Ion thermalization and wave excitation downstream of Earth's bow shock: Theory and observation

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Ion thermalization and wave excitation downstream of Earth's bow shock: Theory and observation

Abstract
It has been well documented that the plasma immediately downstream of Earth's quasi-perpendicular bow shock, which consists of reflected protons and directly transmitted ions with large temperature anisotropies, is unstable to the excitation of ion-cyclotron waves. These waves in turn scatter the protons and ions to marginal stability. A quasilinear theory is presented for the relaxation of the proton and helium distribution functions and the associated excitation of ion cyclotron waves, downstream of the low-Mach-number quasi-perpendicular Earth's bow shock. For a plasma with low density of He 2+ ions, the theory predicts the wave polarization, power and peak frequency, and the proton bulk velocity and temperature anisotropy, sufficiently far downstream of the shock that the ions and waves have relaxed to a quasi-equilibrium, and the time scale for the relaxation. The results except for the time scale are compared with the AMPTE/IRM crossings of the marginally supercritical bow shock documented by Sckopke et al. [1990], for which the number of "reflected" protons is small and the quasilinear approximation is expected to be valid and He2+ ions are negligible. The agreement with the observations except for the total wave power is generally very good if the contribution of the transmitted core protons is included.

Some of He2+ ions in the downstream plasma diffuse in v1 (velocity in the direction perpendicular to the ambient magnetic field) due to stochastic scattering. The second part of the theory predicts the time evolution of temperature anisotropy for the relaxation of the He2+ ions downstream of the shock and estimates the waves spectrum excited by the protons and He2+ ions.

We also present Cluster data following the inbound shock crossing at 17:17:48 on 31 March 2001, which is an event with higher concentration of He2+ ions. The observed results show that some of the alpha particles are heated perpendicular to the magnetic field as predicted. The predicted evolution of temperature anisotropy, the general shape of wave spectrum, and the time scale match the observed quantities remarkably well though some of the detailed feature for the evolution of the wave spectrum needs further work.

Keywords
Physics, Fluid and Plasma, Physics, Astronomy and Astrophysics

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ION THERMALIZATION AND WAVE EXCITATION DOWNSTREAM OF EARTH'S BOW SHOCK: THEORY AND OBSERVATION

BY

YONG LIU

B.S. Wuhan University, 1994
M.S. Wuhan University, 1997

DISSERTATION

Submitted to the University of New Hampshire
In Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy

In
Physics

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ABSTRACT

ION THERMALIZATION AND WAVE EXCITATION DOWNSTREAM OF EARTH’S BOW SHOCK: THEORY AND OBSERVATION

by

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University of New Hampshire, December, 2006

It has been well documented that the plasma immediately downstream of Earth’s quasi-perpendicular bow shock, which consists of reflected protons and directly transmitted ions with large temperature anisotropies, is unstable to the excitation of ion-cyclotron waves. These waves in turn scatter the protons and ions to marginal stability. A quasilinear theory is presented for the relaxation of the proton and helium distribution functions and the associated excitation of ion cyclotron waves, downstream of the low-Mach-number quasi-perpendicular Earth’s bow shock. For a plasma with low density of He$^{2+}$ ions, the theory predicts the wave polarization, power and peak frequency, and the proton bulk velocity and temperature anisotropy, sufficiently far downstream of the shock that the ions and waves have relaxed to a quasi-equilibrium, and the time scale for the relaxation. The results except for the time scale are compared with the AMPTE/IRM crossings of the marginally supercritical bow shock documented by Sckopke et al. [1990], for which the number of “reflected” protons is small and the quasilinear approximation is expected to be valid and He$^{2+}$ ions are negligible. The agreement with
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Some of He\(^{2+}\) ions in the downstream plasma diffuse in \(v_{\perp}\) (velocity in the
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We also present Cluster data following the inbound shock crossing at 17:17:48 on
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results show that some of the alpha particles are heated perpendicular to the magnetic
field as predicted. The predicted evolution of temperature anisotropy, the general shape
of wave spectrum, and the time scale match the observed quantities remarkably well
though some of the detailed feature for the evolution of the wave spectrum needs further
work.
CHAPTER 1

INTRODUCTION

In a homogenous gas, a disturbance propagates at a certain speed $v_s$ determined by the density and temperature of the gas. This is the sound speed. For example, if a rocket moves in a gas with speed $v$, the disturbance caused by the rocket propagates in the gas with speed $v_s$. Figure 1-1a shows that when $v \ll v_s$, a sample of wavefronts caused by the rocket propagate away from the source in all directions with almost equal spacing. The wavefronts of the disturbances are compressed in front of the rocket as shown in Figure 1-1b, however, when $v$ is comparable with, but less than $v_s$. If $v = v_s$, the wavefronts pile up in front of the rocket as shown in Figure 1-1c.

Figure 1-1 The wavefronts generated by a rocket moving with (a) $v \ll v_s$, (b) $v_s \approx v$, (c) $v = v_s$, (d) $v > v_s$. 

-1-
A shock will be formed if \( v > v_s \) as shown in Figure 1-1d. The gas in front of the rocket is compressed and "shocked" while the gas further ahead of the rocket is "unshocked" since it cannot sense the approach of the rocket. The disturbance in front of the shock is forced to propagate at \( v (> v_s) \). If we view the gas flow from the frame of the rocket, the gas moves supersonically against the obstacle, in this case the rocket, and a shock is formed in front of the obstacle. The unshocked gas is upstream of the shock and the shocked and compressed gas is downstream of the shock.

The solar wind is a fully-ionized magnetized plasma flowing approximately radially outwards from the solar corona. The temperature in the solar corona is sufficiently high, and the outward energy flux sufficiently large, that the Sun's gravity cannot retain a static atmosphere. The result in magnetically open regions of the corona is the escape of the solar wind. The wind is comprised of electrons, protons, helium ions and a trace amount of all other ions. The mass density of the protons and helium ions is greater than 99% of the total mass density. The speed of the solar wind at Earth orbit varies from 350 km/s to 800 km/s with an average value of \( \sim 400 \) km/s. Low frequency disturbances in the solar wind propagate at the speeds comparable Alfvén and/or sound speed (which are similar at Earth orbit) since the plasma is generally magnetized. At Earth the solar wind encounters the Earth's magnetic field. Since \( V_f \sim v_s \sim 50 \) km/s in the solar wind of Earth orbit, the solar wind flow is highly supersonic and a shock forms in front of Earth's magnetosphere, which we call Earth's bow shock.

The strength of the shock is determined by the ratio of the solar wind speed \( V \) to the Alfvén speed \( V_A \), which we call the Alfvén Mach number \( M_A \). If \( M_A \) is larger than a certain number \( M_c \) which is less than \( \sim 2.7 \), a small fraction of the protons is reflected
from an electrostatic potential that forms in the shock front. Since the plasma is magnetized, the magnetic field, in particular, the angle between the upstream magnetic field and the shock normal $\theta_{Bn}$, is another important parameter for the structure of Earth's bow shock. Figure 1-2 shows a schematic diagram of Earth's bow shock for a typical orientation of the interplanetary magnetic field. Above the nose of the shock in this diagram generally $\theta_{Bn} < 45^\circ$; we call these portions of the shock quasi-parallel. Below the nose, generally $\theta_{Bn} > 45^\circ$; these portions are quasi-perpendicular. The protons reflected from a quasi-perpendicular shock generally gyrate back to the shock front and pass through the shock plane to the downstream region of the shock. These downstream reflected protons excite waves, and the waves in turn scatter the protons toward a marginally stable distribution function further downstream.

The reflected protons were investigated using data from the ISEE Mission in the late 1970s and early 1980s. The major objective of the ISEE Mission was to understand the origin of the waves and particles at Earth's bow shock. These investigations and subsequent studies have had significant theoretical impact.

In addition to the reflected and transmitted protons, the He$^{2+}$ ions pass through the shock and are decelerated less than the directly transmitted protons. Their resulting distribution function is also unstable to the generation of ion-cyclotron waves. Generally, the Mach number of Earth's bow shock is very large and the ion relaxation/“thermalization” is too rapid to be tracked by instruments onboard a spacecraft. Sckopke et al. [1990] investigated the downstream reflected protons and their interaction with ion-cyclotron waves at a sample of low Mach number shocks observed by the AMPTE/IRM spacecraft. These events were chosen because the percentage of reflected protons and the
excited wave power is low so that the ion thermalization evolves slowly and is resolvable by the spacecraft. The downstream ion temperature anisotropy and the wave power spectrum are documented by Sckopke et al. [1990] for these events.

**Figure 1-2** Earth's bow shock and the interplanetary magnetic field.

We first applied a theory called quasilinear theory to describe the nonlinear wave-ion interaction at all the events documented by Sckopke et al. [1990]. In Section 2 we present their key observations and our theory; the predictions and the observations match remarkably well.

The AMPTE/IRM data, however, have two limitations. One is that the instruments do not have the ability to distinguish different species of ions. The other is
that we cannot determine the distance from the shock to the spacecraft with a single spacecraft. Thus we cannot determine the timescales of the relaxation process.

With the four Cluster spacecraft we can determine the shock speed (and orientation) as it passes the spacecraft, and therefore we can approximate the distance from the shock to the spacecraft when they are downstream of the shock in order to determine the timescale of ion thermalization. In addition, the CODIF (Composition and Distribution Function Analyzer) instrument on board Cluster can distinguish the He$^{2+}$ ions from protons and measure their distribution function separately. We discovered in one inbound shock crossing that the observed temperature and temperature anisotropy of the He$^{2+}$ ions increase downstream of the shock, which shows that the He$^{2+}$ ions are accelerated perpendicular to the ambient magnetic field. We have developed a theory to predict the evolution of the helium temperature anisotropy when these ions interact with the ion-cyclotron waves. We also account for the timescale of the interaction of the waves with the protons and helium ions, and the wave spectrum generated during this process.

The thesis is arranged as follows: In Chapter 2 we present the theory which predicts the quasi-equilibrium configuration of ions and waves which are excited downstream of the shock for the events documented by Sckopke et al. [1990]. The predictions and the observations match very well. The Cluster observations, and the theory for the acceleration and relaxation of helium ions and the excitation of the associated waves, are presented in Chapter 3. Chapter 2 is based on a paper published on Journal of Geophysics Research. [Liu et al., 2005]. Chapter 3 is another paper which will
be submitted to Journal of Geophysics Research. General conclusions about this area of research are drawn in Chapter 4.
CHAPTER 2

A QUASILINEAR THEORY OF ION "THERMALIZATION" AND WAVE EXCITATION DOWNSTREAM OF EARTH'S BOW SHOCK

2.1 Introduction

The main features of the structure of Earth's bow shock are determined by the characteristic shock parameters: the Alfvén Mach number \( M_A \) (at the nose of the shock, the ratio of the incident solar wind speed to the Alfvén speed \( V_A \)), \( \theta_{bn} \) [the angle \( (0 \leq \theta_{bn} \leq 90^\circ) \) between the shock normal and the ambient upstream magnetic field], and the upstream plasma-\( \beta \), which is the ratio of thermal pressure to magnetic pressure. Early studies showed that there exists a critical Mach number \( M_C \) such that when \( M_A > M_C \) resistivity alone cannot provide all the necessary dissipation at the shock. In this case, a fraction of the protons incident on the shock are reflected by a combination of electrostatic and magnetic forces in the shock foot, ramp, and overshoot. The viscosity of these reflected protons provides the additional dissipation. Such a shock is called supercritical [Kennel et al., 1985]. Later studies found that the transition of a shock from subcritical to supercritical is not abrupt; a small number of reflected protons and small magnetic overshoots exist in subcritical shocks with Mach number slightly smaller than \( M_C \) [Greenstadt and Mellott, 1987; Mellott and Livesey 1987]. For a supercritical quasi-perpendicular shock \( (\theta_{bn} \geq 45^\circ) \) most of the reflected protons gyrate...
back toward the shock, gain energy from the motional electric field and are transmitted through the shock ramp into the region downstream of the shock. Since these transmitted reflected protons have a velocity very different from the downstream bulk flow velocity, their motion is dominated by gyration about the magnetic field. The temperature anisotropy $T_{\perp}/T_{\parallel}$ (the subscripts refer to the directions relative to the ambient magnetic field) of the reflected protons just downstream of the shock is therefore large and this distribution is unstable. Waves are generated downstream of the shock which, in turn, scatter the reflected protons and reduce the temperature anisotropy. This process can be viewed as the relaxation of the free energy associated with the temperature anisotropy by wave excitation. At some distance downstream of the shock, the plasma will reach a quasi-equilibrium where the anisotropy and wave spectrum saturate. Early observations [e.g., Sckopke et al., 1983; Livesey et al., 1984], simulations [Papadopoulos et al., 1971; Leroy et al., 1981, 1982] and analytical study [Leroy, 1983] established the existence and basic behavior of the reflected protons (see also reviews by Gosling and Robson [1985] and Goodrich [1985], and references therein). Goodrich [1985] raised the question of how the reflected protons are thermalized downstream, while Tanaka [1985] investigated the electromagnetic waves driven by ion temperature anisotropy using a 1-D hybrid-code simulation. Winske and Quest [1988] extended the investigation of downstream thermalization using a 2-D simulation. These studies established the qualitative description of the downstream relaxation/thermalization process given above.

At Earth's bow shock, which is usually a high-Mach-number shock [Russell et al., 1982], the relaxation process generally progresses too rapidly to be resolved by spacecraft measurements. In order to investigate the relaxation process in more detail,
Sckopke et al. [1990] selected several lower-Mach-number marginally supercritical crossings of Earth's bow shock in the AMPTE/IRM data, which exhibit a slower evolution of the proton-wave interaction in a parcel of downstream plasma. Their data include time profiles of the ambient magnetic field, the magnetic fluctuation power, plasma fluid parameters, and the temperature anisotropy for each crossing. The data show that with increasing distance downstream of the shock the temperature anisotropy decreases to a "residual" value, which is larger than unity, and the magnetic fluctuation intensity, which is dominated by left-hand circularly polarized fluctuations, increases to a stable level.

