Unitarily invariant norms and tensor products of maximal injective von Neumann subalgebras

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Unitarily invariant norms and tensor products of maximal injective von Neumann subalgebras

Abstract

This dissertation consists of four contributions to the study of von Neumann algebras. In the first part, we set up a representation theorem for unitarily invariant norms on finite factor von Neumann algebras. In the second part, we set up a representation theorem for unitarily invariant norms related to infinite factor von Neumann algebras. In the third part, we introduce a notion of completely singular von Neumann subalgebras and characterize the class of completely singular von Neumann subalgebras. In the fourth part, we study a longstanding open question on the tensor product of maximal injective von Neumann subalgebras.

The first part of the dissertation is joint work with Don Hadwin, Eric Nordgren and Junhao Shen. The second part of the dissertation is joint work with Don Hadwin. This dissertation is partially supported by a University of New Hampshire dissertation fellowship.

Keywords

Mathematics

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UNITARILY INVARIANT NORMS AND TENSOR PRODUCTS
OF MAXIMAL INJECTIVE VON NEUMANN SUBALGEBRAS

BY

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DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

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4/25/08

Date
DEDICATION

To my wife, my parents, my grandma, and my teachers
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ABSTRACT

UNITARILY INVARIANT NORMS AND TENSOR PRODUCTS OF MAXIMAL INJECTIVE VON NEUMANN SUBALGEBRAS

by

Junsheng Fang
University of New Hampshire, May, 2008

This dissertation consists of four contributions to the study of von Neumann algebras. In the first part, we set up a representation theorem for unitarily invariant norms on finite factor von Neumann algebras. In the second part, we set up a representation theorem for unitarily invariant norms related to infinite factor von Neumann algebras. In the third part, we introduce a notion of completely singular von Neumann subalgebras and characterize the class of completely singular von Neumann subalgebras. In the fourth part, we study a longstanding open question on the tensor products of maximal injective von Neumann subalgebras.

The first part of the dissertation is joint work with Don Hadwin, Eric Nordgren and Junhao Shen. The second part of the dissertation is joint work with Don Hadwin. This dissertation is partially supported by a University of New Hampshire dissertation fellowship.
INTRODUCTION

F. J. Murray and J. von Neumann [22, 23, 24, 44, 45, 46] introduced and studied certain algebras of Hilbert space operators. Those algebras are now called von Neumann Algebras. They are strong-operator closed self-adjoint subalgebras of the algebra of all bounded linear transformations on a Hilbert space. Factors are von Neumann algebras whose centers consist of scalar multiples of the identity operator. Every von Neumann algebra is a direct sum (or "direct integral") of factors. Thus factors are the building blocks for general von Neumann algebras.

Murray and von Neumann [22] classified factors into type I, II, III factors. Type I factors are full matrix algebras: $M_n(\mathbb{C})$ and $B(H)$. Type II factors are called finite factors. There is a unique faithful normal tracial state on a finite factor. Type II factors are infinite-dimensional finite factors. They are "continuous" matrix algebras. Factors except type III factors are called semi-finite factors. So type I factors, II factors, III factors, II factors are semi-finite factors. A semi-finite factor admits a faithful normal tracial weight. Examples of semi-finite factors and type III factors arise from group representation theory and ergodic theory.

This dissertation consists of four contributions to the study of factors and von Neumann algebras. In the first part, we set up a representation theorem for unitarily invariant norms on finite factors. In the second part, we set up a representation theorem for unitarily invariant norms related to infinite factors. In the third part, we introduce a notion of completely singular von Neumann subalgebras and characterize the class of completely singular von Neumann subalgebras. In the fourth part, we study a longstanding open question on the tensor product of maximal injective von Neumann subalgebras.
PART I
UNITARILY INVARIANT NORMS ON
FINITE FACTORS
CHAPTER 1

BACKGROUND

The unitarily invariant norms were introduced by von Neumann [44] for the purpose of metrizing matrix spaces. Von Neumann, together with his associates, established that the class of unitarily invariant norms of \( n \times n \) complex matrices coincides with the class of symmetric gauge functions of their \( s \)-numbers. These norms have now been variously generalized and utilized in several contexts. For example, Schatten [32, 33] defined norms on two-sided ideals of completely continuous operators on an arbitrary Hilbert space; Ky Fan [9] studied Ky Fan norms and obtained his dominance theorem. The unitarily invariant norms play a crucial role in the study of function spaces and group representations (see e.g. [19]) and in obtaining certain bounds of importance in quantum field theory (see [36]). For historical perspectives and surveys of unitarily invariant norms, see Schatten [32, 33], Hewitt and Ross [16], Gohberg and Krein [13] and Simon [36].

The theory of non-commutative \( L^p \)-spaces has been developed beginning with pioneer work of Segal, Dixmier, and Kunze. Since then the theory has been extensively studied, extended and applied by Nelson, Haagerup, Fack, Kosaki, Junge, Xu, and many others. The recent survey by Pisier and Xu [29] presents a rather complete picture on noncommutative integration and contains a lot of references. This theory is still a very active subject of investigation. Some tools in the study of the usual commutative \( L^p \)-spaces still work in the noncommutative setting. However, most of the time, new techniques must be invented. To illustrate the difficulties one may encounter in studying the noncommutative \( L^p \)-spaces, we mention here one basic well-known fact. Let \( \mathcal{H} \) be a complex Hilbert space, and let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of all bounded linear operators on \( \mathcal{H} \). The basic fact states that the usual triangle inequality for the absolute values of complex numbers is no longer valid for the absolute values of operators, namely, in general, we do not have \( |S + T| \leq |S| + |T| \) for \( S, T \in \mathcal{B}(\mathcal{H}) \), where \( |S| = (S^*S)^{1/2} \).
is the absolute value of $S$. Despite such difficulties, by now the strong parallelism between noncommutative and classical Lebesgue integration is well-known.

Motivated by von Neumann's theorem and the analogies between noncommutative and classical $L^p$-spaces, in this Part, we will set up a structure theorem for unitarily invariant norms on finite factors. For interesting applications of the representation theorem, we refer to [7].
CHAPTER 2

s-NUMBERS OF OPERATORS IN FINITE VON NEUMANN ALGEBRAS

2.1 Nonincreasing rearrangements of functions

Let I be [0, 1] or [0, °°), and m be the Lebesgue measure on I. In the following, a measurable function and a measurable set mean a Lebesgue measurable function and a Lebesgue measurable set. For two measurable sets A and B, A = B means m((A \ B) ∪ (B \ A)) = 0. Let f(x) be a real measurable function on I. The nonincreasing rearrangement function, f*(x), of f(x) is defined by

\[ f^*(x) = \sup\{y : m(\{f > y\}) > x\}, \quad x \in I. \]  

(2.1.1)

We summarize some well-known properties of f*(x) in the following proposition. In this Part, we will be interested in the case that I = [0, 1].

Proposition 2.1. Let f(x), g(x), f_1(x), f_2(x), … be real measurable functions on I, c be a real number. Then we have the following:

1. f*(x) is a nonincreasing, right-continuous function on I;

2. (f + c)* = f* + c;

3. (cf)* = cf* if c ≥ 0;

4. if f(x) is a simple function, then f*(x) is also a simple function;

5. if f(x) ≤ g(x) for almost all x, then f*(x) ≤ g*(x) everywhere;

6. \|f^*(x) - g^*(x)\|_\infty ≤ \|f(x) - g(x)\|_\infty;
7. If \( \lim_{n \to \infty} f_n(x) = f(x) \) uniformly on \( I \), then \( \lim_{n \to \infty} f_n^*(x) = f^*(x) \) uniformly on \( I \);

8. If \( f_n(x) \) converges to \( f(x) \) in measure, then \( \lim \inf_{n \to \infty} f_n^*(x) \geq f^*(x) \) for every \( x \in I \);

9. If \( f_n(x) \) converges to \( f(x) \) in measure, then \( \lim \sup_{n \to \infty} f_n^*(x) \leq f^*(x) \) for every \( x \in I \) such that \( f^* \) is continuous at \( x \);

10. \( f(x) \) and \( f^*(x) \) are equi-measurable, i.e., for any real number \( y \), \( m(\{f > y\}) = m(\{f^* > y\}) \);

11. \( f^* = g^* \) if and only if \( f(x) \) and \( g(x) \) are equi-measurable;

12. If \( f(x) \) and \( g(x) \) are bounded functions and \( \int f^n(x)dx = \int g^n(x)dx \) for all \( n = 0, 1, 2, \ldots \), then \( f^*(x) = g^*(x) \);

13. \( \int f(x)dx = \int f^*(x)dx \) when either integral is well-defined.

### 2.2 Invertible measure-preserving transformations on \([0, 1]\)

Let \( \mathcal{G} = \{ \phi : \phi(x) \) is an invertible measure-preserving transformation on \([0, 1]\}\}. It is well known that \( \mathcal{G} \) acts on \([0, 1]\) ergodically (see [10] page 3-4, for instance), i.e., for a measurable subset \( A \) of \([0, 1]\), \( \phi(A) = A \) for all \( \phi \in \mathcal{G} \) implies that \( m(A) = 0 \) or \( m(A) = 1 \).

**Lemma 2.2.** Let \( A, B \) be two measurable subsets of \([0, 1]\) such that \( m(A) = m(B) \). Then there is a \( \phi \in \mathcal{G} \) such that \( \phi(A) = B \).

**Proof.** We can assume that \( m(A) = m(B) > 0 \). Since \( \mathcal{G} \) acts ergodically on \([0, 1]\), there is a \( \phi \in \mathcal{G} \) such that \( m(\phi(A) \cap B) > 0 \). Let \( B_1 = \phi(A) \cap B \) and \( A_1 = \phi^{-1}(B_1) \). Then \( m(A_1) = m(B_1) \) and \( \phi(A_1) = B_1 \). By Zorn's lemma and maximality arguments, we prove the lemma. \( \square \)

**Corollary 2.3.** Let \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_n \) be disjoint measurable subsets of \([0, 1]\) such that \( m(A_k) = m(B_k) \) for \( 1 \leq k \leq n \). Then there is a \( \phi \in \mathcal{G} \) such that \( \phi(A_k) = B_k \) for \( 1 \leq k \leq n \).

**Proof.** We can assume that \( A_1 \cup \cdots \cup A_n = B_1 \cup \cdots \cup B_n = [0, 1] \). By Lemma 2.2, there is a \( \phi_k \in \mathcal{G} \) such that \( \phi_k(A_k) = B_k \), \( 1 \leq k \leq n \). Define \( \phi(x) = \phi_k(x) \) for \( x \in A_k \). Then \( \phi \in \mathcal{G} \) and \( \phi(A_k) = B_k \) for \( 1 \leq k \leq n \). \( \square \)
For \( f(x) \in L^\infty[0,1] \), define \( \tau(f) = \int_0^1 f(x)\,dx \). The following theorem is a version of the Dixmier's averaging theorem (see [3] or [21]) and it has a similar proof.

**Theorem 2.4.** Let \( f(x) \in L^\infty[0,1] \) be a real function. Then \( \tau(f) \) is in the \( L^\infty \)-norm closure of the convex hull of \( \{ f \circ \phi(x) : \phi \in \Theta \} \).

We end this subsection with the following proposition.

**Proposition 2.5.** Let \( \phi(x) \) be an invertible measure-preserving transformation on \([0,1]\) and let \( \theta(f) = f \circ \phi \). Then \( \theta \) is a \(*\)-automorphism of \( L^\infty[0,1] \) preserving \( \tau \). Conversely, if \( \theta \) is a \(*\)-automorphism of \( L^\infty[0,1] \) preserving \( \tau \), then there is an invertible measure-preserving transformation on \([0,1]\) such that

\[
\theta(f) = f \circ \phi
\]

for all \( f(x) \in L^\infty[0,1] \).

**Proof.** The first part of the proposition is easy to see. Suppose \( \theta \) is a \(*\)-automorphism of \( L^\infty[0,1] \). Let \( \phi(x) = \theta(f)(x) \), where \( f(x) \equiv x \). Then it is easy to see the second part of the proposition.

\[\square\]

### 2.3 \( s \)-numbers of operators in type \( \text{II}_1 \) factors

In [8], Fack and Kosaki give a complete exposition of generalized \( s \)-numbers of \( \tau \)--measurable operators affiliated with semi-finite von Neumann algebras. For the reader's convenience and our purpose, we provide sufficient details on \( s \)-numbers of bounded operators in finite von Neumann algebras in the following. We will define \( s \)-numbers of bounded operators in finite von Neumann algebras from the point of view of non-increasing rearrangement of functions.

The following lemma is well-known. The proof is an easy exercise.

**Lemma 2.6.** Let \((\mathcal{A}, \tau)\) be a separable (i.e., with separable predual) diffuse abelian von Neumann algebra with a faithful normal trace \( \tau \) on \( \mathcal{A} \). Then there is a \(*\)-isomorphism \( \alpha \) from \((\mathcal{A}, \tau)\) onto \( (L^\infty[0,1], \int_0^1 \,dx) \) such that \( \tau = \int_0^1 \,dx \circ \alpha \).

Let \( \mathcal{M} \) be a type \( \text{II}_1 \) factor and \( \tau \) be the unique trace on \( \mathcal{M} \). For \( T \in \mathcal{M} \), there is a separable diffuse abelian von Neumann subalgebra \( \mathcal{A} \) of \( \mathcal{M} \) containing \(|T|\). By Lemma 2.6, there is a
*-isomorphism $\alpha$ from $\mathcal{A}$ onto $L^\infty[0,1]$ such that $\tau = \int_0^1 dx \circ \alpha$. Let $f(x) = \alpha(|x|)$ and $f^*(x)$ be the non-increasing rearrangement of $f(x)$ (see (2.1.1)). Then the $s$-numbers of $T$, $\mu_s(T)$, are defined as

$$\mu_s(T) = f^*(s), \quad 0 \leq s \leq 1.$$  

**Lemma 2.7.** $\mu_s(T)$ does not depend on $\mathcal{A}$ and $\alpha$.

**Proof.** Let $\mathcal{A}'$ be another separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$ containing $|T|$ and $\beta$ be a *-isomorphism from $(\mathcal{A}', \tau)$ onto $(L^\infty[0,1], \int_0^1 dx)$ such that $\tau = \int_0^1 dx \circ \beta$. Let $g(x) = \beta(|T|)$. For every number $n = 0, 1, 2, \cdots$, $\int_0^1 f^n(x)dx = \int_0^1 g^n(x)dx$. Since both $f(x)$ and $g(x)$ are bounded positive functions, by 12 of Proposition 2.1, $f^*(x) = g^*(x)$ for all $x \in [0,1]$. \hfill \Box

**Corollary 2.8.** For $T \in \mathcal{M}$ and $p \geq 0$, $	au(|T|^p) = \int_0^1 \mu_p(T)^p ds$.

Let $\mathcal{P}(\mathcal{M})$ be the set of projections in $\mathcal{M}$. The following lemma says that the above definition of $s$-numbers coincides with the definition of $s$-numbers given by Fack and Kosaki.

**Lemma 2.9.** For $0 < s \leq 1$,

$$\mu_s(T) = \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E) = s\}.$$  

**Proof.** By the polar decomposition and the definition of $\mu_s(T)$, we may assume that $T$ is positive. Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$ containing $T$ and $\alpha$ be a *-isomorphism from $(\mathcal{A}, \tau)$ onto $(L^\infty[0,1], \int_0^1 dx)$ such that $\tau = \int_0^1 dx \circ \alpha$. Let $f(x) = \alpha(T)$ and $f^*(x)$ be the non-increasing rearrangement of $f(x)$. Then $\mu_s(T) = f^*(s)$. By the definition of $f^*$,

$$m(\{f^* > \mu_s(T)\}) = \lim_{n \to \infty} m\left(\left\{f^* > \mu_s(T) + \frac{1}{n}\right\}\right) \leq s$$

and

$$m(\{f^* \geq \mu_s(T)\}) \geq \lim_{n \to \infty} m\left(\left\{f^* > \mu_s(T) - \frac{1}{n}\right\}\right) \geq s.$$  

Since $f^*$ and $f$ are equi-measurable, $m(\{f > \mu_s(T)\}) \leq s$ and $m(\{f \geq \mu_s(T)\}) \geq s$. Therefore, there is a measurable subset $A$ of $[0,1]$, $\{f > \mu_s(T)\} \subset [0,1] \setminus A \subset \{f \geq \mu_s(T)\}$, such that $m([0,1] \setminus A) = s$ and $\|f(x)\chi_A(x)\|_\infty = \mu_s(T)$ and $\|f(x)\chi_B(x)\|_\infty \geq \mu_s(T)$ for all $B \subset [0,1] \setminus A$.  

8
such that \( m(B) > 0 \). Let \( F = \alpha^{-1}(\chi_A) \). Then \( \tau(F^\perp) = s \), \( \|TF\| = \|\alpha^{-1}(f\chi_A)\|_\omega = \mu_s(T) \) and \( \|TF'\| \geq \mu_s(T) \) for all nonzero subprojections \( F' \) of \( F^\perp \). This proves that \( \mu_s(T) \geq \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s\} \). Similarly, for every \( \varepsilon > 0 \), there is a projection \( F_\varepsilon \in \mathcal{M} \) such that \( \tau(F_\varepsilon^\perp) = s + \varepsilon \), \( \|TF_\varepsilon\| = \mu_{s+\varepsilon}(T) \) and \( \|TF'\| \geq \mu_{s+\varepsilon}(T) \) for all nonzero subprojections \( F' \) of \( F_\varepsilon^\perp \). Suppose \( E \in \mathcal{M} \) is a projection such that \( \tau(E^\perp) = s \). Then \( \tau(E \wedge F_\varepsilon^\perp) = \tau(E) + \tau(F_\varepsilon^\perp) - \tau(E \vee F_\varepsilon^\perp) = 1 + \varepsilon - \tau(E \vee F_\varepsilon^\perp) \geq \varepsilon > 0 \). Hence, \( \|TE\| \geq \|T(E \wedge F_\varepsilon^\perp)\| \geq \mu_{s+\varepsilon}(T) \). This proves that \( \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s\} \geq \mu_{s+\varepsilon}(T) \). Since \( \mu_s(T) \) is right-continuous, \( \mu_s(T) \leq \inf\{\|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s\} \). □

**Corollary 2.10.** Let \( S, T \in \mathcal{M} \). Then \( \mu_s(ST) \leq \|S\|\mu_s(T) \) for \( s \in [0, 1] \).

We refer to [5, 8] for other interesting properties of \( s \)-numbers of operators in type \( \text{II}_1 \) factors.

### 2.4 \( s \)-numbers of operators in finite von Neumann algebras

Throughout the Part I, a finite von Neumann algebra \((\mathcal{M}, \tau)\) means a finite von Neumann algebra \(\mathcal{M}\) with a faithful normal tracial state \(\tau\). An embedding of a finite von Neumann algebra \((\mathcal{M}, \tau)\) into another finite von Neumann algebra \((\mathcal{M}_1, \tau_1)\) means a \(*\)-isomorphism \(\alpha\) from \(\mathcal{M}\) to \(\mathcal{M}_1\) such that \(\tau = \tau_1 \circ \alpha\). Let \((\mathcal{L}(\mathcal{F}_2), \tau')\) be the free group factor with the faithful normal trace \(\tau'\). Then the reduced free product von Neumann algebra \(\mathcal{M}_1 = (\mathcal{M}, \tau) \ast (\mathcal{L}(\mathcal{F}_2), \tau')\) is a type \(\text{II}_1\) factor with a (unique) faithful normal trace \(\tau_1\) such that the restriction of \(\tau_1\) to \(\mathcal{M}\) is \(\tau\) (see Lemma 4.8 of [47] for instance). So every finite von Neumann algebra can be embedded into a type \(\text{II}_1\) factor.

**Definition 2.11.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and \(T \in \mathcal{M}\). If \(\alpha\) is an embedding of \((\mathcal{M}, \tau)\) into a type \(\text{II}_1\) factor \((\mathcal{M}_1, \tau_1)\), then the \( s \)-numbers of \(T\) are defined as

\[
\mu_s(T) = \mu_s(\alpha(T)), \quad s \in [0, 1].
\]

Similar to the proof of Lemma 2.7, we can see that \(\mu_s(T)\) is well defined, i.e., does not depend on the choice of \(\alpha\) and \(\mathcal{M}_1\).
Let \( T \in (M_n(\mathbb{C}), \tau_n) \), where \( \tau_n \) is the normalized trace on \( M_n(\mathbb{C}) \). Then \( |T| \) is unitarily equivalent to a diagonal matrix with diagonal elements \( s_1(T) \geq \cdots \geq s_n(T) \geq 0 \). In the classical matrices theory [1, 13], \( s_1(T), \cdots, s_n(T) \) are also called \( s \)-numbers of \( T \). It is easy to see that the relation between \( \mu_s(T) \) and \( s_1(T), \cdots, s_n(T) \) is the following:

\[
\mu_s(T) = s_1(T) \chi_{[0,1/n]}(s) + s_2(T) \chi_{[1/n,2/n]}(s) + \cdots + s_n(T) \chi_{[n-1/n,1]}(s).
\] (2.4.1)

Since no confusion will arise, we will use both \( s \)-numbers for matrices in \( M_n(\mathbb{C}) \). We refer to [1, 13] for other interesting properties of \( s \)-numbers of matrices.

We end this section by the following definition.

**Definition 2.12.** Positive operators \( S \) and \( T \) in a finite von Neumann algebra \( (\mathcal{M}, \tau) \) are equi-measurable if \( \mu_s(S) = \mu_s(T) \) for \( 0 \leq s \leq 1 \).

By 12 of Proposition 2.1 and Corollary 2.8, positive operators \( S \) and \( T \) in a finite von Neumann algebra \( (\mathcal{M}, \tau) \) are equi-measurable if and only if \( \tau(S^n) = \tau(T^n) \) for all \( n = 0, 1, 2, \cdots \).
CHAPTER 3
TRACIAL GAUGE SEMI-NORMS

3.1 Gauge semi-norms

Definition 3.1. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra. A semi-norm \(\| \cdot \|\) on \(\mathcal{M}\) is called gauge invariant if for every \(T \in \mathcal{M}\),
\[
\|T\| = \| |T| \|.
\]

Lemma 3.2. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and let \(\| \cdot \|\) be a semi-norm on \(\mathcal{M}\). Then the following conditions are equivalent:

1. \(\| \cdot \|\) is gauge invariant;
2. \(\| \cdot \|\) is left unitarily invariant, i.e., for every unitary operator \(U \in \mathcal{M}\) and operator \(T \in \mathcal{M}\), \(\|UT\| = \|T\|\);
3. for operators \(A, T \in \mathcal{M}\), \(\|AT\| \leq \|A\| \cdot \|T\|\).

Proof. “3 \(\Rightarrow\) 2” and “2 \(\Rightarrow\) 1” are easy to see. We only prove “1 \(\Rightarrow\) 3”. We need to prove that if \(\|A\| < 1\), then \(\|AT\| \leq \|T\|\). Since \(\|A\| < 1\), there are unitary operators \(U_1, \cdots, U_k\) such that \(A = \frac{U_1 + \cdots + U_k}{k}\) (see [20, 30]). Since \(|U_1 T| = \cdots = |U_k T| = |T|\), \(\|AT\| = \|\frac{U_1 T + \cdots + U_k T}{k}\| \leq \frac{\|U_1 T\| + \cdots + \|U_k T\|}{k} \leq \|T\|\). \(\square\)

Definition 3.3. A normalized semi-norm on a finite von Neumann algebra \((\mathcal{M}, \tau)\) is a semi-norm \(\| \cdot \|\) such that \(\|1\| = 1\).

By Lemma 3.2, we have the following corollary.

Corollary 3.4. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and \(\| \cdot \|\) be a normalized gauge semi-norm on \(\mathcal{M}\). Then for every \(T \in \mathcal{M}\),
\[
\|T\| \leq \|T\|.
\]
A simple operator in a finite von Neumann algebra \((\mathcal{M}, \tau)\) is an operator \(T = a_1E_1 + \cdots + a_nE_n\), where \(E_1, \cdots, E_n\) are projections in \(\mathcal{M}\) such that \(E_1 + \cdots + E_n = 1\).

**Corollary 3.5.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra, and let \(\| \cdot \|_1, \| \cdot \|_2\) be two gauge invariant semi-norms on \(\mathcal{M}\). Then \(\| \cdot \|_1 = \| \cdot \|_2\) on \(\mathcal{M}\) if \(\|T\|_1 = \|T\|_2\) for all positive simple operators \(T \in \mathcal{M}\).

**Proof.** Without loss of generality, assume \(\|1\|_1 = \|1\|_2 = 1\). Let \(T \in \mathcal{M}\) be a positive operator. By the spectral decomposition theorem, there is a sequence of positive simple operators \(T_n \in \mathcal{M}\) such that \(\lim_{n \to \infty} \|T - T_n\| = 0\). By Corollary 3.4, \(\lim_{n \to \infty} \|T - T_n\|_1 = \lim_{n \to \infty} \|T - T_n\|_2 = 0\). By the assumption of the corollary, \(\|T_n\|_1 = \|T_n\|_2\). Hence, \(\|T\|_1 = \|T\|_2\). Since both \(\| \cdot \|_1\) and \(\| \cdot \|_2\) are gauge invariant, \(\| \cdot \|_1 = \| \cdot \|_2\). \(\square\)

### 3.2 Tracial gauge semi-norms

**Definition 3.6.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra. A semi-norm \(\| \cdot \|\) on \(\mathcal{M}\) is called tracial if \(\|S\| = \|T\|\) for every two equi-measurable positive operators \(S, T\) in \(\mathcal{M}\). A semi-norm \(\| \cdot \|\) on \(\mathcal{M}\) is called a tracial gauge semi-norm if it is both tracial and gauge invariant.

Note that for a positive operator \(T\) in a finite von Neumann algebra \((\mathcal{M}, \tau)\),

\[
\|T\| = \lim_{n \to \infty} \left(\tau(T^n)\right)^{\frac{1}{n}}.
\]

Therefore, the operator norm \(\| \cdot \|\) is a tracial gauge norm on \((\mathcal{M}, \tau)\). Another less obvious example of a tracial gauge norm on \((\mathcal{M}, \tau)\) is the non-commutative \(L^1\)-norm: \(\|T\|_1 = \tau(|T|) = \int_0^1 \mu_\tau(T)ds\). The less obvious part is to show that \(\| \cdot \|_1\) satisfies the triangle inequality. The following lemma overcomes this difficulty.

**Lemma 3.7.** \(\|A\|_1 = \sup\{\tau(UA) : U \in \mathcal{U}(\mathcal{M})\}\), where \(\mathcal{U}(\mathcal{M})\) is the set of unitary operators in \(\mathcal{M}\).

**Proof.** By the polar decomposition theorem, there is a unitary operator \(V \in \mathcal{M}\) such that \(A = V|A|\). By the Schwartz inequality, \(\tau(UA) = |\tau(UV|A|)| = |\tau(UV|A|^{1/2}|A|^{1/2})| \leq \tau(|A|)^{1/2} \cdot \tau(|A|)^{1/2} = \tau(|A|)\). Hence \(\|A\|_1 \geq \sup\{\tau(UA) : U \in \mathcal{U}(\mathcal{M})\}\). Let \(U = V^*\), we obtain \(\|A\|_1 \leq \sup\{\tau(UA) : U \in \mathcal{U}(\mathcal{M})\}\). \(\square\)
Corollary 3.8. \[\|A + B\|_1 \leq \|A\|_1 +\|B\|_1.\]

Lemma 3.9. Let \((\mathscr{M}, \tau)\) be a finite von Neumann algebra and \(\|\cdot\|\) be a gauge invariant semi-norm on \(\mathscr{M}\). Then \(\|\cdot\|\) is tracial if \(\|S\| = \|T\|\) for every two equi-measurable positive simple operators \(S, T\) in \(\mathscr{M}\).

Proof. We can assume that \(\|1\| = 1\). Let \(A, B\) be two equi-measurable positive operators in \(\mathscr{M}\). By the spectral decomposition theorem, there are two sequences of positive simple operators \(A_n, B_n\) in \(\mathscr{M}\) such that \(A_n\) and \(B_n\) are equi-measurable and \(\lim_{n \to \infty} \|A - A_n\| = \lim_{n \to \infty} \|B - B_n\| = 0\). By Corollary 3.4, \(\lim_{n \to \infty} \|A - A_n\| = \lim_{n \to \infty} \|B - B_n\| = 0\). By the assumption of the lemma, \(\|A_n\| = \|B_n\|\). Hence, \(\|A\| = \|B\|\). \(\Box\)

### 3.3 Symmetric gauge semi-norms

Definition 3.10. Let \((\mathscr{M}, \tau)\) be a finite von Neumann algebra and \(\text{Aut}(\mathscr{M}, \tau)\) be the set of \(*\)-automorphisms on \(\mathscr{M}\) preserving \(\tau\). A semi-norm \(\|\cdot\|\) on \(\mathscr{M}\) is called symmetric if

\[\|\theta(T)\| = \|T\|, \quad \forall T \in \mathscr{M}, \theta \in \text{Aut}(\mathscr{M}, \tau).\]

A semi-norm \(\|\cdot\|\) on \(\mathscr{M}\) is called a symmetric gauge semi-norm if it is both symmetric and gauge invariant.

Example 3.11. Let \(\mathscr{M} = \mathbb{C}^n\) and \(\tau(T) = \frac{x_1 + \cdots + x_n}{n}\), where \(T = (x_1, \cdots, x_n) \in \mathbb{C}^n\). Then \(\text{Aut}(\mathscr{M}, \tau)\) is the set of permutations on \(\{1, \cdots, n\}\). So a semi-norm \(\|\cdot\|\) on \(\mathscr{M}\) is a symmetric gauge semi-norm if and only if for every \((x_1, \cdots, x_n) \in \mathbb{C}^n\) and a permutation \(\pi\) on \(\{1, \cdots, n\}\),

\[\|(x_1, \cdots, x_n)\| = \|(x_{\pi(1)}, \cdots, x_{\pi(n)})\|,\]

and

\[\|(x_1, \cdots, x_n)\| = \|(x_{\pi(1)}, \cdots, x_{\pi(n)})\|.\]

Lemma 3.12. Let \((\mathscr{M}, \tau)\) be a finite von Neumann algebra and \(\|\cdot\|\) be a semi-norm on \(\mathscr{M}\). If \(\|\cdot\|\) is tracial gauge invariant, then \(\|\cdot\|\) is symmetric gauge invariant.
Proof. Let \( \theta \in \text{Aut}(\mathcal{M}, \tau) \) and \( T \in \mathcal{M} \). We need to prove that \( \| \theta(T) \| = \| T \| \). Since \( |\theta(T)| = \theta(|T|) \) and \( \| \cdot \| \) is gauge invariant, we can assume that \( T \) is positive. Since \( \theta \in \text{Aut}(\mathcal{M}, \tau) \), \( T \) and \( \theta(T) \) are equi-measurable. Hence, \( \| T \| = \| \theta(T) \| \). \( \square \)

Example 3.13. Let \( \mathcal{M} = \mathbb{C} \oplus M_2(\mathbb{C}) \) and \( \tau(a \oplus B) = \frac{a}{2} + \frac{\tau_2(B)}{2} \), where \( \tau_2 \) is the normalized trace on \( M_2(\mathbb{C}) \). Define \( \| a \oplus B \| = \| a \|/2 + \tau_2(|B|) \). Then \( \| \cdot \| \) is a symmetric gauge norm but not a tracial gauge norm. Note that \( 1 \oplus 0 \) and \( 0 \oplus 1 \) are equi-measurable, but \( 1/2 = \| 1 \oplus 0 \| \neq \| 0 \oplus 1 \| = 1 \).

