1}{2}\}$. Then, proposition(4.2.2) shows that $\mathfrak{L}_0$ is a reflexive lattice. This lattice is not minimal, but we will extract a minimal lattice from $\mathfrak{L}_0$.

Theorem 4.3.1. *$\mathfrak{L} = \{(P_t \vee Q) \wedge P_{\frac{1}{2}+s}, \quad 0 \le s \le t \le \frac{1}{2}\}$ is KS with respect to $(\mathfrak{L})'' \subset \mathfrak{M} \subset \mathcal{B}(\mathcal{H})$.*



The lattice $\mathfrak{L}_0$



The lattice $\mathfrak{L}$

The above lattice $\mathfrak{L}_0$ is not minimal generating. To see this, we note that any self-adjoint operator $\tilde{T}$ commuting with $\{P_t\}$ may be written as $\tilde{T} = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$, where T commutes with $\{p_t\}$ in $\mathcal{B}(P_{\frac{1}{2}}\mathcal{H})$ and S commutes with $\{q_t\}$ in $\mathcal{B}((I - P_{\frac{1}{2}})\mathcal{H})$. We note that Q is the projection onto the subspace $\{\begin{pmatrix} \eta \\ \eta \end{pmatrix} : \eta \in (I - P_{\frac{1}{2}})\mathcal{H}\}$. We note that we have the following range identities, for $s,t \in [0,\frac{1}{2}]$,

$$\mathrm{Ran}(P_t \vee Q) = \left(\begin{array}{c} \xi+\eta \\ \eta \end{array}\right) : \xi \in \mathrm{Ran}(p_{2t}) \subset P_{\frac{1}{2}}\mathcal{H}, \eta \in (I-P_{\frac{1}{2}})\mathcal{H}$$

$$\mathrm{Ran}(Q \wedge P_{\frac{1}{2}+s}) = \left(\begin{array}{c} \eta \\ \eta \end{array}\right) : \xi \in P_{\frac{1}{2}}\mathcal{H}, \eta \in \mathrm{Ran}(q_{2s}) \subset (I-P_{\frac{1}{2}})\mathcal{H}$$

$$\mathrm{Ran}((P_t \vee Q) \wedge P_{\frac{1}{2}+s}) = \left(\begin{array}{c} \xi+\eta \\ \eta \end{array}\right) : \xi \in \mathrm{Ran}(p_{2t}) \subset P_{\frac{1}{2}}\mathcal{H}, \eta \in \mathrm{Ran}(q_{2s}) \subset (I-P_{\frac{1}{2}})\mathcal{H}$$

If $\tilde{T} = \left(\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right)$ is self-adjoint and commutes with $(P_t \vee Q) \wedge P_{\frac{1}{2}+s}$, we see that

$$\left(\begin{array}{cc} T & 0 \\ 0 & S \end{array}\right)\left(\begin{array}{c} \xi+\eta \\ \eta \end{array}\right) = \left(\begin{array}{c} \xi'+\eta' \\ \eta' \end{array}\right) \quad \text{where } \xi,\xi' \in \mathrm{Ran}(p_{2t}), \eta,\eta' \in \mathrm{Ran}(q_{2s})$$

Thus,

$$\begin{aligned} T\xi + T\eta &= \xi' + \eta' \\ S\eta &= \eta' \end{aligned}$$

The second equation is satisfied because S commutes with q_t. And we have

$$T\xi + (T-S)\eta = \xi'$$

Since T commutes with p_t, we have that

$$(1-p_{2t})(T-S)q_{2t} = 0 \tag{4.17}$$

If we let $\mathcal{E} = \{(t,s), t \geq \frac{1}{2} \vee s \geq \frac{1}{2}\}$, $\mathfrak{L}_E$ is generated (as a lattice) by $\{P_t, 0 \leq t \leq 1, Q \vee P_{\frac{1}{4}}, Q \wedge P_{\frac{3}{4}}\}$. The above computations give us

$$\begin{aligned} (I-p_{\frac{1}{2}})(T-S) &= 0 \Leftrightarrow (T-S)(I-p_{\frac{1}{2}}) = 0 \\ (T-S)q_{\frac{1}{2}} &= 0 \\ \text{Thus,}(T-S)((I-p_{\frac{1}{2}}) \vee q_{\frac{1}{2}}) &= 0 = (T-S) \end{aligned}$$

In the last step we use that $(I - p_{\frac{1}{2}}) \vee q_{\frac{1}{2}}) = I$, which is true because $p_{\frac{1}{2}}$ and $q_{\frac{1}{2}}$ are free. Thus, $T = S$ and $\tilde{T}$ commutes with Q as well. Thus, $(\mathfrak{L}_{\mathcal{E}})'' = (\mathfrak{L}_0)''$.