Simulations performed by Winske and Quest [1988] and McKean et al. [1995] suggest that there are two possible instabilities responsible for the enhanced fluctuations: the Alfvén ion-cyclotron (AIC) and the mirror instabilities. The instability criterion for the mirror mode generally requires that the plasma-$\beta > 1$. In the events investigated by Sckopke et al. [1990], in which $V_A$ tends to be large, $\beta < 1$. Simulations by Yoon [1992] show that, even if the mirror mode is excited, the ion-cyclotron mode saturates at a much larger intensity compared with the mirror mode though initially the mirror mode may grow at a slightly faster rate. In addition, the mirror mode excites primarily longitudinal fluctuations of the magnetic field, as observed downstream of Earth's bow shock for an event with $\beta > 1$ by Czaykowska et al. [1998]. We conclude that the mirror mode plays a negligible role in the events described by Sckopke et al. [1990] and that the fluctuations are ion-cyclotron waves, consistent with the observed left-hand transverse polarization.
It is worth noting that the interaction between the reflected protons and the downstream ion-cyclotron waves is similar to that of pickup ions in the solar wind with the ambient hydromagnetic turbulence. Initially both of these ion populations have a bulk flow relative to the ambient plasma and a temperature anisotropy. The interaction of MHD waves and pickup ions in the solar wind was investigated by Lee and Ip [1987], who calculate the excited magnetic fluctuation power according to quasilinear theory. In their approach the wave kinetic equation and the proton pitch-angle diffusion equation are considered together under the assumption that the particle speed \( v \gg V_A \). The asymptotic power spectrum is obtained by combining these two equations and noting that the ion distribution is isotropic to zeroth order in \( V_A/v \) as \( t \to \infty \).

The purpose of this chapter is to present a quasilinear theory for the wave excitation and ion “thermalization” (or isotropization) downstream of Earth's bow shock and compare the results with the observations of Sckopke et al. [1990]. Restricting our study to low plasma-\( \beta \), perpendicular and quasi-perpendicular marginally supercritical shocks, we expect quasilinear theory to be valid because the number density of the reflected protons \( n_r \) is small compared with that of the electrons \( N_e \), and the excited wave amplitude is small compared with the ambient magnetic field. We calculate the power, polarization, and peak frequency of the magnetic fluctuation spectrum in the quasi-equilibrium, and the corresponding bulk velocity and residual temperature anisotropy of the protons according to this theory. First we consider the simple case in which wave dispersion is neglected, \( v \gg V_A \), and the reflected protons control the wave excitation. Then we repeat the calculation including wave dispersion and allowing \( v \) to be comparable with \( V_A \). We find that dispersion is an important correction. We also
derive the distribution function of the core protons just downstream of the shock, and calculate the residual temperature anisotropy of the core protons far downstream and the wave spectrum excited by them. The reflected and core protons account for the observed wave spectrum quite well, possibly including a secondary peak in the observed spectrum. We also discuss the contribution of He$^{2+}$ to the downstream wave intensity.

The chapter is arranged as follows: Section 2.2 reviews the observations. Section 2.3 considers the trajectory of the reflected protons. In Section 2.4 we present the simple calculation and discuss the results of this approach. In Section 2.5, for a perpendicular shock, we calculate the temperature anisotropy and wave power allowing $\nu \sim V_A$ which requires wave dispersion. In Section 2.6 we calculate the distribution function of the core protons just downstream of the shock and the wave spectrum excited by the relaxation of the core protons. The waves excited by the He$^{2+}$ ions are estimated in Section 2.7 and conclusions are presented in Section 2.8.

### 2.2 Observation

The data presented by Sckopke et al. [1990] was obtained with the plasma instrument and magnetometer on board the AMPTE/IRM spacecraft in the time period between September 5, 1984 and November 2, 1984. Most of the events chosen during this period are marginally supercritical shock crossings which are characterized by a small relative density ($n_r/N_e$) of the reflected protons in the range 3%-9%. In these events, identified as Events 1, 2, 3, 5 and 9 by Sckopke et al. [1990], the spacecraft crossed the shock near the nose of the bow shock, so that the normal component of the
velocity incident on the shock satisfies $V \equiv V_{sw}$. Data was also presented for one high-Mach-number event (Event X). In Event X the spacecraft crossed the shock at a position where the angle between the solar wind velocity and the inferred shock normal in the GSE frame is $-30^\circ$ so $V = (\sqrt{3}/2)V_{sw}$.

The plasma instrument measures the 3-D ion and electron distribution functions in velocity space. The proton temperature anisotropy is calculated from the measured distribution and is presented as a function of time for all the events. The data shows a large anisotropy just downstream of the shock, which decreases gradually to a residual value greater than unity in the downstream flow. We assume that the residual anisotropy corresponds to the value in the quasi-equilibrium. We shall later show that this assumption is supported by the observations. Several proton phase-space distributions represented by two-dimensional cuts are also presented for each event.

The magnetometer measures the magnetic field with a resolution of 0.1 nT. The magnetic fluctuation power spectrum is decomposed into left-hand- and right-hand-polarized transverse waves, and fluctuations parallel to the ambient magnetic field, and is presented for some of the events. The unstable waves tend to have wavevectors aligned with the magnetic field $\mathbf{B}$ [Gary and Winske, 1986]. Since $\mathbf{B}$ is approximately perpendicular to the plasma flow velocity, the polarization and frequencies measured in the spacecraft frame of reference are very close to those in the plasma rest frame as pointed out by Sckopke et al. [1990]. The spectral densities show that the magnetic fluctuations are mainly enhanced within the frequency band $0.3\Omega - 0.8\Omega$, where $\Omega$ is the downstream proton gyrofrequency. Integration of the magnetic fluctuation intensity $I(\omega)$ over this frequency range is then taken to represent the total power, namely

- 12
\[ P = \int_{0}^{\infty} I(\omega) d\omega \] Adjacent to, and downstream of, the marginally supercritical shocks, the transverse fluctuation power is low. With increasing distance downstream, the wave power increases and the left-hand polarized wave power completely dominates the power.

Sckopke et al. [1990] presented two sets of shock parameters for each Events 1, 3, 5, and 9 as the spacecraft crossed first into and then out of the magnetosheath (See Table 1 in Sckopke et al. [1990]). The parameters include the compression ratio \( X \), \( M_f \), \( \theta_{\beta n} \) and upstream plasma- \( \beta \), where \( M_f \) is the fast Mach number \( V_{sw}/V_f \) and \( V_f \) is the fast speed for perpendicular propagation, \( X \) is the ratio of the downstream electron density to that upstream. Since the plasma- \( \beta \) is very low in most of the events, \( M_f \approx M_A \). We average these two values for each parameter and take the average to be representative of the shock during this time period. Table 2-1 lists these average values for each event. The calculations in the next sections are based on these average values. Table 2-1 also lists the characteristic variation of each parameter during the time period, estimated to be half the difference in the two values. Since the parameters vary within \( \pm 15\% \), they are specified with 1-figure or 2-figure accuracy. For Event X, only one set of parameters are presented.

Table 2-1 also lists the relative density of reflected protons \( n_r/N_e \), peak frequency \( (\omega/\Omega)_o \) of fluctuation power, the left-hand polarized power \( P_{lo} \), and the residual temperature anisotropy \( (T_\perp/T_\parallel)_o \). As we mentioned before, the presumed time of saturation when the spacecraft is furthest downstream of the shock is taken to be when \( T_\perp/T_\parallel \) has its minimum value, \( (T_\perp/T_\parallel)_o \). The wave power \( P_{lo} \) has a larger
variation downstream of the shock than the other quantities and we show the actual range of the observed values. The peak frequencies \((\omega/\Omega)_o\) are simply obtained from the fluctuation power spectrum presented by Sckopke \textit{et al.} [1990]. The wave spectra for Events 1 and 2 were not presented in the paper; they were obtained in a personal communication with N. Sckopke. The wave spectrum in most events appears to display a double-humped structure. The listed peak frequency corresponds to the main hump. The secondary peak frequency is about twice the main peak frequency and contains relatively little power. The error in \(n_r/N_e\) is \(\sim 20\)-30\% according to Sckopke \textit{et al.} [1990].

Among all the events investigated by Sckopke \textit{et al.} [1990], Events 1 and 2 are quasi-perpendicular shocks with a low Mach number, Events 3 and 5 are nearly perpendicular (\(\theta_{bs} > 85^\circ\)) with a low Mach number, Event 9 is quasi-perpendicular with a somewhat higher Mach number, and Event X is a quasi-perpendicular case with a high Mach number.
### (a)

<table>
<thead>
<tr>
<th>Event</th>
<th>$X$</th>
<th>$\theta_{bn}$</th>
<th>$M_f$</th>
<th>$\beta$</th>
<th>$n_r/N_e$</th>
<th>$\nu_0/V_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.8±0.1</td>
<td>73±0</td>
<td>1.9±0</td>
<td>.09±0</td>
<td>0.03</td>
<td>2.3</td>
</tr>
<tr>
<td>2</td>
<td>3.2±0.3</td>
<td>76±4</td>
<td>2.0±0.2</td>
<td>.10±0.1</td>
<td>0.03</td>
<td>2.4</td>
</tr>
<tr>
<td>3</td>
<td>2.7±0.2</td>
<td>87±4</td>
<td>2.4±0.1</td>
<td>.15±0.3</td>
<td>0.05</td>
<td>3.1</td>
</tr>
<tr>
<td>5</td>
<td>2.5±0.2</td>
<td>90±0</td>
<td>2.2±0.2</td>
<td>.15±0</td>
<td>0.05</td>
<td>3.0</td>
</tr>
<tr>
<td>9</td>
<td>2.3±0</td>
<td>63±3</td>
<td>2.8±0.4</td>
<td>.43±0.2</td>
<td>0.09</td>
<td>4.0</td>
</tr>
<tr>
<td>X</td>
<td>3.3</td>
<td>80</td>
<td>4.9</td>
<td>2.35</td>
<td>0.20</td>
<td>6.3</td>
</tr>
</tbody>
</table>

### (b)

<table>
<thead>
<tr>
<th>Event</th>
<th>$(\omega/\Omega)_o$</th>
<th>$(\omega/\Omega)_l$</th>
<th>$P_{Lo}$</th>
<th>$P_{Lt}$</th>
<th>$(T_{\perp}/T_{\parallel})_o$</th>
<th>$(T_{\perp}/T_{\parallel})_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.44</td>
<td>.64</td>
<td>.01-.05</td>
<td>.01</td>
<td>3.0</td>
<td>1.7</td>
</tr>
<tr>
<td>2</td>
<td>.41</td>
<td>.64</td>
<td>.01-.04</td>
<td>.01</td>
<td>2.7</td>
<td>1.8</td>
</tr>
<tr>
<td>3</td>
<td>.4</td>
<td>.48</td>
<td>.01-.04</td>
<td>.03</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>5</td>
<td>.5</td>
<td>.50</td>
<td>.02-.06</td>
<td>.02</td>
<td>1.9</td>
<td>1.7</td>
</tr>
<tr>
<td>9</td>
<td>.5</td>
<td>.37</td>
<td>.01-.04</td>
<td>.08</td>
<td>1.7</td>
<td>1.2</td>
</tr>
<tr>
<td>X</td>
<td>.4</td>
<td>.23</td>
<td>.05</td>
<td>.27</td>
<td>1.5</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Table 2-1 (a) Shock parameters for the events identified by Sckopke et al. [1990]: compression ratio $X$, the angle $\theta_{bn}$, the fast Mach number $M_f$, and plasma-$\beta$. Observed quantities: percentage of reflected protons $n_r/N_e$; (b) normalized peak frequency $(\omega/\Omega)_o$, wave power $P_{Lo}$, and temperature anisotropy $(T_{\perp}/T_{\parallel})_o$. Quantities calculated or predicted by the simple approach: normalized downstream speed of reflected protons $\nu_0/V_A$, normalized peak frequency $(\omega/\Omega)_l$, wave power $P_{Lt}$, and temperature anisotropy $(T_{\perp}/T_{\parallel})_l$. 

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2.3 Trajectory of the Reflected Protons

Simulations show that in perpendicular and quasi-perpendicular supercritical shocks the protons incident on the shock are first deflected by the magnetic force due to the presence of the previously reflected protons and then a fraction is reflected by the electrostatic potential due to charge separation in the shock ramp [Leroy et al., 1982]. The reflected protons are responsible for the foot, overshoot and undershoot magnetic structure of the supercritical shock [Leroy et al., 1982; Leroy, 1983]. This structure has been confirmed by Scudder et al. [1986] in their detailed study of Earth’s bow shock using ISEE data. Scudder et al. [1986] find that the electrostatic potential at the shock is distributed over the shock foot, ramp, and overshoot with a peak at the overshoot. Using numerical simulations, Burgess et al. [1989] investigated on what basis an incoming ion is selected for reflection. They concluded that the reflected protons come from the wings, not the core, of the distribution of the incoming protons. The core protons are transmitted through the shock, first decelerated by the electrostatic potential and then partially reaccelerated following the overshoot in the potential, and form the downstream bulk flow. A later study by Gedalin [1996a] shows that the reflection is due to the induced gyration in the overshoot, returning the gyrating ions back to the ramp.

The trajectories of the reflected protons have been considered in several studies. Gosling et al. [1982] calculated their velocity for a perpendicular shock and Schwartz et al. [1983] calculated the guiding-center velocity of specularly reflected protons for an arbitrary shock geometry. However, their work is restricted to the motion of the reflected protons upstream of the shock. We require the downstream proton velocity as
The calculation is performed most easily in the normal-incidence frame for a perpendicular shock and in the deHoffman-Teller frame for a quasi-perpendicular shock. We make the basic assumption that the shock surface is planar and stationary. This simplification of Earth's bow shock is based on the fact that shock curvature and motion can be ignored locally on the scale of the reflected proton gyroradius and gyroperiod respectively. Specular reflection is assumed for the reflected protons; it has been shown in previous studies that most reflected protons are specularly reflected or nearly specularly reflected [Sckopke et al., 1983; Kucharek et al., 2004]. In the shock foot, where the reflected protons gyrate back into the upstream flow, the modification of the magnetic field is neglected. The electric field in the upstream plasma satisfies the hydromagnetic approximation, $E = -V \times B$; here $B$ is the ambient magnetic field and $V$ is the velocity of the bulk plasma.

