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Planar algebra of families of subfactors

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Planar algebra of families of subfactors

Abstract
The standard invariant is an object that classifies a fairly general class of subfactors and was reformulated by Vaughan Jones as 'planar algebras'. We compute the planar algebra of two families of subfactors arising out of actions of finitely generated groups on a II 1 factor in an attempt to see whether this class can be enlarged. This is a joint work with Dietmar Bisch and Shamindra Ghosh.

Keywords
Mathematics

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PLANAR ALGEBRA OF FAMILIES OF SUBFACTORS

BY

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DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

September 2008
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DEDICATION

To the memory of Bhutum
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ABSTRACT

PLANAR ALGEBRA OF FAMILIES OF SUBFACTORS

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The standard invariant is an object that classifies a fairly general class of subfactors and was reformulated by Vaughan Jones as 'planar algebras'. We compute the planar algebra of two families of subfactors arising out of actions of finitely generated groups on a $II_1$ factor in an attempt to see whether this class can be enlarged. This is a joint work with Dietmar Bisch and Shamindra Ghosh.
Chapter 1

Introduction

The standard invariant is a mathematical object that captures a lot of information about a subfactor and can be portrayed in a number of seemingly quite different ways. For example, it has descriptions as a certain category of bimodules ([25], see also [3]), as a lattice of algebras ([26]), or as a planar algebra ([16]). Jones' planar algebra technology has become a very efficient tool to capture and analyze the standard invariant of a subfactor.

Our aim is to use planar algebra methods to investigate subfactor-theoretic properties like amenability, strong amenability, property $(T)$ etc. The hope is to determine a class of subfactors (possibly larger than the class of amenable subfactors) for which the standard invariant is a complete invariant. It is already known (see [26]) that the standard invariant classifies the amenable subfactors. To achieve this goal, we plan to investigate these properties for two well-known families of subfactors arising out of action of finitely generated groups. Finding the planar algebra seems to be the first step towards this goal, and this is what has been accomplished so far.

A brief chapter-wise summary of the contents of this thesis is as follows. Chapter 2 consists entirely of preliminaries. Section 1 gives a brief outline of subfactor theory. Section 2 is devoted to planar algebras. This is central to the understanding of the planar algebra techniques involved in the proofs given here.

Chapters 3 and 4 constitute the core of the thesis.

In Chapter 3 we find the planar algebra of the diagonal subfactor, which is defined as follows. Starting with a finite set $\{\theta_i\}_{i \in I}$ of automorphisms of a $II_1$ factor $N$ we
consider the inclusion $N \subset M_1(N)$ where an element $x \in N$ sits in $M_1(N)$ diagonally with the $i$-th diagonal element being given by $\theta_i(x)$. It turns out that the action of the planar tangles depends on the 3-cocycle obstruction to lifting the group generated by $\theta_i$ in $Out(N)$ to $Aut(N)$. When the obstruction is trivial, this planar algebra matches with Jones's example of planar algebra associated to finitely generated group.

In Chapter 4 we consider the subfactors constructed by composing the fixed-point subalgebra inclusion and the crossed product algebra inclusion with respect to the outer actions of two finite groups $H$ and $K$ on a $II_1$ factor $P$ - that is, $P^H \subset P \rtimes K$. They are referred as Bisch-Haagerup subfactors in the literature. Assuming that the group $G = \langle H, K \rangle_{Aut(P)}$ intersects trivially with the inner automorphisms there is a nice prescription for defining the standard invariant in terms of the two finite subgroups which is in fact a characterization, i.e., any subfactor $N \subset M$ with such a standard invariant must necessarily look like $P^H \subset P \rtimes K$ for an intermediate subfactor $P$ (on which there are outer actions of the groups $H$ and $K$). The planar algebra of such a subfactor turns out to be an example of IRF planar algebras (IRF stands for Interaction 'Round a Face), i.e., words in $G$ with entries coming from the two subgroups alternately whose product is the identity element of the group form a basis for planar algebra and the action of a tangle can be read off from the boundary of the faces in the tangle. Once we drop the assumption, the planar algebra becomes heavily dependent on the obstruction (which is again a 3-cocycle) that arises when one tries to lift $\langle H, K \rangle_{Out(P)}$ to $Aut(P)$. However, the exact IRF-structure is yet to be determined. After this is accomplished we hope to be able to identify a class of the Bisch-Haagerup subfactors, together with a prescribed action of the group, for which the planar algebra is a complete invariant and then check whether we get something outside the class of amenable subfactors. Eventually, this might lead to classification of these subfactors by looking at the cohomology of the appropriate group.
Chapter 2

Preliminaries

This chapter consists mainly of preliminaries. The first section covers the subfactor theory prerequisites. The second section is a brief overview of Planar Algebras.

2.1 Subfactors

In the first section of this chapter we recapitulate some basic facts in subfactor theory. Most of the material included here can be found in [20], [31] and [19].

2.1.1 $\text{II}_1$ factors

We briefly survey the theory of $\text{II}_1$ factors. In order to define $\text{II}_1$ factors we need to start from von Neumann algebras. A von Neumann algebra $M$ is a self-adjoint subalgebra of bounded linear operators on a Hilbert space which is closed in the topology of point-wise convergence. There is also an abstract definition of a von Neumann algebra which does not depend on its concrete realisation as operators on some Hilbert space. $M$ is said to be a von Neumann algebra if it is a $C^*$ algebra and there exists a Banach space, say $M_*$, such that $M$ is isometrically isomorphic as Banach space to the dual of $M_*$. Projection operators - i.e., elements of $M$ of the form $p = p^2 = p^*$ play a very important role in the theory of von Neumann algebras. The linear span of projections is a dense subspace, and in fact the classification of von Neumann algebras depend heavily on the properties of its projections.
Two projections $p$ and $q$ are said to be (Murray-von Neumann) equivalent - written $p \sim q$ - if there exists an element $u \in M$ such that $uu^* = p$ and $u^*u = q$. There is a natural ordering among projections given by containment of their ranges - i.e., $q \preceq p$ if $\text{ran}(q) \subseteq \text{ran}(p)$. The set of equivalence classes of projections in $M$, denoted by $\mathcal{P}(M)$, has the structure of a lattice.

A factor is a von Neumann algebra with trivial center. In a factor $M$, $\mathcal{P}(M)$ is a totally ordered set - given two projections $p$ and $q$, we can always find two other projections in their respective equivalence classes which can be compared.

There are two important properties of projections that need to be described at this point. A projection $p$ is said to be finite if for any projection $q$, we have $p \sim q \preceq p \Rightarrow q = p$. In other words, a finite projection is one which is not equivalent to any of its proper subprojections.

A projection is said to be minimal if it does not have a proper subprojection.

A factor is said to be of type $II_1$ if it does not contain minimal non-zero projections and if all its non-zero projections are finite.

Factors of type $II_1$ always come equipped with a normalized trace. In fact, there is something more - there is a unique linear functional $tr$ which is

(i) faithful, i.e., $tr(x^*x) \neq 0$ if $x \neq 0$,

(ii) normal, i.e., $\sigma$-weakly continuous ($\sigma$-weak topology is same as the weak * topology on $M$ gotten from $M_*$),

(iii) tracial, i.e., $tr(xy) = tr(yx)$,

(iv) and is a state, i.e., $tr(x^*x) \geq 0$ and $tr(1) = 1$.

This enables us to perform the GNS construction with respect to this trace. We denote the GNS Hilbert space by $L^2(M, tr)$.

In the rest of the thesis we consider only $II_1$ factors. An unital inclusion $N \subset M$ of $II_1$ factors is called a subfactor.
2.1.2 Conditional expectation

The GNS Hilbert space $L^2(M, tr)$ has an $M-M$ bimodule structure. By this we mean that $M$ acts on $L^2(M, tr)$ on the left as well as on the right, and the two actions commute. Let us denote the cyclic vector by $\Omega$. Then we know that the span of $M\Omega$ is dense in $L^2(M, tr)$. The left action is given by $\pi_l(x)\xi = x\xi$ whereas the right action is given by $\pi_r(x)\xi = Jx^*J\xi$ where $J : M \to M$ is the map taking $x\Omega$ to $x^*\Omega$.

The closed subspace spanned by $N\Omega$ can be naturally identified with $L^2(N, tr)$. Let $e_N$ denote the orthogonal projection of $L^2(M, tr)$ onto the subspace $L^2(N, tr)$. It is a fact that this restricts to a Banach space projection $E : M \to N$. This trace-preserving, $N-N$ bilinear map is known as the conditional expectation and satisfies $e_Nxe_N = E(x)e_N$ for all $x \in M$.

2.1.3 Modules over $M$ and their dimensions

Throughout this section we will assume that $M$ is a $II_1$ factor which admits a faithful, normal representation on a separable Hilbert space. We consider separable modules over $M$. It turns out (see § 2.2 of [19]) that there is a unique function $\dim_M : M$-modules $\to [0, \infty]$ which satisfies;

- $\dim_M(L^2(M)) = 1$.
- $\dim_M(\cdot)$ is additive under direct sums.
- For every $d \in [0, \infty]$, there is a unique $M$-module $\mathcal{H}_d$ such that $\dim_M(\mathcal{H}_d) = d$.
- For any $M$-module $\mathcal{H}$, if $\dim_M(\mathcal{H}) < \infty$ then $M'$ is also a $II_1$-factor and $\dim_M(\mathcal{H}) \dim_{M'}(\mathcal{H}) = 1$.

2.1.4 Index

If $N$ is a subfactor of a $II_1$ factor $M$, we will denote by $L^2(M)$ the GNS Hilbert space with respect to the unique trace $tr$ on $M$. The index of $N$ in $M$ is then defined by the expression

$$[M : N] = \dim_N(L^2(M))$$
There is a remarkable result of Jones [15], which proves that the only possible values of the index lie in the set \( \{4\cos^2\frac{n\pi}{6} : n = 3, 4, \ldots \} \cup [4, \infty] \) and further that for each member of the above set there is a subfactor with the given element as index value. For more on the properties of the index the reader may consult [19]. We will mainly be using the two properties listed below.

(a) If \( N \subseteq M \subseteq P \) are \( II_1 \) factors, then \([P : N] = [P : M] [M : N]\)

(b) \( [M : N] < \infty \Rightarrow N' \cap M \) is finite-dimensional.

### 2.1.5 Basic construction

Unless specified otherwise, we need not restrict ourselves to \( II_1 \) factors, and the assertions in this subsection will go through for \( N \subseteq M \), an inclusion of finite von Neumann algebras with a fixed faithful normal tracial state on \( M \). Define \( M_1 = \langle M, e_N \rangle \) = the von Neumann algebra generated by \( M \) and \( e_N \) in \( \mathcal{L}(L^2(M)) \). This way of obtaining a von Neumann algebra from the initial inclusion is called the basic construction. The following proposition is taken from [19] and points out some important features of the basic construction.

**Proposition 2.1.1.** With the same notation as in the previous subsections we have:

(i) \( e_N \in N' \)

(ii) \( N = M \cap \{e_N\}' \)

(iii) \( M_1 = JN'J \)

(iv) Further, if both \( N \) and \( M \) are \( II_1 \) factors, then

(a) \( M_1 \) is a \( II_1 \) factor iff \( [M : N] < \infty \); in this case, we have \([M_1 : M] = [M : N]\)

(b) \( tr_{M_1}(e_N) = [M : N]^{-1} \)

(c) \( E_M(e_N) = tr_{M_1}(e_N) \) - where \( E_M \) denotes the conditional expectation of \( M_1 \) onto \( M \).
In the case of II\_1 factors we can iterate this process. \( N \) and \( M \) will be denoted by \( M\_{-1} \) and \( M_0 \) respectively, for simplicity of the notation. So, \( M_1 = (M_0, e_1) \), where we write \( e_1 \) for the projection we denoted earlier by \( e_N \). Inductively, define \( M_{k+1} = (M_k, e_{k+1}) \), the result of applying the basic construction to the inclusion \( M_{k-1} \subseteq M_k \), and \( e_{k+1} \) denotes the projection implementing the trace-preserving conditional expectation of \( M_k \) onto \( M_{k-1} \). The tower of algebras and the sequence of projections now looks like:

\[
M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k \subseteq \cdots
\]

and will be referred as the \textit{tower of basic construction} of the initial subfactor.

\[2.1.6 \text{ Relative commutants}\]

We start with a finite index and depth two inclusion \( N \subset M \) of II\_1 factors. By properties (a) and (b) of the index mentioned in Subsection 2.1.4, all the algebras \( \{M'_i \cap M_j\}_{i,j \geq 1} \) are finite-dimensional. These are called the relative commutants of the subfactor.

The following is then a grid of finite-dimensional C\(^*\) algebras associated to the subfactor \( N \subset M \).

\[
N' \cap M \subseteq N' \cap M_1 \subseteq N' \cap M_2 \subseteq \cdots \subseteq N' \cap M_n \subseteq \cdots
\]

\[C = M' \cap M \subseteq M' \cap M_1 \subseteq M' \cap M_2 \subseteq \cdots \subseteq M' \cap M_n \subseteq \cdots\]

This grid is called the \textit{standard invariant} of the subfactor \( N \subset M \) and captures various informations about the initial inclusion. In fact there is a deep theorem due to Popa [27] which says that the isomorphism class of this grid is a complete invariant if one starts out with a sufficiently general class of subfactors which he calls 'strongly amenable'. We avoid such technicalities in the present context.

\[2.1.7 \text{ Depth}\]

As mentioned in subsection 2.1.5, it makes sense to talk about basic construction of finite dimensional C\(^*\) algebras. We say that \( N \subseteq M \) is of finite depth if \( \exists \ n \) for which \( N' \cap M_{n-1} \subseteq N' \cap M_n \subseteq N' \cap M_{n+1} \) is an instance of the basic construction. (It is a fact that \( N' \cap M_{n+1} \) always contains a copy of the basic construction of the inclusion
The smallest such integer \( n \), if it exists, is said to be the depth of the subfactor \( N \subseteq M \). If \( N \subseteq M \) is of depth \( n \) then any element of \( N' \cap M_{n+1} \) can be expressed as linear combination of elements of the form \( xe_ny \) where \( x, y \in N' \cap M_n \).

### 2.1.8 Extremality

The inclusion \( N \subseteq M \) is said to be extremal if the trace on \( N' \cap M \) obtained by restricting the two traces - one coming from \( M \) and the other from \( N' \) (in \( \mathcal{L}(L^2(M)) \)) coincide.

### 2.2 Planar Algebras

Planar algebras - originally introduced by Vaughan Jones to reformulate the standard invariant of finite index subfactors - first appeared in [16]. The operadic definition is discussed in [18].

#### 2.2.1 Basic definitions

In order to define planar algebras, we first need to introduce the concept of planar tangles which is central to this discussion.

Consider a picture consisting of:

(i) a disc

(ii) finitely many (possibly none) mutually disjoint numbered discs in the interior of the original disc

(iii) an even number of marked points on the boundary of each disc (internal or external) numbered clockwise

(iv) a collection of smooth disjoint curves - called strings - in the complement (in the external disc) of the interior of union of the internal discs such that:

(a) each of the strings is either closed, or joins a marked point on the boundary of a disc to another such point, meeting the boundary of the disc transversally
and each marked point on every disc must be the endpoint of exactly one string

(b) there is a chequerboard shading on the connected components of the complement (in the interior of the external disc) of the union of the internal discs and the strings in such a way that the component adjacent to both the first and the last marked points of the boundary of a disc or a disc with no marked points remains unshaded.

**Remark 2.2.2.** By the 'color' of a disc (internal or external), we will mean the non-negative integer which is half of the number of marked points on its boundary. We will choose a point on the boundary of every disc between its last and first marked points (moving clockwise) to indicate the clockwise numbering on the marked points; this point will be referred as '*' of the disc.

We also consider planar isotopy of such pictures which preserve the numbering of the internal discs as well as that of all their marked points.

**Definition 2.2.3.** The isotopy class of such a picture described above will be termed as a planar $k$-tangle where $k$ refers to the color of its external disc.

Unless otherwise mentioned, we will denote a tangle by $T$, the color of its external disc by $k_0(T)$ and assume that it has $b(T)$ internal discs of colors $k_1(T), k_2(T), \ldots, k_b(T)$ respectively. As a matter of notation, tangles will be given names including subscripts and a superscript. The subscripts indicate the colors of the internal discs in order and the superscript indicates the color of the tangle. Given the subscripts and the superscript, the entire shading scheme is determined and will not be indicated. If a tangle has only one internal disc we do not show its numbering.

There is a natural way to compose two tangles. This can be done only if the color of the one of the two tangles matches with one of the internal discs of the other.

**Definition 2.2.4.** Given a $k$-tangle $T$ having an internal $k_i$ disc, say $D_i$, and $k_i$-tangle $S$, one can define the $k$-tangle denoted $T \circ_{D_i} S$ first by isotoping $S$ so that its boundary, together with the numbered, marked points coincide with that of $D_i$ and then erasing the boundary of $D_i$. 
Remark 2.2.5. For $k > 0$, the planar $k$-tangle having a unique internal $k$-disc and strings going from marked points on the internal disc to marked points of the same number on the external disc is clearly the identity with respect to the composition defined above and will be denoted by $I_k^k$. For $k = 0$, $I_0^0$ is just an unshaded annulus.

We are now ready to define a planar algebra.

Definition 2.2.6. A planar algebra consists of

(i) a family $P = \{P_k : k \geq 0\}$ of vector spaces

(ii) an action $Z$ of the set of tangles on the vector spaces in the sense that for every $k_0 = k_0(T)$-tangle $T$ with internal discs $D_1(T), D_2(T), \cdots D_{b(T)}(T)$ of colors $k_1(T), k_2(T), \cdots k_{b(T)}(T)$, there is associated a linear map $Z_T : \otimes_{i=1}^{b(T)} P_{k_i(T)} \to P_{k_0(T)}$ (where the empty tensor product is interpreted as the underlying field $\mathbb{C}$ in case the tangle $T$ has no internal discs)

such that

(a) The action of the identity tangle is the identity map, in other words,

$$Z_{I_k^k} = \text{id}_{P_k} \forall k \geq 0$$

(b) The action of tangles is 'compatible with composition of tangles' in the following obvious manner. Fix $1 \leq i \leq b(T)$ and let $S$ be a $k_i(T)$-tangle such that the 'composed' tangle $\tilde{T} = T \circ_{D_i(T)} S$ makes sense; it is a $k_0(T)$-tangle where the $(b(T) + b(S) - 1)$ internal discs are numbered according to the following convention:

$$D_j(\tilde{T}) = \begin{cases} 
   D_j(T) & \text{if } 1 \leq j < i \\
   D_{j-i+1}(S) & \text{if } i \leq j \leq i + b(S) - 1 \\
   D_{j-b(S)+1}(T) & \text{if } i + b(S) \leq j \leq b(T) + b(S) - 1 
\end{cases}$$
In that case the associated maps satisfy the commutativity of the following diagram:

\[
\begin{array}{c}
(\otimes_{j=1}^{k-1} P_{k_j(T)}) \odot (\otimes_{j=1}^{b(S)} P_{k_j(S)}) \odot (\otimes_{j=i+1}^{k(T)} P_{k_j(T)}) & Z_T \downarrow \\
\text{id} \otimes Z_S \otimes \text{id} \downarrow & P_{k_0(T)} \\
(\otimes_{j=1}^{k(T)} P_{k_j(T)}) & Z_T \nearrow \\
\end{array}
\]

Suppose in the above situation \( S \) does not have any internal discs, i.e., \( b(S) = 0 \). Then we have the modified diagram:

\[
\begin{array}{c}
\otimes_{j \neq i} P_{k_j(T)} \simeq \downarrow & Z_T \downarrow \\
(\otimes_{j < i} P_{k_j(T)}) \otimes C \otimes (\otimes_{j > i} P_{k_j(T)}) & P_{k_0(T)} \\
\text{id} \otimes Z_S \otimes \text{id} \downarrow & \\
(\otimes_{j=1}^{k(T)} P_{k_j(T)}) & Z_T \nearrow \\
\end{array}
\]

We also require that the assignment \( T \mapsto Z_T \) is independent of the renumbering of the internal discs of \( T \). Formally, write \( b = b(T) \), let \( \sigma \in S_b \) be a permutation and let \( U_\sigma : V_1 \otimes \cdots \otimes V_b \rightarrow V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(b)} \) be defined by

\[
U_\sigma(\otimes_{i=1}^{b} v_i) = (\otimes_{i=1}^{b} v_{\sigma^{-1}(i)})
\]

Let \( \sigma(T) \) be the tangle which differs from \( T \) only in the numbering of its internal discs - the \( i \)-th disc of \( \sigma(T) \) being the \( \sigma^{-1}(i) \)-th disc of \( T \). Then, we must have

\[
Z_{\sigma^{-1}(T)} = Z_T \circ U_\sigma
\]

**2.2.2 Some useful tangles**

We go on to introduce some more important classes of tangles.

