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Comments on the relation of M-theory and supergravity

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Comments on the relation of M-theory and supergravity

Abstract
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During the past three years, related but independent field theory calculations suggest that N=8 d=4 supergravity scattering amplitudes may be ultraviolet finite. Since supergravity theories represent the low-energy effective actions of string theories and describe the interaction of massless fields in the string theory spectrum, it is noteworthy that it was suggested recently that N=8 supergravity, in the event stringy states are discarded, is inconsistent and thus in the Swampland.

This thesis proposes a different interpretation based on results that integrate BPS-soliton states and strengthen the symbiotic relationship between N=8 supergravity and M-theory, which would lead to mutual development.

Keywords
Physics, Theory
COMMENTS ON THE RELATION OF
M-THEORY AND SUPERGRAVITY

BY

TIMOTHY C. STAMNITZ
B.S.E. Physics, University of Michigan, 1970

THESIS

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Date
DEDICATION

To my wife, Lucy, the true center and strength of our family, and our four grown “boys,” who have become admirable men: Peter, Matthew, Chris, and David. Also, to their wives, Nancy, Christina, Jyoti, and Jessica, respectively; and to the future of our beloved grandchildren: Amber, Michael, Sonya, Raphael, Sohani, the twins Luca and Kirpal, Oliver, Ethan and Maren. May you all have the benefit of an enlightened education!
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Finally, it goes without saying, that any ideas or opinions expressed herein, especially shortcomings or misunderstandings, are solely the responsibility of the author, who is a novice in the difficult subjects of general relativity, quantum field theory, supergravity and string/M-theory. On the other hand, anything stated correctly is likely due to my teachers, for whom I have great admiration and respect!
FOREWORD

Both supergravity and string theory have been studied and developed for about thirty-five to forty years. The fact that explicit experimental confirmation has not occurred to date has caused disillusionment among certain scientists and laymen during the past few years. That is, many have become bitter and lost faith in the idealism of this enterprise. Nevertheless, I continue to believe strongly in the importance of these higher-dimensional theories and that eventually they will lead to a better understanding of the physical world in which we live; and also, to a better understanding of the higher-realities that include mental disciplines, the brain, and spiritual consciousness. Patience and perseverance are needed in these endeavors. At the beginning, therefore, I call to your attention several quotes that have provided inspiration over the years:

“Between the poles of the conscious and the unconscious,
The mind has made a swing:
Thereon hang all beings and all worlds, and the swing never ceases its sway.
Millions of beings are there, the sun and the moon in their courses are there.
Millions of ages pass and the swing goes on. All swing! The sky and the earth
And the air and the water; and the Lord Himself taking form: the sight of this
Has made Kabir a servant.”

--One Hundred Poems of Kabir, translated by Rabindranath Tagore,

“We are Thy Children, O Lord, Grant Thou the gift of right understanding.”
– Guru Ramdas (1534-1581)

“I want to know God’s thoughts...the rest are details.”
–Albert Einstein (1879-1955)
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ABSTRACT

COMMENTS ON THE RELATION OF
M-THEORY AND SUPERGRAVITY

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Timothy C. Stamnitz
University of New Hampshire, September 2009

M-theory was synthesized in 1995 due to observations regarding an intricate “web-of-dualities” that relate the five superstring theories, and recognition that D=11 supergravity is both nonperturbatively dual to type-IIA superstring and the low-energy limit of an 11-dimensional theory. Mutual development of M-theory and supergravity ensued.

During the past three years, related but independent field theory calculations suggest that \( \mathcal{N}=8 \) d=4 supergravity scattering amplitudes may be ultraviolet finite. Since supergravity theories represent the low-energy effective actions of string theories and describe the interaction of massless fields in the string theory spectrum, it is noteworthy that it was suggested recently that \( \mathcal{N}=8 \) supergravity, in the event stringy states are discarded, is inconsistent and thus in the Swampland.

This thesis proposes a different interpretation based on results that integrate BPS-soliton states and strengthen the symbiotic relationship between \( \mathcal{N}=8 \) supergravity and M-theory, which would lead to mutual development.
M-theory was essentially invented in 1995 after Witten [158] interpreted the implications of a relatively small number of papers [153][154][156] and made several important observations regarding: (a) the "intricate web of dualities" that relate the five superstring theories, and (b) the fact that both the Type IIA superstring and the $E_8 \times E_8$ heterotic string exhibit an eleventh dimension at strong coupling—thus, approaching the common 11-dimensional limit now referred to as M-theory. Since 11-dimensional supergravity is nonperturbatively dual to the IIA superstring and represents the "low-energy limit" of M-theory, string/M-theory and supergravity have subsequently experienced a relatively synergistic development.\footnote{The expression "string/M-theory" as used here intends to encompass also the concepts of F-theory [224][225]. Significant advances in understanding the duality between M- and F-theory [219][227] suggest that M- and F-theory are complementary, in that one theory or the other provides more insight as a function of the compactification manifold. The 12$^{th}$-dimension in F-theory is auxiliary in the sense that in the limit, it will not have a physical "length;" nevertheless, the advantages it provides are employed as required for clarity in the presentation. For recent advances in understanding M- and F-theory duality and its use in model building, see for example [226].}

In a related development during the past 2 or 3 years, somewhat independent field theory calculations suggest the possibility that $\mathcal{N}=8$ supergravity scattering amplitudes may be ultraviolet finite (for a recent review see [82]). The latter seems like important news, since supergravity theories represent the low-energy effective actions of string theories in various dimensions, and thus describe the interactions of the massless fields in the string theory spectrum. Further, the various nonperturbative string dualities in many cases can be studied by use of these low-energy effective supergravity actions, as evidenced by many papers (see [153] through [174], for example). Therefore, it is surprising that recent work by Green, Ooguri, and Schwarz [71] concludes, taking into account the conjectures of Vafa [21] and Ooguri/Vafa [22], that $\mathcal{N}=8$ supergravity is in the Swampland [21]. Similarly, they conclude that the many superstring compactifications with $\mathcal{N} < 8$ supersymmetry, in the event string states are discarded, result in further supergravity theories in the Swampland. The primary focus herein will be the case of $\mathcal{N}=8$ supergravity.
The thesis presented below argues instead that $\mathcal{N}=8$ supergravity is consistent, and offers a different interpretation of the facts presented by Green, Ooguri and Schwarz (GOS). In addition, our interpretation suggests that the results obtained by GOS should serve to strengthen the symbiotic relationship between M-theory and supergravity, and thus should lead to further mutual development. A brief outline of the thesis organization follows here.

Chapter II provides brief historical and technical background related to the developments of superstring and supergravity theories. This is intended to support a certain perspective related to the issues discussed later. In Chapter III, we summarize the arguments and conclusions reached in the GOS paper, and also provide an initial indication of our distinctly different interpretation, and some recent results that we believe support this interpretation—if not directly at least indirectly.

Chapter IV offers technical reasons why we believe that supergravity theory should not be viewed as a denizen of the Swampland, but rather, should be considered an essential component in further mutual development of both string/M-theory and supergravity. A technical calculation related to supersymmetrization of an $R^4$ term in $\mathcal{N}=8$ supergravity is given to suggest, in albeit pedestrian manner, the feasibility of changing the outcome of a calculation by introducing a minor BPS extension of $\mathcal{N}=8$ supergravity. The failure to achieve supersymmetrization of this $R^4$ term within pure $\mathcal{N}=8$ supergravity was originally interpreted to support the GOS argument [75][77]. In contradistinction, our calculation supports the notion that an extended $\mathcal{N}=8$ supergravity theory would have the ability to include the additional towers of massless states that remain in the limit proposed by GOS. Our conclusions are summarized in Section V, and speculation regarding possible future investigations is offered in Chapter VI.
CHAPTER II

BACKGROUND

Historical and technical information related to both supergravity and string theory is presented in the following five subsections.

II.1 Supergravity Theory—Historical Background

Although supergravity perhaps dominated theoretical physics from about 1976 through late 1984, since the advent of the Green-Schwarz anomaly cancellation in superstring theory [116], it has been viewed primarily in the context of string/M-theory [232]—substantially in the manner first pointed out earlier by Green and Schwarz in [98]. While significant study of M-theory has occurred in the past fourteen years, the full quantum M-theory is still relatively unknown. It is difficult, for example, to construct solutions of M-theory, type IIA and IIB superstring theories; thus, for many applications one merely studies the solutions and properties of the corresponding effective low-energy supergravity. Progress in understanding M-theory also occurs in many cases by dimensional reduction or compactification of the low-energy effective D=11 supergravity. Since the primary objective is to understand the events and phenomena occurring in 4-dimensional spacetime, the focus of research in all higher-dimensional theories invariably involves compactification to lower dimensions with significant emphasis placed on obtaining a realistic 4-dimensional theory that describes both the macrocosmic universe and the ultra-microscopic structure of matter [232].

Supergravity was invented in the mid-1970’s by incorporating spacetime supersymmetry into Einstein’s 4-dimensional general relativity theory (a good historical review is given in by Duff et. al. in [124]). The key to ensure supersymmetry is to match the number of bosons to the number of fermions. In the simplest case in four-dimensions with four bosonic coordinates, one must have four fermionic partners—i.e., one four-component spinor. This is the setup for $\mathcal{N} = 1$ supergravity. In general, extended supergravity results by increasing the amount of supersymmetry from $\mathcal{N} = 1$ up to the maximal amount of $\mathcal{N} = 8$ in four-dimensions. In the latter case, one supersymmetry relates states differing by $\frac{1}{2}$-unit of helicity, and assuming no massless particles with spin > 2, there are 8 “steps” (supersymmetries) of $\frac{1}{2}$-helicity from -2 to +2. The minimal supersymmetry generator is represented by a Majorana spinor with four

---

off-shell components; hence, there are $4 \times 8 = 32$ spinor components or 32 supersymmetry "charges" total. The supersymmetry charges comprise supersymmetry algebra with an anticommutative product. When supergravity is generalized to spacetimes with dimension $D > 4$ and signature $(1, D-1)$, one finds that the maximum value of $D$ admitting a total of 32 spinor components is $D=11$. Hence, supersymmetry places an upper limit of eleven on the dimension of spacetime [91].

The latter observation inspired the invention of 11-dimensional supergravity [92]; wherein, $\mathcal{N}=1$ supersymmetry corresponds to one 32-component Majorana spinor. It appears the first $\mathcal{N}=8$ supergravity in 4-dimensions was constructed by a Kaluza-Klein type dimensional reduction from $D=11$ supergravity [93]. Earlier, starting from $d=4$ and simple $\mathcal{N}=1$ supergravity theory (van Nieuwenhuizen [124]), the amount of supersymmetry was gradually increased in $d=4$, resulting in the maximally extended $\mathcal{N}=8$ supergravity theory [90]. Afterwards, a great deal of effort was directed towards understanding the relationship of $d=4$ extended supergravity to $D=11$ supergravity compactified on various manifolds, resulting in various amounts of unbroken supersymmetry, to obtain a realistic 4-dimensional theory (Duff et. al. [124]).

The degree of activity in developing supergravity gradually waned after the "anomaly cancellations in supersymmetric $D=10$ gauge theory and superstring theory" occurred in late 1984 [116]. Nevertheless, within a month or so after the latter event, it was shown [120] that gauged $\mathcal{N}=8$ $d=4$ supergravity could be embedded into 11-dimensional $\mathcal{N}=1$ supergravity; subsequently, a relatively small group of dedicated workers continued to develop, in particular, 11-dimensional Kaluza-Klein supergravity [123][124].

Significant advances occurred from 1987 through 1991. For example, the supermembrane was introduced in 1987 [127][140], and the spacetime supersymmetric Green-Schwarz covariant superstring action was extended to the supermembrane in dimensions $d=4$, 5, 7, and 11, corresponding to the superstring action in $d=3$, 4, 6, and 10, respectively [129]. The GS superstring action was applied to $p$-dimensional extended objects ($p$-branes) in supergravity, provided that the on-shell $p$-dimensional Bose and Fermi degrees of freedom were equal—constituting evidence for world-volume supersymmetry in these models.

---

3 The mathematics of supersymmetry demands that particles with spin > 2 appear in spacetimes with $D > 11$. Eventually, one also discovers that supersymmetric extended objects—strings, membranes, or in general $p$-branes—only exist in a maximum of eleven dimensions.

4 The Kaluza-Klein procedure or ansatz was pioneered in the late 1920's/early 1930's in the effort to unify Einstein's general theory of relativity with Maxwell's theory of electromagnetism.

5 See [127-131], [133-134], [136-138], [140-143], and [145-146].
Remarkably, the type IIA superstring in 10-d was formally derived from the supermembrane in 11-d by simultaneous dimensional reduction of the world-volume and spacetime \([128]\); similarly, all super p-brane models were related to branes in superstring theories by “double dimensional” reduction \([131]\). In late 1989 it was shown \([143]\)—after imposition of the appropriate KK-ansatz to obtain a string from the membrane in one higher dimension—that the surviving parts of 3-dimensional diffeomorphism invariance of the supermembrane world-volume, engendered precisely the 2-dimensional conformal invariance (Weyl invariance plus diffeomorphisms) on the string world-sheet \([143]\).

Exact multi-membrane solutions to the field equations of 11-dimensional supergravity were found in 1991 to saturate a Bogomol’nyi bound and thus insure stability \([146]\). In fact, progress related to supergravity continued steadily (perhaps at a slower rate) through January 1995, when the 11-dimensional supermembrane was “revisited” \([156]\) and it was argued that the type IIA superstring is actually a compactified 11-dimensional supermembrane theory!

**II.2 Supergravity Theory—Technical Development**

Supergravity is generally considered an important part of string theory; in the latter, the elementary particles are considered as vibrations of the fundamental string. As stated above, M-theory offers a magnificent framework connecting all superstring and supergravity theories for the study of all particles and interactions. Because many unexpected connections between superstrings and supergravity have emerged over the years, the technical development of supergravity should continue unabated. For many applications of string theory, it is only necessary to study the low-energy limit—that is, the structure of the corresponding supergravity theory.

**D=11 Supergravity Action**

As related in historical and technical discussions of string/M-theory below, D=11 supergravity occupies a unique position. It appears at the strong coupling limit of the type IIA and heterotic superstrings, and is also viewed as the low-energy effective theory of the 11-dimensional M-theory.

D=11 supergravity developed several years before the anomaly cancellation in string theory, and played a pivotal role in pioneering higher-dimensional Kaluza-Klein theories. The bosonic part of the 11-dimensional supergravity action \(S\) is still expressed substantially as originally written \([92]\),

\[
2\kappa_{11}^2 S = \int d^{11}x \sqrt{-g} \left( R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4, \tag{2.1}
\]
where $R$ is the scalar curvature, $F_4 = dA_3$ is the field strength of the potential $A_3$, and $\kappa_{11}$ denotes the 11-dimensional gravitational coupling constant. The latter is related to the Newtonian gravitation constant and 11-dimensional Planck length $\ell_p$ as,

$$16\pi G_{11} = 2\kappa^2_{11} = \frac{1}{2\pi} \left( \frac{2\pi \ell_p}{\sqrt{2}} \right)^9. \quad (2.2)$$

The first term in the action (2.1) is the Einstein-Hilbert scalar, which depends on the "elfbein," i.e., 11-dimensional frame field in the metric combination:

$$G_{MN} = \eta_{AB} E^A_M E^B_N. \quad (2.3)$$

M,N,... indices are for curved base-space vectors and A,B,... for tangent space vectors. The field strength quantity $|F_4|^2$ is defined by the general rule,

$$|F_4|^2 = \frac{1}{n!} G^{M_1 N_1} G^{M_2 N_2} ... G^{M_n N_n} F_{M_1 M_2 ... M_n} F_{N_1 N_2 ... N_n}. \quad (2.4)$$

The last term in (2.1), $A_3 \wedge F_4 \wedge F_4$, is the Chern-Simons structure. It is a topological term independent of the elfbein; i.e., the metric. The above equations are presented in the notation of Becker, Becker and Schwarz (BB&S) [232].

BB&S point out that the complete action, including fermionic terms, is invariant under supersymmetry transformations, where the latter are shown in a sub-section below. Nevertheless, to construct classical solutions, only the bosonic terms shown in (2.1) are required, because a classical solution must always have vanishing fermionic fields.

The fermionic terms in the action, as originally given for 11d supergravity are briefly described. Two fermionic terms are discussed. The first one has form,

$$S_\psi \sim \int \bar{\Psi}_M \Gamma^{MNP} \partial_N \Psi_P d^D x, \quad (2.5)$$

and the corresponding Lagrangian appears in earlier literature as [92][124][230],

$$L_\psi = -\frac{i}{2} e \bar{\Psi}_M \Gamma^{MNP} D_N \left[ \frac{1}{2} (\omega + \bar{\omega}) \right] \Psi_P, \quad (2.6)$$

with

$$D_M (\omega) \Psi_P = \partial_M \Psi_P - \frac{1}{4} \omega^{AB}_M \Gamma_{AB}. \quad (2.7)$$

When the spin connection $\omega^{AB}_M$ includes torsion, in general, it is written,

$$\omega^{MAB}_M = \frac{1}{4} (-\Omega^{MAB}_M + \Omega^{ABM}_M - \Omega^{BMA}_M) + K^{MAB}_M. \quad (2.8)$$

Cremmer, Julia, and Scherk [92] refer to $K^{MAB}_M$ as the "contorsion tensor,"

$$K^{MAB}_M = \frac{1}{4} \left[ \bar{\Psi}_N \Gamma^{NP}_{MAB} \Psi_P + 2(\bar{\Psi}_N \Gamma^N_B \Psi_P - \bar{\Psi}_N \Gamma^N_A \Psi_P + \bar{\Psi}_B \Gamma^N_M \Psi_A) \right] \quad (2.9)$$

And write the torsion tensor as,
\[ T^A_{MN} = K^A_{NM} - K^A_{MN} = \left( \frac{iK^2}{2} \right) [\bar{\Psi}_P \Gamma^{APQ}_M \Psi_Q - 2 \bar{\Psi}_M \Gamma^A \Psi_N] . \] (2.10)

It is understood that \( \omega_{MAB} \) is given by the solution of its field equation, which results from varying it as an independent field [230]. It includes a torsion part containing terms of form \( \bar{\Psi} \Gamma \Psi \). Also, \( \Omega^A_{MN} \) is defined from the antisymmetrized derivative of the elfbein as,

\[ \Omega^A_{MN} = 2 \partial_{[N} E^A_{M]} . \] (2.11)

The **supercovariant connection** is defined so that its supersymmetry variation does not involve derivatives of the infinitesimal Grassman parameter \( \epsilon \),

\[ \bar{\omega}_{MAB} = \omega_{MAB} + \frac{1}{\theta} \bar{\Psi}^P \Gamma_{PMAB} \Psi^Q . \] (2.12)

The second fermionic term in the 11d supergravity action has form [124],

\[ L_{\bar{\Psi} \Gamma \Psi} = \frac{3}{4 \cdot 144} \epsilon [\bar{\Psi}_M \Gamma^{MNMWXY} \Psi^N + \bar{\Psi}^W \Gamma^{WX} \Psi^Z] (F_{WXY} + \bar{F}_{WXYZ}) . \] (2.13)

\( F_{WXY} \) is the curl or invariant field strength of the field rank-3 antisymmetric gauge field \( A_{MNP} \), and to insure its supersymmetry variation does not contain derivatives of the Grassman parameter \( \epsilon \), its supercovariant form is also defined [124],

\[ F_{MNPQ} = 4 \partial_{[M} A_{NP]} , \] (2.14)

\[ \bar{F}_{MNPQ} = F_{MNPQ} - 3 \bar{\Psi}_{[M} \Gamma_{NP} \Psi_{Q]} . \] (2.15)

The conventions in notation above are: \( M, N, P, \ldots \) refer to \( d=11 \) world indices and \( A, B, C, \ldots \) refer to \( d=11 \) tangent space indices. \( \epsilon^{MNP\ldots} \) is a tensor vs. tensor density, and \( e = \epsilon_{12\ldots11} \). The Clifford algebra \( \Gamma \)-matrices satisfy,

\[ \{ \Gamma_A, \Gamma_B \} = -2 \eta_{AB} . \] (2.16)

\( \eta_{AB} \) is thus the metric in the tangent space, and the following convention is used,

\[ \Gamma_{A_1\ldots A_p} \equiv \Gamma_{[A_1\ldots A_p]} \] (2.17)

The spinors in the Lagrangian expressions (2.6) and (2.13) are anticommuting and satisfy the Majorana condition,

\[ \bar{\Psi} = \psi^T C^{-1} , \] (2.18)

where the charge conjugation matrix \( C \) is antisymmetric and defined by,

\[ C^{-1} \Gamma_A C = -\Gamma_A^T . \] (2.19)

Finally, we are ready to summarize the field content and derive the full set of field equations for 11d supergravity.
Field Content

The 11-dimensional supergravity action must be invariant under local supersymmetry transformations. The field content is relatively simple:

1. Gravity is carried by the graviton, a massless spin-2 boson, represented by the elfbein field $E_M^a$;
2. The gauge field for local supersymmetry is the spin-3/2 gravitino field $\Psi_M$, which has an implicit spinor index in addition to the vector index shown;
3. The rank-3 antisymmetric tensor field $A_{MNP}$, a massless spin-1 boson, which can also be represented by the 3-form $A_3$ ("the three-index photon").