We treat the reflected protons as test particles as they move in the laminar electric and magnetic fields specified above. Since the reflected protons are only 3%-9% of the total electron density in the marginally supercritical events we study, this is a good approximation. Under these assumptions we first perform the trajectory calculation for a perpendicular shock and then for a quasi-perpendicular shock. The scattering of the reflected protons out of this trajectory due to the downstream ion-cyclotron waves is considered in Sections 2.4 and 2.5.

2.3.1 Perpendicular Shock

We perform the calculation in the normal-incidence frame. As shown in Figure 2-1, we take the shock plane to be the $y$-$z$ plane and the upstream magnetic field to be in the $z$ direction. The solar wind particles flow in the $+x$ direction with speed $V$. The
electric field in this frame is $VB\hat{y}$. The equations of motion for protons in the upstream plasma ($x < 0$) are:

$$m\dot{v}_x = eBv_y$$

(2.1a)

$$m\dot{v}_y = eB(-v_x + V)$$

(2.1b)

where $m$ is the mass of the proton and $e$ is the charge of the proton.

We take the time when the proton is reflected from the shock to be $t = 0\,$, and neglect the thermal spread of the incident proton distribution. The initial velocity of a specularly reflected proton is $-V\hat{x}$ at $t = 0$. Solving equation (2.1) with this initial condition, we obtain the velocity of the reflected protons and their $x$ displacement upstream of the shock as

$$v_x = V(1 - 2\cos\Omega_u t)$$

(2.2a)

$$v_y = 2V\sin\Omega_u t$$

(2.2b)

$$x = Vt - \frac{2V}{\Omega_u}\sin\Omega_u t$$

(2.2c)

where $\Omega_u (\approx eB/m)$ is the upstream proton gyrofrequency.

Setting $x = 0$ we obtain the time $t_0$ when the reflected protons encounter the shock plane for the second time: $\Omega_u t_0 \approx 1.9$. Since $v_x(t_0) \approx 1.64V$ [Sckopke et al., 1983; Schwartz et al., 1983], these protons can now pass through the shock plane while they lose part of their energy due to the shock potential difference $\phi$. Their velocity adjacent to the shock at $x = 0$ in the downstream plasma frame is:
Figure 2-1 Schematic diagram of a perpendicular shock at the plane $x = 0$.

\[
\dot{v}_{sd} = \sqrt{V^2 \left(1 - 2 \cos(\Omega_s t_0)\right)^2 - \frac{2e\phi}{m} \frac{V}{X}} \tag{2.3a}
\]

\[
v_{yd} = 2V \sin(\Omega_s t_0) \tag{2.3b}
\]

The potential energy $e\phi$ is equal to the kinetic energy difference between the incident and the transmitted core protons. The speed of the reflected protons in the downstream plasma frame is:

\[
v_0 = \sqrt{\dot{v}_{sd}^2 + v_{yd}^2} \tag{2.4}
\]

In the absence of scattering, the reflected protons continue to gyrate in the downstream plasma flow with speed $v_0$ and a pitch angle of 90° unless they encounter the shock.
plane again. We now investigate whether a downstream reflected proton may return to the shock plane. The downstream equations of motion in the frame of the shock are:

\[ m\dot{v}_x = eBv_y \]  
\[ m\dot{v}_y = eB(-v_x + V) \]

The initial \((t = t_0)\) velocity components are \(v_{xd} + V/X\) in the \(x\) direction and \(v_{yd}\) in the \(y\) direction. We calculate the \(x\) displacement and the velocity component in the \(x\) direction as:

\[ x(t') = \frac{V}{X} t' + \frac{v_0}{\Omega_a t} \sin[\Omega_a t't - \psi] + \frac{v_{yd}}{\Omega_a t} \]

\[ \dot{x}(t') = \frac{V}{X} + v_0 \cos[\Omega_a t't - \psi] \]

where \(t' = t - t_0\), and \(\psi\) is the initial phase of the gyration defined by \(\tan \psi = v_{yd} / v_{xd} \) \((0 < \psi < \pi/2)\). The curve in Figure 2-2 shows \(v_0 \cos[\Omega_a t't - \psi]\) as a function of \(t'\) along with the line \(-V/X\). The intersection of these two lines implies \(\dot{x} = 0\). It is clear that the first intersection \((t' > 0)\) corresponds to a local maximum of \(x\) while the second corresponds to a local minimum value of \(x\): \(x_{\text{min}} = x(t_c)\). From Figure 2-2 we note that \(\Omega_a t_c > \pi\). From equation (2.6) we obtain:

\[ x_{\text{min}} = \frac{V}{X} t_c + \frac{v_0}{\Omega_a t} \sin(\Omega_a t_c - \psi) + \frac{v_{yd}}{\Omega_a t} > \frac{1}{\Omega_a t} \left( \frac{V}{X} - v_0 + v_{yd} \right) \]
Figure 2-2 Solution for the time $t_c$ of the minimum value of $x(t)$ within one downstream gyration.

We find that $v_0 < 2.5V$ and $v_{yd} \approx 1.9V$ from equations (2.3a), (2.3b) and (2.4). Inserting these values into the right hand side of equation (2.8) we obtain $x_{\text{min}} > 0$ if $X < \pi/0.6 \approx 5$. Since for a nonrelativistic isotropic monatomic gas $X \leq 4$, we conclude that a reflected proton remains downstream of the shock. Scattering should not alter this conclusion substantially. A characteristic trajectory of a reflected proton is shown in Figure 2-3 using the geometry of Figure 2-1.
2.3.2 Quasi-Perpendicular Shock

We now determine the trajectory of the reflected protons at a quasi-perpendicular shock in the deHoffman-Teller frame in which the motional electric field vanishes [de Hoffman and Teller, 1950]. Figure 2-4 shows the configuration of the shock in the $x$-$z$ plane. The shock plane is the $y$-$z$ plane and the solar wind is incident on the shock plane along the direction of the upstream magnetic field. The upstream magnetic field is $B \sin \theta \, \hat{x} + B \cos \theta \, \hat{z}$, where $|\theta| (-\pi/2 \leq \theta \leq \pi/2)$ is the complement of $\theta_{bn}$. The initial velocity of the reflected protons is $v_{x0} = -V$, $v_{y0} = 0$, $v_{z0} = V \cot \theta$.

The upstream equations of motion for a reflected proton are:

\begin{align*}
\frac{d}{dt} v_x &= -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{(B \sin \theta \, \hat{x} + B \cos \theta \, \hat{z}) \cdot (v_x \, \hat{x} + v_y \, \hat{y} + v_z \, \hat{z})}{B \, B} = -\frac{v_x \, B \sin \theta + v_y \, B \cos \theta}{B^2} = -\frac{v_x \, \sin \theta + v_y \, \cos \theta}{B} = -\frac{v_x}{B} \sin \theta - \frac{v_y}{B} \cos \theta,
\frac{d}{dt} v_y &= -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{(B \sin \theta \, \hat{x} + B \cos \theta \, \hat{z}) \cdot (v_x \, \hat{x} + v_y \, \hat{y} + v_z \, \hat{z})}{B \, B} = -\frac{v_x \, B \sin \theta + v_y \, B \cos \theta}{B^2} = -\frac{v_x \, \sin \theta + v_y \, \cos \theta}{B} = -\frac{v_x}{B} \sin \theta - \frac{v_y}{B} \cos \theta,
\frac{d}{dt} v_z &= -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{\mathbf{E} \cdot \mathbf{v}}{\mathbf{B} \cdot \mathbf{B}} = -\frac{(B \sin \theta \, \hat{x} + B \cos \theta \, \hat{z}) \cdot (v_x \, \hat{x} + v_y \, \hat{y} + v_z \, \hat{z})}{B \, B} = -\frac{v_x \, B \sin \theta + v_y \, B \cos \theta}{B^2} = -\frac{v_x \, \sin \theta + v_y \, \cos \theta}{B} = -\frac{v_x}{B} \sin \theta - \frac{v_y}{B} \cos \theta.
\end{align*}
\[ \dot{v}_x = \Omega_u v_y \cos \theta \]  
(2.9a)
\[ \dot{v}_y = -\Omega_u v_x \cos \theta + \Omega_u v_z \sin \theta \]  
(2.9b)
\[ \dot{v}_z = -\Omega_u v_y \sin \theta \]  
(2.9c)

with the solution:
\[ v_x = -2 V \cos^2 \theta \cos(\Omega_u t) + V \cos 2\theta \]  
(2.10a)
\[ v_y = 2 V \cos \theta \sin(\Omega_u t) \]  
(2.10b)
\[ v_z = V \sin 2\theta[\cos(\Omega_u t) - 1] + V \cot \theta \]  
(2.10c)
\[ x = -\frac{1}{\Omega_u} 2V \cos^2 \theta \sin(\Omega_u t) + Vt \cos(2\theta) \]  
(2.10d)

Setting \( x = 0 \) we obtain the time \( t_0 \) when the particle encounters the shock plane a second time:
\[ 2 \cos^2 \theta \sin(\Omega_u t_0) = \Omega_u t_0 \cos 2\theta \]  
(2.11)

As in Section 2.3.1 for \( \theta = 0 \), this transcendental equation for \( \Omega_u t_0 \) can be solved numerically. It has no real solution for a quasi-parallel shock \((50.1^\circ < |\theta| < 90^\circ)\) [Schwartz et al., 1983]. For \( 50.1^\circ < |\theta| < 90^\circ \), a specularly reflected proton escapes the shock upstream and may return to the shock surface only after scattering. This case is not considered further here. Substituting \( \Omega_u t_0 \) into equations (2.10a-c) we obtain the velocity of a reflected proton when it reencounters the shock plane. We now calculate the reflected proton speed and pitch angle in the downstream plasma frame after it crosses the shock plane.
Figure 2-4 Schematic diagram of a quasi-perpendicular shock at \( x = 0 \) viewed in the deHoffman-Teller frame.

The bulk velocity of the downstream flow in the shock rest frame and the downstream magnetic field follow from the Rankine-Hugoniot relations:

\[
V_{zd} = \frac{V}{X} \tag{2.12a}
\]

\[
V_{zd} = \chi V \cot \theta \tag{2.12b}
\]

\[
B_{zd} = B \sin \theta \tag{2.12c}
\]

\[
B_{zd} = \chi X B \cos \theta \tag{2.12d}
\]

\[
V_{yd} = 0 = B_{yd} \tag{2.12e}
\]

where

\[
\chi = \frac{V^2 - V_{zd}^2 \sin^2 \theta}{V^2 - XV_{zd}^2 \sin^2 \theta} \tag{2.13}
\]
\[ V_{Au}^2 = \frac{B^2}{(\mu_0 \rho)} \] is the upstream Alfvén speed, \( \mu_0 \) is the permeability of free space, and 
\( \rho \approx m N_e \) is the plasma mass density.

In the downstream plasma frame, the velocity of the proton at \( x = 0^+ \) is:

\[
\begin{align*}
V_{\text{sd}} &= \sqrt{(-2V \cos^2 \theta \cos \Omega_\alpha t_0 + V \cos 2\theta)^2 - \frac{2e\phi}{m} \frac{V}{\chi}} \quad (2.14a) \\
V_{\phi d} &= 2V \cos \theta \sin \Omega_\alpha t_0 \quad (2.14b) \\
V_{zd} &= V(\cos \Omega_\alpha t_0 - 1) \sin 2\theta + (1 - \chi)V \cot \theta. \quad (2.14c)
\end{align*}
\]

Similar to the shock potential used in Lee et al. [1996], the shock potential we use is the effective shock potential including both the electrostatic shock potential and the \( x \)-component of the Lorentz force across the shock ramp; we notice that the electrostatic shock potentials in the deHoffman-Teller frame and the normal-incidence frame are different due to the existence of a noncoplanar magnetic field component in the ramp [Goodrich and Scudder, 1984; Thomsen et al., 1987].

The cosine of the pitch angle for the reflected proton just downstream of the shock is determined by \( \mu_{d0} = \cos \alpha \), where \( \mu_{d0} \) is:

\[
\mu_{d0} = \frac{V_{\text{sd}} B_{xd} + V_{zd} B_{zd}}{\sqrt{\left(V_{\text{sd}}^2 + V_{zd}^2 + V_{zd}^2\right)\left(B_{xd}^2 + B_{zd}^2\right)}}. \quad (2.15)
\]

For a perpendicular shock (\( \theta = 0 \)), equations (2.14a-c) reduce to equations (2.3a-b) and \( V_{zd} = 0 \).

Wilkinson [1999] also calculates the trajectory of a reflected ion under the same assumptions for perpendicular and quasiperpendicular shocks. Our conclusion in Section 2.3.1 that the downstream reflected protons do not gyrate back to the shock is consistent.
with their conclusion for a perpendicular shock. They also show that this conclusion holds for $\theta_{bn} > 60^\circ$.