\( \text{Aut}(\mathcal{M}, \tau) \) acts on \( \mathcal{M} \) ergodically if \( \theta(T) = T \) for all \( \theta \in \text{Aut}(\mathcal{M}, \tau) \) implies \( T = \lambda I \).

Lemma 3.14. Let \( (\mathcal{M}, \tau) \) be a finite von Neumann algebra and \( \| \cdot \| \) be a semi-norm on \( \mathcal{M} \). If \( \text{Aut}(\mathcal{M}, \tau) \) acts on \( \mathcal{M} \) ergodically, then the following are equivalent:

1. \( \| \cdot \| \) is a tracial gauge semi-norm;
2. \( \| \cdot \| \) is a symmetric gauge semi-norm.

Proof. “1 \( \Rightarrow \) 2” by Lemma 3.12. We need to prove “2 \( \Rightarrow \) 1”. By Corollary 3.5, we need to prove \( \| S \| = \| T \| \) for two equi-measurable simple operators \( S, T \) in \( \mathcal{M} \). Similar to the proof of Corollary 2.3, there is a \( \theta \in \text{Aut}(\mathcal{M}, \tau) \) such that \( S = \theta(T) \). Hence \( \| S \| = \| T \| \). \( \square \)

Corollary 3.15. A semi-norm on \( (L^\infty[0,1], \int_0^1 dx) \) or \( (\mathbb{C}^n, \tau) \) is a tracial gauge norm if and only if it is a symmetric gauge norm, where \( \tau((x_1, \ldots, x_n)) = \frac{x_1 + \cdots + x_n}{n} \).

3.4 Unitarily invariant semi-norms

Definition 3.16. Let \( (\mathcal{M}, \tau) \) be a von Neumann algebra. A semi-norm \( \| \cdot \| \) on \( \mathcal{M} \) is unitarily invariant if \( \| UTV \| = \| T \| \) for all \( T \in \mathcal{M} \) and unitary operators \( U, V \in \mathcal{M} \).

Proposition 3.17. Let \( \| \cdot \| \) be a semi-norm on \( \mathcal{M} \). Then the following statements are equivalent:

1. \( \| \cdot \| \) is unitarily invariant;
2. \(|\cdot|\) is gauge invariant and unitarily conjugate invariant, i.e., \(|UTU^*| = |T|\) for all \(T \in \mathcal{M}\) and unitary operators \(U \in \mathcal{M}\);

3. \(|\cdot|\) is left-unitarily invariant and \(|T| = |T^*|\) for every \(T \in \mathcal{M}\);

4. for all operators \(T,A,B \in \mathcal{M}\), \(|ATB| \leq |A| \cdot |T| \cdot |B|\).

Proof. “1 \Rightarrow 4” is similar to the proof of Lemma 3.2. “4 \Rightarrow 3”, “3 \Rightarrow 2”, and “2 \Rightarrow 1” are routine.

Corollary 3.18. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and let \(|\cdot|\) be a unitarily invariant semi-norm on \(\mathcal{M}\). If \(0 \leq S \leq T\), then \(|S| \leq |T|\).

Proof. Since \(0 \leq S \leq T\), there is an operator \(A \in \mathcal{M}\) such that \(S = ATA^*\) and \(|A| \leq 1\). By Proposition 3.17,

\[|S| = |ATA^*| \leq |A| \cdot |T| \cdot |A| \leq |T|\]

Corollary 3.19. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and let \(|\cdot|\) be a symmetric, gauge invariant semi-norm on \(\mathcal{M}\). Then \(|\cdot|\) is a unitarily invariant semi-norm on \(\mathcal{M}\).

Example 3.20. Let \(\mathcal{M} = \mathbb{C}^n\), \(n \geq 2\) and \(\tau((x_1, \cdots, x_n)) = \frac{x_1 + \cdots + x_n}{n}\). Define \(|(x_1, \cdots, x_n)| = |x_1|\). Then \(|\cdot|\) is a unitarily invariant semi-norm but not a symmetric gauge semi-norm on \(\mathcal{M}\).

Lemma 3.21. Let \((\mathcal{M}, \tau)\) be a finite factor and let \(|\cdot|\) be a semi-norm on \(\mathcal{M}\). Then the following conditions are equivalent:

1. \(|\cdot|\) is a tracial gauge semi-norm;

2. \(|\cdot|\) is a symmetric gauge semi-norm;

3. \(|\cdot|\) is a unitarily invariant semi-norm.
Proof. "1 ⇒ 2" by Lemma 3.12 and "2 ⇒ 3" by Corollary 3.19. We need to prove "3 ⇒ 1". By Corollary 3.5, we need to prove \( \|S\| = \|T\| \) for two equi-measurable positive simple operators \( S, T \in M \). Suppose \( S = a_1 E_1 + \cdots + a_n E_n \) and \( T = a_1 F_1 + \cdots + a_n F_n \), where \( E_1 + \cdots + E_n = 1 \) and \( F_1 + \cdots + F_n = 1 \) and \( \tau(E_k) = \tau(F_k) \) for \( 1 \leq k \leq n \). Since \( M \) is a factor, there is a unitary operator \( U \in M \) such that \( E_k = UF_kU^* \) for \( 1 \leq k \leq n \). Therefore, \( S = UTU^* \) and \( \|S\| = \|T\| \).

\[ \]

3.5 Weak Dixmier property

**Definition 3.22.** A finite von Neumann algebra \((M, \tau)\) satisfies the weak Dixmier property if for every positive operator \( T \in M \), \( \tau(T) \) is in the operator norm closure of the convex hull of \( \{ S \in M : S \text{ and } T \text{ are equi-measurable} \} \).

A finite factor \((M, \tau)\) satisfies the Dixmier property (see [21]): for every operator \( T \in M \), \( \tau(T) \) is in the operator norm closure of the convex hull of \( \{ UTU^* : U \in \mathcal{U}(M) \} \). Hence finite factors satisfy the weak Dixmier property. In the following, we will characterize finite von Neumann algebras satisfying the weak Dixmier property.

There is a central projection \( P \) in a finite von Neumann algebra \((M, \tau)\) such that \( P M \) is type I and \( (1 - P) M \) is type II. A type II von Neumann algebra is diffuse, i.e., there are no minimal projections in the von Neumann algebra. Furthermore, there are central projections \( P_1, \ldots, P_n, \ldots \) in \( M \), such that \( P_1 + \cdots + P_n + \cdots = P \) and \( P_n M = A_n \otimes M_n(\mathbb{C}) \), \( A_n \) is abelian.

We can decompose \( A_n \) into an atomic part \( A_n^a \) and a diffuse part \( A_n^d \), i.e., there is a projection \( Q_n \) in \( A_n \), \( A_n^a = Q_n A_n \), such that \( Q_n = E_{n1} + E_{n2} + \cdots, E_{nk} \) is a minimal projection in \( A_n^a \) and \( \tau(E_{nk}) > 0 \), and \( A_n^d = (1 - Q_n) A_n \) is diffuse. Let \( M_a = \sum A_n^a \otimes M_n(\mathbb{C}) \) and \( M_c = \sum A_n^c \otimes M_n(\mathbb{C}) \otimes (1 - P) M \). Then \( M = M_a \oplus M_c \). We call \( M_a \) the atomic part of \( M \) and \( M_c \) the diffuse part of \( M \). A finite von Neumann algebra \((M, \tau)\) is atomic if \( M = M_a \) and is diffuse if \( M = M_c \).

**Lemma 3.23.** Let \((M, \tau)\) be a finite dimensional von Neumann algebra such that for every two non-zero minimal projections \( E, F \in M \), \( \tau(E) = \tau(F) \). Then \((M, \tau)\) satisfies the weak Dixmier property.

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Proof. Since $\mathcal{M}$ is finite dimensional, $\mathcal{M} \cong M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_r}(\mathbb{C})$. Since $\tau(E) = \tau(F)$ for every two non-zero minimal projections $E, F \in \mathcal{M}$, $(\mathcal{M}, \tau)$ can be embedded into $(M_n(\mathbb{C}), \tau_n)$, where $n = k_1 + \cdots + k_r$. So we can assume that $(\mathcal{M}, \tau)$ is a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ such that $\mathcal{M}$ contains all diagonal matrices $a_1E_1 + \cdots + a_nE_n$. Now for every positive operator $T \in \mathcal{M}$, there is a unitary operator $U \in \mathcal{M}$ such that $UTU^* = a_1E_1 + \cdots + a_nE_n$, $a_1, \ldots, a_n \geq 0$ and $\tau(T) = \frac{a_1 + \cdots + a_n}{n!}$. Then $\tau(T) = \frac{\sum_{(a_1E_1 + \cdots + a_nE_n)}{a_1}^n}{n!}$. \hfill $\Box$

**Lemma 3.24.** Let $(\mathcal{M}, \tau)$ be a diffuse finite von Neumann algebra. Then $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property.

**Proof.** Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$. By Lemma 2.6, there is a $*$-isomorphism $\alpha$ from $(\mathcal{A}, \tau)$ onto $(L^\infty[0,1], \int_0^1 dx)$ such that $\int_0^1 dx \circ \alpha = \tau$. For a positive operator $T \in \mathcal{M}$, there is an operator $S \in \mathcal{A}$ such that $\alpha(S) = \mu_T(T)$. Hence $\tau(T) = \tau(S) = \int_0^1 \mu_T(T)ds$. By Theorem 2.4, for any $\varepsilon > 0$, there are $S_1, \ldots, S_n$ in $\mathcal{A}$ such that $S, S_1, \ldots, S_n$ are equi-measurable and $\|\tau(S) - \frac{S_1 + \cdots + S_n}{n}\| < \varepsilon$. Hence $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property. \hfill $\Box$

**Lemma 3.25.** Let $(\mathcal{M}, \tau)$ be an atomic finite von Neumann algebra with two minimal projections $E$ and $F$ in $\mathcal{M}$ such that $\tau(E) \neq \tau(F)$. Then $(\mathcal{M}, \tau)$ does not satisfy the weak Dixmier property.

**Proof.** Since $(\mathcal{M}, \tau)$ is an atomic finite von Neumann algebra, $\mathcal{M} \cong M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C}) \oplus \cdots$. Let $E_{ij}$ be minimal projections in $M_{k_i}$ such that $\sum E_{ij} = 1$. Without loss of generality, assume that $\tau(E_{11}) > \tau(E_{21}) \geq \tau(E_{31}) \geq \cdots$. Let $T = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \oplus A$, where

$$A = \begin{pmatrix} \frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2} \end{pmatrix} \oplus \cdots.$$

If $T_1 \in \mathcal{M}$ and $T$ are equi-measurable, then $T_1 = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \oplus A_1$, where $A$ and $A_1$
are equi-measurable. Hence, if \( \tau(T) \) is in the operator norm closure of the convex hull of 
\( \{ S \in \mathcal{M} : S \text{ and } T \text{ are equi-measurable} \} \), then \( \tau(T) = 1 \). It is a contradiction. \( \square \)

Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and \( E \in \mathcal{M} \) be a non-zero projection. The induced finite von Neumann algebra \((\mathcal{M}_E, \tau_E)\) is the von Neumann algebra \( \mathcal{M}_E = E \mathcal{M} E \) with 
a faithful normal trace \( \tau_E(ETE) = \frac{\tau(EET)}{\tau(E)} \). The proof of the following lemma is similar to the proof of Lemma 3.25.

**Lemma 3.26.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra such that \( \mathcal{M}_a \neq 0 \) and \( \mathcal{M}_c \neq 0 \). Then \( \mathcal{M} \) does not satisfies the weak Dixmier property.

**Proof.** Let \( P \) be the central projection such that \( \mathcal{M}_a = P \mathcal{M} \) and \( \mathcal{M}_c = (1 - P) \mathcal{M} \). Let \( \mathcal{A} \) be a separable diffuse abelian von Neumann subalgebra of \((\mathcal{M}_c, \tau_{1-P})\). By Lemma 2.6, there is a positive operator \( A \) in \( \mathcal{M}_c \) such that \( \mu(A) = \frac{1-s}{2} \) with respect to \((\mathcal{M}_c, \tau_{1-P})\). Consider 
\( T = P + A(1-P) \). Then

\[
\mu_s(T) = \begin{cases} 
1, & 0 \leq s < \tau(P); \\
\frac{1-s}{2\tau(1-P)} \leq \frac{1}{2}, & \tau(P) \leq s \leq 1 
\end{cases}
\]

with respect to \((\mathcal{M}, \tau)\). If \( T_1 \in \mathcal{M} \) and \( T \) are equi-measurable, then \( T_1 = P + A_1 \) such that \( A_1 \) and \( A \) are equi-measurable. Hence, if \( \tau(T) \) is in the operator norm closure of the convex hull of \( \{ S \in \mathcal{M} : S \text{ and } T \text{ are equi-measurable} \} \), then \( \tau(T) = 1 \). It is a contradiction. \( \square \)

Summarizing Lemma 3.23, 3.24, 3.25, 3.26, we can characterize finite von Neumann algebras satisfying the weak Dixmier property as the following theorem.

**Theorem 3.27.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra. Then \( \mathcal{M} \) satisfies the weak Dixmier property if and only if \( \mathcal{M} \) satisfies one of the following conditions:

1. \( \mathcal{M} \) is finite dimensional (hence atomic) and for every two non-zero minimal projections \( E, F \in \mathcal{M} \), \( \tau(E) = \tau(F) \), or equivalently, \((\mathcal{M}, \tau)\) can be identified as a von Neumann subalgebra of \((M_n(\mathbb{C}), \tau_n)\) that contains all diagonal matrices;

2. \( \mathcal{M} \) is diffuse.
Corollary 3.28. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and \(E \in \mathcal{M}\) be a non-zero projection. Then \((\mathcal{M}_E, \tau_E)\) also satisfies the weak Dixmier property.

The following example shows that we cannot replace the weak Dixmier property by the following condition: \(\tau(T)\) is in the operator norm closure of the convex hull of \(\{\theta(T) : \theta \in \text{Aut}(\mathcal{M}, \tau)\}\).

Example 3.29. \((C \oplus M_2(\mathbb{C}), \tau)\), \(\tau(a \oplus B) = \frac{1}{2} a + \frac{1}{3} \tau_2(B)\), satisfies the weak Dixmier property. On the other hand, let \(T = 1 \oplus 2 \in C \oplus M_2(\mathbb{C})\). Then for every \(\theta \in \text{Aut}(\mathcal{M}, \tau)\), \(\theta(T) = T\). Hence, \(\tau(T)\) is not in the operator norm closure of the convex hull of \(\{\theta(T) : \theta \in \text{Aut}(\mathcal{M}, \tau)\}\).

3.6 A comparison theorem

The following theorem is the main result of this section.

Theorem 3.30. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property. If \(|\cdot|\) is a normalized tracial gauge semi-norm on \(\mathcal{M}\), then for all \(T \in \mathcal{M}\),

\[ |T|_1 \leq |T| \leq \|T\| . \]

In particular, every tracial gauge semi-norm on \(\mathcal{M}\) is a norm.

Proof. By Corollary 3.4, \(|T| \leq \|T\|\) for every \(T \in \mathcal{M}\). To prove \(|T|_1 \leq |T|\), we can assume \(T \geq 0\). Let \(\epsilon > 0\). Since \((\mathcal{M}, \tau)\) satisfies the weak Dixmier property, there are \(S_1, \ldots, S_k\) in \(\mathcal{M}\) such that \(T, S_1, \ldots, S_k\) are equi-measurable and \(|\tau(T) - \frac{S_1 + \cdots + S_k}{k}| < \epsilon\). By Corollary 3.4, \(|\tau(T) - \frac{S_1 + \cdots + S_k}{k}| \leq \|\tau(T) - \frac{S_1 + \cdots + S_k}{k}\| < \epsilon\). Hence \(|T|_1 = |\tau(T)| \leq \|\frac{S_1 + \cdots + S_k}{k}\| + \epsilon \leq \|\frac{S_1 + \cdots + S_k}{k}\| + \epsilon = \|T\| + \epsilon\).

By Theorem 3.30 and Lemma 3.21, we have the following corollary.

Corollary 3.31. Let \((\mathcal{M}, \tau)\) be a finite factor and \(|\cdot|\) be a normalized unitarily invariant norm on \(\mathcal{M}\). Then

\[ |T|_1 \leq \|T\| \leq \|T\|, \quad \forall T \in \mathcal{M}. \]

In particular, every unitarily invariant semi-norm on a finite factor is a norm.
By Theorem 3.30 and Lemma 3.14, we have the following corollary.

**Corollary 3.32.** Let $\| \cdot \|$ be a normalized symmetric gauge semi-norm on $(L^\infty[0,1], \int_0^1 dx)$ (or $(\mathbb{C}^n, \tau)$, where $\tau((x_1, \cdots, x_n)) = \frac{x_1 + \cdots + x_n}{n}$). Then

$$\|T\|_1 \leq \|T\| \leq \|T\|, \quad \forall T \in L^\infty[0,1] \, (or \, \mathbb{C}^n).$$

In particular, every symmetric gauge semi-norm on $(L^\infty[0,1], \int_0^1 dx)$ (or $(\mathbb{C}^n, \tau)$) is a norm.
CHAPTER 4
UNITAL BALLS

The main result of this chapter is the following theorem.

**Theorem 4.1.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and let \(\| \cdot \|\) be a tracial gauge norm on \(\mathcal{M}\). Then \(\mathcal{M}_{1,\| \cdot \|} \triangleq \{ T \in \mathcal{M} : \| T \| \leq 1 \}\) is closed in the weak operator topology.

To prove Theorem 4.1, we need the following lemmas.

**Lemma 4.2.** Let \(E_1, \ldots , E_n\) be projections in \(\mathcal{M}\) such that \(E_1 + \cdots + E_n = 1\) and \(T \in \mathcal{M}\). Then \(S = E_1TE_1 + \cdots + E_nTE_n\) is in the convex hull of \(\{ UTU^* : U \in \mathcal{U}(\mathcal{M}) \}\).

**Proof.** Let \(T = (T_{ij})\) be the matrix with respect to the decomposition \(1 = E_1 + \cdots + E_n\). Let \(U = -E_1 + E_2 + \cdots + E_n\). Then simple computation shows that

\[
\frac{1}{2}(UTU^* + T) = \begin{pmatrix}
T_{11} & 0 & \cdots & 0 \\
0 & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & T_{n2} & \cdots & T_{nn}
\end{pmatrix} = E_1TE_1 + (1 - E_1)T(1 - E_1).
\]

By induction, \(S = E_1TE_1 + \cdots + E_nTE_n\) is in the convex hull of \(\{ UTU^* : U \in \mathcal{U}(\mathcal{M}) \}\).

**Corollary 4.3.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and \(\| \cdot \|\) be a unitarily invariant norm on \(\mathcal{M}\). Let \(E_1, \ldots , E_n\) be projections in \(\mathcal{M}\) such that \(E_1 + \cdots + E_n = 1\) and \(T \in \mathcal{M}\) and \(S = E_1TE_1 + \cdots + E_nTE_n\). Then \(\| S \| \leq \| T \|\).

Recall that for a (non-zero) finite projection \(E\) in \(\mathcal{M}\), \(\tau_E(ETE) = \frac{\tau(ETE)}{\tau(E)}\) is the induced trace on \(\mathcal{M}_E = E\mathcal{M}E\).
Lemma 4.4. Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and $\| \cdot \|$ be a tracial gauge norm on $\mathcal{M}$. Suppose $T, E_1, \cdots, E_n \in \mathcal{M}$, $T \geq 0$, $E_1 + \cdots + E_n = 1$. Then $\|T\| \geq \|\tau_{E_1}(E_1TE_1)E_1 + \cdots + \tau_{E_n}(E_nTE_n)E_n\|$. 

Proof. We may assume that $\|1\| = 1$. Since $\mathcal{M}$ satisfies the weak Dixmier property, by Corollary 3.28, $(\mathcal{M}_{E_i}, \tau_{E_i})$ also satisfies the weak Dixmier property, $1 \leq i \leq n$. Let $\varepsilon > 0$. There are operators $S_{11}, \cdots, S_{nk}$ in $\mathcal{M}_{E_i}$ such that $E_iTE_i, S_{11}, \cdots, S_{nk}$ are equi-measurable and

$$\left\| \frac{S_{11} + \cdots + S_{nk}}{k} - \tau_{E_i}(E_iTE_i)E_i \right\| < \varepsilon.$$ 

Let $S_j = S_{1j}E_1 + \cdots + S_{nj}E_n$, $1 \leq j \leq k$. Then $T, S_1, \cdots, S_n$ are equi-measurable and

$$\left\| \frac{S_1 + \cdots + S_k}{k} - (\tau_{E_1}(E_1TE_1)E_1 + \cdots + \tau_{E_n}(E_nTE_n)E_n) \right\| < \varepsilon.$$ 

By Corollary 3.4,

$$\left\| \frac{S_1 + \cdots + S_k}{k} - (\tau_{E_1}(E_1TE_1)E_1 + \cdots + \tau_{E_n}(E_nTE_n)E_n) \right\| < \varepsilon.$$ 

Hence, $\|\tau_{E_1}(E_1TE_1)E_1 + \cdots + \tau_{E_n}(E_nTE_n)E_n\| \leq \|T\| + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain the lemma. 

Corollary 4.5. Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and $\| \cdot \|$ be a tracial gauge norm on $\mathcal{M}$. If $\mathcal{A}$ is a finite-dimensional abelian von Neumann subalgebra of $\mathcal{M}$ and $E_{\mathcal{A}}$ is the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$ preserving $\tau$, then for every $T \in \mathcal{M}$, $\|E_{\mathcal{A}}(T)\| \leq \|T\|$. 

Proof. Let $\mathcal{A} = \{E_1, \cdots, E_n\}''$ such that $E_1 + \cdots + E_n = 1$. Then for every $T \in \mathcal{M}$, $E_{\mathcal{A}}(T) = \tau_{E_1}(E_1TE_1)E_1 + \cdots + \tau_{E_n}(E_nTE_n)E_n$. By Corollary 4.3 and Lemma 4.4, $\|E_{\mathcal{A}}(T)\| \leq \|T\|$. 

Proof of Theorem 4.1. By Lemma 3.12 and Corollary 3.19, $\| \cdot \|$ is unitarily invariant. Suppose $T_\alpha$ is a net in $\mathcal{M}_{1, \| \cdot \|}$ such that $\lim_\alpha T_\alpha = T$ in the weak operator topology. Let $T = U|T|$ be the polar decomposition of $T$. Then $\lim_\alpha U^*T_\alpha = |T|$ in the weak operator topology. Since $\| \cdot \|$ is unitarily invariant, $\|UT_\alpha\| \leq 1$ and $\|T\| = \|T\|$. So we may assume that $T \geq 0$ and $T_\alpha = T_\alpha^*$. By the spectral decomposition theorem and Corollary 3.4, to prove $\|T\| \leq 1$, we need to prove $\|S\| \leq 1$ for every positive simple operator $S$ such that $S \leq T$. Let $S = a_1E_1 + \cdots + a_nE_n$. 

...
a_n E_n and ε > 0. Since \( \lim \alpha T_\alpha = T \geq S \), \( \lim \alpha E_i T_\alpha E_i = E_i T E_i \geq a_i E_i \) for \( 1 \leq i \leq n \). Hence, \( \lim \tau E_i (E_i \alpha T \alpha + \varepsilon) E_i \geq a_i + \varepsilon \geq a_i \). So there is a \( \beta \) such that \( \tau E_i (E_i \beta + \varepsilon) E_i \geq a_i + \varepsilon \). By Lemma 4.4 and Corollary 3.18, \( 1 + \varepsilon \geq ||\beta E_i + \varepsilon|| \geq ||\tau (E_i \beta + \varepsilon) E_i || \geq ||S|| \). Since \( \varepsilon > 0 \) is arbitrary, \( ||S|| \leq 1 \).

**Corollary 4.6.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and \( || \cdot || \) be a tracial gauge norm on \( \mathcal{M} \). If \( \mathcal{A} \) is a separable abelian von Neumann subalgebra of \( \mathcal{M} \) and \( \mathcal{E}_{\mathcal{A}, \tau} \) is the normal conditional expectation from \( \mathcal{M} \) onto \( \mathcal{A} \) preserving \( \tau \), then \( ||\mathcal{E}_{\mathcal{A}, \tau}(T)|| \leq ||T|| \) for all \( T \in \mathcal{M} \).

**Proof.** Since \( \mathcal{A} \) is a separable abelian von Neumann algebra, there is a sequence of finite dimensional abelian von Neumann subalgebras \( \mathcal{A}_n \) such that \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A} \) and \( \mathcal{A} \) is the closure of \( \cup_n \mathcal{A}_n \) in the strong operator topology. Let \( \mathcal{E}_{\mathcal{A}, \tau} \) be the normal conditional expectation from \( \mathcal{M} \) onto \( \mathcal{A}_n \) preserving \( \tau \). Then for every \( T \in \mathcal{M} \), \( \mathcal{E}_{\mathcal{A}, \tau}(T) = \lim_{n \to \infty} \mathcal{E}_{\mathcal{A}_n, \tau}(T) \) in the strong operator topology. By Theorem 4.1 and Corollary 4.5, \( ||\mathcal{E}_{\mathcal{A}, \tau}(T)|| \leq ||T|| \).

**Corollary 4.7.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and \( || \cdot || \) be a tracial gauge norm on \( \mathcal{M} \). Suppose \( 0 \leq T_1 \leq T_2 \leq \cdots \leq T \) in \( \mathcal{M} \) such that \( \lim_{n \to \infty} T_n = T \) in the weak operator topology. Then \( \lim_{n \to \infty} ||T_n|| = ||T|| \).

**Proof.** By Corollary 3.18, \( ||T_1|| \leq ||T_2|| \leq \cdots \leq ||T|| \). Hence, \( \lim_{n \to \infty} ||T_n|| \leq ||T|| \). By Theorem 4.1, \( \lim_{n \to \infty} ||T_n|| \geq ||T|| \).

**Corollary 4.8.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property, and let \( || \cdot ||_1 \), \( || \cdot ||_2 \) be two tracial gauge norms on \( \mathcal{M} \). Then \( || \cdot ||_1 = || \cdot ||_2 \) on \( \mathcal{M} \) if \( ||T||_1 = ||T||_2 \) for every operator \( T = a_1 E_1 + \cdots + a_n E_n \) in \( \mathcal{M} \) such that \( a_1, \cdots, a_n \geq 0 \) and \( \tau(E_1) = \cdots = \tau(E_n) = \frac{1}{n} \), \( n = 1, 2, \cdots \).

**Proof.** We need only to prove \( ||T||_1 = ||T||_2 \) for every positive operator \( T \) in \( \mathcal{M} \). By Theorem 3.27, \( \mathcal{M} \) is either a finite dimensional von Neumann algebra such that \( \tau(E) = \tau(F) \) for each pair of nonzero minimal projections in \( \mathcal{M} \) or \( \mathcal{M} \) is diffuse. If \( \mathcal{M} \) is a finite dimensional von Neumann algebra such that \( \tau(E) = \tau(F) \) for each pair of nonzero minimal projections in \( \mathcal{M} \), then the corollary is obvious. If \( \mathcal{M} \) is diffuse, by the spectral decomposition theorem, there is a sequence of operators \( T_n \in \mathcal{M} \) satisfying the following conditions:
1. $0 \leq T_1 \leq T_2 \leq \cdots \leq T$,

2. $T_n = a_{n1}E_{n1} + \cdots + a_{nm}E_{nm}$, $a_{n1}, \ldots, a_{nm} \geq 0$ and $\tau(E_{n1}) = \cdots = \tau(E_{mn}) = \frac{1}{n}$,

3. $\lim_{n \to \infty} T_n = T$ in the weak operator topology.

By the assumption of the corollary, $\|T_n\|_1 = \|T_n\|_2$. By Corollary 4.7, $\|T\|_1 = \|T\|_2$. 

**Corollary 4.9.** Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and $\| \cdot \|_1$ and $\| \cdot \|_2$ be two unitarily invariant norms on $\mathcal{M}$. Then $\| \cdot \|_1 = \| \cdot \|_2$ on $\mathcal{M}$ if $\| \cdot \|_1 = \| \cdot \|_2$ on all type $\mathrm{I}_n$ subfactors of $\mathcal{M}$, $n = 1, 2, \ldots$. 

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CHAPTER 5

KY FAN NORMS

Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and \(0 \leq t \leq 1\). For \(T \in \mathcal{M}\), define the Ky Fan \(t\)-th norm by

\[
\|T\|_{(t)} = \begin{cases} 
\|T\|, & t = 0; \\
\frac{1}{t} \int_0^t \mu_\tau(T) ds, & 0 < t \leq 1.
\end{cases}
\]

Let \(\mathcal{M}_1 = (\mathcal{M}, \tau) \ast (L_\mathcal{F}, \tau')\) be the reduced free product von Neumann algebra of \(\mathcal{M}\) and the free group factor \(L_\mathcal{F}\). Then \(\mathcal{M}_1\) is a type \(\text{II}_1\) factor with a faithful normal trace \(\tau_1\) such that the restriction of \(\tau_1\) to \(\mathcal{M}\) is \(\tau\). Recall that \(\mathcal{U}(\mathcal{M}_1)\) is the set of unitary operators in \(\mathcal{M}_1\) and \(\mathcal{P}(\mathcal{M}_1)\) is the set of projections in \(\mathcal{M}_1\).