The graviton is represented by a symmetric traceless tensor. The "little group in D=11 for this massless spin-2 particle is $SO(9)$ with physical degrees of freedom corresponding to the dimensions:

$$\frac{1}{2}(D-1)(D-2) - 1 = \frac{1}{2}(11)(8) = 44$$  \hspace{1cm} (2.20)

The spin-3/2 gravitino $\Psi_M$ is a 32-component spinor field represented by the covering group of the little group, which is Spin(9), having a real spinor rep of 16-dimensions. Further technical discussion of spinors, which are decomposed upon compactification to lower dimensions, will be given below. In this section only a few facts are needed. For example, group theoretically the Spin(9) Kronecker product consists of a vector and spinor:

$$9 \times 16 = 128 + 16$$  \hspace{1cm} (2.21)

The kinetic term for the free gravitino field in any dimension has the form (2.5),

$$S_{\Psi} \sim \int \bar{\Psi}_M \Gamma^{MNP} \partial_N \Psi_P d^D x. \hspace{1cm} (2.22)$$

In 11-dimensions due to the antisymmetry of $\Gamma^{MNP}$ and a certain local symmetry, there are only 128 degrees of freedom [232].

Supersymmetry requires an equal number of physical bosonic and fermionic degrees of freedom. We have accounted for 44-bosonic and 128-fermionic degrees of freedom. The additional 84-bosonic degrees of freedom are obtained precisely from the rank-3 antisymmetric tensor field $A_{MNP}$. When represented as the 3-form $A_3$, it is invariant under the gauge transformations,

$$A_3 \rightarrow A_3 + d\lambda_2. \hspace{1cm} (2.23)$$

$\lambda_2$ is a 2-form field and gauge invariance ensures that the indices for independent physical polarizations are transverse. For the 3-form field in 11d this implies,
physical degrees of freedom. Including the graviton, we have 44 + 84 bosonic
degrees of freedom to match the 128 fermionic degrees of freedom.

**D=11 Supergravity Field Equations**

Upon variation of the full 11-dimensional action with respect to $E^A_M$, $\bar{\psi}_M$, and $A_{MNP}$
the full set of field equations are obtained as,

$$
R_{MN}(\bar{\omega}) - \frac{1}{2} g_{MN} R(\bar{\omega}) = \frac{1}{3} \left[ \bar{F}^{PQR} \bar{F}_N^{PQR} - \frac{1}{8} g_{MN} \bar{F}^{PQRS} \bar{F}^{PQRS} \right] (2.25)
$$

$$
F^{MNP} D_N(\bar{\omega}) \psi_P = 0 (2.26)
$$

$$
\nabla_M(\bar{\omega}) F^{MNP} = \frac{-1}{4 \times 144} \epsilon^{PQR M_1 \ldots M_8} \bar{F}_{M_1 \ldots M_4} \bar{F}_{M_5 \ldots M_8} (2.27)
$$

The appearance of $\frac{1}{2} (\omega + \bar{\omega})$ in the Lagrangian term (2.6), and $(F_{WXYZ} + \bar{F}_{WXYZ})$ in
the Lagrangian term (2.13) above, ensures that only the supercovariant $\bar{\omega}$ and
$\bar{F}_{WXYZ}$ enter the field equations.

**Supersymmetry Transformations**

The 11d bosonic action (2.1) is invariant under supersymmetry transformations,
which transform the graviton, antisymmetric tensor, and gravitino fields, respectively,

$$
\delta E^A_M = \bar{\epsilon} \Gamma^A \psi_M, \quad \delta A_{MNP} = -3 \bar{\epsilon} \Gamma_{[MN} \psi_P], \quad \delta \psi_M = \nabla_M \epsilon + \frac{1}{12} \left( \Gamma_M F^{(4)} - 3 F_M^{(4)} \right) \epsilon (2.28)
$$

The $F^{(4)}$ quantities in the latter are defined in the notation of [232] by,

$$
F^{(4)} = \frac{1}{4!} F_{MNPQ} \Gamma^{MNPQ} (2.29)
$$

$$
F^{(4)}_M = \frac{1}{2} \left[ \Gamma_M, F^{(4)} \right] = \frac{1}{3!} F_{MNPQ} \Gamma^{NPQ} (2.30)
$$

The Dirac matrices satisfy: $\Gamma_M = E_M^A \Gamma_A$, where $\Gamma_A$ are the numerical, coordinate-
independent matrices obeying the flat-space Dirac algebra. Square brackets above
represent antisymmetrization with weight one; for example, in (2.29) we have,
\[(\Gamma_{MN}\Psi_P) = \frac{1}{3}(\Gamma_{MN}\Psi_P + \Gamma_{NP}\Psi_M + \Gamma_{PM}\Psi_N) \]  
\(2.33\)

In (2.4) above, similar to (2.17), we also made use of the convenient notation,
\[\Gamma^{M_1M_2...M_n} = \Gamma^{[M_1}_M^{,M_2}...^{,M_n]} . \]  
\(2.34\)

The covariant derivative in (2.30) is based on the spin connection \(\omega\) (vs. \(\tilde{\omega}\)),
\[\nabla_M \epsilon = \partial_M \epsilon + \frac{1}{4} \omega_{MAB} \Gamma^{AB} \epsilon . \]  
\(2.35\)

Also, using the elfbein field, we define the quantity in (2.11),
\[\Omega^A = 2\partial_{[N}E^A_{M]} , \]  
\(2.34\)

such that the spin-connection is expressed in terms of the elfbein,
\[\omega_{MAB} = \frac{1}{2}(-\Omega_{MAB} + \Omega_{ABM} - \Omega_{BMA}) , \]  
\(2.35\)

where no torsion term, \(K_{MAB}\), like that in (2.8) arises for the bosonic action.

**Supersymmetric Solutions**

An important point is again emphasized: the Lagrangian must be invariant under supersymmetry transformations. Although we wrote the transformations of both bosonic and fermionic fields under supersymmetry, the action in (2.1) includes only bosonic terms. Only the bosonic terms are required when we are interested in classical solutions, because all fermionic fields *vanish* in the classical solution.

**General Discussion of Supergravity**

The portion of supergravity that describes only the spin-zero fields provides great insight into the structure of supergravity. These scalar fields determine the vacua of the theory. From the standpoint of string theory, the latter are the moduli of deformations of the compactified dimensions. The scalar fields define the geometry, and the structure of the latter determines a large part of the full action due to supersymmetry. We are interested in \(N=8\) supersymmetries in \(d=4\), because the latter provides the richest geometrical structure.

The fundamental symmetries in the laws of nature appear to be *local* rather than global; technically, we say that the laws of nature are *gauged*. This is true, for example, in the Standard Model, \(SU(3)_{\text{color}} \times [SU(2)xU(1)]_{\text{electroweak}}\) of elementary particle theory. If indeed the laws of nature include *supersymmetry*, then it seems
imperative that the latter is also gauged, which leads directly to inclusion of the theory of gravitation. This occurs roughly as follows.

The superalgebra—these are the algebras of transformations that leave the action invariant. The “super” refers to supersymmetry, and the latter is described as symmetry between fermions and bosons, however, it turns out that it is more than just another symmetry of matter. It implies an extension of the Poincare symmetry of spacetime, thus, it is literally an extension of the structure of spacetime. One way to visualize this is to consider a “superspace” with fermionic directions (coordinates) in addition to the usual bosonic directions (coordinates).

All supersymmetry algebras in four-dimensions \(1 \leq \mathcal{N} \leq 8\) contain the Poincare or Anti-de Sitter algebra as a subalgebra. Gauging supersymmetry thus includes gauging the Poincare or Anti-de Sitter algebra, and gauging the former is essentially the same as Einstein’s theory of gravitation! There are several technical steps involved in setting up a supergravity scenario, and since we cannot cover everything here, the following is a brief summary.

Gravity is described in a quantum theory by a massless spin-2 boson. For local \(\mathcal{N}=1\) supersymmetry, this spin-2 graviton must have a supersymmetric partner. On the basis of group representation theoretic grounds, this partner must be a massless fermion of spin-3/2. The supersymmetry charges \(Q_a\) span a spin-1/2 Majorana spinor, and the corresponding gauge field having one vector index beyond that of these charges must describe a spin-3/2 field. Thus, the supergravity multiplet for \(\mathcal{N}=1\) supergravity contains one massless spin-2 boson; i.e., the graviton, and one Majorana spin-3/2 fermion known as the gravitino. Thus, it is necessary to construct an Einstein-Hilbert Lagrangian for the action that couples the resulting supergravity theory to \(\mathcal{N}=1\) supersymmetric matter. A brief description of \(\mathcal{N}=1\) supergravity follows (see references [234][235]).

\(\mathcal{N}=1\) Supergravity

For \(d=4\) supersymmetry, the conventions of Wess & Bagger [234] are predominantly used; for example, two-component Weyl spinors and a Minkowski metric with mostly plus signature. Component multiplets are used instead of the superspace formalism due to time constraints and for brevity. First, the basics of rigid supersymmetry are given to enable introduction of supergravity.

The Dirac \(\gamma\)-matrices \(\gamma^a\) with \(a = 0, \ldots, 3\) satisfying the Clifford algebra,

\[
\{\gamma_a, \gamma_b\} = -2\eta_{ab} \ 1, \tag{2.36}
\]
where \((\eta_{ab}) = \text{diag}(-1, 1, 1, 1)\) is the Minkowski metric. A particular representation of this algebra is the Weyl representation,

\[
\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}.
\] (2.37)

The \(\sigma\)-matrices are given as,

\[
\sigma^a = (-\mathbb{1}, \mathbb{i}), \quad (2.38)
\]

\[
\bar{\sigma}^a = (-\mathbb{1}, -\mathbb{i}). \quad (2.39)
\]

The symbol \(\mathbb{i}\) represents the three Pauli spin matrices. In the Weyl representation ("rep"), one can write:

\[
\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (2.40)

This allows decomposing Dirac spinors \(\Psi_D\) into left- and right-handed two-component spinors with respect to the projectors:

\[
P_L/R = \tfrac{1}{2}(\mathbb{1} \pm \gamma^5),
\]

\[
\Psi_D = \begin{pmatrix} \chi^a \\ \bar{\chi}^\dagger \end{pmatrix}.
\] (2.41)

In this van der Warden type spinor notation, the dotted and undotted Greek indices from the beginning of the alphabet take values 1 and 2. The Weyl spinors \(\chi^a\) and \(\bar{\chi}\) form irreducible and inequivalent reps of the universal covering of the Lorentz group \(SL(2,\mathbb{C})\), where infinitesimally one can write,

\[
\ell_{ab} \Psi_D = -\tfrac{1}{4} [\gamma_a, \gamma_b] \Psi_D = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix} \begin{pmatrix} \chi^a \\ \bar{\chi} \end{pmatrix}.
\] (2.43)

The matrices \(\sigma_{ab}\) are defined as,

\[
\sigma_{ab} = \frac{1}{4}(\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a) \quad \text{and} \quad \sigma_{ab} = \frac{1}{4}(\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a).
\] (2.44)

These matrices satisfy the commutation relations of the Lorentz generators \(\ell_{ab}\), i.e.,

\[
[\sigma_{ab}, \sigma_{cd}] = \eta_{ac} \sigma_{bd} - \eta_{bc} \sigma_{ad} + \eta_{bd} \sigma_{ac} - \eta_{ad} \sigma_{bc},
\] (2.45)

and analogously for the conjugate spinor \(\bar{\sigma}_{ab}\). The \(SL(2,\mathbb{C})\) invariant \(\epsilon\)-tensors have the form,

\[
(\epsilon^{\alpha\beta}) = -(\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\epsilon^{\alpha\beta}) = -(\epsilon_{\alpha\beta}).
\] (2.46)
Spinor indices are raised and lowered with the latter as, for example,
\[ \chi_\alpha = \varepsilon_{\alpha\beta} \chi^\beta \quad \text{and} \quad \bar{\chi}^\dot{\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}}. \]  

(2.47)

In this manner, Lorentz invariants are formed from two Weyl spinors of the same chirality; for example,

\[ \chi \psi = \chi^\alpha \psi_\alpha = \varepsilon^{\alpha\beta} \chi_\alpha \psi_\beta = -\varepsilon^{\alpha\beta} \chi_\alpha \psi_\beta = \text{(etc)} = \psi \chi \]  

(2.48)

Precisely analogous quantities are formed using \( \bar{\lambda} \) and \( \bar{\psi} \), and we have assumed that complex conjugation reverses the order of fields and turns left-handed (LH) spinors into right-handed (RH) spinors and vice versa; that is,

\[ (\chi \psi)^* = (\chi^\alpha \psi_\alpha)^* = (\psi_\alpha)^* (\chi^\alpha)^* = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\psi} \bar{\chi} = \bar{\chi} \bar{\psi} \]  

(2.49)

The Majorana spinors in the Weyl rep have the form,

\[ \Psi_D = \left( \begin{array}{c} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{array} \right), \quad \text{since} \quad \bar{\chi}^{\dot{\alpha}} = (\chi^\alpha)^* \]  

(2.50)

The free action for a massive Majorana spinor is then given as,

\[ \mathcal{L}_0 = -\frac{\gamma}{2} \bar{\Psi}_M (i \gamma^\mu \partial_\mu + m) \Psi_M = -\frac{\gamma}{2} \chi \sigma^\mu \bar{\chi}^{\dot{\alpha}} - \frac{\gamma}{2} m (\chi \chi + \bar{\chi} \bar{\chi}), \]  

(2.51)

Where \( A\partial_\mu B = A \partial_\mu B - \partial_\mu AB \). The \( \gamma \) and \( \sigma \)-matrices carry Greek spacetime indices in the above; also, in flat spacetime with Cartesian coordinates the relation is (recall that \( a, b \) indices are used for the Clifford algebra):

\[ \gamma^\mu = \delta^\mu_\nu \gamma^\nu \quad \text{and analogously for} \quad \sigma^\mu. \]  

Whenever gravity is included, it is necessary to distinguish the two sets of indices.

**Brief Summary : Supersymmetry and Superspace**

1. Chiral invariance is an exact property of the supergravity Lagrangian; therefore, its consequences hold for complete invariants, not just for linearized invariants.

2. In globally supersymmetric models, the energy is always greater than or equal to zero, since the LHS of the expression below is always positive:

\[ \{ Q_\alpha, \bar{Q}_{\dot{\beta}} \} = 2 \sigma^{\mu}_{\alpha\beta}, \]  

(2.52)

Multiply by \( \sigma^0 \) and take the trace to get:

\[ Q_\alpha \bar{Q}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}} Q_\alpha = E. \]  

(2.53)
Since the LHS of (2.53) is positive, we find $E \geq 0$. In global supersymmetry, $E=0$ is a very special case, because the expectation value of the energy is an order parameter for supersymmetry breaking. If supersymmetry is unbroken, $Q_\alpha |0\rangle = 0$, implying that the ground-state energy vanishes if and only if supersymmetry is unbroken.

Regarding the supersymmetry algebra and its representations: supersymmetry generators are spinors. Thus, they do not commute with the Lorentz generators. Supersymmetry algebra involves the translation generators,

$$\{Q^A_\alpha, \bar{Q}^B_\dot{\beta}\} = 2\sigma_{\alpha\dot{\beta}}^\mu \sigma^\mu, \quad (2.54)$$

$$\{Q^A_\alpha, \bar{Q}^B_\dot{\beta}\} = \epsilon_{\alpha\dot{\beta}} Z^{AB}, \quad (2.55)$$

where $Z^{AB}$ are Lorentz scalars antisymmetric in $A, B$, known as central charges.

It appears certain that at low-energy, in the real-world of experimental physics, most supersymmetry is broken. However, for the physics of nature to be relatively stable and calculable, we find that $\mathcal{N}=1$ supersymmetry is a reasonable assumption. Also, only $\mathcal{N}=1$ supersymmetry has chiral representations. In practice, we imagine that chiral matter arises at the point where supersymmetry is broken. Although it is difficult to break $\mathcal{N} > 1$ supersymmetry spontaneously, $\mathcal{N}=1$ supersymmetry is broken relatively easily. The smallest irreducible representations of $\mathcal{N}=1$ supersymmetry that describe massless fields are:

1. Chiral superfields $(\phi, \psi_\alpha)$ consisting of a complex scalar and a chiral fermion;
2. Vector superfields $(\lambda, A_\mu)$ with a chiral fermion and a vector meson, both in general in the adjoint representation of the gauge group;
3. The gravity supermultiplets $(\psi_{\mu\alpha}, g_{\mu\nu})$ a spin-$3/2$ particle, the gravitino, and the spin-$2$ graviton.

Great simplification is achieved by enlarging spacetime to include both commuting and anticommuting coordinates; the resulting space is called superspace.

Please see the Appendix for more detail of the "on-shell" supergravity theory.
11.3 String/M-theory—Historical Background

As stated in the Introduction, M-theory is conceptually an 11-dimensional quantum gravity theory, first constructed from the type IIA superstring by taking the limit of the coupling constant into the strong-coupling regime, and interpreting that increase as growth of an 11th-dimension [158]. Just as 10-dimensional type IIA supergravity is viewed as the effective low-energy theory of the type IIA superstring, analogously, 11-dimensional supergravity is viewed as the effective low-energy theory of a still relatively unknown quantum M-theory.

String theory originated in theoretical high-energy particle physics in the late 1960's/early 1970's as an outcome of several programs that sought to unify the elementary particle forces ([230], Ch. 1 and [231, Ch. 1). One of these programs included the dual resonance models that incorporated the newly invented “supersymmetry” between bosons and fermions. In early 1974, John Schwarz and Joel Scherk noted that superstring theory automatically included a massless spin-2 particle that could be identified with the graviton—suggesting the possibility of a truly unified theory that included not only the fundamental particle forces but also gravitation. Nevertheless, two intervening developments in the mid-1970’s galvanized the attention of the community of physicists: (1) unified gauge theories leveraged from the electro-weak unification, followed by the colored quark theory SU_c(3), which led to the Standard Model of elementary particle physics, SU(3)_c x SU(2)_w x U(1)_y, and thence to SU(5) and SO(10) GUTs (Grand Unified Theories) [233]; and (2) the marriage of supersymmetry with Einstein’s theory of gravitation, which led to D=11 supergravity [92], SO(8) d=4 supergravity [93], and the gauged N=8 supergravity in d=4 [100].