2.4 A Simple Approach

Figure 2-5 shows schematically the spatially-averaged velocity distribution of the unscattered downstream reflected protons in $v_{\perp} - v_{\parallel}$ space relative to the magnetic field direction for a perpendicular (a) and a quasi-perpendicular (b) shock. These reflected protons gyrate in the downstream flow under the Lorentz force with the speed and pitch angle given by equations (2.14) and (2.15). Their distribution is singular when we neglect the thermal spread of the reflected protons and assume that the shock is planar and stationary; however, even including the thermal spread, the resulting downstream temperature anisotropy $T_{\perp}/T_{\parallel}$ is large. This distribution is unstable to the excitation of ion cyclotron waves, which grow as the protons and waves are advected away from the shock. The excited waves in turn scatter the particles and reduce the temperature anisotropy. Sufficiently far downstream from the shock, the reflected protons and waves will attain a quasi-equilibrium where the wave spectrum and the temperature anisotropy approach quasi-asymptotic values. We neglect nonlinear processes occurring on timescales larger than the Alfvén ion-cyclotron instability timescale, so that the values attained are only quasi-asymptotic values. The ion-wave interactions in the post-saturated magnetosheath were investigated by Gary and Winske [1993].
We note that the distribution function of the downstream reflected protons is nongyrotropic. From Figure 2-3, it is clear that a plane downstream of, and parallel to, the shock plane intersects the downstream trajectory at a few discrete points. Thus the actual velocity distribution consists of a few points, each with a specific gyrophase. Figure 2-5 shows the distribution averaged over the spatial interval $0 < x < 2\pi V/\Omega X$, which is gyrotropic. In what follows, we consider only the spatially-averaged distribution. The difference between the ion-cyclotron wave spectrum excited by the actual and the spatially-averaged distribution is small as will be discussed in Section 2.8.

We now calculate the asymptotic wave spectrum, the reflected proton bulk velocity and temperature anisotropy, under the following simplifying approximations: (i) The fluctuation power is small so that quasilinear theory is valid; (ii) Wave dispersion is
negligible; (iii) Reflected protons dominate the wave generation; (iv) The speed of the reflected protons is much larger than the downstream Alfvén speed; (v) The fluctuations are assumed to propagate parallel to the ambient magnetic field.

2.4.1. Wave Intensity

For wavenumbers $|k| > \Omega / v_0$, the asymptotic magnetic fluctuation intensity $I(k) = I_+(k) + I_-(k)$ [$\langle \mathbf{B} \cdot \partial \mathbf{B} \rangle = \int_0^\infty I(k) dk$] excited by the reflected protons is [Lee and Ip, 1987]:

$$I_\pm (k, t \to \infty) = \frac{1}{2} \left( C(k) \right)^2 + 4 I_+(k, 0) I_-(k, 0) \right)^{1/2} \pm \frac{1}{2} C(k), \quad (2.16a)$$

where

$$C(k) = I_+(k, 0) - I_-(k, 0) + \frac{\mu_0 n_m V_d \Omega}{2 k^2} \left[ \frac{\Omega}{k v_0} - \text{sgn} \left( \frac{\Omega}{k v_0} - \mu_0 \right) \right], \quad (2.16b)$$

and $\text{sgn}(x) = x/|x|$. For $|k| < \Omega / v_0$, we have $I_+(k, t \to \infty) = I_+(k, t = 0)$. In this notation, $I_+(k > 0)$ [$I_+(k < 0)$] describes right [left] polarized waves propagating parallel to the magnetic field and $I_-(k > 0)$ [$I_-(k < 0)$] describes left [right] polarized waves propagating antiparallel to the magnetic field.

In accord with the low intensity observed just downstream of the shock, we neglect the ambient waves, $I_\pm (k, 0)$. For quasi-perpendicular shocks $|\mu_0| << 1$ so that the dominant wavevector range of the enhancement satisfies $|\Omega / k v_0| > |\mu_0|$. Under these two conditions, the wave spectra become:

$$I_+(k < 0, t \to \infty) = \frac{\mu_0 n_m V_d \Omega}{2 k^2} \left( \frac{\Omega}{k v_0} + 1 \right) \quad (2.17a)$$

$$I_+(k > 0, t \to \infty) = 0 \quad (2.17b)$$

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In this case, the right-hand polarized waves, \( I_+(k > 0) \) and \( I_-(k < 0) \), are both negligible; the left-hand polarized ion-cyclotron waves, \( I_+(k < 0) \) and \( I_-(k > 0) \), grow to the enhanced level described by equation (2.17). The peak frequency of the wave spectrum in the plasma frame, obtained by requiring that the derivative of equation (2.17a,c) with respect to \( k \) vanishes, satisfies:

\[
\frac{|\omega|}{\Omega} = \frac{3 V_A}{2 v_0}.
\] (2.18)

In order to compare these predictions with the observations, we integrate the fluctuation power over the frequency range from 0.3 \( \Omega \) to 0.8 \( \Omega \). We obtain

\[
\frac{P_L}{B_0^2} = \frac{n_e}{N_e} \left( \frac{\Omega}{\omega} + \frac{V_A \Omega^2}{2 v_0 \omega^2} \right)|_{\omega = \omega_p}^{\omega = 0.8\Omega} \] (2.19)

where \( \omega_p = \min(-V_A \Omega/v_0, -0.3\Omega) \). Equation (2.19) shows that the wave power is proportional to the ratio of the reflected proton density to the electron density.

2.4.2 Estimate of the Relaxation Timescale

The timescale \( \tau \) for the wave growth and the relaxation of the temperature anisotropy is estimated as follows. The process is governed by the pitch angle diffusion equation [Lee and Ip, 1987]:

\[
\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) D_{\mu\mu} \frac{\partial F}{\partial \mu} \right]
\] (2.20)
where $F(v, \mu, t)$ is the spatially-averaged gyrotrropic proton distribution in a parcel of downstream plasma.

$$D_{\mu \nu} = \frac{\pi e^2}{2m^2|\mu|v_0} I(\frac{\Omega}{\nu_\mu})$$

(2.21)

according to quasilinear theory. Substituting equation (2.17) into equation (2.21) we have

$$\tau \sim D_{\mu \nu}^{-1} \sim \frac{N_s V_A}{n_s v_0 \Omega}$$

(2.22)

Unfortunately, it is impossible to compare this result with the timescale of downstream "thermalization" observed by Sckopke et al. [1990] since the speed of the spacecraft relative to the shock, which is in continual motion in response to changing solar wind conditions, is unknown. Nevertheless, it is clear that for Event X, which is a supercritical shock crossing with higher $n_r$ and $v_0/V_A$ (see Table 2-1), the process develops much faster than for the other events.

2.4.3 Anisotropy

The enhanced wave intensity grows to the level described by equation (2.16) or (2.17) at the expense of the "free energy" in the reflected proton distribution. Since the ambient wave intensity is low, the reflected protons relax to a "bispherical" distribution [Galeev and Sagdeev, 1988] in the quasi-equilibrium downstream of the shock.

The ion-wave interaction conserves ion energy in the wave frame. The reflected protons are therefore scattered on a spherical surface centered on the wave phase speed. The initial paths of scattering in velocity space are shown in Figure 2-6a for a perpendicular shock and in Figure 2-7a for a quasi-perpendicular shock when the only
waves present have phase velocity \(-V_A\). The distribution prior to scattering is shown by
the dot; the semicircle shows the path if the scattering is elastic in the fluid frame. Since
the total energy of the reflected protons and the resonant waves is conserved in the fluid
frame, when the protons are scattered into the direction in which they gain energy
(shown by the dotted line in Figures 6a and 7a), the resonant waves will lose energy.
Unless the ambient wave intensity is very large, this process cannot sustain itself since
the wave intensity will decay. In contrast, when particles scatter into the direction in
which they lose energy (shown by the solid curve segment in Figures 6a and 7a), the
waves will gain energy and the process is enhanced. If wave propagation in both
directions exists initially, then the unstable direction will soon grow to dominate the
stable direction. The reflected protons, which are initially localized in velocity space at
the solid dot, will therefore scatter onto the bispherical shell shown in Figures 6b and 7b
for the perpendicular and quasi-perpendicular cases. In the quasi-equilibrium the protons
are uniformly distributed over the nondispersive shell with a resulting temperature
anisotropy:

\[
\frac{T_r}{T_H} = \left[ \frac{1}{\sum_{i=1}^{3} (\nu_i \cos \alpha_i - \frac{1}{3} \nu_i \cos^3 \alpha_i)} \right] \sqrt{2 \sum_{i=1}^{3} \nu_i^4 + (-1)^i \frac{1}{3} \nu_i^4 \cos^3 \alpha_i + (V_A - (-1)^i \langle \nu_i \rangle)^2 (1 + (-1)^i \cos \alpha_i) \nu_i^2}
\]

-31-

(2.23)

where \(\nu_1, \nu_2, \alpha_1,\) and \(\alpha_2\) are defined in Figure 2-5. The average velocity of the reflected
proton distribution along the magnetic field direction in the frame of the core protons is:
\[
\langle v_\parallel \rangle = \frac{\sum_{i=1,2} \frac{1}{2} (-1)^i v_0^i v_i \sin^2 \alpha + (-1)^i V_A v_i + V_A v_i^2 \cos \alpha}{2(v_0^2 + V_A^2) + \sum_{i=1,2} - V_A v_i + (-1)^i v_0 v_i \cos \alpha}.
\] (2.24)

Figure 2-6 (a) Schematic illustration of the proton velocity distribution downstream of a perpendicular shock as the protons are scattered by Alfvén waves with phase speed \(-V_A\); (b) Nondispersive bispherical distribution of protons downstream of a perpendicular shock. The semicircles would result from elastic scattering in the plasma frame.

Equation (2.23) and (2.24) are obtained from the definition

\[
T_\parallel \propto (v_\parallel - \langle v_\parallel \rangle)^2 F d^3 v,
\]

\[
T_\perp \propto (1/2) \int v_\perp^2 F d^3 v \quad \text{and} \quad \langle v_\parallel \rangle = n_{-1}^{-1} \int v_\parallel F d^3 v.
\] The weighting according to \(F\) is equivalent in the nondispersive case to weighting according to the area of the
bispherical distribution. Williams and Zank [1994] also calculated the average velocity of pickup ions on a bispherical distribution. Their result is identical to equation (2.24).

For a perpendicular shock ($\alpha = 0$):

$$\langle v_{\parallel} \rangle = 0$$  \hspace{1cm} (2.25)

$$\frac{T_{i}}{T_{\parallel}} = \frac{2/3 v_{i}^{4} + 2(v_{i}^{4} \cos \alpha - 1/3 v_{i}^{4} \cos^{2} \alpha)}{4(1/3 v_{i}^{4} + 1/3 v_{i}^{4} \cos^{3} \alpha + V_{A}^{2}(1 + \cos \alpha)v_{i}^{2} - V_{A}v_{i}^{3} \sin^{2} \alpha)}$$ \hspace{1cm} (2.26)

Figure 2-7 Schematic illustration as in Figure 2-6 for a quasi-perpendicular shock.

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The bispherical distribution will be distorted if wave dispersion is taken into account, which will modify the temperature anisotropy. This effect will be considered in Section 2.5.

2.4.4 Comparison with Observations

First of all, the observed dominance of left-hand circularly-polarized waves is consistent with the instability of ion-cyclotron waves described in Section 2.4.1 and predicted by equation (2.17). In order to predict the magnetic power spectrum for each event, we first calculate the initial downstream speed $v_0$ of the reflected protons for the prescribed shock parameters; it is listed in Table 2-1 for each event as $v_0/V_A$. The predicted peak frequencies and fluctuation power are then calculated according to equations (2.18) and (2.19). In Table 2-1, they are denoted by subscript “t” for “theory” while the observed values are denoted by “o” for “observation”. The peak frequencies are normalized by the downstream proton gyrofrequency; the magnetic fluctuation power is normalized by the square of the downstream average magnetic field. For Events 1, 2, 3, and 5, the predicted power of the magnetic fluctuations lies in, or nearly in the range of that observed. For Events 9 and X, which have rather higher Mach numbers, the predicted power is larger than that observed by about 100%. The theory predicts that the power of the magnetic fluctuations increases with $M_A$, $n_r/n_e$ and $v_0/V_A$; for Events 1, 2, 3 and 5, the variations in the observed power presumably obscure these trends. The predicted and observed peak frequencies agree qualitatively with discrepancies up to 50%. The predicted spectra for Events 1, 3 and 9 are shown in Figure 2-8. The spectrum of Event 2 (5) is very similar to that of Event 1 (3). We note
that about 50% of the power of the magnetic fluctuations is outside of the frequency band $0.3\Omega - 0.8\Omega$, which is not consistent with the observations.

The residual temperature anisotropies ($T_x/T_\parallel$) given by Sckopke et al. [1990] for all the events are dominated by the reflected protons. They are listed in Table 2-1 together with the predicted temperature anisotropy for each event. The observed and predicted values match quite well with the exception of Events 1 and 2; it is noteworthy that the predicted values are smaller than those observed, a point we return to in Section 2.6.2.2. We also estimate the average downstream velocity of the gyrating reflected protons in Event 9 to be 120 km/s; it is predicted from equation (2.24) to be 140 km/s. The predicted average parallel velocity matches the observation very well.

![Figure 2-8 Normalized magnetic fluctuation power spectra predicted using the simple approach for Events 1, 3 and 9 described by Sckopke et al. [1990].](image)

For Event X the observed anisotropy matches the predicted value but the peak frequency and the fluctuation power show obvious differences. In a high Mach number shock more protons are reflected and the wave intensity is higher; the condition
For the validity of quasilinear theory is not well satisfied. We shall not discuss this case further.

The simple approach presented in this section has the advantage that we can derive analytical expressions for the wave spectrum, peak frequency and temperature anisotropy. However, the theoretical results match the observations only qualitatively. We now consider a few effects neglected in the simple approach in order to improve the agreement between predictions and observations.

2.5. Effects of Dispersion

For protons with $v_\parallel \gg V_A$ dispersive effects are negligible. However, for the marginally supercritical shocks considered here, even $v_\parallel$ is not much larger than $V_A$ (see Table 2-1). Thus, a substantial fraction of the reflected protons satisfies $v_\parallel \sim V_A$, and we expect dispersive effects to be important.

