Fall 2009

Invariant Frechet algebras on bounded symmetric domains

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Abstract
Let D be a bounded domain in the complex vector space Cn . We say that D is symmetric iff, given any two points p, q ∊ D, there is a biholomorphism &phis;, which interchanges p and q. These domains were classified abstractly by Elie Cartan in his general study of symmetric spaces, and were canonically realized in Cn by Harish-Chandra. They include polydisks and Siegel domains.

Let D be a bounded symmetric domain in Cn , and G be the largest connected group of biholomorphic automorphisms of D. The algebra C(D) of all continuous (not necessarily bounded) complex-valued functions on D with compact-open topology is a Frechet algebra. A closed subalgebra of C(D) is called an invariant algebra if it is closed under compositions with elements of G.

We prove that if D is irreducible, then there are only three invariant algebras with identity with maximal ideal space D : C(D), the set of all holomorphic functions H(D) and the set of all antiholomorphic functions H¯(D). This result partially generalizes the Rudin's classification of invariant algebras on unit ball in Cn . For the general symmetric bounded domain D we prove that the only invariant algebras are tensor products of invariant algebras on irreducible factors of D.

Keywords
Mathematics

INVARIANT FRÈCHET ALGEBRAS ON BOUNDED SYMMETRIC DOMAINS

BY

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DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

September 2009

This dissertation has been examined and approved.

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July 23, 2009
Date

DEDICATION

Грише, Жене, Марине, Диме, Саше

ACKNOWLEDGEMENTS

I have been fortunate to meet many great teachers. I am very grateful to Nikolay Shcherbina for teaching me Complex Analysis and for his kind guidance at early years of my mathematical life. I thank Vladimir Dolnikov for teaching me Combinatorial Geometry. I learned Lie Group Theory and Representation Theory from D. Nikshych and E. Grinberg. Don Hadwin spend a lot of time teaching me Operator Algebra.

I would like to express my deep gratitude to my advisor, Professor Eric Grinberg. Without his kind support and constant encouraging this project will never be completed. I am very grateful to the faculty, graduate students and staff at the Department of Mathematics at the University of New Hampshire for their kind help. Thanks to the University of New Hampshire Graduate School for awarding me a Dissertation Fellowship, providing me with the financial means to complete this project.

Finally, thanks to my wife, children, parents, and numerous friends who endured this long process with me, always offering support and love.

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INVARIANT FRÈCHET ALGEBRAS
ON BOUNDED SYMMETRIC DOMAINS

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Let D be a bounded domain in the complex vector space $\mathbb{C}^n$. We say that D is *symmetric* iff, given any two points $p, q \in D$, there is a *biholomorphism* ϕ, which interchanges p and q. These domains were classified abstractly by Élie Cartan in his general study of symmetric spaces, and were canonically realized in $\mathbb{C}^n$ by Harish-Chandra. They include polydisks and Siegel domains.

Let D be a bounded symmetric domain in $\mathbb{C}^n$, and G be the largest connected group of biholomorphic automorphisms of D. The algebra $C(D)$ of all continuous (not necessarily bounded) complex-valued functions on D with compact-open topology is a Fréchet algebra. A closed subalgebra of $C(D)$ is called an invariant algebra if it is closed under compositions with elements of G.

We prove that if D is irreducible, then there are only three invariant algebras with identity with maximal ideal space D : $C(D)$, the set of all holomorphic functions $H(D)$ and the set of all antiholomorphic functions $\overline{H}(D)$. This result partially generalizes the Rudin's classification of invariant algebras on unit ball in $\mathbb{C}^n$. For the general symmetric bounded domain D we prove that the only invariant algebras are tensor products of invariant algebras on irreducible factors of D.

CHAPTER 1

INTRODUCTION

Throughout this document, G will denote a Lie group with at most countably many components, and H will denote a closed subgroup.

Let $M = G/H$ be a homogeneous space and $C(M)$ be the algebra of all continuous complex valued functions on M with the topology of uniform convergence on compact subsets. The homogeneous space $M = G/H$ is an analytic manifold, and by our assumption on G it has a countable base, so $C(M)$ is a Fréchet algebra.

The group G acts on $C(M)$ by the formula

$$(g \cdot f)(z) = f(g^{-1} \cdot z)\,.$$

If A is a closed subalgebra of $C(M)$ we say that A is an *invariant algebra* if whenever f is in A and g is in G, then $g \cdot f$ is in A. The study of invariant subalebras and subspaces of $C(M)$ on a compact homogeneous space has a long history and for many spaces complete classification is available (see, for example [10], [12], [43], [31], [20]).

Invariant algebras of *bounded* continuous functions on noncompact symmetric spaces were considered by several authors. A. Nagel and W. Rudin [31, 37] classified invariant algebras of function continuous on the closed unit ball $\overline{B}$ in $\mathbb{C}^n$. T. Kaptanoğlu [27] characterized invariant algebras on the closed unit polydisk $\overline{U}^n$. Independently, E. Grinberg [22] classified invariant algebras on closed bounded symmetric domains in $\mathbb{C}^n$.

W. Rudin remarked in [37] that "there seems to be no good reason why continuity on the boundary should play a role in the preceding results." Later W. Rudin in [38] proved that there are only four unital invariant subalgebras of $C(B^n)$: $C(B^n)$, the algebra of holomorphic functions $H(B^n)$, the algebra of antiholomorphic functions $\overline{H}(B^n)$, and the algebra of constant functions $\mathbb{C}$.

E. Grinberg conjectured that this result should generalize to an arbitrary irreducible bounded symmetric domain.

A bounded domain M in $\mathbb{C}^n$ is called *symmetric* if every point of M is an isolated fixed point of an involutive biholomorphic automorphism of M. See [23] and [44] as general references for the theory of bounded symmetric domains. Every bounded symmetric domain with Bergman metric is a Hermitian symmetric space of noncompact type. Let G be a connected component of the group of biholomorphic isomorphisms of M. The group G is a semisimple Lie group, acting transitively on $M = G/H$, where compact subgroup H is an isotropy group of a point in M. For example, if $D = \mathbb{U}^n$ is the unit polydisk, then G is the product of n copies of the Möbius group $SU(1,1)$, and H is the product of n copies of $SO(2)$. As shown by Harish-Chandra, every bounded symmetric domain can be realized as circular convex domain in $\mathbb{C}^n$. Recall that a domain M is called circular if $e^{i\phi}z \in M$ for every real ϕ and every $z \in M$. Then $0 \in M$ and without lost of generality we may assume, that H is an isotropy group of 0. We will always assume that a bounded symmetric domain is given in Harish-Chandra realization. The maximal ideal space of Fréchet algebra is the set of all continuous multiplicative functionals. We prove the following.

Theorem 1.1. *Let M be a bounded symmetric domain and A be a unital invariant algebra with the maximal ideal space M.*

1. *If M is irreducible, then A is one of the following*
   a) *the algebra of all continuous functions $C(M)$;*
   b) *the algebra of all holomorphic functions $H(M)$;*
   c) *the algebra of all antiholomorphic functions $\overline{H}(M)$;*
2. *If $M = M_1 \times M_2 \times \cdots \times M_k$ is a decomposition into the product of irreducible components, then $A = A_1 \otimes A_2 \otimes \cdots \otimes A_k$ where A_j is an invariant subalgebra of $C(M_j)$.*

If A is a subalgebra of $C(X)$, and B is a subalgebra of $C(Y)$, the tensor product $A \otimes B$ is a closed subalgebra of $C(X \times Y)$, generated by products $f(x)g(y)$, where $f \in A$, and $g \in B$.

It is an interesting question, if the condition on the maximal ideal space is necessary in the theorem. The last chapter contains some preliminary results toward this problem.

The maximal ideal space $\mathfrak{M}_A$ of an invariant algebra A on a homogeneous manifold $M = G/H$ can be much bigger than M. The group G acts naturally on $\mathfrak{M}_A$. The dimension of an orbit $G \cdot \phi$ in $\mathfrak{M}_A$ can be greater than the dimension of M. However, it may happen only in exchange for an "extra complex structure". We show that the image of M in $\mathfrak{M}_A$ and orbits $G \cdot \phi$ have natural structures of CR-manifolds, and the following inequality holds.

Theorem 1.2. *Let M' be the image of M in $\mathfrak{M}_A$ under the evaluation map $z : f \to f(z)$. If M' is totally real, then for every $\phi \in \mathfrak{M}_A$*

$$dim\, G \cdot \phi - CR\text{-}dim\, G \cdot \phi \leq dim\, M' \leq dim\, M\,.$$

The expression in the left side of the inequality can be interpreted as a dimension over $\mathbb{C}$ of a linear space of differentials of smooth functions in A at a point.

The complete classification of invariant algebras on homogeneous manifolds is out of reach. Nevertheless the author hopes that this theorem can be used to characterize maximal ideal spaces of invariant algebras.

In Chapter 2 we define the homogeneous CR structure on orbits. Theorem 1.1 is proved in Chapter 3. Chapter 4 is based on the author's paper [16]. We prove here that manifolds of Gevrey class are pluripolar. We introduce the notion of Kolmogorov dimension, and estimate the Kolmogorov dimension of manifolds of Gevrey class. Theorem 1.2 is proved in Chapter 5 using the result on pluripolarity of manifolds. If M is a bounded symmetric domain, we show that orbits in maximal ideal space are isomorphic to orbits in the complexification of M, called the complex crown. We show that in this case, if the maximal ideal space $\mathfrak{M}_A \neq M$, than $\mathfrak{M}_A$ must contains a neighborhood of M in the complex crown.

CHAPTER 2

INVARIANT FUNCTION SPACES ON HOMOGENEOUS MANIFOLDS

2.1 Preliminaries

Definition 2.1. A Hausdorff space is called a *k-space* if every subset intersecting every compact subset in a closed set is itself closed.

A locally compact Hausdorff space is an example of a k-space. In particular, a manifold is a k-space.

Definition 2.2. A Hausdorff space X is called *hemicompact* if there is a countable compact exhaustion $K_1 \subset K_2 \subset \ldots$ of X such that for every compact subset $K \subset X$ there is n such that $K \subset K_n$. Such an exhaustion (K_n) is called *admissible.*

Theorem 2.3. *Let X be hemicompact k-space. Then $C(X)$ with the topology of uniform convergence on compact subsets is a Fréchet algebra. The topology on $C(X)$ is generated by the seminorms $||\cdot||_{K_n}$, where*

$$||f||_{K_n} = \sup_{z \in K_n} |f(z)|.$$

Proof. See [21] p. 69. □

Example 2.4. Let M be a manifold with a countable base. Then M is metrizable. Choose a metric on M and a point $p \in M$. A family of closed balls $K_n = \overline{B(p,n)}$ of radius n is an admissible exhaustion of M. Therefore $C(M)$ is a Fréchet algebra.

Definition 2.5. Let A be a Fréchet algebra. The set of all continuous nonzero complex-valued algebra homomorphism of A is called the *maximal ideal space* of A and is denoted by $\mathfrak{M}_A$. The maximal ideal set $\mathfrak{M}_A$ is a closed subset of a topological dual space A'. We endow $\mathfrak{M}_A$ with the relative topology.

If A is a closed point separating subalgebra of $C(M)$, then the evaluation map $M \to \mathfrak{M}_A$ given by $z : f \to f(z)$ is one-to-one.

2.2 Functions of Gevrey Class

Let $M = G/H$ be a homogeneous space. If $f \in C(M)$ and μ is a measure on G with compact support, the convolution

$$\mu * f(z) = \int_G f(g^{-1} \cdot z) d\mu(g)$$

is a well defined continuous function on M. Let dg be a Haar measure on G. If $h \in C(G)$ has compact support, define the convolution

$$h * f = (h\, dg) * f\,.$$

Definition 2.6. Let V be a vector subspace of $C(M)$. The vector space of all smooth functions in V is denoted V^∞.

If V is a closed invariant vector subspace of $C(M)$, then V^∞ is dense in V. Indeed, the vector space V is invariant with respect to convolutions. Convolutions with smooth functions are smooth and dense in V. If M is compact, then V is a Banach space, and, as E. Nelson showed [32], analytic functions are dense in V. Such a result is not available for invariant Fréchet spaces of functions on noncompact homogeneous manifolds. To mitigate this obstacle we will work with ultra-differentiable functions.

We need to introduce some notation first. For a multi-indices

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)\,, \qquad \beta = (\beta_1, \beta_2, \ldots, \beta_m)$$

we define

$$|\alpha| = \sum_{j=1}^{m} \alpha_j\,, \qquad \alpha! = \prod_{j=1}^{m} \alpha_j!\,, \qquad \text{and} \qquad \binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha - \beta)!\beta!}\,.$$

For an integer k we define $\alpha + k = (\alpha_1 + k, \alpha_2 + k, \dots, \alpha_m + k)$. For a point $x \in \mathbb{R}^m$ we define $x^\alpha = \prod_{j=1}^m x_j^{\alpha_j}$. If $f \in C^\infty(\mathbb{R}^m)$ we denote

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_m} x_m} f .$$

Let U be an open set in $\mathbb{R}^m$ and $s \geq 1$. A function $f \in C^\infty(U)$ is said to belong to Gevrey class $G^s(U)$ if for every compact $K \subset U$ there exists a constant $C_K > 0$ such that

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq C_K^{|\alpha|+1} (\alpha!)^s , \tag{2.1}$$

for every multi-index α. The class G^s forms an algebra. The Gevrey class G^s is closed with respect to composition and the Implicit Function Theorem holds for G^s [29], thus one may define manifolds (and submanifolds) of Gevrey class G^s in the usual way. A homogeneous space $M = G/H$ has a natural analytic manifold structure. A function $f \in C^\infty(M)$ is said to belong to Gevrey class $G^s(X)$ if f is in G^s locally on every analytic chart.