**Lemma 5.1.** For \(0 < t \leq 1\) and \(T \in \mathcal{M}\),

\[
t\|T\|_{(t)} = \sup \{ |\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \}.
\]

**Proof.** We may assume that \(T\) is a positive operator. Let \(\mathcal{A}\) be a separable diffuse abelian von Neumann subalgebra of \(\mathcal{M}_1\) containing \(T\) and let \(\alpha\) be a \(*\)-isomorphism from \((\mathcal{A}, \tau_1)\) onto \((L^\infty[0,1], \int_0^1 dx)\) such that \(\tau_1 = \int_0^1 dx \circ \alpha\). Let \(f(x) = \alpha(T)\) and \(f^*(x)\) be the non-increasing rearrangement of \(f(x)\). Then \(\mu_\tau(T) = f^*(s)\). By the definition of \(f^*\) (see (2.1.1)),

\[
m(\{f^* > f^*(t)\}) = \lim_{n \to \infty} m\left(\left\{ f^* > f^*(t) + \frac{1}{n} \right\} \right) \leq t
\]

and

\[
m(\{f^* \geq f^*(t)\}) \geq \lim_{n \to \infty} m\left(\left\{ f^* > f^*(t) - \frac{1}{n} \right\} \right) \geq t.
\]

Since \(f^*\) and \(f\) are equi-measurable, \(m(\{f > f^*(t)\}) \leq t\) and \(m(\{f \geq f^*(t)\}) \geq t\). Therefore, there is a measurable subset \(A\) of \([0,1]\), \(\{f > f^*(t)\} \subset A \subset \{f \geq f^*(t)\}\), such that \(m(A) = t\).

Since \(f(x)\) and \(f^*(x)\) are equimeasurable, \(\int_A f(s) ds = \int_0^t f^*(s) ds\). Let \(E' = \alpha^{-1}(\mathcal{A})\). Then
\[ \tau_1(E') = t \text{ and } \tau_1(TE') = \int_A f(s)ds = \int_0^t f^*(s)ds = t\|T\|_t. \] Hence, \( t\|T\|_t \leq \sup \{ \|\tau_1(UTE)\| : U \in \mathcal{U}(\mathcal{M}), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \}. \)

We need to prove that if \( E \) is a projection in \( \mathcal{M}_1 \), \( \tau_1(E) = t \), and \( U \in \mathcal{U}(\mathcal{M}_1) \), then \( t\|T\|_t \geq |\tau_1(UTE)| \). By the Schwartz inequality,

\[ |\tau_1(UTE)| = \tau_1(U^*E^1/2T^1/2E) \leq \tau_1(U^*EUT)^{1/2} \tau_1(E)^{1/2}. \]

By Corollary 2.8, \( \tau_1(ET) = \int_0^1 \mu_s(ET)ds \). By Corollary 2.10, \( \mu_0(ET) \leq \min \{ \mu_s(T), \mu_s(E) \|T\| \}. \)

Note that \( \mu_s(E) = 0 \) for \( s \geq \tau_1(E) = t \). Hence, \( \tau_1(ET) \leq \int_0^t \mu_s(T)ds = t\|T\|_t \). Similarly,

\( \tau_1(U^*EUT) \leq t\|T\|_t \). So \( |\tau_1(UTE)| \leq t\|T\|_t \). This proves that \( t\|T\|_t \geq \sup \{ |\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \}. \)

**Theorem 5.2.** For \( 0 \leq t \leq 1 \), \( \|\cdot\|_t \) is a normalized tracial gauge norm on \( (\mathcal{M}, \tau) \).

**Proof.** We only prove the triangle inequality, since the other parts are obvious. We may assume that \( 0 < t < 1 \). Let \( S, T \in \mathcal{M} \). By Lemma 5.1, \( t\|S + T\|_t = \sup \{ \|\tau_1(U(S + T))\| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \} \leq \sup \{ \|\tau_1(USE)\| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \} + \sup \{ |\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \} = t\|S\|_t + t\|T\|_t. \)

**Proposition 5.3.** Let \( (\mathcal{M}, \tau) \) be a finite von Neumann algebra and \( T \in (\mathcal{M}, \tau) \). Then \( \|T\|_t \) is a non-increasing continuous function on \( [0, 1] \).

**Proof.** Let \( 0 < t_1 < t_2 \leq 1 \). Then

\[
\|T\|_{t_1} - \|T\|_{t_2} = \frac{1}{t_1} \int_0^{t_1} \mu_s(T)ds - \frac{1}{t_2} \int_0^{t_2} \mu_s(T)ds
\]

\[
= \frac{1}{t_1} \int_0^{t_1} \mu_s(T)ds - \frac{\int_0^{t_2} \mu_s(T)ds - \int_0^{t_1} \mu_s(T)ds}{t_2(t_2 - t_1)} \geq 0.
\]

Since \( \mu_s(T) \) is right-continuous, \( \|T\|_t \) is a non-increasing continuous function on \( [0, 1] \).

**Example 5.4.** The Ky Fan \( \frac{k}{n} \)-th norm of a matrix \( T \in (\mathcal{M}_n(\mathbb{C}), \tau_n) \) is

\[
\|T\|_{(\frac{k}{n})} = \frac{s_1(T) + \cdots + s_k(T)}{k}, \quad 1 \leq k \leq n.
\]
CHAPTER 6
DUAL NORMS

6.1 Dual norms

Let $\|\cdot\|$ be a norm on a finite von Neumann algebra $(\mathcal{M}, \tau)$. For $T \in \mathcal{M}$, define

$$\|T\|^\# = \sup\{|\tau(TX)| : X \in \mathcal{M}, \|X\| \leq 1\}.$$  

When no confusion arises, we simply write $\|\cdot\|^\#$ instead of $\|\cdot\|^\#_\mathcal{M}$.

**Lemma 6.1.** $\|\cdot\|^\#$ is a norm on $\mathcal{M}$.

**Proof.** If $T \neq 0$, $\|T\|^\# \geq \tau(TT^*)/\|T^*\| > 0$. It is easy to see that $\|\lambda T\|^\# = |\lambda| \cdot \|T\|^\#$ and $\|T_1 + T_2\|^\# \leq \|T_1\|^\# + \|T_2\|^\#$. \hfill \Box

**Definition 6.2.** $\|\cdot\|^\#$ is called the *dual norm* of $\|\cdot\|$ on $\mathcal{M}$ with respect to $\tau$.

The next lemma follows directly from the definition of dual norm.

**Lemma 6.3.** Let $\|\cdot\|$ be a norm on a finite von Neumann algebra $(\mathcal{M}, \tau)$ and $\|\cdot\|^\#$ be the dual norm on $\mathcal{M}$. Then for $S, T \in \mathcal{M}$, $|\tau(ST)| \leq \|S\| \cdot \|T\|^\#$.

The following corollary is a generalization of Hölder's inequality for bounded operators in finite von Neumann algebras.

**Corollary 6.4.** Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $\|\cdot\|$ be a gauge norm on $\mathcal{M}$. Then for $S, T \in \mathcal{M}$, $\|ST\|_1 \leq \|S\| \cdot \|T\|^\#$.

**Proof.** By Lemma 3.7, $\|ST\|_1 = \sup\{|\tau(UST)| : U \in \mathcal{U}(\mathcal{M})\}$. By Lemma 6.3 and Lemma 3.2, $|\tau(UST)| \leq \|US\| \cdot \|T\|^\# = \|S\| \cdot \|T\|^\#$. \hfill \Box
Proposition 6.5. If $\| \cdot \|$ is a unitarily invariant norm on a finite von Neumann algebra $(\mathcal{M}, \tau)$, then $\| \cdot \|^\#$ is also a unitarily invariant norm on $\mathcal{M}$.

Proof. Let $U$ be a unitary operator. Then

$$
\|UT\|^\# = \sup\{\tau(UTX) : X \in \mathcal{M}, \|X\| \leq 1\}
= \sup\{\tau(TXU) : X \in \mathcal{M}, \|X\| \leq 1\}
= \sup\{\tau(TX) : X \in \mathcal{M}, \|X\| \leq 1\}
= \|T\|.
$$

Similarly, $\|TU\|^\# = \|T\|$. 

Proposition 6.6. If $\| \cdot \|$ is a symmetric gauge norm on a finite von Neumann algebra $(\mathcal{M}, \tau)$, then $\| \cdot \|^\#$ is also a symmetric gauge norm on $(\mathcal{M}, \tau)$.

Proof. Let $\theta \in \text{Aut}(\mathcal{M}, \tau)$. Then

$$
\|\theta(T)\|^\# = \sup\{\tau(\theta(T)X) : X \in \mathcal{M}, \|X\| \leq 1\}
= \sup\{\tau(\theta(T^{-1}(X))) : X \in \mathcal{M}, \|X\| \leq 1\}
= \sup\{\tau(TX) : X \in \mathcal{M}, \|X\| \leq 1\}
= \|T\|.
$$

Lemma 6.7. Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property, and let $\| \cdot \|$ be a tracial gauge norm on $\mathcal{M}$. If $T \in \mathcal{M}$ is a positive operator, then

$$
\|T\|^\# = \sup\{\tau(TX) : X \in \mathcal{M}, X \geq 0, XT = TX, \|X\| \leq 1\}.
$$

Proof. Let $\mathcal{A}$ be a separable abelian von Neumann subalgebra of $\mathcal{M}$ containing $T$ and $E_{\mathcal{A}}$ be the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$ preserving $\tau$. For every $Y \in \mathcal{M}$ such that $\|Y\| \leq 1$, let $X = E_{\mathcal{A}}(Y)$. By Corollary 4.6, $\|X\| = \|X\| \leq \|Y\| \leq 1$. Furthermore, $|\tau(TY)| = |\tau(E_{\mathcal{A}}(TY))| = |\tau(TE_{\mathcal{A}}(Y))| = |\tau(TX)| \leq \tau(T|X|)$. Hence,

$$
\|T\|^\# = \sup\{\tau(TX) : X \in \mathcal{M}, X \geq 0, XT = TX, \|X\| \leq 1\}.
$$
Lemma 6.8. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property, and let \(\| \cdot \|\) be a tracial gauge norm on \(\mathcal{M}\). Suppose \(T = a_1E_1 + \cdots + a_nE_n\) is a positive simple operator in \(\mathcal{M}\). Then
\[
\| T \| = \sup \{ \tau(TX) : X = b_1E_1 + \cdots + b_nE_n \geq 0 \text{ and } \|X\|^\# \leq 1 \}
= \sup \left\{ \sum_{k=1}^n a_kb_k\tau(E_k) : X = b_1E_1 + \cdots + b_nE_n \geq 0 \text{ and } \|X\|^\# \leq 1 \right\}.
\]

Proof. By Lemma 6.7, \(\|T\|^\# = \sup \{ \tau(TX) : X \in \mathcal{M}, X \geq 0, XT = TX, \|X\| \leq 1 \}\). Let \(\mathcal{A} = \{E_1, \ldots, E_n\}''\) and \(E_{\mathcal{A}}\) be the normal conditional expectation from \(\mathcal{M}\) onto \(\mathcal{A}\) preserving \(\tau\). Then \(S = E_{\mathcal{A}}(X) = \tau_{E_1}(E_1XE_1)E_1 + \cdots + \tau_{E_n}(E_nXE_n)E_n\) is a positive operator, \(\tau(TX) = \tau(E_{\mathcal{A}}(TX)) = \tau(TS)\), and \(\|S\| \leq \|X\|\) by Corollary 4.5. Combining the definition of dual norm, this proves the lemma. \(\Box\)

Corollary 6.9. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and \(\| \cdot \|\) be a tracial gauge norm on \(\mathcal{M}\). Suppose \(S, T\) are equi-measurable, positive simple operators in \(\mathcal{M}\). Then \(\|S\|^\# = \|T\|^\#\).

Theorem 6.10. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and \(\| \cdot \|\) be a tracial gauge norm on \(\mathcal{M}\). Then \(\| \cdot \|^\#\) is also a tracial gauge norm on \(\mathcal{M}\). Furthermore, if \(\|1\| = 1\), then \(\|1\|^\# = 1\).

Proof. By Lemma 3.12, \(\| \cdot \|\) is a symmetric gauge norm on \(\mathcal{M}\). By Proposition 6.6, Corollary 6.9 and Lemma 3.9, \(\| \cdot \|^\#\) is a tracial gauge norm on \(\mathcal{M}\). Note that \(\|1\| = 1\), hence, \(\|1\|^\# \geq \tau(1 \cdot 1) = 1\). On the other hand, by Theorem 3.30, \(\|1\|^\# = \sup \{ |\tau(X)| : X \in \mathcal{M}, \|X\| \leq 1 \}\) \(\leq \sup \{ \|X\| : X \in \mathcal{M}, \|X\| \leq 1 \}\) \(\leq 1\). \(\Box\)

Corollary 6.11. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and \(\| \cdot \|\) be a tracial gauge norm on \(\mathcal{M}\). If \(\mathcal{N}\) is a von Neumann subalgebra of \(\mathcal{M}\) satisfying the weak Dixmier property, then \(\| \cdot \|^\#_{\mathcal{N}}\) is the restriction of \(\| \cdot \|^\#\) to \(\mathcal{N}\).

Proof. Let \(\| \cdot \|_1 = \| \cdot \|^\#_N\) and \(\| \cdot \|_2\) be the restriction of \(\| \cdot \|^\#_{\mathcal{M}}\) to \(\mathcal{N}\). By Theorem 6.10, both \(\| \cdot \|_1\) and \(\| \cdot \|_2\) are tracial gauge norms on \(\mathcal{N}\). By Lemma 3.5, to prove \(\| \cdot \|_1 = \| \cdot \|_2\),
we need to prove \( \|T\|_1 = \|T\|_2 \) for every positive simple operator \( T \in \mathcal{N} \). Let \( \mathcal{A} \) be a finite dimensional abelian von Neumann subalgebra of \( \mathcal{N} \) containing \( T \). By Lemma 6.8, \( \|T\|_\#(\mathcal{A}) = \|T\|_{\mathcal{N}} = \|T\|_\#(\mathcal{A}) \). So \( \|T\|_1 = \|T\|_2 \). \( \Box \)

### 6.2 Dual norms of Ky Fan norms

For \( (x_1, \cdots, x_n) \in \mathbb{C}^n \), \( \tau(x) = \frac{x_1 + \cdots + x_n}{n} \) defines a trace on \( \mathbb{C}^n \). For \( 1 \leq k \leq n \), the Ky Fan \( \frac{k}{n} \)-th norm on \( (\mathbb{C}^n, \tau) \) is \( \| (x_1, \cdots, x_n) \|_{(\frac{k}{n})} = \frac{x_1 + \cdots + x_k}{k} \), where \( (x_1^*, \cdots, x_n^*) \) is the decreasing rearrangement of \( (|x_1|, \cdots, |x_n|) \). Let \( \mathcal{D} = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : x_1 \geq x_2 \geq x_k = x_{k+1} = \cdots = x_n \geq 0, \frac{x_1 + \cdots + x_k}{k} \leq 1 \} \) and \( \mathcal{E} \) be the set of extreme points of \( \mathcal{D} \).

The proof of the following lemma is an easy exercise.

**Lemma 6.12.** The set \( \mathcal{E} \) consists of \( k + 1 \) points: \( (k, 0, \cdots), (\frac{k}{2}, \frac{k}{2}, 0, \cdots), \cdots, (\frac{k}{k-1}, \cdots, 0, \cdots), (1, 1, \cdots, 1) \) and \( (0, 0, \cdots, 0) \).

The following lemma is well-known. For a proof we refer to 10.2 of [15].

**Lemma 6.13.** Let \( s_1 \geq s_2 \geq \cdots \geq s_n \geq 0 \) and \( t_1, \cdots, t_n \geq 0 \). If \( t_1^* \geq t_2^* \geq \cdots \geq t_n^* \) is the decreasing rearrangement of \( t_1, \cdots, t_n \), then \( s_1 t_1^* + \cdots + s_n t_n^* \geq s_1 t_1 + \cdots + s_n t_n \).

**Lemma 6.14.** For \( T \in (M_n(\mathbb{C}), \tau_n) \),

\[
\|T\|_{(\frac{k}{n})} = \max \left\{ k \|T\|_1, \|T\|_2 \right\}.
\]

**Proof.** Let \( \|T\|_1 = \|T\|_{(\frac{k}{n})} \) and \( \|T\|_2 = \max \left\{ \frac{k}{n} \|T\|_2, \|T\|_1 \right\} \). Then both \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are unitarily invariant norms on \( M_n(\mathbb{C}) \). To prove \( \| \cdot \|_1 = \| \cdot \|_2 \), we need only to prove \( \|T\|_1 = \|T\|_2 \) for every positive matrix \( T \) in \( M_n(\mathbb{C}) \). We can assume that \( T = \begin{pmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix} \), where \( s_1, \cdots, s_n \) are \( s \)-numbers of \( T \) such that \( s_1 \geq s_2 \geq \cdots \geq s_n \). By Lemma 6.8 and 6.13,

\[
\|T\|_1 = \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} s_i t_i : (t_1, \cdots, t_n) \in \mathcal{D} \right\} = \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} s_i t_i : (t_1, \cdots, t_n) \in \mathcal{E} \right\}.
\]

Note that \( \|T\| = s_1 \geq s_2 \geq \cdots \geq s_n \geq 0 \). By Lemma 6.12 and simple computations, \( \|T\|_1 = \max \left\{ \frac{k}{n} \|T\|_2, \|T\|_1 \right\} = \|T\|_2 \). \( \Box \)
The next lemma simply follows from the definition of dual norms.

**Lemma 6.15.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and \(\|\cdot\|_1, \|\cdot\|_2\) be norms on \(\mathcal{M}\) such that

\[
\|T\|_1 \leq \|T\| \leq \|T\|_2, \quad \forall T \in \mathcal{M}.
\]

Then

\[
\|T\|_2^p \leq \|T\|^p \leq \|T\|_1^p, \quad \forall T \in \mathcal{M}.
\]

**Corollary 6.16.** Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and \(\|\cdot\|_1, \|\cdot\|_2\) be equivalent norms on \(\mathcal{M}\). Then \(\|\cdot\|_1^p\) and \(\|\cdot\|_2^p\) are equivalent norms on \(\mathcal{M}\).

**Theorem 6.17.** Let \(\mathcal{M}\) be a type II\(_1\) factor and \(0 \leq t \leq 1\). Then

\[
\|T\|_{(t)}^p = \max\{t\|T\|, \|T\|_1\}, \quad \forall T \in \mathcal{M}.
\]

**Proof.** Firstly, we assume \(t = \frac{k}{n}\) is a rational number. Let \(\mathcal{N}\) be a type \(I_n\) subfactor of \(\mathcal{M}\). Then the restriction of \(\|\cdot\|_{(t)}\) to \(\mathcal{N}\) is \(\|\cdot\|_{(\frac{k}{n})}\). By Lemma 6.14 and Corollary 6.11, \(\|T\|_{(t)}^p = \max\{t\|T\|, \|T\|_1\}\) for \(T \in \mathcal{N}\). By Corollary 4.9, \(\|T\|_{(t)}^p = \max\{t\|T\|, \|T\|_1\}\) for all \(T \in \mathcal{M}\). Now assume \(t\) is an irrational number. Let \(t_1, t_2\) be two rational numbers such that \(t_1 < t < t_2\).

By Lemma 6.15, for every \(T \in \mathcal{M}\),

\[
\max\{t_2\|T\|, \|T\|_1\} \leq \|T\|_{(t)}^p \leq \max\{t_1\|T\|, \|T\|_1\}.
\]

Since \(t_1 < t < t_2\) are arbitrary, \(\|T\|_{(t)}^p = \max\{t\|T\|, \|T\|_1\}\). \(\square\)

### 6.3 Second Dual Norms

**Lemma 6.18.** Let \(n \in \mathbb{N}\) and \(\tau\) be an arbitrary faithful state on \(\mathbb{C}^n\). If \(\|\cdot\|\) is a norm on \((\mathbb{C}^n, \tau)\) and \(\|\cdot\|^\#\) is the dual norm with respect to \(\tau\), then \(\|\cdot\|^\## = \|\cdot\|\).

**Proof.** By Lemma 6.3, \(\|T\|^\## = \sup\{\|\tau(TX)\| : X \in \mathbb{C}^n, \|X\|^\# \leq 1\} \leq \|T\|\). We need to prove \(\|T\| \leq \|T\|^\##\). By the Hahn-Banach Theorem, there is a continuous linear functional \(\phi\) on \(\mathbb{C}^n\) with respect to the topology induced by \(\|\cdot\|\) on \(\mathbb{C}^n\) such that \(\|T\| = \phi(T)\) and \(\|\phi\| = 1\). Since all norms on \(\mathbb{C}^n\) induce the same topology, there is an element \(Y \in \mathbb{C}^n\) such that \(\phi(S) = \tau(SY)\).
for all $S \in \mathbb{C}^n$. By the definition of dual norm, $\|Y\|^# = \|\phi\| = 1$. By Lemma 6.3, $\|T\| = \phi(T) = \tau(TY) \leq \|T\|^#$. □

**Theorem 6.19.** Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and $\|\cdot\|$ be a tracial gauge norm on $\mathcal{M}$. Then $\|\cdot\|^#$ is also a tracial gauge norm on $\mathcal{M}$ and $\|\cdot\|^# = \|\cdot\|$. Proof. By Theorem 6.10, both $\|\cdot\|^#$ and $\|\cdot\|$ are tracial gauge norms on $\mathcal{M}$. By Corollary 3.5, to prove $\|\cdot\|^# = \|\cdot\|$, we need to prove that $\|T\| = \|T\|^#$ for every positive simple operator $T \in \mathcal{M}$. Let $\mathcal{A}$ be the abelian von Neumann subalgebra generated by $T$. By Corollary 6.11 and Lemma 6.18, $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}}^# = \|T\|$. □
CHAPTER 7
MAIN RESULT

Let \( n \in \mathbb{N}, \ a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} = 0 \) and \( f(x) = a_1 x_{[0, \frac{1}{n})} + a_2 x_{[\frac{1}{n}, \frac{2}{n})} + \cdots + a_n x_{[\frac{n-1}{n}, 1]}(x) \). Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra. For \( T \in \mathcal{M} \), define \( \|T\|_f = \int_0^1 f(s) \mu_s(T) \, ds \).

Lemma 7.1. Let \( n \in \mathbb{N}, \ a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} = 0 \) and \( f(x) = a_1 x_{[0, \frac{1}{n})} + a_2 x_{[\frac{1}{n}, \frac{2}{n})} + \cdots + a_n x_{[\frac{n-1}{n}, 1]}(x) \). For \( T \in \mathcal{M} \), define

\[
\|T\|_f = \int_0^1 f(s) \mu_s(T) \, ds.
\]

Then

\[
\|T\|_f = \sum_{k=1}^n \frac{k(a_k - a_{k+1})}{n} \|T\|\left(\frac{k}{n}\right).
\]

Proof. Since \( t\|T\|_f = \int_0^t \mu_s(T) \, ds \), summation by parts shows that

\[
\|T\|_f = \int_0^1 f(s) \mu_s(T) \, ds = a_1 \int_0^{\frac{1}{n}} \mu_s(T) \, ds + \cdots + a_n \int_{\frac{n-1}{n}}^1 \mu_s(T) \, ds
\]

\[
= \sum_{k=1}^n \frac{k(a_k - a_{k+1})}{n} \|T\|\left(\frac{k}{n}\right). \quad \Box
\]

Corollary 7.2. The norm \( \| \cdot \|_f \) defined as above is a tracial gauge norm on \( \mathcal{M} \) and \( \|1\|_f = \int_0^1 f(x) \, dx = \frac{a_1 + \cdots + a_n}{n} \).

Lemma 7.3. Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra satisfying the weak Dixmier property and \( \{\| \cdot \|_\alpha\} \) be a set of tracial gauge norms on \((\mathcal{M}, \tau)\) such that \( \|1\|_\alpha \leq 1 \) for all \( \alpha \). For every \( T \in \mathcal{M} \), define

\[
\|T\| = \sup_{\alpha} \|T\|_\alpha.
\]

Then \( \| \cdot \| \triangleq \vee_{\alpha} \| \cdot \|_\alpha \) is also a tracial gauge norm on \((\mathcal{M}, \tau)\).

Proof. By Corollary 3.4, \( \|T\| \leq \|T\| \) is well defined. It is easy to check that \( \| \cdot \| \) is a tracial gauge norm on \((\mathcal{M}, \tau)\).  \( \Box \)
Let $F = \{ f(x) = a_1x(x) + a_2x^{1/2}(x) + \cdots + a_nx^{1/n}(x) : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0, \frac{a_1 + \cdots + a_n}{n} \leq 1, n = 1, 2, \cdots \}$. The following is the main result of this Part.

**Theorem 7.4.** Let $(M, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property. If $\| \cdot \|$ is a normalized tracial gauge norm on $M$, then there is a subset $F'$ of $F$ containing the constant 1 function on $[0, 1]$ such that for every $T \in M$,

$$\| T \| = \sup \{ \| T \|_f : f \in F' \},$$

where $\| T \|_f$ is defined as above.

**Proof.** Let

$$F' = \{ \mu_s(X) : X \in M, \| X \|^s \leq 1, X = b_1F_1 + \cdots + b_kF_k \geq 0, \text{where } F_1 + \cdots + F_k = 1 \text{ and } \tau(F_1) = \cdots = \tau(F_k) = \frac{1}{k}, k = 1, 2, \cdots \}.$$ 

For every positive operator $X \in M$ such that $\| X \|^s \leq 1$, $\int_0^1 \mu_s(X)ds = \tau(X) = \| X \|^s \leq \| X \| \leq 1$ by Theorem 3.30. Hence $F' \subset F$ and $\mu_s(1) = \chi_{[0,1]}(s) \in F'$. For $T \in M$, define

$$\| T \|' = \sup \{ \| T \|_f : f \in F' \}.$$ 

By Corollary 7.2, $\| \cdot \|$ is a tracial gauge norm on $M$. To prove that $\| \cdot \| = \| \cdot \|'$, by Corollary 4.8, we need to prove that $\| T \|' = \| T \|$ for every positive operator $T \in M$ such that $T = a_1E_1 + \cdots + a_nE_n$ and $\tau(E_1) = \cdots = \tau(E_n) = \frac{1}{n}$.

By Lemma 6.8 and Theorem C,

$$\| T \| = \sup \left\{ \frac{1}{n} \sum_{k=1}^n a_kb_k : X = b_1E_1 + \cdots + b_nE_n \geq 0 \text{ and } \| X \|^s \leq 1 \right\}.$$ 

Note that if $X = b_1E_1 + \cdots + b_nE_n$ is a positive simple operator in $M$ and $\| X \|^s \leq 1$, then $\mu_s(X) \in F'$ and $\| T \|_{\mu_s(X)} = \int_0^1 \mu_s(X)\mu_s(T)ds = \frac{1}{n} \sum_{k=1}^n a_k^*b_k^*$, where $\{a_k^*\}$ and $\{b_k^*\}$ are non-increasing rearrangements of $\{a_k\}$ and $\{b_k\}$, respectively. By Lemma 6.13, $\| T \| \leq \sup \{ \| T \|_f : f \in F' \} = \| T \|'$. 

We need to prove $\| T \| \geq \| T \|'$. Let $X = b_1F_1 + \cdots + b_kF_k$ be a positive operator in $M$ such that $F_1 + \cdots + F_k = 1$, $\tau(F_1) = \cdots = \tau(F_k) = \frac{1}{k}$ and $\| X \|^s \leq 1$. We need only prove that $\| T \| \geq \| T \|_{\mu_s(X)}$. Since $(M, \tau)$ satisfies the weak Dixmier property, by Theorem 3.27, $(M, \tau)$
is either a von Neumann subalgebra of \((M_r(\mathbb{C}), \tau_n)\) that contains all diagonal matrices or \(\mathcal{M}\) is a diffuse von Neumann algebra. In either case, we may assume that \(T = \tilde{a}_1 \tilde{E}_1 + \cdots + \tilde{a}_r \tilde{E}_r\) and \(X = \tilde{b}_1 \tilde{F}_1 + \cdots + \tilde{b}_r \tilde{F}_r\), where \(\tilde{E}_1 + \cdots + \tilde{E}_r = \tilde{F}_1 + \cdots + \tilde{F}_r = 1\) and \(\tau(\tilde{E}_i) = \tau(\tilde{F}_i) = \frac{1}{r}\) for \(1 \leq i \leq r\), \(\tilde{a}_1 \geq \cdots \geq \tilde{a}_r \geq 0\) and \(\tilde{b}_1 \geq \cdots \geq \tilde{b}_r \geq 0\). Let \(Y = \tilde{b}_1 \tilde{E}_1 + \cdots + \tilde{b}_r \tilde{E}_r\). Then \(X\) and \(Y\) are two equi-measurable operators in \(\mathcal{M}\) and \(\mu_\delta(X) = \mu_\delta(Y)\). By Theorem 6.10, \(\|Y\|_\delta^\delta \leq 1\). By Lemma 6.3,

\[
\|T\| \geq \tau(TY) = \frac{1}{r} \sum_{i=1}^{r} \tilde{a}_i \tilde{b}_i = \int_0^1 \mu_\delta(Y) \mu_\delta(T) ds = \int_0^1 \mu_\delta(X) \mu_\delta(T) ds = \|T\|_{\mu_\delta(X)}.
\]

Combining Theorem 7.4 and Lemma 3.21, we obtain the following corollary.

**Corollary 7.5.** Let \((\mathcal{M}, \tau)\) be a finite factor and \(\| \cdot \|\) be a normalized unitarily invariant norm on \(\mathcal{M}\). Then there is a subset \(\mathcal{F}'\) of \(\mathcal{F}\) containing the constant 1 function on \([0, 1]\) such that for all \(T \in \mathcal{M}\), \(\|T\| = \sup\{\|T\|_f : f \in \mathcal{F}'\}\).

Combining Theorem 7.4 and Lemma 3.14 we obtain the following corollary.