2.5.1 Dispersive Bispherical Distribution

In the plasma frame, the condition for cyclotron resonance between a proton with velocity component $v_\parallel$ and a transverse wave with frequency $\omega$ and wavenumber $k$ propagating parallel to the ambient magnetic field is [Stix, 1992]:

$$\omega(k) = kv_\parallel - \Omega. \quad \text{(2.27)}$$

The dispersion relation for low frequency waves in a cold plasma is [Stix, 1992]:

$$\frac{\omega}{k} = \pm V_A \sqrt{1 + \frac{\omega}{\Omega}}, \quad \text{(2.28)}$$
where $\omega < 0$ for ion-cyclotron waves and $+-(-)$ corresponds to waves propagating parallel (antiparallel) to the ambient magnetic field.

For a given value of parallel velocity $v_\parallel$, the solution of equations (2.27) and (2.28) yields the resonant wave speed $V_p = \omega/k$. This is shown schematically in Figure 2-9. The two curves passing through the origin represent the dispersion relations of waves propagating parallel (+) and antiparallel (−) to the ambient magnetic field for the sense of circular polarization indicated. The straight line crossing the $\omega$-axis at $\omega = -\Omega$ is the resonance line for specified $v_\parallel$; the intersection(s) of the resonance line and the dispersion curves yields the wavenumber $k$ and frequency $\omega$ of the resonant wave. It is clear that $V_p$ varies with $v_\parallel$, namely $V_p = V_p(v_\parallel)$.

![Figure 2-9 Dispersion relations for parallel-propagating hydromagnetic and ion-cyclotron waves in cold plasma, together with the resonance line for protons with velocity component $v_\parallel$. The intersection of the two gives the wavenumber and frequency of the resonant waves. (R, L) and +/− specifies polarization and propagation direction of each branch of the dispersion relation.](image-url)
Dispersive bispherical distributions of pickup ions in the solar wind have been discussed by Isenberg and Lee [1996] in detail. We revise their analysis and apply it to the reflected protons downstream of the bow shock. If we neglect wave dispersion and set $V_p = \pm V_A$, as shown in Figures 6b and 7b, each branch of the bispherical shell is part of a sphere centered on $+V_A$ or $-V_A$. With dispersion taken into account, the shell is locally tangent to a sphere centered on the phase speed $V_p(v_\parallel)$.

![Diagram of dispersive bispherical shell](image)

**Figure 2-10 Geometry of the dispersive bispherical shell for $v_p > 0$ at a perpendicular shock.**

The geometry of the dispersive bispherical shell is shown in Figure 2-10. The shell satisfies the differential equation:
\[
\frac{dv_\perp}{dv_\parallel} = \tan \theta = \frac{V_p - v_\parallel}{v_\perp}.
\] (2.29)

Figure 2-11 The dispersive bispherical shell (dark solid curve) compared with the corresponding nondispersive shell (light solid curve), and the elastic scattering shell (dotted curve) for (a) \( v_0 = 2V_A \) and (b) \( v_0 = 3V_A \).

A reflected proton just downstream of a perpendicular shock starts with \((v_\perp, v_\parallel) = (v_0, 0)\); the corresponding resonant wave phase speed vanishes so that the proton is scattered initially perpendicular to the \( v_\perp \) axis. As \(|v_\parallel|\) increases and \( v_\perp \) decreases, the phase speed of the resonant wave increases toward \( V_A \). The solution in velocity space describes a closed dispersive bispherical shell when it is extended to \( v_\perp = 0 \). The exact shape of the shell can be calculated by solving equations (2.27) - (2.29) numerically. Figure 2-11 shows two examples of dispersive bispherical shells (dark solid curves) downstream of a perpendicular shock together with the bispherical shells (light solid curves) obtained if dispersion is neglected (Figure 2-6b) and the elastic scattering shell.
(dotted curves) with $v = v_0$ centered on the origin. All velocities are normalized by the downstream Alfvén speed. The initial speed is chosen to be $v_0 = 2V_A$ in Figure 2-11a, which is close to the speed of the reflected protons in Events 1 and 2. In Figure 2-11b, we choose $v_0 = 3V_A$, which is close to the speed of the reflected protons in Events 3 and 5. It is clear that the predicted temperature anisotropy and "free energy" are both reduced when dispersion is taken into account.

![Figure 2-12 Thickness of the region between two dispersive bispherical shells varies with position on the shell.](image)

The calculation of the distribution function for the dispersive bispherical shell needs to take into account the fact that the protons are actually distributed continuously in velocity space. Figure 2-12 shows two dispersive bispherical shells; the inner (outer) shell satisfies $v_\perp = v_0 (v_0 + \Delta v)$ at $v_\parallel = 0$, where $\Delta v$ is much less than the scale of the
initial velocity-space gradients at $x = 0^+$ including thermal spread. The protons which are initially distributed between these two shells become uniformly distributed throughout the volume between the bounding shells as shown in Figure 2-12 after scattering to the quasi-equilibrium dispersive bispherical shell. Note that in contrast with the nondispersive bispherical shell the distribution of protons is not uniform over the surface of the shell since the thickness of the slab varies.

2.5.2 Resonance Gap

The appropriate dispersive bispherical shell for quasi-perpendicular shocks is more complicated. We first consider the case that the unscattered reflected protons satisfy $\cos \alpha > 0$. Figure 2-13(a) shows the dispersion relations, together with the resonance line for $v_\parallel = v_t = (3\sqrt{3}/2)V_A$ which is tangent to the dispersion curve $R_\perp$. It is clear that the resonance lines for $0 < v_\parallel < v_t$ only intersect the dispersion curve of $L_\perp$, which implies that particles with $0 < v_\parallel < v_t$ are resonant only with the waves propagating antiparallel to $B$. If the reflected protons adjacent to the shock satisfy $0 < v_0 \cos \alpha < v_t$, they initially interact with the waves propagating antiparallel to $B$ and are scattered on the corresponding distorted shell. In principle, they can be scattered in both directions of $v_\parallel$; however, scattering in the direction $v_\parallel < v_0 \cos \alpha$ must damp the waves since the protons gain energy in the plasma frame. Soon the waves resonant with the particles with velocity $0 < v_\parallel < v_0 \cos \alpha$ will decay and leave a resonance gap in that domain, which prevents subsequent scattering into the hemisphere $v_\parallel < 0$. If the unscattered reflected protons satisfy $v_0 \cos \alpha > v_t$, they form a bispherical distribution in the domain $v_\parallel > v_t$, but similarly create a resonance
gap in the domain $0 < u_\parallel < u_i$. The actual "bispherical" surface in the domain $u_i < u_\parallel < u_0 \cos \alpha$ is complicated by the existence of two resonances on the $R_+$ branch, a lower frequency mode with phase speed $V_p^-$ and a higher frequency mode with phase speed $V_p^+$. The lower (higher) frequency mode would be expected to have higher (lower) wave intensity initially, but smaller (larger) growth rate since $V_p^- < V_p^+$. Which mode dominates the actual shape of the bispherical surface is unclear. The distinction between the two modes vanishes as $u_\parallel \to u_i$ and the resonance gap starts. However, the case $u_0 \cos \alpha > u_i$ is not relevant for the crossings of Earth's bow shock which we consider, since they satisfy $u_0 \approx u_i$ and $\cos \alpha \ll 1$. For $\cos \alpha < 0$, a mirror image resonance gap appears in the domain $-u_i < u_\parallel < 0$.

In fact, there exist nonlinear processes which can facilitate particle transport across the resonance gap as pointed out by Isenberg and Lee [1996]. Once particles enter the domain $u_\parallel < 0$ (in the case $\cos \alpha > 0$), they again form a complete shell. However, the $u_\perp$-intercept of the resulting shell, or shells if the mechanism for traversing the gap allows it, is unclear as indicated by the dotted line in Figure 2-13b. In the calculations which follow, we assume that the reflected protons start with pitch angle $\alpha = 90^\circ$, appropriate for a rigorously perpendicular shock, and calculate the asymptotic temperature anisotropy based on the dispersive bispherical shell.
2.5.3 Johnstone et al.’s Approach and the Dispersive Wave Spectrum

As discussed in the previous section, the wave power spectra predicted by the simple approach do not match the observations very well. From Figure 2-11, it is clear that dispersion, or equivalently \( v_\parallel \sim V_A \), affects the free energy in the unscattered reflected proton distribution. The wave power spectra described by equation (2.16) assume \( v_\parallel \gg V_A \) and cannot treat dispersion. However, in most of the events documented by Sckopke et al. [1990], typically \( v_0 \approx 2V_A - 3V_A \) (see Table 2-1). Johnstone et al. [1991], and Huddleston and Johnstone [1992] developed a method to calculate the excited wave power spectrum for cases with arbitrary \( v_\parallel /V_A \) but neglecting the dispersive effects of the ion-cyclotron waves. The general drawback of Johnstone’s approach is that it requires that the background wave intensity be negligible in order that stable waves are absent. However, this drawback is not germane here since
the relevant background wave intensity just downstream of, and adjacent to, the shock is negligible in the events documented by Sckopke et al. [1990] as noted in Section 2.2. Isenberg and Lee [1996] modified Johnstone et al.'s approach to calculate the dispersive wave spectrum excited by interstellar pickup ions in the solar wind. We describe this method and apply it to the dispersive wave spectra excited by the reflected protons downstream of a perpendicular shock.

Johnstone et al.'s approach is based on energy conservation of particles and resonant waves when viewed in the plasma frame. A proton moving with parallel velocity $v_\parallel$ loses energy to the resonant wave as it scatters along the dispersive bispherical shell. The differential energy lost is $de/dv_\parallel$. The asymptotic energy density of the waves excited by reflected protons with wavenumber between $k$ and $k+dk$ is equal to the energy lost by a reflected proton as it scatters across the resonant range of $v_\parallel$ times the net number density of protons which scatter across the range of $v_\parallel$. This number density is proportional to $R(v_\parallel)$, the partial volume of the shaded volume shown in Figure 2-12 in which the parallel velocity is larger than $v_\parallel$ (for $v_\parallel > 0$). The energy density spectrum of the waves is then given by:

$$E(k) = -n_e \frac{de}{dv_\parallel} \frac{dv_\parallel}{dk} \frac{R(v_\parallel)}{R_0},$$

(2.30)

where $R_0$ is the total volume of the shaded region in Figure 2-12. The shape of the shells and the thickness of the shaded region can be obtained by solving equations (2.27), (2.28) and (2.29) numerically. The ratio $R(v_\parallel)/R_0$ has the simple form,

$$\frac{R(v_\parallel)}{R_0} = \frac{v_\parallel_f - v_\parallel}{v_\parallel_f},$$

(2.31)
where \( \nu_{//f} \), which must be determined numerically, is the largest value of \( \nu_{//} \) on the dispersive bispherical shell. Equation (2.31) may be obtained by noting that the solution of equation (2.29) yields \( \nu_{//}^2 = \nu^2 - 2 \int_0^{\nu_{//}} dx V_p(x) \). Thus any function of \( \nu \) alone is constant along the dispersive bispherical shells. A shell of small thickness is then well represented by

\[
\delta[\nu^2 - 2 \int_0^{\nu_{//}} dx V_p(x) - \nu_0^2] = (2\nu_\perp)^{-1} \delta(\nu_\perp - [\nu_0^2 + 2 \int_0^{\nu_{//}} dx V_p(x) - \nu_0^2]^{1/2}) \, .
\]

Integrating this distribution function over velocity space we readily obtain equation (2.31).

The energy of a reflected proton is \( e = m(\nu_\perp^2 + \nu_{//}^2)/2 \), and according to equation (2.29):

\[
\frac{de}{d\nu_{//}} = -m|V_p|.
\]  

(2.32)

Solving equations (2.30), (2.31) and (2.32) numerically along with the dispersive bispherical shell we derived in Section 2.5.1, we obtain the wave energy density spectrum. The ratio of the magnetic fluctuation power to the total wave power is \((1 + \omega/\Omega)/(2 + \omega/\Omega)\) for ion-cyclotron waves [Hollweg, 2004]; the power spectrum of magnetic fluctuations for each event can then be calculated from \( E(k) \). We note that the power spectrum obtained by Isenberg and Lee [1996] is not completely correct since they assumed the protons are scattered uniformly over the surface of the bispherical distribution rather than uniformly throughout the volume shown in Figure 2-12. For the dispersionless case considered by Johnstone et al. [1991] and Williams and Zank [1994], the surface and volume expressions are identical.
If dispersion is neglected as in Section 2.4, $|V_p| = V_A$ and the volume $R(\nu_\parallel)$ is part of a spherical shell of fixed thickness. Noting that the magnetic fluctuation power is half the total wave power for the Alfvén wave, the wave intensity spectrum as defined previously is:

$$I(k) = \frac{\mu_0 n_i m V_A \Omega}{2 k^2} \frac{\nu_i}{\nu_i - V_A} (1 - \frac{\Omega}{|k|\nu_i})$$

(2.33)

where $\nu_i$ is the speed of the particles in the wave frame (see Figure 2-5a). When $\nu_0 >> V_A$, equation (2.33) is exactly equivalent to equation (2.17).

2.5.4 A Revised Comparison with the AMPTE/IRM Observations

Table 2-2 shows the revised predictions of the temperature anisotropy, the peak frequency and the wave power for the same events of Table 2-1. For all the low Mach number events, the predicted anisotropy is still smaller than that observed.