Lemma 2.7. *Let V be an invariant vector subspace of $C(M)$. If $s > 1$ then the set of functions of Gevrey class $V^s = V \cap G^s(M)$ is dense in V.*

Proof. By Denjoy-Carleman theorem [24] the Gevrey class is not quasi-analytic. For every nonempty open set $\omega \subset G$ there exists a nonzero function $h \in G^s(G)$ with support in ω. Choose a "delta function sequence" h_n such, that $\lim \operatorname{supp}(h_n) = \{e\}$, and $\int_G h_n \, dg = 1$. Then convolutions $h_n * f$ are in $G^s(M)$ and $\lim h_n * f = f$. □

Remark 2.8. Clearly, if $f \in C^k(M)$, then $h_n * f$ converges to f in the topology of C^k.

2.3 CR Structure on Orbits

In this section we will show that a closed invariant vector subspace $V \subset C(M)$ induces a natural invariant CR structure on orbits of G on a dual space. Let V' be a topological dual to V. The group G acts on V' by the formula

$$(g \cdot \phi)(f) = \phi(g^{-1} \cdot f) .$$

We will always consider orbits of G in V' with a unique homogeneous CR structure induced by V. This is the "largest"[1] CR structure for which functions in V are CR functions on the orbit.

[1]having a maximal CR dimension

Let S be a smooth manifold, and let $\mathcal{V}$ be a subbundle of the complexified tangent bundle $\mathbb{C}T(S)$. Recall, that the pair $(S,\mathcal{V})$ is called a CR manifold if $\mathcal{V}$ is involutive, that is $[\mathcal{V},\mathcal{V}] \subset \mathcal{V}$, and for every $p \in S$

$$\mathcal{V}_p \cap \overline{\mathcal{V}_p} = \{0\}.$$

The complex tangent space $H_p(S) \subset T_p(S)$ at p is the real part of $\mathcal{V}_p$, that is, the vector space of $X + \overline{X}$ for $X \in \mathcal{V}_p$. The complex dimension of the vector space $H_p(S)$ is called the CR dimension , and the real dimension of the factorspace $T_p(S)/H_p(S)$ is called the CR codimension of S. A continuous function f on S is called a CR function if $\overline{L}f = 0$ for every section $\overline{L} \in \overline{\mathcal{V}}$, where $\overline{L}f$ is the derivative in the sense of distribution. See [8] and [5] as references for CR manifolds and CR functions.

A CR structure on a homogeneous manifold $M = G/H$ is called *homogeneous* if G acts on M by CR automorphisms. Let $p \in M$ be a coset H. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras of G and H. Let $\mathfrak{g}_C$ and $\mathfrak{h}_C$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$. We will identify complex vector spaces $\mathbb{C}T_p(S)$ and $\mathfrak{g}_C/\mathfrak{h}_C$, and also real vector spaces $T_p(S)$ and $\mathfrak{g}/\mathfrak{h}$. Suppose $\mathfrak{q}$ be a complex subalgebra of $\mathfrak{g}_C$. The pair $(G/H,\mathfrak{q})$ is called a *homogeneous CR structure*, if

$$\mathfrak{q} \cap \mathfrak{g} = \mathfrak{h}, \quad \text{and} \qquad \mathfrak{q}/\mathfrak{h}_C = \mathcal{V}_p. \tag{2.2}$$

See [3] for the basic facts on homogeneous CR manifolds. The CR structure on G/H is completely determined by $\mathfrak{q}$. For example,

$$H_p(S) = \left((\mathfrak{q} + \overline{\mathfrak{q}}) \cap \mathfrak{g}\right)/\mathfrak{h}. \tag{2.3}$$

The CR dimension of $(G/H,\mathfrak{q})$ is equal to $\dim_{\mathbb{C}} \mathfrak{q} - \dim_{\mathbb{R}} \mathfrak{h}$, and the CR codimension is equal to $\dim_{\mathbb{R}} \mathfrak{g} - \dim_{\mathbb{C}} \mathfrak{q}$.

Let $G\cdot\phi$ be an orbit in V'. Let N be the isotropy group of ϕ. The vector $X - iY \in \mathfrak{g}_C$, where $X, Y \in \mathfrak{g}$, is said to annihilate V if for every $f \in V^{\infty}$

$$\frac{d}{dt}(exp(tX)\cdot\phi)f = i\frac{d}{dt}(exp(tY)\cdot\phi)f. \tag{2.4}$$

The vector space V induces on $G\cdot\phi$ the homogeneous CR structure $(G/N,\mathfrak{q})$, where

$$\mathfrak{q} = \{Z = X + iY : X - iY \text{ annihilates } V\}. \tag{2.5}$$

Remark 2.9. A "typical" functional $\phi \in V'$ has no symmetries. For such ϕ the CR structure on the orbit $G\cdot\phi$ is totally-real, $\dim G\cdot\phi = \dim G$, and CR-$\dim G\cdot\phi = 0$.

An orbit $G\cdot\phi$ can be considered as a smooth submanifold of an infinite-dimensional complex space $(V^\infty)'$. This imbedding gives an equivalent description of the homogeneous CR structure on the orbit. It is more convenient to work with local imbeddings into finite-dimensional spaces. Let

$$m = \dim G\cdot\phi - \text{CR-}\dim G\cdot\phi;$$

then there exist a neighborhood U of $e \in G$ and m functions $f_1, f_2, \ldots, f_m \in V^\infty$ such that the map $T : U\cdot\phi \to \mathbb{C}^m$ given by $T(\psi) = \big(\psi(f_1), \psi(f_2), \ldots, \psi(f_m)\big)$ is a CR imbedding.

Let $Gr_q(\mathfrak{g}_\mathbb{C})$ be the Grassmanian of q-dimensional complex linear subspaces of $\mathfrak{g}_\mathbb{C}$.

Lemma 2.10. *Let $W \subset Gr_q(\mathfrak{g}_\mathbb{C})$ be an open set. Let U be the set of all functionals $\phi \in V'$ such that the orbit $G\cdot\phi$ has the homogeneous CR structure $(G\cdot\phi, \mathfrak{q})$, where $\mathfrak{q}$ is a subspace of an element of W. Then the set U is open.*

Proof. Let $f_1, f_2, \ldots, f_m$ be as above. Let $X_1, X_2, \cdots, X_s$ be a basis for $\mathfrak{g}$. Consider functions f_{jr}, $j = 1, 2, \cdots, m$, $r = 1, 2, \cdots, s$, defined by:

$$f_{jr} = \frac{d}{dt}\big(exp(-tX_r)\cdot f_j\big)\,. \tag{2.6}$$

Clearly these functions are in V. The matrix valued function $F = (f_{jr})$ is continuous on V'. The kernel of the matrix $F(\phi)$ is the complex Lie algebra $\mathfrak{q}$ defining the homogeneous CR structure on the orbit $G\cdot\phi$. Let $\psi \in V'$ and $(G\cdot\psi, \mathfrak{q}')$ be a homogeneous CR structure on an orbit $G\cdot\psi$. Then $\mathfrak{q}'$ is a subspace of a kernel of $F(\psi)$. Because $W \subset Gr_q(\mathfrak{g}_\mathbb{C})$ is an open set, the set Y of $m\times s$ complex-valued matrices B such that $\ker B$ is a subspace of an element of W is an open set. Let Z be the set of functionals ψ, such that $\ker F(\psi) \in Y$. Then Z is open, and $\phi \in Z \subset U$. □

2.4 Decomposition of Bounded Symmetric Domains

The image of M under the evaluation map $M \to V'$ given by $z : f \to f(z)$ is an orbit of G. If functions in V separate points in M, then the evaluation map is a diffeomorphism onto this orbit and V induces a homogeneous CR structure on M. In general, we may identify the image of M with $M' = G/N$, where $N \supseteq H$ is a closed subgroup of G. Functions in V are constant on orbits of N in M and separate points in M'.

The CR structure on M' is especially simple when M is a bounded symmetric domain. In this case the CR structure is Levi-flat and M' is a product of a complex domain and a totally-real domain. The following two lemmas are well known.

Lemma 2.11. *Let $M = G/H$ be a bounded symmetric domain. Let $N \subset G$ be a closed subgroup. If H is a connected component of N, then $H = N$.*

Proof. Recall that we assume that a bounded symmetric domain is given in Harish-Chandra realization, so $M \subset \mathbb{C}^n$ is convex and circular, $0 \in M$ and H is an isotropy group of 0. The group

$$S^1 = \left\{e^{i\phi} : \phi \in \mathbb{R}\right\},$$

acting on M by multiplications, is a subgroup of H. Then for every $z \in N \cdot 0$ the circle $S^1 \cdot z = \{e^{i\phi} z : \phi \in \mathbb{R}\}$ is in $N \cdot 0$. By the assumptions, the image $N \cdot 0 = N/H$ is discrete. Therefore $N \cdot 0 = 0$ and $N = H$. □

Lemma 2.12. *Let $M = G/H$ be a bounded symmetric domain. Let $N \supset H$ be a closed proper subgroup of G. Then there exist bounded symmetric domains $M' = G'/H'$ and $M'' = G''/H''$ such that*

$$M = M' \times M'', \tag{2.7}$$

$$G/N = M', \tag{2.8}$$

$$N = H' \times G''. \tag{2.9}$$

Proof. Let $M = M_1 \times M_2 \times \cdots \times M_k$ be a decomposition of M into the product of irreducible bounded symmetric domains. Let $M_j = G_j/H_j$, and $\mathfrak{g}_j$, $\mathfrak{h}_j$ be Lie algebras of G_j and H_j. If $\mathfrak{g}_j = \mathfrak{h}_j + \mathfrak{m}_j$ is a Cartan decomposition , then each $\mathfrak{m}_j$ is an irreducible representation of $\mathrm{Ad}(H)$ (see [23], ch. VIII). Clearly, for $i \neq j$, a group $\mathrm{Ad}(H_i)$ acts on $\mathfrak{m}_j$ trivially. Hence $\mathrm{Ad}(H)$ representations $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are not equivalent. Because the Lie algebra $\mathfrak{n}$ of the group N is $\mathrm{Ad}(H)$-invariant, it follows that $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2 + \cdots + \mathfrak{n}_k$, where each $\mathfrak{n}_j$ is either $\mathfrak{h}_j$ or $\mathfrak{g}_j$. Put

$$J' = \{j : \mathfrak{n}_j = \mathfrak{h}_j\},$$

and $J'' = \{1, 2, \ldots, k\} - J'$. Choose

$$G' = \prod_{i \in J'} G_i, \qquad G'' = \prod_{j \in J''} G_j, \qquad H' = G' \cap H, \qquad H'' = G'' \cap H.$$

By construction $\mathfrak{n} = \mathfrak{h}' + \mathfrak{g}''$. Because G_j and H_j are connected, it follows that the product $H' \times G''$ is a connected component of N. It remains to show that $N = H' \times G''$. Consider $N' = N \cap G'$. Then H' is a connected component of N' and by Lemma 2.11 we have $N' = H'$, so $N = H' \times G''$. □

Lemma 2.13. *Let M be a bounded symmetric domain.*

1. *If M is irreducible then a homogeneous CR structure on M is one of the following*

   a) *The CR functions are the holomorphic functions;*

   b) *The CR functions are the antiholomorphic functions;*

   c) *The CR functions are the continuous functions (totally real case).*

2. *If $M = M_1 \times M_2 \times \cdots \times M_k$ is a decomposition into the product of irreducible components, then a homogeneous CR structure on M is a product of homogeneous CR structures on components.*

Proof. Let $M_j = G_j / H_j$ be an irreducible factor in the decomposition of M. Let $\mathfrak{g}_j$ and $\mathfrak{h}_j$ be Lie algebras of G_j and H_j correspondingly. Let $\mathfrak{g}_j = \mathfrak{h}_j + \mathfrak{m}_j$ be a Cartan decomposition. Then $\mathfrak{m}_j$ is an irreducible representation of $\mathrm{Ad}(H)$. If H_j is an isotropy group of $p \in M_j$, then we can identify vector spaces $\mathfrak{m}_j$ and $T_p(M_j)$, and therefore there exists an $\mathrm{Ad}(H)$-invariant complex structure J on $\mathfrak{m}_j$. The complex structure J is the $\mathbb{R}$-linear endomorphism J such that $J^2 = -I$ and J commutes with each element in $\mathrm{Ad}(H)$. The endomorphism J can be extended by $\mathbb{C}$-linearity to the complexification $\mathfrak{m}_j^{\mathbb{C}}$. Let W and $\overline{W}$ be eigenspaces of J corresponding to eigenvalues i and $-i$. Then $\mathfrak{m}_j^{\mathbb{C}} = W + \overline{W}$ is a decomposition into sum of irreducible $\mathrm{Ad}(H)$ representations. Let $(M, \mathfrak{q})$ be a homogeneous CR structure. Put $\mathfrak{q}_j = \mathfrak{q} \cap \mathfrak{g}_j^{\mathbb{C}}$. Then $\mathfrak{q}_j \cap \mathfrak{g}_j = \mathfrak{h}_j$ and $\mathfrak{q}_j$ is $\mathrm{Ad}(H)$-invariant. Clearly $\mathfrak{q} = \sum_{j=1}^{k} \mathfrak{q}_j$. There are only three possibilities for $\mathfrak{q}_j$.

1. $\mathfrak{q}_j = \mathfrak{h}_j^{\mathbb{C}}$ (totally real case);

2. $\mathfrak{q}_j = \mathfrak{h}_j^{\mathbb{C}} + W$ (holomorphic case);

3. $\mathfrak{q}_j = \mathfrak{h}_j^{\mathbb{C}} + \overline{W}$ (antiholomorphic case).

□

We can combine these three lemmas into the following result.