Also,

$$\mathrm{Alg}(\mathfrak{L}_{\mathcal{E}}) = \{\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}\} \text{ with } A_1, A_2, A_3 \text{ satisfying}$$

$$\begin{aligned}(I - p_t)A_1 p_t =& 0\\ (I - q_t)A_2 q_t =& 0\\ (I - p_{\frac{1}{2}})(A_1 + A_3 - A_2) =& (A_1 + A_3 - A_2)q_{\frac{1}{2}} = 0\end{aligned}$$

While,

$$\mathrm{Alg}(\mathfrak{L}_0) = \{\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}\} \text{ with } A_1, A_2, A_3 \text{ satisfying}$$

$$\begin{aligned}(I - p_t)A_1 p_t =& 0\\ (I - q_t)A_2 q_t =& 0\\ (A_1 + A_3 - A_2) =& 0\end{aligned}$$

We see that $\begin{pmatrix} 0 & p_{\frac{1}{2}}(I - q_{\frac{1}{2}}) \\ 0 & 0 \end{pmatrix} \in \mathrm{Alg}(\mathfrak{L}_{\mathcal{E}}) \setminus \mathrm{Alg}(\mathfrak{L}_0)$. Thus, $\mathfrak{L}_0$ is not a KS lattice.

Remark 4.3.1. *Recall that given* $(t, s), (t', s') \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$,

$$\max((t, s), (t', s')) = (\max(t, t'), \max(s, s'))$$

and

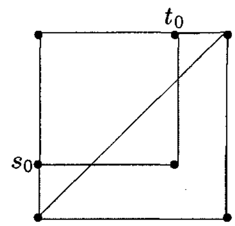
$$\min((t, s), (t', s')) = (\min(t, t'), \min(s, s')).$$

Let $\mathcal{E} \subset [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ *be closed under* max *and* min. *Then* $\mathfrak{L}_{\mathcal{E}} = \{(P_t \vee Q) \wedge P_{\frac{1}{2}+s} : (t, s) \in \mathcal{E}\}$ *is reflexive. The proof is exactly as in the previous paragraph.*

Remark 4.3.2. *The lattice* $\mathfrak{L}_{E_n}$, *where* $E_n = \{(t, s), (t - s) \geq -\frac{1}{n}\}$ *generates* $\mathfrak{M}$ *as well(and is reflexive by the remark above).*

Lemma 4.3.1. *Let $\mathfrak{K} \subsetneq \mathfrak{L}$ be a strict sublattice. Then $(\mathfrak{K})'' \subsetneq (\mathfrak{L})''$*

Proof. It is easy to see that it is enough to prove this for lattices $\mathfrak{K}$ of the form $\mathfrak{K} = \{(P_t \vee Q) \wedge Q_s, \quad t \geq t_0 \text{ or } s \leq s_0, \quad s_0 < t_0\}$. Pictorially,



The lattice $\mathfrak{K}$

Fix $s_0 < t_0$. The lattice $\mathfrak{K}$ is generated by $\{P_t, Q \vee P_{t_0}, Q \wedge P_{\frac{1}{2}+s_0} : t \in [0, \frac{1}{2}]\}$. Since $s_0 < t_0$, $(q_{2s_0} \vee (I - p_{2t_0}) < I$. By the computations above, if $\tilde{T}$ self-adjoint commutes with $\mathfrak{K}$, $\tilde{T} = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$, where T commutes with $\{p_t\}$ in $\mathcal{B}(P_{\frac{1}{2}}\mathcal{H})$, S commutes with $\{q_t\}$ in $\mathcal{B}((I - P_{\frac{1}{2}})\mathcal{H})$ and $(I - p_{2t_0})(T - S) = q_{2s_0}(T - S) = 0$. ie

$$((I - p_{2t_0}) \vee q_{2s_0})(T - S) = 0$$

We must show that we can pick T, S satisfying the equation above such that $T \neq S$. The algebra generated by $\{p_t\}$ and q_{2s_0} is a factor and so is $\{p_t, q_t\}''$, we may find $T \in \{p_t, q_{2s_0}\}' \setminus \{p_t, q_t\}'$ so that $T((I - p_{2t_0}) \vee q_{2s_0})0$ and $T \notin \{p_t, q_t\}'$. Let S be 0. We see that $\tilde{T} = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ lies in $\mathfrak{K}' \setminus \mathfrak{L}'$ and thus, $(\mathfrak{K})'' \subsetneq (\mathfrak{L})''$. □