In late 1984 John Schwarz and Michael Green demonstrated that superstrings enabled an important “anomaly cancellation” within a quantum field theoretic calculation, such that the theory of superstrings was firmly launched [116]. By the end of 1985 five fundamental string theories, Type I, Type IIA, Type IIB, heterotic-O [SO(32) gauge group], and heterotic-E [E_8 x E_8 gauge group] had been established [230][231][232]. All of these theories were 10-dimensional theories; hence, in analogy with and/or in some cases commensurate with procedures in higher-dimensional supergravity theories, compactification schemes were developed to obtain, hopefully, more phenomenological models in four-dimensions [98]. In this conceptual scheme, the six extra dimensions are visualized as “curled up” into a very small diameter, perhaps larger than the Planck length but certainly many orders of magnitude smaller than what would have been observed at high energy particle accelerator-colliders at that time.
Shortly after invention of the heterotic superstring [119], Candelas, Horowitz, Strominger, and Witten [121] proposed compactification of the heterotic string on a six-dimensional Calabi-Yau manifold, an approach that is influential to this day. Eventually, in analogy with the Kaluza-Klein (KK) approach to supergravity, extensive investigations of KK-compactification in string theory led within a few years to recognition that literally thousands (perhaps millions or billions) of distinct superstring vacuums existed as a basis for distinct theories of the physical reality. Some uneasiness may have resulted when it appeared difficult to determine a physical principle to guide selection of the unique internal space that governed the vacuum of our universe; however, there was also great optimism [232].

Two important conceptual changes introduced in string theory are [230]: (1) the concept of an abstract “point particle” was replaced by the concept of an extremely tiny “bit” of string, whose modes of vibration determined the properties of an elementary particle [231]; and (2) consistency of the field-theoretic calculations required that strings live in higher-dimensional spaces; i.e., the bosonic string lives in 26-dimensions and the superstring lives in 10-dimensions. In effect, the string is an extended 1-dimensional object that can be either “open” or “closed,” and the fundamental vibrations determine physical properties such as mass, charge, spin, etc. The bosonic string in 26-dimensions obeys Bose-Einstein statistics; whereas, the superstring includes both bosons and fermions on the string world-sheet, and exhibits manifest spacetime supersymmetry. The heterotic string is an interesting hybrid theory that includes both 26 bosonic coordinates and 10 fermionic coordinates; however, 16 of the 26 bosonic dimensions are compactified on a lattice, such that 10 bosonic degrees of freedom remain to share the 10-dimensional space with the fermions. In one of several approaches to this model, the 10-bosonic modes are “left moving” and the 10-fermionic modes are “right moving” vibrations around the closed string [232].

In all five cases of the superstring described above, six of the ten dimensions are compactified on a Calabi-Yau manifold, an orbifold, or a generalized Calabi-Yau manifold [232]—where the latter could be Kahler or complex. Compactification on a compact manifold, typically by means of the KK-mechanism, was originally developed in five-dimensional spacetime to achieve formal unification of the general theory of relativity with Maxwell’s field equations in the 1920’s [231]. Thus, an effective 4-dimensional spacetime theory can be elaborated irrespective of the number of compactified dimensions available at each ‘point’ of the 4-dimensional spacetime manifold.

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66 Several early references to this fact are given in [14], but after work presented in [11] and [13], it became well known. Further discussion of the “string landscape” occurs at the end of this sub-section.
A major generalization in string theory, which ultimately led to the foundation of M-theory, was the eventual recognition that, analogous to the supermembrane and p-branes introduced in the context of supergravity theories, fundamental higher-dimensional extended objects with \( p > 1 \) also exist in string theory [153][154][156]. That is, the fundamental string theories also contain membranes of various dimensions, and open strings must end on membranes known as Dirichlet or “D-branes” [165][166]. In the type IIA theory, there are D2, D4, D6 and D8 branes; whereas, in type IIB superstring theory there are D3, D5, D7 and D9 branes [232]. Witten utilized the concept of branes to elaborate the concept of “duality” in string theory [158]. Non-trivial duality implies that distinct descriptions of the same physical situation can result in different but complementary physical insights, by means of calculations using mathematical methods of analysis not available in one of the dual theories. In this way, duality becomes a strategy for transcending the perturbative understanding of string theory.

Witten found that as the coupling constant of Type IIA string is increased, the string becomes “stretched” into a two-dimensional tubular-type membrane [158]. Subsequently, Horava and Witten found that increasing the coupling constant of heterotic \( E_8 \times E_8 \) string also caused a new space dimension to appear [167], and the heterotic string itself thus becomes a two-dimensional cylindrical membrane stretched between two 10-dimensional boundaries located at the orbifold points of a new 11th-dimension interval [167][172] (shown schematically in the technical part below [232]).

Two types of duality are required to complete all of the relationships between the five string theories and D=11 supergravity within the (so called) web of dualities [231][232]: (a) S-duality is the strong-weak duality related to the strength of the string coupling constant in perturbation theory; and (b) T-duality is a geometrical duality related to the existence of a smallest possible “quantum of length” for the extended object in spacetime. In the context of S-duality, the weak coupling region of one string theory is nonperturbatively connected to the strong coupling region of its string theory dual [149]. T-duality states that the mass spectrum of one string theory is invariant when \( R \) becomes \( 1/R \), which links the large scale behavior of one string theory to the small-scale structure of the dual theory.

Another important concept to develop string theory beyond the domain of perturbative approximations is that of the “BPS state” [150][232]. This concept says that within a given isolated physical system, constrained to have the property of supersymmetry, the BPS states have the minimum possible mass for a given amount of charge. The properties of BPS states—the mass and the charge (associated to a force) are uniquely determined by supersymmetry. This is a minimalization constraint. This concept
enabled Witten and Horava to understand characteristics of strongly coupled heterotic string [167][172] without need of difficult quantum field theory calculations. Perhaps more surprising is that physicists Horowitz and Strominger [147][148], whose ideas to develop p-branes were enhanced by J. Polchinski with the invention of D-branes [165], demonstrated that: BPS states are precisely related to the whole range of extended objects or branes that are present in higher dimensional theory. One knows not only the mass and charge of BPS states, but also, what they look like: BPS states are branes.

Many dualities between the five string theories and 11-dimensional M-theory represent nonperturbative properties; that is, they cannot be seen in perturbative string theory. When analyzing nonperturbative features that appear in the extrapolation from weak to strong coupling, ordinarily, the results of calculations cannot be controlled. However, important aspects of M-theory, including various nonperturbative string dualities, can be illustrated and studied by means of low-energy effective actions. This occurs precisely because the calculations that involve extrapolations can be restricted to quantities protected by supersymmetry; i.e., the quantities known as BPS states [232].

In summary, string/M-theory has become the dominant paradigm in the development of quantum gravity for the past twenty-five years. Many successes have occurred, including the statistical interpretation of black hole entropy and resolution of the information paradox problem (for an excellent summary see Becker, Becker and Schwarz [232], Ch. 11, Sections 11.2 and 11.4). Whereas, it was originally hoped that string/M-theory would provide a unique, precise description of all fundamental interactions, in practice, it provides an exponentially large number of vacuum solutions, known as the “string landscape” ([11]through [25]).

An early estimate of the number of string vacua was $10^{1500}$ [14]; more recently, it appears that $10^{500}$ distinct flux vacuum compactifications [11][15] is realistic. Since the latter numbers would correspond to the same number of distinct physical theories, this implies a loss of predictability. This was apparently first noted by Strominger in 1986 [125]. As a result, there are now perhaps hundreds of potentially viable models in both cosmology and high-energy elementary particle physics, but none have the ability to make precise experimental predictions. Many aspects of known physics, for example, the standard model in high-energy physics, can be modeled closely, but none of the string-theory models have made compelling predictions of new experimental phenomena up to the present day. This situation may change, however, within the next few years with the analyses of data soon to be generated by the Large Hadron Collider (LHC) at CERN in Geneva, Switzerland.
II.4 String/M-theory—Technical Development

Most introductory presentations start with the description of the bosonic string. Zwiebach's book [231] is an excellent starting point to gain a fundamental understanding of the bosonic string, and a very brief introduction to the superstring is also provided. Since bosonic string theory is not directly related to our purpose here, we start with a brief summary of superstring theory with an emphasis directed towards understanding the strong-coupling limit of the type IIA superstring [158]. This leads, as stated in the historical background, to D=11 supergravity and the conjectured 11-dimensional quantum M-theory.

Fermions are added to the bosonic string world-sheet to obtain a supersymmetric theory. Fermions must be included in the theory to enable the possibility of a realistic physical theory, as all elementary particles of matter have fermionic characteristics, represented mathematically by spinors. In more detail, the spectrum of quantum states of the bosonic string contain a tachyon, which corresponds to having an unstable vacuum. Supersymmetry is introduced in string theory in two primary ways.

The Ramond-Neveu-Schwarz (RNS) formalism is the first approach that arose in the context of high-energy particle physics. It employs supersymmetry on the two-dimensional world-sheet, and requires the "GSO projection" to realize spacetime supersymmetry (see [232], p. 133-135). The spectrum is free of tachyons and the theory exhibits modular invariance, a requirement of all consistent string theories ([232], p. 94).

The Green-Schwarz (GS) light-cone formalism emerged later, as a result of efforts to make spacetime supersymmetry explicit; thus, it avoids the need for the GSO projection. The GS superstring, however, cannot be readily quantized in a fully covariant manner, because it relies on light-cone gauge fixing of the spacetime coordinates. It can be quantized in the light-cone gauge, which is sufficient to analyze the physical spectrum.

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7 For an in depth treatment of the bosonic string, see Polchinski [236], Volume 1, which is difficult going until one first studies other more intermediate level texts. Although we've relied primarily on BB&S [236], there are important introductions to the mathematics of string theory in Green, Schwarz, and Witten, Volume 2; also, Kaku [237] provides a succinct high-level discussion that includes aspects of string theory not available elsewhere. Szabo [238] provides a condensed overview that includes D-brane dynamics.
The RNS Superstring

Bosonic fields $X^\mu(\sigma, \tau)$ of the two-dimensional world-sheet are paired up with fermionic partners $\psi^\nu(\sigma, \tau)$, where the latter are two-component spinors on the world sheet, but transform as vectors under Lorentz transformations in the D-dimensional spacetime. The spinors anticommute, which is consistent with spin and statistics for spinors in the two-dimensional sense [232]. One introduces D Majorana spinors (fermions) as internal degrees of freedom on the world-sheet, that belong to the vector representation of the Lorentz group SO(D−1, 1). In the two-dimensional Clifford (Dirac) algebra, a Majorana spinor is equivalent to a real spinor. By adding the standard Dirac action for D free massless fermions to the free theory of D massless bosons, one obtains the action,

$$S = \frac{1}{2\pi} \int d^2 \sigma \left( \partial_\alpha X^\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right)$$

(2.56)

$\rho^\alpha$ with $\alpha = 0, 1$ represents the two-dimensional Clifford matrices obeying,

$$\{\rho_\alpha, \rho_\beta\} = 2\eta_{\alpha\beta},$$

(2.57)

Where the basis is chosen as real for a Majorana representation,

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(2.58)

The classical fermionic world-sheet field $\psi_\mu$ is made of Grassmann numbers and satisfies the anticommutation relations,

$$\{\psi^\mu, \psi^\nu\} = 0,$$

(2.59)

which changes after quantization [232]. The spinor $\psi^\mu$ has two components,

$$\psi^\mu = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix},$$

(2.60)

Such that the Dirac conjugate spinor is defined as,

$$\bar{\psi} = \psi^\dagger \beta, \quad \text{with} \ \beta = i\rho^0$$

(2.61)

For a Majorana spinor, the above is merely $\psi^T \beta$, and $\psi^\mu$ are real in the sense appropriate for Grassmann numbers: $\psi^\mu_+ = \psi_+$ and $\psi^\mu_- = \psi_-$. Thus, suppressing the Lorentz index, the fermionic part of the above action can be expressed simply as,

$$S_f = \frac{i}{2\pi} \int d^2 \sigma (\psi_- \partial_+ \psi_+ + \psi_+ \partial_- \psi_+),$$

(2.62)
where $\partial_{\pm}$ refers to the world-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. The equations of motion for the spinor components is the Dirac equation,

$$\partial_+ \psi_- = 0 \quad \text{and} \quad \partial_- \psi_+ = 0,$$

which describes the left-movers and right-movers (waves), respectively. For spinor in two-dimensions, these are equivalent to the Weyl conditions; thus, the fields $\psi_\pm$ are actually Majorana-Weyl spinors.

**Global World-Sheet Supersymmetry**

The above action is invariant under the following supersymmetry transformations, where $\epsilon$ is a constant infinitesimal Majorana spinor consisting of anticommuting Grassmann numbers,

$$\delta X^\mu = \bar{\epsilon} \psi^\mu,$$

$$\delta \psi^\mu = \rho^a \partial_a X^\mu \epsilon.$$

When written in components, the supersymmetry transformations take the form,

$$\epsilon = \left( \begin{array}{c} \epsilon_- \\ \epsilon_+ \end{array} \right) \rightarrow \delta X^\mu = i(\epsilon_+ \psi_+^\mu - \epsilon_- \psi_+^\mu)$$

$$\delta \psi_+^\mu = -2\partial_- X^\mu \epsilon_+, \quad (2.67)$$

$$\delta \psi_-^\mu = -2\partial_+ X^\mu \epsilon_-.$$

**Superspace and Superfields**

The RNS action as written above is in component form, such that the supersymmetry is not manifest. The action can be rewritten in superspace, which is the extension of ordinary spacetime including additional anticommuting Grassmann coordinates and superfields. Superfields are essentially fields defined on superspace instead of just spacetime.

In the superfield formalism, one must add an off-shell degree of freedom to the world-sheet theory, without changing the physical content. The supersymmetry transformation algebra then closes off-shell without use of the equations of motion. For any setup with a small number of conserved supercharges, the superfield formalism makes the supersymmetry manifest and simplifies the calculations.

Whenever the number of supercharges is greater than four, which occurs for spacetime dimensions greater than four, a superfield formalism becomes difficult or
nearly impossible. The world-sheet coordinates in superspace, \((\sigma^a, \theta_A)\), include \(\theta_A\), anticommuting Grassmann coordinates which form a Majorana spinor:

\[
\theta_A = \left( \theta^-_A \right) \quad \text{and} \quad \{\theta_A, \theta_A\} = 0.
\] (2.69)

The upper and lower indices are not distinguished, and are not usually displayed.

The bosonic coordinates on the world-sheet are defined \(\sigma^0=\tau\) and \(\sigma^1=\sigma\), and the most general superfield function has a series expansion in \(\theta\) of the form,

\[
Y^\mu(\sigma^a, \theta) = X^\mu(\sigma^a) + \bar{\theta}\psi^\mu(\sigma^a) + \frac{1}{2}\bar{\theta}\theta F^\mu(\sigma^a),
\] (2.70)

where \(F\) is an auxiliary field that does not change the physical content of the theory. \(F\) is the field that makes the supersymmetry manifest. The anticommutation properties of the Grassmann numbers insure that the above expansion terminates as shown, since higher powers of \(\theta\) vanish automatically. For Majorana spinors, we also have \(\bar{\psi}\theta = \bar{\theta}\psi\), such that terms linear in \(\theta\) are equivalent to terms linear in \(\bar{\theta}\).

The supercharges are the generators of supersymmetry transformations of the world-sheet coordinates in superspace,

\[
Q_A = \frac{\delta}{\delta \theta_A} - (\rho^a \theta)_A \partial_a
\] (2.71)

The world-sheet supersymmetry transformations can be expressed in terms of \(Q_A\), as \(\bar{\epsilon}Q\) acts on superspace to generate transformation of the superspace coordinates,

\[
\delta \theta^A = [\bar{\epsilon}Q, \theta^A] = \epsilon^A \quad \text{and} \quad \delta \sigma^a = [\bar{\epsilon}Q, \sigma^a] = -\bar{\epsilon}\rho^a \theta = \bar{\theta}\rho^a \epsilon.
\] (2.72)

A supersymmetry transformation is thus a geometrical transformation of superspace. The supercharge acts on the superfield \(Y^\mu(\sigma^a, \theta)\) as,

\[
\delta Y^\mu = [\bar{\epsilon}Q, Y^\mu] = \bar{\epsilon}Q Y^\mu.
\] (2.73)

This can be expanded in components analogous to the series expansion in \(\theta\) above, such that the supersymmetry transformations take the form,

\[
\delta X^\mu = \bar{\epsilon}\psi^\mu,
\] (2.74)

\[
\delta \psi^\mu = \rho^a \partial_a X^\mu \epsilon + F^\mu \epsilon,
\] (2.75)

\[
\delta F^\mu = \bar{\epsilon}\rho^a \partial_a \psi^\mu.
\] (2.76)

For the equation of motion \(F^\mu = 0\), these equations reduce to (2.64) and (2.65) above.
Using the *supercovariant derivative*, the action can be expressed in superfield language. The supercovariant derivative is written,

\[ D_A = \frac{\partial}{\partial \theta A} + (\rho^\alpha\theta) \partial_\alpha, \quad (2.77) \]

and we note that \( \{ D_A, Q_B \} = 0 \). For an arbitrary superfield \( \Phi \), the supercovariant derivative \( D_A \Phi \) transforms under supersymmetry the same way as \( \Phi \) itself. We are now in a position to express the action in terms of superfields,

\[ S = \frac{i}{4\pi} \int d^2\sigma d^2\theta \bar{Y}^\mu D\gamma^\mu. \quad (2.78) \]

The superspace action has manifest supersymmetry, as seen by the variation,

\[ \delta S = \frac{i}{4\pi} \int d^2\sigma d^2\theta [\varepsilon Q] \bar{D}Y^\mu D\gamma^\mu. \quad (2.79) \]

Looking at the definition of \( Q_A \) above, both terms are *total* derivatives; i.e., one term is a total derivative in \( \sigma^\alpha \) and the other in \( \theta^A \). The \( \sigma^\alpha \) boundary conditions determine whether or not world-sheet supersymmetry is broken or unbroken. It is necessary to learn integration over Grassmann coordinates (see for example [230][232] or [236]); we note merely that the \( \theta \) integral of a \( \sigma \) derivative is zero, and that there are no boundary terms associated with Grassmann integration. Substituting the component expansion for the superfield \( Y^\mu(\sigma^\alpha, \theta) \), and the corresponding expansions for \( D Y^\mu \) and \( \bar{D}Y^\mu \), and executing the Grassmann integrations, the action above can be written:

\[ S = -\frac{1}{2\pi} \int d^2\sigma \left( \partial_\alpha X^\mu \partial^\alpha X^\mu + \bar{\psi}^\alpha \rho^\alpha \partial_\alpha \psi_\mu - F^\mu F_\mu \right). \quad (2.80) \]

This form of the action shows that indeed the equation of motion for \( F^\mu \) is \( F^\mu = 0 \). When the auxiliary field \( F \) is eliminated, we recover exactly the original RNS action. Elimination, however, results in loss of manifest supersymmetry and loss of off-shell closure of the supersymmetry algebra.

**M-Theory and String Duality**

The five different 10-dimensional superstrings are unified by means of an intricate web of T-dualities and S-dualities. The S-dualities relate various string theories at strong coupling to a corresponding dual string description at weak coupling. More importantly for our purposes, the type IIA superstring (and \( E_8 \times E_8 \) heterotic string) exhibit an 11th-dimension at strong coupling, and in the limit approach a common 11-dimensional theory called M-theory. In the previous technical section on supergravity, we discussed the fact that D=11 supergravity is the low-energy limit of the full quantum M-theory; but also, D=11 supergravity is...
dual to the type IIA string at strong coupling. An abbreviated technical discussion of these issues is given here.

First, we discuss \textit{T-duality}. In ordinary point particle field theory, compactification of one dimension leads to a periodic momentum: \( p = (n/R) \), for a circle of radius \( R \) and an integer \( n \). In the limit of \( R \to \infty \), the momentum becomes continuous, and the uncompactified theory is recovered. Next, consider the limit \( R \to 0 \), and note that the momentum must become either 0 or \( \infty \), and the compactified dimension effectively \textit{decouples} from the theory. Therefore, these two limits are completely different in point particle field theory.