*The inclusion tangles:* For every \( k \geq 0 \), there is an associated \( k + 1 \)-tangle \( I_k^{k+1} \) with one internal disc of color \( k \). The associated maps \( Z_{I_k^{k+1}} : P_k \rightarrow P_{k+1} \) are indeed injective in case of planar algebras with non-zero 'modulus', a term we will define in the next subsection. These 'inclusion' tangles are formed by adding an extra vertical line at the right as illustrated in the picture.
The multiplication tangles: For each $k \geq 0$, consider the $k$-tangle $M_{k,k}^k$ with two internal $k$-discs as given in the following picture. This tangle equips $P_k$ with a multiplication given by

$$x_1 x_2 = Z_{M_{k,k}^k} (x_1 \otimes x_2)$$

It should be noted that $P_0$ is a commutative algebra.

The unit tangles: These are devoid of any internal discs and have $k$ strings going straight down in case $k > 0$, as shown in the picture. The unit tangle of color 0 is just the empty disc.

A pleasant verification shows that given a planar algebra $P$, each $P_k$ has the structure

Figure 2.2: Inclusion tangles: $I_0^1, I_2^3$

Figure 2.3: Multiplication tangles: $M_{0,0}^0, M_{2,2}^2$

Figure 2.4: The unit tangles: $1^0, 1^2, 1^3$
of an associative, unital algebra where the multiplication is given by $Z_{M_k}$ and the unit by $1_k = Z_{1^k}(1)$ (for instance, the fact that $1_k$ is the left identity for $P_k$ follows from the tangle equation $M_k \circ D_2 1^k = I_k^k$) and that the $Z_{I_{k+1}}$ are algebra homomorphisms from $P_k$ to $P_{k+1}$.

The conditional expectation tangles: We have two families of tangles $\{\mathcal{E}_{k+1}^k : k \geq 0\}$ and $\{(\mathcal{E}')_k : k \geq 1\}$ where $\mathcal{E}_{k+1}^k$ is a $k$-tangle with only one internal disc of color $k+1$, with the property that the $k$-th marked point is connected to the $k+1$-st point by a string. In the case of $(\mathcal{E}')_k$, the ‘capping’ is to the left instead of the right, and there is an extra strand at the left to ensure that the shading is done in a consistent manner.

![Figure 2.5: Conditional expectation tangles: $\mathcal{E}_1^0, \mathcal{E}_3^2$](image)

![Figure 2.6: Conditional expectation tangles: $(\mathcal{E}')_2^1, (\mathcal{E}')_3^3$](image)

Jones projection tangles: The Jones projection tangles $E_k$ are $k$-tangles with no internal discs, with the property that the first $k-2$ strings go down straight, the strings coming out from the $(k-1)$-th and the $k$-th points are connected (same for the $(k+1)$-th and $(k+2)$-th points).

Rotation tangles: For every $k \geq 2$, we have a $k$-tangle $R_k^k$ with one internal $k$-disc, and as the name suggests, the effect of applying the tangle is to shift the $*$ of the internal disc clockwise by two places. The cases $k = 2, 3$ have been illustrated in Figure 2.8.
2.2.3 Additional structure on planar algebras

Definition 2.2.7. A planar algebra is said to be finite-dimensional if

$$\dim P_k < \infty \ \forall k \geq 0.$$ 

Definition 2.2.8. A planar algebra is said to have modulus $\delta \in \mathbb{C} \setminus \{0\}$ if for any tangle $T$ having a contractible loop $l$ we have

$$Z_T = \delta Z_{\tilde{T}}$$

where $\tilde{T}$ is the tangle obtained from $T$ by removing the loop $l$.

Definition 2.2.9. A planar algebra is said to be connected if $\dim P_0 = 1 = \dim (\text{Ran}((\mathcal{E}')_1))$. Since they are unital $\mathbb{C}$-algebras, it follows that there exist unique algebra isomorphisms $P_0 \cong \mathbb{C}$ identifying $1_0$ with $1_{\mathbb{C}}$.

Remark 2.2.10. In a connected planar algebra with modulus $\delta$, one can define a linear functional $\tau$ on $P_k$ for every $k \geq 0$ as follows:

$$\tau(x) \cdot 1 = \delta^{-k} Z_{\nu_k}(x)$$
for all $x \in P_k$, where $tr^0_k$ is as in Figure 2.9. This can be easily checked to be a trace, i.e., $\tau(xy) = \tau(yx)$ for all $x, y \in P_k$. The multiplicative factor of $\delta^{-k}$ ensures that $\tau$ can be consistently defined over the whole filtered algebra associated to $P$ taking $1_P$ to 1.

![Figure 2.9: The trace tangle: $tr^0_3$](image)

We digress a little bit to introduce the notion of planar networks and partition functions. Planar networks are similar to planar tangles except that they do not have an external disc in their picture. More precisely, consider a picture comprising of:

(i) finitely many (possibly none) mutually disjoint colored discs that are numbered

(ii) a collection of smooth disjoint curves - called strings - in the complement of the interior of union of the discs such that:

(a) each of the strings is either closed, or joins a marked point on the boundary of a disc to another such point, meeting the boundary of the disc transversally and each marked point on every disc must be the endpoint of exactly one string

(b) there is a chequerboard shading on the connected components of the complement of the union of the discs and the strings in such a way that the component adjacent to both the first and the last marked points of the boundary of a disc or a disc with no marked points remains unshaded.

The isotopy class of such a picture described above will be termed as a positive (resp., negative) planar network depending on whether the unbounded region is shaded (resp., unshaded).

Note that one can associate a 0-tangle (resp., 1-tangle) $T_N$ to every positive (resp., negative) network $N$ by embedding it inside an empty 0-tangle (resp., the shaded region.
of the 1-tangle having no internal discs).

**Remark 2.2.11.** If $P$ is a connected planar algebra, then each network $N$ induces a multilinear linear functional $Z_N$ via $Z_{T_N} = Z_N \cdot 1$. This functional is called the 'partition function' of the planar algebra.

**Definition 2.2.12.** A connected planar algebra is said to be spherical if for every network $N$, the multilinear functional $Z_N$ is not just planar isotopy invariant but also an isotopy invariant of the network regarded as embedded on the 2-sphere.

We can define an involution on the set of tangles in the following way. Given a tangle $T$, the adjoint $T^*$ is the equivalence class of a picture obtained from a representative of $T$ by reflecting it along a non-intersecting line. (the $*$ on each of the discs of $T^*$ can be placed at the reflections of the $*$ of the corresponding discs of $T$.)

A planar algebra is said to be a $*$-planar algebra if there is an involution $*: P_n \to P_n$ such that for every $k_0$-tangle $T$ with $b$ internal discs of colors $k_1, k_2, \ldots, k_b$ and $x_i \in P_{k_i}$, we have,

$$Z_T(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*)$$  \hspace{1cm} (2.2.1)

**2.2.4 Relation to subfactors**

Among planar algebras, the ones that we will be interested in are the 'subfactor planar algebras', in the sense of the Definition below.

**Definition 2.2.13.** A finite-dimensional, spherical $*$-planar algebra $P$ is said to be a subfactor planar algebra provided $P$ is such that the pictorial trace $\tau$ defined above is a positive definite functional on $P_k$ for every $k$.

We now state the fundamental theorem of Jones [16] relating subfactors and subfactor planar algebras which is the key to all our results.

**Theorem 2.2.14.** Let

$$N \subset M = M_0 \subset M_1 \subset \cdots \subset M_k \subset \cdots$$

be the tower of the basic construction associated to an extremal subfactor with $[M:N] = \delta^2 < \infty$. Then there exists a unique subfactor planar algebra $P = P^{N\subset M}$ of modulus $\delta$ satisfying the following conditions:
(0) $P_k = N' \cap M_{k-1} \ \forall k \geq 1$ - where this is regarded as an equality of $*$-algebras which is consistent with the inclusions on the two sides;

(1) $Z_{E_{k+1}}(1) = \delta \ v_k \ \forall k \geq 1$;

(2) $Z_{(E')_k}(x) = \delta \ E_{M' \cap M_{k-1}}(x)$ for all $x \in N' \cap M_{k-1}, \ \forall k \geq 1$;

(3) $Z_{E_{k+1}}(x) = \delta \ E_{N' \cap M_{k-1}}(x)$ for all $x \in N' \cap M_k$; and this is required to hold for all $k \geq 0$, where for $k = 0$, the equation is interpreted as

$$Z_{E_0}(x) = \delta \ tr_M(x) \ \forall x \in N' \cap M.$$ 

Conversely, any subfactor planar algebra $P$ with modulus $\delta$ arises from an extremal subfactor of index $\delta^2$ in this fashion.

We note that one consequence of this theorem is that for $x \in P_k = N' \cap M_{k-1}$, the pictorial trace $\tau(x)$ agrees with the canonical trace $tr_{M_{k-1}}(x)$.

We should mention here that the fact that every (appropriate) planar algebra can be realized by a subfactor is a theorem of Popa proved in [26]. Recently Guionnet, Jones and Shlyakhtenko in [13] came up with an alternative proof. They used a graded algebra structure of a planar algebra and certain traces inspired by random matrix theory to construct $II_1$ factors and subfactors whose standard invariant realizes the given planar algebra.
Chapter 3

Diagonal subfactor

3.1 Basics of group cohomology

In this section, we recall the definition of a cocycle of a group which can be found in any standard text book.

Let $G$ be a group with $e$ as its identity. Define $C^n = \text{Fun}(G^n, S^1)$ = space of functions from $G^n$ to $S^1$ and $\partial^n : C^n \rightarrow C^{n+1}$ by

$$\partial^n(\phi)(g_1, \ldots, g_{n+1})$$
$$= \phi(g_2, \ldots, g_{n+1}) \phi^{-1}(g_1g_2, g_3, \ldots, g_{n+1}) \phi(g_1, g_2g_3, g_4, \ldots, g_{n+1}) \cdots$$
$$\cdots \phi^{(-1)}(g_1, \ldots, g_{n-1}, g_n) \phi^{(-1)}(g_1, \ldots, g_n)$$

It follows that $(\partial^{n+1} \circ \partial^n)(\cdot) = 1_{C^{n+2}}$ where 1 denoted the constant function 1. Denote $\ker(\partial)$ by $Z^n(G, S^1)$ (whose elements are called n-cocycles) and $\text{Im}(\partial^{n-1})$ by $B^n(G, S^1)$ (whose elements are called coboundaries). Note that $B^n(G, S^1) \subset Z^n(G, S^1)$.

In the thesis, we will be dealing with a 3-cocycle $\omega$. So, $\omega$ satisfies

$$\omega(g_1, g_2, g_3) \omega(g_1, g_2g_3, g_4) \omega(g_2, g_3, g_4) = \omega(g_1, g_2, g_3, g_4) \omega(g_1, g_2, g_3g_4)$$

(3.1.1)

The cocycle $\omega$ will be called normalized if $\omega(g_1, g_2, g_3) = 1$ whenever either of $g_1, g_2, g_3$ is $e$.

Any cocycle $\omega$ is coboundary equivalent to a normalized cocycle; in particular, $(\omega \cdot \partial^2(\phi))$ is a normalized 3-cocycle where $\phi \in C^2$ is defined as $\phi(g_1, g_2) = \omega(g_1, e, e) \omega(e, e, g_2)$
for all \( g_1, g_2 \in G \).

## 3.2 Finitely generated group planar algebra

In this section, we will abstractly define a planar algebra which, in Section 3.3, is shown to be isomorphic to the planar algebra of the diagonal subfactor.

Starting with a group \( G \) (possibly infinite) generated by a finite subset \( \{ g_i \}_{i \in I} \) (where \( I \) denotes a finite indexing set) and a normalized 3-cocycle \( \omega \in Z^3(G, S^1) \), we define the planar algebra \( P^{(g_i \in I)}|\omega \) \( (= P) \). Let \( e \) denote the identity of \( G \). The map \( \text{alt} \) is defined as

\[
\prod_{n \geq 0} I^n \ni i = (i_1, \ldots, i_n) \xrightarrow{\text{alt}} g_{i_1}^{-1} g_{i_2} \cdots g_{i_n}^{(-1)^n} = \text{alt}(i) \in G
\]

where \( \text{alt} \) of the empty multi-index is defined to be the identity element of the group.

### The vector spaces:

For \( n \geq 0 \), define \( P_n \) as

\[
P_n = \begin{cases} 
\mathbb{C}\{i \in I^{2n} : \text{alt}(i) = e\} & \text{if } n > 0 \\
\mathbb{C} & \text{if } n = 0.
\end{cases}
\]

### Action of tangles:

Let \( T \) be an \( n_0 \)-tangle having internal discs \( D_1, \ldots, D_b \) with colors \( n_1, \ldots, n_b \) respectively. A state \( \sigma \) on \( T \) is a map from \{strings in \( T \)\} to \( I \) such that \( \text{alt}(\sigma|_{\partial D_c}) = e \) for all \( 1 \leq c \leq b \) where \( \sigma|_{\partial D_c} \) denotes the element of \( I^{2n_c} \) obtained by reading the elements of \( I \) induced at the marked points on the boundary of \( D_c \) by the strings via the map \( \sigma \), starting from the first marked point and moving clockwise. This has been illustrated in Figure 3.1, where \( \text{alt}(\sigma|_{\partial D_1}) \) and \( \text{alt}(\sigma|_{\partial D_2}) \) are just the products \( g_{i_1}^{-1} g_{i_2} g_{i_3}^{-1} g_{i_4} g_{i_5}^{-1} \) and \( g_{i_1}^{-1} g_{i_6} g_{i_7}^{-1} g_{i_8} \), respectively, and are thus required to be the identity element. It is a consequence that \( \text{alt}(\sigma|_{\partial D_0}) = g_{i_8}^{-1} g_{i_6} g_{i_7}^{-1} g_{i_4} g_{i_5}^{-1} g_{i_9} = e \) holds for the external disc.

Let \( S(T) \) denote the states on \( T \). In order to define the action \( Z_T \) of \( T \), it is enough to define the coefficient \( \langle Z_T(k^1, \ldots, k^b) | k^0 \rangle \) of \( k^0 \) in the linear expansion of \( Z_T(k^1, \ldots, k^b) \) where \( k^c \in I^{2n_c} \) such that \( \text{alt}(k^c) = e \) for \( 1 \leq c \leq b \). For this, we choose a picture \( T_1 \) in
the isotopy class of $T$ and then choose a simple path $p_c$ in $D_0 \setminus \bigcup_{c=1}^{b} \text{Int}(D_c)$ starting from the $\ast$ \footnote{Recall $\ast$ of a disc $D$ is a point chosen on the boundary of $D$ strictly between the last and the first marked points, moving clockwise.} of $D_0$ to that of $D_c$ for $1 \leq c \leq b$ such that:

(i) $p_c$ intersects the strings of $T_i$ transversally for $1 \leq c \leq b$,

(ii) $p_{c_1}$ and $p_{c_2}$ intersects exactly at the $\ast$ of $D_0$ for $1 \leq c_1 \neq c_2 \leq b$.

Note that any state $\sigma$ on $T$ gives an element $\sigma|_{p_c} \in I^{m_c}$ obtained by reading the elements of $I$ induced by $\sigma$ at the crossings of the path $p_c$ and the strings along the direction of the path where $m_c$ (necessarily even) is the number of strings cut by $p_c$.

Define

$$\langle Z_T(k^1, \ldots, k^b) | k^{0}\rangle = \sum_{\sigma \in S(T) \text{ s.t. } c=1}^{b} \prod_{c=1}^{b} \lambda_{\sigma|_{p_c}}(k^c)$$

where $\lambda_{\hat{i}}(\hat{i}) = \prod_{s=1}^{b} \lambda_{\hat{i}}(\hat{i}, s)$ and $\lambda_{\hat{i}}(\hat{i}, s) = \begin{cases} \overline{\omega}(\text{alt}(j), \text{alt}(i_1, \ldots, i_s), g_i) & \text{if } s \text{ is odd} \\
\omega(\text{alt}(j), \text{alt}(i_1, \ldots, i_{s-1}), g_i) & \text{if } s \text{ is even} \end{cases}$ for $i \notin I^n, j \notin I^m$. If there is no compatible state on $T$, then we take the coefficient to be 0 and if there is no internal disc in $T$, then the scalar inside the sum is considered to be 1. (Note that $\lambda_{\hat{i}}(\hat{i})$ depends only on alt$(j)$ and $\hat{i}$.)

The first natural question to ask is whether the action $Z_T$ is well-defined. Two con-
figurations of paths \( \{p_c\}_{c=1}^b \) and \( \{p'_c\}_{c=1}^b \) in \( T_1 \) can be obtained from each other using a finite sequence of the following moves:

I. isotopy

\[ A \xrightarrow{p} B \quad \sim \quad A \xrightarrow{p} B \]

II. cap-sliding moves

\[ A \quad \sim \quad A \]

III. disc-sliding moves

\[ D_0 \quad \sim \quad D_0 \]

IV. rotation moves

\[ D_0 \quad \sim \quad D_0 \]

It is enough to check that the action is independent under each of the above moves. Invariance under isotopy moves are the easiest to check since \( \sigma|_{p_c} = \sigma|_{p'_c} \) for all \( c \in \{1, \ldots, b\} \). Under the remaining three moves, we show that \( \text{alt}(\sigma|_{p_c}) = \text{alt}(\sigma|_{p'_c}) \) for \( 1 \leq c \leq b \). For a cap-sliding move, note that the cap induces the same index at the
two consecutive crossing with the path but after applying the \( \text{alt} \) map, the corresponding
group elements cancel each other since they inverses of each other. For the disc-sliding
(resp., rotation moves), the invariance follows from the fact that \( \text{alt}(\sigma|_{\partial D_x}) = e \) (resp.,
\( \text{alt}(\sigma|_{\partial D_x}) = e \)).