The predicted power in Events 1, 2, 3, 5 and 9 is almost the same as the power predicted in the simple approach in Section 2.4.1. We may have expected the wave power to be substantially smaller than that predicted by the simple approach because of two effects caused by wave dispersion. One is the reduction of the "free energy" as pointed out in Section 2.5.1; the other is that dispersion results in less than half of the wave energy density appearing as magnetic fluctuation energy density. The explanation for the similar power predicted is that the dispersive power spectra have more of the wave power in the frequency range $0.3\Omega - 0.8\Omega$ in comparison with the nondispersive spectra. The dispersive power spectra of the magnetic fluctuations predicted for 3 of the events are plotted in Figure 2-14, which shows that more than 90% of the wave power
occurs in the frequency range $0.3 \Omega - 0.8 \Omega$ as observed. In contrast, about 50% of the magnetic fluctuation power is outside of this range for the nondispersive spectra (see Figure 2-8). The predicted peak frequencies for the excited waves match the observations with a discrepancy less than 30% for the low-Mach-number events, which is better agreement than in the simple approach. We conclude that for these events with $v_0$ comparable with $V_A$, dispersion is an important correction for the wave spectrum

<table>
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<th>Event</th>
<th>$\left(\frac{\omega}{\Omega}\right)_o$</th>
<th>$\left(\frac{\omega}{\Omega}\right)_i$</th>
<th>$P_{L_o}$</th>
<th>$P_{L_i}$</th>
<th>$\left(\frac{T_{\perp}}{T_{\parallel}}\right)_o$</th>
<th>$\left(\frac{T_{\perp}}{T_{\parallel}}\right)_i$</th>
<th>$\left(\frac{T_{\perp}}{T_{\parallel}}\right)_{i2}$</th>
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<td>.33</td>
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<td>1.3</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Table 2-2 Observed quantities for the events identified by Sckopke et al. [1990]: normalized peak frequency $\left(\frac{\omega}{\Omega}\right)_o$, wave power $P_{L_o}$ and temperature anisotropy $\left(\frac{T_{\perp}}{T_{\parallel}}\right)_o$. Predicted quantities including effects of wave dispersion: normalized peak frequency $\left(\frac{\omega}{\Omega}\right)_i$, wave power $P_{L_i}$ and temperature anisotropy $\left(\frac{T_{\perp}}{T_{\parallel}}\right)_i$. Predicted temperature anisotropy including core protons as well $\left(\frac{T_{\perp}}{T_{\parallel}}\right)_{i2}$. 

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2.6. Waves Excited by the Core Protons at a Perpendicular Shock

As we mentioned in Section 2.2, the observed spectrum of the waves downstream of the bow shock appears to display a double-humped structure. Based on numerical evaluation of the ion-cyclotron wave growth rate, Brinca et al. [1990] suggested that the reflected protons generate the main peak while the directly transmitted core protons generate the secondary peak. The wave excitation by the downstream reflected protons, which produces the main peak, has been discussed in Sections 2.4 and 2.5; we now calculate the spectrum of waves excited by the directly transmitted core protons.

In the marginally supercritical events investigated by Sckopke et al. [1990], more than 90% of the protons are directly transmitted through the shock to form the bulk of

Figure 2-14 Normalized magnetic fluctuation power spectra predicted using dispersive ion-cyclotron waves for Events 1, 3 and 9.
the downstream plasma. It has been shown by simulations [Burgess *et al*., 1989; Wilkinson, 1991; Gedalin, 1996b] that these downstream core protons are initially spread out in the direction perpendicular to the magnetic field and have a large temperature anisotropy just downstream of the shock. Similar to the reflected protons, this distribution is unstable and redistribution of these protons generates ion-cyclotron waves. In the following subsection, we first describe the dynamics of the core protons within the shock ramp and in the region just downstream of the shock; in Section 2.6.2 we calculate the wave spectrum generated by these protons.

**2.6.1 Core Proton Dynamics in the Shock Ramp**

In this Section, we limit our discussion to the protons which are directly transmitted through the shock. Within the very narrow shock ramp in which the magnetic force on the protons can be neglected, the proton distribution function in the shock frame is governed by the Vlasov equation

$$u_x \frac{\partial f}{\partial x} + \frac{eE_x}{m} \frac{\partial f}{\partial v_x} = 0,$$

where $E_x$ is the electric field in the shock ramp. The solution is $f = g(W, v_y, v_z)$, where $g$ is an arbitrary function of the 3 independent variables and $W = 1/2 m v_x^2 + e\Phi(x)$. The electric field $E_x = -d\Phi(x)/dx$, where $\Phi$ increases from 0 to $\phi$ as $x$ increases through the ramp and the overshoot. The distribution function of the core protons just across the shock ramp and overshoot is $f_0(\sqrt{v_x^2 + 2e\phi/m}, v_y, v_z)$, where $f_0(v_x, v_y, v_z)$ is the distribution function of the upstream core protons and we neglect the small fraction of reflected protons. If the distribution of upstream protons is
Maxwellian, namely \( f_0 = \bar{n}_e \exp\left\{-\left[ (v_x - V)^2 + v_y^2 + v_z^2 \right]/2v_{th}^2 \right\}/\left(\sqrt{2\pi v_{th}}\right)^3 \), where \( \bar{n}_e \) is the upstream core proton density, the downstream distribution function is
\[
f = \bar{n}_e \exp\left\{-\left[ (\sqrt{v_x^2 + 2e\phi/m - V})^2 + v_y^2 + v_z^2 \right]/2v_{th}^2 \right\}/\left(\sqrt{2\pi v_{th}}\right)^3 .
\]
Assuming that the thermal speed satisfies \( v_{th} \ll \sqrt{V^2 - 2e\phi/m} \) as appropriate for these low-\( \beta \) bow shock crossings, the downstream distribution function adjacent to the shock ramp is
\[
f = (n_e/X)\exp\left\{-\left[ (v_x - V_d)^2/X^2 + v_y^2 + v_z^2 \right]/2v_{th}^2 \right\}/\left(\sqrt{2\pi v_{th}}\right)^3 ,
\]
where \( n_e = X\bar{n}_e \) is the downstream core proton density. Clearly the thermal speed in the \( x \) direction increases by a factor of the compression ratio \( X \), while the \( v_y \) and \( v_z \) dependence of the distribution does not change in the shock ramp.

The distribution function of the downstream transmitted protons is governed by
\[
(V_d + \delta v_x) \frac{\partial f}{\partial x} + \Omega(v_y \frac{\partial f}{\partial v_x} - \delta v_x \frac{\partial f}{\partial v_y}) = 0 \tag{2.36}
\]
where \( \delta v_x = v_x - V_d \) and we first neglect the proton scattering and impose \( f(x = 0^+) \) as the boundary condition. Replacing variables \( \delta v_x \) and \( v_y \) with perpendicular velocity \( v_\perp(v_\perp^2 = \delta v_x^2 + v_y^2) \) and phase angle \( \psi \), equation (2.36) becomes
\[
(V_d + \delta v_x) \frac{\partial f}{\partial x} - \Omega \frac{\partial f}{\partial \psi} = 0 . \tag{2.37}
\]
The general solution is \( f = g(v_\perp, v_x, L) \), where \( L = x + (V_d/\Omega)\psi + (v_\perp/\Omega)\sin\psi \), which describes a simple gyration of the ions. Since the dependence of \( f \) on \( \psi \) must be periodic with period \( 2\pi \), \( g(L) \) must be periodic with period \( 2\pi V_d/\Omega \). Viewed in the
downstream flow the spatially-averaged proton distribution function, which controls the AIC instability, can be obtained by averaging $g$ over the period of $L$:

$$
\frac{\Omega}{2\pi v_d} \int_0^{2\pi v_d} f(v_{\perp}, v_{\parallel}, \psi, x) dx = \frac{\Omega}{2\pi v_d} \int_0^{2\pi v_d} g(v_{\perp}, v_{\parallel}, L) dL.
$$

(2.38)

The integral over $L$ can be replaced by an integral over $\psi$:

$$
\frac{\Omega}{2\pi v_d} \int_0^{2\pi v_d} f(v_{\perp}, v_{\parallel}, \psi, x) dx = \frac{\Omega}{2\pi v_d} \int_0^{2\pi} \left[ f(v_{\perp}, v_{\parallel}, \psi, x_0) \frac{dL}{d\psi} d\psi \right]
$$

(2.39)

where $x_0$ is an arbitrary positive value which may be chosen as $x_0 = 0^\circ$. The integral involving the second term of $dL/d\psi$, $(v_{\perp}/\Omega)\cos\psi$, vanishes by definition of the downstream plasma velocity $V_d$. Thus, the spatially-averaged distribution is gyrotropic and given by the $\psi$-average of equation (2.35):

$$
f(v_{\parallel}, v_{\perp}) = \frac{n_c}{2\pi X^3(\sqrt{2\pi \nu_{th}})^3} \exp\left(-\frac{\nu_{th}^2}{2\nu_{th}^2}\right) \int_0^{2\pi} \exp\left[-\frac{\nu_{\perp}^2}{2\nu_{th}^2} \left( \cos^2 \psi + \sin^2 \psi \right) \right] d\psi.
$$

(2.40)

The downstream plasma-$\beta$ predicted by this analysis matches the observed $\beta$ within the observational accuracy for all the marginally supercritical events.

By analyzing the simulated behavior of the distribution of core protons, Wilkinson [1991] concludes that their distribution is elongated in the $v_x$ direction in the shock ramp; these protons gyrate and evolve into a highly anisotropic distribution once they reach the stable magnetic field downstream of the shock. His conclusion is consistent with our simple analysis. A potential shortcoming of our analysis is that when the core and reflected protons gyrate about each other, they change the average flow velocity which must be balanced by the magnetic field and proton pressure gradient, and
the motional electric field gradient. A varying electromagnetic field is contradictory with our assumption that the downstream fields are constant. However, in an analysis similar to ours, Gedalin [1996a] shows that after the initial overshoots the downstream magnetic field varies by only 10% in a shock with Mach number 2.5, in support of our assumption.

The gyration of the core protons about the proton center of mass contributes to the “thermal” broadening of the core distribution. We can estimate the scale of the gyration speed \( v_{gc} \) of the core protons in terms of the gyration speed \( v_{gr} \approx v_0 \) of the reflected protons as \( v_{gc} = v_{gr} n_r / n_c \); we find that \( v_{gc} \ll Xv_{sh} \). Therefore, the contribution of this gyration to the core proton anisotropy can be neglected.

The heating of the directly transmitted protons is also investigated by Ellacott and Wilkinson [2003]. Our analysis is consistent with their result for shocks with low incident temperature that the distribution function is stretched in the shock normal direction but remains unaffected in the directions transverse to the shock normal.

2.6.2 The Power Spectrum of the Waves Excited by the Core Protons

We calculate the wave spectrum excited by the core protons downstream of a perpendicular shock according to the method appropriate for the dispersive bispherical distribution described in Section 2.5.3. We first illustrate the calculation in the simple case that we replace the thermal spread of the core protons enhanced perpendicular to the magnetic field with a ring with \( v_{\perp} = v_0 \approx v_{th} \) and \( v_{\parallel} = 0 \) just as for the reflected protons. Then we perform the calculation under the initial condition that the core protons are distributed as described by equation (2.40). Although the interaction of the waves and protons downstream of the shock is a nonlinear process, in this case the waves
excited by each species of protons (reflected or core) do not affect the asymptotic
distribution of the other. Thus we may calculate the wave spectrum excited by each
species and then add them to obtain the total wave spectrum.

2.6.2.1 Ring Distribution

Since the thermal speed downstream is generally much smaller than the Alfvén
speed for these low-$\beta$ shocks, we may calculate the wave power spectrum and the
dispersive shell in the limit $v_0 \ll V_A$. This allows the analysis of Sections 2.4.1 and
2.4.3 to be performed analytically. We note that since we consider the thermal spread of
the core protons, the cold plasma dispersion relation may not be valid, especially when
we discuss the waves resonant with protons of small $v_\parallel$.

The interaction of waves and protons is governed by equations (2.27) and (2.28),
which may be rewritten as follows:

\[ p = lq - 1 \quad (2.41) \]
\[ p = \pm q \sqrt{1 + p} \quad (2.42) \]

where $l = v_\parallel / V_A$, $p = \omega / \Omega$, and $q = kV_A / \Omega$. For $l \to 0$ as appropriate for $v_0 \ll V_A$,
and only considering waves propagating in the negative direction appropriate for $l > 0$,
we obtain

\[ p = -1 + l^{2/3} \quad (2.43) \]
\[ q = l^{-1/3} \quad (2.44) \]

where we retain only the leading power in $l$. Combining equations (2.43), (2.44) and
(2.29), we obtain for the shape of the dispersive bispherical surface
\[ \frac{v^2_1}{v^2_A} = \frac{v^2_0}{v^2_A} - \frac{3}{2} l^{4/3} . \]  

According to the shape of the surface, the net number of the protons per unit volume scattered across \( l = v_{||}/V_A \) is \( n_c[l - (l/l_0)]/2 \), where \( l_0 = [(2v^2_0)/(3V^2_A)]^{3/4} \) is the maximum normalized parallel velocity on the bispherical shell. Using equations (2.30) – (2.32) (replacing \( n_r \) with \( n_c \)) and noting that the power of the magnetic fluctuations is \((1 + \omega/\Omega)/(2 + \omega/\Omega)\) times the total wave power, we obtain the power spectrum of the magnetic fluctuations as 

\[ I(q) = \frac{3n_c}{N_c q^7} \left[ 1 - \left( \frac{1}{l_0 q^3} \right) \right] B_0^2 , \]  

where \( 2 \int dq I(q) = \left\langle |\mathbf{B}|^2 \right\rangle \). The peak frequency is \( \Omega(1 - [4/(7l_0)]^{3/3}) \); the total wave power of the magnetic fluctuations is \((n_c/N_c)l_0^2 B_0^2\).

Figure 2-15 shows the magnetic power spectrum generated by reflected (dotted line) and core protons (solid line) for Event 1. The magnetic power spectrum is obtained by the numerical solution of equations (2.29)-(2.32), (2.41) and (2.42); the value of \( \nu_0/V_A \approx \nu_m/V_A \) is such that the approximation leading to equation (2.46) is inaccurate by more than 50%. The peak frequency of the waves generated by the core protons is reasonably consistent with the observed location of the secondary peak, but the secondary peak has a larger magnitude, which is contradictory with the observations. The core protons are redistributed in a narrow range of \( v_{||} \) for two reasons: \( v_0 \approx v_m \) is small for the core protons so that \( v_{||f} << v_0 \), and the thickness and area of the slab shown in Figure 2-12 are smaller in the region near \( v_{||f} \) so that fewer protons are...
scattered to larger $v_\parallel$. This makes the wave spectrum, which is calculated according to the redistribution, sensitive to the initial core proton distribution at $x = 0^+$. Thus we cannot neglect the initial thermal spread of the downstream core protons.