In closed string theory, consider compactifying the 9\textsuperscript{th} dimension. An added complication appears that did not exist in the point particle case. The string can wind around the compactified dimension, and the momentum operators take the following values \cite{237}:

\[
(P_L, P_R) = \left( \frac{n}{2R} + mR, \frac{n}{2R} - mR \right),
\]

where \( n \) arises from the Kaluza-Klein excitations of the circle, and \( m \) labels the number of times the string winds around the circle. Because \( R \) appears in the numerator and the denominator of the momentum operators, when we interchange \( n \leftrightarrow m \) and simultaneously substitute,

\[
R \leftrightarrow \frac{1}{2R},
\]

the mass spectrum for \( M^2 \) is invariant. This is an unusual symmetry in the context of classical physics, because we have linked the large-scale behavior of string theory to its small-scale structure. It is claimed that "the string cannot differentiate between these two regions." This duality symmetry interchanges the Kaluza-Klein modes with the winding modes, which is only possible due to the geometry associated with the extended nature of the string.

This duality transformation can be written in the language of conformal field theory, where it is equivalent to a substitution that requires a "sign" change for one set of movers as \cite{237},

\[
\partial X \to \partial X \quad \text{and} \quad \bar{\partial} X \to -\bar{\partial} X.
\]

When this duality is applied to the superstring in the RNS formalism, we find for the left- and right-moving fermions,

\[
\psi^9_L \to -\psi^9_L \quad \text{and} \quad \psi^9_R \to \psi^9_R.
\]
The above transformation of the left-moving oscillator in the 9th-direction also reverses the sign of the ten-dimensional left-moving chirality operator constructed from the fermionic zero modes, which "flips" the chirality of the left-movers:

\[ \Gamma_{11} = \psi_L^0 \psi_L^1 \psi_L^2 ... \psi_L^9 \rightarrow -\Gamma_{11}. \] (2.85)

Due to the T-duality transformation, then, type IIA spinors, which consist of two types, one with positive and one with negative chirality, are transformed into a theory where both spinors will have the same chirality. That is, we obtain the type IIB chiral superstring and *vice versa*:

T-duality: type IIA \( \leftrightarrow \) type IIB. \[ \text{ (2.86)} \]

The type IIA and type IIB superstrings are thus viewed as two extreme points along the same continuum of vacua labeled by R. As \( R \rightarrow \infty \) or as \( R \rightarrow 0 \), one recovers the two type II string theories in ten-dimensions [237].

An analogous T-duality transformation exists——although complicated by the necessity to consider the Narain lattice on which 16-left movers are compactified—that connects the \( E_8 \times E_8 \) heterotic string with the \( SO(32) \) heterotic string.

Consider next the *S-duality transformation*. T-duality is perturbative in the string coupling constant, and thus provides no insight into the nature of non-perturbative phenomena in string theory. S-duality links the weak coupling regime of one theory to the strong coupling region of another (dual) string theory. This is an inherently nonperturbative duality, such that completely new possibilities arise.

The *dilaton field* is central to the discussion of nonperturbative effects, because the expectation value of the dilaton is directly related to the string coupling constant. The first quantized string Lagrangian contains not only the usual term describing the area of the string world sheet, but also a coupling to the Riemann curvature tensor on the world-sheet, where \( \phi \) represents the dilaton [237],

\[ \sqrt{g} (g^{ab} \partial_a X^\mu \partial_b X_\mu + \phi R^2). \] (2.87)

The Euler number for a two-dimensional Riemann surface of genus \( g \) is,

\[ \chi = \frac{1}{4\pi} \int d^2 \sigma \sqrt{-g} R^{(2)}. \] (2.88)

Making the substitution \( \phi \rightarrow \phi + \langle \phi \rangle \), the Euclidean path-integral \( e^{-S} \) gains a term,

\[ e^{-S} \rightarrow e^{-S} e^{\langle \phi \rangle (2g-2)}. \] (2.89)
Each \( n \)-point string amplitude is multiplied by the coupling constant factor \( g_s^{2g-2+n} \), where \( n \) is the number of boundaries or external strings. Placing \( e^{\langle \phi \rangle} \) at each vertex function on the Riemann surface, this factor is absorbed into the string coupling constant \( g_s \) by defining it as,

\[
g_s = e^{\langle \phi \rangle}. \tag{2.90}
\]

The coupling constant is therefore directly related to the vacuum expectation value of the dilaton field. The key point is that S-duality changes \( \langle \phi \rangle \) into \(-\langle \phi \rangle\) to connect the strong coupling region of one string theory to the weak-coupling region of a dual string theory [237].

**Type IIA String at Strong Coupling and D=11 Supergravity**

Type IIA supergravity is the low-energy limit of the type IIA superstring theory, but it can also be obtained by dimensional reduction of D=11 supergravity. The correspondence between type IIA superstring theory and M-theory is much deeper than that [158], however, which can be seen by study of the strong-coupling limit of the type IIA superstring [232].

Keep in mind that D0-branes are stable, nonperturbative excitations in the type IIA superstring spectrum with mass given by \( (\ell_s g_s)^{-1} \) in the string frame.

S-duality reveals a profound counterintuitive result that relates the type IIA string directly to D=11 supergravity, which led Witten to postulate the existence of the mysterious 11-dimensional theory called M-theory [158].

Consider the original 11-dimensional supergravity theory [92] discussed in the technical section above. This theory was dismissed historically because it was viewed as non-renormalizable, and also incapable of generating chiral fermions. From the perspective of the duality discussion given in the previous sub-section, however, both of these problems are cured by the addition of new terms [237].

Start with the bosonic terms in the Lagrangian as written previously,

\[
2\kappa_{11}^2 S = \int d^{11}x \sqrt{-g} \left( R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4, \tag{2.91}
\]

where \( F_4 \) represents the field tensor constructed out of antisymmetrized derivatives of \( A_{MNP}; \) that is,

\[
F_{MNPQ} = 4 \delta_{[M} A_{NPQ]} . \tag{2.92}
\]
We want to compare this action with the type IIA action in terms of massless fields, but after we integrate out over higher fields. The string variables are coupled to the massless fields and treated as background fields. Thus, in addition to the graviton $g_{\mu\nu}$ and the dilaton $\phi$, the antisymmetric 2nd-rank tensor $B_{\mu\nu}$ arises from the product of two Neveu-Schwarz fields. In the RNS formalism, the bosonic sector results from the product of two NS-operators or two R-operators corresponding to the left- and right-movers, such that the bosonic spectrum is spanned by the states: $[\text{NS}_L \times \text{NS}_R] \text{or} [\text{RS}_L \times \text{RS}_R]$. Both type IIA and IIB have the same set of massless fields in the NS-NS sector:

$$\text{NS-NS: } \{\phi, g_{\mu\nu}, B_{\mu\nu}\}. \quad (2.93)$$

However, the type IIA contains the additional fields,

$$\text{R-R: } \{C^\mu, A_{\mu\nu}\}. \quad (2.94)$$

The effective action for the type IIA theory to lowest order in the massless fields is 10-dimensional non-chiral supergravity with $\mathcal{N}=2$ supersymmetry. Letting $K=dC$, $H=dB$, and $G=dA$, this action can be written [237],

$$S = \int d^{10} \left\{ \sqrt{-g} e^{-2\phi} \left[ R + 4|d\phi|^2 - \frac{1}{3} |H|^2 \right] - \sqrt{-g} \left[ |K|^2 + \frac{1}{12} |G|^2 \right] \right\} + \frac{1}{144} \int G \wedge G \wedge B \quad (2.95)$$

When 11-dimensional supergravity is compactified on a circle $S^1$, and the radius of the circle goes to zero, one obtains type IIA supergravity theory. The claim [158] is that the true physical meaning of this correspondence can be understood more deeply in the following way.

The actions of 11-dimensional supergravity and type IIA string theory are identical to lowest order upon decomposition of the 11-dimensional metric tensor $g_{MN}$ and the 3-form $A_{MNP}$ into the 10-dimensional fields,

$$g_{MN} \rightarrow (g_{\mu\nu}, C^\mu, \phi) \quad (2.96)$$
$$A_{MNP} \rightarrow (A_{\mu\nu\rho}, B_{\mu\nu}) \quad (2.97)$$

and the radius of the 11-th dimension is $e^{2\phi/3}$. In detail the decomposition is [237],

$$ds^2 = g_{MN} dx^M dx^N = e^{\frac{2\phi}{3}} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4\phi}{3}} (dy - dx^\mu C_\mu)^2, \quad (2.98)$$

$$A = \frac{1}{6} dx^\mu \wedge dx^\nu \wedge dx^\rho A_{\mu\nu\rho} + \frac{1}{2} dx^\mu \wedge dx^\nu \wedge dy B_{\mu\nu}. \quad (2.99)$$
Since 11-dimensional supergravity compactified on a circle has Kaluza-Klein states that cannot be seen in 10-dimensional type IIA string theory, one may question this identification. These Kaluza-Klein states, however, correspond to soliton solutions in the 10-dimensional string theory. Thus, the 10-dimensional type IIA with soliton states is the same as the 11-dimensional theory.

Type IIA string theory expressed in terms of massless fields is highly non-polynomial in the curvature tensors, while 11-dimensional supergravity is not. Thus, Witten concluded that the type IIA string rewritten in 11-dimensions must have D=11 supergravity as its low-energy limit. He then postulated, in addition, the existence of a new 11-dimensional theory that he called M-theory. The claim is, in effect, that the strong-coupling limit of the type IIA string is M-theory.

II.5 The Basic Relation between M-theory and Supergravity

In all five string theories the tension of the string becomes very large in the weak-coupling limit when \( \alpha' \to 0 \), and the masses of all string states (except for the massless states) become very large. In a Minkowski space background, the \( \alpha' \to 0 \) limit corresponds to the low-energy limit of string theory. This occurs because the quantity \( \alpha'E^2 \) is the only dimensionless parameter in the theory [232]. Since the massive modes are too heavy to be observed in this limit, they are "decoupled," such that the interactions of the remaining massless modes are described to a good approximation by the corresponding supergravity theory. In this way the various supergravity theories arise as the low-energy limits of the respective string theories; and by analogy, it is believed that 11-dimensional supergravity accurately represents the low-energy limit of M-theory [232].

The relationship of M-theory and supergravity can also be understood in many respects by consideration of holonomy groups in Riemannian geometry, which determine those geometrical structures on a manifold \( M \) that are compatible with the Riemannian metric \( g \) [232]. The focus is then classical field theory, whose basic mathematical framework is the partial differential equations of differential geometry and Lie group theory. An essential generalization required in supersymmetric theories, however, is the inclusion of anticommuting variables—which goes beyond the realm of standard general relativity and Yang-Mills theory [232], such that one employs graded Lie algebras and groups, Clifford and Grassman algebras, and Berezin integration.
Recall from the preceding technical section that the basic relationship of string/M-theory and supergravity was established when Witten found that increasing the coupling constant of the IIA superstring resulted in D=11 supergravity. Subsequently, it was determined that compactification of the IIA superstring on $T^6$ results in a 4-dimensional theory dual to $\mathcal{N}=8$ d=4 supergravity compactified on $T^7$. This basic relationship is analyzed in the claim that $\mathcal{N}=8$ supergravity is in the swampland [71], and in our discussion of that claim.

After the birth of 11-dimensional M-theory, the event catalyzed renewed interest in 11-dimensional supergravity. By June 1995, compactification of D=11 supergravity to d=5, 4, and 3 on spaces of exceptional holonomy; i.e., on compact manifolds of SU(3), G2, and Spin (7), respectively, occurred [162]. So much activity related to the development of superstrings and supermembranes has occurred by means of the low-energy supergravity approximation, it would be difficult to recount every aspect here. Instead, we focus on two aspects of that development that attracted considerable attention over the past two or three years: (1) M2-branes in the context of the AdS/CFT correspondence (see references [27] through [61]); and (2) renormalization of pure $\mathcal{N}=8$ supergravity (references [62] through [82]), and its possible implications for the relationship between supergravity and string/M-theory.

The same set of papers that gave birth to M-theory ([153] [156][158][167][172]) simultaneously established that D=11 supergravity plays a key role in the long-distance approximation to M-theory. This also revived interest in the supermembrane of D=11 supergravity, which evolved into the M2-brane in the context of the full quantum M-theory.

The M2-brane concept has undergone significant development recently in terms of the Bagger-Lambert-Gustavsson (BLG) theory [29][30][33][34][35][36] [38][51][61] for a d=3 $\mathcal{N}=8$ superconformal Chern-Simons action in an AdS$_4 \times M^7$ setup related to the AdS/CFT correspondence; and also, the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [41] for $\mathcal{N}=6$ super-conformal Chern-Simons matter theories and their AdS$_4$/CFT$_3$ gravity duals [42-47][54][55][60]). Several studies have focused on generalizations that provide insight into the relationship of the $\mathcal{N}=8$ and $\mathcal{N}=6$ superconformal theories, see for example [38][48][50][52][57][59]. These developments underscore the important synergistic relationship of D=11 supergravity to the advancement of M-theory, the M2-brane concept; all of which enables a deeper understanding of the AdS/CFT correspondence.
There are many open questions, nevertheless, as to whether for example the M2-brane has massless states. The existence of massless states was first recognized via identification of Kaluza-Klein states of M-theory compactified on $S^1$ with the Dirichlet particles and their bound states (D0-branes) in type-IIA string theory. One can speculate that an improved understanding of the M2-brane will ultimately lead to recognition of additional massless states [189]. Perhaps one may eventually find an analogy or correspondence of the latter to the towers of massless states that remain in the "decoupling limit" of $\mathcal{N}=8$ supergravity, as defined by Green, Ooguri and Schwarz [71].

In the context of the recent M2-brane development, one can also speculate on whether the geometric and algebraic structure is telling us something new about the intrinsic properties of physical reality. With respect to the question of whether or not $\mathcal{N}=8$ supergravity is finite, we review the recent argument that concludes it is in the "swampland" and therefore inconsistent [71][75][77]. In the following we hope to convince the reader instead that $\mathcal{N}=8$ supergravity is an important compactified sector of a more complete 11-dimensional theory, which should be mapped in direct correspondence to the 11-dimensional M-theory. Furthermore, in certain scenarios, it may perhaps be better understand as a 12-dimensional theory [201][202] that could be mapped in relation to Vafa’s F-theory [224][225].
CHAPTER III

IS $\mathcal{N} = 8$ SUPERGRAVITY FINITE AND/OR IS IT IN THE SWAMPLAND?

During the past two to three years, significant progress has occurred towards demonstrating the ultraviolet (UV) finiteness of pure $\mathcal{N}=8$ supergravity (see the review in [82]). This is the result of "unexpected cancellations" [67, 74]. Computations of scattering amplitudes show that $\mathcal{N}=8$ supergravity is surprisingly well behaved in the ultraviolet and may be UV-finite in perturbation theory, as discussed in III.1, the first subsection below. However, in an interesting paper [71], which has attracted considerable attention during the past two years, Green, Ooguri and Schwarz (GOS) demonstrate that, contrary to widespread belief, perturbative maximal ($\mathcal{N}=8$) supergravity in $d=4$ does not arise as the decoupling limit of Type II superstring theory compactified on $T^6$, the six-torus.\(^8\)

The authors of [71] attempt to define a limit in which the only finite-mass states that remain are the 256 massless graviton states of $\mathcal{N}=8$ supergravity; however, they find infinite towers of additional massless and 'light' finite-mass states in $d=4$. Therefore, they conclude—taking into account conjectures made by Vafa and Ooguri [21][22]—that $\mathcal{N}=8$ supergravity is indeed in the Swampland [71]. Since the latter concept implies theories that are "consistent-looking semi-classical effective field theories, which are actually inconsistent" [21], we are compelled to study this matter further. The meaning of "inconsistency" and our view that pure $\mathcal{N}=8$ supergravity may be merely "incomplete" is taken up in section III.2.

We find that the facts presented by GOS are essentially correct, but an argument is made for a distinctly different interpretation. It is suggested that, in contradistinction, $\mathcal{N}=8$ supergravity is not only not in the swampland, but also, it should be considered as a complimentary aspect of a complete theory that might provide an improved understanding of M-theory. It is hoped that the completion of M-theory will, for example, ultimately aid in the identification of preferred vacua within the string/M-theory landscape.

\(^8\) This paper actually discusses more generally, the conditions necessary to obtain perturbative maximal supergravity in $d$-dimensions as a decoupling limit of type II superstring theory compactified on a (10-$d$)-torus; however, for our limited purpose we focus only on toroidal compactification to 4-dimensions.
III.1 Unexpected Cancellations in N=8 Supergravity Scattering Amplitudes

Recent activity suggests the possibility that N=8 supergravity is finite. The progress achieved is documented, for example, in references [67] through [82]; although several earlier references in [62-66] are germane to this discussion. The two papers shown as reference [67] appear to have sparked the recent activity.

The unexpected cancellations are surprising from the point of view of power counting, which together with standard integral reduction formulas imply that “triangle integrals” should appear at five points and bubble integrals at six points [74].

Dimensionally regularized amplitudes in four dimensions are usually expressed as a linear combination of scalar box, triangle and bubble integrals together with rational terms. Schematic diagrams are shown, respectively, in Figure 1. The “no-triangle hypothesis” states that all one-loop amplitudes in N=8 supergravity are expressed solely in terms of scalar box integrals [74]. Triangle integrals, bubble integrals, and additional rational terms do not appear.

Cancellation of triangle and bubble integrals was first observed in “maximally helicity violating (MHV) amplitudes of the N=8 theory, and recently extended to obtain the same cancellations in all N=8 one-loop amplitudes (see [67] and earlier references therein). The latter are now expressed solely in terms of scalar box integrals.

Modern unitarity methods enable multiloop calculations that exploit a remarkable relation between tree-level gravity and gauge-theory amplitudes. Kawai-Lewellen-Tye (KLT) developed the original form of this relationship some time ago [62], but it was recently clarified to provide a more transparent form (Bern et. Al. [79]) to facilitate calculation.

An investigation of the origin of the unexpected cancellations observed in gravity scattering amplitudes suggest they are linked to [78][79]: (1) general coordinate invariance of the gravitational action (“gauge invariance from diffeomorphism symmetries”); and (2) the summation over all orderings of external legs (i.e., the amplitudes are colorless and exhibit crossing symmetry).

The following provides a brief overview of some of the above technical issues related to demonstrating that N=8 supergravity is finite.
Figure 1. Basis of one-loop scalar integrals given by (a) a scalar box, (b) scalar triangle, and (c) a scalar bubble integral. In $D = (4-2\epsilon)$ dimensions, these diagrams carry all the UV- and IR-divergences of the amplitudes.

The recent calculations of four-point scattering amplitudes are based upon the maximally symmetric ($N=8$) supergravity of Cremmer and Julia [92]. Novel ultraviolet (uv) cancellations that could lead to perturbative finiteness of the theory exist at three and higher loops.

String dualities have also been used to argue for uv-finiteness of $N=8$ supergravity, however, several works have suggested caution in this respect [74], as a result of the recent work by Green, Ooguri, and Schwarz [71]. For example, Kallosh [76] states that it's not clear one can trust the finiteness arguments for $N=8$ $d=4$ supergravity derived from string theory, and references the non-decoupling of non-pertubative states as found in [71].

Beyond six and seven points, scaling and factorization properties of the amplitudes provide strong evidence that the no-triangle hypothesis holds for the remaining amplitudes in $N=8$ supergravity (Bjerrum-Bohr et. al. [67]). At three loops an improved uv-behavior has been confirmed by explicit calculation of the complete four-point scattering amplitude [70]. In addition, consistency of the Regge limit with improved uv-behavior was recently discussed in [69] and [73].

Supersymmetry has been studied for many years for its ability to reduce the degree of divergence of gravity theories, as documented in Green, Russo, and Vanhove [68]. All superspace “counting arguments,” however, merely delay the onset of divergences by a finite number of loops only; depending on assumptions in the types of superspaces and associated invariants that can be constructed.
The real question seems to be the reason for the remainder of the observed cancellations? Bern et. al. proposed in [74] that the extra cancellations are generic to any quantum theory of gravity based on the Einstein-Hilbert action.