**Corollary 7.6.** Let \(\| \cdot \|\) be a normalized symmetric gauge norm on \((L^\infty[0, 1], \int_0^1 dx)\). Then there is a subset \(\mathcal{F}'\) of \(\mathcal{F}\) containing the constant 1 function on \([0, 1]\) such that for all \(T \in L^\infty[0, 1]\), \(\|T\| = \sup\{\|T\|_f : f \in \mathcal{F}'\}\).
PART II

UNITARILY INVARIANT NORMS
RELATED TO INFINITE FACTORS
CHAPTER 8
BACKGROUND

F.J. Murray and J. von Neumann [22, 23, 24, 45, 46] introduced and studied certain algebras of Hilbert space operators. Those algebras are now called “Von Neumann algebras.” They are strong-operator closed self-adjoint subalgebras of all bounded linear transformations on a Hilbert space. Factors are von Neumann algebras whose centers consist of scalar multiples of the identity. Every von Neumann algebra is a direct sum (or “direct integral”) of factors. Thus factors are the building blocks for all von Neumann algebras. Murray and von Neumann [22] classified factors into type I_n, II_1, II_∞, III factors. Type I_n and I_∞ factors are full matrix algebras: $M_n(\mathbb{C})$ and $\mathcal{B}(\mathcal{H})$. Type I_n and II_1 factors are called finite factors. Factors except type III factors are called semi-finite factors. An semi-finite factor admits a faithful normal tracial weight $\tau$.

The unitarily invariant norms on type I_n factors were introduced by von Neumann [44] for the purpose of metrizing matrix spaces. Von Neumann, together with his associates, established that the class of unitarily invariant norms of type I_n factors coincides with the class of symmetric gauge norms on $C^n$. These norms have now been variously generalized and utilized in several contexts. For example, Schatten [32, 33] defined unitarily invariant norms on two-sided ideals of completely continuous operators in type I_n factors; Ky Fan [9] studied Ky Fan norms and obtained his dominance theorem. The unitarily invariant norms play a crucial role in the study of function spaces and group representations (see e.g. [19]) and in obtaining certain bounds of importance in quantum field theory (see [36]). For historical perspectives and surveys of unitarily invariant norms, see Schatten [32, 33], Hewitt and Ross [16], Gohberg and Krein [13] and Simon [36].

In the first Part, we set up a representation theorem for unitarily invariant norms on finite factors. The main purpose of this Part is to set up a representation theorem for unitarily invariant
norms related to infinite factors. For interesting applications of the representation theorem, we refer to [6].
CHAPTER 9

s-NUMBERS OF OPERATORS IN SEMI-FINITE VON NEUMANN ALGEBRAS

9.1 s-numbers of operators in type II∞ factors

In [8], Fack and Kosaki give a complete exposition of generalized s-numbers of τ—measurable operators affiliated with semi-finite von Neumann algebras. For the reader's convenience and our purpose, we provide sufficient details on s-numbers of bounded operators in semi-finite von Neumann algebras in this section. We will define s-numbers of bounded operators in semi-finite von Neumann algebras from the point of view of non-increasing rearrangements of functions. The following lemma is well known.

Lemma 9.1. Let (𝒜, τ) be a separable (i.e., with separable predual) diffuse abelian von Neumann algebra with a faithful normal tracial weight τ on 𝒜 such that τ(1) = ∞. Then there is a *-isomorphism α from (𝒜, τ) onto $(L^{∞}[0,∞), \int_{0}^{∞} dx)$ such that $\tau = \int_{0}^{∞} dx \cdot \alpha$.

Let $M$ be a type II∞ factor and τ be a faithful normal tracial weight on $M$. For $T \in M$, there is a separable diffuse abelian von Neumann subalgebra 𝒜 of $M$ containing $|T|$. By Lemma 9.1, there is a *-isomorphism α from (𝒜, τ) onto $(L^{∞}[0,∞), \int_{0}^{∞} dx)$ such that $\tau = \int_{0}^{∞} dx \cdot \alpha$. Let $f(x) = \alpha(|T|)$ and $f^{*}(x)$ be the non-increasing rearrangement of f(x) (see (2.1.1)). Then the s-numbers of T, $\mu_{s}(T)$, are defined as

$$\mu_{s}(T) = f^{*}(s), \ 0 \leq s < \infty.$$ 

Similar to the proof of Lemma 2.7, we can prove the following lemma.

Lemma 9.2. $\mu_{s}(T)$ does not depend on 𝒜 and α.
Corollary 9.3. For $T \in \mathcal{M}$ and $p \geq 0$, $\tau(|T|^p) = \int_0^\infty \mu_s(T)^p \, ds$.

Lemma 9.4. Let $E, F$ be two projections in $\mathcal{M}$. If $\tau(E^\perp) < \tau(F^\perp) < \infty$, then $\tau(E \land F^\perp) > 0$.

Proof. By Proposition 2.5.14 of [8], $R(F^\perp E^\perp) = F^\perp - E \land F^\perp$, where $R(F^\perp E^\perp)$ is the range projection of $F^\perp E^\perp$. Therefore, $\tau(E \land F^\perp) = \tau(F^\perp) - \tau(R(F^\perp E^\perp)) \geq \tau(F^\perp) - \tau(E^\perp) > 0$.

Combining Lemma 9.4 and the proof of Lemma 2.9, we can prove the following lemma.

Recall that $\mathcal{P}(\mathcal{M})$ is the set of projections in $\mathcal{M}$.

Lemma 9.5. For $0 < s < \infty$,

$$\mu_s(T) = \inf \{ \|TE\| : E \in \mathcal{P}(\mathcal{M}), \tau(E^\perp) = s \}.$$

Corollary 9.6. Let $S, T \in \mathcal{M}$. Then $\mu_s(ST) \leq \|S\|\mu_s(T)$ for $s \in [0, \infty)$.

### 9.2 s-numbers of operators in semi-finite von Neumann algebras

Throughout Part II, a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ means a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial weight $\tau$. An embedding of a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ into another semi-finite von Neumann algebra $(\mathcal{M}_1, \tau_1)$ means a $*$-isomorphism $\alpha$ from $\mathcal{M}$ to $\mathcal{M}_1$ such that $\tau = \tau_1 \cdot \alpha$. Every semi-finite von Neumann algebra can be embedded into a type $\text{II}_\infty$ factor.

Definition 9.7. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and $T \in \mathcal{M}$. If $\alpha$ is an embedding of $(\mathcal{M}, \tau)$ into a type $\text{II}_\infty$ factor $(\mathcal{M}_1, \tau_1)$, then the s-number of $T$ is defined as

$$\mu_s(T) = \mu_s(\alpha(T)).$$

Similar to the proof of Lemma 2.7, we can see that $\mu_s(T)$ is well defined, i.e., does not depend on the choice of $\alpha$ and $\mathcal{M}_1$.

Let $T \in (\mathcal{B}(\mathcal{H}), \text{Tr})$ be a finite rank operator, where $\mathcal{H}$ is the separable infinite dimensional complex Hilbert space and $\text{Tr}$ is the classical tracial weight on $\mathcal{B}(\mathcal{H})$. Then $|T|$ is
unitarily equivalent to a diagonal operator with diagonal elements $s_1(T) \geq s_2(T) \geq \cdots \geq 0$. In the classical operator theory [13], $s_1(T), s_2(T), \cdots$ are also called $s$-numbers of $T$. It is easy to see that the relation between $\mu_e(T)$ and $s_1(T), s_2(T), \cdots$ is the following

$$\mu_e(T) = s_1(T)\chi_{[0,1)}(s) + s_2(T)\chi_{[1,2)}(s) + \cdots.$$  \hspace{1cm} (9.2.1)

Since no confusion will arise, we will use both $s$-numbers for a finite rank operator in $(B(ℋ), \text{Tr})$.

We end this section by the following definition.

**Definition 9.8.** Two positive operators $S, T$ in a semi-finite von Neumann algebra $(M, \tau)$ are **equi-measurable** if $\mu_e(S) = \mu_e(T)$ for $0 \leq s < \infty$.

By 12 of Proposition 2.1, positive operators $S$ and $T$ in a semi-finite von Neumann algebra $(M, \tau)$ are equi-measurable if and only if $\tau(S^n) = \tau(T^n)$ for all $n = 0, 1, 2, \cdots$. 

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CHAPTER 10

SEMI-NORMS $\mathcal{J}(\mathcal{M})$

In this section, $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra with a faithful normal tracial weight $\tau$. Let $\mathcal{J}(\mathcal{M})$ be the set of operators $T \in \mathcal{M}$ such that $T = ETE$ for some finite projection $E \in \mathcal{M}$. If $\mathcal{M}$ is a finite von Neumann algebra, then $\mathcal{J}(\mathcal{M}) = \mathcal{M}$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we simply write $\mathcal{J}(\mathcal{H})$ instead of $\mathcal{J}(\mathcal{B}(\mathcal{H}))$. Note that $\mathcal{J}(\mathcal{H})$ is the set of bounded linear operators $T$ on $\mathcal{H}$ such that both $T$ and $T^*$ are finite rank operators.

10.1 Gauge invariant semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 10.1. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra. A semi-norm $\| \cdot \|$ on $\mathcal{J}(\mathcal{M})$ is gauge invariant if $\|T\| = \|UT\|$ for all $T \in \mathcal{J}(\mathcal{M})$. A semi-norm $\| \cdot \|$ on $\mathcal{J}(\mathcal{M})$ is called left unitarily invariant if for all unitary operators $U$ in $\mathcal{M}$ and all $T$ in $\mathcal{J}(\mathcal{M})$, $\|UT\| = \|T\|$.

Similar to the proof of Lemma 3.2 and Corollary 3.18, we can prove the following results.

Lemma 10.2. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and $\| \cdot \|$ be a left unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$. If $T \in \mathcal{J}(\mathcal{M})$ and $A \in \mathcal{M}$, then $AT \in \mathcal{J}(\mathcal{M})$ and $\|AT\| \leq \|A\| \cdot \|T\|$.

Lemma 10.3. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and $\| \cdot \|$ be a semi-norm on $\mathcal{J}(\mathcal{M})$. Then $\| \cdot \|$ is gauge invariant if and only if $\| \cdot \|$ is left unitarily invariant.

10.2 Unitarily invariant semi-norms on $\mathcal{J}(\mathcal{M})$

Definition 10.4. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra. A semi-norm $\| \cdot \|$ on $\mathcal{J}(\mathcal{M})$ is unitarily invariant if $\|UTV\| = \|T\|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators
$U, V \in \mathcal{M}$. 

Similar to the proof of Proposition 3.17, we can prove the following proposition.

**Proposition 10.5.** Let $\| \cdot \|$ be a semi-norm on $\mathcal{J}(\mathcal{M})$. Then the following statements are equivalent:

1. $\| \cdot \|$ is unitarily invariant;

2. $\| \cdot \|$ is gauge invariant and unitarily conjugate invariant, i.e., $\|UTU^*\| = \|T\|$ for all $T \in \mathcal{J}(\mathcal{M})$ and unitary operators $U \in \mathcal{M}$;

3. $\| \cdot \|$ is gauge invariant and $\|T\| = \|T^*\|$ for all $T \in \mathcal{J}(\mathcal{M})$;

4. for all operators $A, B \in \mathcal{M}$ and $T \in \mathcal{J}(\mathcal{M})$, $\|ATB\| \leq \|A\| \cdot \|T\| \cdot \|B\|$.

**Corollary 10.6.** Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and $\| \cdot \|$ be a unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$. If $T \in \mathcal{J}(\mathcal{M})$ and $0 \leq S \leq T$, then $S \in \mathcal{J}(\mathcal{M})$ and $\|S\| \leq \|T\|$.

**Corollary 10.7.** Let $\| \cdot \|$ be a unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$ and $E, F$ be two equivalent projections in $\mathcal{J}(\mathcal{M})$. Then $\|E\| = \|F\|$.

### 10.3 Symmetric gauge semi-norms on $\mathcal{J}(\mathcal{M})$

**Definition 10.8.** Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra, and let $\text{Aut}(\mathcal{M}, \tau)$ be the set of $*$-automorphisms on $\mathcal{M}$ preserving $\tau$. A semi-norm $\| \cdot \|$ on $\mathcal{J}(\mathcal{M})$ is called symmetric if

$$\|\theta(T)\| = \|T\|, \forall T \in \mathcal{J}(\mathcal{M}), \theta \in \text{Aut}(\mathcal{M}, \tau);$$

a semi-norm $\| \cdot \|$ on $\mathcal{J}(\mathcal{M})$ is called a symmetric gauge semi-norm if it is both symmetric and gauge invariant on $\mathcal{J}(\mathcal{M})$.

**Corollary 10.9.** Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra, and let $\| \cdot \|$ be a symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$. Then $\| \cdot \|$ is a unitarily invariant semi-norm on $\mathcal{J}(\mathcal{M})$. 

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10.4 Symmetric gauge norms on \((\mathcal{M}_E, \tau_E)\)

In this paper we are interested symmetric gauge semi-norms on \(\mathcal{J}(\mathcal{M})\), where \((\mathcal{M}, \tau)\) is one of the following semi-finite von Neumann algebras:

- \(\mathcal{M} = \mathcal{B}(\mathcal{H})\) and \(\tau = \text{Tr}\) on \(\mathcal{B}(\mathcal{H})\), where \(\mathcal{H}\) is the separable infinite dimensional complex Hilbert space;
- \(\mathcal{M} = l^\infty(N)\) and \(\tau((x_1, x_2, \cdots)) = x_1 + x_2 + \cdots\);
- \(\mathcal{M}\) is a type II_{\infty} factor and \(\tau\) is a faithful normal tracial weight on \(\mathcal{M}\);
- \(\mathcal{M} = L^\infty[0, \infty)\) and \(\tau = \int_0^\infty dx\).

Note that in each case, \(\text{Aut}(\mathcal{M}, \tau)\) acts on \(\mathcal{M}\) ergodically. Recall that \(\text{Aut}(\mathcal{M}, \tau)\) acts on \(\mathcal{M}\) ergodically if \(\theta(T) = T\) for all \(\theta \in \text{Aut}(\mathcal{M}, \tau)\) implies \(T = \lambda 1\). Let \(E, F\) be finite projections in \(\mathcal{M}\) such that \(\tau(E) = \tau(F)\). If \(\text{Aut}(\mathcal{M}, \tau)\) acts on \(\mathcal{M}\) ergodically, there is a \(\theta \in \text{Aut}(\mathcal{M}, \tau)\) such that \(\theta(E) = F\). Furthermore, if \(\|\cdot\|\) is a symmetric gauge semi-norm on \(\mathcal{J}(\mathcal{M})\), then \(\|E\| = \|F\|\). In this case, a semi-norm \(\|\cdot\|\) on \(\mathcal{J}(\mathcal{M})\) is called a normalized symmetric gauge semi-norm if \(\|E\| = 1\) whenever \(\tau(E) = 1\).

Let \((\mathcal{M}, \tau)\) be one of the above semi-finite von Neumann algebras. For every finite projection \(E(\neq 0)\) in \(\mathcal{M}\), let \(\mathcal{M}_E = E \mathcal{M} E\) and \(\tau_E(ETE) = \frac{\tau(ETE)}{\tau(E)}\). Then \((\mathcal{M}_E, \tau_E)\) is a finite von Neumann algebra satisfying the weak Dixmier property (see Definition 3.22), i.e., for every positive operator \(T \in \mathcal{M}_E\), \(\tau_E(T)E\) is in the operator norm closure of the convex hull of \(\{S \in \mathcal{M}_E : S\text{ and }T\text{ are equi-measurable}\}\). So in the following sections we will always make the following assumptions on \((\mathcal{M}, \tau)\):

- \((\mathcal{M}, \tau)\) is a semi-finite von Neumann algebra such that \(\text{Aut}(\mathcal{M}, \tau)\) acts on \(\mathcal{M}\) ergodically;
- for every non-zero finite projection \(E\) in \(\mathcal{M}\), \((\mathcal{M}_E, \tau_E)\) is a finite von Neumann algebra satisfying the weak Dixmier property.

With above assumptions, it is easy to show that if \(E\) is a finite projection of \(\mathcal{M}\), then \(\text{Aut}(\mathcal{M}_E, \tau_E)\) acts on \(\mathcal{M}_E\) ergodically.
Lemma 10.10. Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra satisfying above assumptions and \(\| \cdot \|\) be a symmetric gauge semi-norm on \(\mathcal{J}(\mathcal{M})\). If \(E \in \mathcal{M}\) is a finite projection, then the restriction of \(\| \cdot \|\) to \((\mathcal{M}_E, \tau_E)\) is also a symmetric gauge semi-norm on \((\mathcal{M}_E, \tau_E)\).

Proof. It is obvious that the restriction of \(\| \cdot \|\) to \((\mathcal{M}_E, \tau_E)\) is also a gauge semi-norm on \((\mathcal{M}_E, \tau_E)\). Let \(\theta \in \text{Aut}(\mathcal{M}_E, \tau_E)\). Define \(\theta(S) = \| \theta(S) \|\) for \(S \in \mathcal{M}_E\). We need to prove \(\| \cdot \|\) is a symmetric gauge norm on \(\mathcal{M}_E, \tau_E\). Let \(T = a_1E_1 + \cdots + a_nE_n\) be a simple positive operator in \(\mathcal{M}_E\), where \(E_1 + \cdots + E_n = E\). Then \(\theta(T) = a_1\theta(E_1) + \cdots + a_n\theta(E_n)\). Since \(\theta \in \text{Aut}(\mathcal{M}_E, \tau_E)\), \(\tau(E_k) = \tau(\theta(E_k))\) for \(1 \leq k \leq n\). By the assumption of the lemma, \(\text{Aut}(\mathcal{M}, \tau)\) acts on \(\mathcal{M}\) ergodically. Therefore, there is a \(\theta' \in \text{Aut}(\mathcal{M}, \tau)\) such that \(\theta'(E_k) = \theta(E_k)\) for \(1 \leq k \leq n\). Hence, \(\theta'(T) = \theta(T)\). Since \(\| \cdot \|\) is a symmetric gauge semi-norm on \(\mathcal{J}(\mathcal{M})\), \(\| T \| = \| \theta'(T) \| = \| \theta(T) \| = \| T \|_2\). By Corollary 3.5, \(\| \cdot \|\) is also a symmetric gauge semi-norm on \((\mathcal{M}_E, \tau_E)\). This implies that the restriction of \(\| \cdot \|\) to \((\mathcal{M}_E, \tau_E)\) is also a symmetric gauge semi-norm on \((\mathcal{M}_E, \tau_E)\). \(\square\)

Corollary 10.11. Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra satisfying the above conditions. If \(\| \cdot \|\) is a normalized symmetric gauge semi-norm on \(\mathcal{J}(\mathcal{M})\), then \(\| \cdot \|\) is a symmetric gauge norm on \(\mathcal{J}(\mathcal{M})\).

Proof. Let \(T \in \mathcal{J}(\mathcal{M})\). Then there is a finite projection \(E \in \mathcal{M}\) such that \(T = ETE \in (\mathcal{M}_E, \tau_E)\). By Lemma 10.10, the restriction of \(\| \cdot \|\) to \((\mathcal{M}_E, \tau_E)\) is also a symmetric gauge semi-norm. If \(T \neq 0\), then by Corollary 3.32, \(\| T\| \geq \tau_E([T])\cdot \| E \| > 0\). So \(\| \cdot \|\) is a symmetric gauge norm on \(\mathcal{J}(\mathcal{M})\). \(\square\)

A simple operator in a semi-finite von Neumann algebra \((\mathcal{M}, \tau)\) is an operator \(T = a_1E_1 + \cdots + a_nE_n\), where \(E_1, \cdots, E_n\) are mutually orthogonal projections.

Lemma 10.12. Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra such that \(\text{Aut}(\mathcal{M}, \tau)\) acts on \(\mathcal{M}\) ergodically and \((\mathcal{M}_E, \tau_E)\) satisfies the weak Dixmier property for every finite projection \(E\) in \(\mathcal{M}\). Suppose \(\| \cdot \|_1\) and \(\| \cdot \|_2\) are two symmetric gauge norms on \(\mathcal{J}(\mathcal{M})\). Then \(\| \cdot \|_1 = \| \cdot \|_2\) on \(\mathcal{J}(\mathcal{M})\) if \(\| T\|_1 = \| T\|_2\) for every simple positive operator \(T\) in \(\mathcal{J}(\mathcal{M})\) such that \(T = a_1E_1 + \cdots + a_nE_n\) and \(\tau(E_1) = \cdots = \tau(E_n)\).

Proof. Suppose \(\| T\|_1 = \| T\|_2\) for every simple operator \(T\) in \(\mathcal{J}(\mathcal{M})\). Let \(S \in \mathcal{J}(\mathcal{M})\). Then there is a finite projection \(E \in \mathcal{M}\) such that \(S = ESE \in \mathcal{M}_E\). By Lemma 10.10, the restrictions
of \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) to \( (\mathcal{M}_E, \tau_E) \) are two symmetric gauge norms. Since \( \| T \|_1 = \| T \|_2 \) for every simple operator \( T \) in \( \mathcal{M}_E \) such that \( T = a_1E_1 + \cdots + a_nE_n \) and \( \tau(E_1) = \cdots = \tau(E_n) \), \( \| \cdot \|_1 = \| \cdot \|_2 \) by Corollary 4.8.

**Proposition 10.13.** Let \( (\mathcal{M}, \tau) \) be a semi-finite factor and \( \| \cdot \| \) be a norm on \( \mathcal{J}(\mathcal{M}) \). Then the following conditions are equivalent:

1. \( \| \cdot \| \) is a symmetric gauge norm;
2. \( \| \cdot \| \) is a unitarily invariant norm.

**Proof.** "1 \( \Rightarrow \) 2" is obvious. We only prove "2 \( \Rightarrow \) 1". We need to prove that for every positive operator \( T \in \mathcal{J}(\mathcal{M}) \) and \( \theta \in \text{Aut}(\mathcal{M}, \tau) \), \( \| \theta(T) \| = \| T \| \). Let \( S = \theta(T) \). Then \( S \in \mathcal{J}(\mathcal{M}) \).

Therefore, there is a finite projection \( E \) in \( \mathcal{M} \) such that \( S, T \in \mathcal{M}_E \). By the spectral decomposition theorem, there is a sequence of simple positive operators \( T_n \in \mathcal{M}_E \) such that \( S_n = \theta(T_n) \in \mathcal{M}_E \) and \( \lim_{n \to \infty} \| T_n - T \| = \lim_{n \to \infty} \| S_n - S \| = 0 \). By Theorem 3.30, \( \| T - T_n \| \leq \| T - T_n \| \cdot \| E \| \) and \( \| S - S_n \| \leq \| S - S_n \| \cdot \| E \| \). Hence, \( \lim_{n \to \infty} \| T - T_n \| = \lim_{n \to \infty} \| S - S_n \| = 0 \). We need only prove \( \| T_n \| = \| S_n \| \) for all \( n = 1, 2, \ldots \). Suppose \( T_n = a_1E_1 + \cdots + a_mE_m \).

Then \( S_n = \theta(T_n) = a_1F_1 + \cdots + a_mE_m \), where \( \theta(E_k) = F_k \) for \( 1 \leq k \leq m \). Since \( \theta \in \text{Aut}(\mathcal{M}, \tau) \), \( \tau(E_k) = \tau(F_k) \) for \( 1 \leq k \leq m \). Since \( \mathcal{M} \) is a factor, there is a unitary operator \( U \in \mathcal{M} \) such that \( E_k = UF_kU^* \) for \( 1 \leq k \leq m \). Therefore, \( S_n = UT_nU^* \) and \( \| T_n \| = \| S_n \| \). \( \Box \)

### 10.5 Semi-norms associated to von Neumann algebras

**Definition 10.14.** Let \( \mathcal{M} \) be a von Neumann algebra (not necessarily semi-finite). A (generalized) semi-norm associated to \( \mathcal{M} \) is a map \( \| \cdot \| \) from \( \mathcal{M} \) to \([0, \infty)\) satisfying the following properties:

1. \( \| \lambda T \| = |\lambda| \cdot \| T \| \),
2. \( \| S + T \| \leq \| S \| + \| T \| \)

for all \( S, T \in \mathcal{M} \) and \( \lambda \in \mathbb{C} \). To make the definition nontrivial, we always make the following assumption: \( 0 < \| T \| < \infty \) for some non-zero element \( T \in \mathcal{M} \).

Let \( \mathcal{I} = \{ T \in \mathcal{M} : \| T \| < \infty \} \). Then \( \mathcal{I} \) is called the domain of the semi-norm \( \| \cdot \| \).
Definition 10.15. Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra. A semi-norm \(\| \cdot \|\) associated to \(\mathcal{M}\) is called gauge invariant if for all \(T \in \mathcal{M}\), \(\|T\| = \|T\|\); a semi-norm \(\| \cdot \|\) associated to \(\mathcal{M}\) is unitarily invariant if \(\|UTU^*\| = \|T\|\) for all \(T \in \mathcal{M}\) and unitary operators \(U, V \in \mathcal{M}\); a semi-norm \(\| \cdot \|\) associated to a semi-finite von Neumann algebra \((\mathcal{M}, \tau)\) is called symmetric if
\[\|\theta(T)\| = \|T\|, \; \forall T \in \mathcal{M}, \; \theta \in \text{Aut}(\mathcal{M}, \tau);\]
a semi-norm \(\| \cdot \|\) associated to \((\mathcal{M}, \tau)\) is called a symmetric gauge semi-norm if it is both symmetric and gauge invariant.

Similar to the proof of Proposition 3.17, we can prove the following proposition.

Proposition 10.16. Let \(\| \cdot \|\) be a semi-norm associated to \(\mathcal{M}\). Then the following statements are equivalent:

1. \(\| \cdot \|\) is unitarily invariant;

2. \(\| \cdot \|\) is gauge invariant and unitarily conjugate invariant, i.e., \(\|UTU^*\| = \|T\|\) for all \(T \in \mathcal{M}\) and unitary operators \(U \in \mathcal{M}\);

3. \(\| \cdot \|\) is gauge invariant and \(\|T\| = \|T^*\|\) for all \(T \in \mathcal{M}\);

4. for all operators \(T, A, B \in \mathcal{M}\), \(\|ATB\| \leq \|A\| \cdot \|T\| \cdot \|B\|\).

Corollary 10.17. Let \(\| \cdot \|\) be a unitarily invariant semi-norm associated to \(\mathcal{M}\). If \(S, T \in \mathcal{M}\) and \(0 \leq S \leq T\), then \(\|S\| \leq \|T\|\).

Corollary 10.18. Let \(\| \cdot \|\) be a unitarily invariant semi-norm associated to \(\mathcal{M}\) and \(E, F\) be two equivalent projections in \(\mathcal{M}\). Then \(\|E\| = \|F\|\).

Lemma 10.19. Let \(\| \cdot \|\) be a unitarily invariant semi-norm associated to \(\mathcal{M}\) and \(T \in \mathcal{M}\) be a nonzero element such that \(\|T\| < \infty\). Then there is a nonzero projection \(E\) in \(\mathcal{M}\) such that \(\|E\| < \infty\).

Proof. Since \(\| \cdot \|\) is unitarily invariant, we may assume \(T > 0\). By the spectral decomposition theorem, there exist a \(\lambda > 0\) and a nonzero projection \(E\) in \(\mathcal{M}\) such that \(T \geq \lambda E\). By Corollary 10.6, \(\|E\| < \infty\). □
The following theorem shows that, up to a scale $a > 0$, the operator norm $\| \cdot \|$ is the unique unitarily invariant semi-norm associated to a type III factor.

**Theorem 10.20.** Let $\mathcal{M}$ be a type III factor and $\| \cdot \|$ be a unitarily invariant semi-norm associated to $\mathcal{M}$. Then there is an $a > 0$ such that $\| \cdot \| = a\| \cdot \|$, i.e., $\|T\| = a\|T\|$ for all $T \in \mathcal{M}$.

**Proof.** By Lemma 10.19, there is a nonzero projection $E$ in $\mathcal{M}$ such that $\|E\| < \infty$. If $\|E\| = 0$, then $\|1\| = 0$ by Corollary 10.18. By Proposition 10.16, for every $T$ in $\mathcal{M}$, $\|T\| \leq \|T\| \cdot \|1\| = 0$. In our definition of semi-norm, we assume that $\|T\| > 0$ for some projection $E$ in $\mathcal{M}$. We may assume that $\|E\| = 1$. By Corollary 10.18, $\|F\| = 1$ for every non-zero projection in $\mathcal{M}$. In particular, $\|1\| = 1$. By Proposition 10.16, for every $T$ in $\mathcal{M}$, $\|T\| \leq \|T\| \cdot \|1\| = \|T\|$. On the other hand, let $T \in \mathcal{M}$ be a positive operator and $\varepsilon > 0$. By the spectral decomposition theorem, there is a nonzero projection $F$ in $\mathcal{M}$ such that $T \geq (\|T\| - \varepsilon)F$. By Corollary 10.17, $\|T\| \geq (\|T\| - \varepsilon) \cdot \|F\| = \|T\| - \varepsilon$. Hence $\|T\| \leq \|T\| - \varepsilon$. This proves that $\|T\| = \|T\|$ for every positive operator $T$ in $\mathcal{M}$ and therefore for every operator $T$ in $\mathcal{M}$. □

We end this section with the following lemma.

**Lemma 10.21.** Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra such that $\text{Aut}(\mathcal{M}, \tau)$ acts on $\mathcal{M}$ ergodically. If $\| \cdot \|$ is a normalized symmetric gauge semi-norm associated to $\mathcal{M}$ with domain $\mathcal{I}$, then $\mathcal{I} \supseteq \mathcal{J}(\mathcal{M})$ and $\| \cdot \|$ is a normalized symmetric gauge semi-norm on $\mathcal{J}(\mathcal{M})$.

**Proof.** Let $E$ be a finite projection in $\mathcal{M}$ such that $\tau(E) = 1$. Then $\|E\| = 1$. Suppose that $F$ is a finite projection in $\mathcal{M}$ such that $n \leq \tau(F) < n + 1$. Since $\text{Aut}(\mathcal{M}, \tau)$ acts on $\mathcal{M}$ ergodically, by induction, there are mutually orthogonal finite projections $E_1, E_2, \ldots, E_{n+1}$ in $\mathcal{M}$, $\tau(E_1) = \cdots = \tau(E_{n+1}) = 1$, such that $E_1 + \cdots + E_n \leq F \leq E_1 + \cdots + E_{n+1}$. By Corollary 10.17, $\|F\| \leq \|E_1 + \cdots + E_{n+1}\| \leq n + 1$. So every finite projection is in $\mathcal{I}$. Hence $\mathcal{I} \supseteq \mathcal{J}(\mathcal{M})$. □
CHAPTER 11
KY FAN NORMS ON $\mathcal{J}(\mathcal{M})$

Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann subalgebra of a type $\text{II}_\infty$ factor $(\mathcal{M}_1, \tau_1)$ and $0 \leq t \leq \infty$. For $T \in \mathcal{M}$, define $\|T\|_{(t)}$, the Ky Fan $t$-th norm of $T$, by

$$
\|T\|_{(t)} = \begin{cases} 
\|T\|, & t = 0; \\
\frac{1}{t} \int_0^t \mu_s(T) \, ds, & 0 < t \leq 1; \\
\int_0^t \mu_s(T) \, ds, & 1 < t \leq \infty.
\end{cases}
$$

Recall that $\mathcal{U}(\mathcal{M})$ is the set of unitary operators in $\mathcal{M}$ and $\mathcal{P}(\mathcal{M})$ is the set of projections in $\mathcal{M}$. Similar to the proof of Lemma 5.1, we can prove the following lemmas.