**Action preserves composition:**

Let \( S \) be an \( m_0 \)-tangle containing the internal discs \( D_1', \cdots, D_a' \) with colors \( m_1, \cdots, m_a \)
and \( T \) be an \( m_1 \)-tangle containing internal discs \( D_1, \cdots, D_{b} \) with colors \( n_1, \cdots, n_{b} \). Let
\( D_0 \) and \( D_0 \) denote the external discs of \( S \) and \( T \) respectively. We need to show that for
all \( i'^{c'} \in I_{2m_{c'}}^{l} \) and \( j'^{c} \in I_{2n_{c}}^{l} \) where \( c' \in \{0, 2, \cdots, a\} \) and \( c \in \{1, \cdots, b\} \)

\[
(Z_{S}(Z_{T}(i^{1}, \cdots, i^{b}), j^{2}, \cdots, i^{a}))|_{i'^{0}} = (Z_{S_{D}}(Z_{T}(j^{1}, \cdots, j^{b}), i^{2}, \cdots, i^{a}))|_{j'^{0}} \tag{3.2.2}
\]

The left side can be expanded as

\[
\sum_{i^{t} \in I_{2m_{t}}^{l}, \text{s.t.} \; \; \; \; \text{alt}(i^{t}) = e} (Z_{S}(i^{1}, i^{2}, \cdots, i^{a}))|_{i'^{0}} (Z_{T}(j^{1}, \cdots, j^{b}))|_{j'^{0}} = \sum_{\sigma \in S(S), \tau \in S(T) \text{ s.t.} \; \; \; \; \sigma|_\partial D' = i'^{c'} = i^{c'} \text{ for } c' \in \{0, 2, \cdots, a\}, c \in \{1, \cdots, b\} \text{ and } \sigma|_{\partial D_1} = \tau|_{\partial D_0} = i^{t}} \left( \prod_{c=1}^{a} \lambda_{\sigma|_{\partial D'}} (i^{c'}) \right) \left( \prod_{c=1}^{b} \lambda_{\tau|_{\partial D_1}} (j^{c}) \right)
\]

where \( p_{c'} \)'s and \( p_{c} \)'s are paths in the tangles \( S \) and \( T \) respectively required to define their
actions. For the action of \( S \circ_{D_1} T \), we consider the paths \( p_{c} \) for \( 2 \leq c' \leq a \) and \( p_{c} \circ p_{c} \)
for \( 1 < c < b \). (Strictly speaking - one has to make the \( p_{c} \)'s portion of the paths \( (p_{c} \circ p_{c}) \)
for different values of \( c \) - disjoint, in order to define the action of \( S \circ_{D_1} T \).) So, the right
side of the equation 3.2.2 becomes

\[
\sum_{\gamma \in S(S \circ_{D_1} T) \text{ s.t.} \; \; \; \; \gamma|_{\partial D'} = i'^{c'}, \gamma|_{\partial D_x} = i^{c} \text{ for } c' \in \{0, 2, \cdots, a\}, c \in \{1, \cdots, b\}} \left( \prod_{c=2}^{a} \lambda_{\gamma|_{\partial D'}} (i^{c'}) \right) \left( \prod_{c=1}^{b} \lambda_{\gamma|_{\partial D_1}} (j^{c}) \right)
\]

Observe that \( \left\{ (\sigma, \tau) \in S(S) \times S(T) : \begin{array}{c}
\sigma|_{\partial D'} = i'^{c'} \text{ for } c' \in \{0, 2, \cdots, a\} \\
\tau|_{\partial D_x} = j^{c} \text{ for } 1 \leq c \leq b, \sigma|_{\partial D_1} = \tau|_{\partial D_0}
\end{array} \right\} \) has an
obvious bijection with \( \left\{ \gamma \in S(S \circ_{D_1} T) : \begin{array}{c}
\gamma|_{\partial D'} = i'^{c'} \text{ for } c' \in \{0, 2, \cdots, a\}, \\
\gamma|_{\partial D_x} = j^{c} \text{ for } 1 \leq c \leq b
\end{array} \right\} \). This is
obtained by sending \((\sigma, \tau)\) to the state defined by \(\sigma\) (resp. \(\tau\)) on the \(S\)-part (resp. \(T\)-part) of \(S \circ D\), and the fact that such a state is indeed well-defined is a consequence of the condition \(\sigma|_{\partial D'} = \tau|_{\partial D_0}\); we denote this state by \(\sigma \circ \tau\). If these sets are empty, the equation \(3.2.2\) trivially holds since both sides have value 0. Let us assume that the sets are nonempty. It is enough to prove that if \(\sigma\) and \(\tau\) are states on \(S\) and \(T\) respectively with \(\sigma|_{\partial D'} = \tilde{\mathbf{i}}^c\) for \(0 \leq c' \leq a\), \(\tau|_{\partial D_e} = \tilde{\mathbf{j}}^c\) for \(1 \leq c \leq b\) and \(\tau|_{\partial D_0} = \tilde{\mathbf{i}}^1\), then

\[
\lambda_{\sigma|_{\partial D'}}(\tilde{\mathbf{i}}^1) \prod_{c=1}^{b} \lambda_{\tau|_{\partial D_e}}(\tilde{\mathbf{j}}^c) = \prod_{c=1}^{b} \lambda(\sigma \circ \tau)|_{(\partial D_0')} (\tilde{\mathbf{j}}^c) \quad (3.2.3)
\]

We prove this in two cases.

**Case 1:** \(T\) has no internal disc or closed loop, that is, \(T\) is a Temperley-Lieb diagram. The right side of equation \(3.2.3\) is 1. It remains to show \(\lambda_{\sigma|_{\partial D'}}(\tilde{\mathbf{i}}^1) = 1\). This follows from the next lemma and the fact \(\tilde{\mathbf{i}}^1 = Z_T(1)\) is a sequence of non-crossing matched pairings of indices from \(I\).

**Lemma 3.2.15.** If \(i \in I\), \(j \in I_m\), \(\mathbf{i} = (i_1, \ldots, i_n) \in I^n\) and \(0 \leq s \leq n\), then we have

\[
\lambda_\mathbf{i}(i) = \lambda_\mathbf{i}(i_1, \ldots, i_s, i, i_{s+1}, \ldots, i_n).
\]

**Proof:** Note that

(i) \(\lambda_\mathbf{i}(i, r) = \lambda_\mathbf{i}((i_1, \ldots, i_s, i, i_{s+1}, \ldots, i_n), r)\) for \(1 \leq r \leq s\),

(ii) \(\lambda_\mathbf{i}(i, r) = \lambda_\mathbf{i}((i_1, \ldots, i_s, i, i_{s+1}, \ldots, i_n), r + 2)\) for \(s + 1 \leq r \leq n\).

Now,

\[
\lambda_\mathbf{i}((i_1, \ldots, i_s, i, i_{s+1}, \ldots, i_n), s + 1) \lambda_\mathbf{i}((i_1, \ldots, i_s, i, i_{s+1}, \ldots, i_n), s + 2) = \begin{cases} 
\omega(\text{alt}(j), \text{alt}(i_1, \ldots, i_s, i), g_i) \omega(\text{alt}(j), \text{alt}(i_1, \ldots, i_s, i), g_i) & \text{if } s \text{ is even} \\
\omega(\text{alt}(j), \text{alt}(i_1, \ldots, i_s), g_i) \omega(\text{alt}(j), \text{alt}(i_1, \ldots, i_s, i), g_i) & \text{if } s \text{ is odd} 
\end{cases}
\]

= 1.

\[\square\]

**Remark 3.2.16.** In the Lemma 3.2.15, if \(\mathbf{i}\) is a sequence of indices with non-crossing matched pairings, then we can apply Lemma 3.2.15 several times to reduce all the consecutive matched pairings and finally get \(\lambda_\mathbf{i}(\mathbf{i}) = 1\). \[\square\]
Case 2: $T$ has at least one internal disc. Any unlabelled tangle $T$ can be expressed as composition of elementary annular tangles of four types as described in 4, namely, capping, cap-inclusion, left-inclusion and disc-inclusion tangles. It is enough to prove equation 3.2.3 for any tangles $S$ and compatible $T$ in $\mathcal{E}$ (= the set of all elementary tangles). If $T$ is of capping or cap-inclusion type annular tangle, the proof directly follows from Lemma 3.2.15 and is left to the reader.

If $T$ is a left-inclusion annular tangle, the equation 3.2.3 is implied from the following lemma.

**Lemma 3.2.17.** For all $i \in I^{2m}$, $j \in I^{2n}$ and $k \in I^{2n_1}$ such that $alt(i) = e$, we have

$$\lambda_{k,j}(i) = \lambda_k(j, i, \tilde{j}) \lambda_j(i)$$

where $\tilde{j}$ is the sequence of indices from $j$ in the reverse order.

**Proof:** We rearrange the terms of the right side in the following way:

$$\lambda_k(j, i, \tilde{j}) \lambda_j(i)$$

$$= \left( \prod_{r=1}^{2m} \lambda_k((j, i, \tilde{j}), 2(m + 2n) - r + 1) \right) \left( \prod_{s=1}^{2m} \lambda_k((j, i, \tilde{j}), 2n + s) \lambda_j(i, s) \right)$$

Let $i = (i_1, \cdots, i_{2m})$ and $j = (j_1, \cdots, j_{2n})$. Note that for $1 \leq r \leq 2n,$

$$\lambda_k((j, i, \tilde{j}), 2(m + 2n) - r + 1)$$

$$= \left\{ \begin{array}{ll}
\omega(alt(k), alt(j, i, \cdots, i_r), g_{j_r}) & \text{if } r \text{ is odd} \\
\overline{\omega}(alt(k), alt(j_1, \cdots, j_{r-1}), g_{j_r}) & \text{if } r \text{ is even}
\end{array} \right.$$ (since $alt(i) = e$ and $alt(j, i, \tilde{j}) = alt(j)alt(i)(alt(j))^{-1} = e$)

$$= \lambda_k((j, i, \tilde{j}), r)$$

Thus the first product of the terms in the rearrangement vanishes. For the second product, if $s \in \{1, \cdots, 2m\}$ is odd, then

$$\lambda_k((j, i, \tilde{j}), 2n + s) \lambda_j(i, s)$$

$$= \overline{\omega}(alt(k), alt(j)alt(i_1, \cdots, i_s), g_{i_s}) \overline{\omega}(alt(j), alt(i_1, \cdots, i_s), g_{i_s})$$
\[
= \varpi(\text{alt}(k), \text{alt}(j), \text{alt}(i_1, \cdots, i_{s-1})) \varpi(\text{alt}(k, j), \text{alt}(i_1, \cdots, i_s), g_u).
\]
\[
\omega(\text{alt}(k), \text{alt}(j), \text{alt}(i_1, \cdots, i_s))
\]
(\text{using the defining equation (Equation 3.1.1) of the 3-cocycle } \omega)
\]
\[
= \varpi(\text{alt}(k), \text{alt}(j), \text{alt}(i_1, \cdots, i_{s-1})) \lambda_{[k,j]}(i, s) \omega(\text{alt}(k), \text{alt}(j), \text{alt}(i_1, \cdots, i_s))
\]

Similarly, if \(s \in \{1, \cdots, 2m\}\) is even, then
\[
\lambda_{k}((j, i, j), 2n + s) \lambda_{j}(i, s)
\]
\[
= \varpi(\text{alt}(k), \text{alt}(j), \text{alt}(i_1, \cdots, i_{s-1})) \lambda_{[k,j]}(i, s) \omega(\text{alt}(k), \text{alt}(j), \text{alt}(i_1, \cdots, i_s))
\]

Thus,
\[
\prod_{s=1}^{2m} \lambda_{k}((j, i, j), 2n + s) \lambda_{j}(i, s)
\]
\[
= \prod_{t=1}^{m} \left( \lambda_{k}((j, i, j), 2n + 2t - 1) \lambda_{j}(i, 2t - 1) \right) \left( \lambda_{k}((j, i, j), 2n + 2t) \lambda_{j}(i, 2t) \right)
\]
\[
= \prod_{t=1}^{m} \left( \varpi(\text{alt}(k), \text{alt}(j), \text{alt}(i_1, \cdots, i_{2t-2})) \lambda_{[k,j]}(i, 2t - 1) \right)
\]
\[
= \varpi(\text{alt}(k), \text{alt}(j), e) \left( \prod_{t=1}^{m} \lambda_{[k,j]}(i, 2t - 1) \right) \omega(\text{alt}(k), \text{alt}(j), \text{alt}(i))
\]
\[
= \lambda_{[k,j]}(i)
\]
(since \(\text{alt}(i) = e\) and \(\omega\) is normalized). \(\Box\)

\textbf{Remark 3.2.18.} The proof of Lemma 3.2.17 also gives the following relation
\[
\lambda_{k}((j, i, j), 2n+1) = \prod_{s=2n+1}^{2m+2n} \lambda_{k}((j, i, j), s)
\]

Now, suppose \(T\) is a disc-inclusion tangle as shown in Figure 3.2. Note that from the configuration it follows that \(n_1 = m_1\). Without loss of generality, let \(T\) be given by the following picture in which we also indicate the paths \(p_1\) and \(p_2\). Note that since \(p_1\) does not intersect any string, \(\tau|_{p_1}\) is the empty multi-index. So it is enough to prove
\[
\lambda_{\sigma|_{p_1'}}((\tilde{j})^1) \lambda_{\tau|_{p_2}}((\tilde{j})^2) = \lambda_{\sigma|_{p_1'}}((\tilde{j})^1) \lambda_{\sigma|_{p_1'}}((\tilde{j})^2)
\]
(3.2.4)
Let us denote $\tau|_{p_2}$ by $k \in T^{2r}$. Since $\tau$ is a state, the following relations clearly follow from the picture:

(i) $i_s^1 = j_s^1$ for $1 \leq s \leq 2r$ and $2r + n_2 + 1 \leq s \leq 2n_1$,
(ii) $i_{2r+s}^1 = j_s^2$ for $1 \leq s \leq n_2$
(iii) $i_s^1 = k_s = j_s^1$ for $1 \leq s \leq 2r$,
(iv) $j_{2r+s}^1 = j_{2n_2-s+1}^2$ for $1 \leq s \leq n_2$.

We now express $\lambda_{\tau|p_1}^{(i_1)}$ as a product of three terms with which we work separately.

\[
\lambda_{\sigma|p_1}^{(i_1)} = \left( \prod_{s=1}^{2r} \lambda_{\sigma|p_1}^{(i_1^s, s)} \right) \left( \prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|p_1}^{(i_1^s, s)} \right) \left( \prod_{s=2r+n_2+1}^{2n_1} \lambda_{\sigma|p_1}^{(i_1^s, s)} \right)
\]

First term: For $1 \leq s \leq 2r$,

\[
\lambda_{\sigma|p_1}^{(i_1^s, s)} = \begin{cases} 
\omega(\text{alt}(\sigma|p_1), \text{alt}(i_1^s, \ldots, i_{s-1}^s), g_{i_s^1}) & \text{if } s \text{ is odd} \\
\omega(\text{alt}(\sigma|p_1), \text{alt}(i_1^s, \ldots, i_{s-1}^s), g_{i_s^1}) & \text{if } s \text{ is even}
\end{cases}
\]

Second term: For $2r + 1 \leq s \leq 2r + n_2$,

\[
\lambda_{\sigma|p_1}^{(i_1^s, s)} = \begin{cases} 
\omega(\text{alt}(\sigma|p_1), \text{alt}(i_1^s, \ldots, i_{s-1}^s), g_{i_s^1}) & \text{if } s \text{ is odd} \\
\omega(\text{alt}(\sigma|p_1), \text{alt}(i_1^s, \ldots, i_{s-1}^s), g_{i_s^1}) & \text{if } s \text{ is even}
\end{cases}
\]

(applying (i))
Third term: First note that
\[
\text{alt}(i_{2r+1}, \ldots, i_{2r+n_2}) = \text{alt}(j_1^2, \ldots, j_{n_2}^2) \quad \text{(using (ii))}
\]
\[
= \text{alt}(j_1^2, \ldots, j_{n_2}^2) \quad \text{(since \text{alt}(j^2) = e)}
\]
\[
= \text{alt}(j_{2r+1}^3, \ldots, j_{2r+n_2}^3) \quad \text{(using (iv))}
\]
Thus, for \(2r + n_2 + 1 \leq s \leq 2n_1\),
\[
\lambda_{\sigma|\pi'_1}(i^1, s)
\]
\[
= \left\{ \begin{array}{ll}
\bar{\omega}(\text{alt}(\sigma|\pi'_1), \text{alt}(i_1^1, \ldots, i_{2r}^1)\text{alt}(i_{2r+1}^3, \ldots, i_{2r+n_2}^3, i_{s-1}^1, g_{j_1}) & \text{if } s \text{ is odd} \\
\omega(\text{alt}(\sigma|\pi'_1), \text{alt}(i_1^1, \ldots, i_{2r}^1)\text{alt}(i_{2r+1}^3, \ldots, i_{2r+n_2}^3, i_{s-1}^1, g_{j_1}) & \text{if } s \text{ is even}
\end{array} \right.
\]
\[
= \left\{ \begin{array}{ll}
\bar{\omega}(\text{alt}(\sigma|\pi'_1), \text{alt}(j_1^1, \ldots, j_{s-1}^1), g_{j_1}) & \text{if } s \text{ is odd} \\
\omega(\text{alt}(\sigma|\pi'_1), \text{alt}(j_1^1, \ldots, j_{s-1}^1), g_{j_1}) & \text{if } s \text{ is even}
\end{array} \right. \quad \text{(using (i) and (v))}
\]
Combining the three terms, we get
\[
\lambda_{\sigma|\pi'_1}(i^1, s)
\]
\[
= \left( \prod_{s=1}^{2r} \lambda_{\sigma|\pi'_1}(i^1, s) \right) \left( \prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|\pi'_1}(j_1^1, s) \right) \left( \prod_{s=2r+n_2+1}^{2n_1} \lambda_{\sigma|\pi'_1}(j_1^1, s) \right) 
\]
\[
= \lambda_{\sigma|\pi'_1}(j_1^1) \left( \prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|\pi'_1}(j_1^1, s) \right) \left( \prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|\pi'_1}(j_1^1, s) \right)
\]
Now, for \(2r + 1 \leq s \leq 2r + n_2\),
\[
\bar{\lambda}_{\sigma|\pi'_1}(j_1^1, s)
\]
\[
= \left\{ \begin{array}{ll}
\omega(\text{alt}(\sigma|\pi'_1), \text{alt}(j_1^1, \ldots, j_{2r}^1), g_{j_1}) & \text{if } s \text{ is odd} \\
\bar{\omega}(\text{alt}(\sigma|\pi'_1), \text{alt}(j_1^1, \ldots, j_{2r}^1), g_{j_1}) & \text{if } s \text{ is even}
\end{array} \right.
\]
\[
= \left\{ \begin{array}{ll}
\omega(\text{alt}(\sigma|\pi'_1), \text{alt}(j_2^{2n_2+2r-s+1}), g_{j_2^{2n_2+2r-s+1}}) & \text{if } s \text{ is odd} \\
\bar{\omega}(\text{alt}(\sigma|\pi'_1), \text{alt}(j_2^{2n_2+2r-s+2}), g_{j_2^{2n_2+2r-s+2}}) & \text{if } s \text{ is even}
\end{array} \right.
\]

(using (iii) and (iv))

\[
\begin{cases}
\omega(\text{alt}(\sigma_{l'}^j), \text{alt}(k)\text{alt}(j^2, \cdots, j_{2n+2r-s}, g_{j_{2n+2r-s+s+1}})) & \text{if } s \text{ is odd} \\
\overline{\omega}(\text{alt}(\sigma_{l'}^j), \text{alt}(k)\text{alt}(j^2, \cdots, j_{2n+2r-s+s+1}), g_{j_{2n+2r-s+s+1}}) & \text{if } s \text{ is even}
\end{cases}
\]

(since \(\text{alt}(j^2) = e\))

\[= \lambda_{\sigma|_{l'}^j}((k, j^2, \bar{k}), 2n_2 + 4r - s + 1)\]

Hence,

\[
\lambda_{\sigma|_{l'}^j}(j^1) = \lambda_{\sigma|_{l'}^j}(j^1) \left( \prod_{s=2r+n_2+1}^{2r+2n_2} \lambda_{\sigma|_{l'}^j}((k, j^2, \bar{k}), s) \right) \left( \prod_{s=2r+1}^{2r+n_2} \lambda_{\sigma|_{l'}^j}((k, j^2, \bar{k}), s) \right)
\]

\[= \lambda_{\sigma|_{l'}^j}(j^1) \lambda_{\sigma|_{l'}^j}(k, j^2, \bar{k}) \text{ (by Remark 3.2.18)}\]

Getting back to the proof of Equation 3.2.4

\[
\lambda_{\sigma|_{l'}^j}(j^1) \lambda_{\sigma|_{l'}^j}(j^2) = \lambda_{\sigma|_{l'}^j}(j^1) \lambda_{\sigma|_{l'}^j}(k, j^2, \bar{k}) \lambda_{\sigma|_{l'}^j}(j^2)
\]

\[= \lambda_{\sigma|_{l'}^j}(j^1) \lambda_{\sigma|_{l'}^j}(j^2) \text{ (applying Lemma 3.2.17)}\]

This completes the proof of action of tangles preserving composition. Hence, \(P^{(g_i; i \in I)}\) is a planar algebra.