The Feynman diagram approach, wherein, naïve power counting of each individual diagram is executed, does not display the subject cancellations. The latter are manifest only in "carefully chosen" representations of the amplitudes [74]. In supersymmetric theories, the thought is that supersymmetric cancellations are in addition to the primary cancellations under investigation.

String theory provides a consistent framework for quantum gravity and its supersymmetric extensions. Within this formalism various gravity amplitudes can be computed, and expressions for field theory amplitudes preserving supersymmetry can be derived in the infinite tension limit of the string; i.e., $\alpha' \to 0$. String theory combines the effect of a hard uv-momentum cutoff, determined by extension of the string while keeping gauge invariance, and the decoupling of unphysical states thanks to the modular invariance of its world-sheet. Although string theory is perturbatively finite, its complete degrees of freedom are provided by the non-perturbative U-duality symmetries [153][158].

Power counting arguments based on known symmetries indicate that supergravity theories have uv-divergences in 4-dimensions and candidates for explicit counter-terms at three-loop order have been constructed [97][76]. Contrary to the results of power counting, however, explicit computation of one-loop amplitudes in $N=8$ supergravity [67][70][74] demonstrates they can be constructed from the same basis of scalar integrals as $N=4$ super Yang-Mills theory. Divergences in four-dimensions in maximal supergravity are explicitly absent until three-loop order by direct computation [70].

The discrepancy between power counting and explicit computation underscores the lack of knowledge of the consequence of physical effects such as gauge invariance associated with diffeomorphisms.

Recent work focused on order-by-order finiteness of the perturbative series of $N=8$ supergravity [79] does not consider the non-perturbative issues; i.e., how one might construct a realistic finite theory. There is interest in the perturbative finiteness of gravity, because the existence of cancellations sufficient to render the theory finite would imply a new symmetry or dynamical mechanism. A correct understanding of the mechanism behind the cancellations will have profound impact on our understanding of gravity [79].
Feynman Diagram Approach

Some insight can be gained quickly by surveying the Feynman diagrams for gravity in comparison to those of gauge theory (see Figures 2 and 3).

![Feynman diagrams](image1)

Figure 2. Gauge theories have three- and four-point vertices in a Feynman diagram description.

![Feynman diagrams](image2)

Figure 3. Gravity theories have an infinite number of higher-point contact interactions in a Feynman diagram description.

Consider the Einstein-Hilbert and Yang-Mills Lagrangians,

\[ L_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \quad \text{and} \quad L_{EH} = \frac{\kappa^2}{2} \sqrt{-g} R. \]  

(3.1)

In standard Feynman diagram methods, one fixes the gauge and then expands the Lagrangians in a set of vertices. As shown in Figure 2, there are three- and four-point interactions with standard gauge-fixing in gauge theory, but in the case of gravity, as depicted in Figure 3, there are an infinite number of contact interactions. Nevertheless, the recent claim in [79] is that all interactions beyond three-points are unnecessary provided one used on-shell methods. In reference [82], one can see the three-graviton vertex written in excruciating detail, as compared to the relatively simple three-gluon vertex in Feynman gauge theory.

Power Counting in Gravity Theories

Newton’s constant is dimensionful in four-dimensions, implying that gravity is non-renormalizable by power-counting. The fact that gravity theories
are badly behaved in the uv-region follows simply from loop-level Feynman diagrams, as shown in the three-loop Feynman diagram in Figure 4.

![Feynman Diagram](image)

Figure 4. If the three-loop Feynman diagram represents a three-gluon vertex in YM-theory, the vertex is linear in the momenta; whereas, a three-graviton vertex is quadratic in the momenta (adapted from [82]).

If the diagram above represents gluon scattering in Yang-Mills theory, the diagram yields a Feynman integral of form [82],

\[ \int \prod_{j=1}^{L} \frac{d^D p_j}{(2\pi)^D} \prod_{m} \frac{\cdots (g f^{abc} p_i^\mu) \cdots}{m! (i \pi_m^2 + i\epsilon)} . \quad (3.2) \]

The numerator \( (g f^{abc} p_i^\mu) \) stands for the vertex factor given by a coupling \( g \), a color factor \( f^{abc} \) and momentum \( p_i^\mu \). The denominators are the Feynman propagators in the diagram, which carry momenta that depend on the independent loop and external momenta \( p_i \) and \( k_i \).

In contradistinction, consider the corresponding expression for the gravity diagram, whose form is similar, except the vertices have two powers of momenta in the numerator for each vertex,

\[ \int \prod_{j=1}^{L} \frac{d^D p_j}{(2\pi)^D} \prod_{m} \frac{\cdots (\kappa p_i^\mu p_j^\gamma) \cdots}{m! (i \pi_m^2 + i\epsilon)} . \quad (3.3) \]

The momenta in the vertices are generically loop momenta. Since there are a large number of loop momenta in the numerators, each gravity Feynman integral will behave badly in the ultraviolet as compared to the corresponding gauge diagram. Based on these simple power-counting arguments, unless there are non-trivial cancellations, we expect the behavior will become worse as the number of loops increases.
Given a divergence at a particular loop order, one can readily determine the structure of a counterterm: since every loop gains an extra power of $G_N \sim 1/M_{Pl}^2$, each additional loop must gain two powers of mass dimension to compensate. This corresponds to an additional power of the Riemann curvature tensor, $R^\mu_{\nu\rho\sigma}$, or to a covariant derivative $D^2$. At one-loop, pure-gravity counterterms in the Lagrangian must have two powers of the Riemann tensor. By 'accident' in four-dimensions, however, the Gauss-Bonnet theorem eliminates the potential on-shell counterterm.

If matter is added into the scenario, a one-loop divergence generically appears, and at two-loops the potential counterterm has the form:

$$\text{CT} \sim R^\rho_{\mu\nu} R^\mu_{\nu\lambda\sigma} R^\lambda_{\sigma\tau} R^\tau_{\rho\sigma} \ldots$$  \hspace{1cm} (3.4)

Due to calculations performed many years ago, it is considered a fact that pure gravity diverges and that the coefficient of this counterterm does not vanish [82].

In d=4 supergravity theory, the first divergence occurs at three- or more loops, since the potential one- and two-loop counterterms are inconsistent with supersymmetry (see [97] and references therein). The potential counterterm consistent with supersymmetry, is an $R^4$-term with indices contracted, corresponding to the Bel-Robinson tensor [82]. The 'new' $R^4$-term that we discuss in Chapter IV, which appears at third-order, and the debate as to whether it can be supersymmetrized seems relatively unknown.

The loop order at which a divergence in $N=8$ supergravity is first expected can be raised by means of additional assumptions. If an off-shell superspace with $N=6$ exists with the supersymmetries manifest, then the potential divergences in four dimensions are delayed to at least five loops [65]. Assuming the existence of a superspace with $N=7$ supersymmetries manifest, one can go beyond and delay the first potential divergence to at least six loops [65]. Continuing in this manner, if one assumes the existence of a fully covariant off-shell superspace with $N=8$ supersymmetries manifest, then the first potential divergence would be seven loops or greater [82].

It is important to realize that the above sequence of assumptions is not firmly based in theory at this time. For example, no off-shell superspace beyond $N=4$ has been constructed to date. (This issue is discussed extensively in [235],
which was published in 1989. It does not appear this fact has changed in the past 20-years, although I did not find an explicit recent reference for confirmation. Howe and Lindstrom, and independently Kallosh (see references in [97]), constructed an eight loop potential counterterm, which suggests a divergence may occur at this loop order, if not earlier.

In brief, power counting arguments cannot prove the existence of a divergence, but only that a divergence cannot appear prior to a certain loop order. The power counting bound may suggest the appearance of a divergence at a certain order, when in fact, due to hidden symmetries, no divergence occurs.

**On-shell Methods**

The elementary building blocks for obtaining new amplitudes are the on-shell amplitudes obtained previously. There are two primary methods: (1) on-shell recursion, and (2) the modern unitarity method (see [67][70] and references therein). The on-shell recursion is used for obtaining complete tree-level amplitudes, while the unitarity method is suitable at loop level. At one loop it is possible to use a hybrid bootstrap method comprised of both unitarity and on-shell recursion. The methods have become widespread in recent years due to their computational efficiency [67-70 and 72-74].

Proceeding naively with an on-shell formalism, one encounters a difficulty: the on-shell three vertex for gravitons or gluons actually vanishes [82]. This is because the process is kinematically forbidden. The solution uses complex momenta, which allows satisfaction of the on-shell conditions and momentum conservation, while allowing definition of a non-vanishing vertex.

It is best to express gauge and gravity amplitudes in four-dimensions in terms of spinors. The two helicity configurations for gluons are as,

\[
\epsilon^\mu_\pm(k_i,q_i) = \pm \frac{(q_i^\pm p^\rho |k^-)}{\sqrt{2}(q_i^\pm |k^-)}.
\]

(3.5)

The \(|k^-\rangle \equiv u_{\pm(k_i)}\) are Weyl spinors and the \(q_i\) are arbitrary null “reference momenta.” The graviton polarization tensors are products of these,

\[
\epsilon^{\mu\nu}_\pm(k_i,q_i) = \epsilon^\mu_\pm(k_i,q_i)\epsilon^\nu_\pm(k_i,q_i).
\]

(3.6)
Antisymmetric spinor inner products are defined, and for massless momenta, the spinor inner products are complex square roots of Lorentz inner products, satisfying [82],

$$\langle i j \rangle [j i] = 2k_i \cdot k_j.$$ (3.7)

Tree-level scattering amplitudes can be recursively constructed by means of the on-shell recursion relations, starting from three-point vertices. This can be carried out for pure gravity in D dimensions. These tree amplitudes are all that is needed to systematically construct all higher-loop amplitudes using the unitarity method [82]. The construction does not use four- and higher-point vertices in any step for either gauge or gravity theories. In fact, the four- and higher-point vertices displayed in Figures 2 and 3 above are irrelevant for scattering amplitudes to any loop order.

**Kawai-Lewellen-Tye Tree-level Relations**

It is difficult to discern simple relations between gravity and gauge-theory amplitudes, either from the perspective of Lagrangians or off-shell Feynman rules. Tree-level gravity amplitudes can be rewritten in terms of gauge-theory amplitudes, as first noticed in [62]. These relations hold in field theory as the low-energy limit of string theory. In the limit, KLT relations for four- and five-point amplitudes are,

$$M_4^{tree}(1,2,3,4) = -is_{12}A_4^{tree}(1,2,3,4)\tilde{A}_4^{tree}(1,2,3,4)$$ (3.8)

$$M_5^{tree}(1,2,3,4,5) = -is_{12}s_{34}A_5^{tree}(1,2,3,4,5)\tilde{A}_5^{tree}(2,1,4,3,5) + is_{13}s_{24}A_5^{tree}(1,3,2,4,5)\tilde{A}_5^{tree}(3,1,4,2,5)$$ (3.9)

The gravity theory states are direct products of gauge-theory states for each external leg. Explicit formulae for n-point amplitudes may be found in [79]; in particular, see [arXiv:0805.3993]. In fact, the KLT relations have been clarified in the latter to provide a more transparent form of the relation.

An interesting simple example consists of the four-graviton amplitude, wherein, one uses the four-gluon color-ordered amplitudes, which results in a much easier, stream-lined calculation [82]. Although the KLT relations are used merely as a technical trick to efficiently evaluate gravity amplitudes, they signify a non-trivial field-theory unification of gravity and gauge theory. String theory automatically encodes this unification.
Modern Unitarity Method to Obtain Loop Amplitudes

The modern unitarity method used in [67][70][72][74], and references to earlier works therein, provides a systematic means to construct loop amplitudes using on-shell tree amplitudes as input. One first constructs an initial ansatz in terms of integral functions that reproduces one cut. Then, subsequent cuts of the amplitude are compared against the corresponding cuts of the ansatz. If any discrepancy is found in a later cut, additional terms that vanish when all the earlier cut conditions are imposed are added to the ansatz [82]. When a complete set of cuts have been checked, an integral representation of the loop amplitude with the correct cuts in all channels is obtained. The complete amplitude thus results, essentially equivalent to what would have been obtained by Feynman diagrams.

Massless amplitudes in supersymmetric gauge theories are completely determined at one loop by their four-dimensional cuts. This has not been demonstrated at higher loops. A version of dimensional regularization compatible with supersymmetry must be used in four-dimensions, due to the presence of infrared singularities—even though the theory may be uv-finite. The unitarity cuts must be evaluated in D dimensions to insure that no terms are dropped in the construction. This makes the calculation significantly more difficult, because powerful 4-dimensional spinor methods cannot be used [82].

For N=4 super Yang-Mills theory, some of the additional complexity is avoided by calculating internal-state sums by means of the gauge supermultiplet of the D=10 N=1 theory instead of the D=4 N=4 multiplet.

In practical calculations, one first constructs an ansatz for the amplitude based on four-dimensional momenta in the generalized cuts, due to the availability of powerful spinor methods. Considerable progress has been made recently in keeping track of the superpartners crossing unitarity cuts in four-dimensions [82]. After the latter is achieved, based on four-dimensional cuts, one must compare against the D-dimensional cuts to insure that no terms have been dropped. For four-point amplitudes in N=4 super Yang-Mills, the evidence suggests that all terms are detected through five loops by four-dimensional cuts.

In summary, the KLT relations provide an efficient means to evaluate gravity generalized cuts, that reduce the amplitude to products of tree amplitudes summed over intermediate states. The cuts of gravity amplitudes are
re-expressed as sums of products of cuts of gauge-theory amplitudes. The gauge-theory cuts are generally much simpler to evaluate. Once the superpartner sums are performed for $N=4$ super Yang-Mills generalized cuts, the corresponding super-partner sum in $N=8$ supergravity follows directly from the KLT relations. Simplifications obtained on the gauge-theory generalized cuts can be carried over immediately to gravity cuts. An example of generalized cuts is shown in Figure 5.

![Generalized cuts to determine a three-loop four-point amplitude. Each 'blob' represents an on-shell tree amplitude. The intermediate lines are all on-shell [82].](image)

**Is N=8 Supergravity Finite?**

The above provides some insight into the current status of the calculations that attempt to prove that $N=8$ supergravity is finite. However, the question posed above cannot yet be answered with any degree of conviction. Whether or not it is ultimately shown to be finite, it is hoped that this summary has provided a sense of the tremendous depth and mystery of $N=8$ supergravity. The latter theory appears just as mysterious and unknown as M-theory itself. It is our view that the two theories represent precisely the same reality, and that eventually, further progress will be made in theoretical physics when the correspondence between these two theories is fully completed and understood.
III.2 The Argument that $\mathcal{N}=8$ Supergravity cannot be decoupled from M-theory

As a first step, we summarize the argument of GOS [71] as it relates specifically to $\mathcal{N}=8$ Supergravity in four-dimensions. Having recently become aware of a concise version of the argument given online by J. Distler, we follow the latter in the context of $D=11$ M-theory compactified on $T^n$, but focus only on the case of $n=7$ for the $\mathcal{N}=8$ d=4 scenario, due to the interesting possibility that maximal $\mathcal{N}=8$ supergravity might be finite in d=4.

$\mathcal{N}=8$ d=4 supergravity arises as the low-energy limit of Type-II string theory compactified on $T^6$. In [71] the question asked is whether a "decoupling limit" exists, such that one can hold $M_4$, the Planck mass in 4 dimensions, fixed, while decoupling all degrees of freedom except the massless supergravity multiplet. Recall that massless scalar fields in the $\mathcal{N}=8$ supergravity multiplet take values on $M=\mathbb{E}_7/K_7$, because we consider only the $n=(11-d)=(11-4)=7$ case.

Type II string theory compactified on $T^6$ does not quite yield the same multiplet, because of the massive charged states in the theory. In particular, the continuous $\mathbb{E}_7$ symmetry is broken to the discrete subgroup $\mathbb{E}_7(\mathbb{Z})$, which is a gauge symmetry, such that the true moduli space is $M/\mathbb{E}_7(\mathbb{Z})$.

The question can now be stated: as one approaches the boundary of the moduli space, is there a decoupling limit in which all of the massive degrees of freedom decouple, leaving only the supergravity multiplet whose moduli space, in the limit, looks like $M$? Green, Ooguri and Schwarz say no for $d\geq 4$. We consider only the $d=4$ case, such that the argument in M-theory language is as follows.

In M-theory on $\mathbb{R}_4 \times T^7$, BPS p-branes exist in the 4-dimensional theory with tensions (masses, for $p=0$) as shown in the table at the top of the next page (adapted from Distler's more general table). $M_{11}$ represents the 11-dimensional Planck mass and $R$ is a typical radius of the $T^7$. The last row corresponds to $\mathbb{R}_1 \times (\text{Taub-NUT}) \times T^6$, which is a D6-brane in $d=10$, but represents the KK monopole in the case of $d=4$. The last column writes the tensions in terms of the Planck mass, $M_4$.

The overall size of the torus, the radius $R$, is now varied while all remaining moduli are fixed. When $R\to 0$, we check whether the Kaluza-Klein modes decouple, while holding the 4-dimensional Planck mass, $2M_4 = M_{11} \times R_7$ fixed.

http://golem.ph.utexas.edu/~distler/blog/archivest. My thanks to Per Berglund for sending this link via email (7/24/09). After having reached certain conclusions stated herein, it was comforting to learn that, perhaps for different reasons, some of J. Distler's viewpoints agree with portions of my own ideas.
Table III.1. BPS p-branes in the compactified 4-dimensional theory

As can be seen in the table for the d=4 case, the KK monopoles and the wrapped M5-brane go to zero mass. The decoupling fails. (The same result occurs for dimensions d > 4, whereas, one finds that for d=2, 3 all of the particles and branes with p≥0 have masses that go to infinity in this limit, suggesting they do decouple).

One could approach the boundary of moduli space differently, for example, by scaling different radii of the toroidal compactification to zero at different rates, but the conclusion is still the same. In fact, the limit used in the GOS paper [71] represents an asymmetrical limit of this type.

The reason for the failure to decouple is straightforward. Consider a p-brane and its magnetic dual p'=(D—4—p)-brane; for example, KK modes are dual to KK monopoles and the M5 and M2-branes are dual. The product of their tensions for d=11 is \( T_p T_{7-p} = 2M_4 \); that is, twice the Planck mass, but the latter is held fixed. Therefore, if one tension goes to infinity, the other tension goes to zero. In d=4, particles are dual to particles. The BPS charges form a 56 of \( \tilde{E}_7(\mathbb{Z}) \); thus, in this limit, essentially 28 “electric” states have been sent to infinite mass, which simultaneously sends the 28 “magnetic” states to zero.

(A loophole exists in dimensions d=2, 3; because electric-magnetic duality does not exist. For example, in d=3, particles are dual to instantons).

Returning now to the original question: what can we conclude about \( \mathcal{N}=8 \) supergravity in d=4? Is it or is it not inconsistent, and thus in the swampland, as claimed [71]? As stated earlier, we believe the failure to decouple suggests pure \( \mathcal{N}=8 \) supergravity is perhaps incomplete, but not inconsistent. First, we take up the issue of what is meant by “inconsistent” versus “incomplete.”
The word and concept of inconsistency generally implies that a theory is illogical, incoherent, and contradictory. The rigorous mathematical structure of \( \mathcal{N}=8 \) supergravity suggests that none of the above would apply. Experimentally, none of the supersymmetric theories have been validated; hence, it would be improper, essentially impossible, to conclude inconsistency based upon physical evidence. On the other hand, a theory may be considered incomplete when parts of the theory appear to be missing; i.e., the theory is unfinished. Thus, claiming that a theory is incomplete does not imply that existing parts of the theory are illogical or contradictory. Rather, one essentially reserves judgment, regarding consistency, until such time as further evidence is accumulated.