**Lemma 11.1.** For $0 < t \leq 1$, $\|T\|_{(t)} = \sup \{ |\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \}$.

**Lemma 11.2.** For $1 \leq t \leq \infty$, $\|T\|_{(t)} = \sup \{ |\tau_1(UTE)| : U \in \mathcal{U}(\mathcal{M}_1), E \in \mathcal{P}(\mathcal{M}_1), \tau_1(E) = t \}$.

Similar to the proof Theorem 5.2, we can prove the following theorem.

**Theorem 11.3.** For $0 \leq t \leq \infty$, $\| \cdot \|_{(t)}$ is a normalized symmetric gauge norm associated to $(\mathcal{M}, \tau)$.

**Corollary 11.4.** Let $T \in \mathcal{M}$ and $\delta > 0$. If $\|T\|_{(1)} < \delta$, then $\tau(\chi_{(\delta, \infty)}(\|T\|)) \leq \frac{\|T\|_{(1)}}{\delta}$.

**Proof.** We may assume that $\mathcal{M}$ is a type $\text{II}_\infty$ factor and $T \geq 0$. By the proof of Lemma 11.1,

$$
\|T\|_{(1)} = \sup \{ |\tau(UTE)| : U \in \mathcal{U}(\mathcal{M}), E \in \mathcal{P}(\mathcal{M}), \tau(E) \leq 1 \}.
$$

If $\tau(\chi_{(\delta, \infty)}(T)) > 1$, then there is a sub-projection $E$ of $\chi_{(\delta, \infty)}(T)$ such that $\tau(E) = 1$. Then $TE \geq \delta E$. Hence, $\|T\|_{(1)} \geq \tau(TE) \geq \tau(\delta E) = \delta$. This contradicts the assumption that $\|T\|_{(1)} < \delta$. Therefore, $\tau(\chi_{(\delta, \infty)}(T)) \leq 1$. So $\|T\|_{(1)} \geq \tau(T\chi_{(\delta, \infty)}(T)) \geq \tau(\delta \chi_{(\delta, \infty)}(T)) \geq \delta \tau(\chi_{(\delta, \infty)}(T))$. This implies the corollary. \qed

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Proposition 11.5. Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra and \(T \in \mathcal{M}\). Then \(\|T\|_{(t)}\) is a non-increasing continuous function on \([0, 1]\) and a non-decreasing continuous function on \([1, \infty]\).

Proof. Let \(0 < t_1 < t_2 \leq 1\). Then

\[
\|T\|_{(t_1)} - \|T\|_{(t_2)} = \frac{1}{t_1} \int_0^{t_1} \mu_s(T) ds - \frac{1}{t_2} \int_0^{t_2} \mu_s(T) ds
\]

\[
= \frac{1}{t_1} \int_0^{t_1} \mu_s(T) ds - \frac{1}{t_2} \int_0^{t_2} \mu_s(T) ds + \frac{1}{t_1} \int_{t_1}^{t_2} \mu_s(T) ds - \frac{1}{t_1} \int_{t_1}^{t_2} \mu_s(T) ds \
\leq 0.
\]

Since \(\mu_s(T)\) is right-continuous, \(\|T\|_{(t)}\) is a non-increasing continuous function on \([0, 1]\). Since \(\mu_s(T) \geq 0\) for \(s \in [0, \infty)\), \(\|T\|_{(t)}\) is a non-decreasing continuous function on \([1, \infty]\). \(\square\)

Proposition 11.6. Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra satisfying the following conditions:

1. \(\text{Aut}(\mathcal{M}, \tau)\) acts on \(\mathcal{M}\) ergodically;
2. for every finite projection \(E\) in \(\mathcal{M}\), \((\mathcal{M}E, \tau_E)\) satisfies the weak Dixmier property,

and let \(\|\cdot\|\) be a normalized symmetric gauge norm on \(\mathcal{J}(\mathcal{M})\). Then for every \(T \in \mathcal{J}(\mathcal{M})\),

\[
\|T\|_{(1)} \leq \|T\|.
\]

Proof. We can assume that \(T\) is a positive operator in \(\mathcal{J}(\mathcal{M})\). Then there is a finite projection \(F\) in \(\mathcal{M}\) such that \(T = FTF \in \mathcal{M}_F\). We can assume that \(\tau(F) = k\) is a positive integer. By the assumption of the proposition, \((\mathcal{M}_F, \tau_F)\) satisfies the weak Dixmier property. By Theorem 3.27, either \((\mathcal{M}_F, \tau_F)\) is a diffuse von Neumann algebra or \((\mathcal{M}_F, \tau_F)\) is \(*\)-isomorphic to a von Neumann subalgebra of \((M_n(\mathbb{C}), \tau_n)\) that contains all diagonal matrices. In either case, there is a projection \(E\) in \(\mathcal{M}\), \(E \leq F\), such that \(\tau(E) = 1\) and \(\|T\|_{(1)} = \|ETE\|_{(1)}\). By Corollary 10.9 and Proposition 10.5, \(|\|ETE\| - \|T\|| \leq \|T\|\). By Theorem 3.30, \(\|ETE\|_{(1)} \leq \|ETE\| \leq \|T\|\). \(\square\)

Example 11.7. The Ky Fan \(n\)-th norm of a compact operator \(T \in (B(\mathcal{H}), \text{Tr})\) is

\[
\|T\|_{(n)} = s_1(T) + \cdots + s_n(T)
\]

and

\[
\|T\|_{(\infty)} = s_1(T) + s_2(T) + \cdots.
\]
Corollary 11.8. Let \( \| \cdot \| \) be a normalized unitarily invariant norm on \( \mathcal{B}(\mathcal{H}) \). Then for every \( T \in \mathcal{J}(\mathcal{H}) \),

\[
\sigma_1(T) \leq \| T \| \leq \sigma_1(T) + \sigma_2(T) + \cdots.
\]

Proof. By Proposition 11.6, \( \sigma_1(T) = \| T \|_{(1)} \leq \| T \| \). On the other hand, we may assume that \( T \) is a positive operator in \( \mathcal{J}(\mathcal{H}) \). Then \( T \) is unitarily equivalent to a diagonal operator \( \sigma_1(T)E_1 + \cdots + \sigma_n(T)E_n \). Hence, \( \| T \| = \| \sigma_1(T)E_1 + \cdots + \sigma_n(T)E_n \| \leq \sigma_1(T) + \cdots + \sigma_n(T) \). \( \square \)
CHAPTER 12

DUAL NORMS OF SYMMETRIC GAUGE NORMS ON $\mathcal{J} (\mathcal{M})$

Throughout this section, we assume that $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra satisfying the following assumptions.

- $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra such that $\text{Aut}(\mathcal{M}, \tau)$ acts on $\mathcal{M}$ ergodically;
- for every finite projection $E$ in $\mathcal{M}$, $(\mathcal{M}_E, \tau_E)$ satisfies the weak Dixmier property, i.e., for every positive operator $T \in \mathcal{M}_E$, $\tau_E(T)E$ is in the operator norm closure of the convex hull of $\{S \in \mathcal{M}_E : S$ and $T$ are equimeasurable\}.

Recall that $\mathcal{J} (\mathcal{M})$ is the subset of $\mathcal{M}$ consisting of operators $T$ in $\mathcal{M}$ such that $T = ETE$ for some finite projection $E \in \mathcal{M}$.

12.1 Dual norms

Let $\| \cdot \|$ be a norm on $\mathcal{J} (\mathcal{M})$. For $T \in \mathcal{J} (\mathcal{M})$, define

$\|T\|^{\#, \tau}_{\mathcal{M}} = \sup \{ |\tau(TX)| : X \in \mathcal{J} (\mathcal{M}), \|X\| \leq 1 \}.$

When no confusion arises, we simply write $\| \cdot \|^\#$ or $\| \cdot \|^\#_{\mathcal{M}}$ instead of $\| \cdot \|^{\#, \tau}_{\mathcal{M}}$.

Lemma 12.1. $\| \cdot \|^\#$ is a norm on $\mathcal{J} (\mathcal{M})$.

Proof. Note that if $T \in \mathcal{J} (\mathcal{M})$ is not 0, then $\|T\|^\# \geq \frac{\tau(TT^*)}{\|T\|^2} > 0$. It is easy to check that $\| \cdot \|^\#$ satisfies the other conditions for a norm. \qed

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Definition 12.2. \( \| \cdot \|^{\#} \) is called the dual norm of \( \| \cdot \| \) on \( \mathcal{J}(\mathcal{M}) \) with respect to \( \tau \).

The following lemma follows simply from the definition of dual norm.

**Lemma 12.3.** Let \( \| \cdot \| \) be a norm on \( \mathcal{J}(\mathcal{M}) \) and \( \| \cdot \|^{\#} \) be the dual norm on \( \mathcal{J}(\mathcal{M}) \). Then for \( S, T \in \mathcal{J}(\mathcal{M}) \), \( |\tau(ST)| \leq \|S\| \cdot \|T\|^{\#} \).

For \( T \in \mathcal{M} \), define \( \|T\|_1 = \tau(|T|) \). Then \( \|T\|_1 = \|T\|_{(\infty)} \). The following corollary is the Hölder’s inequality for operators in \( \mathcal{J}(\mathcal{M}) \).

**Corollary 12.4.** Let \( \| \cdot \| \) be a gauge invariant norm on \( \mathcal{J}(\mathcal{M}) \) and \( \| \cdot \|^{\#} \) be the dual norm. Then for \( S, T \in \mathcal{J}(\mathcal{M}) \), \( \|ST\|_1 \leq \|S\| \cdot \|T\|^{\#} \).

**Proof.** By Lemma 11.2, \( \|ST\|_1 = \|ST\|_{(\infty)} = \sup\{|\tau(UST)| : U \in \mathcal{U}(\mathcal{M})\} \). By Lemma 12.3 and Lemma 10.3, \( |\tau(UST)| \leq \|US\| \cdot \|T\|^{\#} = \|S\| \cdot \|T\|^{\#} \).

Let \( E \) be a (non-zero) finite projection in \( \mathcal{M} \). Recall that \( \mathcal{M}_E = E \mathcal{M} E \) is a finite von Neumann algebra with a faithful normal tracial state \( \tau_E \) such that \( \tau_E(T) = \frac{\tau(T)}{\tau(E)} \) for \( T \in \mathcal{M}_E \). If \( \| \cdot \| \) is a norm on \( \mathcal{M}_E \), the dual norm of \( T \in \mathcal{M}_E \) with respect to \( \tau_E \) is defined by

\[
\|T\|^{\#}_{\mathcal{M}_E, \tau_E} = \sup\{|\tau_E(TX)| : X \in \mathcal{M}_E, \|X\| \leq 1\}.
\]

**Lemma 12.5.** Suppose \( \| \cdot \| \) is a unitarily invariant norm on \( \mathcal{J}(\mathcal{M}) \). Let \( E \) be a non-zero finite projection in \( \mathcal{M} \) and \( T \in \mathcal{M}_E \). Then

\[
\|T\|^{\#}_{\mathcal{M}_E, \tau_E} = \tau(E) \cdot \|T\|^{\#}_{\mathcal{M}_E, \tau_E}.
\]

**Proof.** Since \( T = ETE \), for every \( X \in \mathcal{J}(\mathcal{M}) \), \( \tau(TX) = \tau(ETEX) = \tau(ETEEXE) = \tau(E) \cdot \tau_E(ETEEXE) \). If \( \|X\| \leq 1 \), then \( \|EXE\| \leq \|X\| \) by Proposition 10.5. This implies that

\[
\|T\|^{\#}_{\mathcal{M}_E, \tau} = \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}), \|X\| \leq 1\} = \tau(E) \cdot \|T\|^{\#}_{\mathcal{M}_E, \tau_E}.
\]
Note that for two operators $S, T$ in $\mathcal{J}(\mathcal{M})$, there is a finite projection $E$ in $\mathcal{M}$ such that $S, T \in \mathcal{M}_E$. By Lemma 12.5, Proposition 6.5 and Proposition 6.6, we have the following proposition.

**Proposition 12.6.** Let $\| \cdot \|$ be a norm on $\mathcal{J}(\mathcal{M})$. We have the following:

1. if $\| \cdot \|$ is a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$, then $\| \cdot \|^\#$ is also a unitarily invariant norm on $\mathcal{J}(\mathcal{M})$;

2. if $\| \cdot \|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then $\| \cdot \|^\#$ is also a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$.

Furthermore, if $\| \cdot \|$ is a normalized symmetric norm, i.e., $\|E\| = 1$ if $\tau(E) = 1$, then $\| \cdot \|^\#$ is also a normalized symmetric norm.

By Lemma 12.5, and Lemma 6.8, we have the following Lemma.

**Lemma 12.7.** Let $\| \cdot \|$ be a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. If $T = a_1E_1 + \cdots + a_nE_n$ is a positive simple operator in $\mathcal{J}(\mathcal{M})$, then

$$\|T\|^\# = \sup \left\{ \sum_{k=1}^n a_kb_k\tau(E_k) : S = b_1E_1 + \cdots + b_nE_n \geq 0, \|S\| \leq 1 \right\}.$$ 

### 12.2 Dual norms of Ky Fan norms

**Theorem 12.8.** For $T \in \mathcal{J}(\mathcal{H})$ and $k = 1, 2, \cdots, \infty$,

$$\|T\|^\#_{(k)} = \max \left\{ \|T\|, \frac{1}{k} \|T\|_1 \right\},$$

where $\|T\|_{(k)} = s_1(T) + \cdots + s_k(T)$, $\|T\|_1 = \text{Tr}(|T|) = s_1(T) + s_2(T) + \cdots$ and $\frac{1}{\infty} = 0$.

**Proof.** For $T \in \mathcal{J}(\mathcal{H})$, there is a finite rank projection $E$ such that $T = ETE \in \mathcal{B}(\mathcal{H})_E$. Let $\text{Tr}(E) = n$. Then $\mathcal{B}(\mathcal{H})_E \cong M_n(\mathbb{C})$. First assume $k < \infty$. We may assume that $n \geq k$. Then $\|T\|_{(k)} = k\|T\|_{(\frac{k}{n}), \tau_n}$. By Lemma 12.5 and Lemma 6.14,

$$\|T\|^\#_{(k)} = \text{Tr}(E) \cdot (k\|T\|_{(\frac{k}{n}), \tau_n})_{M_n(\mathbb{C})} = \frac{n}{k} \max \left\{ \frac{k}{n} \|T\|_1, \|T\|_1, \tau_n \right\}.$$
If \( k = \infty \), then \( \|T\|_{(\infty)} = \|T\|_{(n)} \) by Lemma 12.7. Since \( \frac{1}{n} \|T\|_1 \leq \|T\|_1 \), \( \|T\|_{(n)} = \max \{ \frac{1}{n} \|T\|_1 \} = \|T\|_1 \).

**Theorem 12.9.** Let \( \mathcal{M} \) be a type II\(_{\infty} \) factor and \( 0 \leq t \leq \infty \). Then for all \( T \in \mathcal{J}(\mathcal{M}) \),

\[
\|T\|_{(t)}^\# = \max \left\{ \frac{t}{n} \|T\|_1, \frac{1}{t} \|T\|_1 \right\},
\]

if \( 0 < t < \infty \), then \( \|T\|_{(t)}^\# = \frac{1}{t} \|T\|_1 \).

**Proof.** Let \( T \in \mathcal{J}(\mathcal{M}) \) and \( 0 < t < \infty \). There is a finite projection \( E \) in \( \mathcal{M} \) such that \( T = ETE \) is in \( \mathcal{M}_E \). We can assume that \( \tau(E) = n > 1 \). Let \( \tau_E (ESE) = \frac{\tau(ESE)}{\tau(E)} \). Then \( (\mathcal{M}_E, \tau_E) \) is a type II\(_1 \) factor and \( \tau_E \) is the unique tracial state on \( \mathcal{M}_E \). If \( 0 < t \leq 1 \), by Lemma 11.1,

\[
t\|T\|_{(t)} = \tau(E) \cdot \sup \{ |\tau_E(UET)| : U \in \mathcal{U}(\mathcal{M}_E), E' \in \mathcal{D}(\mathcal{M}_E), \tau(E') = t \}
\]

\[
= \frac{n}{t} \|T\|_{(t)}, \tau_E \cdot \tau(E) = t\|T\|_{(t)}, \tau_E \cdot \tau(E).
\]

Hence, \( \|T\|_{(t)} = \|T\|_{(t)}, \tau_E \). By Lemma 12.5 and Theorem 6.17,

\[
\|T\|_{(t)}^\# = \tau(E) \cdot \left( \frac{t}{n} \|T\|_{(t)}, \tau_E \right)_{\mathcal{M}_E, \tau_E}^{\#} = n \max \left\{ \frac{t}{n} \|T\|_1, \|T\|_1, \tau_E \right\}
\]

\[
= \max \{ t\|T\|_1, \|T\|_1 \}.
\]

If \( 1 < t < \infty \), then \( \|T\|_{(t)}^\# = t\|T\|_{(t)}, \tau_E \). By Lemma 12.5 and Theorem 6.17,

\[
\|T\|_{(t)}^\# = \frac{n}{t} \max \left\{ \frac{t}{n} \|T\|_1, \|T\|_1, \tau_E \right\}
\]

\[
= \max \left\{ \|T\|_1, \|T\|_1 \right\}.
\]

Similar to the proof of Theorem 6.17, \( \|T\|_{(\infty)}^\# = \|T\|_1 \).

**12.3 Second dual norms**

**Theorem 12.10.** Let \( (\mathcal{M}, \tau) \) be a semi-finite von Neumann algebra satisfying the following conditions:
1. $\text{Aut}(\mathcal{M}, \tau)$ acts on $\mathcal{M}$ ergodically;

2. for every finite projection $E$ in $\mathcal{M}$, $(\mathcal{M}_E, \tau_E)$ is a finite von Neumann algebra satisfying the weak Dixmier property.

If $\| \cdot \|$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$, then $\| \cdot \|^\#$ is also a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$ and $\| \cdot \|^\# = \| \cdot \|$ on $\mathcal{J}(\mathcal{M})$.

Proof. By Proposition 12.6, $\| \cdot \|^\#$ is a symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Furthermore, both $\| \cdot \|^\#$ and $\| \cdot \|$ are symmetric gauge norms on $\mathcal{J}(\mathcal{M})$. We need to prove that $\| T \| = \| T \|^\#$ for every positive operators $T \in \mathcal{J}(\mathcal{M})$. Let $E$ be a finite projection in $\mathcal{M}$ such that $T \in \mathcal{M}_E$. By Lemma 12.5 and Theorem 6.19,

$$\| T \|^\#_{\mathcal{M}, \tau} = \sup\{|\tau(TX)| : X \in \mathcal{J}(\mathcal{M}), \|X\|^\#_{\mathcal{M}, \tau} \leq 1\}$$

$$= \sup\{|\tau(E) : \tau(E)(TX)| : X \in \mathcal{M}_E, \|X\|^\#_{\mathcal{M}_E, \tau} \leq 1\}$$

$$= \sup\{|\tau_E(T(\tau(E)X))| : X \in \mathcal{M}_E, \|\tau(E)X\|^\#_{\mathcal{M}_E, \tau} \leq 1\}$$

$$= \| T \|^\#_{\mathcal{M}_E, \tau_E} = \| T \|_{\mathcal{M}_E, \tau_E} = \| T \|.$$
CHAPTER 13
A REPRESENTATION THEOREM

Throughout this section, we assume that \((\mathcal{M}, \tau)\) is a semi-finite von Neumann algebra.

**Lemma 13.1.** Let \(f(x) = \sum_{k=1}^{n} a_k \chi_{[a_{k-1}, a_k]}(x)\), where \(a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 (= a_{n+1})\) and \(0 = a_0 < a_1 < \cdots < a_n < \infty\). For \(T \in \mathcal{M}\), define
\[
\|T\|_f = \int_0^{\infty} f(s) \mu_s(T) ds.
\]
Then
\[
\|T\|_f = \sum_{k=1}^{n} \min\{a_k, 1\} (a_k - a_{k+1}) \|T\|_{(a_k)}.
\]

**Proof.** Since \(t \|T\|_{(t)} = \int_0^{t} \mu_s(T) ds\) for \(0 < t < 1\) and \(\|T\|_{(s)} = \int_0^{s} \mu_s(T) ds\) for \(1 \leq s < \infty\), summation by parts shows that
\[
\|T\|_f = \int_0^{\infty} f(s) \mu_s(T) ds = a_1 \int_0^{a_1} \mu_s(T) ds + \cdots + a_n \int_{a_{n-1}}^{a_n} \mu_s(T) ds
\]
\[
= \sum_{k=1}^{n} \min\{a_k, 1\} (a_k - a_{k+1}) \|T\|_{(a_k)}.
\]

**Corollary 13.2.** The norm \(\| \cdot \|_f\) defined as above is a symmetric gauge associated to \(\mathcal{M}\) and therefore a symmetric gauge norm on \(\mathcal{F}(\mathcal{M})\). Furthermore, if \(\tau(E) = 1\) then \(\|E\|_f = \int_0^1 f(x) dx\).

**Lemma 13.3.** Let \((\mathcal{M}, \tau)\) be a semi-finite von Neumann algebra and \(E \in \mathcal{M}\) be a (non-zero) finite projection. Suppose \(\mathcal{M}_E\) is a diffuse von Neumann algebra and \(T, X \in \mathcal{M}_E\) are positive operators such that \(T = a_1 E_1 + \cdots + a_n E_n\), \(E_1 + \cdots + E_n = E\), and \(\tau(E_1) = \cdots = \tau(E_n)\). Then there is a sequence of simple positive operators \(X_n \in \mathcal{M}_E\) satisfying the following conditions:

1. \(0 \leq X_1 \leq X_2 \leq \cdots \leq X\) and hence \(0 \leq \mu_s(X_1) \leq \mu_s(X_2) \leq \cdots \leq \mu_s(X)\) for all \(s \in [0, \infty)\);
2. \( \lim_{n \to \infty} \mu_s(X_n) = \mu_s(X) \) for almost all \( s \in [0, \infty) \);

3. there exists an \( r_n \in \mathbb{N} \) such that \( T = a_{n,1}E_{n,1} + \cdots + a_{n,1}E_{n,r_n} \) and \( X = b_{n,1}F_{n,1} + \cdots + b_{n,1}F_{n,r_n} \), where \( E_{n,1} + \cdots + E_{n,r_n} = F_{n,1} + \cdots + F_{n,r_n} = E \) and \( \tau(E_{n,i}) = \tau(F_{n,i}) \) for \( 1 \leq i, j \leq r_n \).

Proof. Since \( M_E \) is diffuse, there is a separable diffuse abelian von Neumann subalgebra \( \mathcal{A} \) of \( M_E \) such that \( X \in \mathcal{A} \). By Lemma 2.6, there is a \(*\)-isomorphism \( \theta \) from \( \mathcal{A} \) onto \( L^\infty[0, 1] \) such that \( \tau_E = \int_0^1 dx \cdot \theta \). Let \( f(x) = \theta(X) \). We can choose a sequence of simple functions \( f_n(x) \) in \( L^\infty[0, 1] \) such that \( 0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f(x) \) and \( \lim_{n \to \infty} f_n(x) = f(x) \) for almost all \( x \). Let \( X_n = \theta^{-1}(f_n(s)) \). Then \( X_n \in M_E \) and \( 0 \leq X_1 \leq X_2 \leq \cdots \leq X \). By Lemma 9.5,

\[
\mu_s(T) = \inf\{\|TF\| : F \in \mathcal{P}(\mathcal{M}), \ \tau(F^{-1}) = s\}
\]

\[
= \inf\{\|TF\| : F \in \mathcal{P}(M_E), \ \tau_E(F^{-1}) = s \tau(E)\} = f^*(\tau(E)s),
\]

where \( f^*(x) \) is the non-increasing rearrangement of \( f(x) \). Therefore, we obtain 1 and 2. To obtain 3, we need only to construct \( f_n(x) = \alpha_{n,1}x_{I_{n,1}}(x) + \cdots + \alpha_{n,1}x_{I_{n,r_n}}(x) \) such that \( m(I_{n,1}) = \cdots = m(I_{n,r_n}) = \frac{\tau(E)}{k_n} \), for some \( k_n \in \mathbb{N} \).

Let \( \mathcal{F} \) be the set of non-increasing, non-negative, right continuous simple functions \( f(x) \) on \( [0, \infty) \) with compact supports such that \( \int_0^1 f(x)dx \leq 1 \).

**Theorem 13.4.** Let \( (\mathcal{M}, \tau) \) be a semi-finite von Neumann algebra satisfying the following conditions:

1. \( \text{Aut}(\mathcal{M}, \tau) \) acts on \( \mathcal{M} \) ergodically;

2. for every finite projection \( E \) in \( \mathcal{M} \), \( (\mathcal{M}_E, \tau_E) \) is a finite von Neumann algebra satisfying the weak Dixmier property.

If \( \| \cdot \| \) is a normalized symmetric gauge norm on \( \mathcal{J}(\mathcal{M}) \), then there is a subset \( \mathcal{F}' \) of \( \mathcal{F} \) containing the characteristic function on \( [0, 1] \) such that for all \( T \in \mathcal{J}(\mathcal{M}) \), \( \| T \| = \sup\{\| T \|_f : f \in \mathcal{F}'\} \).

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Proof. Suppose \( \| \cdot \| \) is a normalized symmetric gauge norm on \( \mathcal{M} \). Let
\[
\mathcal{F}' = \{ \mu_s(X) : X \text{ is a simple positive operator in } \mathcal{F}(\mathcal{M}), \| X \|^\# \leq 1 \}.
\]
For every positive operator \( X \in \mathcal{F}(\mathcal{M}) \) such that \( \| X \|^\# \leq 1 \), by Proposition 11.6, \( \int_0^1 \mu_s(X) ds = \| X \|_{(1)} \leq \| X \|^\# \leq 1 \). Hence \( \mathcal{F}' \subset \mathcal{F} \) and \( \mu_s(E) = \chi_{[0,1]}(s) \in \mathcal{F}' \) if \( E \) is a projection such that \( \tau(E) = 1 \). For \( T \in \mathcal{F}(\mathcal{M}) \), define
\[
\| T \|' = \sup \{ \| T \|_f : f \in \mathcal{F}' \}.
\]
By Corollary 13.2, \( \| \cdot \|' \) is a symmetric gauge norm on \( \mathcal{F}(\mathcal{M}) \). By Lemma 10.12, to prove that \( \| \cdot \|' = \| \cdot \| \), we need to prove \( \| T \|' = \| T \| \) for every positive simple operator \( T \in \mathcal{F}(\mathcal{M}) \) such that \( T = a_1 E_1 + \cdots + a_n E_n \) and \( \tau(E_1) = \cdots = \tau(E_n) = c > 0 \).

By Lemma 12.7 and Theorem 12.10,
\[
\| T \| = \sup \left\{ c \sum_{k=1}^n a_k b_k : X = b_1 E_1 + \cdots + b_n E_n \geq 0, \| X \|^\# \leq 1 \right\}.
\]
Note that if \( X = b_1 E_1 + \cdots + b_n E_n \) is a simple positive operator in \( \mathcal{F}(\mathcal{M}) \) and \( \| X \|^\# \leq 1 \), then \( \mu_s(X) \in \mathcal{F}' \) and \( \| T \|_{\mu_s(X)} = \int_0^\infty \mu_s(X) \mu_s(T) ds = c \sum_{k=1}^n a_k^* b_k^* \), where \( \{a_k^*\} \) and \( \{b_k^*\} \) are nondecreasing rearrangements of \( \{a_k\} \) and \( \{b_k\} \), respectively. By the Hardy-Littlewood-Polya Theorem, \( \sum_{k=1}^n a_k b_k \leq \sum_{k=1}^n a_k^* b_k^* \). Hence,
\[
\| T \| = \sup \left\{ c \sum_{k=1}^n a_k b_k : X = b_1 E_1 + \cdots + b_n E_n \geq 0, \| X \|^\# \leq 1 \right\}
\leq \sup \{ \| T \|_f : f \in \mathcal{F}' \} = \| T \|'.
\]

We need to prove that \( \| T \|' \leq \| T \| \). Let \( X \in \mathcal{F}(\mathcal{M}) \) be a positive simple operator such that \( \| X \|^\# \leq 1 \). We need to prove that \( \| T \|_{\mu_s(X)} \leq \| T \| \). Since \( T, X \in \mathcal{F}(\mathcal{M}) \), there is a finite projection \( E \in \mathcal{M} \) such that \( T, X \in \mathcal{M}_E \).

First, we assume that \( T = a_1 E_1 + \cdots + a_r E_r \) and \( X = b_1 F_1 + \cdots + b_r F_r \), where \( E_1 + \cdots + E_r = F_1 + \cdots + F_r = E \), \( \tau(E_i) = \tau(F_j) = c \) for \( 1 \leq i, j \leq r \), \( a_1 \geq \cdots \geq a_r \), and \( b_1 \geq \cdots \geq b_r \). Let \( Y = b_1 E_1 + \cdots + b_r E_r \). Then \( \mu_s(Y) = \mu_s(X) \). Since \( \tau(E_i) = \tau(F_j) = c \) for \( 1 \leq i, j \leq r \) and \( \text{Aut}(\mathcal{M}, \tau) \) acts on \( \mathcal{M} \) ergodically, there is a \( \theta \in \text{Aut}(\mathcal{M}, \tau) \) such that \( \theta(E_i) = F_i \) for \( 1 \leq i \leq r \). Hence \( \theta(Y) = X \). Since \( \| \cdot \|' \) is a symmetric gauge norm, \( \| Y \|^\# = \| X \|^\# \leq 1 \). By Corollary 12.4,
\[
\| T \| \geq \tau(TY) = c \sum_{k=1}^r a_k b_k = \int_0^\infty \mu_s(Y) \mu_s(T) ds = \int_0^\infty \mu_s(X) \mu_s(T) ds = \| T \|_{\mu_s(X)}.
\]
Now we consider the general case. Since $\mathcal{M}_E$ satisfies the weak Dixmier property, by Theorem 3.27, $\mathcal{M}_E$ is either a finite dimensional von Neumann algebra such that $\tau(F) = \tau(F')$ for every two minimal projections $F$ and $F'$ or $\mathcal{M}_E$ is a diffuse von Neumann algebra. The first case is obvious. If $\mathcal{M}_E$ is a diffuse von Neumann algebra, by Lemma 13.3, we can construct a sequence of simple positive operators $X_n \in \mathcal{J} (\mathcal{M})$ satisfying the following conditions:

1. $0 \leq X_1 \leq X_2 \leq \cdots \leq X$ and hence $0 \leq \mu_s(X_1) \leq \mu_s(X_2) \leq \cdots \leq \mu_s(X)$ for all $s \in [0, \infty)$;
2. $\lim_{n \to \infty} \mu_s(X_n) = \mu_s(X)$ for almost all $s \in [0, \infty)$;
3. there exists an $r_n \in \mathbb{N}$ such that $T = \tilde{a}_{n,1} E_{n,1} + \cdots + \tilde{a}_{n,r_n} E_{n,r_n}$ and $X = b_{n,1} F_{n,1} + \cdots + b_{n,r_n} F_{n,r_n}$ where $E_{n,1} + \cdots + E_{n,r_n} = F_{n,1} + \cdots + F_{n,r_n} = E$ and $\tau(E_{n,i}) = \tau(F_{n,j})$ for $1 \leq i, j \leq n$.