Theorem 2.14. *Let M be a bounded symmetric domain and V be a closed invariant subspace of $C(M)$. There exists a unique decomposition*

$$M = M_r \times M_h \times M_a \times M_c, \tag{2.10}$$

where all factors are bounded symmetric domains satisfying the following properties.

1. *All functions in V are holomorphic on M_h, antiholomorphic on M_a, and constant on M_c.*

2. *Functions in V separate points in $M' = M_r \times M_h \times M_a$.*

3. *Put $m = dim_{\mathbb{R}} M_r + dim_{\mathbb{C}} M_h + dim_{\mathbb{C}} M_a$,. For every $p \in M$ there exist functions $f_1, f_2, \ldots, f_m \in V^\infty$ with $\mathbb{C}$-linearly independent differentials $df_j(p)$.*

Remark 2.15. By changing the complex structure on M_a factor to opposite, we may assume that the decomposition in (2.10) is $M = M_r \times M_h \times M_c$.

CHAPTER 3

INVARIANT ALGEBRAS ON BOUNDED SYMMETRIC DOMAINS

3.1 Invariant Algebras and Decomposition

In this section we use the explicit description of a homogeneous CR structure on a bounded symmetric domain $M = G/H$ to characterize invariant algebra A. An invariant algebra is an invariant vector space, and so it induces a decomposition as in Theorem 2.14.

The restriction of A on M_h is $H(M_h)$ by the following theorem.

Theorem 3.1. *Let M be a bounded symmetric domain and A be an invariant algebra on M. If holomorphic functions in A separate points on M then A contains all holomorphic functions.*

Proof. We follow [37], Proposition 13.4.7. Without the lost of generality we may assume that $A \subset H(M)$. Suppose that M is irreducible. For every $f \in A$ the function

$$\tilde{f}(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}z)e^{-i\theta}\,d\theta \tag{3.1}$$

is linear and belongs to A. At 0 we have $df(0) = d\tilde{f}(0)$. This and the property (3) of the decomposition in Theorem 2.14 imply, that A contains all coordinate functions $z_1, z_2, \ldots, z_n$. These functions generate $H(M)$. □

The local characterization of A on M_r immediately follows from the Approximation Theorem [33]. Let X^n be a smooth n-dimensional manifold with countable base. Then

$C(X^n)$ is a Fréchet algebra. Let B be a closed subalgebra of $C(X^n)$. Let B^∞ be the subalgebra of C^∞ functions in B.

Definition 3.2. Algebra B is called a *totally real algebra* if for every point $p \in X^n$ there exist functions $f_1, f_2, \ldots, f_n$ in B^∞ with $\mathbb{C}$-linearly independent differentials $df_j(p)$.

This condition implies that there exists a neighborhood W of p such that the image $F(W) \subset \mathbb{C}^n$ is a totally real manifold, where $F(z) = \big(f_1(z), f_2(z), \ldots, f_n(z)\big)$. For an invariant algebra on a homogeneous space it is sufficient to check the condition at one point. By Theorem 2.14 an invariant totally real algebra separates points.

Proposition 3.3. *A totally real algebra B on a smooth n-dimensional manifold X^n is locally dense in $C(X^n)$, that is for every $p \in X^n$ there exists a neighborhood $U \ni p$ such that the algebra B_U of restrictions of functions in B on U is dense in $C(U)$.*

Proof. Let W and F be as above. By the Corollary 7.2 [33], the point p has a neighborhood $p \in U \subset W$ such that the algebra generated by functions $f_1, f_2, \ldots, f_n$ is dense in $C(U)$. □

The next section and Chapter 5 will be devoted to globalization of this result for invariant algebras, to show that an invariant totally real algebra A on a bounded symmetric domain M is the algebra of all continuous functions $C(M)$. Now we will show that such result implies the complete classification of algebras on bounded symmetric domains.

Let M_1 and M_2 be bounded symmetric domains. Let A be an invariant algebra on $M = M_1 \times M_2$. Because M_2 is homogeneous and A is invariant, the restrictions of A on slices $M_1 \times p$ and $M_1 \times q$ are naturally isomorphic for all $p, q \in M_2$. We will call such algebra *a restriction of A on M_1*.

Proposition 3.4. *Let M_r and M_h be bounded symmetric domains. Let A be an invariant algebra on $M = M_r \times M_h$. Suppose that the restriction of A on M_r is dense in $C(M_r)$ and the restriction of A on M_h is $H(M_h)$. Then $A = C(M_r) \otimes H(M_h)$.*

The proof of this proposition is based on Bishop-Stone-Weierstrass theorem. Usually this theorem is stated for Banach algebras. For the reader's convenience we prove the version of Bishop-Stone-Weierstrass theorem for Fréchet algebras.

Let X be a hemicompact k-space (see Section 2.1). Let B be a closed subalgebra of $C(X)$ containing constants.

Definition 3.5. A subset F of X is called a *set of antisymmetry* of B if every function in B which is real valued on F is constant on F.

Every point in X is a set of antisymmetry. By Zorn's lemma every point is contained in a maximal set of antisymmetry. Continuity of functions in B implies that maximal sets of antisymmetry are closed. If E and F are sets of antisymmetry and $E \cap F$ is not empty, then $E \cup F$ is a set of antisymmetry. Therefore maximal sets of antisymmetry constitute a closed pairwise disjoint cover for X.

Theorem 3.6 (Bishop-Stone-Weierstrass). *If $f \in C(X)$ and $f|_F \in B|_F$ for every maximal set of antisymmetry F then $f \in B$.*

Proof. Let K be a compact subset of X. Consider a Banach algebra $\overline{B|_K}$ which is the closure of the restriction of B on K. Every maximal set of antisymmetry for $\overline{B|_K}$ is an intersection of K and a maximal set of antisymmetry for B. By Bishop-Stone-Weierstrass theorem for Banach algebras (see for example [19]) $f \in \overline{B|_K}$. Because it holds for every compact set K, function f is in B. □

Proof of Proposition 3.4. First we will show that the restriction of A on M_r is $C(M_r)$. Let $\pi_r : M \to M_r$ be the projection on first factor. The closed algebra $B = \pi_r^{-1}(C(M_r)) \subset C(M)$ is the algebra of all continuous functions on M which are constant on M_h. We claim that B is a subalgebra of A. Let $(z, w) \in M$, $z \in M_r$, $w \in M_h$. Define the following operator $P : A \to A$ of "averaging" over M_h

$$P(f)(z,w) = \frac{1}{2\pi}\int_0^{2\pi} f(z, e^{i\theta} w)\, d\theta. \tag{3.2}$$

Then $P(f)(z, w) = f(z,0)$. Clearly $P(A)$ is a subset of B. By the assumptions the algebra $P(A)$ is dense in B. On the other hand, the operator P is a projection and the image $P(A)$ is a closed subalgebra of A. Therefore $B \subset A$.

For every $p \in M_r$ the set $p \times M_h$ is a set of antisymmetry for A. These are maximal sets of antisymmetry for A, because real-valued functions in B separate such sets. Now Bishop-Stone-Weierstrass theorem implies that A contains all continuous functions that are holomorphic on $p \times M_h$ for every $p \in M_r$. Hence $A = C(M_r) \otimes H(M_h)$. □

Theorem 3.7. *Let M be a bounded symmetric domain and B be an invariant algebra on M. Suppose that $M = M_r \times M_h \times M_a \times M_c$ is a decomposition induced by A. If the restriction of A on M_r is dense in $C(M_r)$ then*

$$A = C(M_r) \otimes H(M_h) \otimes \overline{H}(M_a). \tag{3.3}$$

Proof. We may assume that A separates points in M. Also by Remark 2.15 we may assume that the decomposition has form $M = M_r \times M_h$. Then the theorem follows from Proposition 3.4. □

3.2 Approximation in Totally Real Algebras

Example 6.1 in [26] shows that a point-separating totally real algebra on a bounded domain D can be a proper subalgebra of $C(D)$. It is a classical result that if a *finitely-generated* totally real algebra B on a smooth manifold X is such that the maximal ideal space $\mathfrak{M}_B$ is X, then $B = C(X)$ (see [33], [26]).

Question 3.8. Suppose that B is a totally real algebra on a smooth manifold X such that the maximal ideal space $\mathfrak{M}_B$ is X. Does it follow that $B = C(X)$?

We show that the answer is positive for *an invariant algebra* on a bounded symmetric domain.

In this section we assume that A is an invariant totally real algebra on a bounded symmetric domain $M \subset \mathbb{C}^n$,

The main result of this section is the following.

Theorem 3.9. *If the maximal ideal space* $\mathfrak{M}_A$ *is* M, *then* $A = C(M)$.

Definition 3.10. Let f be a continuous function on M, and h be a continuous function on M with compact support. Because H is compact, a convolution

$$(h * f)(z) = \int_G h(g\cdot 0) f(g^{-1}\cdot z)\, dg,$$

is well-defined.

Definition 3.11. Let X and Y be subsets of M. A convolution $X * Y$ is defined as

$$X * Y = \left\{g_1 g_2 \cdot 0 : \, g_1 \cdot 0 \in X, \, g_2 \cdot 0 \in Y\right\}.$$

If X and Y are compact, then $X * Y$ is a support of a convolution of characteristic functions $X * Y = \operatorname{supp} \xi_X * \xi_Y$.

Definition 3.12. Let f_j be a sequence of continuous functions on M. Let K be a compact subset of M. The sequence f_j is called a *delta function sequence on K* if for every $h \in C(M)$ with $\operatorname{supp} h \subset K$ the sequence $h * f_j$ uniformly converges to h on K.

To prove the theorem we will show that if $\mathfrak{M}_A = M$, then for every compact $K \subset M$ there exists a *delta function sequence* $f_j \in A$. We need several lemmas first.

Definition 3.13. Let E be a subset of M. A point $p \in M$ is a *peak point* for $f \in C(M)$ on E if $f(p) = 1$ and $|f| < 1$ on $E \backslash p$.

Definition 3.14. A point $p \in M$ is a *local peak point* for $f \in C(M)$ if there exists a neighborhood U of p in M, such that p is a peak point for f on U.

Definition 3.15. A point $p \in M$ is a *local peak point* for an algebra A if p is a local peak point for some $f \in A$.

Remark 3.16. Our definition of local peak point is equivalent to the usual one (see for example [19]), because we assume that $\mathfrak{M}_A = M$.

Lemma 3.17. *For every point $p \in M$ there is a smooth function $h \in A$ such that p is a local peak point for f.*

Proof. The condition that the algebra A is totally real is equivalent to the existence of smooth functions $f_1, f_2, \dots f_{2n}$ in A, such that

$$df_j(p) = dz_j, \quad \text{and } df_{n+j}(p) = d\overline{z_j} \quad \text{for } j = 1, 2, \dots, n. \tag{3.4}$$

We may assume that $f_1(p) = \dots = f_{2n}(p) = 0$. Then p is a local peak point for

$$h = 1 - \sum_{j=1}^{n} f_j f_{n+j} . \tag{3.5}$$

□

For Banach algebras local peak points are global peak points. However it is not true in general for Fréchet algebras. See Section (9.2) in [21] for a counterexample. For this reason we work with peak points on compact sets.

Lemma 3.18. *Let K be a compact subset of M. There exists a smooth function $f \in A$ such that*

1. *the origin is a peak point for f on K,*
2. *the function f is H-invariant,*

3. *near* 0 *we have*

$$f(z) = 1 - Q(z,\overline{z}) + O\left(|z|^4\right), \tag{3.6}$$

where $Q(z,\overline{z}) = \sum b_{jk} z_j \overline{z}_k$ *is a bilinear form, and the real part* $\operatorname{Re} Q(z,\overline{z})$ *is positive-definite.*

Proof. We may assume that K is H-invariant and that $0 \in K$. Let h be as in the previous lemma. Let U be a neighborhood of 0 such that 0 is a peak point for h on U. We may assume that U is totally bounded. For a subset X of M we define $X^1 = X$ and $X^{m+1} = X * X^m$. Choose a neighborhood W of 0 such that $W^2 \subset U$. Let X be a closure of $U * K$. Let $B = \overline{A|_X}$ be the closure of the restriction of A on X. By Rossi's Local Peak Set Theorem (Theorem 8.1 in [19]) there exists a function $f_0 \in B$, such that 0 is a peak point for f_0 on X. Because A^∞ is dense in A and $A|_X$ is dense in B, we may approximate f_0 on X by functions in A^∞. Hence there exists $f_1 \in A^\infty$ such that $|f_1|$ has maximum on X at $p \in W$ and

$$\sup_{z \in X \setminus W} |f_1(z)| < |f_1(p)|. \tag{3.7}$$

We may assume that $f_1(p) = 1$. Let $g \cdot 0 = p$, then $f_2 = g \cdot f_1$ satisfies the following properties.

$$\max_{z \in K} |f_2(z)| = 1 = f_2(0), \tag{3.8}$$

$$\sup_{z \in K \setminus U} |f_2(z)| < 1. \tag{3.9}$$

Let dh be the normalized Haar measure on H. Then for sufficiently small $\varepsilon > 0$ the function $f = \frac{1}{1+\varepsilon}(dh * f_2 + \varepsilon h)$ satisfies conditions of the lemma. Clearly f is H-invariant and has a peak point at 0. We just need to check (3.6). Recall that the group H contains $S^1 = \{e^{i\phi} : \phi \in \mathbb{R}\}$. A Taylor expansion of an S^1 invariant function has form (3.6), and $\operatorname{Re} Q$ is positive-definite by the construction of h. □

If f is as in Lemma 3.18, then the sequence $f_m = c_m f^m$ is a delta function sequence on K, where c_m are appropriate constants. This result follows from the arguments in [22] and [27]. Let $\mathbb{U}^n$ be a unit polydisk in $\mathbb{C}^n$, and $d\mu$ be a Lebesgue measure on $\mathbb{C}^n$. The following is essentially Lemma (3.3) in [27].