One may readily question, in our view, whether it is appropriate to take a limit where all the BPS masses go to infinity, even if such a limit is formally possible. Consider such a limit, for example, in the case of charged blackhole solutions in supergravity theory. As Distler notes, sending their masses to infinity would correspond to sending the electric gauge coupling to infinity.

In contradistinction to taking such a limit, our interpretation suggests that an infinite number of finite but low-mass BPS-soliton states should be included on an equal footing with the usual massless states in calculations with \( \mathcal{N}=8 \) supergravity. This approach allows the possibility of directly mapping the correspondence between extended \( \mathcal{N}=8 \) supergravity and string/M-theory. In this manner, one could investigate explicitly how to “bootstrap” from \( \mathcal{N}=8 \) supergravity into the full string/M-theory. As Distler noted online before this thesis was written, starting from supergravity theory with charged blackhole solutions, it should be possible to “bootstrap ourselves up to the full string theory, by demanding that [charged blackhole solutions] have a consistent quantum-mechanical interpretation.” This contention agrees in essence with the interpretation presented herein.

There is one additional point to be made with respect to the proposed limit and the contention that supergravity theory is inconsistent. Namely, the inverse relationship between the tensions of dual-branes, when the Planck mass is held constant, was derived purely within supergravity theory without stringy input. Therefore, in the event that supergravity theory is shown UV-complete, then all the degrees of freedom that remain in the limit proposed by GOS in [71] may already be intrinsically present. This possibility is suggested by the calculation on Reggeization of \( \mathcal{N}=8 \) supergravity amplitudes discussed briefly in II.4 below, and was pointed out by Distler. In brief, one might say that \textit{the full quantum-mechanical theory of N=8 supergravity is M-theory compactified on T^7}. To reach this conclusion, however, one must include BPS-states, which cannot occur within perturbation theory alone.
In conclusion, we agree formally with the results of the limit proposed by GOS; i.e., that the additional towers of massless or light massive states will not decouple and thus remain. However, our argument below is essentially that this phenomenon is expected in the context of a “complete” supergravity theory—whether completeness results from proving that $\mathcal{N}=8$ supergravity is finite or by recognizing that it should be extended to include BPS (non-perturbative) states. The latter type of supergravity theory partially exists already, although apparently not in a form accepted by most physicists.\(^\text{10}\)

It is also noteworthy that one of the arguments in [71] seems to undermine the conclusion that “supergravity is inconsistent and in the swampland.” It is stated that the inclusion of BPS particles in $\mathcal{N}=8$ supergravity would result in string theory. They imagine an “alternative history” in which type II superstring theory and M-theory are discovered by properly interpreting the BPS solitons of $\mathcal{N}=8$ supergravity. It is almost as if the equivalence of BPS-extended supergravity and string/M-theory is taken for granted.

This supports our contention that pure supergravity with the addition of BPS solitons would have the consistency needed to predict the existence of string/M-theory! The possibility of such equivalence has, in our view, potentially profound implications. The explicit mapping of the correspondence between M-theory and BPS-extended $\mathcal{N}=8$ supergravity, for example, could result in significant progress in our understanding of a completely unified theory, and perhaps eventually lead to a vacuum selection principle.

### III.3 Interpretation of the Failure to Decouple

There are two related aspects to the conclusion that “failure to decouple” implies $\mathcal{N}=8$ supergravity is in the swampland.

*First*, one must assume that pure $\mathcal{N}=8$ supergravity is defined only by the 256 graviton states contained within the one multiplet identified in the original papers related to D=11 supergravity [92], which include SO(8) supergravity [93][94] and gauged SU(8) supergravity [120], respectively. If we restrict ourselves to the original concept of $\mathcal{N}=8$ Supergravity, then one must agree with GOS that it likely does not take into account the “new light particles” that appear near “infinite distance points” in the moduli space. As suggested by Distler (footnote 7), however, in certain recent

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\(^\text{10}\) The many references include: I. Bars [168], [171], [173]; a review by Obers and Pioline [192]; deWit and/or Nicolai [193], [198], [201] and [202]. See also Bergshoeff and Van Proeyen [200].
developments the results seem to imply that $\mathcal{N}=8$ supergravity is intrinsically capable of generating the additional states (see also Schnitzer et. al. [69][73], as discussed below). In any case, there is good reason to question the restriction to pure $\mathcal{N}=8$ supergravity.

Secondly, to accept that $\mathcal{N}=8$ supergravity is in the swampland, one must agree that the appearance of additional massless states represents an inconsistency. In our view, the additional states more likely signify that the theory under consideration is incomplete rather than inconsistent. Several arguments are presented below to support this position.
III.4 Reggeization of $\mathcal{N}=8$ Supergravity and Additional Massless States

One of the arguments that $\mathcal{N}=8$ supergravity is inconsistent, and thus resides in the swampland, centers around the notion that its moduli space contains infinite distance points, but it does not take into account “new light particles” that appear near these points [71]. For this reason, it is interesting that a recent paper, “Reggeization of $\mathcal{N}=8$ supergravity and $\mathcal{N}=4$ Yang-Mills Theory” [73], demonstrates in an explicit calculation of graviton-graviton scattering in $\mathcal{N}=8$ supergravity, that the graviton lies on a Regge trajectory, and a multiple exchange of Regge poles in nonplanar graphs engenders a countable infinity of moving Regge cuts that accumulate near $s=0$.

Schnitzer conjectures [73] that the above states could be the infinite number of non-perturbative massless states that remain in the type II superstring “decoupling limit” proposed by Green, Ooguri and Schwarz [71]. Although Schnitzer offers that it would be difficult to prove such an assertion in detail; nevertheless, it is our view that such an appearance of an infinite number of non-perturbative massless states in the context of a technical field theory calculation, based specifically on $\mathcal{N}=8$ supergravity, is likely to represent more than a mere coincidence. As a minimum it demonstrates that under specific circumstances, $\mathcal{N}=8$ supergravity has the capacity to include an infinite number of additional massless states.

The above facts can be interpreted as compatible with the notion that the original pure $\mathcal{N}=8$ supergravity is perhaps merely a sector of a more complete BPS-extended supergravity; wherein, under the right circumstances additional towers of an infinite number of massless particles may be excited. In brief, contrary to the GOS claim [71], $\mathcal{N}=8$ supergravity has the capacity to take into account additional massless (and/or low-mass) states; furthermore, the possibility exists that one could map the correspondence of these states to Kaluza-Klein charges and monopoles, wound strings and/or branes wrapping cycles of the toroidally compactified extra dimensions in the context of string/M-theory.

III.5 M-theory, the U-duality Group, and BPS-Extended Supergravity

Shortly after the introduction of M-theory by Witten [158], and leveraging the introduction of the U-duality group by Hull and Townsend [153], a series of papers by I. Bars [163][168][171][173] explores the relationship of M-theory to D=11 supergravity. Noting that nonperturbative black hole or monopole-type states, which are alleged to arise in the strongly coupled D=10 string theory, are precisely the massive Kaluza-Klein states of D=11 supergravity, he proposes a formula for identifying nonperturbative states in D=10 type IIA superstring theory, such that, together with the elementary excited string states, they form D=11 supersymmetric multiplets in SO(10)
representations [163]. Bars essentially employs an algebraic approach to unify perturbative and non-perturbative states on an equal footing in the form of U-duality multiplets [168]. A global superalgebra with 32 supercharges and all possible central extensions is studied to extract general properties of duality and hidden dimensions in a theory based on the democracy of p-branes [170], and having a maximum of 12-dimensions with signature (10,2); wherein, one space and one time dimension are hidden from the point of view of perturbative 10-dimensional string theory and its compactification. (Under a Wick-type rotation, the extra time-dimension may play a role more similar to the extra space dimension employed by Vafa [224][225] in the 12-dimensional F-theory introduced about the same time period).

Later deWit and Nicolai, expanding their earlier work on “hidden symmetries” [193] in extended supergravity, generalize their approach to define a “BPS-extended supergravity” that explicitly takes into account its relation to M-theory duality [202]. It is noted that one knows about the existence of massive and massless states from perturbative string theory, and that the latter are captured in supergravity where one can also explore solitonic solutions, which correspond to the branes that arise in nonperturbative string theory.

In spacetimes with compactified dimensions there exist winding and momentum states, which are BPS states that can in principle be incorporated into supergravity as matter supermultiplets. While massive string states typically decouple in the zero-slope limit, it is noted that more “daring ideas” about the underlying degrees of freedom of M-theory must take into account supermembranes and matrix theories [201]. In discussing the conjectured discrete U-duality group of M-theory, which contains the S- and T-dualities, the authors point out that it is “the arithmetic subgroup of the group of nonlinearly realized symmetries of the maximal supergravity theories [93][94].” The discrete U-dualities, therefore, have precise correspondence to the geometrical symmetries of supergravity. In fact, they are conjectured [202] to be exact symmetries of toroidally compactified M-theory, and therefore act on the BPS states.

The U-duality group is expected to be broken to an integer-valued subgroup that respects the perturbatively and/or non-perturbatively generated charge lattice of the BPS states. The inclusion of BPS states unavoidably leads back in special limits to eleven-dimensional supergravity or extensions thereof. The simple reason is that BPS states include the Kaluza-Klein states needed to elevate the theory to eleven dimensions. Going one step further, it can be shown that under certain conditions this theory must live in more than eleven spacetime dimensions with maximal
supersymmetry, but only partial Lorentz invariance. The authors provide a specific example interpreted as a theory living in twelve dimensions [202].

The above examples and references are given merely to support the notion that a more complete supergravity theory, perhaps having an even stronger correspondence to M-theory than exploited in the past, may yet be identified. Although we are not in a position to present the details of such a theory herein, we attempt to demonstrate below that even a relatively minor extension of the \( \mathcal{N}=8 \) supergravity theory could change the outcome of a particular technical calculation [75], interpreted originally as supporting the GOS argument that \( \mathcal{N}=8 \) Supergravity is in the swampland [77].
CHAPTER IV

SUPERSYMMETRIZING THE $\alpha'^3 R^4$ TERM IN $\mathcal{N}=8$ SUPERGRAVITY

In a recent paper [77], F. Moura attempts to supersymmetrize a “new” $R^4$ correction term required in the context of a one-loop effective action for the type IIA superstring compactified on $T^6$. The effective action for this compactification is the $\mathcal{N}=8$ supergravity action. The correction term is required by M-theory and upon compactification must be supersymmetrized for consistency. However, Moura finds that, within the constraints of pure $\mathcal{N}=8$ supergravity, supersymmetrization is not possible [75]. The latter fact is interpreted as providing support or evidence [77] for the claim that $\mathcal{N}=8$ supergravity is inconsistent, and consequently in the swampland [71].

Moura first demonstrates that the new $R^4$ term is present in type II, heterotic superstrings, and M-theory actions at order $\alpha'^3$. Thus, these correction terms should be added to the corresponding low-energy effective supergravity actions, they should be supersymmetric, and included in the corresponding low-energy 4-dimensional supergravity theory after compactification.

After Moura recognized the existence of the new independent $R^4$ term at one loop in the type IIA effective action, he determined that it must be taken into account, after toroidal compactification to four dimensions, along with the better-known $R^4$ correction term; i.e., the square of the Bel-Robinson tensor [229]. The latter $R^4$ term can be directly supersymmetrized in both simple and extended supergravity, as shown in four earlier papers listed in [97]. However, Moura concluded—after a great amount of effort documented in [75]—that the new $R^4$ term cannot be supersymmetrized in the context of $\mathcal{N}=8$ supergravity.

Commensurate with our assertion that $\mathcal{N}=8$ supergravity is incomplete, but not inconsistent, we intend to take into account the U-duality group for $\mathcal{N}=8$ supergravity, which leads essentially to a larger R-symmetry group [192]. A simple U(1) factor along with the usual SU(8) R-symmetry enables a calculation below that supersymmetrizes the new $R^4$ term in the context of an apparently minimally extended $\mathcal{N}=8$ supergravity. That is, the calculation seems to imply that we are, in effect, working beyond the confines of the conventional superspace typically used in $\mathcal{N}=8$ supergravity calculations [83][84]. This possible interpretation is discussed further after the calculation is completed.
IV.1 Origin of the New $R^4$ Term

The 'new' $R^4$ term was initially recognized in the work of Peeters, Vanhove and Westerberg [64], related to supersymmetric higher-derivative actions in superspace. In their Appendix B.2 on Riemann tensor polynomials and the $t_8$ tensor, they summarize a mathematical study of Fulling et. al. (also listed in Ref. [64]) on the normal forms for polynomials of the Riemann tensor. The latter is decomposed in $d$-spacetime dimensions into the Weyl tensor, $\mathcal{W}_{mnpq}$, the Ricci Tensor $R_{mn}$, and the Ricci scalar $R$.

\[
R_{mnpq} = \mathcal{W}_{mnpq} - \frac{1}{d-2} (g_{mp}R_{nq} - g_{np}R_{mq} + g_{nq}R_{mp} - g_{mq}R_{np}) \\
+ \frac{1}{(d-1)(d-2)} (g_{mp}R_{nq} - g_{np}R_{mq}) R \tag{4.1}
\]

Fulling et al. [64] show that seven independent real scalar polynomials can be constructed in $d=10$ from the irreducible components of the Weyl tensor raised to the fourth power, which Peeters et. al. [64] indicate as follows:

\[
R_{41} = \mathcal{W}_{mnpq} \mathcal{W}^{mnpq}, \tag{4.2}
\]
\[
R_{42} = \mathcal{W}_{mnpq} \mathcal{W}^{mnpq} \mathcal{W}_{stu} \mathcal{W}^{stu}, \tag{4.3}
\]
\[
R_{43} = \mathcal{W}_{mnpq} \mathcal{W}_{stu} \mathcal{W}^{stu}, \tag{4.4}
\]
\[
R_{44} = \mathcal{W}_{mnpq} \mathcal{W}_{stuv} \mathcal{W}^{stu}, \tag{4.5}
\]
\[
R_{45} = \mathcal{W}_{mnpq} \mathcal{W}_{rstu} \mathcal{W}^{stu}, \tag{4.6}
\]
\[
R_{46} = \mathcal{W}_{mnpq} \mathcal{W}_{rstu} \mathcal{W}^{stu}, \tag{4.7}
\]
\[
A_7 = \mathcal{W}_{mn} \mathcal{W}_{pq} \mathcal{W}_{stu} \mathcal{W}^{stu}, \tag{4.8}
\]

As stated above, the $R^4$ corrections terms enter the type-IIA superstring effective actions at order $\alpha'^3$ in the form of two independent bosonic terms, which Peeters et. al. [64] express at linear order as,

\[
I_X = t_8 s_8 R^4 + \frac{1}{2} \varepsilon_{10} s_8 B R^4, \tag{4.9}
\]
\[
I_Z = -\varepsilon_{10} s_{10} R^4 + 4\varepsilon_{10} t_8 B R^4. \tag{4.10}
\]

The $s_8$ tensor has eight free spacetime indices and acts in four two-index antisymmetric tensors, as defined by Gross and Witten [87]. In a purely
gravitational IIA superstring action to order \( \alpha'^3 \) in the string frame, the perturbative terms occur at string tree and one loop levels, and according to Green, Gutperle and Vanhove [63], there are no higher-loop contributions. The perturbative part of the effective type-IIA action can thus be written [64][75] in the string frame as,

\[
\frac{1}{\sqrt{-g}} \mathcal{L}_{\text{IIA}} |_{\alpha'^3} = -e^{-2\phi} \alpha'^3 \left( I_X - \frac{1}{8} I_Z \right) - a'^3 \frac{1}{3n^2 \times 2^{16}} \left( I_X + \frac{1}{8} I_Z \right).
\]  

(4.11)

The purely gravitational parts of the two independent terms \( I_X \) and \( I_Z \) are denoted as \( X \) and \( Z \), and written in terms of seven independent real scalar polynomials in [64],

\[
X := t_8 t_8 \mathcal{W}^4 = 192R_{41} + 384R_{42} + 24R_{43} + 12R_{44} - 192R_{45} - 96R_{46}, \quad (4.12)
\]

\[
\frac{1}{8} Z := -\frac{1}{8} \varepsilon_{10} \varepsilon_{10} \mathcal{W}^4 = X + 192R_{46} - 768A_7. \quad (4.13)
\]

By substituting the above expressions into the type IIA-action in place of \( I_X \) and \( I_Z \), respectively, and writing the \( R_{mn} \) components in terms of the Weyl tensor \( \mathcal{W}_{mnpq} \), we obtain the action in terms of the Weyl tensor. Everything above is expressed the 10-dimensional theory, but we are interested in compactification to 4-dimensions. Before proceeding, we focus on reduction of the above expressions, including the effective action, to four-dimensions. One further note: the type IIA theory is obtained from compactification of M-theory on \( S^1 \), but in [64] we see that its tree level \( \alpha'^3 \) term vanishes in the 11-dimensional limit. The one-loop type-IIA \( R^4 \)-term is therefore the compactification of the D=11 \( R^4 \) term. It was shown by Angulova, Grassi and Vanhove [65] that this term is unique and its coefficient can be determined directly without using any anomaly cancellation argument.

**IV.2 Reduction of the IIA Action and \( R^4 \) Term to \( d=4 \)**

To obtain the above expressions in four-dimensional form, we write the Weyl tensor using Greek indices, where \( \mu, \nu, \rho, \sigma \) (etc.)= 0, 1,2, 3. For distinction, we also drop use of the script \( \mathcal{W} \) in favor of a *slight-italic* capital \( \mathcal{W} \). The Weyl tensor is also decomposed into its self-dual and antiself-dual parts by Penrose and Rindler [229]:

\[
W_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma}^+ + W_{\mu\nu\rho\sigma}^-,
\]  

(4.14)

where,

\[
W_{\mu\nu\rho\sigma}^+ = \frac{1}{2} \left( W_{\mu\nu\rho\sigma} - \frac{i}{2} \epsilon^{\lambda\tau}_{\mu\nu} W_{\lambda\tau\rho\sigma} \right) \quad (4.15)
\]

\[
W_{\mu\nu\rho\sigma}^- = \frac{1}{2} \left( W_{\mu\nu\rho\sigma} + \frac{i}{2} \epsilon^{\lambda\tau}_{\mu\nu} W_{\lambda\tau\rho\sigma} \right). \quad (4.16)
\]

These tensors obey the following properties [229],

52
\[ W_{\mu_1 \nu_1 \rho_1 \sigma_1} W_{\mu_2 \nu_2 \rho_2 \sigma_2}^{-\mu_2 \nu_2 \rho_2 \sigma_2} = 0, \]  
(4.17)

\[ W_{\mu_1 \nu_1 \rho_1 \sigma_1} W_{\mu_2 \nu_2 \rho_2 \sigma_2}^+ = \frac{1}{4} g_{\mu_1 \nu_1} W_+^2, \]  
(4.18)

\[ W_{\mu_1 \nu_1 \rho_1 \sigma_1} W_{\mu_2 \nu_2 \rho_2 \sigma_2}^- = \frac{1}{4} g_{\mu_1 \nu_1} W_2^2. \]  
(4.19)

In addition to the usual Bianchi identities, the Weyl tensor in four dimensions obeys the Schouten identities including [229],

\[ W^\rho_\mu W^\mu_\nu = \frac{1}{4} (g_{\rho \sigma} g_{\nu \lambda} - g_{\rho \lambda} g_{\nu \sigma}) W^2 + 2(W^\mu_\nu W^\nu_\lambda - W^\mu_\mu W^\nu_\nu). \]  
(4.20)

The Bel-Robinson tensor in four dimensions is given by,

\[ W^+_{\mu \rho, \nu} W^-_{\nu \lambda} = \]  
(4.21)

which is totally antisymmetric in the indices.