To complete the proof, it is thus enough to show that $\mathfrak{L}$ generates $\mathfrak{M} = M_2(LF_2)$.

Theorem 4.3.2. *$\mathfrak{L}$ generates $\mathfrak{M}$.*

Proof. If a self-adjoint operator $\tilde{T}$ commutes with $\mathfrak{L}$, we have that $\tilde{T} = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$ where T, S commute with $\{p_t\}, \{q_t\}$ respectively and

$$(1 - p_t)(T - S)q_t = 0 \qquad 0 \leq t \leq 1$$

Let us write $R = U^*SU$. Noting that $p_t = U^*q_tU$, we have that R commutes with p_t and thus, for $t \in [0, \frac{1}{2}]$,

$$(1 - p_{2t})(T - URU^*)Up_{2t}U^* = 0$$
$$\Rightarrow \quad T(1 - p_{2t})Up_{2t} - (1 - p_{2t})Up_{2t}R = 0$$

Therefore,

$$T(1 - p_t)Up_t = (1 - p_t)Up_tR \qquad 0 \leq t \leq 1$$

By the Putnam-Fuglede theorem[10, Chapter 6], noting that U is a self-adjoint unitary,

$$Tp_tU(1 - p_t) = p_tU(1 - p_t)R \qquad 0 \leq t \leq 1$$

We see that if $s > t$,

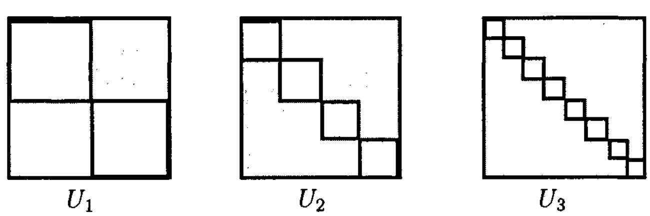
$$T(1 - p_s)Up_t = (1 - p_s)T(1 - p_t)Up_t = (1 - p_s)(1 - p_t)Up_tR = (1 - p_s)Up_tR \tag{4.20}$$

And similarly, if $s < t$,

$$Tp_tU(1 - p_s) = p_tU(1 - p_s)R \tag{4.21}$$

Let $U_0 = 0$, $U_1 = U - p_{\frac{1}{2}}Up_{\frac{1}{2}} - (I - p_{\frac{1}{2}})U(I - p_{\frac{1}{2}})$ and

$$U_n = U - \sum_{i=1}^{2^n}(p_{\frac{i}{2^n}} - p_{\frac{i-1}{2^n}})U(p_{\frac{i}{2^n}} - p_{\frac{i-1}{2^n}})$$



We see that $||U - U_n|| \leq ||\sum_{i=1}^{2^n}(p_{\frac{i}{2^n}} - p_{\frac{i-1}{2^n}})U(p_{\frac{i}{2^n}} - p_{\frac{i-1}{2^n}})|| \leq 1$ and thus $||U_n|| \leq 2$. We also have, by (4.20),(4.21) that $TU_n = U_nR$ and it is checked that $U_n \rightarrow_{\text{WOT}} U$. Thus, we have that $TU = UR$, ie $T = S$.

Therefore, $\tilde{T} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ and $\tilde{T}$ commutes with Q. Thus, $(\mathfrak{L})'' = (\mathfrak{L}_0)''$, i.e. $\mathfrak{L}$ generates $\mathfrak{M}$.