\textit{*-structure on } \(P^{(g_i; i \in I)}\)

Note that if \(i \in I^{2n}\), then \(\text{alt}(i) = e \iff \text{alt}(\bar{i}) = e\). Extend \(\sim\) conjugate linearly to define \(*\) on \(P_n^{(g_i; i \in I)}\). Clearly \(*\) is an involution. We need to check whether the action of tangles preserve *, that is, \(Z_T \circ (* \times \cdots \times *) = * \circ Z_T\). It is enough to check this equation for the cases when \(T\) has no internal disc or closed loops, and when \(T\) is an elementary annular tangle.

If \(T\) has no internal disc or closed loops, then it is a Temperley-Lieb diagram and hence \(Z_T\) is the sum of all sequences of indices from \(I\) which have non-crossing matched pairings where the position of the pairings are given by the numberings of the marked points on the boundary of \(T\) which are connected by a string. Now, in the tangle \(T^*\), the \(m\)-th and the \(n\)-th marked points are connected by a string if and only if the \(m\)-th and the \(n\)-th marked points starting from the last point in \(T\) reading anticlockwise, are
connected; so, $Z_T^*$ is indeed the sum of all sequences featuring in the linear expansion of $Z_T$ in the reverse order (that is, applying $\sim$).

If $T$ is an elementary annular tangle of capping (resp. cap-inclusion) type with $m$-th and $(m+1)$-th marked points of the internal (resp. external) disc being connected by a string, then $T^*$ is also same kind of elementary annular tangle but the ‘capping’ occurs at the $m$-th and $(m+1)$-th marked points of the internal (resp. external) disc starting from the last point reading anticlockwise. The equation easily follows from this observation and the verification is left to the reader.

Next, if $T$ is an elementary tangle of left-inclusion type, then $T = T^*$. Then, the equation follows from the following lemma.

**Lemma 3.2.19.** If $\vec{i} \in I^m$ and $\vec{i} = (i_1, \cdots, i_{2n}) \in I^{2n}$ such that alt$(\vec{i}) = e$, then $\lambda_j(\vec{i}) = \overline{\lambda}_j(\vec{i})$.

**Proof:** The proof is an immediate consequence of

$$alt(i_1, \cdots, i_s) = alt(i_{2n}, \cdots, i_{s+1}) \text{ (since alt}(i) = e)$$

Lastly, if $T$ is the disc-inclusion type elementary annular tangle given by Figure 3.2, then we need to show $\langle Z_T^*(\vec{i}, \vec{j}), \vec{k} \rangle = \langle Z_T(\vec{i}, \vec{j}), \vec{k} \rangle$ for $\vec{i}, \vec{k} \in I^{2n_1}, \vec{j} \in I^{2n_2}$ such that $alt(\vec{i}) = alt(\vec{j}) = alt(\vec{k}) = e$. It is easy to verify that $\vec{i}$, $\vec{j}$ and $\vec{k}$ defines a state on $T$ if and only if $\vec{\tilde{i}}$, $\vec{\tilde{j}}$ and $\vec{\tilde{k}}$ defines the same on $T^*$; if they fail to define a state, then both sides are zero. If they define a state, then the scalars appearing in the two sides can be made equal by applying Lemma 3.2.19.

**Remark 3.2.20.** If $\omega$ is trivial, that is, a 3-coboundary, then $P^{(g: i \in I)}\omega$ is isomorphic to Jones's example of constructing a planar algebra from a finitely generated group (Example 2.7 in [16]). Jones constructed this example by considering a certain subspace of the tensor planar algebra (TPA) over the vector space with the indexing set $I$ as a basis and then showed that this subspace is closed under TPA-action of tangles, and thereby showing the subspace is a planar algebra. One can view our planar algebra $P^{(g: i \in I)}\omega$ as a subspace of TPA in an obvious way but the action induced by TPA will not be the same as our action which involve the extra data of a 3-cocycle.
With above remark in mind, we ask the question:

**Question 3.2.21.** Can $P^{(g_i:i \in I)}$ be viewed as a planar subalgebra of tensor planar algebra over the vector space with $I$ as a basis?

### 3.3 Diagonal subfactors

In this section, we will compute the relative commutants of the diagonal subfactors based on a model of its tower of basic construction, and find the basic structures, namely, the filtered algebra structure, Jones projection and the conditional expectations. Most of these were already computed in the past (see [2], [27]). What we were able to achieve here is to make appropriate choice of the basis of the relative commutants such that the action of tangles on these relative commutants matches with that of the last section.

Let $N$ be a $II_1$ factor, $I$ be a finite set and for $i \in I$, choose $\theta_i \in \text{Aut}(N)$. Set $M = M_I \otimes N$ where $M_I$ denotes the matrix algebra whose rows and columns are indexed by $I$. Consider the subfactor $N \subset M$ given by

$$N \ni x \mapsto \sum_{i \in I} E_{i,i} \otimes \theta_i(x) \in M$$

that is, an element $x$ of $N$ sits in $M$ as a diagonal matrix whose $i$th diagonal is $\theta_i(x)$. This is known as a diagonal subfactor. If we have another collection of automorphisms $\theta'_i \in \text{Aut}(N)$ for $i \in I$ such that $\theta_i = \theta'_i$ for all $i \in I$ (or up to a permutation of $I$), then the diagonal subfactors arising from the two collections are isomorphic. So, it is more natural to start with a collection in $\text{Out}(N)$, instead of $\text{Aut}(N)$, and consider the diagonal subfactor with respect to a lift of the collection to $\text{Aut}(N)$.

Given a collection $g_i \in \text{Out}(N)$ for $i \in I$, we consider the subgroup $G = \langle g_i : i \in I \rangle$ of $\text{Out}(N)$. Choose a lift

$$G \ni g \xrightarrow{\alpha} \alpha_g \in \text{Aut}(N)$$

such that $\alpha_e = id_N$. Set $\alpha_i = \alpha_{g_i}$ for all $i \in I$. Consider the diagonal subfactor $N \subset M = M_I \otimes N$ where the $i$th diagonal entry of an element of $N$ viewed in $M$ is twisted by the action of $\alpha_i$; the index of this subfactor is $|I|^2$. Set $M_n = M_{I^{n+1}} \otimes N$
for \( n \geq 0 \). We will often identify \( E_{i,j} \otimes E_{k,l} \in M_{I^m} \otimes M_{I^n} \) with \( E_{(i,k),(j,l)} \in M_{I^{m+n}} \) for \( i, j \in I^m \) and \( k, l \in I^n \). Now, \( M_{n-1} \) is included in \( M_n \) in the following way:

\[
M_{n-1} = M_{I^n} \otimes N \ni E_{i,j} \otimes x \mapsto E_{i,j} \otimes \psi_n(x) \in M_{I^n} \otimes M_I \otimes N = M_{I^{n+1}} \otimes N = M_n
\]

for all \( i, j \in I^n \) and \( x \in N \) where \( \psi_n : N \to M_I \otimes N \) is defined as:

\[
\psi_n(x) = \begin{cases} 
\sum_{k \in I} E_{k,k} \otimes \alpha_k(x) & \text{if } n \text{ is even} \\
\sum_{k \in I} E_{k,k} \otimes \alpha_k^{-1}(x) & \text{if } n \text{ is odd}
\end{cases}
\]

It is easy to check (see [2]) that \( 1 \in N \subset M \subset M_1 \subset M_2 \subset \cdots \) is isomorphic to the tower of basic construction of \( N \subset M \) where the Jones projections and conditional expectation maps are given by:

\[
e_n = \frac{1}{|I|} \sum_{k \in I^{n-1}, i,j \in I} E_{(k,i),(k,j)} \otimes 1 \in M_n
\]

\[
E_{M_{n-1}}^n(E_{(k,i),(j,l)} \otimes x) = \delta_{k,l} |I|^{-1} E_{i,j} \otimes \alpha_k^{(-1)^n-1}(x)
\]

for all \( i, j \in I^n, k, l \in I, x \in N \). Moreover, \( \{\sqrt{|I|}(E_{i,j} \otimes 1) : i, j \in I\} \) forms a basis of \( M \) over \( N \); this basis will be used to get the conditional expectation of commutant of \( N \) onto the commutant of \( M \).

To find the relative commutant \( N' \cap M_{n-1} \), first note that \( N \) is included \( M_{n-1} \) by the following map:

\[
N \ni x \mapsto \sum_{i \in I^n} E_{i,i} \otimes \text{alt}_{\alpha}^{-1}(i)(x) \in M_{n-1}
\]

where \( \text{alt}_{\alpha}(i) = \alpha_{i_1}^{-1} \cdot \alpha_{i_2}^{-1} \cdots \alpha_{i_n}^{-1} \in Aut(N) \) for \( \bar{i} = (i_1, \cdots, i_n) \in I^n \). Now, if \( \sum_{i,j \in I^n} x_{i,j}(E_{i,j} \otimes 1) \in N' \cap M_{n-1} \), then

\[
\sum_{i,j \in I^n} x_{i,j}(E_{i,j} \otimes 1) y = y \sum_{i,j \in I^n} x_{i,j}(E_{i,j} \otimes 1) \text{ for all } y \in N
\]

\[
\Leftrightarrow x_{i,j}(\text{alt}_{\alpha}(i) \text{alt}_{\alpha}^{-1}(j))(y) = y x_{i,j} \text{ for all } y \in N, i, j \in I^n
\]

\[
\Leftrightarrow x_{i,j}(\text{alt}_{\alpha}(i,j))(y) = y x_{i,j} \text{ for all } y \in N, i, j \in I^n
\]

So, for \( i, j \in I^n \), if \( x_{i,j} \neq 0 \), then \( x_0 = \frac{x_{i,j}}{\|x_{i,j}\|} \in U(N) \) and \( Ad_{x_0} \circ \text{alt}_{\alpha}(i,j) = id_N \) which implies \( \text{alt}_{\alpha}(i,j) = e \). Similarly, if there exist \( i, j \in I^n \) such that \( \text{alt}_{\alpha}(i,j) = e \), then \( u(E_{i,j} \otimes 1) \in N' \cap M_{n-1} \) where \( u \in U(N) \) satisfies \( Ad_u \circ \text{alt}_{\alpha}(i,j) = id_N \). Thus,

\[
N' \cap M_{n-1} = \text{span}\left\{ u(E_{i,j} \otimes 1) \in M_{n-1} \mid \begin{array}{l}
i, j \in I^n \text{ and } u \in U(N) \\
\text{s.t. } Ad_u \circ \text{alt}_{\alpha}(i,j) = id_N \end{array}\right\}
\]
The set is not yet a basis since $N$ being a factor, the unitary $u$ can be chosen up to a scalar of absolute value 1. However, the choice of $u$ to get a basis of $N' \cap M_{n-1}$ has to be done in such a way that the planar algebra associated to $N \subset M$ matches with the abstract one defined in Section 3.2.

We now digress a little bit to set up some notations. Let $u : G \times G \to U(N)$ be an unitary obstruction to $\alpha$ being a homomorphism, that is,

$$\alpha_{g_1} \alpha_{g_2} = Ad_{u(g_1, g_2)} \circ \alpha_{g_1 g_2} \quad \text{for all } g_1, g_2 \in G$$

such that $u(g_1, g_2) = 1$ whenever either of $g_1$ or $g_2$ is $e$. For $i = (i_1, \ldots, i_n) \in I^n$, define

$$v_n(i) = \begin{cases} u^*(\text{alt}(i_1, \ldots, i_m), g_{i_m}) & \text{if } m \text{ is odd} \\ u(\text{alt}(i_1, \ldots, i_{m-1}), g_{i_m}) & \text{if } m \text{ is even} \end{cases}$$

and set $v(i) = v_1(i) \cdots v_n(i)$. Next, we prove a couple of useful lemmas involving $v$. The first lemma motivates our choice of the basis.

**Lemma 3.3.22.** $\text{alt}_\alpha(i) = Ad_{v(i)} \circ \alpha_{\text{alt}(i)}$ for all $i \in I^n$.

**Proof:** Using the definition of $u$, note that for all $m \geq 1$,

$$Ad_{v_n(i)} = \begin{cases} \alpha_{\text{alt}(i_1, \ldots, i_{m-1})}^{-1} \alpha_{i_m}^{-1} \alpha_{\text{alt}(i_1, \ldots, i_m)} & \text{if } m \text{ is odd} \\ \alpha_{\text{alt}(i_1, \ldots, i_{m-1})} \alpha_{i_m} \alpha_{\text{alt}(i_1, \ldots, i_m)}^{-1} & \text{if } m \text{ is even} \end{cases}$$

So,

$$Ad_{v(i)} = Ad_{v_1(i)} \cdots Ad_{v_n(i)} = \alpha e \alpha_{i_1}^{-1} \alpha_{i_2} \cdots \alpha_{i_n}^{(-1)^n} \alpha_{\text{alt}(i_1, \ldots, i_n)}^{-1} = \text{alt}_\alpha(i) \alpha_{\text{alt}(i)}^{-1}$$

\qed

**Lemma 3.3.23.** For any $k = (k_1, \ldots, k_{2n}) \in I^{2n}$ such that $\text{alt}(k) = e$, we have $v(k) v(k) = 1$.

**Proof:** First expand $v(k)$ and $v(k)$ into products of unitaries arising from the definition of $v$, and then we consider that the product of the $p$-th unitary of $v(k)$ from the right and $p$-th unitary of $v(k)$ from the left, that is,

$$v_{2n-p+1}(k) v_p(k) = \begin{cases} u(\text{alt}(k_2n, \ldots, k_{p+1}), k_p) u^*(\text{alt}(k_1, \ldots, k_p), k_p) & \text{if } p \text{ is odd} \\ u^*(\text{alt}(k_2n, \ldots, k_p), k_p) u(\text{alt}(k_1, \ldots, k_{p-1}), k_p) & \text{if } p \text{ is even} \end{cases}$$
\[
\begin{cases}
  u(alt(k_1, \cdots, k_p), k_p) u^*(alt(k_1, \cdots, k_p), k_p) & \text{if } p \text{ is odd} \\
  u^*(alt(k_1, \cdots, k_{p-1}), k_p) u(alt(k_1, \cdots, k_{p-1}), k_p) & \text{if } p \text{ is even}
\end{cases}
\]

for \(1 \leq p \leq 2n\).

**Lemma 3.3.24.** For \(i = (i_1, \cdots, i_n), j = (j_1, \cdots, j_n), k = (k_1, \cdots, k_n) \in I^n\) such that \(alt(i, j) = e = alt(j, k)\), we have \(v(i, j) v(j, k) = v(i, k)\).

**Proof:** Using argument similar to the proof of Lemma 3.3.23, one can prove that the product of the \(p\)-th unitary of \(v(i, j)\) from the right and the \(p\)-th unitary of \(v(j, k)\) from the left, is 1 for \(1 \leq p \leq n\). Again, for \(n+1 \leq p \leq 2n\,

\[
v_p(j, k) = \begin{cases}
  u^*(alt(j)k_{2n}^{(-1)^{n+1}} \cdots k_{2n-p+1}^{-1}, k_{2n-p+1}) & \text{if } p \text{ is odd} \\
u(alt(j)k_{2n}^{(-1)^{n+1}} \cdots k_{2n-p+2}^{-1}, k_{2n-p+1}) & \text{if } p \text{ is even}
\end{cases}
\]

Thus,

\[
v(i, j) v(j, k) = v_1(i, j) \cdots v_n(i, j) v_{n+1}(j, k) \cdots v_{2n}(j, k) \\
= v_1(i, k) \cdots v_n(i, k) v_{n+1}(i, k) \cdots v_{2n}(i, k) = v(i, k)
\]

Applying Lemma 3.3.22, we identify the set \(\left\{v^*(i, j)(E_{i,j} \otimes 1) \mid i, j \in I^n \text{ s.t. } alt(i, j) = e\right\}\) as basis for \(N' \cap M_{n-1}\). We will use this basis to establish an isomorphism between the planar algebra associated to \(N \subset M\) and the one defined in Section 3.2, but before that we need the 3-cocycle obstruction \(\omega : G \times G \times G \to S^1\) for \((G, \alpha, u)\), that is, for all \(g_1, g_2, g_3 \in G\)

\[
u(g_1, g_2)u(g_1, g_2, g_3) = \omega(g_1, g_2, g_3)u(g_1, g_2)u(g_1, g_2, g_3)
\]

This is a consequence of associativity of the multiplication in \(G\). We now prove another useful lemma relating \(v\) and \(\omega\).
Lemma 3.3.25. If $i \in I$ and $k = (k_1, \ldots, k_{2n}) \in I^{2n}$ such that $\text{alt}(k) = e$ and $k_1 = k_{2n}$, then $\text{alt}_\alpha(i, k_1)(v(k)) = \overline{\lambda}_{(i, k_1)}(k) v(i, k_2, \ldots, k_{2n-1}, i)$.

Proof: We expand $\text{alt}_\alpha(i, k_1)(v(k))$ as a product of unitaries and work with them separately. For $1 \leq p \leq n$,

(i) $\text{alt}_\alpha(i, k_1)(v_{2p-1}(k))$

\begin{align*}
= (\alpha_{g_i^{-1}, g_{k_1}})(u^*(\text{alt}(k_1, \ldots, k_{2p-1}, g_{k_{2p-1}}))) \\
= \left(\text{Ad}_{u^*(g_i^{-1}, g_{k_1})} \circ \alpha_{g_i^{-1}, g_{k_1}}\right)(u^*(\text{alt}(k_1, \ldots, k_{2p-1}, g_{k_{2p-1}})))
\end{align*}

(Using definition of $u$)

\begin{align*}
= \omega(\text{alt}(i, k_1), \text{alt}(k_1, \ldots, k_{2p-1}, g_{k_{2p-1}}))
\end{align*}

(Using the definition of $\omega$)

(ii) $\text{alt}_\alpha(i, k_1)(v_{2p}(k))$

\begin{align*}
= (\alpha_{g_i^{-1}, g_{k_1}})(u(\text{alt}(k_1, \ldots, k_{2p-1}, g_{k_{2p}}))) \\
= \left(\text{Ad}_{u^*(g_i^{-1}, g_{k_1})} \circ \alpha_{g_i^{-1}, g_{k_1}}\right)(u(\text{alt}(k_1, \ldots, k_{2p-1}, g_{k_{2p}})))
\end{align*}

(Using definition of $u$)

\begin{align*}
= \overline{\omega}(\text{alt}(i, k_1), \text{alt}(k_1, \ldots, k_{2p-1}, g_{k_{2p}}))
\end{align*}

(Using the definition of $\omega$)

Multiplying (i) and (ii), we get,

\begin{align*}
\text{alt}_\alpha(i, k_1)(v_{2p-1}(k)) \text{alt}_\alpha(i, k_1)(v_{2p}(k))
= & \overline{\lambda}_{(i, k_1)}(k, 2p - 1) \overline{\lambda}_{(i, k_1)}(k, 2p) \\
& \text{Ad}_{u^*(g_i^{-1}, g_{k_1})} \circ \alpha_{g_i^{-1}, g_{k_1}}
\end{align*}

\begin{align*}
= & \overline{\lambda}_{(i, k_1)}(k, 2p - 1) \overline{\lambda}_{(i, k_1)}(k, 2p)
\end{align*}
Thus, \[ \text{alt}_\alpha(i, k_1)(v(k)) = W_1 \cdots W_n = \overline{\lambda}_{(i,k_1)}(k). \]

Let us recall a well-known fact about isomorphisms of two planar algebras which will be used in the next theorem.