By 1 and Corollary 10.6, $\|X_n\| \leq \|X\| \leq 1$ for all $n = 1, 2, \cdots$. By 3 and above arguments, for every $n$, $\|T\| \geq \|T\|_{\mu_s(X_n)}$. By 1, 2 and the Monotone Convergence theorem,

$$\|T\|_{\mu_s(X)} = \int_0^\infty \mu_s(X) \mu_s(T) ds = \lim_{n \to \infty} \int_0^\infty \mu_s(X_n) \mu_s(T) ds$$

$$= \lim_{n \to \infty} \|T\|_{\mu_s(X_n)} \leq \|T\|.$$

Corollary 13.5. Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra as in Theorem 13.4, and let $\| \cdot \|$ be a normalized symmetric gauge norm on $\mathcal{J}(\mathcal{M})$. Then $\| \cdot \|$ can be extended to a normalized symmetric gauge norm $\| \cdot \|'$ associated to $\mathcal{M}$.

**Proof.** For $T \in \mathcal{M}$, define $\|T\|' = \max \{\|T\|_f : f \in \mathcal{F}'\}$. Then $\| \cdot \|'$ is an extension of $\| \cdot \|$. □

**Remark 13.6.** In Corollary 13.5, the extension is not unique. Indeed, define $\| \cdot \|$ on $\mathcal{B}(\mathcal{H})$ by $\|T\| = \|T\|$ if $T$ is a finite rank operator and $\|T\| = \infty$ if $T$ is an infinite rank operator. It is easy to see that $\| \cdot \|$ defines a unitarily invariant norm associated to $\mathcal{B}(\mathcal{H})$ such that the restriction of $\| \cdot \|$ on $\mathcal{J}(\mathcal{H})$ is the operator norm.

Corollary 13.7. Let $\| \cdot \|$ be a normalized symmetric norm on $L^\infty(0, \infty)$. Then there is a subset $\mathcal{F}'$ of $\mathcal{F}$ containing the characteristic function on $[0, 1]$ such that for all $T \in \mathcal{J}(L^\infty(0, \infty))$,

$$\|T\| = \sup \{\|T\|_f : f \in \mathcal{F}'\}.$$
CHAPTER 14

UNITARILY INVARIANT NORMS RELATED TO FACTORS

Recall that $\mathcal{F}$ is the set of non-increasing, non-negative, right continuous simple functions $f(x)$ on $[0, \infty)$ with compact supports such that $\int_0^1 f(x) dx \leq 1$.

Theorem 14.1. Let $\mathcal{M}$ be a semi-finite factor and let $\| \cdot \|$ be a normalized unitarily invariant norm on $\mathcal{F}(\mathcal{M})$. Then there is a subset $\mathcal{F}'$ of $\mathcal{F}$ containing the characteristic function on $[0, 1]$ such that for all $T \in \mathcal{F}(\mathcal{M})$, $\| T \| = \sup \{ \| T \|_f : f \in \mathcal{F}' \}$.

Proof. Combining Theorem 13.4 and Proposition 10.13, we obtain the theorem. \qed

Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \text{Tr}$. By the proof of Theorem 13.4, if $f \in \mathcal{F}'$, then $f(s) = \mu_s(X)$ for some finite rank operator $X \in \mathcal{B}(\mathcal{H})$, $X \geq 0$ and $\| X \|^\mu \leq 1$. Write $\mu_s(X) = s_1(X)X_{[0,1)}(s) + s_2(X)X_{[1,2)}(s) + \cdots$, where $s_1(X), s_2(X), \cdots$, are $s$-numbers of $X$. Since $\int_0^1 \mu_s(X) \leq 1, s_1(X) \leq 1$. By Lemma 7.1 and simple computations, for every $T \in \mathcal{F}(\mathcal{H})$,

$$\| T \|_{\mu_s(X)} = s_1(X)s_1(T) + s_2(X)s_2(T) + \cdots,$$

where $s_1(T), s_2(T), \cdots$, are $s$-numbers of $T$.

Let $\mathcal{G} = \{ (a_1, a_2, \cdots) : 1 \geq a_1 \geq a_2 \geq \cdots \geq 0 \text{ and } a_n \neq 0 \text{ for finitely many } n \}$. For $(a_1, a_2, \cdots) \in \mathcal{G}$ and $T \in \mathcal{F}(\mathcal{H})$, define

$$\| T \|_{(a_1, a_2, \cdots)} = a_1s_1(T) + a_2s_2(T) + \cdots. \quad (14.0.1)$$

Then $\| T \|_{(a_1, a_2, \cdots)} = \| T \|_f$ is a unitarily invariant norm on $\mathcal{F}(\mathcal{H})$, where $f(x) = a_1X_{[0,1)}(x) + a_2X_{[1,2)}(x) + \cdots$. By identifying $\mu_s(X)$ with $(s_1(X), s_2(X), \cdots)$ in $\mathcal{G}$, we obtain the following corollary.
**Corollary 14.2.** Let $\| \cdot \|$ be a unitarily invariant norm on $\mathcal{F}(\mathcal{H})$. Then there is a subset $\mathcal{G}'$ of $\mathcal{G}$, $(1, 0, \cdots) \in \mathcal{G}'$, such that for all $T \in \mathcal{F}(\mathcal{H})$,

$$\| T \| = \sup \{ a_1 s_1(T) + a_2 s_2(T) + \cdots : (a_1, a_2, \cdots) \in \mathcal{G}' \},$$

where $s_1(T), s_2(T), \cdots$ are s-numbers of $T$.

Similar to the proof of Corollary 14.2, we have the following corollary.

**Corollary 14.3.** Let $\| \cdot \|$ be a normalized symmetric gauge norm on $\mathcal{F}(l^\infty(\mathbb{N}))$. Then there is a subset $\mathcal{G}'$ of $\mathcal{G}$, $(1, 0, \cdots) \in \mathcal{G}'$, such that for all $(x_1, x_2, \cdots) \in \mathcal{F}(l^\infty(\mathbb{N}))$,

$$\|(x_1, x_2, \cdots)\| = \sup \{ a_1 x_1^* + a_2 x_2^* + \cdots : (a_1, a_2, \cdots) \in \mathcal{G}' \},$$

where $(x_1^*, x_2^*, \cdots)$ is the nonincreasing rearrangement of $(|x_1|, |x_2|, \cdots)$. 
PART III

COMPLETELY SINGULAR VON NEUMANN SUBALGEBRAS
CHAPTER 15
BACKGROUND

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ is singular if the only unitary operators in $\mathcal{M}$ satisfying the condition $U\mathcal{N}U^* = \mathcal{N}$ are those in $\mathcal{N}$. The study of singular von Neumann subalgebras has a long and rich history (see for instance [3, 17, 26, 27, 28]). Recently, there is a remarkable progress on singular maximal abelian von Neumann subalgebras in type II$_1$ factors (see [37, 31, 39]).

In [37], Allan Sinclair and Roger Smith introduced a concept of asymptotic homomorphism property. In [31], a concept of weak asymptotic homomorphism property is introduced. Remarkably, in [39], it was shown that every singular maximal abelian von Neumann subalgebra in a type II$_1$ factor satisfies the weak asymptotic homomorphism property. As a corollary, the tensor product of singular maximal abelian von Neumann subalgebras in type II$_1$ factors is proved to be a singular maximal abelian von Neumann subalgebra in the tensor product of type II$_1$ factors (see [39]), which was a long-standing problem.

It is very natural to ask the following question: if $\mathcal{N}_1$ and $\mathcal{N}_2$ are singular von Neumann subalgebras of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, is $\mathcal{N}_1 \otimes \mathcal{N}_2$ singular in $\mathcal{M}_1 \otimes \mathcal{M}_2$? It turns out this is not always true. Let $\mathcal{M}_1 = M_3(\mathbb{C})$ and $\mathcal{N}_1 = M_2(\mathbb{C}) \oplus \mathbb{C}$. Then $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 \end{pmatrix}$ are central projections in $\mathcal{N}_1$ and $\mathcal{N}_1 = \{P, Q\}'$. Suppose $U \in \mathcal{M}_1$ is a unitary matrix such that $U\mathcal{N}_1U^* = \mathcal{N}_1$. Then $UPU^* = P$ and $UQU^* = Q$ (because $adU$ preserves the center of $\mathcal{N}_1$ and $\tau(P) = \frac{2}{3}$, $\tau(Q) = \frac{1}{3}$, where $\tau$ is the normalized trace on $M_3(\mathbb{C})$). So $U \in \{P, Q\}' = \mathcal{N}_1$. This implies that $\mathcal{N}_1$ is singular in $\mathcal{M}_1$. Let $\mathcal{M}_2 = B(l^2(\mathbb{N}))$ and

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$\mathcal{N}_2 = \mathcal{M}_2$. Then $\mathcal{N}_1 \otimes \mathcal{B}(l^2(\mathbb{N})) = M_2(\mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N})) = M_3(\mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N}))$ is not singular in $\mathcal{M}_1 \otimes \mathcal{B}(l^2(\mathbb{N})) = M_3(\mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N}))$. Indeed, let $V$ be an isometry from $l^2(\mathbb{N})$ onto $\mathbb{C}^2 \otimes l^2(\mathbb{N})$. Then $U = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$ is a unitary operator in $M_3(\mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N}))$ such that

$$U(\mathcal{N}_1 \otimes \mathcal{B}(l^2(\mathbb{N})))U^* = \mathcal{N}_1 \otimes \mathcal{B}(l^2(\mathbb{N})).$$

Since $U$ is not in $\mathcal{N}_1 \otimes \mathcal{B}(l^2(\mathbb{N}))$, $\mathcal{N}_1 \otimes \mathcal{B}(l^2(\mathbb{N}))$ is not singular in $\mathcal{M}_1 \otimes \mathcal{B}(l^2(\mathbb{N}))$. Indeed, $\mathcal{N}_1 \otimes \mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{M}_1 \otimes \mathcal{B}(l^2(\mathbb{N}))$.

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. If the von Neumann algebra $\mathcal{N} \otimes \mathcal{B}(\mathcal{H})$ is singular in $\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$ for every Hilbert space $\mathcal{H}$, then $\mathcal{N}$ is said to be completely singular in $\mathcal{M}$. In this Part, we prove that if $\mathcal{N}$ is a singular maximal abelian von Neumann subalgebra or if $\mathcal{N}$ is a singular subfactor of a type $\text{II}_1$ factor $\mathcal{M}$, then $\mathcal{N}$ is completely singular in $\mathcal{M}$. For every type $\text{II}_1$ factor $\mathcal{M}$, we construct a singular von Neumann subalgebra $\mathcal{N}$ which is not completely singular. The main result of this Part is a characterization of complete singularity. We give some applications of the main result in the last chapter of this Part.
CHAPTER 16
NORMALIZER AND NORMALIZING GROUPOID

Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ denotes the normalizer of $\mathcal{N}$ in $\mathcal{M}$:

$$\mathfrak{N}_{\mathcal{M}}(\mathcal{N}) = \{ U \in \mathcal{M} : U \text{ is a unitary operator, } U \mathcal{N} U^* = \mathcal{N} \},$$

and $\mathfrak{G}_{\mathcal{M}}^{(2)}(\mathcal{N})$ denotes the (two-sided) normalizing groupoid of $\mathcal{N}$ in $\mathcal{M}$:

$$\mathfrak{G}_{\mathcal{M}}^{(2)}(\mathcal{N}) = \{ V \in \mathcal{M} : V \text{ is a partial isometry with initial space } E \text{ and final space } F \text{ such that } E, F \in \mathcal{N} \text{ and } V E V^* = N_F \},$$

where $N_E = E N E$ and $N_F = F N F$. $\mathcal{N}$ is singular in $\mathcal{M}$ if and only if $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$, the von Neumann algebra generated by $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})$, is $\mathcal{N}$. Recall that $\mathcal{N}$ is regular in $\mathcal{M}$ if $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{M}$.

If $\mathcal{M}$ is a finite von Neumann algebra and $\mathcal{N}$ is a maximal abelian von Neumann subalgebra of $\mathcal{M}$, then $V \in \mathfrak{G}_{\mathcal{M}}^{(2)}(\mathcal{N})$ if and only if there is a unitary operator $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ and a projection $E \in \mathcal{N}$ such that $V = UE$ ([17], Theorem 2.1). In other words: any partial isometry that normalizes $\mathcal{N}$ extends to a unitary operator that normalizes $\mathcal{N}$. As a corollary, we have $\mathfrak{G}_{\mathcal{M}}^{(2)}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$, i.e., the von Neumann algebra generated by the normalizing groupoid of $\mathcal{N}$ in $\mathcal{M}$ is the von Neumann algebra generated by the normalizer of $\mathcal{N}$ in $\mathcal{M}$. If $\mathcal{M}$ is an infinite factor, e.g., type III, and $\mathcal{N} = \mathcal{M}$, then there is an isometry in $\mathcal{M}$ which can not be extended to a unitary operator in $\mathcal{M}$. The following example tells us that even the weak form $\mathfrak{G}_{\mathcal{M}}^{(2)}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ can fail. Let $\mathcal{M} = M_3(\mathbb{C})$ and $\mathcal{N} = M_2(\mathbb{C}) \oplus \mathbb{C}$. As we point out in the introduction, $\mathcal{N}$ is singular in $\mathcal{M}$, i.e., $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{N}$. Simple computations show
that $V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is in $\mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$. Note that $V$ is not in $\mathcal{N}$.

Let $V_1, V_2 \in \mathcal{M}$ be two partial isometries in $\mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$ and $E_i = V_i^* V_i \in \mathcal{N}$, $i = 1, 2$. We say $V_1 \preceq V_2$ if $E_1 \preceq E_2$ and $V_1 = V_2 E_1$. It is obvious that $\preceq$ is a partial order on the set of partial isometries in $\mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$. Let $\{V_\alpha\}$ be a totally ordered subset of $\mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$. Then $V = \lim_\alpha V_\alpha$ (in strong operator topology) exists and $V \in \mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$.

**Lemma 16.1.** If $\mathcal{M}$ is a finite von Neumann algebra and $\mathcal{N}$ is a subfactor of $\mathcal{M}$, then for every $V \in \mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$, there is a unitary operator $U \in \mathfrak{M}_\mathcal{M} (\mathcal{N})$ such that $V \preceq U$. In particular, $\mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})'' = \mathfrak{M}_\mathcal{M} (\mathcal{N})''$.

**Proof.** By Zorn's lemma, there is a maximal element $W \in \mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$ such that $V \preceq W$. Let $E = W^* W$ and $F = WW^*$. Then $E,F \neq 0$ and $E,F \in \mathcal{N}$. We need to prove $E = I$. If $E \neq I$, then $F \neq I$ since $\mathcal{M}$ is finite. So $I - E, I - F \in \mathcal{N}$ are not 0. Since $\mathcal{N}$ is a factor, there is a partial isometry $V_1 \in \mathcal{N}$ with initial space $E_1$, a non-zero subprojection of $I - E$, and final space $E_2$, a non-zero subprojection of $E$. Let $F'$ be the range space of $WE_2$. Then $F' = W E_2 W^* \in \mathcal{N}$. Since $\mathcal{N}$ is a factor, there is a partial isometry $V_2 \in \mathcal{N}$ with initial space $F_2$, a non-zero subprojection of $F'$, and final space $F_1$, a non-zero subprojection of $I - F$. Now $W' = V_2 W V_1$ is a partial isometry with initial space $E_1 \preceq I - E$ and final space $F_1 \leq I - F$. Simple computation shows that $W + W' \in \mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})$. Note that $V \preceq W \preceq W + W'$ and $W \neq W + W'$. This contradicts the maximality of $W$. \qed

**Lemma 16.2.** Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{N}$ be an abelian von Neumann subalgebra of $\mathcal{M}$. Then $\mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})'' = \mathfrak{M}_\mathcal{M} (\mathcal{N})''$.

**Proof.** Let $\mathfrak{M}_1 = \mathfrak{M}_\mathcal{M} (\mathcal{N})''$. We only need to prove that $\mathfrak{M}_\mathcal{M}^{(2)}(\mathcal{N})'' \subseteq \mathfrak{M}_1$. For $V \in \mathfrak{M}$ a partial isometry, define $\mathcal{F}(V) = \{W \in \mathfrak{M}_1 : W$ is a partial isometry and $W \preceq V\}$. Suppose $V \notin \mathfrak{M}_1$. By Zorn's lemma, we can choose a maximal element $W \in \mathcal{F}(V)$ such that $V - W \neq 0$ and $\mathcal{F}(V - W) = \{0\}$. Since $W \in \mathfrak{M}_1$, $V \in \mathfrak{M}_1$ if and only if $V - W \in \mathfrak{M}_1$. Therefore, we can assume that $V \neq 0$ and $\mathcal{F}(V) = \{0\}$. Let $E = V^* V$ and $F = V V^*$. Then $E \neq 0$ and $F \neq 0$. 

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If $E = F$, let $U = V + (I - E)$. Then $U \in \mathfrak{M}_d(\mathcal{N})$ and $V = UE \in \mathcal{M}_1$. It is a contradiction.

If $E \neq F$, we can assume that $E_1 = E(I - F) \neq 0$ (otherwise consider $V^*$). Let $V_1 = VE_1$ and $F_1$ be the final space of $V_1$. Then $V_1 \in \mathfrak{M}_d(\mathcal{N})$ with initial space $E_1 \leq I - F$ and final space $F_1 \leq F$ such that $0 \neq V_1 \leq V$. Let $U = V_1 + V_1^* + (I - E_1 - F_1)$. Then $U \in \mathfrak{M}_d(\mathcal{N})$ and $V_1 = UE_1 \in \mathcal{M}_1$. Note that $V_1 \neq 0$ and $V_1 \leq V$. $\mathcal{S}(V) \neq \{0\}$. It is a contradiction. □

If $\mathcal{N}$ is singular in $\mathcal{M}$ and $E \in \mathcal{N}$ is a projection, $\mathcal{N}_E (:= E \mathcal{N} E)$ may be not singular in $\mathcal{M}_E$. For example, let $\mathcal{M} = M_3(\mathbb{C})$ and $\mathcal{N} = M_2(\mathbb{C}) \oplus \mathbb{C}$ and

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{N}.$$  

Then $\mathcal{N}_E$ is not singular in $\mathcal{M}_E$. But we have the following result.

**Lemma 16.3.** Let $\mathcal{N}$ be a singular von Neumann subalgebra of $\mathcal{M}$ and $E \in \mathcal{N}$ be a projection. If $\mathcal{N}$ is a countably decomposable, properly infinite von Neumann algebra, then $\mathcal{N}_E$ is singular in $\mathcal{M}_E$.

**Proof.** Let $P$ be the central support of $E$ relative to $\mathcal{N}$. Then there are central projections $P_1$, $P_2$ of $\mathcal{N}$ such that $P_1 + P_2 = P$ and $P_1 E$ is finite and $P_2 E$ is properly infinite. Let $E_1 = P_1 E$ and $E_2 = P_2 E$. Then the central supports of $E_1$ and $E_2$ are $P_1$ and $P_2$, respectively. Since $P_1$ is a properly infinite countably decomposable projection and $E_1$ is a finite projection in $\mathcal{N}_P_1$ and the central support of $E_1$ is $P_1$, it follows that $P_1$ is a countably infinite sum of projections $\{E_{1n}\}$ in $\mathcal{N}$, where each $E_{1n}$ is equivalent to $E_1$ in $\mathcal{N}_P_1$ (see for instance, Corollary 6.3.12 of [21], volume 2). For $n \in \mathbb{N}$, let $W_{1n}$ be a partial isometry in $\mathcal{N}_{P_1}$ such that $W_{1n}^* W_{1n} = E_{1n}$ and $W_{1n} W_{1n}^* = E_1$. Since $P_2$ and $E_2$ are properly infinite projections in $\mathcal{N}_{P_2}$ with same central support $P_2$ and $\mathcal{N}_{P_2}$ is countably decomposable, $P_2$ and $E_2$ are equivalent in $\mathcal{N}_{P_2}$ (see for instance, Corollary 6.3.5 of [21], volume 2). Since $P_2$ is properly infinite in $\mathcal{N}$, it can be decomposed into a countably infinite sum of projections $\{E_{2n}\}$, each $E_{2n}$ is equivalent to $P_2$ and hence to $E_2$. For $n \in \mathbb{N}$, let $W_{2n}$ be a partial isometry in $\mathcal{N}_{P_2}$ such that $W_{2n}^* W_{2n} = E_{2n}$ and $W_{2n} W_{2n}^* = E_2$. Let $W_n = W_{1n} + W_{2n} \in \mathcal{N}$. Then $W_n^* W_n = E_{1n} + E_{2n}$ and $W_n W_n^* = E_1 + E_2 = E$. 

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Suppose $V$ is a unitary operator in $\mathcal{M}_E$ such that $V\mathcal{N}_E V^* = \mathcal{N}_E$. Define $U = \sum_{n=1}^{\infty} W_n V W_n + (I - P_1 - P_2)$. Then $U$ is a unitary operator and $U^* = \sum_{n=1}^{\infty} W_n^* V^* W_n + (I - P_1 - P_2)$. For any $T \in \mathcal{N}$,

$$U T U^* = \sum_{m,n=1}^{\infty} W_n^* V W_m T W_n^* V^* W_n + (I - P_1 - P_2)T.$$ 

Note that $W_n T W_n^* \in \mathcal{N}_E$, $V W_n T W_n^* V^* \in \mathcal{N}_E$. So $U T U^* \in \mathcal{N}$. Similarly, $U^* T U \in \mathcal{N}$. Thus $U \in \mathcal{N}_E(\mathcal{N})$. Since $\mathcal{N}$ is singular in $\mathcal{M}$, $U \in \mathcal{N}$. Therefore, $W_1^* V W_1 = U(E_1 + E_3) \in \mathcal{N}$. So $V = E_1 V E_1 = W_1^* V W_1 W_1^* \in \mathcal{N}_E$. This implies that $\mathcal{N}_E$ is singular in $\mathcal{M}_E$. \qed
CHAPTER 17
EXAMPLES OF COMPLETELY SINGULAR SUBALGEBRAS

Theorem 17.1. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ and $\mathcal{K}$ be a Hilbert space. If $\mathfrak{N}(\mathcal{N})'' = \mathfrak{N}(\mathcal{K})''$, then $\mathfrak{N}(\mathcal{M})'' = \mathfrak{N}(\mathcal{N})'' \otimes \mathfrak{B}(\mathcal{K})$.

Combining Theorem 17.1, Lemma 16.1 and Lemma 16.2, we have the following corollaries.

Corollary 17.2. If $\mathcal{M}$ is a type II$_1$ factor and $\mathcal{N}$ is a singular subfactor of $\mathcal{M}$, then $\mathcal{N}$ is completely singular in $\mathcal{M}$.

Corollary 17.3. If $\mathcal{N}$ is a singular MASA of a von Neumann algebra $\mathcal{M}$, then $\mathcal{N}$ is completely singular in $\mathcal{M}$.

To prove Theorem 17.1, we need the following lemmas. We consider dim$\mathcal{K} = 2$ first, which motivates the general case.

Lemma 17.4. Let $U = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ be a unitary operator in $\mathcal{M} \otimes M_2(\mathbb{C})$. Then the following conditions are equivalent:

1. $U(\mathcal{N} \otimes M_2(\mathbb{C}))U^* = \mathcal{N} \otimes M_2(\mathbb{C})$;

2. $A_i X A_j^* \in \mathcal{N}$ and $A_i^* X A_j \in \mathcal{N}$ for all $X \in \mathcal{N}$, $1 \leq i, j \leq 4$.

Proof. It is easy to see that $U(\mathcal{N} \otimes M_2(\mathbb{C}))U^* = \mathcal{N} \otimes M_2(\mathbb{C})$ if and only if $U(\mathcal{N} \otimes M_2(\mathbb{C}))U^* \subseteq \mathcal{N} \otimes M_2(\mathbb{C})$ and $U^*(\mathcal{N} \otimes M_2(\mathbb{C}))U \subseteq \mathcal{N} \otimes M_2(\mathbb{C})$. Note that $U(\mathcal{N} \otimes M_2(\mathbb{C}))U^* \subseteq \mathcal{N} \otimes M_2(\mathbb{C})$ if and only if

$$U \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U^*, U \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} U^*, U \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} U^*, U \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} U^* \in \mathcal{N}$$

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for all $X \in \mathcal{N}$. Simple computations show that $U(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U^* \subseteq \mathcal{N} \bar{\otimes} M_2(\mathbb{C})$ if and only if $A_iXA_j^* \in \mathcal{N}$ for all $X \in \mathcal{N}, 1 \leq i, j \leq 4$. \hfill \Box

Similarly, we can prove the following lemma.

**Lemma 17.5.** Let $U = (A_{ij})$ be a unitary operator in $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$. Then the following conditions are equivalent:

1. $U(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$;

2. $A_iXA_j^* \in \mathcal{N}$ and $A_i^*XA_j \in \mathcal{N}$ for all $X \in \mathcal{N}, 1 \leq i, j \leq \dim \mathcal{K}$.

Let $X = I$ and $i = j$ in 2 of Lemma 17.5. We have the following corollary.

**Corollary 17.6.** Let $U = (A_{ij})$ be a unitary operator in $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$ such that

$$U(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}).$$

If $A_{ij} = V_{ij}H_{ij}$ is the polar decomposition of $A_{ij}$, then $H_{ij} \in \mathcal{N}, 1 \leq i, j \leq \dim \mathcal{K}$.

**Lemma 17.7.** Let $\mathcal{N}$ be a von Neumann algebra and $H$ be a positive operator in $\mathcal{N}$ and $E$ be the closure of the range space of $H$. Then the strong-operator closure of $\mathcal{T} = \{HXH : X \in \mathcal{N}\}$ is $\mathcal{N}_E(= E\mathcal{N}E)$.

**Proof.** It is easy to see $\mathcal{T} \subseteq \mathcal{N}_E$. Let $H = \int_\mathbb{R} \lambda dE(\lambda)$ and $E_n = E([1/n, \infty))$. Then $\lim_{n \to \infty} E_n = E$ in strong-operator topology. Set $H_n = E_nH + (I - E_n)$. Then $H_n$ is invertible in $\mathcal{N}$. For $X \in \mathcal{N}_E$, let $X_n = H_n^{-1}(E_nXE_n)H_n^{-1} \in \mathcal{N}$. Then

$$HX_nH = HHH_n^{-1}E_nXE_nH_n^{-1}H = E_nXE_n \to EXE = X$$

in the strong-operator topology. Hence, the strong-operator closure of $\mathcal{T}$ contains $\mathcal{N}_E$. \hfill \Box

**Lemma 17.8.** Suppose $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and $A \in \mathcal{M}$ satisfies $AN A^* \subseteq \mathcal{N}$ and $A^* \mathcal{N} A \subseteq \mathcal{N}$. Let $A = VH$ be the polar decomposition and $E = V^*V, F = VV^*$. Then $H, E, F \in \mathcal{N}$ and $V \in \text{Gin}(^2_\mathcal{M})(\mathcal{N})$.\[71\]
Proof. By the assumption, \( A^*A = H^2 \in \mathcal{N} \). So \( H \in \mathcal{N} \) and \( E = R(H) \in \mathcal{N} \), where \( R(H) \) is the closure of range space of \( H \). By symmetry, \( F \in \mathcal{N} \). Note that \( A^*AX \subseteq \mathcal{N} \) for all \( X \in \mathcal{N} \). By Lemma 17.7, \( V_N \subseteq \mathcal{N} \). By \( A^*AX \subseteq \mathcal{N} \) for all \( X \in \mathcal{N} \) and similar arguments, \( V_N \subseteq \mathcal{N} \). Thus \( V_N \subseteq \mathcal{N} \). By Lemma 17.8, \( V_H = \mathcal{N} \), i.e., \( V \in \mathcal{M}^{(2)}(\mathcal{N}) \). □

The proof of Theorem 17.1. Let \( U_1 \in \mathcal{M}(\mathcal{N}) \) and \( V \) be a unitary operator in \( \mathfrak{B}(\mathcal{H}) \). Then \( U_1 \otimes V \in \mathcal{M} \otimes \mathfrak{B}(\mathcal{H})(\mathcal{N} \otimes \mathfrak{B}(\mathcal{H})) \). This implies that

\[
\mathcal{M} \otimes \mathfrak{B}(\mathcal{H})(\mathcal{N} \otimes \mathfrak{B}(\mathcal{H}))' \supseteq \mathcal{M} (\mathcal{N})' \otimes \mathfrak{B}(\mathcal{H}).
\]

Conversely, suppose \( U = (A_{ij}) \) is a unitary operator in \( \mathcal{M} \otimes \mathfrak{B}(\mathcal{H}) \) such that

\[
U(\mathcal{N} \otimes \mathfrak{B}(\mathcal{H})))U^* = \mathcal{N} \otimes \mathfrak{B}(\mathcal{H}).
\]

Let \( A_{ij} = V_{ij}H_{ij} \) be the polar decomposition of \( A_{ij} \). By Lemma 17.5, Corollary 17.6 and Lemma 17.8, \( H_{ij} \in \mathcal{N} \) and \( V_{ij} \in \mathcal{M}^{(2)}(\mathcal{N}) \). By the assumption of Theorem 17.1, \( V_{ij} \in \mathcal{M} \). So \( U \in \mathcal{M} \). So \( U \in \mathcal{M} \). Therefore,\( \mathcal{M} \) and \( \mathfrak{B}(\mathcal{H}) \), i.e.,

\[
\mathcal{M} \otimes \mathfrak{B}(\mathcal{H})(\mathcal{N} \otimes \mathfrak{B}(\mathcal{H}))' \subseteq \mathcal{M} \otimes \mathfrak{B}(\mathcal{H}).
\]

□
CHAPTER 18

SINGULARITY AND COMPLETE SINGULARITY

Proposition 18.1. If \( \mathcal{N} \) is a singular but not a completely singular von Neumann subalgebra of \( \mathcal{M} \), then there is a von Neumann subalgebra \( \mathcal{M}_1 \) of \( \mathcal{M} \) and a Hilbert space \( \mathcal{H} \) such that \( \mathcal{N} \subseteq \mathcal{M}_1 \), \( \mathcal{N} \) is singular in \( \mathcal{M}_1 \) and \( \mathcal{N} \otimes \mathcal{B}(\mathcal{H}) \) is regular in \( \mathcal{M}_1 \otimes \mathcal{B}(\mathcal{H}) \).