Therefore, $\mathfrak{L}$ is a KS lattice for $\mathfrak{M} = M_2(LF_2) \cong LF_{1.25}$. $\square$

CHAPTER 5

TENSOR PRODUCTS AND DIRECT SUMS OF KADISON-SINGER ALGEBRAS

Tensor products and direct sums 10 pages

Theorem 5.0.3. *Let* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *be two von Neumann algebras and let* $\mathfrak{M}_1 \subset \mathfrak{A}_1 \subset \mathcal{B}(\mathcal{H}_1)$ *and* $\mathfrak{M}_2 \subset \mathfrak{A}_2 \subset \mathcal{B}(\mathcal{H}_2)$ *be Kadison-Singer algebras. Let* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. *Then the algebra* $\mathfrak{A} \subset \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ *given by* $\mathfrak{A} = \mathfrak{A}_1|_{\mathcal{H}_1} \oplus \mathfrak{A}_2|_{\mathcal{H}_2} \oplus P_{\mathcal{H}_1}\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)P_{\mathcal{H}_2}$, *i.e.* $\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_1 & \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \\ 0 & \mathfrak{A}_2 \end{pmatrix}$ *is a Kadison-Singer algebra with diagonal* $\mathfrak{M}_1 \oplus \mathfrak{M}_2$.

Proof. It is easy to see that $\mathrm{Lat}(\mathfrak{A}) = \{P_\lambda \oplus 0_{\mathcal{H}_2}, I_{\mathcal{H}_1} \oplus Q_\mu : P_\lambda \in \mathrm{Lat}(\mathfrak{A}_1), Q_\mu \in \mathrm{Lat}(\mathfrak{A}_2)\}$. If $\mathfrak{L} \subsetneq \mathrm{Lat}(\mathfrak{A})$, then there are reflexive sublattices $\mathfrak{L}_1, \mathfrak{L}_2$ of $\mathrm{Lat}(\mathfrak{A}_1), \mathrm{Lat}(\mathfrak{A}_2)$ respectively, one of them a strict sublattice so that $\mathfrak{L} = \{P_\lambda, Q_\mu : P_\lambda \in \mathrm{Lat}(\mathfrak{A}_1), Q_\mu \in \mathrm{Lat}(\mathfrak{A}_2)\}$. Then, $(\mathfrak{L})'' = (\mathrm{Lat}(\mathfrak{A}_1))'' \oplus (\mathrm{Lat}(\mathfrak{A}_2))'' \subsetneq (\mathfrak{L})''$. Thus, $\mathfrak{L}$ is a Kadison-Singer lattice and $\mathfrak{A}$ is a Kadison-Singer algebra. Also,

$$(\mathfrak{L})' = (\mathrm{Lat}(\mathfrak{A}_1))' \oplus (\mathrm{Lat}(\mathfrak{A}_2))' = \mathfrak{M}_1 \oplus \mathfrak{M}_2.$$

The theorem is proved. □

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two von Neumann algebras and let $\mathfrak{M}_1 \subset \mathfrak{A}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathfrak{M}_2 \subset \mathfrak{A}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be Kadison-Singer algebras. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be the associated Kadison-Singer lattices.

Theorem 5.0.4. *If $\mathfrak{M}_i \subsetneq \mathfrak{A}_i \subsetneq \mathcal{B}(\mathcal{H}_i)(i = 1, 2)$, then the spatial tensor product $\mathfrak{A}_1 \overline{\otimes} \mathfrak{A}_2$ is not a Kadison-Singer algebra.*

Proof. Given two lattices $\mathfrak{L} \subset \mathcal{B}(\mathcal{H}), \mathfrak{K} \subset \mathcal{B}(\mathcal{K})$, let $\mathfrak{L} \otimes \mathfrak{K}$ denote the lattice in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ generated by $\{P \otimes I_{\mathcal{K}}, I_{\mathcal{H}} \otimes Q : P \in \mathfrak{L}, Q \in \mathfrak{K}\}$. The question of when we may have the equality

$$\mathrm{Alg}(\mathfrak{L} \otimes \mathfrak{K}) = \mathrm{Alg}(\mathfrak{L}) \otimes \mathrm{Alg}(\mathfrak{K})$$

has been extensively studied and is called the ATPF(Algebra tensor product formula). This question is the non-selfadjoint version of Tomita's double commutant theorem for von Neumann algebras. Kraus[17] showed that the formula may fail even when one of the algebras on the left hand side is a von Neumann algebra. For Kadison-Singer algebras for which the ATPF is invalid, we have immediately that the tensor product is not Kadison Singer. However, the following simple argument shows that the validity or the invalidity of the ATPF is unnecessary for our theorem.