**Fact:** Let \( P^1 \) and \( P^2 \) be two planar algebras. Then \( P^1 \cong P^2 \) if and only if there exist a vector space isomorphism \( \psi : P^1_n \rightarrow P^2_n \) such that:

(i) \( \psi \) preserves the filtered algebra structure,

(ii) \( \psi \) preserves the actions of all possible Jones projection tangles and the (two types of) conditional expectation tangles.
If $P_1$ and $P_2$ are *-planar algebras, then we consider $\psi$ to be *-preserving to get a *-planar algebra isomorphism.

**Theorem 3.3.26.** The planar algebra $P^{\text{sf}}$ associated to the diagonal subfactor obtained from a II$_1$ factor $N$ and a finite collection of automorphisms $\alpha_i \in \text{Aut}(N)$ for $i \in I$, is isomorphic to $P^{(g_i;i \in I)}$ where $g_i = [\alpha_i] \in \text{Out}(N)$ for all $i \in I$.

**Proof:** Let $G = \langle g_i : i \in I \rangle$ and without loss of generality, let us assume that $\alpha$ is a lift of $G$ such that $\alpha_i = \alpha g_i$. By Theorem 4.2.1 in [16], $P^{\text{sf}}_n = N' \cap M_{n-1}$ for all $n \geq 0$. Define the map $\phi : P^{\text{sf}} \rightarrow P = P^{(g_i;i \in I)}$ by first defining on basis elements as $\phi(v*(i,j)(E_{k,l} \otimes 1)) = (i,j)$ for all $i,j \in I^n$ such that $\text{alt}(i,j) = e$, and then extending linearly. Clearly, $\phi$ is a vector space isomorphism. We appeal to the fact mentioned above to show $\phi$ is *-planar algebra isomorphism. First, we make the following note and then do a series of verifications.

**Note:** For $i \in I$, $\bar{i} = (i_1, \ldots, i_n) \in I^n$ and $0 \leq s \leq n$, we have the relation $v(\bar{i}) = v(i_1, \ldots, i_s, i, i, i, \ldots, i_n);$ the proof is similar to Lemma 3.2.15. Thus, if $\bar{i}$ is a sequence of indices with non-crossing matched pairings, then using this fact several times to reduce all consecutive matched pairings, we get $v(\bar{i}) = 1.$

(a) $\phi$ is unital: Since $1_{P^{\text{sf}}_n} = \sum_{i \in I^n}(E_{i,i} \otimes 1)$ and $v(\bar{i}, \bar{j}) = 1$ (by the above note), we get $\phi(1_{P^{\text{sf}}_n}) = \sum_{i \in I^n}(i, j) = 1_{P_n}$.

(b) $\phi$ preserves Jones projection tangle: By Theorem 4.2.1 in [16], the $n$-th Jones projection tangle $E_n$ acts as $Z_n^{\text{sf}} = \sum_{k \in I^{-1}, i,j \in I}(E_{(k,i,i),(k,j,j)} \otimes 1).$ Since $(k,i,i,j,j,k)$ is a sequence of indices with non-crossing matched pairings, by the above note $v(k,i,i,j,j,k) = 1$. So, $\phi(Z_n^{\text{sf}}) = \sum_{k \in I^{-1}, i,j \in I}(k,i,i,j,j,k) = Z_n^{E_n}.$

(c) $\phi$ preserves the action of conditional expectation tangle: By Theorem 4.2.1 in [16], the action of conditional expectation tangle $E_n^{+1}$ is given by $Z_{n+1}^{\text{sf}} = \sum_{i \in I^n, k,l \in I}(E_{(i,k),(i,l)} \otimes 1).$ So,

\[
Z_{n+1}^{\text{sf}}(v^*(i,k,l,j)(E_{(i,k),(i,l)} \otimes 1)) = v^*(i,k,l,j)(E_{i,j} \otimes 1) = \delta_{k,l} v^*(i,k,l,j)(E_{i,j} \otimes 1) = \delta_{k,l} v^*(i,j)(E_{i,j} \otimes 1) \mapsto \delta_{k,l} (i,j) \in P_n
\]

for all $i,j \in I^n, k,l \in I$ such that $\text{alt}(i,j) = e$. From the action of tangles defined in Section 3.2, it is easy to check $Z_n^{\text{sf}}(i,k,l,j) = \delta_{k,l} (i,j).$
(d) \( \phi \) preserves \( * \): Applying \( * \) on a basis element of \( P_n^{sf} \), we get
\[
\left( v^*(i, j)(E_{id} \otimes 1) \right)^* = (E_{id} \otimes 1)v(i, j) = (alt_a(j)alt_a^{-1}(i))(v(i, j))(E_{id} \otimes 1)
\]
\[
= alt_a(j, i)(v(i, j))(E_{id} \otimes 1)
\]
(\text{using Lemma 3.3.22})
\[
= v(j, i) v(i, j) v^*(j, i)(E_{id} \otimes 1)
\]
(\text{using Lemma 3.3.23})
\[
= v^*(j, i)(E_{id} \otimes 1) \rightarrow (j, i) = (i, j)^*
\]
for \( i, j \in I^n \) such that \( alt(i, j) = e \).

(e) \( \phi \) preserves multiplication: Suppose \( i, j, k, l \in I^n \) such that \( alt(i, j) = e = alt(k, l) \). Then,
\[
\left( v^*(i, j)(E_{id} \otimes 1) \right) \cdot \left( v^*(k, l)(E_{kd} \otimes 1) \right)
\]
\[
= v^*(i, j) \left( alt_a(i, j) \right) v^*(k, l)(E_{id} \otimes 1)(E_{kd} \otimes 1)
\]
\[
= v^*(i, j) v(k, l) v^*(i, j)\delta_{j,k} (E_{id} \otimes 1) \quad (\text{using Lemma 3.3.22})
\]
\[
= \delta_{j,k} \left( v(i, j) v(j, i) \right)^* \quad (E_{id} \otimes 1) = \delta_{j,k} v^*(i, j)(E_{id} \otimes 1) \quad (\text{using Lemma 3.3.24})
\]
On the other hand, one can easily deduce from the action of multiplication in \( P \) that \( (i, j) \cdot (k, l) = \delta_{j,k} (i, l) \).

(f) \( \phi \) preserves the action of left conditional expectation tangle: By Theorem 4.2.1 in [16],
the action of left conditional expectation tangle \( \mathcal{E}'_n \) is given by \( Z_{\mathcal{E}'_n}^{P_n^{sf}} = |I| \mathbb{E}_M' \cap M_{n-1} \).
Again, using the basis of \( M \) over \( N \) mentioned before, the conditional expectation onto \( M \)-commutant can be expressed as
\[
\mathbb{E}_M' \cap M_{n-1}(x) = |I|^{-2} \sum_{i,j \in I} \left( \sqrt{|I|}(E_{i,j} \otimes 1) \right) x \left( \sqrt{|I|}(E_{j,i} \otimes 1) \right)
\]
\[
= |I|^{-1} \sum_{i,j \in I} (E_{i,j} \otimes 1)x(E_{j,i} \otimes 1)
\]
for \( x \in N' \cap M_{n-1} \). So, for \( i = (i_1, \ldots, i_{n-1}), j = (j_1, \ldots, j_{n-1}) \in I^{n-1} \) and \( k, l \in I \) such that \( alt(k, i, j, l) = e \), we have
\[
Z_{\mathcal{E}'_n}^{P_n^{sf}} \left( v^*(k, i, j, l)(E_{k,d} \otimes 1) \right)
\]
\[
= \sum_{i,j \in I} (E_{i,j} \otimes 1)v^*(k, i, j, l)(E_{k,d} \otimes 1)(E_{j,i} \otimes 1)
\]
\[
= \sum_{i,j \in I} alt_a(i, j)(v^*(k, i, j, l))(E_{i,j} \otimes 1)(E_{k,d} \otimes 1)(E_{j,i} \otimes 1)
\]
\[
= \sum_{i,j \in I} alt_a(i, j)(v^*(k, i, j, l))(E_{i,j} \otimes 1)(E_{k,d} \otimes 1)(E_{j,i} \otimes 1)
\]

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On the other hand, one can easily check that the action of $E_n'$ is given by

$$Z_{E_n'}(k, \tilde{i}, \tilde{j}, l) = \delta_{k,l} \sum_{i \in I} \lambda_{i,k}(k,i,j,k) u^* (i, \tilde{i}, \tilde{j}, i) (E_{(i,j),(i,j)} \otimes 1)$$

(8.1)

Corollary 3.3.27. Given a group $G$ generated by a finite collection $g_i$ for $i \in I$ and a normalized 3-cocycle $\omega \in Z^3(G, S^1)$, there exists a hyperfinite subfactor whose associated planar algebra is isomorphic to $P^{(g_i:i \in I)} \omega$.

Proof: The proof follows from the fact (see [33], [17]) that any such cocycle $\omega$ can be realized as an obstruction the lifting of finitely group $G$ sitting in $Out(R)$ and Theorem 3.3.26. 

Remark 3.3.28. Note that the isomorphism $\phi$ in the proof of Theorem 3.3.26 does not use the obstruction till the very last step involving the conditional expectation onto the $M$-commutant. In particular, the filtered $*$-algebra structure does not involve the obstruction at all.

Principal Graph

Analyzing the filtered $*$-algebra structure, one can easily find that the principal graph $\Gamma$ of $N \subset M$ is a specialized Cayley graph. If $G_n = \{ alt(i) : i \in I^n \}$ for $n \geq 1$, and $G_0 = \{ e \}$, then $V_n(\Gamma) = G_n \setminus G_{n-1}$ denotes the set of vertices of $\Gamma$ which are at a distance $n$ from the root for $n \geq 1$, and $V_0(\Gamma) = \{ e \}$. The number of edges between $g \in V_n(\Gamma)$ and $h \in V_{n+1}(\Gamma)$ is $\sum_{i \in I} \delta_{g,h,i}$ (resp. $\sum_{i \in I} \delta_{g,h,i}$) if $n$ is odd (resp. even).

Remark 3.3.29. The most elegant feature of the planar algebra $P^{(g_i:i \in I)} \omega$ is that the distinguished basis forms the 'loop-basis' of the filtered $*$-algebra arising out of random walk on the principal graph. This is because the 3-cocycle $\omega$ does not feature at all in the actions (defined in Section 3.2) of multiplication, inclusion and unit tangles or applying $*$. However, the way we came up with such a planar algebra is in the reverse order as

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presented here, that is, we first found the action of tangles on the relative commutants and then came up with the abstract prescription of the associated planar algebra.

Any filtered \(*\)-algebra structure of a ‘subfactor planar algebra’ can be obtained from random walks on the principal graph; the only extra information encoded in the planar algebra which the principal graph cannot provide, is the action of left conditional expectation tangle (equivalently, the rotation tangle or the left-inclusion tangle with even number of strings). The main difficulty here had been the choice of the unitaries \(v(i)\) satisfying the conclusion of Lemma 3.3.22 in such a way that the basis elements of \(N' \cap M_n\) correspond to loops in the principal graph where the correspondence extends to a filtered \(*\)-algebra isomorphism. Note that this choice of \(v(i)\) is unique up to a scalar in \(S^1\); this choice is delicate in the sense that any other choice could bring in the involvement of the obstruction in the filtered \(*\)-algebra structure whereas the obstruction does not feature in the algebra of random walks on \(\Gamma\).
Chapter 4

Bisch-Haagerup subfactor

In this chapter we study the standard invariant of a class of subfactors constructed by composing the fixed-point subalgebra inclusion and the crossed product algebra inclusion with respect to the outer action of two finite groups $H$ and $K$ on a $II_1$ factor $P$ - that is, $P^H \subset P \rtimes K$. This class of subfactors, referred as the Bisch-Haagerup subfactor throughout this chapter, was introduced by the first author in a joint work with Uffe Haagerup in [4] where subfactors obtained by composition in general and this class in particular was studied extensively; he proved subsequent results with Sorin Popa in [6]. Several properties of these subfactors were characterized in terms of analogous properties of the group $G$ generated by the images of $H$ and $K$ in $Out(P)$. For instance it was proved in [4] that

(i) The subfactor is of finite depth iff $G$ is a finite group.

(ii) If $P$ is hyperfinite, then the subfactor is amenable iff $G$ is an amenable group.

(iii) If $H$ and $K$ act freely on $G$ and $P$ is hyperfinite, then the subfactor is strongly amenable iff the entropy of the group $G$ with respect to a particular probability measure is zero.

It was proved in [6] that (iv) The subfactor has property(T) in the sense of Popa (see [29]) iff $G$ has Kazhdan’s property (T).

It is natural to expect a description of the standard invariant of the subfactor $P^H \subset P \rtimes K$, in terms of the two groups $H$ and $K$. The intrinsic planar structure possessed by standard invariants of subfactors of type $II_1$ as discovered by Jones and formalized as planar algebras was mentioned in Chapter 2. Our aim in this chapter is to abstractly characterize
the standard invariant, or equivalently, the planar algebra associated to the subfactor \( P^H \subset P \times K \), under the assumption that the group generated by \( H \) and \( K \) in \( Aut(P) \) intersects trivially with \( Inn(P) \).

We begin by identifying the concrete model of the tower of basic construction of the subfactor \( P^H \subset P \times K \) using a Lemma characterizing the basic construction:

\[
P^H \subset P \times K \subset M_K(\mathbb{C}) \otimes (P \times H) \subset M_{K \times H}(\mathbb{C}) \otimes (P \times K) \subset M_{K \times H \times K}(\mathbb{C}) \otimes (P \times H) \subset \cdots
\]

Then, we compute the relative commutants and exhibit a nice basis which is in one-to-one correspondence with finite sequences of group elements coming alternately from \( K \) and \( H \) such that their product is identity in \( Aut(P) \); the key lemma used in this part is that every Galois automorphism of \( P^H \subset P \) (that is, an automorphism of \( P \) fixing \( P^H \) pointwise) is in \( H \). In order to define the planar algebra abstractly, we investigated how a set of 'generating tangles' act on the relative commutants in the framework of Jones' Theorem (Theorem 4.2.1 in [16]). By generating tangles, we mean a class of tangles which can generate any tangle by composition. Interestingly, the abstract description of the planar algebra can be viewed as an IRF (Interactions 'Round a Face) model in Statistical Mechanics. For convenience, we will refer to this as IRF planar algebra. After defining the action of tangles, we showed that it is indeed well-defined and respects composition of tangles. The biggest obstacle was to prove the latter part which we achieved by considering different configurations that could arise out of composition of tangles. We also found that even if we twist the actions of \( H \) and \( K \) by conjugation by inner automorphisms, the planar algebra remains the same. Conversely, if we start with a subfactor whose planar algebra is isomorphic to the IRF planar algebra, then the subfactor is indeed one of the Bisch-Haagerup type; to prove this, we use the results on intermediate subfactors from [1].

### 4.1 Abstract IRF Planar Algebra

In this section we will abstractly define the planar algebra corresponding to the Bisch-Haagerup subfactor \( P^H \subset P \times K \) (as described in [4]).
Let $G$ be a group generated by two of its finite subgroups $H$ and $K$. Let us define

$$S_n = \begin{cases} 
\{e\} & \text{if } n = 0 \\
K \times H \times K \times H \times \cdots & \text{if } n \geq 1
\end{cases}$$

$$S = \prod_{n \geq 0} S_n$$

$$L_n = \begin{cases} 
K, & \text{if } n \text{ is even} \\
H, & \text{otherwise}
\end{cases}$$

where $e$ denotes the identity of the group $G$. Let $\mu : S \to G$ be the multiplication map.

**Terminology:**

By a *face* in an unlabelled tangle $T$, we will mean a connected component of $D_0 \setminus \left( \bigcup_{i=1}^{b} D_i \cup S \right)$ where $D_0$ is the external disc, $D_i$ is the $i$-th internal disc for $i = 0, 1, 2 \cdots b$ and $S$ is the set of strings of (an element in the isotopy class of) $T$. By an *opening* in a tangle, we will mean the subset of the boundary of a disc lying between two consecutive marked points. An opening will be called *internal* (resp., *external*) if it is a subset of the boundary of the internal (resp., external) disc. Note that the boundary of a generic face may be disconnected due to the presence of loops or networks inside it (see Figure 4.1).

The set of connected components of the boundary of each face will have a single *outer component* and several (possibly none) *inner component(s).*

![Figure 4.1: Example of faces in a tangle](image)
Definition 4.1.30. A state \( f \) on a tangle \( T \) is a function \( f : \{ \text{all openings in } T \} \rightarrow H \amalg K \) such that following holds:

(i) \( f(\alpha) \in \begin{cases} K, & \text{if the face containing } \alpha \text{ is shaded,} \\ H, & \text{otherwise.} \end{cases} \)

(ii) Triviality on the outer component of the boundary of a face:

Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be the openings on the outer component of the boundary of a face counted clockwise, then we must have

\[
f(\alpha_1)^{\eta_1} f(\alpha_2)^{\eta_2} \cdots f(\alpha_m)^{\eta_m} = e
\]

where \( \eta_i = \begin{cases} +1, & \text{if } \alpha_i \text{ is an external opening,} \\ -1, & \text{otherwise.} \end{cases} \)

(iii) Triviality on internal discs:

\( f \) induces a map \( \partial f : \{ D_0, D_1, \ldots, D_b \} \rightarrow \coprod_{n \geq 0} S_n \) defined by

\[
\partial f(D_i) = (f(\alpha_1^{(i)}), f(\alpha_2^{(i)}), \ldots, f(\alpha_{2n_i}^{(i)})) \in S_{2n_i}
\]

where \( \alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{2n_i}^{(i)} \) are consecutive openings counted clockwise such that \( \alpha_1^{(i)} \) is the opening between the first and the second marked points of \( \partial D_i \).

We demand that \( \mu(\partial f(D_i)) = e \), for all \( i = 1, 2, \ldots b \).

The above definition also applies to networks (positive or negative).

Figure 4.2 illustrates conditions (ii) and (iii) of Definition 4.1.30; triviality on internal discs give \( b_1 b_2 b_3 b_4 = c_1 c_2 c_3 c_4 = d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8 = f_1 f_2 = e \) and triviality on the outer component of the boundaries of faces give \( a_1 b_1^{-1} = a_2 b_2^{-1} = a_3 d_3^{-1} d_3^{-1} b_3^{-1} = c_4^{-1} f_2^{-1} = c_2 = d_2^{-1} d_4^{-1} = d_3 = a_4 a_6 b_4^{-1} d_5^{-1} d_6^{-1} = d_7 = a_5 = e \). Note that the above relations also imply \( a_1 a_2 a_3 a_5 a_6 = e \), and \( c_1^{-1} c_3^{-1} f_1^{-1} = e \).

As the computation involving Figure 4.2 suggested, triviality on inner boundaries of a face and on the external disc are actually consequences of the definition of a state. This has been made precise in the following remark.

Remark 4.1.31. Let \( f \) be a state on a tangle or a network. Then the following conditions hold:
(ii) Triviality on every inner component of the boundary of a face:

For every inner component of the boundary of a face with openings $\alpha_1, \alpha_2, \ldots, \alpha_m$ counted clockwise, we have $f(\alpha_1)f(\alpha_2)\cdots f(\alpha_m) = e$.

(iii) Triviality on the external disc:

This is just a restatement of condition (iii) in Definition 4.1.30 applied to the external disc only if its color is greater than zero.