Lemma 3.19. *Let* $K \subset \mathbb{C}^n$ *be a compact neighborhood of origin. Suppose*

(a) $f : K \to \mathbb{C}$ *is continuous,*

(b) $|f(z)| < 1$ *if* $z \neq 0$,

(c) *near origin* $f(z) = 1 - Q(z,\overline{z}) + O(|z|^4)$, *where* $Q(z,\overline{z})$ *is a bilinear form and* $\operatorname{Re} Q$ *is positive-definite.*

Fix $p \in \left(\dfrac{n+1}{2n+4}, \dfrac{1}{2}\right)$. *For* $m = 2,3,\ldots$, *let* $\varepsilon_m = m^{-p}$, *and take* c_m *so as to satisfy* $\int_{\varepsilon_m \mathbb{U}^n} c_m f^m \, d\mu = 1$. *Then as* $m \to \infty$,

(i) $\lim_{n\to\infty} |c_m| m^{-n} > 0$,

(ii) $\int_{\varepsilon_m \mathbb{U}^n} |c_m f^m| \, d\mu = O(1)$,

(iii) $k_m = \sup\{|c_m f^m(z)| : z \in K \setminus \varepsilon_m \mathbb{U}^n\} \to 0$.

Proof. In [27] (Lemma 3.3) this result is proved for $K = \overline{\mathbb{U}}^n$ for "polyradial" functions (i.e. depending on $|z_1|, |z_2|, \ldots, |z_n|$ only). We will reduce the lemma to this special case. Replacing, if necessary function $f(z)$ by the dilatation $f(rz)$ we may assume, that K is a subset of $\overline{\mathbb{U}}^n$. Let $\widetilde{f}$ be a continuous extension of f on $\overline{\mathbb{U}}^n$ such that $|\widetilde{f}(z)| < 1$ if $z \neq 0$. Let $\phi(z)$ and $\Phi(z)$ be *polyradial* functions defined by

$$\phi(z) = \min\{\operatorname{Re} \widetilde{f}(w) : |w_j| = |z_j| \text{ for } j = 1,2,\ldots,n\} \tag{3.10}$$

$$\Phi(z) = \max\{|\widetilde{f}(w)| : |w_j| = |z_j| \text{ for } j = 1,2,\ldots,n\} \tag{3.11}$$

Functions ϕ and Φ satisfy the conditions of the Lemma with K replaced by $\overline{\mathbb{U}}^n$ and by Lemma (3.3) in [27], the result holds for these functions. Let b_m and d_m be defined by $\int_{\varepsilon_m \mathbb{U}^n} b_m \phi^m \, d\mu = 1$ and $\int_{\varepsilon_m \mathbb{U}^n} d_m \Phi^m \, d\mu = 1$. Then $d_m \le c_m \le b_m$ and (i) follows. Clearly

$$\int_{\varepsilon_m \mathbb{U}^n} |c_m f^m| \, d\mu \le \int_{\varepsilon_m \mathbb{U}^n} |c_m \Phi^m| \, d\mu, \quad \text{and}$$

$$|c_m f^m(z)| \le |c_m| \Phi^m(z).$$

The result follows. □

Remark 3.20. If k is a compact subset of M, then the Lemma holds if we replace Lebesgue measure by G-invariant dm measure on M. Indeed, $dm(z) = \Omega(z) d\mu(z)$, where $\Omega(z)$ is a smooth positive function.

Lemma 3.21. *Let* K *be compact subset of* M. *There exists a delta function sequence on* K.

Proof. Without lost of generality we may assume that the interior of K contains origin. Let $K^2 = K * K$. By Lemma 3.18 there exists function f such that origin is a peak point for f on K^2 and f satisfies (3.6). Let ε_m be as in Lemma 3.19. Let $\pi : G \to G/H = M$ be the natural mapping and $W_m = \pi^{-1}(\varepsilon_m \mathbb{U}^n)$. Clearly $\cap_{m=1}^{\infty} W_m = \pi^{-1}(0) = H$. We claim that $f_m = c_m f^m$ is a delta function sequence on K, where c_m is defined so as to satisfy

$$\int_{\varepsilon_m \mathbb{U}^n} c_m f^m(z) dm(z) = \int_{W_m} c_m f^m(g \cdot 0) dg = 1. \tag{3.12}$$

By Lemma 3.19 (see also Remark 3.20) $f_m \to 0$ uniformly on $K^2 \setminus \varepsilon_m \mathbb{U}^m$, and

$$\int_{W_m} \left| c_m f^m(g \cdot 0) \right| dg = O(1). \tag{3.13}$$

Let h be a continuous function on M with $\operatorname{supp} h \subset K$. We need to show that convolutions $h * f_m$ converge to h uniformly on K. Let $z = x \cdot 0 \in K$. Pick $\varepsilon > 0$. Let

$$V = \{w : |h(w) - h(z)| < \varepsilon\}$$

be a neighborhood of z. For sufficiently large m we have $x W_m^{-1} \cdot 0 \subset V$, were

$$W_m^{-1} = \{g^{-1} : g \in W_m\}.$$

Then

$$\begin{aligned}
&(h * f_m)(z) - h(z) \\
&\quad = \int_G h(g \cdot 0) f_m(g^{-1} \cdot z) dg - h(z) = \int_G h(x g^{-1} \cdot 0) f_m(g \cdot 0) dg - h(z) \\
&\quad = \int_{W_M} \left(h(x g^{-1} \cdot 0) - h(z) \right) f_m(g \cdot 0) dg + \int_{G \setminus W_M} h(x g^{-1} \cdot 0) f_m(g \cdot 0) dg.
\end{aligned} \tag{3.14}$$

The first integral is bounded by

$$\varepsilon \int_{W_m} \left| c_m f^m(g \cdot 0) \right| dg = O(\varepsilon).$$

The second integral is bounded by

$$\sup_{K^2 \setminus \varepsilon_m \mathbb{U}^m} |f_m| \int_G |h(g \cdot 0| dg = o(1).$$

□

Proof of Theorem 3.9. It is sufficient to show that for any compact $X \subset M$ the restriction of A on X is dense in $C(X)$. Let $K \subset M$ be a compact such that an interior of k contains X. The space of continuous functions on K with compact support is dense in $C_0(K)$, the Banach space of continuous functions vanishing on ∂K. Therefore, by the previous Lemma, the restriction of A on K is dense in $C_0(K)$. The space $C_0(K)$ is dense in $C(X)$ and the result follows. □

Proof of Theorem 1.1. This theorem is immediately follows from Theorem 3.7 and 3.9. □

CHAPTER 4

PLURIPOLARITY OF MANIFOLDS

4.1 Pluripolar Sets

A set $E \subset \mathbb{C}^n$ is called *pluripolar* if there exists a non-constant plurisubharmonic function ϕ such that $\phi \equiv -\infty$ on E. Pluripolar sets form a natural category of "small" sets in complex analysis. Pluripolar sets are polar, so they have Lebesgue measure zero, but there are no simple criteria to determine pluripolarity. In this chapter we discuss the conditions that ensure pluripolarity for smooth manifolds. This problem has a long history. S. Pinchuk [35], using Bishop's "gluing disks" method, proved that a generic manifold of class C^3 is non-pluripolar. Recall that the manifold $M \subset \mathbb{C}^n$ is called *generic at a point* $p \in M$, if the tangent space T_pM is not contained in a proper complex subspace of $\mathbb{C}^n$. A. Sadullaev [39], using the same method proved that a subset of positive measure of a generic manifold of class C^3 is non-pluripolar. It would be of interest to reduce the required smoothness in this results.. In this direction, N. V. Shcherbina [40] proved that the pluripolar graphs of continuous functions over domains are holomorphic. The generalization of this result for graphs of functions on generic CR manifolds seems to be an open problem.

In the opposite direction E. Bedford [6] showed that a real-analytic nowhere generic manifold is pluripolar. For some applications to harmonic analysis the condition of real-analyticity is too restrictive. However, the result does not hold for merely smooth manifolds. K. Diederich and J. E. Fornaess [13] found an example of a non-pluripolar smooth curve in $\mathbb{C}^2$. They constructed a function $f \in C^\infty[0,1]$ such that the graph of this function is not pluripolar. In this example the derivatives $f^{(k)}$ grow very fast as $k \to \infty$.

Recently, D. Coman, N. Levenberg and E. A. Poletsky [11] proved that curves of Gevrey class G^s, $s < n+1$ in $\mathbb{C}^n$ are pluripolar. We generalize this result to higher dimensional manifolds. Recall that the submanifold $M \subset \mathbb{C}^n$ is called *totally real* if for every $p \in M$ the tangent space $T_p M$ contains no complex line.

Theorem 4.1. *Let $M \subset \mathbb{C}^n$ be a totally real submanifold of Gevrey class G^s. If* $\dim M = m$ *and $ms < n$, then M is pluripolar.*

In fact we prove a stronger result. It follows from Theorem 2.1 in [2] that a compact set $X \subset \mathbb{C}^n$ is pluripolar if and only if for any bounded domain D containing X and $\varepsilon > 0$ there exists polynomial P such that

$$\sup\{|P(z)| : z \in D\} \geq 1$$

and

$$\sup\{|P(z)| : z \in X\} \leq \varepsilon^{\deg P} .$$

Theorem 4.2. *Let $M \subset \mathbb{C}^n$ be a totally real submanifold of Gevrey class G^s. Let X be a compact subset of M. If* $\dim M = m$ *and $ms < n$, then for every $h < \frac{n}{ms}$ and every $N > N_0 = N_0(h)$ there exists a non-constant polynomial $P \in \mathbb{Z}[z_1, z_2, \ldots, z_n]$,* $\deg P \leq N$ *with coefficients bounded by $exp(N^h)$, such that*

$$\sup\{|P(z)| : z \in X\} < \exp(-N^h) . \tag{4.1}$$

This result is similar to the construction of an auxiliary function in transcendental number theory (cf.[41] Proposition 4.10).

The Theorem 4.1 gives some information about polynomially convex hulls of manifolds of Gevrey class. Recall, that the *polynomially convex hull* $\widehat{X}$ of X consists of all $z \in \mathbb{C}^m$ such that

$$|P(z)| \leq \sup_{\zeta \in X} |P(\zeta)| ,$$

for all polynomials P. It is well known, that the polynomially convex hull of a pluripolar compact set is pluripolar (this follows immediately from Theorem 4.3.4 in [25]).

We also introduce the notion of *Kolmogorov dimension* for a compact subset $X \subset \mathbb{C}^n$ (denoted $\mathscr{K}$-dim X) with the following properties.

1. $0 \leq \mathscr{K}\text{-dim}\, X \leq n$.

2. $\mathscr{K}\text{-dim}\, \widehat{X} = \mathscr{K}\text{-dim}\, X$.

3.

$$\mathscr{K}\text{-dim} \bigcup_{j=1}^{m} X_j = \max\{\mathscr{K}\text{-dim } X_j : j = 1, \ldots, m\}.$$

4. If $D \subset \mathbb{C}^n$ is a domain, $X \subset D$ and $\phi : D \to \mathbb{C}^k$ is a holomorphic map, then $\mathscr{K}$-dim $\phi(X) \le$ $\mathscr{K}$-dim X.

5. If $\mathscr{K}$-dim $X < n$, then X is pluripolar.

The main result of this chapter is the following estimate of the Kolmogorov dimension of totally real submanifolds of Gevrey class.

Theorem 4.3. *Let $M \subset \mathbb{C}^n$ be a totally real submanifold of Gevrey class G^s. Let X be a compact subset of M. If* $\dim M = m$ *then $\mathscr{K}$-dim $X \le ms$.*

Remark 4.4. This estimate is sharp. Similar estimates hold for more general class of CR-manifolds. These issues will be addressed in the forthcoming paper.

We defined the Kolmogorov dimension of X in terms of ε-entropy of traces on X of bounded holomorphic functions. The definition and basic properties of ε-entropy are given in the next Section. In Section 4.3 we discuss the notion of Kolmogorov dimension. The proof of Theorem 4.3 is given in Section 4.4.

4.2 The notion of ε-entropy

Let (E, ρ) be a totally bounded metric space. A family of sets $\{C_j\}$ of diameter not greater than 2ε is called an *ε-covering* of E if $E \subseteq \bigcup C_j$. Let $N_\varepsilon(E)$ be the smallest cardinality of the ε-covering.

A set $Y \subseteq E$ is called *ε-distinguishable* if the distance between any two points in Y is greater than ε: $\rho(x, y) > \varepsilon$ for all $x, y \in Y$, $x \ne y$. Let $M_\varepsilon(E)$ be the largest cardinality of an ε-distinguishable set.

For a nonempty totally bounded set E the natural logarithm

$$\mathscr{H}_\varepsilon(E) = \log N_\varepsilon(E)$$

is called the *ε-entropy.*

The notion of ε-entropy was introduced by A. N. Kolmogorov in the 1950's. Kolmogorov was motivated by Vitushkin's work on Hilbert's 13th problem and Shannon's information theory. Note that Kolmogorov's original definition (see [28]) is slightly different from ours (he used the logarithm to base 2). Here we follow [30].

We will need some basic properties of the ε-entropy.

Lemma 4.5. *([28], Theorem IV) For each totally bounded space E and each $\varepsilon > 0$*

$$M_{2\varepsilon}(E) \leq N_\varepsilon(E) \leq M_\varepsilon(E) \tag{4.2}$$

Lemma 4.6. *Let $\{(E_j, \rho_j) : j = 1, 2, \ldots, k\}$ be a family of totally bounded metric spaces. Let (E, ρ) be a Cartesian product with a sup-metric, i.e.*

$$E = E_1 \times E_2 \times \cdots \times E_k\,,$$

$$\rho((x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k)) = max_j \rho_j(x_j, y_j)\,.$$

Then

$$\mathcal{H}_\varepsilon(E) \leq \sum_j \mathcal{H}_\varepsilon(E_j)$$

.