Figure 2-15  Magnetic fluctuation power spectra of waves excited by core (solid line) and reflected (dashed line) protons. The core protons just downstream of the perpendicular shock are assumed to have a ring distribution with $v_0 = v_{th}$.

2.6.2.2 Broad Distribution

We now calculate the wave excitation assuming that the core protons just downstream have a broad distribution as described by equation (2.40). The redistribution of bimaxwellian distributed protons has been investigated by Isenberg [2003]. We modify his analysis and apply it to the core proton redistribution. The downstream core protons are redistributed onto a family of shells, the shapes of which are given by the integration of equation (2.29) [Isenberg and Lee, 1996; Isenberg, 2003]:

$$v_\parallel^2 + v_\perp^2 - 2 \int_{0}^{\nu_p} \nu_p' (v_p') dv_p' = \text{const} = \eta^2$$  \hspace{3cm} (2.47)
where $\eta$ specifies the shell. The distribution function is constant along each shell. The asymptotic distribution $f(\eta)$ is given by averaging the initial distribution between shells $\eta$ and $\eta + d\eta$. The differential volume for variables $(\eta, \nu_\parallel)$, integrated over gyrophase, is $2\pi \eta d\eta d\nu_\parallel$; integrating equation (2.40) we then obtain

$$f(\eta) = \frac{1}{\nu_{\parallel f}(\eta)} \int_0^{\nu_{\parallel f}(\eta)} d\nu_\parallel \, f(\sqrt{\eta^2 - \nu_\parallel^2 + 2 \int_0^{\nu_{\parallel f}(\eta)} V_p(\nu_\parallel') d\nu_\parallel', \nu_\parallel})$$

(2.48)

where $f(\nu_\perp, \nu_\parallel)$ is given explicitly by equation (2.40). The thermal spread is

$$\langle \nu_\parallel^2 \rangle = \frac{2\pi}{n_c} \int_0^{\nu_{\parallel f}(\eta)} d\eta \int_0^{\nu_{\parallel f}(\eta)} d\nu_\parallel \nu_\parallel^2 \propto T_\parallel$$

(2.49a)

$$\langle \nu_\perp^2 \rangle = \frac{2\pi}{n_c} \int_0^{\nu_{\parallel f}(\eta)} d\eta \int_0^{\nu_{\parallel f}(\eta)} d\nu_\parallel \nu_\perp^2 \propto 2T_\perp$$

(2.49b)

The total downstream temperature anisotropy includes both the reflected protons and the core protons: $(n_c T_\perp + n_r T_\perp)/(n_c T_\perp + n_r T_\perp)$.

We calculate the total temperature anisotropy $(T_\perp/T_\parallel)_{12}$ for all the events; the new results are shown in the final column of Table 2-2. Comparing these results with those we obtained in Section 2.5, it is obvious that including the anisotropy of the core protons makes the discrepancy between the predicted anisotropy and the observed anisotropy much smaller. For most of the events the discrepancy is less than 10%. Event 1 has the largest discrepancy of about 20%. The agreement is excellent considering the uncertainties involved in the measurements, and is consistent with our assumption that the quasi-equilibrium has been reached at the furthest point from the shock. We note that the reflected protons are not resolved in the observed proton distribution functions and
the observed anisotropy is actually the composite anisotropy of the reflected and core protons.

We now derive the net number of protons scattered across $v_{\parallel}$ to obtain the wave spectrum. For the shell distribution (2.48), the number density of the protons satisfying $v_{\parallel}' < v_{\parallel}$ is

$$N_{S}(v_{\parallel}) = 2\pi \int_{0}^{\infty} f(\eta) \eta d\eta \int_{0}^{\min(v_{\parallel}, v_{\parallel}')} d\nu_{\parallel}' . \quad (2.50a)$$

For the initial distribution this density is

$$N_{I}(v_{\parallel}) = n_{c} \frac{1}{\sqrt{2\pi v_{\parallel}}{\nu_{th}}} \int_{0}^{v_{\parallel}} \exp \left[-\left(\frac{v_{\parallel}'}{2v_{\parallel}^{2}}\right)^{2}\right] d\nu_{\parallel}' . \quad (2.50b)$$

The number of protons scattered across $v_{\parallel}'$ is $N_{I}(v_{\parallel}') - N_{S}(v_{\parallel})$. The magnetic power spectrum of the waves excited by the core protons is then

$$[(1 + \omega/\Omega)/(2 + \omega/\Omega)](N_{I} - N_{S})mV_{p} (dv_{\parallel}/dk),$$

where $\omega$ and $v_{\parallel}$ are expressed in terms of $k$ by solving for the intersection of the resonance line and the dispersion relation as we have done for the reflected protons in Section 2.5.3. The composite spectrum for Event 1 is shown in Figure 2-16 along with the wave spectrum contributed by each component. The predicted spectrum displays a double-humped structure. It shows that the secondary peak, excited by the core protons, has a peak frequency about twice the main peak frequency derived in Section 2.5.3. However, the secondary peak contains less power by a factor of ~0.3. These features match the observed spectrum very well except for the fact that the observed secondary peak is partly right-hand-polarized and contains ~1% of the power of the main peak.
Figure 2-16 Magnetic power spectra of waves excited by core protons (dashed line), reflected protons (dotted lines) and the sum (solid line). The core protons just downstream of the shock have a broad distribution as described by equation (2.40).

An interesting feature of Figure 2-16 is the frequency range of the waves excited by the core protons. For a ring distribution with $v_\perp = v_{ih}$ and $v_\parallel = 0$ as discussed in Section 2.6.2.1, the corresponding bispherical shell leads to excitation of waves predominantly by protons with small $v_\parallel$. Thus, in this case the power of the excited waves resides mainly in the vicinity of frequency $\Omega$ as shown in Figure 2-15. For the broad distribution discussed above, the redistribution of protons onto the family of dispersive bispherical shells extends to larger $v_\parallel$ corresponding to the lower frequency range $\sim(0.6\Omega - 0.7\Omega)$ as shown in Figure 2-16. In addition, for the inner shells with $v_\perp < v_{ih}$ the redistribution of protons is toward smaller $v_\parallel$, which leads to wave stability and reduced wave power at frequencies near $\Omega$. These features are evident in Figure 2-17 which shows contours of the initial broad distribution (solid lines) and the family of
Figure 2-17 Representative dispersive bispherical shells (dashed lines) and the contours (solid lines) of the spatially-averaged distribution function of the core protons just downstream of the shock corresponding to equation (2.40). Redistribution of protons along shells 1 and 2 leads to a net transfer of protons to smaller speed and therefore wave growth, whereas redistribution on the inner shells leads to a net transfer to larger speed and wave damping.

dispersive bispherical curves (dashed lines). In fact, the damping of waves in the frequency range $\omega \geq 0.75\Omega$ will lead to a decrease in the total wave power in that range, and a modification of Figure 2-16; we have not included this small effect.
2.7 Waves Excited by Helium

An additional possible origin of the enhanced fluctuation power downstream of the shock is the heavy ions, dominated by He$^{2+}$. The He$^{2+}$ ions behave very differently from the protons at the bow shock ramp because of their different mass per charge. Virtually all of the incident solar wind He$^{2+}$ ions will be transmitted through the shock potential since they have approximately twice the kinetic energy per charge of the protons [Fuselier and Schmidt, 1994]. Thus all He$^{2+}$ ions form a ring distribution downstream of the shock, which gyrates in the frame of the directly transmitted protons (as do the reflected protons); they should generate waves as well.

2.7.1 Trajectory of Helium

For a perpendicular shock, the speed of He$^{2+}$ in the downstream plasma frame is:

$$v_{He} = \sqrt{V^2 - \frac{e\phi}{m} \frac{V}{X}}$$

(2.51)

where the helium mass is $4m$.

For a quasi-perpendicular shock, the velocity of He$^{2+}$ in the downstream plasma frame is:

$$v_x = \sqrt{V^2 - \frac{e\phi}{m} \frac{V}{X}}$$

(2.52a)

$$v_z = -(\chi - 1)V \cot \theta$$

(2.52b)

The corresponding speed of He$^{2+}$ in the downstream plasma frame for every event described by Sckopke et al. [1990] is listed in Table 2-3. We can see that they are smaller than the downstream Alfvén speeds.
Table 2-3 Calculated downstream normalized helium speed for the events identified by Sckopke et al. [1990]. Power contributed by helium in the helium-dominant case \( P_{L1} \) and the proton-dominant case \( P_{L2} \). Total estimated magnetic fluctuation power \( P_{total} \) and the observed range of \( P_{Lo} \).

<table>
<thead>
<tr>
<th>Event</th>
<th>( \frac{v_{He}}{V_A} )</th>
<th>( P_{L1} )</th>
<th>( P_{L2} )</th>
<th>( P_{total} )</th>
<th>( P_{Lo} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.003</td>
<td>0.0006</td>
<td>0.012</td>
<td>.01-.05</td>
</tr>
<tr>
<td>2</td>
<td>0.45</td>
<td>0.003</td>
<td>0.0006</td>
<td>0.012</td>
<td>.01-.04</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>0.006</td>
<td>0.0006</td>
<td>0.03</td>
<td>.01-.04</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>0.006</td>
<td>0.0006</td>
<td>0.03</td>
<td>.02-.06</td>
</tr>
<tr>
<td>9</td>
<td>0.35</td>
<td>0.003</td>
<td>0.0006</td>
<td>0.072</td>
<td>.01-.04</td>
</tr>
<tr>
<td>X</td>
<td>0.7</td>
<td>0.01</td>
<td>0.0006</td>
<td>0.33</td>
<td>.05</td>
</tr>
</tbody>
</table>

2.7.2 Distribution of Helium and its Wave Excitation

The condition for cyclotron resonance between a \( \text{He}^{2+} \) ion with velocity component \( v_\parallel \) and a wave with frequency \( \omega \) and wavenumber \( k \) is [Stix, 1992]:

\[
\omega(k) = kv_\parallel - \frac{\Omega}{2}
\]

We shall neglect the effect of the \( \text{He}^{2+} \) ions on the dispersion relation, so that the dispersion relation is described by equation (2.28). The solution of equations (2.28) and (2.53) yields the resonant wave speed for a given value of \( v_\parallel \). The solution is again shown schematically in a \( \omega-k \) plot in Figure 2-18. The straight line crossing the point \((k = 0, \omega = -\Omega/2)\) is the resonant line for \( v_\parallel > 0 \); the intersection of the resonant line and the dispersion curve yields the wavenumber \( k \) and frequency \( \omega \) of the resonant wave. For small \( v_\parallel \) (\( |v_\parallel| < 0.17V_A \)), the resonant line intersects both dispersion curves \( L_+ \) and \( L_- \). Thus \( \text{He}^{2+} \) with small \( v_\parallel \) is resonant with left-hand-polarized waves propagating in both directions, but not with the right-hand-polarized wave. Actually the
line for small positive $v_\parallel$ intersects the $L_+$ curve at two points; presumably the interaction is dominated by the intersection with smaller $|k|$. 

![Figure 2-18 Dispersion relations and the resonance line for He$^{2+}$](image)

For a perpendicular shock, the He$^{2+}$ ions just downstream of the shock have $v_\parallel = 0$. Figure 2-19 shows schematically the scattering of He$^{2+}$ ions from $v_\parallel = 0$ into the domain $v_\parallel > 0$. The solid line corresponds to ions scattered by $L_-$ waves. In this case the waves, which satisfy $\omega > -0.5\Omega$, are unstable since the ions lose energy. The dotted line corresponds to ions scattered by $L_+$ waves. In this case the waves, which satisfy $\omega < -0.5\Omega$, lose energy since the ions gain energy. However, scattering by $L_-$ waves only occurs for $v_\parallel < v_{\parallel 1}$ as indicated in Figure 2-19.
Figure 2-19 Schematic diagram showing surfaces followed by scattered He\(^{2+}\) in velocity space for \(v_{//} > 0\) downstream of a perpendicular shock in two cases: (i) He\(^{2+}\) dominates wave damping/excitation in the joint (with protons) frequency range corresponding to \(v_{//} < v_{//1}\) (lower solid curve, see text); (ii) protons dominate (dashed curve; see text). The dotted curve shows the path of He\(^{2+}\) if scattered by the waves damped by He\(^{2+}\).

If the contribution of the He\(^{2+}\) is neglected, the downstream reflected protons excite the left-hand-polarized waves propagating in both directions. Combining the effects of He\(^{2+}\) and the downstream reflected protons yields a wave configuration, depending on shock parameters, intermediate between two limiting cases: He\(^{2+}\) dominates the proton excitation of the waves in the frequency range resonant with...
both species, or the reverse. Here we neglect the possible process in which the He\textsuperscript{2+} ions are scattered stochastically by the waves propagating in both directions.

In the He\textsuperscript{2+} -dominant case, the waves damped by He\textsuperscript{2+} are of negligible intensity and the He\textsuperscript{2+} ions are scattered along the shell locally tangent to the sphere centered on the phase speed of the resonant unstable wave (lower solid curve in Figure 2-19). An interesting feature in this case is that the He\textsuperscript{2+} ions will probably affect the proton distribution since some protons will not be scattered across the velocity range corresponding to the shared frequency range if the resonant waves are damped. This possible effect is neglected in the calculations we have presented in Sections 2.4 and 2.5.

For the proton-dominant case, the wave excitation by the downstream reflected protons dominates the damping or growth due to the He\textsuperscript{2+} ions, which are scattered by these resonant waves as test particles. Since He\textsuperscript{2+} ions moving with $|v_\parallel| < v_{\parallel 1}$ interact with waves propagating in both directions, they are scattered onto a shell which is locally tangent to a sphere centered on the average phase speed (weighted by the wave intensities) of the resonant waves according to quasilinear theory [Bogdan et al., 1991]. This path in velocity space is shown as a dashed line in Figure 2-19. They shift to a shell centered on the single resonant wave if $|v_\parallel| > v_{\parallel 1}$ (solid continuation of dashed curve).