We prove the above remark using planar graphs. Without loss of generality, we may start with a tangle or a network which is connected. By a network or a tangle (with non-zero color on its external disc) being connected, we mean that the union of the boundaries of all the discs and strings is a connected set; a 0-tangle is said to be connected if the network obtained after removing its external disc is connected. To each tangle or network, we associate a planar graph with vertex set as the set of all marked points on the internal and external discs, and edges being the openings and the strings. Further, we make this graph a directed one in such a way that the directions on the edges arising from the openings are induced by clockwise orientation on the boundary of the discs, and the remaining edges (coming from the strings) are free to have any direction. Any state $f$ assigns group elements to edges arising from openings; we label each of the remaining
edges by $e$ and that is why we did not put any restriction on the direction of such edges. Note that the definition of the state implies the following condition on the group labelled planar directed graph:

*Triviality on the boundary each bounded face of the graph*:

If $g_1, g_2, \cdots, g_m$ are the group elements assigned to consecutive edges around a face read clockwise, then

$$g_1^{\eta_1} g_2^{\eta_2} \cdots g_m^{\eta_m} = e$$

where $\eta_i = \begin{cases} +1, & \text{if } i\text{-th edge induces clockwise orientation in the face}, \\ -1, & \text{otherwise}. \end{cases}$

To establish Remark 4.1.31, it is enough to prove *triviality on the boundary of the unbounded face*. To see this, we first consider a pair of bounded faces which have at least one vertex or edge in common. If this pair is considered as a separate graph, then using triviality on each face, it is easy to check triviality on the boundary of the unbounded face of this pair. One can then use this fact inductively to deduce the desired result for the whole graph.

Next, we move on to the definition of the planar algebra. Let the set of states on a tangle $T$ be denoted by $\mathcal{S}(T)$.

**The vector spaces:**

For $n \geq 0$, define $P_n = \mathbb{C}\{\bar{s} \in S_{2n} : \mu(\bar{s}) = e\}$.

**Action of tangles:**

Let $T$ be an unlabelled tangle with (possibly zero) internal disc(s) $D_1, D_2, \cdots, D_b$ and external disc $D_0$ where the color of $D_i$ is $n_i$. Then $T$ defines a multilinear map, denoted by $Z_T : P_{n_1} \otimes P_{n_2} \cdots \otimes P_{n_b} \to P_{n_0}$. We will define $Z_T(s_1, s_2, \cdots, s_b) \in P_{n_0}$, where $s_i \in S_{2n_i}$ such that $\mu(s_i) = e$. In fact, we will just prescribe the coefficient of $\bar{s}_0 \in S_{2n_0}$ (such that $\mu(\bar{s}_0) = e$) in the expansion of $Z_T(s_1, s_2, \cdots, s_b)$ in terms of the canonical basis mentioned above.

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1Since we started with a connected tangle or network, the associated planar graph will be connected; in particular, boundary of each face of the graph will be connected.
We choose and fix a representative in the isotopy class of $T$ and call it the *standard form* of $T$, denoted by $\bar{T}$. It is assumed to satisfy the following properties:

- $\bar{T}$ is in rectangular form - meaning that - all of its discs are replaced by boxes and it is placed in $\mathbb{R}^2$ in such a way that the boundaries of the boxes are parallel to the co-ordinate axes.

- The first marked point on the boundary of each box is on the top left corner.

- The collection of strings have finitely many local maxima or minima.

- The external box can be sliced by horizontal lines in such a way that each maximum, minimum, internal box is in a different slice.

To each local maximum or minimum of a string with end-points, we assign a scalar according to Figure 4.3. Let $p(T)$ denote the product of all scalars arising from the local maxima or minima and $n_+(T)$ (resp., $n_-(T)$) be the number of non-empty connected positive (resp., negative) network(s) in the tangle $\bar{T}$. Then, the coefficient $\langle Z_T(s_1, s_2, \cdots s_b)|s_0\rangle$ of $s_0$ in $Z_T(s_1, s_2, \cdots s_b)$ is given by:

$$p(T) \left| H \right|^{n_+(T)} \left| K \right|^{n_-(T)} \left| \{ f \in \mathcal{S}(T) : \partial f(D_i) = s_i \text{ for all } i = 0, 1, \cdots b \} \right|$$

Note that there could be several standard form representatives for $T$. However, one
standard form representative for $T$ can be transformed into another by a finite sequence of moves of the following three types:

I. Horizontal or vertical sliding of boxes,

II. Wiggling of the strings,

III. Rotation of an internal box by a multiple of 360°.

It is easy to check that the above three moves do not alter the number of connected networks and keeps $|\{f \in S(T) : \partial f(D_i) = s_i \text{ for all } i = 0, 1, \ldots b\}|$ unaltered. So, it remains to show that $p(T)$ is unchanged under the moves. Type I moves are the easiest because they do not generate any new local maximum or minimum. In each of the moves of type II and III, there arises pair(s) of local maxima and minima in such a way that the two scalars assigned to each pair are inverses of each other; as a result, $p(T)$ remains unchanged.

**Action of the tangles preserve composition:**

For $S$ an $n_0$-tangle with internal discs $D_1, D_2, \ldots, D_b$ having colors $n_1, n_2, \ldots, n_b$ respectively, and $T$ an $n_j$-tangle for some $j \in \{1, 2, \ldots b\}$, we would like to show $Z_{S \circ D_j} T = Z_S \circ (id_{P_{n_1}} \times \cdots \times Z_T \cdots \times id_{P_{n_b}})$. For this, we first identify a set $E$ of annular tangles (with the distinguished internal disc as $D_1$) which we call *elementary tangles*, namely,

(i) Capping tangles:

![Diagram](attachment:diagram.png)

with $\text{col}(D_1) = n \geq 1$, $\text{col}(D_0) = n - 1$, and $1 \leq i \leq 2n - 1$ ($\text{col}(D_1) = 1$ just means that there are no strings connecting the internal disc to the external disc),
(ii) Cap inclusion tangles:

![Cap inclusion tangle diagram]

with $\text{col}(D_1) = n \geq 0$, $\text{col}(D_0) = n + 1$, and $1 \leq i \leq 2n + 1$.

(iii) Left inclusion tangles:

![Left inclusion tangle diagram]

with $\text{col}(D_1) = n \geq 0$, $\text{col}(D_0) = n + 2$.

Disc inclusion tangles:

(iv) $p$ strings $q$ strings $r$ strings

![Disc inclusion tangle diagram]

with $\text{col}(D_1) = \text{col}(D_0) = n \geq 0$, $\text{col}(D_2) = q$, where $p \geq 0, q \geq 0, r \geq 0$ such that $p + q + r = n$ and $p$ is even.

Note that any annular tangle can be expressed as a composition of the elementary tangles. To see this, express the annular tangle in standard form and cut it into horizontal strips each of which contains at most one internal disc or one local maxima or minima. Now, the strip containing the distinguished internal disc inside the annular tangle can be obtained by composition of elementary tangles of type (iii) and type (ii) (more specifically,
the inclusion tangles); one can then glue the other strips consecutively one after the other along the lines of cutting to back the original tangle. Each such gluing operation is given by composition of an elementary tangle of types (i), (ii), (iv) or (iv'). So, to prove that the action of the tangles preserve composition, it is enough to prove \( Z_{E \circ D, T} = Z_E \circ Z_T \) (resp., \( Z_{E \circ D, T} = Z_E \circ (Z_T \times id_{D'_a}) \)) for any tangle \( T \) and any \( E \in \mathcal{E} \) of type (i), (ii) or (iii) (resp. (iv) or (iv')) whenever the composition makes sense.

We fix an \( n \)-tangle \( T \) with internal discs \( D'_1, D'_2, \ldots, D'_b \) with colors \( n_1, n_2, \ldots, n_b \) respectively, and an \( n_0 \)-tangle \( E \in \mathcal{E} \) such that both \( T \) and \( E \) are in standard forms and color of \( D_1 \) in \( E \) is \( n \). Let us consider the standard form on \( E \circ D_1 \ T \) induced by the standard forms of \( E \) and \( T \). Our goal is to show:

\[
\langle Z_{E \circ D, T}(s_1, s_2, \ldots, s_b) | s_0 \rangle = \sum_{s \in S_{2n} \text{ s.t. } \mu(s) = e} \langle Z_E(s) | s_0 \rangle \langle Z_T(s_1, s_2, \ldots, s_b) | s \rangle
\]

if \( E \) is of type (i), (ii) or (iii), and

\[
\langle Z_{E \circ D, T}(s_1, s_2, \ldots, s_b, t) | s_0 \rangle = \sum_{s \in S_{2n} \text{ s.t. } \mu(s) = e} \langle Z_E(s, t) | s_0 \rangle \langle Z_T(s_1, s_2, \ldots, s_b) | s \rangle
\]

if \( E \) is of type (iv) or (iv') where \( s_j \in S_{2n_j} \) for \( 0 \leq j \leq b \) and \( t \in S_q \) such that \( \mu(s_j) = e = \mu(t) \) for all \( j \). An interesting situation arises when we pick elementary tangles of type (i), since composition of tangles in this case may lead to a change in the number of connected networks. The reasoning in the other cases is either similar or straightforward.

For \( E \) being type (i) elementary tangle, the above equation is equivalent to:

\[
p(E \circ D_1 \ T) | H |^{n_+ (E \circ D_1 \ T)} | K |^{n_- (E \circ D_1 \ T)} \left\{ \begin{array}{l} f \in \mathcal{S}(E \circ D_1 \ T) \\ \partial f(D'_j) = s_j \\ \text{for } 1 \leq j \leq b, \\ \partial f(D_0) = s_0 \end{array} \right\}
\]

\[
= p(E) p(T) | H |^{n_+ (E) + n_+ (T)} | K |^{n_- (E) + n_- (T)}
\]

\[
\cdot \sum_{s \in S_{2n} \text{ s.t. } \mu(s) = e} \left\{ \begin{array}{l} f \in \mathcal{S}(E) \\ \partial f(D'_1) = s_j \\ \partial f(D_0) = s_0 \end{array} \right\} \left\{ \begin{array}{l} f \in \mathcal{S}(T) \\ \partial f(D'_j) = s_j \\ \text{for } 1 \leq j \leq b, \\ \partial f(D'_0) = s_0 \end{array} \right\}
\]
where $D'_0$ denotes the external disc of $T$. First, observe that $p(E \circ D, T) = p(E) p(T)$. To show the equality of the remaining scalars, we consider the following two cases.

**Case 1:** The string which connects the $i$-th and the $(i + 1)$-th points on $D_1$ in $E$, does not produce any new network in $E \circ D, T$ other than those which are already present in $T$.

Clearly, $n_{\epsilon}(E \circ D, T) = n_{\epsilon}(T)$ (since no new network appears in $E \circ D, T$) and $n_{\epsilon}(E) = 0$ for $\epsilon \in \{+, -\}$.

A typical example of such a case can be viewed in the following picture where we label the openings on the internal discs of $T$ by group elements coming from the coordinates of $s_j$ for $1 \leq j \leq b$, and the openings on $D_0$ of $E$ by coordinates of $s_0 = (g_1, g_2, \cdots, g_{2n-2})$.

![Figure 4.4:](image)

Using triviality on the boundary of faces in the definition of a state, we get

$$
\left\{ \begin{array}{l}
  f \in \mathcal{S}(E) \\
  \partial f(D_1) = s \\
  \partial f(D_0) = s_0
\end{array} \right\} \neq 0 \iff \left\{ \begin{array}{l}
  f \in \mathcal{S}(E) \\
  \partial f(D_1) = s \\
  \partial f(D_0) = s_0
\end{array} \right\} = 1
$$

\[ \iff s = (g_1, \cdots, g_{i-2}, g_{i-1}g, e, g^{-1}, g_i, \cdots, g_{2n-2}) \in S_{2n} \text{ for some } g \in I_i. \]
Define $s^g = (g_1, \ldots, g_{i-2}, (g_{i-1}g), e, g^{-1}, g_i, \ldots, g_{2n-2})$ for $g \in L_i$. So, it is enough to check

$$\left\{ \begin{array}{l} f \in S(E \circ D_1, T) \quad \partial f(D_j') = s_j \\
\text{for } 1 \leq j \leq b, \\
\partial f(D_0) = s_0 \\
\end{array} \right\} = \sum_{g \in L_i} \left\{ \begin{array}{l} f \in S(T) \quad \partial f(D_j') = s_j \\
\text{for } 1 \leq j \leq b, \\
\partial f(D_0) = s_0 \\
\end{array} \right\}$$

Carefully observing Figure 4.4 and using triviality on the boundary of faces once again, we get

$$\left\{ \begin{array}{l} f \in S(E \circ D_1, T) \quad \partial f(D_j') = s_j \\
\text{for } 1 \leq j \leq b, \\
\partial f(D_0) = s_0 \\
\end{array} \right\} \neq 0, \text{ equivalently, equals to 1}$$

$$\Longrightarrow \left\{ \begin{array}{l} f \in S(T) \quad \partial f(D_j') = s_j \\
\text{for } 1 \leq j \leq b, \\
\partial f(D_0) = s_0 \\
\end{array} \right\} = \delta_{g, b_1 b_2 \cdots b_n}$$

where $\eta_j = \pm 1$ according as the corresponding opening is external or internal. Conversely, if $\sum_{g \in L_i} \left\{ \begin{array}{l} f \in S(T) \quad \partial f(D_j') = s_j \\
\text{for } 1 \leq j \leq b, \\
\partial f(D_0) = s_0 \\
\end{array} \right\}$ is non-zero, then it has to equal to 1 since $g$ must be $b_1 b_2 \cdots b_n$ by triviality on the boundary of faces in $T$; from the unique state on $T$ which makes the above sum non-zero, one can easily induce a well-defined state on $E \circ D_1, T$, and hence $\left\{ \begin{array}{l} f \in S(E \circ D_1, T) \quad \partial f(D_j') = s_j \\
\text{for } 1 \leq j \leq b, \\
\partial f(D_0) = s_0 \\
\end{array} \right\} = 1$. This completes the proof of Case 1.

**Case 2:** The string which connects the $i$-th and the $(i+1)$-th points on $D_1$ in $E$, produces a new network in $E \circ D_1, T$ other than those which are already present in $T$.

First, let us assume that the new network is positive, equivalently, $i$ is odd. Clearly, $n_-(E \circ D_1, T) = n_-(T)$ (since no new negative network appears in $E \circ D_1, T$), and $n_+(E \circ D_1, T) = n_+(T) + 1$.

Further, assume that $col(D_0) \geq 1$. In this case, $n_+(E) = 0$ for $\epsilon \in \{+, -\}$. A typical example of this case can be viewed in the following picture where we label the openings on
the internal discs of $T$ by group elements coming from the coordinates of $s_j$ for $1 \leq j \leq b$, and the openings on $D_0$ of $E$ by coordinates of $s_0 = (g_1, g_2, \ldots, g_{2n-2})$. So, in this case,

\[ \begin{array}{c}
E \quad g_{i-2} \quad g_{i-1} \quad g_i \\
T \quad a_i \quad a_{i+1} \quad D_1 \\
D_0 \\
\end{array} \]

Figure 4.5:

it is enough to check

\[ \left| H \right| \begin{cases} f \in S(E \circ D, T) & \text{if } 1 \leq j \leq b, \\
& \partial f(D_j) = s_j \\
& \partial f(D_0) = s_0 \end{cases} = \sum \begin{cases} f \in S(T) & \text{if } 1 \leq j \leq b, \\
& \partial f(D_j) = s_j \\
& \partial f(D_0) = s_0 \end{cases} \]

If \( f \in S(E \circ D, T) \) is non-empty (equivalently, singleton), from Figure 4.5, we have \( a_1 a_2 \cdots a_k = e \) and \( g_{i-1} b_{i}^{\eta_j} b_{2}^{\eta_j} \cdots b_{i}^{\eta_j} = e \) where \( \eta_j = \pm 1 \) according as the corresponding opening is external or internal. For any \( g \in H \), define \( f^g \) by setting \( \partial f(D_j) = s_j \) for \( 1 \leq j \leq b \) and \( \partial f(D_0) = s_0 \). To check whether \( f^g \) is a state, we consider the face in \( T \) appearing in Figure 4.5; triviality on the boundary of this face is given by the equation \( g_{i-1} b_{i}^{\eta_j} b_{2}^{\eta_j} \cdots b_{i}^{\eta_j} (g_{i-1} g) a_1 a_2 \cdots a_k = e \) which indeed holds. Triviality on all the other discs or faces is induced by the existence of the state on \( E \circ D, T \).

Thus, \( \left\{ f \in S(T) \bigg| \begin{array}{c}
\partial f(D_j) = s_j \\
\text{for } 1 \leq j \leq b \\
\partial f(D_0) = s_0 
\end{array} \right\} = 1 \) for all \( g \in H \). Conversely, if the right hand side is non-zero, then there exists \( g \in H \) such that \( f^g \) (defined earlier) is a state on \( T \). Analyzing Figure 4.5, we get \( g_{i-1} b_{i}^{\eta_j} b_{2}^{\eta_j} \cdots b_{i}^{\eta_j} (g_{i-1} g) a_1 a_2 \cdots a_k^{-1} a_1^{-1} = e \). Note that the opening between the \( i \)-th and the \( (i + 1) \)-th points of the disc \( D_0 \) of \( T \), is assigned
Now, if we consider the network appearing in Figure 4.5 separately, then we have triviality on each of its internal faces and discs (induced by $f^g$ being a state); by Remark 4.1.31 (ii), we also have $a_1 a_2 \cdots a_k = e$ (triviality on the internal boundary of the external face of the network). This implies $b_1^{q_1} b_2^{q_2} \cdots b_l^{q_l} g_{i-1} = e$; as a result, $f^h$ is a state for every $h \in H$. So, if right hand side is non-zero, then it has to be $|H|$; moreover, we get $b_1^{q_1} b_2^{q_2} \cdots b_l^{q_l} g_{i-1} = e$ which plays an important role in showing that
\[
\begin{cases}
  f \in \mathcal{S}(E \circ D_1 T) & \partial f(D'_j) = s_j \\
  for \ 1 \leq j \leq b,
\end{cases}
\]
\[
\begin{cases}
  f \in \mathcal{S}(E \circ D_0 T) & \partial f(D_0) = s_0
\end{cases}
\]

This finishes the proof for the case where $i$ is odd. For $i$ even, the proof is similar, except that one has to interchange $|H|$ and $|K|$.

The subcase that deserves separate treatment is the one in which $\text{col}(D_0) = 0$. In this case, $n_+(E) = 1$ and $|\{ f \in \mathcal{S}(E) : \partial f(D_1) = s_1 \}| = \delta_{s_1(e,e)}$. Therefore, it is enough to show
\[
\begin{cases}
  f \in \mathcal{S}(E \circ D_1 T) & \partial f(D'_j) = s_j \\
  for \ 1 \leq j \leq b
\end{cases}
\]
\[
\begin{cases}
  f \in \mathcal{S}(T) & \partial f(D'_j) = s_j \\
  for \ 1 \leq j \leq b,
\end{cases}
\]

The proof of the equality of the two sides is similar and is left to the reader.

We now analyze the filtered $\ast$-algebra structure of $P$ and the action of Jones projection tangles and conditional expectation tangles which will be useful in Section 4.3 to show that the $P$ is isomorphic to the planar algebra arising from the Bisch-Haagerup subfactor. We start with laying some notations.

**Notation:** Define
\[
\overline{S}_n = \begin{cases}
  \{ e \} & \text{if } n = 0 \\
  \cdots \times H \times K \times H \times K & \text{if } n \geq 1
\end{cases}
\]
\[
T_n = \begin{cases}
  \{ e \} & \text{if } n = 0 \\
  H \times K \times H \times K \times \cdots & \text{if } n \geq 1
\end{cases}
\]

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Define ~ : $S_n \to \widetilde{S_n}$ by $(s_1, s_2, \ldots, s_n) = (s_n^{-1}, s_{n-1}^{-1}, \ldots, s_2^{-1}, s_1^{-1})$ for $(s_1, s_2, \ldots, s_n) \in S_n$ and let $^{-1} : \widetilde{S_n} \to S_n$ denote its inverse.