Proof. Let $\{C_{jl}\}$ $l = 1, \ldots N_j$ be an ε-covering of E_j. Then the family

$$\{C_{1l_1} \times C_{2l_2} \times \cdots \times C_{kl_k} : l_j = 1, \ldots, N_j\}$$

is an ε-covering of E. □

We also need upper bounds for ε-entropy of a ball in finite-dimensional ℓ^∞ space. Let $\mathbb{R}^n_\infty$ be $\mathbb{R}^n$ with the sup-norm:

$$||(x_1, x_2, \ldots, x_n)||_\infty = \max_j |x_j|\,.$$

Lemma 4.7. *Let B_r be a ball of radius r in $\mathbb{R}^n_\infty$. Then*

$$\mathcal{H}_\varepsilon(B_r) < n \log\left(\frac{r}{\varepsilon} + 1\right).$$

Proof. The inequality is obvious for $n = 1$. The general case then follows from Lemma (4.6). □

4.3 Kolmogorov Dimension

Let X be a compact subset of a domain $D \subset \mathbb{C}^n$. Let A_X^D be a set of traces on X of functions analytic in D and bounded by 1. So $f \in A_X^D$ if and only if there exists a function F holomorphic on D such that

$$\sup_{z \in D} |F(z)| \leq 1$$

and $f(z) = F(z)$ for every $z \in X$. By Montel's theorem A_X^D is a compact subset of $C(X)$.

The connections between the asymptotics of ε-entropy and the pluripotential theory were predicted by Kolmogorov, who conjectured that in the one dimensional case,

$$\lim_{\varepsilon \to 0} \frac{\mathcal{H}_\varepsilon(A_X^D)}{\log^2(1/\varepsilon)} = \frac{C(X,D)}{(2\pi)},$$

where $C(X,D)$ is the condenser capacity. This conjecture was proved simultaneously by K. I. Babenko [4] and V. D. Erokhin [14] for a simply-connected domain D and connected compact X (see also [15]). For more general pairs (X,D) the conjecture was proved by Widom [42]. A simplified proof can be found in [18].

In the multidimensional case Kolmogorov asked for a proof of the existence of the limit

$$\lim_{\varepsilon \to 0} \frac{\mathcal{H}_\varepsilon(A_X^D)}{\log^{n+1}(1/\varepsilon)}$$

and a calculation of its explicit value. V. P. Zahariuta [45] showed how the solution of Kolmogorov problem will follow from the existence of the uniform approximation of the relative extremal plurisubharmonic function $u_{X,D}^*$ by multipole pluricomplex Green functions with logarithmic poles in X ("Zahariuta conjecture"). Later this conjecture was proved by Nivoche [34] for a "nice" pairs (D,X). Therefore it is established that for such pairs

$$\lim_{\varepsilon \to 0} \frac{\mathcal{H}_\varepsilon(A_X^D)}{\log^{n+1}(1/\varepsilon)} = \frac{C(X,D)}{(2\pi)^n},$$

where $C(X,D)$ is the relative capacity (see [7]).

The pluripolarity of X is equivalent to the condition $C(X,D) = 0$ ([7]). If $\mathcal{H}_\varepsilon(A_X^D) = o(\log^{n+1}(\frac{1}{\varepsilon}))$ then X is "small" (pluripolar) and the asymptotics of ε-entropy can be used to determine how "small" X is. We will use the function

$$\Psi(X,D) = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{H}_\varepsilon(A_X^D)}{\log\log\frac{1}{\varepsilon}} - 1 \tag{4.3}$$

to characterize the "dimension" of X.

For a compact subset $X \subset \mathbb{C}^n$ we define the *Kolmogorov dimension* $\mathcal{K}\text{-}\dim X = \Psi(X, D)$, where D is a bounded domain containing X. A. N. Kolmogorov proposed in [28] to use $\Psi(X, D)$ as a *functional dimension* of space of holomorphic functions on X. The idea to use $\Psi(X, D)$ to characterize the "size" of compact X seems to be new. We proceed to prove that $\Psi(X, D)$ is independent of the bounded domain containing X.

Lemma 4.8. *Let $D \subset \mathbb{C}^n$ be a bounded domain. If $X_1, X_2, \ldots, X_k$ are compact subsets of D, then*

$$\Psi(\bigcup X_j, D) = \max \Psi(X_j, D) .$$

Proof. Let $X = \bigcup X_j$. Embeddings $X_j \to X$ generate a natural isometric embedding $C(X) \to C(X_1) \times C(X_2) \times \cdots \times C(X_k)$. Restriction of this embedding on A_X^D gives an isometric embedding $A_X^D \to A_{X_1}^D \times A_{X_2}^D \times \cdots \times A_{X_k}^D$. By Lemma 4.6

$$\mathcal{H}_\varepsilon(A_X^D) \leq \sum_{j=1}^{k} \mathcal{H}_\varepsilon(A_{X_j}^D) \leq k \max \mathcal{H}_\varepsilon(A_{X_j}^D) , \tag{4.4}$$

and the result follows. □

For a point $a \in \mathbb{C}^n$ and $R > 0$, we denote $\Delta(a, R)$ the open polydisk of radius R with center at a

$$\Delta(a, R) = \left\{ z = (z_1, z_2, \ldots, z_n) : |z_j - a_j| < R, \ j = 1, \ldots, n \right\} .$$

The following well known result follows directly from Cauchy's formula.

Lemma 4.9. *Let R and r be real numbers, $R > r > 0$. Let $a \in \mathbb{C}^n$ and f be a bounded analytic function on the polydisk $\Delta(a, R)$. Then for any positive integer k there exists a polynomial P_k of degree k such that*

$$\sup_{w \in \Delta(a,r)} |f(w) - P_k(w)| \leq \frac{1}{R - r} \left(\frac{r}{R}\right)^k \sup_{z \in \Delta(a,R)} |f(z)| . \tag{4.5}$$

Lemma 4.10. *Let R and r be real numbers, $R > r > 0$. If a polynomial P of degree k satisfies the inequality $|P(w)| \leq A$ for every $w \in \Delta(a, r)$, then for $z \in \Delta(a, R)$ we have*

$$|P(z)| \leq A \left(\frac{R}{r}\right)^k . \tag{4.6}$$

Proof. Let $Q(\lambda) = P((\lambda(z - a) + a)$. The inequality follows from the application of maximum modulus principle to $Q(\lambda)/\lambda^k$. □

Theorem 4.11. *Let X be a compact in $\mathbb{C}^n$. If $D_1, D_2 \subset \mathbb{C}^n$ are bounded domains containing X, then $\Psi(X, D_1) = \Psi(X, D_2)$.*

Proof. Without loss of generality we may assume that $D_1 \subset D_2$. Then $A_X^{D_1} \supset A_X^{D_2}$ and $\Psi(X, D_1) \geq \Psi(X, D_2)$. We establish the special case of two polydisks first. Suppose $D_1 = \Delta(a, r)$ and $D_2 = \Delta(a, R)$, where $R > r > 0$. Choose $r' > 0$ such that $r > r'$ and $X \subset \Delta(a, r')$. Let $\{f_1, f_2, \ldots, f_N\} \subset A_X^{D_1}$ be a maximal ε-distinguishable set with $N = M_\varepsilon(A_X^{D_1})$. By Lemma 4.9 there exist a positive integer k and polynomials $\{p_1, p_2, \ldots, p_N\}$ of degree k such that

$$\sup_{z \in X} |f_j(z) - p_j(z)| < \varepsilon/3 \qquad \text{for } j = 1, 2, \ldots, N$$

and $k \leq L \log \frac{1}{\varepsilon}$, where L depends only on R and r'. Then polynomials $\{p_1, p_2, \ldots, p_N\}$ are $\varepsilon/3$-distinguishable. By Lemma 4.10, polynomials

$$q_j = \left(\frac{r}{R}\right)^k p_j$$

are bounded on D_2 by 1. There exist positive constants c, λ, which depend only on R, r and r', but not on ε, such that polynomials $\{q_1, q_2, \ldots, q_N\}$ are δ-distinguishable (as points in $A_X^{D_2}$), where $\delta = c\varepsilon^\lambda$. Hence

$$N_\delta(A_X^{D_2}) \leq M_\delta(A_X^{D_2}) \leq N_\varepsilon(A_X^{D_1}), \tag{4.7}$$

and $\Psi(X, D_1) = \Psi(X, D_2)$.

If $D_1 = \Delta(a, r)$ and D_2 is an arbitrary bounded domain containing D_1, then there exists $R > 0$, such that $D_3 = \Delta(a, R) \supset D_2$. In this case the theorem follows from the inequalities

$$\Psi(X, D_1) \geq \Psi(X, D_2) \geq \Psi(X, D_3) = \Psi(X, D_1).$$

Now consider the general case. Let polydisks $\Delta_1, \Delta_2, \ldots, \Delta_s \subset D_1$ form an open cover of X. There exist compact sets $X_1, X_2, \ldots X_s$ such that $X_j \subset \Delta_j$ for $j = 1, 2, \ldots, s$ and $\bigcup X_j = X$. Then

$$\Psi(X_j, D_1) = \Psi(X_j, \Delta_j) = \Psi(X_j, D_2), \qquad \text{for } j = 1, 2, \ldots, s.$$

The theorem follows now from Lemma 4.8. □

Example 4.12. Let $X = \overline{\Delta(0, r)}$. Let $R > r$ and $D = \Delta(0, R)$. Kolmogorov [28] (see also [30]) showed that

$$\mathscr{H}_\varepsilon(A_X^D) = C(n, r, R)\left(\log\frac{1}{\varepsilon}\right)^{n+1} + O\left(\left(\log\frac{1}{\varepsilon}\right)^n \log\log\frac{1}{\varepsilon}\right). \tag{4.8}$$

Therefore $\Psi(X, D) = n$ and $\mathscr{K}$-dim $X = n$.

Theorem 4.13. *Let X be a compact subset of $\mathbb{C}^n$. The Kolmogorov dimension $\mathcal{K}$-dim X satisfies the following properties.*

(1) $0 \le \mathcal{K}\text{-}dim\, X \le n$.

(2) $\mathcal{K}\text{-}dim\, \{z\} = 0$.

(3) $\mathcal{K}\text{-}dim\, \widehat{X} = \mathcal{K}\text{-}dim\, X$.

(4) If $Y \subset X$ *then* $\mathcal{K}\text{-}dim\, Y \le \mathcal{K}\text{-}dim\, X$.

(5) If $\{X_j\}$ *is a finite family of compact subsets of* $\mathbb{C}^n$, *then*

$$\mathcal{K}\text{-}dim \bigcup_{j=1}^{m} X_j = \max\{\mathcal{K}\text{-}dim\, X_j : j = 1, \ldots, m\}.$$

(6) If $D \subset \mathbb{C}^n$ *is a domain,* $X \subset D$ *and* $\phi : D \to \mathbb{C}^k$ *is a holomorphic map, then* $\mathcal{K}\text{-}dim\, \phi(X) \le \mathcal{K}\text{-}dim\, X$.

(7) If $\mathcal{K}\text{-}dim\, X < n$, *then X is pluripolar.*

Remark 4.14. Property (5) does not hold for countable unions. There exists a countable compact set X such that $n = \mathcal{K}$-dim X (see Example 4.17). Such set X also provides a counterexample to the converse of (7).

Proof. Properties (4) and (6) follow immediately from the definition. Property (2) follows from Lemma 4.7. Lemma 4.8 implies (5). The inequality $\mathcal{K}$-dim $X \ge 0$ immediately follows from the definition. From (4.8) follows that Kolmogorov dimension of a closed polydisk equal n. Therefore (4) implies the second part of (1).

To show (3) consider a Runge domain D containing X. Let $W = \widehat{X}$. Then A_X^D and A_W^D are isometric, hence $\Psi(X, D) = \Psi(\widehat{X}, D)$ and (3) follows.

The property (7) follows from the following theorem and Lemma 4.18. □

Theorem 4.15. *Let X be a compact subset of $\mathbb{C}^n$, such that $\mathcal{K}$-dim $X = s < n$. Then for every $1 < h < \frac{n}{s}$ and every $N > N_0 = N_0(h)$ there exists a non-constant polynomial $P \in \mathbb{Z}[z_1, z_2, \ldots, z_n]$, $\deg P \le N$ with coefficients bounded by $\exp(N^h)$, such that*

$$\sup\{|P(z)| : z \in X\} < \exp(-N^h). \tag{4.9}$$

Proof. The result follows from Dirichlet principle. Let $D = \Delta(0,R)$ be a polydisk containing X. Assume that $R > 1$. Let $\varepsilon = \frac{1}{2}exp(-2N^h - N\log R - n\log N)$. Choose t such that $s < t < \frac{n}{h}$. For large enough N there exists an ε-covering of A_X^D with cardinality $\le \exp\left\{(\log\frac{1}{\varepsilon})^{t+1}\right\}$.

Let $T = [exp(N^h)]$ (integer part of $exp(N^h)$). There are

$$M = T^{\binom{N+n}{n}}$$

polynomials of degree at most N with coefficients in $\{1,2,\dots T\}$. Let $\{p_1, p_2, \dots p_M\}$ be a list of all such polynomials. Clearly the polynomial

$$q_j = \frac{1}{N^n R^N \exp(N^h)} p_j$$

belongs to A_X^D. By our choice of t, $h(t+1) < n+h$, therefore there are more polynomials q_j than the cardinality of the ε-covering, so there are two polynomials, let say q_1 and q_2 such that

$$|q_1(z) - q_2(z)| \le 2\varepsilon \qquad \text{for every } z \in X.$$

Then $P = p_1 - p_2$ satisfies (4.9). □

Let $\mathscr{P}_N$ be the set of all polynomials (with complex coefficients) on $\mathbb{C}^n$ of the degree $\le N$, whose supremum on a unit polydisk $\Delta(0,1)$ is at least 1.