Assume first that both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are nest algebras. Then, it is easy to see that $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is a CSL algebra that is not a nest algebra. And in view of remark (2.0.2), $\mathfrak{A}_1 \overline{\otimes} \mathfrak{A}_2$ cannot be a Kadison-Singer algebra.

Therefore, we may assume that one of $\mathfrak{A}_1$, $\mathfrak{A}_2$ is not a nest algebra. Without loss of generality, we may assume that $\mathfrak{A}_1$ is not a nest algebra. Now, it is easy to see that

$$(\mathfrak{A}_1 \overline{\otimes} \mathfrak{A}_2) \cap (\mathfrak{A}_1 \overline{\otimes} \mathfrak{A}_2)^* = \mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2$$

It is clear that

$$\mathfrak{A}_1 \overline{\otimes} \mathfrak{A}_2 \subset \mathrm{Alg}(\mathfrak{L}_1 \otimes \mathfrak{L}_2)$$

And that

$$(\mathfrak{L}_1 \otimes \mathfrak{L}_2)'' = \mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2$$

By the assumptions in the theorem, we may find a projection $P_0 \in \mathfrak{L}_1$, so that P_0 is *not* order comparable with every element in $\mathfrak{L}_1$. We may also pick $Q_0 \in \mathfrak{L}_2$, $Q_0 \neq 0$ or $I_{\mathcal{H}_2}$. Let $\mathfrak{L}_3$ denote the lattice generated by

$$\mathcal{S} = \{P \otimes Q_0 : P \in \mathfrak{L}_1\} \cup \{P \otimes I \vee I \otimes Q_0 : P \in \mathfrak{L}_1\} \cup \{P_0 \otimes Q : Q \in \mathfrak{L}_2\} \cup \{I \otimes Q \vee P_0 \otimes I : Q \in \mathfrak{L}_2\}.$$

We claim that

Claim 1 $(\mathfrak{L}_3)'' = \mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2$

Claim 2 $\mathrm{Alg}(\mathfrak{L}_3) \supsetneq \mathrm{Alg}(\mathfrak{L}_1 \otimes \mathfrak{L}_2) \supseteq \mathfrak{A}_1 \otimes \mathfrak{A}_2$

Proof of Claim 1: Let $P \in \mathfrak{L}_1$. It is easy to see that

$$P \otimes I = P \otimes I \vee I \otimes Q_0 + P \otimes Q_0 - I \otimes Q_0$$

And since each of the projections on the right hand side belong to $(\mathfrak{L}_3)''$, so does $P \otimes I$.

And if $Q \in \mathfrak{L}_2$,

$$I \otimes Q = I \otimes Q \vee P_0 \otimes I + P_0 \otimes Q - P_0 \otimes I$$

And thus, $I \otimes Q$ belongs to $(\mathfrak{L}_3)''$ as well. Thus,

$$\begin{aligned}(\mathfrak{L}_3)'' &\subseteq (\mathfrak{L}_1 \otimes \mathfrak{L}_2)'' \\ &= \mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2 \\ &\subseteq (\{P \otimes I, I \otimes Q : P \in \mathfrak{L}_1, Q \in \mathfrak{L}_2\})'' \\ &\subseteq (\mathfrak{L}_3)''\end{aligned}$$

Claim 1 is proved.

Proof of Claim 2: Let $\mathfrak{K}_1$ be the lattice generated by $\{P \otimes Q_0 : P \in \mathfrak{L}_1\} \cup \{P \otimes I \vee I \otimes Q_0 : P \in \mathfrak{L}_1\}$. And let $\mathfrak{K}_2$ be the lattice generated by $\{P_0 \otimes Q : Q \in \mathfrak{L}_2\} \cup \{I \otimes Q \vee P_0 \otimes I : Q \in \mathfrak{L}_2\}$. We note that

$$\mathrm{Alg}(\mathfrak{L}_3) = \mathrm{Alg}(\mathfrak{K}_1) \cap \mathrm{Alg}(\mathfrak{K}_2)$$