**Remark 4.1.32.** We describe below the main structural features of the planar algebra $P$.

(i) **Identity:**

$$1P_n = \begin{cases} e & \text{if } n = 0 \\ \sum_{s \in S_{n-1}} (s, e, s, e) & \text{if } n \geq 1 \end{cases}$$

(ii) **$*$-structure:** Define $*$ on $P$ by defining on the basis as

$$s^* = (s_{n-1}^{-1}, \ldots, s_2^{-1}, s_1^{-1}, s_2n^{-1})$$

where $s = (s_1, s_2, \ldots, s_{2n}) \in S_{2n}$ such that $\mu(s) = e$ for $n \geq 1$, and then extend conjugate linearly. Clearly, $*$ is an involution. One also needs to verify whether the action of a tangle $T$ preserves $*$, that is, $Z_T \circ (\ast \times \ast \times \ast) = \ast \circ Z_T$; in particular, $\langle Z_T(s_1^*, \ldots, s_{2n}^*) | s_0^* \rangle = \langle Z_T(s_1, \ldots, s_{2n}) | s_0 \rangle$. It is enough to check this equation for the cases when $T$ has no internal disc or closed loops, and when $T$ is an elementary tangle. The actual verification in each of these cases is completely routine and is left to the reader.

(iii) **Multiplication:**

$$(a_1, l_1, b_1, h_1) \cdot (a_2, l_2, b_2, h_2) = \delta_{b_1} \cdot \frac{1}{a_2} \cdot (a_1, l_1l_2, b_2, h_2h_1)$$

where $a_i \in S_{n-1}$, $b_i \in \widetilde{S_{n-1}}$, $l_i \in L_{n-1}$, $h_i \in H$ such that $\mu(a_i, l_i, b_i, h_i) = e$ for $i = 1, 2$ and $n \geq 1$ (where we consider the elements $a_i$ and $b_i$ to be void in the case of $n = 1$).

(iv) **Inclusion:**

$$P_n \ni s \mapsto \sum_{l_1, l_2 \in L_{n-1}} (s_1, s_2, \ldots, s_{n-1}, l_1, e, l_2, s_{n+1}, \ldots s_{2n}) \in P_{n+1}$$

where $s = (s_1, s_2, \ldots, s_{2n}) \in S_{2n}$ such that $\mu(s) = e$ for $n \geq 1$.

(v) **Jones Projection Tangle:**

For $P_2$,
and for $P_{n+1}$ for $n > 1,$

$\begin{array}{c}
\begin{array}{c}
1 \\
\hline \\
2 \\
\hline \\
n \\
\hline \\
n+1
\end{array}
\end{array}
= \sqrt{\frac{|K|}{|H|}} \sum_{h \in H} (e, h, e, h^{-1})$

(vi) Conditional Expectation Tangle from $P_{n+1}$ onto $P_n$: 

For $n \geq 1,$ let $s_1 \in S_{n-1}, s_2 \in \widetilde{S}_{n-1}, m_1, m_2 \in L_{n-1}, l \in L_n, h \in H$ such that 

$\mu(s_1, m_1 m_2, s_2, h) = e.$ Then,

$\begin{array}{c}
\begin{array}{c}
1 \\
\hline \\
2 \\
\hline \\
* \\
\hline \\
* \\
\hline \\
n \\
\hline \\
\vdots \\
\hline \\
(s_1, m_1, l, m_2, s_2, h)
\end{array}
\end{array}
= \delta_{l,e} \sqrt{\frac{|L_n|}{|L_{n-1}|}} (s_1, m_1 m_2, s_2, h)$

and for $n = 0,$

$\begin{array}{c}
\begin{array}{c}
* \\
\hline \\
(k, h)
\end{array}
\end{array}
= \delta_{k,e} \delta_{k,e} \sqrt{\frac{|K|}{|H|}} e$

(vii) Conditional Expectation Tangle from $P_n$ onto $P_{1,n}$:

For $n \geq 2$ let $k_1, k_2 \in K, l \in T_{2n-3}, h \in H$ such that $\mu(k_1, l, k_2, h) = e.$ Then,
Let $P$ be a $II_1$ factor and $G$ be a subgroup of $\text{Aut}(P)$ intersecting trivially with $\text{Inn}(P)$ (in other words, $G$ acts outerly on $P$) such that $G$ is generated by two of its finite subgroups $H$ and $K$. Consider the associated Bisch-Haagerup subfactor $P^H \subset P \times K$.

In this section we will give a concrete realization of the tower of basic construction of this subfactor. First, let us recall a characterization of the basic construction of a finite index subfactor, the proof of which, being similar to that of Lemma 5.3.1 in [19] in the case of inclusion of finite-dimensional $C^*$-algebras, is left to the reader.

**Lemma 4.2.33.** Let $N \subset M$ be a finite index subfactor with $\mathbb{E} : M \to N$ being the trace-preserving conditional expectation, $B$ be a $II_1$ factor containing $M$ as a subfactor and $f$ be a projection in $B$ satisfying:

(i) $fxf = \mathbb{E}(x)f$ for all $x \in M$

(ii) $B$ is the algebra generated by $M$ and $f$.

Then, $B$ is isomorphic to the basic construction $M_1$ of $N \subset M$. 

4.2 Tower of basic construction and the relative commutants
Before going into the next lemma, we recall some basic facts and notations for the crossed product construction. Unless otherwise specified, we will reserve the symbol \( e \) for the identity element of a group. The crossed product \( P \times K \) can be realized as the von Neumann subalgebra of \( \mathcal{L}(L^2(K) \otimes L^2(P)) \) \( \cong M_K(\mathbb{C}) \otimes \mathcal{L}(L^2(P)) \) generated by the images of \( P \) and \( K \) in the following way:

\[
P \ni x \mapsto \sum_{k \in K} E_{k,k} \otimes k^{-1}(x) \in M_K(\mathbb{C}) \otimes \mathcal{L}(L^2(P)) \tag{4.2.1}
\]

\[
K \ni k \mapsto \lambda_k \otimes 1 \in M_K(\mathbb{C}) \otimes \mathcal{L}(L^2(P)) \tag{4.2.2}
\]

where we set the convention of considering \( k(x) \) as the element of \( P \) obtained by applying the automorphism \( k \) on \( x \) (in \( P \)) and \( \lambda_k \) is the matrix in \( M_K(\mathbb{C}) \) corresponding to left multiplication by \( k \). Consequently, the following commutation relation holds in \( P \times K \):

\[
k x k^{-1} = k(x) \quad \text{for all} \quad x \in P, k \in K. \tag{4.2.3}
\]

However, \( P \times K \) can also be viewed as the vector space generated by elements of the form \( \sum_{k \in K} x_k k \) where \( x_k \in P \) where the multiplication structure is given by the relation (4.2.3). The unique trace on \( P \times K \) is given by:

\[
\text{tr}\left( \sum_{k \in K} x_k k \right) = \text{tr}(x_e)
\]

and the unique trace-preserving conditional expectation is given by:

\[
\mathbb{E}_{P \times K}^{\mathbb{P}}(\sum_{k \in K} x_k k) = x_e.
\]

If \( P^K \) denotes the fixed point subalgebra of \( P \), then \( P \times K \) is isomorphic to the basic construction of \( P^K \subset P \) where the Jones projection is given by:

\[
e_1 = \frac{1}{|K|} \sum_{k \in K} k \in P \times K
\]

implementing the conditional expectation:

\[
\mathbb{E}_{P^K}^{P}(x) = \frac{1}{|K|} \sum_{k \in K} k(x) \in P^K \quad \text{for all} \quad x \in P.
\]

The basic construction of \( P \subset P \times K \) is isomorphic to \( M_K(\mathbb{C}) \otimes P \) where the inclusion \( P \times K \hookrightarrow M_K(\mathbb{C}) \otimes P \) is given by the maps (4.2.1), (4.2.2), and the corresponding
Jones projection is given by $e_2 = E_{e,e} \otimes 1 \in M_K(C) \otimes P$ implementing the conditional expectation $E^{P \times K}_P$. The next element in the tower of basic construction is given by $M_K(C) \otimes (P \times K)$ where the inclusion $M_K(C) \otimes P \to M_K(C) \otimes (P \times K)$ is induced by the inclusion $P \subset P \times K$ and the Jones projection is given by:

$$e_3 = \frac{1}{|K|} \sum_{k \in K} \rho_{k^{-1}} \otimes k \in M_K(C) \otimes (P \times K) \quad (4.2.4)$$

implementing the conditional expectation:

$$E^{M_K(C) \otimes P}_{P \times K}(E_{k_1,k_2} \otimes x) = \frac{1}{|K|} k_1 x k_2^{-1} \in P \times K \text{ for all } x \in P, k_1, k_2 \in K \quad (4.2.5)$$

where $\rho_k$ is the matrix in $M_K(C)$ corresponding to right multiplication by $k$.

Coming back to the context of Bisch-Haagerup subfactor we consider the unital inclusions $P^H \hookrightarrow P \times K \hookrightarrow M_K(C) \otimes (P \times H)$ where the second inclusion factors through $M_K(C) \otimes P$ in the obvious way.

**Lemma 4.2.34.** $M_K(C) \otimes (P \times H)$ is the basic construction for $P^H \subset P \times K$ with Jones projection $e_1 = E_{e,e} \otimes \frac{1}{|H|} \sum_{h \in H} h$.

**Proof:** We need to show that conditions (i) and (ii) of Lemma 4.2.33 are satisfied. To show (i), let us assume that $\tilde{x} = \sum_{k \in K} x_k k$ denotes a typical element of $P \times K$.

$$e_1 \tilde{x} e_1 = \left( E_{e,e} \otimes \frac{1}{|H|} \sum_{h \in H} h \right) \left( E_{e,e} \otimes \frac{1}{|H|} \sum_{h' \in H} h' \right)$$

$$= \left( E_{e,e} \otimes \frac{1}{|H|} \sum_{h \in H} h \right) \left( \sum_{k' \in K} E_{k',k} \lambda_k \otimes k'^{-1}(x_k) \right) \left( E_{e,e} \otimes \frac{1}{|H|} \sum_{h' \in H} h' \right)$$

$$= \left( \sum_{k \in K} E_{e,e} \lambda_k E_{e,e} \otimes \frac{1}{|H|^2} \sum_{h \in H} h x_k h' \right)$$

$$= \left( E_{e,e} \otimes \frac{1}{|H|^2} \sum_{h \in H} h(x_e) h h' \right)$$

whereas

$$E(\tilde{x}) e_1 = \left( \sum_{k \in K} E_{k,k} \otimes \frac{1}{|H|} \sum_{h \in H} k^{-1}(h(x_e)) \right) \left( E_{e,e} \otimes \frac{1}{|H|} \sum_{h' \in H} h' \right)$$

$$= \left( E_{e,e} \otimes \frac{1}{|H|^2} \sum_{h \in H} h(x_e) h h' \right)$$
Therefore, LHS = RHS.

To show (ii), it is enough to show that elements of the form \( E_{k_1,k_2} \otimes x h \) for \( x \in P, \ h \in H, \ k_1, k_2 \in K \) are in the algebraic span of \( P \times K \) and \( e_1 \). Let us denote the Jones projection in \( P \times H \) corresponding to the inclusion \( P^H \subset P \) by \( f = \frac{1}{|H|} \sum_{h \in H} h. \) Thus,

\[
e_1 = E_{e,e} \otimes f \in M_K(\mathbb{C}) \otimes P \times H.
\]

This implies

\[
P e_1 P = E_{e,e} \otimes P f P = E_{e,e} \otimes P \times H \subset M_K(\mathbb{C}) \otimes P \times H
\]

where \( P \) in the left hand side is identified with its image inside \( M_K(\mathbb{C}) \otimes P \times H \) (namely, the prescription given by (4.2.1)). Thus the algebraic span of \( P \times K \) and \( e_1 \) contains elements of the type \( E_{e,e} \otimes x h \) for \( x \in P, \ h \in H \). To obtain elements of the form \( E_{k_1,k_2} \otimes x h \), note that the relation \( \lambda_{k_1} E_{e,e} \lambda_{k_2}^{-1} = E_{k_1,k_2} \) holds in \( M_K(\mathbb{C}) \).

**Lemma 4.2.35.** \( M_K(\mathbb{C}) \otimes M_H(\mathbb{C}) \otimes (P \times K) \) is the basic construction for \( P \times K \subset M_K(\mathbb{C}) \otimes (P \times H) \) where the Jones projection is given by \( e_2 = \frac{1}{|K|} \sum_{k \in K} \rho_k^{-1} \otimes E_{e,e} \otimes k \).

**Proof:** The conditional expectation \( \mathbb{E}^{M_K(\mathbb{C})\otimes(P\times K)}_{P \times K} \) is the composition \( \mathbb{E}^{M_K(\mathbb{C})\otimes(P\times H)}_{P \times K} \circ \mathbb{E}^{M_K(\mathbb{C})\otimes(P\times K)}_{P \times K} \).

Therefore, \( \mathbb{E}(E_{k_1,k_2} \otimes x h) = \delta_{h,e} \frac{1}{|k|} k_1 x k_2^{-1} \).

To show condition (i) of Lemma 4.2.33,

\[
e_2(E_{k_1,k_2} \otimes x h)e_2
\]

\[
= \frac{1}{|K|^2} \left( \sum_{k' \in K} \rho_{k'^{-1}} \otimes E_{e,e} \otimes k' \right) \left( \sum_{h' \in H} E_{k_1,k_2} \otimes E_{h',h'} \lambda_h \otimes h'^{-1}(x) \right) \left( \sum_{k'' \in K} \rho_{k''^{-1}} \otimes E_{e,e} \otimes k'' \right)
\]

\[
= \delta_{h,e} \frac{1}{|K|^2} \left( \sum_{k',k'' \in K} \rho_{k'^{-1}} E_{k_1,k_2} \rho_{k''^{-1}} \otimes E_{e,e} \lambda_h \otimes k' x k'' \right)
\]

whereas

\[
\mathbb{E}(E_{k_1,k_2} \otimes x h)e_2
\]

\[
= \delta_{h,e} \frac{1}{|K|^2} \left( \sum_{h' \in H} \sum_{k \in K} \lambda_{k_1} E_{k,k} \lambda_{k_2}^{-1} \otimes E_{h',h'} \otimes h'^{-1}(k^{-1}(x)) \right) \left( \sum_{k' \in K} \rho_{k'^{-1}} \otimes E_{e,e} \otimes k' \right)
\]

\[
= \delta_{h,e} \frac{1}{|K|^2} \left( \sum_{k \in K} \lambda_{k_1} E_{k,k} \lambda_{k_2}^{-1} \rho_{k'^{-1}} \otimes E_{e,e} \otimes k^{-1}(x) k' \right)
\]
and the two sides are the same after renaming the indices.

To show condition (ii), note that $M_K(C) \otimes (P \times K)$ is algebraically generated by $M_K(C) \otimes P$ and $\frac{1}{|K|} \sum_{k \in K} \rho_{k^{-1}} \otimes k^r$ (by the remarks preceding Equation (4.2.4)). The following holds in $M_K(C) \otimes M_H(C) \otimes (P \times K)$ because of this fact and the way $P$ sits inside $M_H(C) \otimes P$ (since in the second tensor component we get expressions of the form $\sum_{h,h' \in H} E_{h,h'} E_{h',h'}$ which reduces to $E_{e,e}$):

\begin{equation}
(M_K(C) \otimes P)e_2(M_K(C) \otimes P) = M_K(C) \otimes E_{e,e} \otimes (P \times K)
\end{equation}

\begin{equation}
\Rightarrow (M_K(C) \otimes P \times H)e_2(M_K(C) \otimes P \times H) = M_K(C) \otimes M_H(C) \otimes (P \times K)
\end{equation}

where the last implication is again due to the relation $\lambda_{h_1} E_{e,e} \rho_{h_2} = E_{h_1,h_2}$ in $M_H(C)$. □

Thus, we have the first two levels in the tower of basic construction:

\begin{align*}
P^H \subset P \times K \subset M_K(C) \otimes (P \times H) \subset M_{K \times H}(C) \otimes (P \times K)
\end{align*}

where we identify $M_{K \times H}(C)$ with $M_K(C) \otimes M_H(C)$. The next levels in the tower are obvious generalizations and we gather everything in the following proposition.

**Proposition 4.2.36.** Let $G$ be a group acting outerly on the II$_1$ factor $P$ such that $G$ is generated by two of its finite subgroups $H$ and $K$. Then the $n$th element of the tower of basic construction of the Bisch-Haagerup subfactor $N = P^H \subset P \times K = M$ is given by:

\begin{equation}
M_n \cong M_{S_n}(C) \otimes (P \times L_n)
\end{equation}

where the inclusion of $M_n$ inside $M_{n+1}$ is as follows:

\begin{equation}
M_n \ni E_{s,l} \otimes x \mapsto \sum_{i \in L_n} E_{s,i} \otimes E_{l,i} \otimes l^{-1}(x) \in M_{n+1}
\end{equation}

for all $x \in P$, $s,l \in S_n$

\begin{equation}
M_n \ni E_{s,l} \otimes l \mapsto E_{s,l} \otimes \lambda_l \otimes e \in M_{n+1}
\end{equation}

for all $l \in L_n$, $s,l \in S_n$

and the $n$th Jones projection is:

\begin{equation}
M_n \ni e_n = \begin{cases} \frac{1}{|L_n|} \sum_{i \in L_n} I_{M_{S_{n-2}}} \otimes \rho_{l^{-1}} \otimes E_{e,e} \otimes l, & \text{if } n > 1 \\ \frac{1}{|H|} \sum_{h \in H} E_{e,e} \otimes h, & \text{if } n = 1 \end{cases}
\end{equation}

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Proof: We use induction. The case of $n = 1$ is a little different from the rest and is proved in Lemma 4.2.34 and the $n = 2$ case is proved in Lemma 4.2.35. Suppose the statement of the above proposition holds up to a level $n > 2$. Now, the subfactor $M_{n-1} \subset M_n$ is isomorphic to:

$$M_{S_{n-1}} \otimes P \times L_{n-1} \subset M_{S_{n-1}} \otimes M_{L_{n-1}} \otimes P \times L_n$$

where we identify $M_{S_{n-1}} \otimes M_{L_{n-1}}$ with $M_S$ and the inclusion is induced by identity over $M_{S_{n-1}}$ tensored with the inclusion of the subfactor $P \times L_{n-1} \subset M_{L_{n-1}} \otimes P \times L_n$. Using Lemma 4.2.35 for $K = L_{n-1}$ and $H = L_n$, it is clear that the statement of the proposition holds for the level $n + 1$.

Remark 4.2.37. The unique trace-preserving conditional expectation is given by:

$$\mathbb{E}_{M_{n+1}}^M(E_{s_1,s_2} \otimes E_{m_1,m_2} \otimes x l) = \delta_{l,e} \frac{1}{[L_n]} E_{s_1,s_2} \otimes m_1 x m_2^{-1}$$

where $s_1, s_2 \in S_{n-1}$, $m_1, m_2 \in L_{n-1}$, $l \in L_n$ and $x \in P$.

We will now compute the relative commutants with the above model of basic construction. For this, we need the following two lemmas, where we denote the set of automorphisms of $M \supset N$ that fixes elements of $N$ pointwise by $Gal(N \subset M)$.

Lemma 4.2.38. Let $N \subset M$ be a finite index subfactor and $\theta \in Gal(N \subset M)$, then the bimodule $M L^2(\theta)_M$ (where the module is $L^2(M)$ with usual left action of $M$ but right action is twisted by $\theta$) is a 1-dimensional irreducible sub-bimodule of $M L^2(M_1)_M$.

Proof: Define $u_\theta : L^2(M) \rightarrow L^2(M)$ by $u_\theta(x \Omega) = \theta(x) \Omega$ for $x \in M$ where $\Omega$ is the cyclic and separating vector in the GNS construction with respect to the canonical trace $tr$. Note that $u_\theta(n_1 \cdot x \Omega \cdot n_2) = u_\theta(n_1 x n_2 \Omega) = n_1 \cdot \theta(x) \Omega \cdot n_2$ for $n_1, n_2 \in N$, $x \in M$. This implies $u_\theta \in N' \cap M_1$. Now define $T : M L^2(\theta)_M \rightarrow M L^2(M_1)_M$ by $T(x \Omega) = x u_\theta x \Omega$ for $x \in M$. It is completely routine to check that $T$ is a well-defined $M$-$M$ linear isometry and we leave this to the reader.