Corollary 4.16. *If $\mathcal{K}$-dim $X = s < n$, then for every $1 < h < \frac{n}{s}$ and every $N > N_0 = N_0(h)$ there exists polynomial $P \in \mathscr{P}_N$, such that*

$$\sup\{|P(z)| : z \in X\} < \exp(-N^h). \tag{4.10}$$

Corollary 4.16 may be used to bound Kolmogorov dimension from below.

Example 4.17. Given $0 < r < 1$ and a positive integer N there exists a finite set $X_{r,N} \subset \Delta(0,r)$, such that for any polynomial $P \in \mathscr{P}_N$, the following inequality holds

$$\max_{z \in X_{r,N}} |P(z)| \ge \frac{1}{2} r^N. \tag{4.11}$$

For example, if $\varepsilon = \frac{1-r}{2N} r^N$, then a maximal ε-distinguishable subset of $\Delta(0,r)$ satisfies condition (4.11). Let

$$X = \bigcup_{k=2}^{\infty} X_{1/k,k}.$$

Then X is compact and for any $N > 2$ and $P \in \mathscr{P}_N$

$$\sup_{z\in X}|P(z)| \geq \max_{z\in X_{1/N,N}} |P(z)| \geq \frac{1}{2}\left(\frac{1}{N}\right)^N .$$

Therefore by Corollary 4.16 $\mathscr{K}$-dim $X = n$.

To finish the proof of Theorem 4.13 we need the following well-known result.

Lemma 4.18. *Let X be a compact subset of $\mathbb{C}^n$. If there exists a sequence $\{a_k\}$, $a_k > 0$ and a family of polynomials $P_k \in \mathscr{P}_k$ such that*

$$\sup_{z\in X}|P_k(z)| \leq e^{-a_k}, \qquad \textit{and} \tag{4.12}$$

$$\lim \frac{a_k}{k} = \infty, \tag{4.13}$$

then X is pluripolar.

Proof. Let $v_k(z) = \frac{1}{a_k}\log P_k(z)$ and $v(z) = \limsup v_k(z)$. We will show that $v \geq -2/3$ on a dense set. Let $\zeta \in \mathbb{C}^n$ and $0 < \delta < 1$. Suppose that $\Delta(\zeta, R) \supset \Delta(0,1)$. We will show that there exists a nested sequence of closed polydisks $\Delta_m = \overline{\Delta(w_m, \delta_m)}$ with $\Delta_1 = \overline{\Delta(\zeta,\delta)}$, and an increasing sequence of positive integers $k_1 = 1 < k_2 < \cdots < k_m < \ldots$ such that $v_{k_m} \geq -2/3$ on Δ_m for $m > 1$. Given $\Delta = \Delta_m = \overline{\Delta(w_m, \delta_m)}$ by Lemma 4.10 for any given k there exists $w \in \Delta$ such that

$$|P_k(w)| \geq \left(\frac{\delta_m}{R+\delta}\right)^k . \tag{4.14}$$

Choose $k = k_{m+1} > k_m$ such that

$$\frac{a_k}{k} \geq 2\log\frac{R+\delta}{\delta_m} .$$

Then by (4.14) $v_k(w) \geq -1/2$. Choose $w_{m+1} = w$. Because the function v_k is continuous at w, there exists a closed polydisk $\Delta_{m+1} = \overline{\Delta(w_{m+1}, \delta_{m+1})} \subset \Delta_m$, such that $v_k \geq -2/3$ on Δ_{m+1}. Therefore $v \geq -2/3$ on a dense set. By (4.12), $v|_X \leq -1$ and so X is a negligible set. By [7], negligible sets are pluripolar and result follows. □

Remark 4.19. This lemma and the converse follow from Theorem 2.1 in [2].

4.4 Manifolds of Gevrey Class

In view of Theorem 4.15 and Theorem 4.13 (6), Theorems 4.2 and 4.1 are corollaries of Theorem 4.3. In this section we prove Theorem 4.3.

Let $M \subset \mathbb{C}^n$ be an m-dimensional totally real submanifold of Gevrey class G^s. Let $X \subset M$ be a compact subset. Fix $p \in M$. There exist holomorphic coordinates $(z, w) = (x+iy, w) \in \mathbb{C}^n$, $x, y \in \mathbb{R}^m$, $w \in \mathbb{C}^{n-m}$ near p, vanishing at p, real-valued functions of class G^s $h_1, h_2, \ldots, h_m$, and complex valued functions of class G^s $H_1, H_2, \ldots, H_{n-m}$ such that $h_1'(0) = h_2'(0) = \cdots = h_m'(0) = 0$, $H_1'(0) = H_2'(0) = \cdots = H_{n-m}'(0) = 0$, and locally

$$M = \left\{(x+iy, w) : y_j = h_j(x), w_k = H_k(x)\right\} . \tag{4.15}$$

For a smooth manifold the existence of such coordinates is well known (see, for example [5], Proposition 1.3.8). Note, that functions h_j and H_k are defined by Implicit Function Theorem, and so by [29] are of class G^s.

We fix such coordinates and choose r sufficiently small. In view of Theorem 4.13 (5), it is sufficient to prove Theorem 4.3 for $X \subset \Delta(p, r)$. Put $D = \Delta(p, 1)$. To estimate $\Psi(X, D)$ we will cover X by small balls, approximate functions in A_X^D by Taylor polynomials, and then replace within these polynomials terms w^λ and y^ν by Taylor polynomials of functions H^λ and h^ν. To estimate the Taylor coefficients for powers of functions of Gevrey class we need the following lemma.

Lemma 4.20. *If $f \in G^s(K)$ and $|f| \leq 1$ on K, then there exists a constant C such that for any positive integer k and any multi-index α the following inequality holds on K:*

$$|\partial^\alpha f^k| \leq C^{|\alpha|} \binom{\alpha + k - 1}{\alpha} (\alpha!)^s . \tag{4.16}$$

Recall, that $\alpha + k = (\alpha_1 + k, \alpha_2 + k, \ldots, \alpha_m + k)$.

Proof. We will argue by induction on k. Because $|f| \leq 1$, there exists a constant C, such that

$$|\partial^\alpha f| \leq C^{|\alpha|} (\alpha!)^s$$

and (4.16) holds for $k = 1$. Suppose (4.16) holds for $1, 2, \dots, k$, then

$$|\partial^\alpha f^{k+1}| = \left|\sum_{\nu\le\alpha}\binom{\alpha}{\nu}\partial^\nu f^k \partial^{\alpha-\nu} f\right| \le C^{|\alpha|}\sum_{\nu\le\alpha}\binom{\alpha}{\nu}\binom{\nu+k-1}{\nu}(\nu!)^s((\alpha-\nu)!)^s$$
$$= C^{|\alpha|}\sum_{\nu\le\alpha}\binom{\nu+k-1}{\nu}\binom{\alpha}{\nu}\nu!(\alpha-\nu)!(\nu!)^{s-1}((\alpha-\nu)!)^{s-1}$$
$$\le C^{|\alpha|}\sum_{\nu\le\alpha}\binom{\nu+k-1}{\nu}\alpha!(\alpha!)^{s-1} = C^{|\alpha|}\binom{\alpha+k}{\alpha}(\alpha!)^s$$

□

Remark 4.21. The same proof holds for the product of k different functions, provided that they satisfy the Gevrey class condition (2.1) with the same constant C_K.

Let $t > s \ge 1$ and N be a large integer, which will tend to infinity later. Fix positive $a < t - s$. Put $\delta = N^{1-t}$ and $\varepsilon = N^{-aN}$. We may cover X by fewer than $(1/\delta)^m$ balls of radius δ. Let Q be one of these balls and let K be the set of restrictions on Q of functions in A_X^D. We claim that any function f in K may be approximated by polynomials in $x_1, x_2, \dots, x_m$ of the degree $\le N$ with coefficients bounded by $C^N(N!)^{s-1}$ with error less than 2ε, where the constant C depends on X and r only. Let us show how the theorem follows from this claim. The real dimension of the space of polynomials of the degree $\le N$ is $T = 2\binom{N+m}{N}$. Consider in the T-dimensional space with the sup-norm $\mathbb{R}^T_\infty$ the ball B of a radius $C^N(N!)^{s-1}$. By Lemma 4.7,

$$\mathscr{H}_\varepsilon(B) \le 2\binom{N+m}{N}\log\left(\frac{C^N(N!)^{s-1}}{\varepsilon}+1\right) = O(N^{m+1}\log N)\,.$$

By the claim, an ε-covering of B generates 3ε-covering of K, therefore

$$\mathscr{H}_{3\varepsilon}(K) = O(N^{m+1}\log N)\,.$$

Then by (4.4),

$$\mathscr{H}_\varepsilon(A_X^D) = O\left(\left(\frac{1}{\delta}\right)^m N^{m+1}\log N\right) = O(N^{mt+1}\log N)\,. \tag{4.17}$$

Now we let N tend to infinity. By (4.17), $\mathscr{K}$-dim $X = \Psi(X, D) \le mt$. The only restriction imposed on t so far was $t > s$. Hence $\mathscr{K}$-dim $X \le ms$.

It remains to prove the claim. We approximate a function f in K in two steps. Consider the Taylor polynomial P of f centered at the center of the ball Q of the degree N. By Cauchy's formula

$$\sup_Q |f - P| < \frac{1}{1-r-\delta}\left(\frac{\delta}{1-r}\right)^N < \varepsilon$$

for sufficiently large N. Suppose $P(z,w) = \sum c_{\lambda\mu\nu} x^\lambda y^\mu w^\nu$. Because $f \in A_X^D$, we have $|c_{\lambda\mu\nu}| \leq 1$.

On the next step we approximate y^μ and w^ν by the Taylor polynomials of the degree N of h^μ and H^ν. Let (x_0, y_0, w_0) be the center of the ball Q. Let g be one of the functions $h_1, h_2, \dots h_m, H_1, H_2, \dots, H_{n-m}$ and $L \leq N$. Then by the Taylor formula

$$g^L(x_0+h) = \sum_{|\alpha|\leq N} \partial^\alpha g(x_0)\frac{h^\alpha}{\alpha!} + R_N(x,h)\,.$$

By Lemma 4.20 for $||h||_\infty < \delta$

$$|R_N(x,h)| \leq C^{N+1}\delta^N \sum_{|\alpha|=N+1} \binom{\alpha+N-1}{\alpha} (\alpha!)^{s-1}\,.$$

Therefore $\log|R_N(x,h)| = (s-t+o(1))N\log N$. For sufficiently large value of N, $\log|R_N(x,h)| < \varepsilon/2$ and the claim follows.

CHAPTER 5

ORBITS IN MAXIMAL IDEAL SPACE

5.1 Dimension of Orbits

Let $M = G/H$ be a homogeneous manifold and A be an invariant algebra. The group G acts on dual space A' and there exists a homogeneous CR structure on orbits. The maximal ideal space $\mathfrak{M}_A \subset A'$ is G-invariant. If $\phi \in \mathfrak{M}_A$, then $G \cdot \phi \subset \mathfrak{M}_A$. The dimension of the orbit $G \cdot \phi \subset \mathfrak{M}_A$ may be larger than $\dim M$.

Example 5.1. Let $G = SU(1,1)$ be Möbius group. Let $M = \partial\mathbb{U}$ be a boundary of a unit disk. Let A be a closure in $C(M)$ of an algebra of polynomials. Then $\mathfrak{M}_A = \overline{\mathbb{U}}$ and there are two orbits in $\mathfrak{M}_A$: M and $\mathbb{U}$.

Theorem 1.2 shows that for totally real invariant algebra A on a homogeneous space M "extra" dimension of an orbit in $\mathfrak{M}_A$ is always compensated by "extra" complex structure.

Proof of Theorem 1.2. Without lost of generality we may assume, that the algebra A separates points. Then A is an invariant totally-real algebra.

Let $\phi \in \mathfrak{M}_A$. By [21] Theorem (4.1.6), there exists compact $K \subset M$, such that $\phi \in \mathfrak{M}_{A_K}$, where A_K is a completion of the algebra $A|_K$. It means that the functional ϕ has a *representing measure* μ supported on K (see [19] section II.2). Let $X \subset M$ be a compact neighborhood of K. Then for sufficiently small neighborhood U of the identity element $e \in G$, for every $g \in U$ we have $g \cdot \phi \in \mathfrak{M}_{A_X}$.

Suppose that $n = \dim M$. Choose $s > 1$, such that $ns < n+1$. For example, we may choose $s = \frac{n+2}{n+1}$. The algebra A is totally-real, which means that for every point $p \in M$ there exist n smooth functions $f_1, f_2, \ldots, f_n \in A^\infty$, such that differentials $df_j(p)$ are $\mathbb{C}$-linearly independent. By Lemma 2.7 (see also Remark 2.8) we may choose functions

$f_1, f_2, \dots, f_n$ of Gevrey class G^s. The differentials $df_j(z)$ are $\mathbb{C}$-linearly independent for every z in a neighborhood U_p of p.

Suppose that $m = \dim G \cdot \phi - \text{CR-dim}\, G \cdot \phi$. As in proof of Lemma 2.10 there exists smooth functions $f_1, f_2, \dots, f_m$ with $\mathbb{C}$-linear independent differentials $df_j(\phi)$. Again, we may choose functions f_j in $G^S \cap A$.

Because X is compact, there exists a finite family of functions $f_1, f_2, \dots, f_N \in G^s \cap A$ and a neighborhood U_X of X, such that the following conditions hold.