Any projection in $\mathfrak{K}_1$ is comparable with $I \otimes Q_0$ and it is easy to see that

$$(I \otimes Q_0)\mathcal{B}(\mathcal{H} \otimes \mathcal{K})(I \otimes I - I \otimes Q_0) = (I \otimes Q_0)\mathcal{B}(\mathcal{H} \otimes \mathcal{K})(I \otimes (I - Q_0)) \in \mathrm{Alg}(\mathfrak{K}_1)$$

Similarly, noting that every projection in $\mathfrak{K}_2$ is comparable with $P_0 \otimes I$,

$$(P_0 \otimes I)\mathcal{B}(\mathcal{H} \otimes \mathcal{K})(I \otimes I - P_0 \otimes I) = (P_0 \otimes I)\mathcal{B}(\mathcal{H} \otimes \mathcal{K})((I - P_0) \otimes I) \in \mathrm{Alg}(\mathfrak{K}_2)$$

Thus,

$$(P_0 \otimes Q_0)\mathcal{B}(\mathcal{H} \otimes \mathcal{K})((I - P_0) \otimes (I - Q_0)) \in \mathrm{Alg}(\mathfrak{K}_1) \cap \mathrm{Alg}(\mathfrak{K}_2)$$

By the choice of P_0, there is a projection $P \in \mathfrak{L}_1$ such that P, P_0 are incomparable. Thus, neither $(I \otimes I - P \otimes I)(P_0 \otimes Q_0)$ nor $((I - P_0) \otimes (I - Q_0))(P \otimes I)$ are zero. Therefore,

$$(I \otimes I - P \otimes I)(P_0 \otimes Q_0)\mathcal{B}(\mathcal{H} \otimes \mathcal{K})((I - P_0) \otimes (I - Q_0))(P \otimes I) \neq 0$$

And

$$(P_0 \otimes Q_0)\mathcal{B}(\mathcal{H} \otimes \mathcal{K})((I - P_0) \otimes (I - Q_0)) \nsubseteq \mathrm{Alg}(\mathfrak{L}_1 \otimes \mathfrak{L}_2)$$

Therefore,

$$\mathrm{Alg}(\mathfrak{L}_3) \supsetneq \mathrm{Alg}(\mathfrak{L}_1 \otimes \mathfrak{L}_2)$$

and the assertion is proved.

□

Remark 5.0.3. *If $\mathfrak{M} \subset \mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is a Kadison-Singer algebra, then $\mathfrak{A} \otimes \mathcal{B}(\mathcal{K})$ is a Kadison-Singer algebra with diagonal $\mathfrak{M}\overline{\otimes}\mathcal{B}(\mathcal{K})$. This follows from the easy observation that any reflexive algebra containing $I \otimes \mathcal{B}(\mathcal{K})$ splits as a tensor product $\mathfrak{B} \otimes \mathcal{B}(\mathcal{K})$.*

In what follows, we suggest a version of the tensor product for a class of Kadison-Singer algebras. Given $\mathfrak{M}_i \subset \mathfrak{A}_i \subset \mathcal{B}(\mathcal{H}_i)(i = 1, 2)$, we will construct a canonical Kadison-Singer algebra $\mathfrak{A}$ from $\mathfrak{A}_1, \mathfrak{A}_2$, with diagonal $\mathfrak{M}_1\overline{\otimes}\mathfrak{M}_2$.

Theorem 5.0.5. *Let $\mathfrak{M}_i \subset \mathfrak{A}_i \subset \mathcal{B}(\mathcal{H}_i)(i = 1, 2)$ be Kadison-Singer algebras with associated KS lattices $\mathfrak{L}_1 = \{p_\lambda : \lambda \in \Lambda\}$, $\mathfrak{L}_2 = \{p_\mu : \mu \in \Delta\}$, where $\mathfrak{L}_1$ is totally ordered and $\mathfrak{L}_2$ is arbitrary. Let $\mathcal{K} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{K}) \otimes M_2(\mathbb{C})$ and $(e_{ij})_{i=1,2}$ a system of matrix units in $M_2(\mathbb{C})$. Define $P_\lambda = p_\lambda \otimes e_{11}(\lambda \in \Lambda)$, $P_\mu = e_{11} + p_\mu \otimes e_{22}$ and $Q = \frac{1}{2}(e_{11} + e_{22} + e_{12} + e_{21})$. Note that Q is a projection. Also denote e_{11} by $P_{\frac{1}{2}}$. In matrix form, for $\lambda \in \Lambda$, $\mu \in \Delta$,*

$$P_\lambda = \begin{pmatrix} p_\lambda & 0 \\ 0 & 0 \end{pmatrix} \; P_\mu = \begin{pmatrix} I & 0 \\ 0 & p_\mu \end{pmatrix} \; P_{\frac{1}{2}} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \; Q = \tfrac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

Then the lattice $\mathfrak{L}$ generated by $\{P_\lambda, P_\mu, Q : \lambda \in \Lambda, \mu \in \Delta\}$ is a KS lattice.