Proof. Since \( \mathcal{N} \) is not completely singular in \( \mathcal{M} \), there is a Hilbert space \( \mathcal{K} \) such that \( \mathcal{L} = \mathcal{N} \otimes \mathcal{B}(\mathcal{H}) \subseteq \mathcal{L} \subseteq \mathcal{M} \otimes \mathcal{B}(\mathcal{H}) \), \( \mathcal{L} = \mathcal{M}_1 \otimes \mathcal{B}(\mathcal{H}) \) for some von Neumann algebra \( \mathcal{M}_1 \), \( \mathcal{N} \subseteq \mathcal{M}_1 \subseteq \mathcal{M} \). Since \( \mathcal{N} \) is singular in \( \mathcal{M} \), \( \mathcal{N} \) is singular in \( \mathcal{M}_1 \).

Since \( \mathcal{N} \otimes \mathcal{B}(\mathcal{H}) \) is regular in \( \mathcal{M}_1 \otimes \mathcal{B}(\mathcal{H}) \).

Proposition 18.2. If \( \mathcal{M} \) is a type II_1 factor, then there is a singular von Neumann subalgebra \( \mathcal{N} \) of \( \mathcal{M} \) such that \( \mathcal{N} \neq \mathcal{M} \) and \( \mathcal{N} \otimes \mathcal{B}(L^2(\mathbb{N})) \) is regular in \( \mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{N})) \). In particular, \( \mathcal{N} \) is not completely singular.

Proof. Let \( \mathcal{M}_1 \) be a type I_3 subfactor of \( \mathcal{M} \) and \( \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M} \). Then \( \mathcal{M}_2 \) is a type II_1 factor. We can identify \( \mathcal{M} \) with \( M_2(\mathbb{C}) \otimes \mathcal{M}_2 \) and \( \mathcal{M}_1 \) with \( M_3(\mathbb{C}) \otimes \mathbb{C} I \). With this identification, let \( \mathcal{N} = (M_2(\mathbb{C}) \otimes \mathbb{C}) \otimes \mathcal{M}_2 \). Then

\[
P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} \otimes I,
\]

and

\[
Q = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \otimes I.
\]

are central projections in \( \mathcal{N} \). \( \mathcal{N} = \{P, Q\} \cap \mathcal{M} \) and \( \{P, Q\}'' \) is the center of \( \mathcal{N} \). Let \( U \in \mathcal{M} \) be a unitary operator such that \( UNU^* = \mathcal{N} \). Then \( U\{P, Q\}''U^* = \{P, Q\}'' \). Let \( \tau \) be the unique
tracial state on $\mathcal{M}$. Then $\tau(P) = \frac{2}{3}$ and $\tau(Q) = \frac{1}{3}$. So $UPU^* = P$ and $UQU^* = Q$. This implies that $U \in \{P, Q\}' \cap \mathcal{M} = \mathcal{N}$ and $\mathcal{N}$ is singular in $\mathcal{M}$.

To see $\mathcal{N} \otimes \mathcal{B}(l^2(\mathbb{N}))$ is not singular in $\mathcal{M} \otimes \mathcal{B}(l^2(\mathbb{N}))$, we identify $\mathcal{M} \otimes \mathcal{B}(l^2(\mathbb{N}))$ with $M_3(\mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N})) \otimes M_2$ and $\mathcal{N} \otimes \mathcal{B}(l^2(\mathbb{N}))$ with $(M_2(\mathbb{C}) \oplus \mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N})) \otimes M_2$. Let $V$ be an isometry from $l^2(\mathbb{N})$ onto $\mathbb{C}^2 \otimes l^2(\mathbb{N})$. Then $U = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$ is a unitary operator in $M_3(\mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N}))$ such that $U((M_2(\mathbb{C}) \oplus \mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N})))U^* = (M_2(\mathbb{C}) \oplus \mathbb{C}) \otimes \mathcal{B}(l^2(\mathbb{N}))$. Therefore, $U \otimes I$ is a unitary operator in the normalizer of $\mathcal{N} \otimes \mathcal{B}(l^2(\mathbb{N}))$. Note that $U \otimes I \notin \mathcal{N} \otimes \mathcal{B}(l^2(\mathbb{N}))$.

By Proposition 18.1, there is a von Neumann subalgebra $\mathcal{L}$ of $\mathcal{M}$ such that $\mathcal{N} \subset \subset \mathcal{L}$ and $\mathcal{N} \otimes \mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{L} \otimes \mathcal{B}(l^2(\mathbb{N}))$. Since $(M_2(\mathbb{C}) \oplus \mathbb{C}) \otimes M_2 \subset \subset \mathcal{L} \subset M_3(\mathbb{C}) \otimes M_2$, by Ge-Kadison's splitting theorem (see [13]), $\mathcal{L} = \mathcal{L}_1 \otimes M_2$ for some von Neumann algebra $\mathcal{L}_1$ such that $M_2(\mathbb{C}) \oplus \mathbb{C} \subset \subset \mathcal{L}_1 \subseteq M_3(\mathbb{C})$. Since $M_3(\mathbb{C})$ is the unique von Neumann subalgebra of $M_3(\mathbb{C})$ satisfies the above condition, $\mathcal{L}_1 = M_3(\mathbb{C})$. This implies that $\mathcal{N} \otimes \mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{M} \otimes \mathcal{B}(l^2(\mathbb{N}))$. \hfill \Box
CHAPTER 19
MAIN THEOREM

Theorem 19.1. Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then the following conditions are equivalent.

1. $\mathcal{N}$ is completely singular in $\mathcal{M}$;
2. $\mathcal{N} \otimes \mathcal{B}(l^2(\mathbb{N}))$ is singular in $\mathcal{M} \otimes \mathcal{B}(l^2(\mathbb{N}))$;
3. If $\theta \in \text{Aut}(\mathcal{N}')$ and $\theta(X) = X$ for all $X \in \mathcal{M}'$, then $\theta(Y) = Y$ for all $Y \in \mathcal{N}'$.

Proof. “3 $\Rightarrow$ 1”. Let $\mathcal{K}$ be a Hilbert space and $U \in \mathcal{M} \otimes \mathcal{B}(\mathcal{K})$ be a unitary operator such that $U(\mathcal{N} \otimes \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \otimes \mathcal{B}(\mathcal{K})$. Note that $(\mathcal{M} \otimes \mathcal{B}(\mathcal{K}))(\mathcal{N} \otimes \mathcal{B}(\mathcal{K})) = \mathcal{N}' \otimes \mathcal{K}$, $\theta = \text{ad}U \in \text{Aut}(\mathcal{N} \otimes \mathcal{B}(\mathcal{K}))$. Since $U \in \mathcal{M} \otimes \mathcal{B}(\mathcal{K})$, $\theta(X \otimes I_{\mathcal{K}}) = X \otimes I_{\mathcal{K}}$ for all $X \in \mathcal{M}'$. By the assumption of 3, $Y \otimes I_{\mathcal{K}} = \theta(Y \otimes I_{\mathcal{K}}) = U(Y \otimes I_{\mathcal{K}})U^*$ for all $Y \in \mathcal{N}' \otimes \mathcal{K}$. This implies that $U \in \mathcal{N} \otimes \mathcal{B}(\mathcal{K})$. Therefore, $\mathcal{N} \otimes \mathcal{B}(\mathcal{K})$ is singular in $\mathcal{M} \otimes \mathcal{B}(\mathcal{K})$.

“1 $\Rightarrow$ 2” is trivial.

“2 $\Rightarrow$ 3”. By [14], there is a separable Hilbert space $\mathcal{K}_1$ and a faithful normal representation $\phi$ of $\mathcal{N}'$ such that $\phi(\mathcal{N})$ acts on $\mathcal{K}_1$ in standard form. Let $\theta_1 = \phi \cdot \theta \cdot \phi^{-1}$. Then $\theta_1 \in \text{Aut}(\mathcal{N}')$ and $\theta_1(\phi(X)) = \phi(X)$ for all $X \in \mathcal{M}'$. Now there is a unitary operator $U_1 \in \mathcal{B}(\mathcal{K}_1)$ such that $\theta_1(\phi(Y)) = U_1 \phi(Y)U_1^*$ for all $Y \in \mathcal{N}'$. Let $\mathcal{M}_1$ and $\mathcal{M}_1'$ be the commutants of $\phi(\mathcal{N}')$ and $\phi(\mathcal{M}')$ relative to $\mathcal{K}_1$, respectively. Then $\mathcal{M}_1$ is a von Neumann subalgebra of $\mathcal{M}_1$. Since $\theta_1(\phi(X)) = U_1 \phi(X)U_1^* = \phi(X)$ for all $X \in \mathcal{M}'$, $U_1 \in \mathcal{M}_1$. Since $\theta = \text{ad}U_1 \in \text{Aut}(\mathcal{N}')$, $\theta = \text{ad}U_1 \in \text{Aut}(\mathcal{M}_1)$. Now we only need to prove that $\mathcal{N}_1$ is a singular von Neumann subalgebra of $\mathcal{M}_1$. Then $U_1 \in \mathcal{N}_1$ and $\theta_1(\phi(Y)) = U_1 \phi(Y)U_1^* = \phi(Y)$ for all $Y \in \mathcal{N}'$. This implies that $\theta(Y) = Y$ for all $Y \in \mathcal{N}'$. 

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By [3] (Theorem 3, page 61), \( \phi = \phi_3 \cdot \phi_2 \cdot \phi_1 \), where \( \phi_1(\mathcal{N}') = \mathcal{N}' \otimes \mathcal{C}_L, \mathcal{H} = l^2(\mathbb{N}) \), \( \phi_2(\mathcal{N}' \otimes \mathcal{C}_L) = (\mathcal{N}' \otimes \mathcal{C}_L)E, E \in (\mathcal{N}' \otimes \mathcal{C}_L)' = \mathcal{N} \otimes \mathcal{B}(\mathcal{H}) \) and \( \phi_3 \) is a spacial isomorphism. We may assume that \( \phi_3 = \text{id} \). Then \( \mathcal{M}_1 = E(\mathcal{N} \otimes \mathcal{B}(\mathcal{H}))E \) and \( \mathcal{M}_1 = E(\mathcal{M} \otimes \mathcal{B}(\mathcal{H}))E \), where \( E \in \mathcal{N} \otimes \mathcal{B}(\mathcal{H}) \). By 2, \( \mathcal{N} \otimes \mathcal{B}(\mathcal{H}) \) is singular in \( \mathcal{M} \otimes \mathcal{B}(\mathcal{H}) \). Note that \( \mathcal{N} \otimes \mathcal{B}(\mathcal{H}) \) is a countably decomposable, properly infinite von Neumann algebra. By Lemma 16.3, \( \mathcal{M}_1 \) is singular in \( \mathcal{M}_1 \). \( \Box \)

Note that in the proof of “\( 3 \Rightarrow 1 \)” of Theorem 19.1, we do not need the assumption that \( \mathcal{H} \) is a separable Hilbert space.
CHAPTER 20
APPLICATIONS

20.1 Complete singularity is stable under tensor product

The proof of the following lemma is similar to the proof of Lemma 6.6 of [41]

Lemma 20.1. Let $\mathcal{M}$ be a separable von Neumann algebra and $\mathcal{N}$ be a singular von Neumann subalgebra of $\mathcal{M}$. If $\mathcal{A}$ is an abelian von Neumann algebra, then $\mathcal{N} \otimes \mathcal{A}$ is a singular von Neumann subalgebra of $\mathcal{M} \otimes \mathcal{A}$.

Proof. We can assume that $\mathcal{M}$ acts on a separable Hilbert space $\mathcal{H}$ in standard form and $\mathcal{A}$ is countably decomposable. Then there is a *-isomorphism from $\mathcal{A}$ onto $L^\infty(\Omega, \mu)$ with $\mu$ a probability Radon measure on some compact space $\Omega$. To the *-isomorphism $\mathcal{A} \to L^\infty(\Omega, \mu)$ corresponds canonically a *-isomorphism $\Phi$ from the von Neumann algebra $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$ onto $L^\infty(\Omega, \mu; \mathcal{B}(\mathcal{H}))$. Note that $\Phi(\mathcal{M} \otimes \mathcal{A}) = \mathcal{M}$ and $\Phi(\mathcal{N} \otimes \mathcal{A})(\omega) = \mathcal{N}$ for almost all $\omega \in \Omega$. Let $U \in \mathcal{M} \otimes \mathcal{A}$ be a unitary operator such that $U(\mathcal{N} \otimes \mathcal{A})U^* = \mathcal{N} \otimes \mathcal{A}$. Then $\Phi(U) = U(\omega)$ such that $U(\omega)$ is a unitary operator in $\mathcal{M}$ for almost all $\omega \in \Omega$. Because $U(\mathcal{N} \otimes \mathcal{A})U^* = \mathcal{N} \otimes \mathcal{A}$, we have $U(\omega)N^*U(\omega)^* = \mathcal{N}$ for almost all $\omega \in \Omega$. Since $\mathcal{N}$ is singular in $\mathcal{M}$, $U(\omega) \in \mathcal{N}$ for almost all $\omega \in \Omega$. Hence $U \in \mathcal{N} \otimes \mathcal{A}$. 

Note that for every Hilbert space $\mathcal{H}$, the von Neumann algebra $\mathcal{M} \otimes \mathcal{A} \otimes \mathcal{B}(\mathcal{H})$ is canonically isomorphic to $\mathcal{M} \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$. We have the following corollary.

Corollary 20.2. Let $\mathcal{M}$ be a separable von Neumann algebra and $\mathcal{N}$ be a completely singular von Neumann subalgebra. If $\mathcal{A}$ is an abelian von Neumann algebra, then $\mathcal{N} \otimes \mathcal{A}$ is a completely singular von Neumann subalgebra of $\mathcal{M} \otimes \mathcal{A}$.

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Theorem 20.3. Let $\mathcal{M}$ be a separable von Neumann algebra, and let $\mathcal{N}$ be a completely singular von Neumann subalgebra. Then $\mathcal{N} \otimes \mathcal{L}$ is completely singular in $\mathcal{M} \otimes \mathcal{L}$ for every separable von Neumann algebra $\mathcal{L}$.

Proof. We can assume that $\mathcal{M}$ and $\mathcal{L}$ act on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{N}$ in standard form, respectively. Let $\theta$ be in $\text{Aut}(\mathcal{N'} \otimes \mathcal{L})$ such that $\theta(X \otimes Z) = X \otimes Z$ for all $X \in \mathcal{M'}$ and $Z \in \mathcal{L'}$. Let $\mathcal{A}$ be the center of $\mathcal{L'}$. Then

\[
(\mathcal{N'} \otimes \mathcal{L'})' \cap (\mathcal{N'} \otimes \mathcal{L'}) = (\mathcal{B}(\mathcal{H}) \otimes \mathcal{L})' \cap (\mathcal{N'} \otimes \mathcal{L'}) = (\mathcal{B}(\mathcal{H}) \cap \mathcal{N'}) \otimes (\mathcal{L} \cap \mathcal{L'}) = \mathcal{N'} \otimes \mathcal{A}.
\]

So for $T \in \mathcal{N'} \otimes \mathcal{A}$ and $Z \in \mathcal{L'}$, $T(I_{\mathcal{H}} \otimes Z) = (I_{\mathcal{H}} \otimes Z)T$ and $\theta(T)\theta(I_{\mathcal{H}} \otimes Z) = \theta(I_{\mathcal{H}} \otimes Z)\theta(T)$. Since $\theta(I_{\mathcal{H}} \otimes Z) = I_{\mathcal{H}} \otimes Z$, $\theta(T)(I_{\mathcal{H}} \otimes Z) = (I_{\mathcal{H}} \otimes Z)\theta(T)$. This implies that $\theta(T) \in \mathcal{N'} \otimes \mathcal{A}$. So $\theta \in \text{Aut}(\mathcal{N'} \otimes \mathcal{A})$ when $\theta$ is restricted on $\mathcal{N'} \otimes \mathcal{A}$ such that $\theta(X \otimes Z) = X \otimes Z$ for all $X \in \mathcal{M'}$ and $Z \in \mathcal{A}$.

Consider the standard representation $\phi$ of $\mathcal{A}$ on a separable Hilbert space $\mathcal{K}_1$. Then $\phi(\mathcal{A}) = \phi(\mathcal{A})$. By Corollary 4.2, $\mathcal{N} \otimes \phi(\mathcal{A})$ is completely singular in $\mathcal{M} \otimes \phi(\mathcal{A})$. On $\mathcal{H} \otimes \mathcal{K}_1$, $(\mathcal{N} \otimes \phi(\mathcal{A}))' = \mathcal{N'} \otimes \phi(\mathcal{A})$ and $(\mathcal{M} \otimes \phi(\mathcal{A}))' = \mathcal{M'} \otimes \phi(\mathcal{A})$. Note that $\theta_1 = (id \otimes \phi) \cdot \theta \cdot (id \otimes \phi^{-1}) \in \text{Aut}(\mathcal{N'} \otimes \phi(\mathcal{A}))$ and $\theta_1(X \otimes Z') = (id \otimes \phi) \cdot \theta(X \otimes \phi^{-1}(Z')) = (id \otimes \phi)(X \otimes \phi^{-1}(Z')) = X \otimes Z'$ for all $X \in \mathcal{M}$ and $Z' \in \phi(\mathcal{A})$. By Theorem 19.1, $\theta_1(Y \otimes Z') = Y \otimes Z'$ for all $Y \in \mathcal{M}$ and $Z' \in \phi(\mathcal{A})$. This implies that $\theta(Y \otimes \phi^{-1}(Z')) = Y \otimes \phi^{-1}(Z')$ for all $Y \in \mathcal{N}$ and $Z' \in \phi(\mathcal{A})$. Let $Z' = I_{\mathcal{K}_1}$. Then $\theta(Y \otimes I_{\mathcal{K}}) = Y \otimes I_{\mathcal{K}}$ for all $Y \in \mathcal{N}$. Hence $\theta(Y \otimes Z) = Y \otimes Z$ for all $Y \in \mathcal{N}$ and $Z \in \mathcal{L'}$. By Theorem 19.1, $\mathcal{N} \otimes \mathcal{L}$ is completely singular in $\mathcal{M} \otimes \mathcal{L}$.

\[\Box\]

20.2 Tensor product with singular subfactor

Theorem 20.4. Let $\mathcal{M}_i$ be a separable von Neumann algebra and $\mathcal{N}_i$ be a completely singular von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{N}_i$ is a factor, then $\mathcal{N}_1 \otimes \mathcal{N}_2$ is completely singular in $\mathcal{M}_1 \otimes \mathcal{M}_2$. 

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Proof. We can assume that \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) act on separable Hilbert space \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) in standard form, respectively. Let \( \theta \) be in \( \text{Aut}(\mathcal{M}_1' \otimes \mathcal{M}_2') \) such that \( \theta(X_1 \otimes X_2) = X_1 \otimes X_2 \) for all \( X_1 \in \mathcal{M}_1' \) and \( X_2 \in \mathcal{M}_2' \).

Since \( \mathcal{N}_1 \) is a singular subfactor in \( \mathcal{M}_1 \), \( \mathcal{N}_1' \cap \mathcal{M}_1 = \mathcal{N}_1' \cap \mathcal{N}_1 = \mathcal{C}I_{\mathcal{M}_1} \). Note that

\[
(\mathcal{M}_1' \otimes C I_{\mathcal{M}_2})' \cap (\mathcal{N}_1' \otimes \mathcal{N}_2') = (\mathcal{M}_1 \otimes \mathcal{B}(\mathcal{H}_2)) \cap (\mathcal{N}_1' \otimes \mathcal{N}_2')
\]

\[
= (\mathcal{M}_1 \cap \mathcal{N}_1') \otimes (\mathcal{B}(\mathcal{H}_2) \cap \mathcal{N}_2')
\]

\[
= C I_{\mathcal{M}_1} \otimes \mathcal{N}_2'.
\]

We have

\[
\theta(C I_{\mathcal{M}_1} \otimes \mathcal{N}_2') = \theta((\mathcal{M}_1' \cap \mathcal{M}_1') \otimes (\mathcal{B}(\mathcal{H}_2) \cap \mathcal{N}_2'))
\]

\[
= \theta((\mathcal{M}_1 \cap \mathcal{B}(\mathcal{H}_2)) \cap (\mathcal{N}_1' \otimes \mathcal{N}_2'))
\]

\[
= \theta((\mathcal{M}_1' \otimes C I_{\mathcal{M}_2})' \cap (\mathcal{N}_1' \otimes \mathcal{N}_2'))
\]

\[
= \theta((\mathcal{M}_1' \otimes C I_{\mathcal{M}_2})' \cap \theta(\mathcal{N}_1' \otimes \mathcal{N}_2'))
\]

\[
= (\mathcal{M}_1' \otimes C I_{\mathcal{M}_2})' \cap (\mathcal{N}_1' \otimes \mathcal{N}_2')
\]

\[
= C I_{\mathcal{M}_1} \otimes \mathcal{N}_2'.
\]

Since \( \mathcal{N}_2 \) is completely singular in \( \mathcal{M}_2 \) and \( \theta(I_{\mathcal{M}_1} \otimes X_2) = I_{\mathcal{M}_1} \otimes X_2 \) for all \( X_2 \in \mathcal{M}_2' \), \( \theta(I_{\mathcal{M}_1} \otimes Y_2) = I_{\mathcal{M}_1} \otimes Y_2 \) for all \( Y_2 \in \mathcal{M}_2' \) by Theorem 19.1. Therefore, \( \theta(X_1 \otimes Y_2) = X_1 \otimes Y_2 \) for all \( X_1 \in \mathcal{M}_1' \) and \( Y_2 \in \mathcal{N}_2' \). By Theorem 4.3, \( \mathcal{N}_1' \otimes \mathcal{M}_2 \) is completely singular in \( \mathcal{M}_1' \otimes \mathcal{M}_2 \). Since \( \theta(X_1 \otimes Y_2) = X_1 \otimes Y_2 \) for all \( X_1 \in \mathcal{M}_1' \) and \( Y_2 \in \mathcal{N}_2' \), by Theorem 3.1, \( \theta(Y_1 \otimes Y_2) = Y_1 \otimes Y_2 \) for all \( Y_1 \in \mathcal{N}_1' \) and \( Y_2 \in \mathcal{N}_2' \). By Theorem 19.1 again, \( \mathcal{N}_1' \otimes \mathcal{N}_2' \) is completely singular in \( \mathcal{M}_1 \otimes \mathcal{M}_2 \).

Combining Theorem 20.4 and Corollary 17.2, we obtain the following corollary, which generalizes Corollary 4.4 of [40].

**Corollary 20.5.** If \( \mathcal{N}_1 \) is a singular subfactor of a type II\(_1\) factor \( \mathcal{M}_1 \) and \( \mathcal{N}_2 \) is a completely singular von Neumann subalgebra of \( \mathcal{M}_2 \), then \( \mathcal{N}_1' \otimes \mathcal{N}_2' \) is completely singular in \( \mathcal{M}_1' \otimes \mathcal{M}_2 \).
PART IV

TENSOR PRODUCTS OF MAXIMAL
INJECTIVE VON NEUMANN
SUBALGEBRAS
CHAPTER 21
BACKGROUND

In [24], Murray and von Neumann introduced and studied a family of factors of type II$_1$ very closely related to matrix algebras. Murray and von Neumann called these factors approximately finite since they are the ultraweak closure of the ascending union of a family of finite-dimensional self-adjoint subalgebras. They proved that all “approximately finite” factors of type II$_1$ are * isomorphic. Since these factors are finite, J. Dixmier [3] considered the term “approximately finite” inappropriate and called them hyperfinite. However, for infinite factors possessing the same property, the term “hyperfinite” is also inappropriate. So later on the name approximately finite dimensional (AFD) was introduced for these factors.

A von Neumann algebra $\mathcal{B}$ acting on a Hilbert space $\mathcal{H}$ is called injective if there is a norm one projection from $B(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$, onto $\mathcal{B}$. Since the intersection of a decreasing sequence of injective algebras is injective, and the commutant of an injective algebra is injective, every AFD factor is injective. In [2], A. Connes proved that a separable injective von Neumann algebra (von Neumann algebra with separable predual) is approximately finite dimensional. As a corollary, this shows that the hyperfinite type II$_1$ factor $\mathcal{B}$ is the unique separable injective factor of type II$_1$. The proof of Connes’ result is so deep and rich in ideas and techniques that it remains a basic resource in the subject.

Compared with injective factors, non-injective factors (even non-injective type II$_1$ factors) are far from being understood. A standard method of investigation in the study of general factors is to study the injective von Neumann subalgebras of these factors. Along this line, we have R. Kadison’s question (Problem 7 in [18]): Does each self-adjoint operator in a II$_1$ factor lie in some hyperfinite subfactor? Since every separable abelian von Neumann algebra is generated by a single self-adjoint operator, Kadison’s question has an equivalent form: Is each separable abelian von Neumann algebra of a II$_1$ factor contained in some hyperfinite subfactor?
This problem was answered in the negative in a remarkable paper [27] by S. Popa. In [27], Popa showed that if $\mathcal{L}(F_n)$ is the type II$_1$ factor associated with the left regular representation $\lambda$ of the free group $F_n$ on $n$ generators, $2 \leq n \leq \infty$, and $a$ is one of the generators of $F_n$, then the abelian von Neumann subalgebra generated by the unitary $\lambda(a)$ is a maximal injective von Neumann subalgebra of $\mathcal{L}(F_n)$. So quite surprisingly, a diffuse abelian von Neumann algebra can be embedded in a type II$_1$ factor as a maximal injective von Neumann subalgebra!

Popa raised the following question in [27]: If $\mathcal{M}_1, \mathcal{M}_2$ are type II$_1$ factors and $\mathcal{B}_1 \subseteq \mathcal{M}_1, \mathcal{B}_2 \subseteq \mathcal{M}_2$ are maximal injective von Neumann subalgebras, is $\mathcal{B}_1 \bar{\otimes} \mathcal{B}_2$ maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$? He also asked if this is true when we assume that $\mathcal{B}_1 = \mathcal{M}_1$ is the hyperfinite II$_1$ factor. This question, if answered in the affirmative, would considerably enlarge our class of examples.

For partial results on Popa’s question, we refer to [13, 41, 35]. In this Part we will develop a new approach to deal with Popa’s question, which enables us to answer Popa’s question affirmatively in a very general setting. Our main result is the following. Let $\mathcal{M}_i$ be a von Neumann algebra, and $\mathcal{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{M}_1$ has separable predual and the center of $\mathcal{B}_1$ is atomic, e.g., $\mathcal{B}_1$ is a factor, then $\mathcal{B}_1 \bar{\otimes} \mathcal{B}_2$ is maximal injective in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. 
CHAPTER 22
PRELIMINARIES

22.1 Injective von Neumann algebras

A conditional expectation $E$ from a von Neumann algebra $M$ onto a von Neumann subalgebra $N$ is a positive, linear mapping such that $E(S_1 T S_2) = S_1 E(T) S_2$ for all $S_1, S_2$ in $N$ and all $T$ in $M$. J. Tomiyama [42] showed that an idempotent of norm 1 from $M$ onto $N$ is a conditional expectation. A von Neumann algebra $B$ acting on a Hilbert space $H$ is called injective if there is a conditional expectation from $B(H)$ onto $B$. If $B$ is a von Neumann subalgebra of a von Neumann algebra $M$ and $B$ is injective, there is a conditional expectation from $M$ onto $B$.

Let $B$ be a von Neumann algebra acting on a Hilbert space $H$. Then $B$ is injective if and only if the commutant $B'$ of $B$ is injective. Recall that if $E$ is a projection in $B$, then the reduced von Neumann algebra of $B$ with respect to $E$ is the algebra $B_E = EBE$. If $B$ is injective and $E$ is a conditional expectation from $B(H)$ onto $B$, then $E$ induces a conditional expectation $E_E$ from $B(EH)$ onto $B_E$ by $E_E(T) = E(ETE)$ for any $T \in B(EH)$.

22.2 Maximal injective von Neumann subalgebras

Let $M$ be a von Neumann algebra. A von Neumann subalgebra $B$ of $M$ is called maximal injective if it is injective and if it is maximal with respect to inclusion in the set of all injective von Neumann subalgebras of $M$. If $\{B_\alpha\}$ is a family of injective von Neumann subalgebras of $M$ which is inductively ordered by inclusion, then the weak operator closure of $\bigcup_\alpha B_\alpha$ is an injective von Neumann subalgebra of $M$ which contains all $B_\alpha$. By Zorn's lemma, maximal injective von Neumann subalgebras of $M$ exist.
If \( \mathcal{M} \) is a separable type II\(_1\) factor, then \( \mathcal{M} \) contains a hyperfinite subfactor \( \mathcal{R} \) such that \( \mathcal{R} \cap \mathcal{M} = CI \) (Corollary 4.1 of [26]). If \( \mathcal{B} \) is a maximal injective von Neumann subalgebra of \( \mathcal{M} \) which contains \( \mathcal{R} \), then \( \mathcal{B}' \cap \mathcal{M} \subseteq \mathcal{R} \cap \mathcal{M} = CI \). In particular, \( \mathcal{B} \) is an injective factor. By [2], \( \mathcal{B} \) is hyperfinite. So every separable type II\(_1\) factor contains a hyperfinite subfactor as a maximal injective von Neumann subalgebra.