1. For every $z \in U_X$, the dimension of the complex linear space spanned by differentials $df_j(z)$ is n.

2. The dimension of the complex linear space spanned by differentials $df_j(\phi)$ is m.

Consider a map $F : \mathfrak{M}_A \to \mathbb{C}^N$

$$F(\psi) = \left(f_1(\psi), f_2(\psi), \dots, f_N(\psi)\right). \tag{5.1}$$

Let $\iota : M \to \mathfrak{M}_A$ be the evaluation map. By the choice of $f_1, f_2, \dots, f_N$, the map $\Phi = F \circ \iota$ is a totally real immersion of Gevrey class G^s of U_X into $\mathbb{C}^N$. Adding functions if necessary, we may assume, that Φ is an embedding of U_X. By Theorem 4.3, $\mathcal{K}\text{-dim}\, \Phi(X) \leq ns < n+1$. Let U be a sufficiently small neighborhood U of the identity element $e \in G$, such that for every $g \in U$ we have $g \cdot \phi \in \mathfrak{M}_{A_X}$. By result of S. I. Pinchuk [35], $\mathcal{K}\text{-dim}\, F(U \cdot \phi) \geq m$. By the choice of U, $F(U \cdot \phi) \subset F(\mathfrak{M}_{A_X}) \subset \widehat{\Phi(X)}$. Therefore, by Theorem 4.13, properties (3) and (4)

$$m \leq \mathcal{K}\text{-dim}\, F((U \cdot \phi) \leq \mathcal{K}\text{-dim}\, \widehat{\Phi(X)} = \mathcal{K}\text{-dim}\, \Phi(X) \leq ns < n+1. \tag{5.2}$$

Hence $m \leq n$ as claimed. □

Let A be an invariant totally real algebra on a homogeneous manifold $M = G/H$.

Corollary 5.2. *Let the homogeneous CR structure on the orbit $G \cdot \phi$ for $\phi \in \mathfrak{M}_A$ be determined by Lie algebra $\mathfrak{q}$, then*

$$\dim_{\mathbb{C}} \mathfrak{q} \geq \dim_{\mathbb{R}} H. \tag{5.3}$$

Proof. By the definition of homogeneous CR structure

$$\dim G \cdot \phi - \text{CR-dim}\, G \cdot \phi = \dim_{\mathbb{R}} G - \dim_{\mathbb{C}} \mathfrak{q}.$$

Clearly

$$\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} G - \dim_{\mathbb{R}} H,$$

and the claim follows. □

Corollary 5.3. *There exists a neighborhood U of M in $\mathfrak{M}_A$, such that for every $\phi \in U$, if the homogeneous structure on the orbit $G \cdot \phi$ is determined by Lie algebra $\mathfrak{q}$, then*

$$\dim_{\mathbb{C}} \mathfrak{q} = \dim_{\mathbb{R}} H. \tag{5.4}$$

Proof. This follows immediately from the previous corollary and Lemma 2.10. □

5.2 Rigidity of Lie Algebras

From now on we assume that $M = G/H$ is a bounded symmetric domain. In this section we study invariant CR structures on orbits satisfying (5.4). For a functional $\phi \in A'$ denote $\mathfrak{q}_\phi$ the Lie algebra defining the homogeneous CR structure on the orbit $G \cdot \phi$. Let $\iota : M \to A'$ be the evaluation map $\iota(z) : f \to f(z)$. Put $k = \dim H$. Let W be a neighborhood of $\mathfrak{h}_{\mathbb{C}}$ in the Grassmanian $Gr_k(\mathfrak{g}_{\mathbb{C}})$ of k-dimensional complex subspaces of $\mathfrak{g}_{\mathbb{C}}$. According to Lemma 2.10, there exists a neighborhood $U \subset A'$ of $\iota(0)$, such that for every $\phi \in U$ with $\dim_{\mathbb{C}} \mathfrak{q}_\phi = k$ we have $\mathfrak{q}_\phi \in W$. By the result of R. W. Richardson [36], Proposition 12.2, the algebra $\mathfrak{h}_{\mathbb{C}}$ is *rigid* with respect to G. This means that there exists a neighborhood W of $\mathfrak{h}_{\mathbb{C}}$ in the Grassmanian $Gr_k(\mathfrak{g}_{\mathbb{C}})$ with the following property: Every complex Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{q} \in W$ is a conjugate of $\mathfrak{h}_{\mathbb{C}}$, i.e. there exists $s \in G_{\mathbb{C}}$, such that

$$\mathfrak{q} = \mathrm{Ad}(s)\mathfrak{h}_{\mathbb{C}}.$$

Here $G_{\mathbb{C}} \supset G$ is the complexification of G. This proves

Lemma 5.4. *There exists a neighborhood U of $\iota(0) \in A'$ with the following property. For every $\phi \in U$ with $\dim_{\mathbb{C}} \mathfrak{q}_\phi = \dim_{\mathbb{C}} \mathfrak{h}_{\mathbb{C}}$ there exists $s \in G_{\mathbb{C}}$, such that*

$$\mathfrak{q}_\phi = \mathrm{Ad}(s)\mathfrak{h}_{\mathbb{C}}. \tag{5.5}$$

5.3 Complex Crown

Consider the *complexification* $M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ of the bounded symmetric domain $M = G/H$. Note, that the group G acts on $M_{\mathbb{C}}$. Let $x_0 \in M_{\mathbb{C}}$ and $H_{\mathbb{C}}$ be the isotropy group of x_0. Then we may identify M with the orbit $G \cdot x_0 \subset M_{\mathbb{C}}$. The complex structure on $M_{\mathbb{C}}$ induces the homogeneous CR structure on orbits $G \cdot x$ in $M_{\mathbb{C}}$. This structure is *universal* in the following sense. Let $s \in G_{\mathbb{C}}$ and $x = s \cdot x_0$. The homogeneous CR structure on the orbit $G \cdot x$ is generated by the complex Lie algebra $\mathfrak{q} = \mathrm{Ad}(s)\mathfrak{h}_{\mathbb{C}}$. The isotropy group

$$G_x = \{g \in G : g \cdot x = x\} = G \cap sH_{\mathbb{C}}s^{-1}$$

is the analytic subgroup of G with Lie algebra $\mathfrak{g} \cap \mathfrak{q}$. By (2.2), the isotropy group of ϕ in $G\cdot\phi$ also has Lie algebra $\mathfrak{g}\cap\mathfrak{q}$. Therefore the map $F_\phi : G\cdot x \to G\cdot\phi$ given by $F_\phi(g\cdot x) = g\cdot\phi$ is a well-defined surjective CR map.

The complexification $M_{\mathbb{C}}$ is too large for our purpose. For example, in general the isotropy subgroup G_x is not compact. However for x sufficiently close to x_0 the isotropy subgroup is compact. The largest connected neighborhood Ω of $G\cdot x_0$ with the property, that the isotropy subgroup G_x for every $x \in \Omega$ is compact is called *the complex crown* of M. Detailed information about the complex crown and the action of G on the complexification $M_{\mathbb{C}}$ can be found in [1], [9], [17].

We need some basic information on *restricted root systems.* The book [23] contains all necessary information with complete proofs. Let $\mathfrak{g} = \mathfrak{h}+\mathfrak{m}$ be a Cartan decomposition of the real semisimple algebra $\mathfrak{g}$. Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{m}$. Then there exist a Cartan subalgebra of the form $\mathfrak{t} = \mathfrak{k} + \mathfrak{a}$ of $\mathfrak{g}$. The complexification $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the corresponding set of roots. Denote $\Delta(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}'$ the set of restrictions of roots on $\mathfrak{a}$. The root space decomposition for $\mathfrak{g}_{\mathbb{C}}$ induces the decomposition

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{l} + \sum_{\lambda\in\Delta(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\lambda, \tag{5.6}$$

where $\mathfrak{l}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. Every $\lambda \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ takes real value on $\mathfrak{a}$, so $\mathrm{ad}(\mathfrak{a})$ are simultaneously diagonalizable over $\mathbb{R}$. The spaces $\mathfrak{g}_\lambda$ are the joint eigenspaces.

The Killing form $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$ induces a positive definite bilinear form on a dual space $\mathfrak{a}'$, also denoted $\langle\cdot,\cdot\rangle$. For $\lambda \in \mathfrak{a}'$ define H^λ by $\langle X, H^\lambda\rangle = \lambda(X)$. The bilinear form on $\mathfrak{a}'$ is defined by $\langle\lambda,\mu\rangle = \langle H^\lambda, H^\mu\rangle$. The restricted root system has the following properties.

1. Restricted roots $\Delta(\mathfrak{g}, \mathfrak{a})$ span the vector space $\mathfrak{a}'$.

2. If $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$ then the reflection

$$\alpha \to \alpha - \frac{2\langle\alpha,\lambda\rangle}{\langle\lambda,\lambda\rangle}\lambda$$

   preserves the restricted root system $\Delta(\mathfrak{g}, \mathfrak{a})$.

3. For $\lambda, \mu \in \Delta(\mathfrak{g}, \mathfrak{a})$ the number $2\frac{\langle\lambda,\mu\rangle}{\langle\lambda,\lambda\rangle}$ is integer.

For computations it is convenient to consider

$$h^\lambda = \frac{2H^\lambda}{\langle H^\lambda, H^\lambda\rangle}.$$

For every reduced root λ there exist bases $\{E_1^\lambda, E_2^\lambda, \dots, E_k^\lambda\}$ of $\mathfrak{g}_\lambda$ and $\{E_1^{-\lambda}, E_2^{-\lambda}, \dots, E_k^{-\lambda}\}$ of $\mathfrak{g}_{-\lambda}$, such that for every $j = 1, 2, \dots, k$

1. The vectors $E_j^\lambda, E_j^{-\lambda}, h^\lambda$ forms an $\mathfrak{sl}_2$ triple, i.e.
$$[h^\lambda, E_j^\lambda] = 2E_j^\lambda, \quad [h^\lambda, E_j^{-\lambda}] = -2E_j^\lambda, \quad \text{and} \quad [E_j^\lambda, E_j^{-\lambda}] = h^\lambda.$$
2. The vectors $X_j^\lambda = E_j^\lambda - E_j^{-\lambda}$ are in $\mathfrak{h}$ and the vectors $Y_j^\lambda = E_j^\lambda + E_j^{-\lambda}$ are in $\mathfrak{m}$.

Then we have
$$[X_j^\lambda, Y_j^\lambda] = 2h^\lambda, \quad [h^\lambda, X_j^\mu] = \mu(h^\lambda) Y_j^\mu, \quad \text{and} \quad [h^\lambda, Y_j^\mu] = \mu(h^\lambda) X_j^\mu \tag{5.7}$$

We need the following elementary fact about the complex crown. Let $\omega \ni 0$ be an open set in $\mathfrak{a}$ defined by
$$\omega = \left\{\xi \in \mathfrak{a} : |\lambda(\xi)| < \frac{\pi}{2} \quad \text{for all } \lambda \in \Delta(\mathfrak{g}, \mathfrak{a})\right\}. \tag{5.8}$$
The complex crown is defined by $\Omega = G\exp(i\omega) \cdot x_0$.

We can summarize this discussion as follows.

Proposition 5.5. *There exists a neighborhood U of $\iota(0) \in A'$ with the following property. For every $\phi \in U$ with $\dim_{\mathbb{C}} \mathfrak{q}_\phi = \dim_{\mathbb{C}} \mathfrak{h}_{\mathbb{C}}$ there exist $\xi \in \omega$ and a surjective CR map $F_\phi : G\exp(i\xi) \cdot x_0 \to G \cdot \phi$. The map F_ϕ is a local isomorphism.*

Proof. Choose U in Lemma (5.4), such that $s \cdot x_0 \in \Omega$. Then $s \cdot x_0 = g_0 \exp(i\xi) \cdot x_0$, where $g_0 \in G$ and $\xi \in \omega$. Put $F_\phi(g\exp(i\xi) \cdot x_0) = gg_0^{-1} \cdot \phi$. As above, the map F_ϕ is a well-defined surjective CR map. Because $\dim G \cdot \phi = \dim G\exp(i\xi) \cdot x_0$, this map is a local isomorphism. □

5.4 Levi Form and Holomorphic Extension

In this section we are using the Boggess-Polking theorem on holomorphic extension of CR functions to show that every CR function on $G \cdot \exp(i\xi) \cdot x_0$ for *regular* ξ extends to a holomorphic function on an open set $G \cdot \exp(iW) \cdot x_0$, where $W \subset \mathfrak{a}$ and $\xi \in \overline{W}$.

Let $(S, \mathcal{V})$ be a CR manifold. Recall, that $\mathcal{V}$ is a subbundle of the complexified tangent bundle $\mathbb{C}T(S)$. For $p \in S$, let
$$\pi_p : \mathbb{C}T_p(S) \to \mathbb{C}T_p(S) \,/\, (\mathcal{V}_p \oplus \overline{\mathcal{V}}_p)$$
be the projection map.

Definition 5.6. The intrinsic vector-valued Levi form of S at a point p is the map

$$L_p : \mathcal{V}_p \to \mathbb{C}T_p(S) \,/\, (\mathcal{V}_p \oplus \overline{\mathcal{V}}_p)$$

defined by

$$L_p(Z_p) = \frac{1}{2i}\pi_p[\overline{Z}, Z],$$

where $Z \in \mathcal{V}$ is an arbitrary extension of Z_p to a vector field.

Simple computations (see [8], section 10.1) show that L_p is a well-defined real valued form.