Proof. By theorem(4.0.3),

$$\mathfrak{L} = \{(P_\lambda \vee Q) \wedge P_\mu : \lambda \in \Lambda, \mu \in \Delta\}$$

An identical computation to proposition(4.2.2) shows that $\mathfrak{L}$ is reflexive.

Let $\mathfrak{K}$ be a strict sublattice. There are three cases:

Case 1: $P_{\frac{1}{2}}$ and Q are in $\mathfrak{K}$. Then there are sublattices $\mathfrak{L}_1', \mathfrak{L}_2'$ of $\mathfrak{L}_1, \mathfrak{L}_2$ such that

$$\mathfrak{K} = \{(P_\lambda \vee Q) \wedge P_\mu : p_\lambda \in \mathfrak{L}_1', p_\mu \in \mathfrak{L}_2''\}$$

We see that

$$(\mathfrak{K})'' = M_2((\mathfrak{L}_1')''\overline{\otimes}(\mathfrak{L}_2')'') \subsetneq M_2(\mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2) = (\mathfrak{L})''$$

Case 2: Q does not belong to $\mathfrak{K}$. By arguments similar to Case 2 of theorem(4.2.3), we see that there exist $\lambda_0 \in \Lambda$, $\mu_0 \in \Delta$ such that

$$\mathfrak{K} \subsetneq \{(P_\lambda \vee Q) \wedge P_\mu : \lambda \in \Lambda, \mu \in \Delta, \lambda \geq \lambda_0 \text{ or } \mu \leq \mu_0\}$$

By the computations of (4.11), any operator T that commutes with $\mathfrak{K}$ can be written in the form $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where

a) T_1 commutes with $\{p_\lambda : \lambda \in \Lambda\}$

b) T_2 commutes with $\{p_\mu : \mu \in \Delta\}$

c) $(I - p_{\lambda_0})(T_1 - T_2) = 0$

d) $(T_1 - T_2)p_{\mu_0} = 0$.

Since $\mathfrak{L}_1$ and $\mathfrak{L}_2$ commute, $(I - p_{\lambda_0}) \wedge p_{\mu_0} = (I - p_{\lambda_0})p_{\mu_0}$ and $\tau((I - p_{\lambda_0}) \wedge p_{\mu_0}) = \tau(I - p_{\lambda_0})\tau(p_{\mu_0})$. Therefore,

$$\begin{aligned} \tau((I - p_{\lambda_0}) \vee p_{\mu_0}) &= \tau(I - p_{\lambda_0}) + \tau(p_{\mu_0}) - \tau((I - p_{\lambda_0}) \wedge p_{\mu_0}) \\ &= 1 - \tau(p_{\lambda_0}) + \tau(p_{\mu_0}) - (1 - \tau(p_{\lambda_0}))\tau(p_{\mu_0}) \\ &= 1 - \tau(p_{\lambda_0})(1 - \tau(p_{\mu_0})) \\ &< 1 \end{aligned}$$

Thus, $(I - p_{\lambda_0}) \vee p_{\mu_0}$ is a non-trivial projection.

The operator $T = \begin{pmatrix} (I - p_{\lambda_0}) \vee p_{\mu_0} & 0 \\ 0 & 0 \end{pmatrix}$ is an operator in $(\mathfrak{K})'$. Note that if an operator $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ is in $(\mathfrak{L})'$, then $T_1 = T_2$. Thus, T is a operator in $(\mathfrak{K})' \setminus (\mathfrak{L})'$ and $\mathfrak{K}$ is not generating for $M_2(\mathfrak{M})$.

Case 3: $P_{\frac{1}{2}}$ does not belong to $\mathfrak{K}$. The proof is similar to the proof of Case 2 and we omit it.

□

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