In [27], Popa exhibited concrete examples of maximal injective von Neumann subalgebras of type II\(_1\) factors. Popa showed that if \( \mathcal{L}(F_n) \) is the type II\(_1\) factor associated with the left regular representation \( \lambda \) of the free group \( F_n \) on \( n \) (\( 2 \leq n < \infty \)) generators, and if \( a \) is one of the generators of \( F_n \), then the abelian von Neumann algebra generated by the unitary \( \lambda(a) \) is a maximal injective von Neumann subalgebra of \( \mathcal{L}(F_n) \). In [11], Ge showed that each non-atomic injective von Neumann algebra with separable predual is maximal injective in its free product with any von Neumann algebra associated with a countable discrete group. Note that any maximal injective von Neumann subalgebra of a type II\(_1\) factor must be non-atomic.

If \( \mathcal{B} \) is a maximal injective von Neumann subalgebra of \( \mathcal{M} \), then \( \mathcal{B} \) is singular in \( \mathcal{M} \), i.e., its normalizers in \( \mathcal{M} \) are unitary elements in \( \mathcal{B} \). Indeed, if \( U \) is a unitary element in \( \mathcal{M} \) and \( UBU^* = \mathcal{B} \), then the von Neumann subalgebra of \( \mathcal{M} \) generated by \( \mathcal{B} \) and \( U \) is also injective. Since \( \mathcal{B} \) is maximal injective in \( \mathcal{M} \), \( U \in \mathcal{B} \). In particular, it follows that \( \mathcal{B}' \cap \mathcal{M} \subseteq \mathcal{B} \). Let \( \mathcal{Z} \) be the center of \( \mathcal{B} \). We have \( \mathcal{Z} \subseteq \mathcal{B}' \cap \mathcal{M} \subseteq \mathcal{B}' \cap \mathcal{B} = \mathcal{Z} ', \) which implies that \( \mathcal{Z} = \mathcal{B}' \cap \mathcal{M} \). We summarize these facts in the following lemma.

**Lemma 22.1.** Let \( \mathcal{B} \) be a maximal injective von Neumann subalgebra of \( \mathcal{M} \). Then \( \mathcal{B} \) is singular in \( \mathcal{M} \). In particular, \( \mathcal{Z} = \mathcal{B}' \cap \mathcal{B} = \mathcal{B}' \cap \mathcal{M} \).

### 22.3 Minimal injective von Neumann algebra extensions

Let \( \mathcal{N} \) be a von Neumann algebra. An injective von Neumann algebra \( \mathfrak{A} \) is called a *minimal injective von Neumann algebra extension* of \( \mathcal{N} \) if \( \mathfrak{A} \supseteq \mathcal{N} \) and if \( \mathfrak{A} \) is minimal with respect to inclusion in the set of all injective von Neumann algebras which contain \( \mathcal{N} \).

Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \) and \( \mathcal{B} \) be a maximal injective von Neumann subalgebra of \( \mathcal{M} \). Then \( \mathcal{B}' \), the commutant of \( \mathcal{B} \), is a minimal injective

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von Neumann algebra extension of $\mathcal{M}$. Indeed, if $\mathcal{L}$ is an injective von Neumann algebra such that $\mathcal{M} \subseteq \mathcal{L} \subseteq \mathcal{B}$, then $\mathcal{L} = (\mathcal{L}')'$ is an injective von Neumann algebra such that $\mathcal{B} \subseteq \mathcal{L} \subseteq \mathcal{M}$. Since $\mathcal{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, $\mathcal{B} = \mathcal{L}$. By von Neumann's double commutant theorem [45], $\mathcal{B}' = \mathcal{L}'$.

Let $\mathcal{A}$ be a minimal injective von Neumann algebra extension of a von Neumann algebra $\mathcal{N}$. Let $\varphi$ be a faithful normal representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Then $(\varphi(\mathcal{A}))'$ is a maximal injective von Neumann subalgebra of $(\varphi(\mathcal{N}))'$. Indeed, if $\mathcal{L}$ is an injective von Neumann algebra such that $(\varphi(\mathcal{A}))' \subseteq \mathcal{L} \subseteq (\varphi(\mathcal{N}))'$, then $\varphi(\mathcal{N}) \subseteq \mathcal{L} \subseteq \varphi(\mathcal{A})$ and $\mathcal{L}$ is injective. Thus $\mathcal{N} \subseteq \varphi^{-1}(\mathcal{L}) \subseteq \mathcal{A}$ and $\varphi^{-1}(\mathcal{L})$ is injective. Since $\mathcal{A}$ is a minimal injective von Neumann algebra extension of $\mathcal{N}$, $\varphi^{-1}(\mathcal{L}) = \mathcal{A}$. Hence $\mathcal{L} = \varphi(\mathcal{A})$ and $\mathcal{L}' = (\varphi(\mathcal{A}))'$.

In [14], U. Haagerup proved that any von Neumann algebra is *-isomorphic to a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, such that there is a conjugate linear, isometric involution $J$ of $\mathcal{H}$ and a self-dual cone $\mathcal{P}$ in $\mathcal{H}$ with the properties:

1. $J\mathcal{M}J = \mathcal{M}$,
2. $JZJ = Z^*$, for $Z$ in the center of $\mathcal{M}$,
3. $J\xi = \xi$, $\xi \in \mathcal{P}$,
4. $XJXJ(\mathcal{P}) \subseteq \mathcal{P}$ for all $X \in \mathcal{M}$.

A quadruple $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ satisfying the conditions 1-4 is called a standard form of the von Neumann algebra $\mathcal{M}$. Recall that a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ is said to be standard if there exists a conjugation $J : \mathcal{H} \to \mathcal{H}$, such that the mapping $X \to JX^*J$ is a * anti-isomorphism from $\mathcal{M}$ onto $\mathcal{M}'$. If $\mathcal{M}$ is standard on $\mathcal{H}$, we can choose $J$ and $\mathcal{P}$ in $\mathcal{H}$, such that $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ is a standard form (cf [14], Theorem 1.1). Let $\mathcal{M}$ be standard on $\mathcal{H}$, and $\theta$ be a *-automorphism of $\mathcal{M}$, then there is a unitary operator $U$ on $\mathcal{H}$ such that $\theta(X) = UXU^*$ for all $X \in \mathcal{M}$ (cf [14], Theorem 3.2).

The following lemma, which is of independent interest, is a key lemma.
Lemma 22.2. Let $\mathcal{N}$ be a von Neumann algebra and $\mathfrak{A}$ be a minimal injective von Neumann algebra extension of $\mathcal{N}$. If $\theta \in \text{Aut}(\mathfrak{A})$ (the group of all *-automorphisms of $\mathfrak{A}$) satisfies $\theta(X) = X$ for all $X \in \mathcal{N}$, then $\theta(Y) = Y$ for all $Y \in \mathfrak{A}$.

Proof. We can assume that $\mathfrak{A}$ is standard on a Hilbert space $\mathcal{H}$. Then there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\theta(Y) = UYU^*$ for all $Y \in \mathfrak{A}$. Since for all $X \in \mathcal{N}$ we have $\theta(X) = X$, $UXU^* = X$. Thus, $U \in \mathcal{N}'$. Define $\theta'(Y') = UY'U^*$ for $Y' \in \mathfrak{A}'$. Note that for all $Y \in \mathfrak{A}, Y' \in \mathfrak{A}'$, $\theta'(Y')\theta(Y) = \theta(Y)\theta'(Y')$. Since $\theta(\mathfrak{A}) = \mathfrak{A}$, $\theta'(Y') \in \mathfrak{A}'$ for all $Y' \in \mathfrak{A}'$, i.e., $U\mathfrak{A}'U^* \subseteq \mathfrak{A}'$. Note that $\theta^{-1}(Y) = U^*YU$ is also a *-isomorphism of $\mathfrak{A}$. Same arguments as above show that $U^*\mathfrak{A}U \subseteq \mathfrak{A}'$. So $U\mathfrak{A}'U^* = \mathfrak{A}'$. This implies that $U \in \mathcal{N}'$ is in the normalizer of $\mathfrak{A}'$. Note that $\mathfrak{A}'$ is maximal injective in $\mathcal{N}'$. By Lemma 22.1, $U \in \mathfrak{A}'$. So $\theta(Y) = UYU^* = Y$ for all $Y \in \mathfrak{A}$. \(\square\)

Corollary 22.3. If $\mathfrak{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, then $\mathfrak{B}$ is completely singular in $\mathcal{M}$.

Proof. We can assume that $\mathcal{M}$ acts on $\mathcal{H}$ in standard form. Then $\mathfrak{B}'$ is a minimal injective von Neumann algebra extension of $\mathfrak{B}'$. Let $\theta \in \text{Aut}(\mathfrak{B}')$ such that $\theta(X) = X$ for all $X \in \mathfrak{B}'$. Then $\theta(Y) = Y$ for all $Y \in \mathfrak{B}'$ by Lemma 22.2. By Theorem 19.1, $\mathfrak{B}$ is completely singular in $\mathcal{M}$. \(\square\)

Lemma 22.4. If $\mathfrak{A}$ is a minimal injective von Neumann algebra extension of a von Neumann algebra $\mathcal{N}$, and $P, Q$ are non-zero central projections in $\mathfrak{A}$ such that $PQ = 0$, then there does not exist a *-isomorphism $\phi$ from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$ such that $\phi(PX) = QX$ for all $X \in \mathcal{N}$.

Proof. Otherwise, assume $\phi$ is a *-isomorphism from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$ such that $\phi(PX) = QX$ for all $X \in \mathcal{N}$. For any $Y \in \mathfrak{A}$, $Y = PY + QY + (I - P - Q)Y$. Define $\theta$ from $\mathfrak{A}$ to $\mathfrak{A}$ by: $\theta(Y) = \phi(PY) + \phi^{-1}(QY) + (I - P - Q)Y$. Since $P, Q$ are mutually orthogonal central projections in $\mathfrak{A}$ and $\phi$ is a *-isomorphism from $\mathfrak{A}_P$ onto $\mathfrak{A}_Q$, $\theta \in \text{Aut}(\mathfrak{A})$. Note that for any $X \in \mathcal{N}$, $\theta(X) = \phi(PX) + \phi^{-1}(QX) + (I - P - Q)X = PX + QX + PX + (I - P - Q)X = X$. Since $\mathfrak{A}$ is a minimal injective von Neumann algebra extension of $\mathcal{N}$, by Lemma 22.2, $\theta(Y) = Y$ for all $Y \in \mathfrak{A}$. Therefore, $P = \theta(P) = \phi(P) = Q$. Now we have $P = PQ = 0$. This contradicts the assumption that $P \neq 0$. \(\square\)
Corollary 22.5. Let $\mathcal{A}$ be a minimal injective von Neumann algebra extension of a von Neumann algebra $\mathcal{N}$, and $P, Q$ be central projections in $\mathcal{A}$. If there is a $*$-isomorphism $\phi$ from $\mathcal{A}_P$ onto $\mathcal{A}_Q$ such that $\phi(PX) = QX$ for all $X \in \mathcal{N}$, then $P = Q$ and $\phi PY = PY$ for all $Y \in \mathcal{A}$.

Proof. Suppose $P \neq Q$. Let $R = PQ$. Without loss of generality, we can assume that $P_1 = P - R > 0$. Let $Q_1 = \theta(P_1) \leq Q$. Then $P_1, Q_1$ are non-zero central projections in $\mathcal{A}$ and $P_1Q_1 = Q_1P_1 = 0$. Since $\phi$ is a $*$-isomorphism from $\mathcal{A}_P$ onto $\mathcal{A}_Q$, $\phi$ induces a $*$-isomorphism $\psi$ from $\mathcal{A}_{P_1}$ onto $\mathcal{A}_{Q_1}$ such that $\psi(P_1 Y) = \phi(P_1 Y)$ for all $Y \in \mathcal{A}$. Since for any $X \in \mathcal{N}$, $\psi(P_1 X) = \phi(P_1)\phi(PX) = Q_1 QX = Q_1 X$. This contradicts Lemma 22.4. Thus $P = Q$. Define $\theta(Y) = \phi(PY) + (I - P)Y$, then $\theta \in Aut(\mathcal{A})$ and $\theta(X) = X$ for any $X \in \mathcal{N}$. Since $\mathcal{A}$ is a minimal injective von Neumann algebra extension of $\mathcal{N}$, by Lemma 22.2, $\theta(Y) = Y$ for all $Y \in \mathcal{A}$. Hence, $PY = \theta(PY) = \phi(PY)$ for all $Y \in \mathcal{A}$. \[\square\]

22.4 Maximal injective subfactors

In [8], B. Fuglede and Kadison established the existence of maximal hyperfinite subfactors of a type II$_1$ factor. Since a separable type II$_1$ factor is injective if and only if it is hyperfinite, a subfactor of a separable type II$_1$ factor is a maximal injective subfactor if and only if it is a maximal hyperfinite subfactor. Therefore, every separable type II$_1$ factor has maximal injective subfactors.

The following lemmas show the relation between maximal injective von Neumann subalgebras and maximal injective subfactors.

Lemma 22.6. Let $\mathcal{M}$ be a factor, and let $\mathcal{B}$ be a maximal injective subfactor of $\mathcal{M}$. Then $\mathcal{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$ if and only if $\mathcal{B}$ is irreducible in $\mathcal{M}$, i.e., $\mathcal{B} \cap \mathcal{M} = CI$.

Proof. If $\mathcal{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, then by Lemma 1.1, the center $\mathcal{Z}$ of $\mathcal{B}$ is $\mathcal{B} \cap \mathcal{M}$. Since $\mathcal{B}$ is a factor, $\mathcal{B} \cap \mathcal{M} = CI$. Conversely, suppose $\mathcal{B} \cap \mathcal{M} = \mathcal{C}$. For any injective von Neumann algebra $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{B} \subseteq \mathcal{M}$, we have $\mathcal{B} \cap \mathcal{M} = \mathcal{B} \cap \mathcal{M} = \mathcal{C}$. Therefore, $\mathcal{B}$ is an injective subfactor of $\mathcal{M}$. Since $\mathcal{B}$ is a maximal injective subfactor of $\mathcal{M}$, $\mathcal{B} = \mathcal{B}$. \[\square\]
Lemma 22.7. Let $\mathcal{M}$ be a factor, and let $\mathcal{B}$ be a maximal injective von Neumann subalgebra of $\mathcal{M}$. Let $\mathcal{Z}$ be the center of $\mathcal{B}$. Assume that $\mathcal{Z}$ is atomic and $P_1, P_2, \cdots$ are minimal projections in $\mathcal{Z}$ such that $\sum P_i = I$. Then $\mathcal{B}_i = \mathcal{B}_{P_i}$ is a maximal injective subfactor of $\mathcal{M}_{P_i}$ such that $\mathcal{B}_i \cap \mathcal{M}_{P_i} = CP_i$ for all $i$.

Proof. Since $\mathcal{Z}$ is atomic and $P_1, P_2, \cdots$, are minimal projections in $\mathcal{Z}$, $\mathcal{B}_i$ is a subfactor of $\mathcal{M}_{P_i}$. Since $\mathcal{B}$ is injective, $\mathcal{B}_i$ is injective. If $\mathcal{L}_i$ is an injective von Neumann algebra such that $\mathcal{B}_i \subseteq \mathcal{L}_i \subseteq \mathcal{M}_{P_i}$, then $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{L}_i \oplus \cdots$ is an injective von Neumann subalgebra of $\mathcal{M}$ such that $\mathcal{B} \subseteq \mathcal{B}_1 \oplus \cdots \oplus \mathcal{L}_i \oplus \cdots \subseteq \mathcal{M}$. Since $\mathcal{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, $\mathcal{B} = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{L}_i \oplus \cdots$. This implies that $\mathcal{B}_i = \mathcal{L}_i$. So $\mathcal{B}_i$ is a maximal injective von Neumann subalgebra of $\mathcal{M}_{P_i}$. Since $\mathcal{B}_i$ is a factor, $\mathcal{B}_i$ is irreducible in $\mathcal{M}_{P_i}$ by Lemma 22.6. □

22.5 On tensor products of von Neumann algebras

In [13], Ge and Kadison proved the following basic theorem for tensor products of von Neumann algebras.

Ge-Kadison's Splitting Theorem If $\mathcal{M}_1$ is a factor and $\mathcal{M}_2$ is a von Neumann algebra, and $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ which contains $\mathcal{M}_1 \bar{\otimes} \mathcal{N}_2$, then $\mathcal{M} = \mathcal{M}_1 \bar{\otimes} \mathcal{N}_2$ for some $\mathcal{N}_2$, a von Neumann subalgebra of $\mathcal{M}_2$.

The slice-map technique of Tomiyama [43] plays a key role in the proof of Ge-Kadison's splitting theorem. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras. With $\rho$ in $(\mathcal{M}_1)_\#$ (the predual of $\mathcal{M}_1$), $\sigma$ in $(\mathcal{M}_2)_\#$ and $T \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, the mapping $\sigma \to (\rho \bar{\otimes} \sigma)(T)$ is a bounded linear functional on $(\mathcal{M}_2)_\#$, hence, an element $\Psi_\rho(T)$ in $\mathcal{M}_2$. Symmetrically, we construct an operator $\Phi_\sigma(T)$ in $\mathcal{M}_1$. The mappings $\Psi_\rho$ and $\Phi_\sigma$ are referred to as slice mappings (of $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ onto $\mathcal{M}_2$ and $\mathcal{M}_1$ corresponding to $\rho$ and $\sigma$, respectively). Tomiyama's Slice Mapping Theorem [43] says that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are von Neumann subalgebras of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and $T \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, then $T \in \mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$ if and only if $\Phi_\sigma(T) \in \mathcal{N}_1$ and $\Psi_\rho(T) \in \mathcal{N}_2$ for each $\sigma \in (\mathcal{M}_2)_\#$ and $\rho \in (\mathcal{M}_1)_\#$. For a generalization of Ge-Kadison's splitting theorem, we refer to [41].

The following lemma is well-known.
Lemma 22.8. If $\mathcal{M}_1, \mathcal{N}_1$ are von Neumann algebras acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}_2, \mathcal{N}_2$ are von Neumann algebras acting on a Hilbert space $\mathcal{K}$, then $(\mathcal{M}_1 \otimes \mathcal{M}_2) \cap (\mathcal{N}_1' \otimes \mathcal{N}_2') = (\mathcal{M}_1 \cap \mathcal{N}_1') \otimes (\mathcal{M}_2 \cap \mathcal{N}_2')$. 
CHAPTER 23

INDUCED CONDITIONAL EXPECTATIONS

Lemma 23.1. Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{L}, \mathcal{N}$ be von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{L} \subseteq \mathcal{N}$. If $E$ is a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$, then $E$ induces a conditional expectation from $\mathcal{L} \cap \mathcal{M}$ onto $\mathcal{L} \cap \mathcal{N}$.

Proof. For $S \in \mathcal{L} \cap \mathcal{M}$ and $T \in \mathcal{L}$, $ST = TS$. Apply the conditional expectation $E$ to both sides of $ST = TS$ and note that $\mathcal{L} \subseteq \mathcal{N}$. We have $E(S)T = TE(S)$. Thus $E(S) \in \mathcal{L} \cap \mathcal{N}$. Since $\mathcal{L} \cap \mathcal{N} \subseteq \mathcal{L} \cap \mathcal{M}$, $E$ is a conditional expectation from $\mathcal{L} \cap \mathcal{M}$ onto $\mathcal{L} \cap \mathcal{N}$ when $E$ is restricted on $\mathcal{L} \cap \mathcal{M}$. □

The following is another key lemma.

Lemma 23.2. Let $\mathcal{A}_i$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_i$, and $\mathcal{N}_i$ be a von Neumann subalgebra of $\mathcal{A}_i$, $i = 1, 2$. Let $\mathcal{A}$ be a von Neumann algebra such that $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2 \subseteq \mathcal{A} \subseteq \mathcal{A}_1 \bar{\otimes} \mathcal{A}_2$. If there is a conditional expectation $E$ from $\mathcal{A}_1 \bar{\otimes} \mathcal{A}_2$ onto $\mathcal{A}$, then $E$ induces a conditional expectation from $(\mathcal{N}_1' \cap \mathcal{A}_1) \bar{\otimes} \mathcal{A}_2$ onto $((\mathcal{N}_1' \cap \mathcal{A}_1) \bar{\otimes} \mathcal{A}_2) \cap \mathcal{A}$.

Proof. By Lemma 22.8, $(\mathcal{A}_1 \bar{\otimes} \mathcal{A}_2) \cap (\mathcal{N}_1 \bar{\otimes} \mathcal{N}_1)' = (\mathcal{A}_1 \bar{\otimes} \mathcal{A}_2) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_1) = (\mathcal{N}_1' \cap \mathcal{A}_1) \bar{\otimes} \mathcal{A}_2$. By Lemma 23.1, $E$ induces a conditional expectation from $(\mathcal{N}_1' \cap \mathcal{A}_1) \bar{\otimes} \mathcal{A}_2$ onto $((\mathcal{N}_1' \cap \mathcal{A}_1) \bar{\otimes} \mathcal{A}_2) \cap \mathcal{A}$. □

Corollary 23.3. Assume the conditions of Lemma 23.2 and $\mathcal{N}_1' \cap \mathcal{A}_1 = CI$. Let $\mathcal{L}_2 = \{T \in \mathcal{A}_2 : I \otimes T \in \mathcal{A}\}$. Then $E$ induces a conditional expectation from $\mathcal{A}_2$ onto $\mathcal{L}_2$.

As an application of Lemma 23.2 and Corollary 23.3, we give a new proof of Ge-Kadison's splitting theorem in the case when $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite. Let $\mathcal{N}$ be a von Neumann algebra such that $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \subseteq \mathcal{N} \subseteq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. Then there is a normal conditional expectation $E$ from $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ onto $\mathcal{N}$. By Corollary 2.3, $E$ induces a conditional expectation, denoted by
\( E_2 \), from \( \mathcal{M}_2 \) onto \( \mathcal{N}_2 \triangleq \{ T \in \mathcal{M}_2 : I \otimes T \in \mathcal{N} \} \). Now for any \( S \in \mathcal{M}_1, T \in \mathcal{M}_2 \), we have \( E(S \otimes T) = S \otimes E_2(T) \in \mathcal{M}_1 \otimes \mathcal{N}_2 \). Since \( E \) is normal, \( \mathcal{N} = E(\mathcal{M}_1 \otimes \mathcal{M}_2) \subseteq \mathcal{M}_1 \otimes \mathcal{N}_2 \). Since \( \mathcal{N} \supseteq \mathcal{M}_1 \otimes \mathcal{N}_2, \mathcal{N} = \mathcal{M}_1 \otimes \mathcal{N}_2 \).

As another application of Lemma 23.2 and Corollary 23.3, we give a new proof of Theorem 6.7 of [41].

**Lemma 23.4.** Let \( \mathfrak{A} \) be an abelian von Neumann algebra, and \( \mathfrak{B}_2 \) be a minimal injective von Neumann algebra extension of a von Neumann algebra \( \mathcal{N} \). Suppose \( \mathfrak{B}_2 \) has separable predual. If \( \mathfrak{A} \) is an injective von Neumann algebra such that \( \mathfrak{A} \otimes \mathcal{N} \subseteq \mathfrak{A} \otimes \mathfrak{B}_2 \), then \( \mathfrak{A} = \mathfrak{A} \otimes \mathfrak{B}_2 \).

**Proof.** We can assume that \( \mathfrak{A} \) and \( \mathfrak{B}_2 \) are von Neumann algebras acting on Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), respectively, in standard form. Then \( \mathfrak{A} = \mathfrak{A} \) and \( \mathcal{N} \) is a separable Hilbert space. By \( \mathfrak{A} \otimes \mathcal{N} \subseteq \mathfrak{A} \subseteq \mathfrak{A} \otimes \mathfrak{B}_2 \), we have \( \mathfrak{A} \otimes \mathfrak{B}_2 \subseteq \mathfrak{A} \subseteq \mathfrak{A} \otimes \mathcal{N} \). Note that \( \mathfrak{B}'_2 \) is a maximal injective von Neumann subalgebra of \( \mathcal{N} \). By Lemma 6.6 of [S-Z], \( \mathfrak{A}' = \mathfrak{A} \otimes \mathcal{N}' \). Therefore, \( \mathfrak{A} = \mathfrak{A} \otimes \mathfrak{B}_2 \).

Lemma 23.4 is almost obvious in the case when \( \mathfrak{A} \) is atomic. If \( \mathfrak{A} \) is diffuse, it is natural to consider direct integrals. The proof of Lemma 6.6 of [41] is based on direct integrals. It would be interesting if there is a “global proof” of Lemma 23.4. Is Lemma 23.4 true without the assumption that \( \mathfrak{B}_2 \) has separable predual?

**Theorem 23.5.** Let \( \mathcal{M}_1 \) be an injective von Neumann algebra and \( \mathcal{M}_2 \) be a von Neumann algebra with separable predual. If \( \mathcal{B}_2 \) is a maximal injective von Neumann subalgebra of \( \mathcal{M}_2 \), then \( \mathcal{M}_1 \otimes \mathcal{B}_2 \) is a maximal injective von Neumann subalgebra of \( \mathcal{M}_1 \otimes \mathcal{M}_2 \).

**Proof.** We can assume that \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are von Neumann algebras acting on Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), respectively. Then \( \mathcal{N} \) is a separable Hilbert space. Let \( \mathfrak{A} \) be the center of \( \mathcal{M}_1 \). Suppose \( \mathcal{B} \) is an injective von Neumann algebra such that \( \mathcal{M}_1 \otimes \mathcal{B}_2 \subseteq \mathcal{B} \subseteq \mathcal{M}_1 \otimes \mathcal{M}_2 \). Then we have \( \mathcal{M}_1 \otimes \mathcal{B}_2 ' \supseteq \mathcal{B} ' \supseteq \mathcal{M}_1 \otimes \mathcal{M}_2 ' \). Since \( \mathcal{B} ' \) is an injective von Neumann subalgebra of \( \mathcal{M}_1 \otimes \mathcal{B}_2 ' \), there is a conditional expectation \( E \) from \( \mathcal{M}_1 \otimes \mathcal{B}_2 ' \) onto \( \mathcal{B} ' \). By Lemma 23.2, \( E \) induces a conditional expectation from \( \mathfrak{A} \otimes \mathcal{B}_2 ' \) onto \( \mathfrak{A} \triangleq (\mathfrak{A} \otimes \mathcal{B}_2 ') \cap \mathcal{B} ' \). So \( \mathfrak{A} \) is an injective von Neumann algebra such that \( \mathfrak{A} \otimes \mathcal{B}_2 ' \supseteq \mathfrak{A} \supseteq \mathfrak{A} \otimes \mathcal{M}_2 ' \). Since \( \mathcal{B}_2 \) is a maximal injective...
von Neumann subalgebra of $\mathcal{M}_2$, $\mathcal{B}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2$. By Lemma 23.4, $\mathcal{A} = \mathcal{A} \bar{\otimes} \mathcal{B}_2'$. Thus $\mathcal{C} \bar{\otimes} \mathcal{B}_2' \subseteq \mathcal{A} \subseteq \mathcal{B}'$. So $\mathcal{B}' = \mathcal{M}_1 \bar{\otimes} \mathcal{B}_2'$ and $\mathcal{B} = \mathcal{M}_1 \bar{\otimes} \mathcal{B}_2$. $\square$
CHAPTER 24

MAIN THEOREM

Theorem 24.1. Let $\mathcal{M}_i$ be a von Neumann algebra, and $\mathcal{B}_i$ be a maximal injective von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{M}_1$ has separable predual and the center of $\mathcal{B}_1$ is atomic, then $\mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$ is maximal injective in $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.

Proof. Let $\mathcal{B}$ be an injective von Neumann algebra such that $\mathcal{B}_1 \overline{\otimes} \mathcal{B}_2 \subseteq \mathcal{B} \subseteq \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$. We can assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. To prove the theorem, we need to show that $\mathcal{B} = \mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$. Using Theorem 23.5, it is sufficient to prove that $\mathcal{B} \subseteq \mathcal{M}_1 \overline{\otimes} \mathcal{B}_2$ and so, it is sufficient to prove that $\mathcal{B}_1 \overline{\otimes} \mathcal{B}_2' \subseteq \mathcal{B}'$.

Denote $\mathcal{A} = \mathcal{B}_1 \cap \mathcal{B}_2' = \mathcal{M}_1 \cap \mathcal{B}_2'$, which is atomic, with the set of minimal projections $\{P_n : n = 1, 2, \ldots\}$. Set $\mathfrak{A} = \mathcal{B}' \cap (\mathcal{A} \overline{\otimes} \mathcal{B}_2') = \mathcal{B}' \cap (\mathcal{M}_1 \overline{\otimes} \mathcal{A})$. Then $\mathfrak{A}$ is injective and $\mathcal{C} \overline{\otimes} \mathcal{M}_1 \subseteq \mathfrak{A} \subseteq \mathcal{A} \overline{\otimes} \mathcal{B}_2'$.

Note that $P_n \otimes I \in \mathfrak{A}'$ for every $n$. Since $\mathcal{B}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2'$, we have $\mathfrak{A}(P_n \otimes I) = P_n \otimes \mathcal{B}_2'$ for every $n$. Denote by $Z_n$ the smallest projection in $\mathcal{L}(\mathfrak{A})$ satisfying $P_n \otimes I \leq Z_n$. We get $\ast$-isomorphisms $\theta_n : \mathcal{B}_2' \to \mathfrak{A}Z_n$ uniquely determined by the formula

$$\theta_n(Y)(P_n \otimes I) = P_n \otimes Y \quad \text{for all } Y \in \mathcal{B}_2'.$$

Since $\mathcal{C} \overline{\otimes} \mathcal{M}_1 \subseteq \mathfrak{A}$, it follows that $\theta_n(X) = (I \otimes X)Z_n$ for all $X \in \mathcal{M}_1$.

So, for every $n, m$ and all $X \in \mathcal{M}_2'$, we have $\theta_n(X)Z_m = \theta_m(X)Z_n$. Since $\mathcal{B}_2'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}_2'$, by Corollary 22.5, the same formula holds for all $Y \in \mathcal{B}_2'$ and all $n, m$. This compatibility formula yields for every $Y \in \mathcal{B}_2'$ an element $A \in \mathfrak{A}$ such that $AZ_n = \theta_n(Y)$ for all $n$. In particular, $A(P_n \otimes I) = P_n \otimes Y$, i.e., $A = I \otimes Y$. So, $\mathcal{C} \overline{\otimes} \mathcal{B}_2' \subseteq \mathfrak{A}$, ending the proof. \qed
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