Lemma 5.7. *On a homogeneous CR manifold* $(G/H, \mathfrak{q})$, *where* H *is the isotropy group of* p, *the intrinsic Levi form at* p *is the map*

$$L_p : \mathfrak{q} \to \mathfrak{g} \,/\, (\mathfrak{g} \cap (\mathfrak{q} + \overline{\mathfrak{q}}))$$

given by

$$L_p(Z) = -\frac{1}{2i}\pi_p[\overline{Z}, Z], \tag{5.9}$$

where $Z \in \mathfrak{q}$, *and the bracket* $[\overline{Z}, Z]$ *is the Lie algebra bracket operation.*

Proof. Let $Z = X + iY$, where $X, Y \in \mathfrak{g}$. Extend X and Y to the right-invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$ on G/H. Then $\widetilde{Z}$ is in $\mathcal{V}$ and formula follows from $[\widetilde{X}, \widetilde{Y}]_p = -[X, Y]$ (see [23], Lemma 3.5, Chapter II). □

If S a CR submanifold of a Hermitian manifold X [1] then it is more convenient to work with the extrinsic Levi form. Recall, that the complex tangent space $H_p(S)$ is the real part of $\mathcal{V}_p$. If J is the complex structure on X, then $H_p(S) = T_p(S) \cap JT_p(S)$. Let $X_p(S)$ be the orthogonal complement of $H_p(S)$ in $T_p(S)$. Let $\pi_p : T_p(S) \to X_p(S)$ be the orthogonal projection. Let $N_p(S)$ be the orthogonal complement to $T_p(S)$ in $T_p(X)$.

Definition 5.8. The *extrinsic vector-valued Levi form* of S at a point p is the map

$$\widehat{L}_p : \mathcal{V} \to N_p(S)$$

given by

$$\widehat{L}_p(Z_p) = -J\pi_p\left(\frac{1}{2i}[\overline{Z}, Z]\right).$$

[1] A Hermitian manifold is a complex manifold with a Riemannian metric that preserves the complex structure.

Definition 5.9. The *Levi cone* Γ_p is the convex hall of the image of $\widehat{L}$ in $N_p(S)$. Clearly Γ_p is a convex cone.

Definition 5.10. An analytic disc in X is a continuous map $T : \overline{\mathbb{U}} \to X$ which is holomorphic on $\mathbb{U}$. We say that the disc T attached to a manifold $S \subset X$ if T maps the circle $\partial\mathbb{U}$ to S.

The Bogges-Polking theorem asserts, that if the Levi cone $\Gamma_p \subset N_p(S)$ has non-empty interior, then for every smaller cone $\Gamma < \Gamma_p$ there exist a neighborhood ω of p and a wedge W with a tangent cone Γ such that each point in $W \cap \omega$ is contained in the image of an analytic disc attached to S. Locally we may identify X with $\mathbb{C}^n$ and consider the wedge $S + \Gamma$.

To apply this result for an orbit $G\exp(i\xi) \cdot x_0$ in the complex crown Ω, we need to find the Levi cone. We are using notations from Section 5.3. Let $a = \exp(i\xi)$. By homogeneity it is enough to find the Levi cone at $x = a \cdot x_0$. Choose an ordering on the set of restricted roots, such that $\lambda(\xi) \geq 0$ for every positive root λ. Such ordering is not unique if $\lambda(\xi) = 0$ for some root λ. Let $\Delta^+(\mathfrak{g}, \mathfrak{a})$ be the set of all positive roots. Computations in [17], Section 10.4, shows that for $\xi \in \omega$, the homogeneous CR structure on $Ga \cdot x_0$ is generated by complex Lie algebra spanned by vectors $Z_j^\lambda = \mathrm{Ad}(a)Y_j^\lambda = \cos\lambda(\xi)Y_j^\lambda + i\sin\lambda(\xi)X_j^\lambda$, where $\lambda \in \Delta^+(\mathfrak{g}, \mathfrak{a})$. By Lemma 5.7 we have

$$\widehat{L}(Z_j^\lambda) = -2\cos\lambda(\xi)\sin\lambda(\xi)ih^\lambda. \tag{5.10}$$

By (5.8), $0 \leq \lambda(\xi) < \frac{\pi}{2}$. Hence, Levi cone contains vectors $-iH^\lambda$ for all $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$, such that $\lambda(\xi) > 0$.

Definition 5.11. A vector $\xi \in \mathfrak{a}$ is called *regular*, if $\lambda\xi \neq 0$ for all $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$. The set of regular vectors is a complement to a finitely many hyperplanes.

Lemma 5.12. *Let $M = G/H$ be a bounded symmetric domain. Let $\mathfrak{g}$ be a Lie algebra of G. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a Cartan decomposition and let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{m}$. Suppose that $\xi \in \omega$ is a regular vector. Let $\mathcal{W}$ be the Weyl group. Let $L \subset \mathfrak{a}$ be the interior of the convex hull of $\mathcal{W} \cdot \xi$ in $\mathfrak{a}$. Then for each smooth CR function on $S = G \cdot \exp(i\xi)$ there exist a unique function F holomorphic on $G \cdot L$ and continuous on $G \cdot L \cup S$, with $F = f$ on S.*

Proof. For a regular vector $\xi \in \omega$ the Levi cone $\Gamma_p(S)$ at $p = \exp(i\xi) \cdot x_0)$ contains $-iH^\lambda$ for every $\lambda \in \Delta^+(\mathfrak{g}, \mathfrak{a})$. By Bogges-Polking Theorem there exists a unique holomorphic extension F of f on some wedge $G \cdot W$. Each of this functions F is holomorphic on the holomorphic hull of $G \cdot W$. By Theorem 6.3.1 in [17] (see also [9]) this hull contains $G \cdot L$. □

We now return to the study of the maximal ideal space $\mathfrak{M}_A$ of an invariant algebra on a bounded domain. Suppose that $\mathfrak{M}_A \neq \imath(M)$, where $\imath$ is the evaluation map. Choose a neighborhood U of $\imath((0)$ as in Proposition 5.5. By the Shilov Idempotent Theorem, the maximal ideal space is connected. Therefore there exist $\phi \in U \cap \mathfrak{M}_A$ such that $\phi \ni \imath(M)$. Then there exists $\xi \in \omega$ and a CR map from $G\exp(i\xi) \cdot x_0$ onto $G \cdot \phi$. Suppose, that ξ is regular. For every $f \in A$, the function $\tilde{f} = f \circ F$ is a CR function on $G\exp(i\xi) \cdot x_0$. By Lemma 5.12, the function $\tilde{f}$ has the holomorphic extension on a neighborhood W of $G \cdot x_0$ in the complex crown Ω. The map F can be extended to a map $W \to \mathfrak{M}_A$ by

$$F(x) : f \to \tilde{f}(x).$$

This map is a locally one-to-one. Under these assumptions the maximal ideal space contains a manifold of the dimension $2\dim M$. Theorem 4.3 shows that this is the maximal possible dimension of the maximal ideal space. The author is unaware of any examples of the point-separating subalgebras of $C(M)$, not necessary invariant, having such large maximal ideal space. However, the author failed to prove that such subalgebras does not exist.

It is still an open problem, if the Rudin's classification of invariant algebras on the unit ball [38] can be extended to bounded symmetric domains, or is it the artifact of the rank-one case.

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BIBLIOGRAPHY

[1] D. N. Akhiezer and S. G. Gindikin. On Stein extensions of real symmetric spaces. *Math. Ann.*, 286(1-3):1–12, 1990.

[2] H. J. Alexander and B. A. Taylor. Comparison of two capacities in $\mathbf{C}^n$. *Math. Z.*, 186(3):407–417, 1984.

[3] A. Altomani and C. Medori. On homogeneous CR manifolds and their CR algebras. *Int. J. Geom. Methods Mod. Phys.*, 3(5-6):1199–1214, 2006.

[4] K. I. Babenko. On the entropy of a class of analytic functions. *Nauchn. Dokl. Vyssh. Shkol. Ser. Fiz.-Mat. Nauk*, (2):9–16, 1958.

[5] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild. *Real submanifolds in complex space and their mappings*, volume 47 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1999.

[6] E. Bedford. The operator $(dd^c)^n$ on complex spaces. In *Seminar Pierre Lelong-Henri Skoda (Analysis), 1980/1981, and Colloquium at Wimereux, May 1981*, volume 919 of *Lecture Notes in Math.*, pages 294–323. Springer, Berlin, 1982.

[7] E. Bedford and B. A. Taylor. A new capacity for plurisubharmonic functions. *Acta Math.*, 149(1-2):1–40, 1982.

[8] A. Boggess. *CR manifolds and the tangential Cauchy-Riemann complex*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991.

[9] D. Burns, S. Halverscheid, and R. Hind. The geometry of Grauert tubes and complexification of symmetric spaces. *Duke Math. J.*, 118(3):465–491, 2003.

[10] E. Cartan. Sur la détermination d'un système orthogonal complet dans un espace de riemann symétrique clos. *Rend. Circ. Mat. Palermo*, 53:217–252, 1929.

[11] D. Coman, N. Levenberg, and E. A. Poletsky. Quasianalyticity and pluripolarity. *J. Amer. Math. Soc.*, 18(2):239–252, 2005.

[12] K. de Leeuw and H. Mirkil. Rotation-invariant algebras on the n-sphere. *Duke Math. J.*, 30:667–672, 1963.

[13] K. Diederich and J. E. Fornæss. A smooth curve in $\mathbf{C}^2$ which is not a pluripolar set. *Duke Math. J.*, 49(4):931–936, 1982.

[14] V. D. Erohin. Asymptotic theory of the ε-entropy of analytic functions. *Dokl. Akad. Nauk SSSR*, 120:949–952, 1958.

[15] V. D. Erohin. Best linear approximation of functions analytically continuable from a given continuum to a given region. *Uspehi Mat. Nauk*, 23(1 (139)):91–132, 1968.

[16] O. Eroshkin. Pluripolarity of manifolds. In *Banach spaces of analytic functions*, volume 454 of *Contemp. Math.*, pages 23–33. Amer. Math. Soc., Providence, RI, 2008.

[17] G. Fels, A. Huckleberry, and J. A. Wolf. *Cycle spaces of flag domains*, volume 245 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2006. A complex geometric viewpoint.

[18] S. D. Fisher and C. A. Micchelli. The n-width of sets of analytic functions. *Duke Math. J.*, 47(4):789–801, 1980.

[19] T. W. Gamelin. *Uniform algebras*. Prentice-Hall Inc., Englewood Cliffs, N. J., 1969.

[20] V. M. Gichev and I. A. Latypov. Polynomially convex orbits of compact Lie groups. *Transform. Groups*, 6(4):321–331, 2001.

[21] H. Goldmann. *Uniform Fréchet algebras*. North-Holland Publishing Co., Amsterdam, 1990.

[22] E. L. Grinberg. Moebius-invariant function algebras on bounded symmetric domains in $\mathbb{C}^n$. unpublished, 1995.

[23] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Academic Press Inc., New York, 1978.

[24] L. Hörmander. *The analysis of linear partial differential operators. I.* Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[25] L. Hörmander. *An introduction to complex analysis in several variables.* North-Holland Publishing Co., Amsterdam, third edition, 1990.

[26] L. Hörmander and J. Wermer. Uniform approximation on compact sets in C^n. *Math. Scand.*, 23:5–21 (1969), 1968.

[27] H. T. Kaptanoğlu. Möbius-invariant spaces and algebras in polydiscs. *Indiana Univ. Math. J.*, 41(2):339–362, 1992.

[28] A. N. Kolmogorov and V. M. Tihomirov. ε-entropy and ε-capacity of sets in functional space. *Amer. Math. Soc. Transl. (2)*, 17:277–364, 1961.

[29] H. Komatsu. The implicit function theorem for ultradifferentiable mappings. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(3):69–72, 1979.

[30] G. G. Lorentz, M. v. Golitschek, and Y. Makovoz. *Constructive approximation.* Springer-Verlag, Berlin, 1996. Advanced problems.

[31] A. Nagel and W. Rudin. Moebius-invariant function spaces on balls and spheres. *Duke Math. J.*, 43(4):841–865, 1976.

[32] E. Nelson. Analytic vectors. *Ann. of Math. (2)*, 70:572–615, 1959.

[33] R. Nirenberg and J. Wells. Approximation theorems on differentiable submanifolds of a complex manifold. *Trans. Amer. Math. Soc.*, 142:15–35, 1969.

[34] S. Nivoche. Proof of a conjecture of Zahariuta concerning a problem of Kolmogorov on the ϵ-entropy. *Invent. Math.*, 158(2):413–450, 2004.

[35] S. I. Pinčuk. A boundary uniqueness theorem for holomorphic functions of several complex variables. *Mat. Zametki*, 15:205–212, 1974.

[36] R. W. Richardson, Jr. A rigidity theorem for subalgebras of Lie and associative algebras. *Illinois J. Math.*, 11:92–110, 1967.

[37] W. Rudin. *Function theory in the unit ball of $\mathbf{C}^n$.* Springer-Verlag, New York, 1980.

[38] W. Rudin. Moebius-invariant algebras in balls. *Ann. Inst. Fourier (Grenoble)*, 33(2):19–41, 1983.

[39] A. Sadullaev. A boundary uniqueness theorem in $\mathbf{C}^n$. *Mat. Sb. (N.S.)*, 101(143)(4):568–583, 639, 1976.

[40] N. Shcherbina. Pluripolar graphs are holomorphic. *Acta Math.*, 194(2):203–216, 2005.

[41] M. Waldschmidt. *Diophantine approximation on linear algebraic groups.* Springer-Verlag, Berlin, 2000.

[42] H. Widom. Rational approximation and n-dimensional diameter. *J. Approximation Theory*, 5:343–361, 1972.

[43] J. A. Wolf. Self adjoint function spaces on Riemannian symmetric manifolds. *Trans. Amer. Math. Soc.*, 113:299–315, 1964.

[44] J. A. Wolf. Fine structure of Hermitian symmetric spaces. In *Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970)*, pages 271–357. Dekker, New York, 1972.

[45] V. P. Zahariuta. *Spaces of analytic functions and maximal plurisubharmonic functions.* Doc. Sci. Thesis, Rostov-on-Don, 1984.