Following Penrose and Rindler [229], it is convenient to introduce the van der Warden notation, and rewrite the above decomposition of the Weyl tensor into self-dual and antiself-dual parts using spinorial indices,

\[ W^\mu_\nu W_{\mu \nu} := -2 \epsilon^\mu_{AB} \epsilon^\nu_{CD} W_{ABCD} - 2 \epsilon^\mu_{AB} \epsilon^\nu_{CD} W_{ABCD}. \]  
(4.22)

Both \( W_{ABCD} \) and \( W_{ABCD} \) are totally symmetric and can be written as,

\[ W_{ABCD} = -\frac{1}{8} W^+_{\mu \nu \rho \sigma} \sigma_{AB}^{\mu \nu} \sigma_{CD}^{\rho \sigma}, \]  
(4.23)

\[ W_{ABCD} = -\frac{1}{8} W^-_{\mu \nu \rho \sigma} \sigma_{AB}^{\mu \nu} \sigma_{CD}^{\rho \sigma}. \]  
(4.24)

This simplifies calculations with the Weyl tensor. The Bel-Robinson tensor in spinorial indices is written as,

\[ W_{ABCD} W_{ABCD}. \]  
(4.25)

In four-dimensions, Fulling et. al [64] have shown that only two independent real scalar polynomials can be constructed with the Weyl tensor to the 4\textsuperscript{th}-power,

\[ W^2 = W_{ABCD} W_{ABCD} W^{ABCD} W^{ABCD}, \]  
(4.26)

\[ W^4 + W^4 = (W_{ABCD} W_{ABCD})^2 + (W_{ABCD} W_{ABCD})^2. \]  
(4.27)
These are the two $R^4$ terms written in spinorial indices. The first is the square of the Bel-Robinson tensor, which can be supersymmetrized; whereas, the latter $R^4$ term is the one we seek to supersymmetrize in the context of $\mathcal{N}=8$ supergravity.

To see how (4.26) and (4.27) arise in the four-dimensional Lagrangian, refer back to the ten-dimensional Lagrangian equation (4.11). This displays the two independent bosonic terms, $I_1$ and $I_2$, containing the $R^4$ terms as expressed in equations (4.9) and (4.10). The gravitational parts given in (4.12) and (4.13) are written in terms of the seven fundamental polynomials, equations (4.2) to (4.8), in ten-dimensions. To express these in four-dimensional form we use the Weyl tensor as given in van der Warden notation in equation (4.22), and find:

\[
R_{41} = \frac{1}{24} W_+^4 + \frac{1}{24} W_-^4 - \frac{5}{8} W_+^2 W_-^2 \quad (4.28)
\]

\[
R_{42} = \frac{1}{12} W_+^4 + \frac{1}{12} W_-^4 + \frac{11}{8} W_+^2 W_-^2 \quad (4.29)
\]

\[
R_{43} = \frac{1}{6} W_+^4 + \frac{1}{6} W_-^4 - 4 W_+^2 W_-^2 \quad (4.30)
\]

\[
R_{44} = W_+^4 + W_-^4 + 2 W_+^2 W_-^2 \quad (4.31)
\]

\[
R_{45} = \frac{1}{4} W_+^4 + \frac{1}{24} W_-^4 - \frac{5}{8} W_+^2 W_-^2 \quad (4.32)
\]

\[
R_{46} = -\frac{1}{6} W_+^4 + \frac{1}{6} W_-^4 - \frac{3}{2} W_+^2 W_-^2 \quad (4.33)
\]

\[
A_7 = -\frac{1}{24} W_+^4 - \frac{1}{24} W_-^4 - \frac{1}{4} W_+^2 W_-^2 \quad (4.34)
\]

Substituting into (4.12) and (4.13), we find,

\[
X = 24(W_+^4 + W_-^4) + 384 W_+^2 W_-^2 \quad (4.35)
\]

\[
\frac{1}{8} Z = 24(W_+^4 + W_-^4) + 288 W_+^2 W_-^2 \quad (4.36)
\]

Therefore,

\[
(I_X - \frac{1}{8} I_Z)_{\text{grav}} = (X - \frac{1}{8} Z) = 96 W_+^2 W_-^2 \quad (4.37)
\]

\[
(I_X + \frac{1}{8} I_Z)_{\text{grav}} = (X + \frac{1}{8} Z) = 48(W_+^4 + W_-^4) + 672 W_+^2 W_-^2 \quad (4.38)
\]
Before substituting these terms into the string frame Lagrangian (4.11), there is one more technicality to resolve. For the purpose of the supergravity analysis, we must redefine the metric through a conformal transformation to reach the Einstein frame,

\[ g_{mn} \rightarrow \exp \left( \frac{4}{d-2} \phi \right) g_{mn}, \quad (4.39) \]

\[ R_{mn}^{pq} \rightarrow \exp \left( - \frac{4}{d-2} \phi \right) \tilde{R}_{mn}^{pq}, \quad (4.40) \]

\[ I_i(R, M) \rightarrow \exp \left( \frac{4}{d-2} \phi \right) I_i(R, M), \quad (4.41) \]

where,

\[ \tilde{R}_{mn}^{pq} = R_{mn}^{pq} - \delta_{[m}^{[p} \nabla_{n]} \nabla^{q]} \phi. \quad (4.42) \]

The third equation in this transformation represents an arbitrary term in the string frame Lagrangian, which has the functional form of \( I_i(R, M) \) with conformal weight \( w_i \), where \( R \) is the Riemann tensor and \( M \) represents any other fields; i.e., gauge fields, scalar fields, or fermion matter fields. Also, taking into account the transformation of the metric determinant factor \( \sqrt{-g} \), the string frame Lagrangian is converted into the Einstein frame Lagrangian:

\[
\frac{1}{2} \sqrt{-g} e^{-2\phi} \left( -R + 4(\partial^m \phi) \partial_m \phi + \sum_i I_i(R, M) \right) \rightarrow \\
\frac{1}{2} \sqrt{-g} e^{-2\phi} \left( -R - \frac{4}{d-2} (\partial^m \phi) \partial_m \phi + \sum_i e^\frac{4}{d-2} (1+w_i) \phi I_i(\tilde{R}, M) \right). \quad (4.43)
\]

Finally, we are in a position to translate the 10d equation (4.11) written in string frame into the effective action with \( R^4 \) terms in 4-dimensions in the Einstein frame:

\[
\frac{\kappa^2}{\sqrt{-g}} |_{R^4} = - \frac{\zeta(3)}{32} e^{-6\phi} c' W_4^2 W_4^2 - \frac{1}{2^{12} \pi^5} e^{-4\phi} c'^3 [(W_4^4 + W_4^4) + 224 W_4^2 W_4^2] 
\quad (4.44)
\]

As stated after equation (4.11), the complete action for every distinct compactification includes moduli-dependent terms ignored for our purpose here; also, all terms are multiplied by the volume of the compactification manifold. The above equation (4.44) clearly shows that the \( R^4 \) term derived in equation (4.27) above, \( W_4^4 + W_4^4 \), is present in the effective Lagrangian in 4-dimensions. This is the term that we seek to supersymmetrize in the case of \( N=8 \) supergravity.
IV.3 A No-Go Theorem For Supersymmetrization of the R^4 Term

The square of the Bel-Robinson tensor $W_+^2 W_-^2$, as stated earlier, was supersymmetrized many years ago in both simple and extended supergravity [97]. However, the "no-go" theorem in reference [85] makes it obvious the R^4 term of interest, $W_+^4 + W_-^4$, cannot be supersymmetrized. This is stated (to effect):

Theorem [85]. For a polynomial $I(W)$ of the Weyl tensor to be supersymmetrized, each of its terms must contain equal powers of $W_{\mu\nu\rho\sigma}^+$ and $W_{\mu\nu\rho\sigma}^-$. It is obvious the Bel-Robinson tensor satisfies that criteria. The new R^4 does not. To understand how one might circumvent the no-go theorem, we consider briefly the assumptions upon which it is based [85].

The theorem applies specifically to a given term on a stand-alone basis; thus, one might consider coupling to this term, for example, other multiplets within a specific supergravity theory, which would change the form of this term within the Lagrangian such that the term becomes supersymmetric [75].

The theorem was originally formulated in a simple $\mathcal{N}=1$ supergravity theory, where the supersymmetry transformation preserves the R-symmetry. For an $\mathcal{N}=1$ theory, this corresponds to U(1) symmetry and is equivalent to chirality. In general, in the context of the relatively recent AdS/CFT correspondence, an R-symmetry is a global symmetry of the dual gauge theory that by definition does not commute with the supersymmetries. In extended supergravity theories with $\mathcal{N} > 1$, one might consider matter couplings or other extra couplings that would conceivably violate the U(1) R-symmetry component (of a presumably larger R-symmetry); but simultaneously, result in a term that is supersymmetric [77]. Our goal is to supersymmetrize the $(W_+^4 + W_-^4)$-term in the Lagrangian of $\mathcal{N}=8$ supergravity, however, which was shown by Moura to be a non-trivial task [75].

Moura surmised that symmetrization might be possible if one could couple to a scalar chiral multiplet; however, the latter possibility is a major problem in the case of pure $\mathcal{N}=8$ supergravity. In that theory, there is one unique graviton multiplet and no scalar chiral or other multiplet available. After considering all possible approaches to achieve supersymmetrization of the new R^4 term, but staying within the confines of conventional $\mathcal{N}=8$ superspace, Moura concluded that "N=8 supersymmetrization [of the new R^4 term] may not be possible at all, which may be another argument favoring the hypothesis that N=8 supergravity is in the swampland [77]."

Since the paper [75] by Moura documents an exhaustive investigation within conventional $\mathcal{N}=8$ superspace, our purpose is to explore supersymmetrization
"outside the box" so to speak; i.e., within the context of what we call for the moment a *minimally* extended \( \mathcal{N}=8 \) supergravity.

IV.4 Symmetrization of the \( (W^4 + W^*) \)-Term in \( \mathcal{N}=8 \) Supergravity

Moura suggests the possibility that to circumvent the no-go theorem in \( \mathcal{N}>1 \) supergravity, one should try to construct a superinvariant that violates *some* of the R-symmetry [75]. Recalling that \( \mathcal{N} \leq 6 \) theories have a \( U(\mathcal{N}) \) that can be split into \( SU(\mathcal{N}) \times U(1) \), an 'extra' \( U(1) \) is then available for coupling. Also, in \( \mathcal{N} \leq 6 \) theories there are other multiplets. The extra \( U(1) \) represents a portion of the R-symmetry, which could conceivably be violated by an appropriate coupling that also does not interfere with the \( \mathcal{N} \leq 6 \) supersymmetry. The problem in the case of the \( \mathcal{N}=8 \) theory is that there is no \( U(8) \) to make an extra \( U(1) \) available, but rather, only the \( SU(8) \) symmetry, which cannot be split without interfering with the basic structure. Thus, we reach the conclusion that the \( R^4 \) term cannot be supersymmetrized in pure \( \mathcal{N}=8 \) supergravity formulated in *conventional* superspace [75]. This begs the question, however: what is possible if we consider, for the sake of theoretical argument, a small step beyond conventional superspace?

Extending Conventional \( \mathcal{N}=8 \) Superspace

Extending conventional \( \mathcal{N}=8 \) superspace is consistent with taking into account the additional massless particles that remain according to [71] and our assertion that \( \mathcal{N}=8 \) supergravity is incomplete (and *not* inconsistent). We investigate, therefore, the related issue of a *larger* R-symmetry group that arises within the concept of U-duality [153][192][186][171], and/or within the "hidden symmetries" approach to supergravity/M-theory [173][193][198][201][202]. For our purpose here, we investigate only a minimal extension of the U-duality group for \( \mathcal{N}=8 \) supergravity, which leads to the following scenario. Upon compactification of M-theory (in practice the low-energy effective D=11 supergravity) on the seven-dimensional torus, which corresponds to type IIA string on the \( T^6 \), there are automorphisms beyond the obvious \( SO(7) \) symmetry [192]. Symmetry enhancement is observed at the level of the Clifford algebra, and the generators of the \( SO(7) \) R-symmetry can be supplemented by generators that form the Lie algebra of the larger R-symmetry group \( SO(8) \). Nevertheless, it is precisely the latter group that Moura concluded does *not* allow supersymmetrization of the \( R^4 \) term [75].  

According to Obers and Pioline, however, *the R-symmetry group can actually be made larger* [192]. For \( d=7 \) toroidal compactification, the Clifford generator \( \Gamma_7 \) can be used (subscript=\( \text{no. of antisymmetric internal indices} \)), such that one obtains an \( SU(8) \times U(1) \) R-symmetry group (reference [192], p. 41). Inclusion of the latter \( U(1) \), essentially as an additional hidden symmetry in \( \mathcal{N}=8 \) supergravity, is the approach to supersymmetrization of the \( W_4 + W^* \) term that we investigate below. Also, as implied above, we assume that the infinite number of additional massless or
light-mass states that remain in the limit defined by Green, Ooguri and Schwarz correspond to additional matter available in the superspace of $\mathcal{N}=8$ supergravity \textit{in addition to} the one unique graviton multiplet ordinarily considered.

One could think of these additional nonperturbative states demanded by the type IIA superstring, as also existing in the context of an (perhaps BPS) extended $\mathcal{N}=8$ supergravity in the sense discussed by de Wit and Nicolai [201]. Partly due to time-constraints for this project, however, we take a pedestrian approach that is naive in comparison to \textit{de rigueur} of supergravity calculations. It is hoped that in a rough sense this will convey the possibility of working beyond the confines of the \textit{conventional} superspace typically employed for pure $\mathcal{N}=8$ supergravity calculations [83][84]. Our approach could perhaps be interpreted as a sort of minimally extended $\mathcal{N}=8$ supergravity, although it is understood that a more sophisticated approach will be required in the future to justify the concept.

Let us recall briefly the pure $\mathcal{N}=8$ supergravity setup. An interesting phenomenon discovered in the SO(8) supergravity papers by Cremmer and Julia [93] is the appearance of \textit{hidden} gauged SU(8) symmetry. Although there are only 28 massless vector bosons in the $\mathcal{N}=8$ supergravity multiplet, whereas, a local SU(8) symmetry requires 63 massless vector particles, the gauging of SU(8) occurs nonlinearly via composite combinations of scalar fields. This is the approach to \textquoteleft pure\textquoteright{} $\mathcal{N}=8$ developed extensively by de Wit & Nicolai [100]. More recently, the latter have conjectured the existence of effective supergravity field theories encompassing nonperturbative degrees of freedom, referred to as \textquoteleft BPS-extended supergravities,\textquoteright{} that are compatible with the hidden symmetries, a subset of which are the Kaluza-Klein states. They are conceived to live in a higher-dimensional space, where the central charges originate from extra dimensions analogous to the way central charges result from internal momenta in KK-compactifications [193][201][202].

\textbf{Supersymmetric $\mathcal{N}=8$ Coupling}

Inspired partly by the above ideas we pursue only a naïve \textquoteleft baby step\textquoteright{} extension of $\mathcal{N}=8$ supergravity. We consider an amorphous \textquoteleft background\textquoteright{} of the infinite towers of additional massless and finite-mass states. These may correspond from the point of view of the type II superstring theory to wound strings or branes wrapping around cycles of the compactified toroidal dimensions. However, we visualize them now as analogous to a \textquoteleft pool\textquoteright{} of random, minute chaotic vibrations, without any shape or quality beyond the potential coupling to the $\mathcal{N}=8$ supergravity multiplet.\footnote{In our view, $\mathcal{N}=8$ supergravity in four-dimensions is most likely uv-finite analogous to the $\mathcal{N}=4$ super Yang-Mills theory in $d=4$, which provides a potentially stable reality.} In this setup, for example, the infinite towers of massless states may play a role perhaps analogous to dark energy, while the infinite tower of finite mass states may perhaps be related to dark matter.
Most of the mass in our Universe apparently occurs in the form of dark matter. A “weakly interacting massive particle” (WIMP) is one of the leading candidates for the composition of this dark matter, and thus, many believe strongly that supersymmetry could be the source of such a WIMP particle.

The Minimal Supersymmetric Standard Model (MSSM) predicts the existence of spin 1/2 fermions called neutralinos, which are the fermionic superpartners of the neutral gauge bosons and Higgs scalars. Neutralinos would have a mass, but interact very weakly with other particles. In fact, they could make up a significant portion of the mass density of the Universe without emitting light; hence, they are reasonably good candidates for the mysterious dark matter in the Universe.

For the purpose of a thought-experiment, assume for the moment that the source of dark matter is the infinite tower of finite-mass states that exist in addition to the usual massless states of $\mathcal{N}=8$ supergravity. Assume further that at least some portion of this matter has precisely the right properties to couple weakly with $\mathcal{N}=8$ supergravity via the non-supersymmetric $W_+^4 + W_-^4$ term.\(^\text{12}\)

Recall that all consistent supergravities with $\mathcal{N} \leq 6$ can be obtained by dimensional reduction from $\mathcal{N}=8$ supergravity. The pure $\mathcal{N}=8$ supergravity states are in some sense immersed in or penetrate into the amorphous pool of states; i.e., the dark matter, such that coupling may occur in any of a full range of supersymmetric possibilities. In a sort of cartoon schematic, think of the $\mathcal{N}=8$ supersymmetries as corresponding to the eight pedals of a lotus, completely immersed in a pool of amorphous states, but initially, each pedal is `dry.’ That is, in a sort of pre-potential state, none of the $\mathcal{N}=8$ supersymmetries have coupled to the amorphous states. The coupling must and will occur either sequentially in a series of $\mathcal{N}=1$ supergravity couplings, or in a sequence of $\mathcal{N}=2$ supersymmetric pairs, or in two sets of $\mathcal{N}=4$ couplings, or in one simultaneous $\mathcal{N}=8$ coupling, or in any sequence of combinations of the above supersymmetric couplings that lead to the full $\mathcal{N}=8$ supersymmetric coupling. This proposed supersymmetric sequence, however, is more of a heuristic tool than a series of factual events, meant to underscore a belief in the simplicity of the coupling mechanism—since each $\mathcal{N}<8$ supergravity is a consistent sector within the full $\mathcal{N}=8$ supergravity.

The Simplest Coupling

This brings us to the simplest coupling in this extended $\mathcal{N}=8$ superspace. Within the amorphous pool of additional states, consider that an $\mathcal{N}=1$ sector of the

\(^{12}\) In this scenario one may expect weak coupling, because the additional finite-mass states described by Green, Ooguri and Schwarz [71] appear in the limit. If the limiting occurred in the past as an event related to inflation or expansion of the universe, then conditions in this era may not be conducive to coupling.
supergravity couples to one simple chiral multiplet represented by the superfield \( \Phi(y, \theta) \), which is analogous to a simple Wess-Zumino type superfield: \( \phi = \Phi|_{\theta=0} \) is a scalar field, \( \psi = D\Phi \) is a spin-1/2 field, and \( F = -1/2 \, D\Phi^2 \) an auxiliary field [233][234]. The superfield \( \Phi \) couples to the \( \mathcal{N}=1 \) supergravity sector by means of a superpotential that can be written\(^\text{13}\)

\[
U(\Phi) = \alpha \Phi + 1/2 \, m\phi^2 + 1/3 \, \lambda \phi^3. \tag{4.45}
\]

The \( \mathcal{N}=1 \) sector has several off-shell formulations and a multiplet with auxiliary fields such as [234][235]: a scalar \( M \), pseudo-scalar \( N \), and a vector field \( A_{\alpha\dot{\alpha}} \), which are \( \theta=0 \) components of the superfields \( R, \bar{R} \) and \( G_{\alpha\dot{\alpha}} \), respectively,

\[
R|_{\theta=0} = 4 \, (M - iN) \tag{4.46}
\]

\[
\bar{R}|_{\theta=0} = 4 \, (M + iN) \tag{4.47}
\]

\[
G_{\alpha\dot{\alpha}}|_{\theta=0} = 1/3 \, A_{\alpha\dot{\alpha}}. \tag{4.48}
\]

There is also a chiral superfield and its hermitian conjugate that is usually represented by "\( W_{ABC} \)"; which conflicts with our symbol for the Weyl tensor. We propose to use \( Y_{ABC} \) and \( Y_{A\dot{A} B\dot{B}} \) here instead, which together at \( \theta=0 \) constitute the field strength of the gravitino. The Weyl tensor is the first term in the \( \theta \) expansion of the 3-form chiral superfield [235],

\[
D_{\alpha} Y_{BCD} = 1/8 \, R_{\mu
u\rho\sigma} \sigma_{\mu\alpha}^{\nu} \sigma_{\rho\sigma}^{\nu} + \ldots = -1/2 \, W_{ABCD} + \ldots \tag{4.49}
\]

where the underlined indices are symmetrized with weight one, and henceforth the vertical 'slash' is understood to mean the \( \theta=0 \) component.