Corollary 4.2.39. $H = Gal(P^H \subset P)$
Proof: Clearly \( H \subset \text{Gal}(P^H \subset P) \).

Let \( \theta \in \text{Gal}(P^H \subset P) \). Note that \( pL^2(P \rtimes H)_P \cong \oplus_{h \in H} pL^2(h)_P \). Thus by Lemma 4.2.38, \( pL^2(\theta)_P \cong pL^2(h)_P \) for some \( h \in H \). This implies \( \theta h^{-1} \in \text{Inn}(P) \cap \text{Gal}(P^H \subset P) = \{ id_P \} \) since \( P^H \subset P \) is irreducible. Hence, \( \theta \in H \). \( \square \)

Lemma 4.2.40. Let \( N \subset M \) be an irreducible subfactor, i.e \( N' \cap M \cong \mathbb{C} \) and \( \theta \in \text{Aut}(M) \). For \( x \in M \), the following are equivalent:

(i) \( x \neq 0 \) and \( x\theta(y) = yx \) for all \( y \in N \),

(ii) \( x_0 := \frac{\theta}{\|x\|} \in \mathcal{U}(M) \) and \( \text{Ad}_{x_0} \circ \theta \in \text{Gal}(N \subset M) \).

Proof (ii)\( \Rightarrow \)(i) part is easy.

For (i)\( \Rightarrow \)(ii), note that we also have \( \theta(y)x^* = x^*y \) for all \( y \in N \). Thus, \( x^*x \in \theta(N)' \cap M \) and \( xx^* \in N' \cap M \). Since \( N' \cap M \cong \mathbb{C} \), \( xx^* = x^*x = \|x\|^2 \). Hence, \( x_0 \in \mathcal{U}(M) \) and \( x_0 \theta(y) x_0^* = y \) for all \( y \in N \). This implies \( \text{Ad}_{x_0} \circ \theta \in \text{Gal}(N \subset M) \). \( \square \)

Proposition 4.2.41. For the Bisch-Haagerup subfactor \( N = P^H \subset P \rtimes K = M \), the relative commutants \( N' \cap M_n \) and \( M' \cap M_n \) are given by:

\[
N' \cap M_n \cong \begin{cases} \mathbb{C} & \text{if } n = -1 \\ \text{span} \left\{ E_{s_1,s_2} \otimes l \left| \begin{array}{l} s_1, s_2 \in S_n, \\ l \in L_n, \\ \mu(s_1)l\mu(s_2)^{-1} \in H \end{array} \right. \right\} & \text{if } n \geq 0 \end{cases}
\]

\[
M' \cap M_n \cong \begin{cases} \mathbb{C} & \text{if } n = 0 \\ \text{span} \left\{ \sum_{k \in K} E_{h,k_0} \otimes E_{t_1,t_2} \otimes l \left| \begin{array}{l} t_1, t_2 \in T_{n-1}, \\ \mu(t_1)l\mu(t_2)^{-1} \in K, \\ k_0 = \mu(t_1)l\mu(t_2)^{-1} \end{array} \right. \right\} & \text{if } n \geq 1 \end{cases}
\]

Proof: We compute the relative commutants in relation to the concrete model of the basic construction described in Proposition 4.2.36. Consider the inclusion

\[
N = P^H \ni x \mapsto \sum_{s \in S_n} E_{s,s} \otimes \mu(s)^{-1}(x) \in M_{S_n}(\mathbb{C}) \otimes (P \rtimes L_n) = M_n.
\]

Let \( w = \sum_{s_1,s_2 \in S_n} E_{s_1,s_2} \otimes x_{s_1,s_2}^l l \in N' \cap M_n \)
\[ wy = yw \text{ for all } y \in N \]
\[ \sum_{s_1, s_2 \in S_n} \sum_{l \in L_n} E_{s_1, s_2} \otimes (x_{s_1, s_2}^l) \mu(s_2)^{-1}(y) = \sum_{s_1, s_2 \in S_n} \sum_{l \in L_n} E_{s_1, s_2} \otimes \mu(s_1)^{-1}(y) x_{s_1, s_2}^l \text{ for all } y \in N \]
\[ x_{s_1, s_2}^l(y) = \mu(s_1)^{-1}(y) x_{s_1, s_2}^l \text{ for all } y \in N, s_1, s_2 \in S_n, l \in L_n \]
\[ \mu(s_1)(x_{s_1, s_2}^l) = \mu(s_1)(x_{s_1, s_2}^l) \text{ for all } y \in N, s_1, s_2 \in S_n, l \in L_n \]

Now, by Lemma 4.2.40 and Corollary 4.2.39, for \( s_1, s_2 \in S_n \) and \( l \in L_n \),
\[ x_{s_1, s_2}^l \neq 0 \iff Ad_{x_0} \circ \mu(s_1)^{-1}(x_0) \in H \text{ where } x_0 = \frac{\mu(s_1)(x_{s_1, s_2}^l)}{\|\mu(s_1)(x_{s_1, s_2}^l)\|}. \]

Moreover, \( x_{s_1, s_2}^l \neq 0 \iff Ad_{x_0} \in G \cap Inn(P) = \{id_P\}. \) Since \( P \) is a factor, \( x_0 \in C1 \) implying \( \). Thus, \( x_{s_1, s_2}^l \) will be nonzero only if \( \mu(s_1)^{-1}(x_0) \in H \) and in such cases \( x_{s_1, s_2}^l \) will be a scalar multiple of identity. Hence, \( N' \cap M \) is spanned by the linearly independent set \( \{E_{s_1, s_2} \otimes l : s_1, s_2 \in S_n, l \in L_n, \mu(s_1)^{-1}(x_0) \in H \} \).

For \( M' \cap M_n \) where \( n \geq 1 \), we consider the inclusion
\[ M = P \times K \supset P \ni x \iff \sum_{g \in S_n} E_{g, g} \otimes \mu(g)^{-1}(x) \in M_{S_n}(\mathbb{C}) \otimes (P \times L_n) = M_n \]
\[ M = P \times K \supset K \ni k \iff \lambda_k \otimes I_{T_{n-1}} \otimes 1 \in M_{S_n}(\mathbb{C}) \otimes (P \times L_n) = M_n \]

Let
\[ w = \sum_{s_1, s_2 \in S_n} \sum_{l \in L_n} E_{s_1, s_2} \otimes x_{s_1, s_2}^l \in M' \cap M_n \]

Commutation of elements of \( M_n \) with that of \( P \) yields:

there exist \( w \) with the above form and \( s_1, s_2 \in S_n, l \in L_n \) such that \( x_{s_1, s_2}^l \neq 0 \iff \mu(s_1)^{-1}(x_0) = e \)

via calculations shown for the case of \( N' \cap M_n \); further, in such cases, \( x_{s_1, s_2}^l \) is a scalar multiple of 1. Now,
\[ wk = kw \text{ for all } k \in K \]
\[ k^{-1}wk = w \text{ for all } k \in K \]
\[ \sum_{s_1, s_2 \in S_n} \sum_{l \in L_n} (\lambda_k^{-1} \otimes I_{T_{n-1}} \otimes 1)(E_{s_1, s_2} \otimes x_{s_1, s_2}^l)(\lambda_k \otimes I_{T_{n-1}} \otimes 1) \]

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Finally, combining the conditions that we get from considering commutation of \( w \) with elements of \( P \) and \( K \), we can express \( w \) as a linear combination of elements of the form:

\[
\sum_{k \in K} E_{kk_1, kk_2} \otimes E_{t_1, t_2} \otimes l
\]

where \( k_1, k_2 \in K, t_1, t_2 \in T_{n-1}, l \in L_n \) such that \( k_1 \mu(t_1) \mu(t_2)^{-1} k_2^{-1} = e \). Equivalently, \( w \) can be realized as a linear combination of:

\[
\sum_{k \in K} E_{k_{k_1}, k_{k_2}} \otimes E_{t_1, t_2} \otimes l = \sum_{k \in K} E_{k, k_0} \otimes E_{t_1, t_2} \otimes l
\]

where \( t_1, t_2 \in T_{n-1}, l \in L_n \) such that \( \mu(t_1) \mu(t_2)^{-1} \in K \) and \( k_0 = \mu(t_1) \mu(t_2)^{-1} \).

\[\square\]

**Remark 4.2.42.** The unique trace-preserving conditional expectation from \( N' \cap M_n \) onto \( M' \cap M_n \) is given by:

\[
\mathbb{E}^{N' \cap M_n}_{M' \cap M_n}(E_{k_1, k_2} \otimes E_{t_1, t_2} \otimes l) = \delta_{k_1, \mu(t_1)} \delta_{k_2, \mu(t_2)} \frac{1}{|K|} \sum_{k \in K} E_{k, k_1^{-1} k_2} \otimes E_{t_1, t_2} \otimes l
\]

where \( k_1, k_2 \in K, t_1, t_2 \in T_{n-1}, l \in L_n \) such that \( k_1 \mu(t_1) \mu(t_2)^{-1} k_2^{-1} \in H \).

**Remark 4.2.43.** The set \( \left\{ E_{s_1, s_2} \otimes l \mid s_1, s_2 \in S_n, \mu(s_1) \mu(s_2)^{-1} \in H \right\} \) (resp. \( \left\{ E_{s_1, s_2} \otimes l \mid l \in L_n, \mu(s_1) \mu(s_2)^{-1} \in H \right\} \)
4.3 Main result

In this section, our main goal is to show that the planar algebra defined in Section 4.1, is isomorphic to the one arising from the Bisch-Haagerup subfactor \( P^H \subset P \times K \). Conversely, any subfactor whose standard invariant is given by such planar algebra, is indeed of the Bisch-Haagerup type.

We will use the following well-known fact regarding isomorphism of two planar algebras and Theorem 4.2.1 of [16] describing the planar algebra arising from an extremal finite index subfactor.

**Fact:** Let \( P^1 \) and \( P^2 \) be two planar algebras. Then \( P^1 \cong P^2 \) if and only if there exists a map \( \psi : P^1 \to P^2 \) such that:

1. \( \psi \) is a filtered algebra isomorphism,
2. \( \psi \) preserves the actions of all possible Jones projection tangles and the (two types of) conditional expectation tangles.

If \( P^1 \) and \( P^2 \) are *-planar algebras, then we consider the above isomorphisms to be *-preserving.

Let us denote the planar algebra in Section 4.1 by \( P^{BH} \) and the one arising from the Bisch-Haagerup subfactor \( N = P^H \subset P \times K = M \) by \( P^{NCM} \).

**Theorem 4.3.44.** \( P^{NCM} \cong P^{BH} \).

**Proof:** Theorem 4.2.1 of [16] says: \( P^{NCM}_n = N' \cap M_n \) where the \( n \)th Jones projection tangle is given by \( \delta e_n \), the conditional expectation tangle from \( P^{NCM}_n \) onto \( P^{NCM}_{n-1} \) is \( \delta E_{N' \cap M_{n+1}} \) and the conditional expectation tangle from \( P^{NCM}_n \) onto \( P^{NCM}_{1,n} \) is \( \delta E_{M' \cap M_{n+1}} \).
Define the map
\[ \psi : P_{n+M}^{NC} \rightarrow P_B^{BH} \]

\[ \cup \quad \cup \quad \psi_n(E_{s_1,s_2} \otimes l) = (s_1,l,s_2,h) \]

\[ \psi_n : P_{n+M}^{NC} \rightarrow P_B^{BH} \]

where \( n \geq 0, s_1, s_2 \in S_n, l \in L_n \) such that \( \mu(s_1)l\mu(s_2)^{-1} \in H \) and \( \mu(s_1)l\mu(s_2)^{-1}h = e. \)

Clearly, \( \psi \) is a vector space isomorphism by definition. In order to check that \( \psi \) is a filtered \(*\)-algebra isomorphism, we use Remark 4.1.32 (i), (ii), (iii) and (iv). For instance, to show that \( \psi_n \) is an algebra homomorphism, we need to show

\[ \psi_n(E_{s_1,s_2} \otimes l_1 \cdot E_{s_3,s_4} \otimes l_2) = \psi_n(E_{s_1,s_2} \otimes l_1) \cdot \psi_n(E_{s_3,s_4} \otimes l_2) \]

\[ \iff \delta_{s_2,s_3} (s_1,l_1l_2,s_4,\mu(s_4)l_2^{-1}l_1^{-1}\mu(s_1)^{-1}) \]

\[ = (s_1,l_1,s_2,\mu(s_2)l_1^{-1}\mu(s_1)^{-1}) \cdot (s_3,l_2,s_4,\mu(s_4)l_2^{-1}\mu(s_3)^{-1}) \]

which indeed holds by Remark 4.1.32 (iii).

Now, it remains to show that \( \psi \) preserves the action of Jones projection tangles and the two types of conditional expectation tangles. For this, we use Remark 4.1.32 (v), (vi) and (vii). Proof of each of the three kinds of tangles is completely routine; however, we will discuss the action of conditional expectation tangle in details. Let us consider the conditional expectation tangle from \( n+1 \) (colour of the internal disc) to colour \( n \) (colour of the external disc) applied to the element \( E_{s_1,s_2} \otimes E_{m_1,m_2} \otimes l \in P_{n+1}^{NC} = N' \cap M_n \); by Jones's theorem (4.2.1 of [16]) and Remark 4.2.37, the output should be

\[ \delta_{l,s} \sqrt{\frac{\ell}{\ell_{n+1}}} E_{s_1,s_2} \otimes m_1m_2^{-1} \]

\[ \psi_n \rightarrow \delta_{l,s} \sqrt{\frac{\ell}{\ell_{n+1}}} (s_1, m_1m_2^{-1}, s_2, \mu(s_2)m_2m_1^{-1}\mu(s_1)^{-1}) \]

On the other hand, by Remark 4.1.32 (vi), the conditional expectation tangle applied to \( \psi_{n+1}(E_{s_1,s_2} \otimes E_{m_1,m_2} \otimes l) = (s_1,m_1,l^{-1},m_2^{-1},s_2,\mu(s_2)m_2lm_1^{-1}\mu(s_1)^{-1}) \) is given by

\[ \delta_{l,s} \sqrt{\frac{\ell}{\ell_{n+1}}} (s_1, m_1m_2^{-1}, s_2, \mu(s_2)m_2m_1^{-1}\mu(s_1)^{-1}) \]

This completes the proof. \( \square \)
Corollary 4.3.45. Given any group $G$ generated by two of its finite subgroups, there exists a hyperfinite subfactor with standard invariant described by $P^{BH}$. Moreover, $P^{BH}$ is a spherical $C^*$ algebra.

Proof: The proof follows from the fact that any finitely generated group $G$ has an outer action on the hyperfinite $II_1$ factor. \qed

Theorem 4.3.46. Given a subfactor $N \subset M$ with standard invariant isomorphic to $P^{BH}$, there exists an intermediate subfactor $N \subset P \subset M$ and outer actions of $H$ and $K$ on $P$ such that $N \subset M \cong P^{H} \subset P \rtimes K$.

Proof: Let $P^{NC \subset M}$ denote the planar algebra of $N \subset M$ formed by its relative commutants and $\phi : P^{NC \subset M} \rightarrow P^{BH}$ be a planar algebra isomorphism. Consider the element $q = \phi(e, e, e, e) \in N' \cap M_1$. Clearly, (i) $q$ is a projection, (ii) $qe_1 = e_1$ and (iii) $E_M(q) = |K|^{-1}$. Using action of tangles and the planar algebra isomorphism $\phi$, it also follows that

\[
(iv) \quad \begin{array}{c}
\begin{array}{c}
q
\end{array}
\end{array} = \sqrt{\frac{|H|}{|K|}} \cdot \begin{array}{c}
\begin{array}{c}
q
\end{array}
\end{array}
\]

The conditions (i) - (iv) asserts that $q$ is an intermediate subfactor projection as described in [1]. Define $P = M \cap \{q\}'$. To show $P$ is a factor, first note that

\[
P' \cap P \subset N' \cap P = N' \cap M \cap \{q\}' = \phi(P^{BH}_1 \cap \{(e, e, e, e)\}')
\]

So, it is enough to show that $P^{BH}_1 \cap \{(e, e, e, e)\}' = C1$. If $x = \sum_{g \in K \cap H} (g, g^{-1}) \in P^{BH}_1 \cap \{(e, e, e, e)\}'$, then

\[
(e, e, e, e) \cdot x = x \cdot (e, e, e, e) \Rightarrow \sum_{g \in K \cap H} \lambda_g(e, e, g, g^{-1}) = \sum_{g \in K \cap H} \lambda_g(g, e, e, g^{-1})
\]

Hence, $\lambda_g = \delta_{g,e} \lambda_e$ for all $g \in K \cap H$ and $x \in C1$.

It remains to establish that $N$ (resp. $M$) is the fixed-point subalgebra (resp. crossed-product algebra) of $P$ with respect to an outer action of the group $H$ (resp. $K$). It is easy to prove (see [16]) that if the standard invariant of a subfactor $\tilde{N} \subset \tilde{M}$ is given by the planar algebra corresponding to the fixed-point subfactor (resp. crossed-product...
subfactor) with respect to a finite group $\hat{G}$, then there exist an outer action of $\hat{G}$ on $\hat{M}$ (resp. $\hat{N}$) such that $\hat{N}$ (resp. $\hat{M}$) is isomorphic to fixed-point subalgebra (crossed-product algebra) of the action. Again, if $\hat{N} \subset \hat{P} \subset \hat{M}$ is an intermediate subfactor and $\hat{q}$ is its corresponding intermediate subfactor projection, then the planar algebra of $\hat{N} \subset \hat{P}$ (resp. $\hat{P} \subset \hat{M}$) is given by the range of the idempotent tangle

\[
\begin{array}{c}
\begin{array}{c}
* \hat{q} \quad \ldots \quad * \hat{q} \\
* \quad 1 \\
\hat{q} \quad \ldots \quad \hat{q} \\
\end{array}
\end{array}
\text{or}
\begin{array}{c}
\begin{array}{c}
* \hat{q} \quad \ldots \quad * \hat{q} \\
\hat{q} \quad 1 \\
* \quad \hat{q} \\
\end{array}
\end{array}
\]

or

\[
\begin{array}{c}
\begin{array}{c}
* \hat{q} \quad \ldots \quad * \hat{q} \\
\hat{q} \quad 1 \\
* \quad \hat{q} \\
\end{array}
\end{array}
\text{or}
\begin{array}{c}
\begin{array}{c}
* \hat{q} \quad \ldots \quad * \hat{q} \\
\hat{q} \quad 1 \\
* \quad \hat{q} \\
\end{array}
\end{array}
\]

according as $n$ is even or odd. (This result first appeared in [1] in an algebraic form and later expressed in the pictorial formalism of planar algebras in [8].)

Getting back to our context, to get the planar algebra of $N \subset P$ (resp. $P \subset M$), the elements in the image of the such idempotent tangles are given by the words with letters coming from $K$ and $H$ alternately where every element coming from $K$ (respectively $H$) must necessarily be $e$. Such a planar algebra is the same as $P^{BH}$ with $K = \{e\}$ (resp. $H = \{e\}$); by Theorem 4.3.44 this is indeed the planar algebra corresponding to fixed-point subfactor (resp. crossed-product subfactor) with respect to $H$ (resp. $K$).

In Theorem 4.3.46, it is natural to ask whether the group generated by $H$ and $K$ in $Out(P)$ is isomorphic to $G$. The effort to prove this fact is ongoing where we consider Bisch-Haagerup type subfactor in its full generality, that is, we discard the assumption that the group generated by $H$ and $K$ in $Out(P)$ lifts to $Aut(P)$ without any obstruction.
Bibliography


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