Using solutions to the Bianchi identities, a fairly lengthy calculation [235] involving (4.49) above leads to an expression for \( D_{\alpha} Y_{BCD} \) showing that the \( \theta^2 \) component of \( Y^2 \) has a term with \( W^2 \), that is, the Weyl tensor squared. We expect that a supersymmetric action including a \( W^4 \) term could have the Bel-Robinson form \( W^2 \bar{W}^2 \). Taking into account the above calculation, we also find a term of form:

\[
W_{+}^4 + W_{-}^4 \cong [(D_{\alpha} Y_{2})^2 + \text{h.c.}] |_{\theta=0} \tag{4.50}
\]

This term cannot result from a superspace integration. Note that the term \( D_{\alpha} Y_{BCD} \) in (4.49) is \( U(1) \) R-symmetric, as pointed out in discussion of the "no-go" theorem in reference [235]. Also, in reference [198] we find the following "weights" used in our expressions: \( D_{\alpha} \rightarrow +1 \), \( R \rightarrow +2 \), \( G_{\alpha\dot{\alpha}} \rightarrow 0 \), and \( Y_{ABC} \rightarrow -1 \). The components of the Weyl tensor are therefore \( U(1) \) R-neutral, and the full term \( (W_{+}^4 + W_{-}^4) \) is \( U(1) \) R-symmetric. This confirms that indeed an extra coupling is needed to break the \( U(1) \) R-symmetry of \( W_{+}^4 + W_{-}^4 \).

\(^{13}\) The superpotential is usually written as \( W \), but having used \( W \) for the Weyl tensor, we use \( U \) instead.
The Form of the Action for $W^4 + W^4$

We are now ready to write an effective action within the $\mathcal{N}=1$ sector of the full $\mathcal{N}=8$ theory, which includes the $W^4_+ + W^4_-$ term. It should be recognized, however, that associated with the superfield $\Phi$ and the superpotential $U$ in (4.45), there is also a Kahler potential written as:

$$e^{K(\Phi,\bar{\Phi})} \sim \Omega(\Phi, \bar{\Phi}) = -3 + \Phi \bar{\Phi} + c\Phi + c\bar{\Phi}$$  \hspace{1cm} (4.51)$$

This form of the Kahler potential, along with that of the superpotential in (4.45) is, according to the “six-authors” in [94], the most general renormalizable coupling of a chiral multiplet in pure supergravity.

Looking back at (4.26) and (4.27), we write for simplicity,

$$W^2 \bar{W}^2 = W^2_+ W^2_-$$  \hspace{1cm} (4.52)$$

$$W^4 + \bar{W}^4 = W^4_+ + W^4_-.$$  \hspace{1cm} (4.53)$$

In our naïve investigation, we’ve assumed the $\mathcal{N}=1$ sector can be investigated within the full theory, because the $\mathcal{N}=1$ supergravity action in four-dimensions is known [94][234][235]. The Wess-Zumino type action can be written in schematic form as:

$$I = \int \int d^4\theta d^4x \mathcal{L} = \frac{1}{2\kappa^2} \int \int E d^4\theta d^4x,$$  \hspace{1cm} (4.54)$$

where $E = s\det E^M_A$ is the superdeterminant of the supervielbein. Including the $R^4$ correction terms, this becomes,

$$I = \frac{1}{2\kappa^2} \int \int E[1 + \alpha W^2 \bar{W}^2 + \beta(W^4 + \bar{W}^4)] d^4\theta d^4x.$$  \hspace{1cm} (4.55)$$

As stated at the outset, the first correction term, the square of the Bel-Robinson tensor, was supersymmetrized a long time ago. As explained in the discussion of the “no-go” theorem, it has the correct form to insure this fact. Therefore, we focus only on the last term in the integrand of (4.55). Many valuable insights for this action are given by the “six authors” in [94], and Moura provides the final form in [77]. We provide additional details and insight in the following; however, there are also several lengthy calculations in the background, some of which will be included in the Appendix.

Taking into account expression (4.50) and the associated discussion, one can verify by staring at it for a few minutes that the effective action can be written as:
\[ \mathcal{L} = -\frac{1}{6\kappa^2} \int E \left[ \Omega(\Phi, \Phi) + \alpha^3 \left( b\Phi(D_A^2 Y^2)^2 + \overline{b\Phi(D_A^2 Y^2)^2} \right) \right] d^4\theta \\
\quad - \frac{1}{\kappa^2} \left( \int \epsilon U(\Phi) d^2\theta + \text{h.c.} \right) \quad (4.56) \]

\[ = \frac{1}{4\kappa^2} \int \epsilon \left[ \left( \overline{D_A^2} + \frac{1}{3} \overline{R} \right) \left\{ \Omega(\Phi, \Phi) + \alpha^3 \left( b\Phi(D_A^2 Y^2)^2 + \overline{b\Phi(D_A^2 Y^2)^2} \right) \right\} - 8U(\Phi) \right] d^2\theta + \text{h.c.} \quad (4.57) \]

In the above, \( \epsilon \) is the chiral density, and \( \left( \overline{D_A^2} + \frac{1}{3} \overline{R} \right) \) is the chiral projector for the scalar field. The explicit \( \theta \) expansions for \( \epsilon \) and for \( D_A^2 Y^2 \) are also included in the appendix. When (4.57) is expanded in components using these expansions, and taking into account that from (4.49) we also have,

\[ D_A^2 Y_{BCD}^2 |_{\theta=0} = -2W_+^2 + \ldots, \quad (4.58) \]

one then finds the form of \( W^+_4 + W^A_4 \) given in (4.27) [namely, \( (W^{ABCD}W_{ABCD})^2 + (W^{ABCD}W_{ABCD})^2) \)]. This form does not appear immediately, because of the auxiliary field term \( F \) that gets higher derivatives in its equation of motion. However, there are simpler terms that also include this auxiliary field, and following the derivation performed by the six-authors in [94], one can obtain the form that makes this explicit. It requires two to three pages of calculation; hence, the latter has also been relegated to the Appendix. To summarize here, the elimination of \( F \) and \( \overline{F} \) results in a nonlocal, nonpolynomial action written explicitly in [94], and expanded in powers of the chiral fields \( \Phi \) and \( \Phi \). When the latter multiply terms of form \( (D_A^2 Y^2)^2 \) and \( (\overline{D_A^2 Y^2})^2 \) in the Lagrangian, respectively, supersymmetrization of the \( (W^+_4 + W^A_4) \)-term is consequently achieved by means of coupling to a chiral multiplet.

**IV.5 Conclusions Regarding Supersymmetrization of the R^4 Term**

Taking into account the results of the above calculation, however rough, we have shown that it is possible to consider a background for the pure \( N=8 \)-supergravity, that could originate from the infinite tower of massless and finite-mass terms explicated by Green, Ooguri and Schwarz in [71]. The fact that it is possible to extend the theory in such a way that additional states may be accommodated, suggests the interpretation that assigns \( N=8 \)-supergravity to the swampland is perhaps premature. By means of a minor extension of the R-symmetry group of \( N=8 \)-supergravity, which frees up an additional \( U(1) \) for supersymmetric coupling while breaking the R-symmetry—such that the \( SU(8) \) associated with \( N=8 \) supersymmetry is preserved—indicates a possible path to demonstrate consistency of an extended theory.
Since a compelling argument has been made [75] for the presence of this term and for the constraint that it be supersymmetric, it seems logical to look for a more rigorous completion of $\mathcal{N}=8$ supergravity, such that consistency is assured. The type of extension of $\mathcal{N}=8$ supergravity theory shown above, in conjunction with the possibility that $\mathcal{N}=8$ supergravity, similar to the case of $\mathcal{N}=4$ super Yang-Mills theory in $d=4$, may ultimately be proven UV-finite, suggests that it may be an important component or sector of any 'unified' theory of quantum gravity. An approach of the type sketched above, while in bare-bones form, may further reflect a synergistic correspondence to M-theory. Again, we emphasize the possibility that identification of a correspondence between M-theory and extended supergravity may, perhaps along the lines suggested in [193][201][202], be the most important reason for continued development of an extended $\mathcal{N}=8$ supergravity theory.
CHAPTER V

CONCLUSIONS

The comments presented above on M-theory and supergravity underscore the importance of their mutually synergistic development. We've attempted to provide some insight into the close alignment between the development of both theories throughout the past two to three decades, and also an indication of how recent developments in both fields are related.

The cross-fertilization between these two theories, in particular, the major role of the Kaluza-Klein mechanism in both theories, suggests that mutual development should continue in the future. For the latter reason, a significant portion of this thesis seeks to demonstrate that pure $\mathcal{N}=8$ supergravity—although possibly an incomplete system, since it does not directly take into account an infinite number of additional massless states—should continue to be developed and expanded in relationship to string/M-theory.

We have shown that it is possible via a minor extension of $\mathcal{N}=8$ supergravity to change the result of a technical calculation; namely, the supersymmetrization of an $\mathbf{R}^4$ correction term in the effective supergravity action. Although this is considered a relatively minor point, it suggests that further extensions of $\mathcal{N}=8$ supergravity in the future could lead to a better understanding of the intricate relationship between M-theory and supergravity, a relationship that we believe is at the heart of a completely unified theory of quantum gravity.
CHAPTER VI

RECOMMENDED FUTURE INVESTIGATIONS

In view of the synergistic relationship that exists between M-theory and supergravity, it seems important that further studies in the future specifically map the correspondence between BPS-extended supergravity and string/M-theory. During the past fourteen-years, the important study of string/M-theory dualities—i.e., T-duality, S-duality, U-duality, and M-theory/F-theory duality—have enabled significant advances in our understanding. In particular, study of the AdS/CFT Correspondence, which is a type of duality between the bulk space and its boundary, has revitalized the important role of supergravity in string/M-theory. Our proposal for future studies is most closely aligned to the latter development, but perhaps with a subtle shift of emphasis that recognizes the importance of an expanded or extended \( \mathcal{N}=8 \) supergravity that is mapped in direct relationship to string/M-theory.
On-shell Supergravity

We consider here on-shell supergravity only for the d=4 and $\mathcal{N}=1$ graviton-gravitino multiplet. The on-shell number of bosonic and fermionic degrees of freedom is equal to two. Off the mass-shell, there are 12 fermionic degrees of freedom, but only 6 bosonic. Thus, one must find a suitable set of bosonic auxiliary fields for the off-shell formulation.

When considering the free action for the gravitino field, one replaces partial derivatives with Lorentz-covariant derivatives. The Christofel connection $\Gamma^\rho_{\mu\nu}$ is not needed for the convector index of $\psi_\mu$, because the antisymmetrized derivatives are GL(4,R)-covariant even without a connection. The kinetic term for the graviton can only be provided by the Einstein-Hilbert action; thus, we consider as a trial Lagrangian,

$$\mathcal{L}(e, \psi) = -\frac{e}{2k^2} E^\mu E^\nu_{\beta} R^{ab}_{\mu\nu} (\omega) + \epsilon^{uv\rho\sigma} (D_\mu \psi_\nu \sigma_\rho \bar{\psi}_\sigma + \psi_\sigma \sigma_\rho D_\mu \bar{\psi}_\nu),$$

Where the spin connection is determined from its algebraic equation of motion, which follows from the Lagrangian considered at first-order with independent $\omega^{ab}_\mu$, which is given below.

The action with the above Lagrangian is invariant under the supersymmetry transformations,

$$\delta_Q(e)e_\mu^a = i\kappa(\epsilon \sigma^a \bar{\psi}_\mu - \psi_\mu \sigma^a \epsilon)$$

$$\delta_Q(e)\psi_\mu = \kappa^{-1} D_\mu \epsilon, \quad \delta_Q(e)\bar{\psi}_\mu = \kappa^{-1} D_\mu \bar{\epsilon}.$$

Using an identity, the Lagrangian can be put into the following form,

$$\mathcal{L} = \epsilon^{uv\rho\sigma} \left( \frac{1}{8k^2} \epsilon_{abcd} e^a_{\rho} e^c_{\sigma} e^d_{\mu} R^{ab}_{\mu\nu} (\omega) + D_\mu \psi_\nu \sigma_\rho \bar{\psi}_\sigma + \psi_\sigma \sigma_\rho D_\mu \bar{\psi}_\nu \right)$$

This form makes variation easier, which can be demonstrated by computing the torsion induced by the gravitino. Upon variation of the spin connection in the 1st-order Lagrangian, the curvature changes and the variation yields,

$$\delta \mathcal{L} = -\frac{1}{2} \epsilon^{uv\rho\sigma} \left( \frac{1}{8k^2} \epsilon_{abcd} e^a_{\rho} e^c_{\sigma} e^d_{\mu} \delta \omega^{ab}_\mu + \delta \omega^{ab}_\mu \psi_\nu (\sigma_\sigma \sigma_\nu + \sigma_\sigma \sigma_\nu) \bar{\psi}_\sigma e^c_{\rho} \right)$$

Integrating this by parts and using another identity, we obtain,
\[ \delta \mathcal{L} = -\frac{1}{2} \delta \omega^a_{\mu ab} e_{abcd} e_{\mu \nu \rho} e^c_{\nu d} \left( \frac{1}{\kappa^2} D_\nu e^a - i \psi_\nu \sigma^d \bar{\psi}_\mu \right) \]

From this one can read off the equation of motion for the spin connection, which provides the torsion relation,

\[ D_{[\mu \nu \rho]}^a = i \kappa^2 \psi_{[\mu \sigma a \bar{\nu} \bar{\rho}]} \]

One can now verify invariance of the action expression given at the outset, under the supersymmetry transformations. The fields which \( \delta \) act on in the 1.5 order formalism are marked with an arrow above,

\[ \delta_+ = e^{\nu \rho \sigma} \delta + \left( \frac{1}{8 \kappa^2} \epsilon_{abcd} e_{\rho e_{\sigma}} e^d_{\mu \nu} (\omega) + D_\mu \psi_{\nu} \sigma_{\rho} \bar{\psi}_\sigma + \psi_{\sigma} \sigma_{\rho} D_\mu \bar{\psi}_\nu \right) \]

The \( \sigma_\rho \) is field-dependent and has a non-trivial variation given as,

\[ \delta_+ \sigma_{\rho a \bar{\alpha}} = \delta_+ e^a_{\rho} \sigma_{a a \bar{\alpha}} = i \kappa (\epsilon \sigma^a \bar{\psi}_\rho) \sigma_{a a \bar{\alpha}} = -2 i \kappa \epsilon a \bar{\psi}_\rho a \]

In the 1.5 formalism after variation of the action the auxiliary fields are replaced by their on-shell expressions, and using a Fierz rearrangement, we conclude the proof that the action is invariant under the supersymmetry transformations. After a series of lengthy calculations,

\[ \delta_+ \mathcal{L} = -i \kappa \epsilon^{\mu \nu \rho \sigma} \psi_{\sigma} (2 \epsilon \bar{\psi}_\rho + \sigma^a \bar{\psi}_\rho \epsilon \sigma_a) D_\mu \bar{\psi}_\nu = 0 \]

This concludes the proof of invariance.

It is necessary to determine that the algebra of symmetry transformations closes on-shell. The equations are shown as follows (please see, for example, Reference [235]):

\[ \left[ \delta_Q (\epsilon_1), \delta_Q (\epsilon_2) \right] e^a_{\mu} = D_\mu \xi^a, \quad \xi^a = i (\epsilon_2 \sigma^a \bar{\epsilon}_1 - \epsilon_1 \sigma^a \bar{\epsilon}_2) \]

\[ D_\mu \xi^a = \xi^v \partial_\mu e^a_v + \partial_\mu \xi^v e^a_v + \xi^v \omega^a_{\mu v} \]

\[ D_\mu \xi^a = L_\xi e^a_{\mu} + 2 \xi^v \partial_{[\mu \nu \rho]}^a + \xi^v \omega^a_{\mu v} \]

\[ D_\mu \xi^a = L_\xi e^a_{\mu} + \xi^v \omega^a_{\mu v} + 2 i \kappa^2 \xi^v \psi_{[\mu a \bar{v} \bar{\rho}]} \]

\[ \left[ \delta_Q (\epsilon_1), \delta_Q (\epsilon_2) \right] e^a_{\mu} = \delta_\rho (\xi) e^a_{\mu} + \delta_L (\epsilon) e^a_{\mu} + \delta_Q (\epsilon_1) e^a_{\mu}. \]
The following is the computation of the supersymmetry commutator on $\psi_\mu$. On-shell one can find,

$$\left[ \delta_Q(\epsilon_1), \delta_Q(\epsilon_2) \right] \psi_\mu \approx -\frac{1}{2} (\psi_{ab} \sigma_\mu \bar{\epsilon}_1) \sigma^{ab} \epsilon_2 - (\epsilon_1 \leftrightarrow \epsilon_2).$$

The RHS can be rewritten with a Fierz identity, and using certain identities for the gravitino field strength, and then dropping the conjugates of $R^\mu$-terms, we obtain:

$$\left[ \delta_Q(\epsilon_1), \delta_Q(\epsilon_2) \right] \psi_\mu \approx -\psi_{\mu\nu} \xi^\nu = \xi^\nu D_\nu \psi_\mu - \xi^\nu D_\mu \psi_\nu$$

$$= L_\xi \psi_\mu + \frac{1}{2} (\xi^\nu \omega_\nu)^{ab} \sigma_{ab} \psi_\mu - D_\mu (\xi^\nu \psi_\nu).$$

Thus, we conclude,

$$\left[ \delta_Q(\epsilon_1), \delta_Q(\epsilon_2) \right] \psi_\mu \approx \delta_P(\xi) \psi_\mu + \delta_L(-\xi^\nu \omega_\nu) + \delta_Q(-\kappa \xi^\nu \omega_\nu) \psi_\mu.$$ 

This is the same commutator as found for the vierbein, but now using the gravitino field equations. The other commutators all close off-shell on both fields. They are summarized as,

$$\left[ \delta_L(\epsilon), \delta_Q(\epsilon) \right] = \delta_Q \left( -\frac{1}{2} \epsilon^{ab} \epsilon \sigma_{ab} \right)$$

$$\left[ \delta_P(\xi), \delta_Q(\epsilon) \right] = \delta_Q \left( -\xi^\mu \partial_\mu \epsilon \right)$$

$$\left[ \delta_P(\xi), \delta_L(\epsilon) \right] = \delta_L \left( -\xi^\mu \partial_\mu \epsilon \right)$$

$$\left[ \delta_P(\xi_1), \delta_P(\xi_2) \right] = \delta_P \left( -\xi_1^\mu \partial_\mu \xi_2 + \xi_2^\mu \partial_\mu \xi_1 \right)$$

$$\left[ \delta_L(\epsilon_1), \delta_L(\epsilon_2) \right] = \delta_L \left( -[\epsilon_1, \epsilon_2] \right).$$

To achieve off-shell closure of the gauge algebra, one must amend the supergravity multiplet by a suitable set of auxiliary fields, as is done for chiral or vector multiplets. In the component formalism employed here, it is difficult to find a suitable set. It is best to set up a tensor calculus for gauge theories, and then apply it to N=1 supergravity. This enables one to derive the field content of the multiplet, its supersymmetry transformations, and the off-shell commutation relations from a few basic constraints on the geometry of superspace. Please see references [234] and [235] for details.
References


[152] "S\textsuperscript{7} and S\textsuperscript{7}," Martin Cederwall and Christian R. Preitschopf [hep-th/9309030v1] 6 Sep 1993