To obtain the He\textsuperscript{2+} shell we integrate equation (2.29) numerically, replacing $V_p$ with the appropriate phase velocity. In the He\textsuperscript{2+} -dominant case, we replace $V_p$ with the phase speed of the resonant unstable wave. In the proton dominant case, we replace $V_p$ with $(I_+ V_+ + I_- V_-)/(I_+ + I_-)$. Here $V_\parallel (V_\perp)$ is the phase velocity, and $I_+ (I_-)$ the
magnetic fluctuation power, of the resonant waves propagating in the +(-) direction. The total wave power excited by transmitted He\(^{2+}\) is then

\[
E_{\text{He}}(k) = -n_{\text{He}} \frac{de}{dv_{\parallel}} \frac{dv_{\parallel}}{dk} \frac{R(v_{\parallel})}{R_0}
\]

(2.54)

where \(\frac{de}{dv_{\parallel}} = -m_{\text{He}} V_p\), \(V_p\) is the appropriate resonant wave speed for the case considered, and \(\frac{R(v_{\parallel})}{R_0}\) is given by equation (2.31). The power spectrum \(I_{\text{He}}(k)\) of magnetic fluctuations excited by He\(^{2+}\) for each event is then calculated numerically from \(E_{\text{He}}(k)\). Integrating the power spectra over the frequency range 0.3 \(\Omega\) - 0.8 \(\Omega\), we obtain the wave power contributed by helium.

The He\(^{2+}\) distributions in the two extreme cases are shown in Figure 2-20 for Event 1. The predicted wave power contributed by He\(^{2+}\) to each event for the two extreme cases, \(P_{L1}\) for dominant He\(^{2+}\) and \(P_{L2}\) for dominant protons, is listed in Table 2-3. Clearly the He\(^{2+}\) power is dominated by the reflected proton power; although \(n_{\text{He}} \sim n_r, v_0\) is much smaller for helium. We expect the contribution of He\(^{2+}\) to lie somewhere between the two extreme cases. Assuming that the actual free energy is the average of the two extreme cases, we estimate the total power of magnetic fluctuations by adding the average helium power to the power excited by the reflected protons. The results are shown together with the observed power in Table 2-3. These values match the observed power range for Events 1, 2, 3, and 5. Event 9 has a higher Mach-number and we expect that nonlinear processes may affect the ion-wave interaction so that the observed power is substantially lower than that predicted.
Figure 2-20  Helium distribution in the two extreme cases (See Fig. 19, caption): (i) $\text{He}^{2+}$ dominates (dotted curve); (ii) protons dominate (solid curve).

2.8 Conclusions

Using a model based on quasilinear theory and simple trajectories of reflected and transmitted protons, and transmitted helium, we have calculated the temperature anisotropy and bulk velocity of the protons and the peak frequency and power of the excited ion-cyclotron waves, downstream of the quasi-perpendicular bow shock. The
results apply to the quasi-equilibrium "plateau" toward which the downstream plasma evolves. In an initial simple analytical approach we neglect the effects of wave dispersion, the transmitted core protons, and the minor ion helium; then we take these effects into account.

We also estimate the timescale of the quasilinear relaxation process, which is inversely proportional to the number of reflected protons and their speed downstream of the shock. This result is qualitatively consistent with the fast relaxation observed downstream of the high Mach number event presented by Sckopke et al. [1990] and the relatively slow relaxation for all the low Mach number events. It is impossible to determine the timescale quantitatively from observations with a single spacecraft measurement. Cluster measurements should provide data to test the predicted timescale using the 4 spacecraft.

The estimate of the wave power generated by He\textsuperscript{2+} downstream of the shock is only illustrative. Here the calculation is complicated by the fact that in one frequency range the waves are excited by protons and damped by helium. The predicted wave power depends on which species controls the wave intensity in that frequency range. Our estimate assumes the average of the two extremes, in which the protons dominate He\textsuperscript{2+} or the reverse, to represent the predicted power excited by the helium. However, this assumption may not represent the wave power accurately due to the complexity of the interplay between the protons and He\textsuperscript{2+}; further investigation of the He\textsuperscript{2+} contribution using simulations and/or comparison with Cluster data is required in the future.
Comparing all these calculations and observations, we draw the following conclusions:

1. Quasilinear theory works remarkably well. The predicted temperature anisotropies match the observations very well when core protons are taken into account. The predicted power spectra of magnetic fluctuations match the observations of the polarization (predominantly left-circular polarized) and the frequency range in which most of the power is contained. In the marginally supercritical events, the predicted power also matches the observations very well.

2. Wave dispersion is important in predicting the power spectra of the magnetic fluctuations. The predicted frequency range of the enhanced magnetic fluctuations matches the data very well only when effects associated with dispersion are taken into account. Since the reflected proton speeds are comparable with $V_a$ in the marginally supercritical events Sckopke et al. [1990] investigated, these protons resonate mainly with waves of higher wavenumber; dispersive effects for these waves are substantial as described by equation (2.28). An interesting and important feature of ion-cyclotron waves in this context is that the kinetic energy density of the fluctuations exceeds the magnetic energy density.

3. The transmitted core protons downstream of the shock contribute to the temperature anisotropy and the excited wave intensity. Although the power of the core proton contribution to the wave intensity is small, it occurs at about double the peak frequency of the reflected proton contribution; thus, the total predicted power spectrum has a double-humped structure. This feature matches the observations very well, and it is also consistent with the linear growth rate analysis presented by Brinca et al. [1990].
4. The minor ions, predominantly $\text{He}^{2+}$, contribute to the ion-cyclotron instability downstream of the shock. The approximate calculations indicate that the power excited by $\text{He}^{2+}$ is 20%~30% of the power excited by the reflected protons, but this number awaits more detailed calculations.

There are other processes or features neglected in our calculations which could affect the ion-wave interaction downstream of the shock:

a) We use the cold plasma dispersion relation as the ion-cyclotron wave dispersion relation in the downstream plasma. As we pointed out in Section 2.6, this dispersion relation is probably not appropriate for the waves interacting with the core protons since these waves should be affected by the thermal spread of the core proton distribution. We also neglect the contribution of $\text{He}^{2+}$ ions to the dispersion relation. A more precise model for this wave/ion interaction process should use a dispersion relation based on the distribution functions of warm protons and the $\text{He}^{2+}$ ions.

b) According to our theory, the predicted velocity distribution of the reflected protons in the downstream region is a thin shell; however, the observed distributions are broad and do not even resolve core and reflected proton components. The origin of the broad distribution is unclear. We do neglect the upstream temperature of the reflected protons and we neglect the turbulence in the shock ramp, both of which could lead to a thicker predicted shell. Quasi-reflected protons [Zilbersher et al., 1998] may fill in the gap between the reflected and core proton components.

c) We noted at the beginning of Section 2.4 that the reflected protons are not gyrotropic as we assume in the quasilinear analysis since their motion is coherent in gyrophase. However, Motschmann et al. [1999] investigated the stability of a
nongyrotropic ion distribution and concluded that the gradients in gyrophase do not introduce new strong instabilities to compete with that which we consider here.

d) For a high-Mach-number event, like Event X investigated by Sckopke et al. [1990], the wave intensity is sufficiently large that quasilinear theory is not valid. The features of this event are completely different from those of the low-Mach-number events. The ion distributions evolve very rapidly in the downstream flow and the left-hand-polarized, right-hand-polarized, and longitudinal waves grow with similar amplitudes. Obviously nonlinear processes, and waves generated very close to the shock ramp and overshoot, dominate the downstream wave/ion interaction processes; further investigation of such events will be conducted in future work.

e) The large variations of the wave power at the stage when the temperature anisotropy is saturated is puzzling. We expect the wave power to be nearly constant at that stage. Further investigations of similar shock crossing data using the CLUSTER spacecraft may provide useful information to explain the puzzle.
CHAPTER 3

ION THERMALIZATION AND WAVE EXCITATION DOWNSTREAM OF EARTH'S BOW SHOCK: A THEORY FOR CLUSTER OBSERVATIONS OF HE\(^{2+}\) ACCELERATION

3.1 Introduction

At Earth's bow shock where the interplanetary magnetic field is nearly perpendicular to the shock normal direction, a small fraction of the incident solar wind protons are reflected from the shock while most of the protons are transmitted through the shock directly. The reflected protons gyrate back to the shock, and then cross the shock plane to the downstream region. These reflected protons have a different velocity from the downstream bulk flow which is dominated by the transmitted protons; they gyrate around the magnetic field in the frame of the transmitted protons with a large temperature anisotropy \(T_\perp/T_\parallel\), where \(\perp\) and \(\parallel\) refer to the directions perpendicular and parallel to the average magnetic field. The directly transmitted protons are also heated more strongly in the direction perpendicular to the magnetic field during their transversal of the shock potential [Ellacott and Wilkinson, 2005; Liu et al., 2005]. The resulting anisotropic distribution is unstable to the excitation of ion-cyclotron waves or mirror-mode waves, depending on the value of the plasma-\(\beta\), which is the ratio of the thermal pressure to the magnetic pressure. These waves, in turn, scatter the protons into a marginally stable distribution with smaller temperature anisotropy as the plasma convects further.
downstream. The ion-wave interaction reaches a quasi-equilibrium at some distance downstream of the shock where temperature anisotropy and wave power saturate. These reflected protons and their interaction with waves downstream has been investigated extensively using satellite observations [Sckopke et al., 1983, 1990], simulations [McKean et al., 1995] and linear and quasilinear theory [Gary et al., 1993, 1997; Liu et al., 2005]. Generally, Alfvén ion-cyclotron instabilities dominate the ion-wave interaction for plasma-\(\beta < 1\) and mirror-mode waves dominate only if \(\beta > 1\) [Winske and Quest, 1988; Gary et al., 1992; Yoon, 1992; Cazykowska et al., 1998]. Sckopke et al. [1990] selected several low Mach number and low \(\beta\) shock crossings to investigate Alfvén ion-cyclotron instabilities downstream of the shock. For these events the instabilities evolve slowly since the wave intensity is low and quasilinear theory is expected to be valid. The calculations presented in Liu et al. [2005] are based on the simplifying assumptions that the reflected protons have a ring distribution function just downstream of the shock and are scattered onto a bispherical shell as the plasma moves further downstream if wave dispersion is neglected. The bispherical shell is comprised of two spherical “caps” in velocity space, each of which is centered on the opposite wave phase velocity as shown in Figures 6 and 10 in Liu et al. [2005]. The form of the bispherical shell is due to ion energy conservation in the resonant wave frame as the ions are scattered by the unstable waves [Galeev and Sagdeev, 1988; Isenberg and Lee, 1996]. The saturated temperature anisotropy and wave spectra predicted by quasilinear theory match the observations very well.

The incident solar wind also contains \(\text{He}^{2+}\) ions and other minor ions which are directly transmitted at the shock due to their higher energy per charge. It has been
observed that these downstream helium ions initially have a ring distribution as expected [Fuselier and Schmidt, 1994]. Gary et al. [1993] calculated wave growth rates in a plasma consisting only of protons and electrons, and in a plasma consisting of protons, electrons and He\textsuperscript{2+} ions. The calculations are based on a numerical evaluation of the wave dispersion relation assuming bimaxwellian particle distribution functions. For plasma with electrons, protons and a small admixture of He\textsuperscript{2+} ions, the growth rate for waves in the frequency range \( v_{ga} < v < v_g \) \((v_{ga})\) is the downstream proton (He\textsuperscript{2+}) gyrofrequency\), which are claimed to be excited by a proton instability in their paper, is smaller than the growth rate in the plasma with no He\textsuperscript{2+} ions. Although their study shows that the helium ions diminish the "proton cyclotron instability", it does not describe how the ions are scattered by the excited waves. Other recent simulations have also investigated the wave-ion interaction including He\textsuperscript{2+} [Lu and Wang, 2005; Gary et al., 2006; Lu et al., 2006]. Using quasilinear theory Liu et al. [2005] calculated the saturated He\textsuperscript{2+} distribution functions and the excited wave power only under two extreme assumptions: that the helium growth rate dominates that of the protons, and vice-versa.

It is worth noting that these ion-wave interaction processes also operate in the solar wind. Lee and Ip [1987] predicted the wave spectra generated by interstellar pickup ions in the solar wind. Galeev and Sagdeev [1988] predicted that cometary ions form a bispherical distribution function. Tu and Marsch [2002] found that the protons in the solar wind are distributed along bispherical shells as predicted by quasilinear theory. In the solar corona heavy ions such as O\textsuperscript{5+} are accelerated in the direction perpendicular to the magnetic field as shown by remote sensing [Kohl et al., 1998; Esser et al., 1999]. Isenberg [2001], Hollweg and Isenberg [2002], and Isenberg and Vasquez [2006] showed
that these heavy ions are accelerated by ion-cyclotron waves with which these particles may have multiple resonances. Can He\(^{2+}\) ions downstream of Earth’s bow shock, which have multiple resonances with ion-cyclotron waves, also be accelerated? If they are accelerated, does the energy going to He\(^{2+}\) acceleration cause the reduction of the proton instability growth rate as shown by Gary et al. [1993]?

In this chapter we present Cluster data for a crossing of Earth’s bow shock on March 31, 2001. At this time and location the shock is nearly perpendicular with a lower Mach number than typical for Earth’s bow shock. In this case we expect the excited wave intensity to be low and quasilinear theory to be valid. The data shows that the He\(^{2+}\) perpendicular temperature just downstream of the shock increases more rapidly than the parallel temperature. Obviously some of the He\(^{2+}\) ions are accelerated primarily in the direction perpendicular to the magnetic field, so that the temperature anisotropy increases.

We develop a quasilinear theory to describe the evolution of the He\(^{2+}\) ion distribution function. An evolutionary equation is derived based upon several reasonable assumptions. The resulting temperature anisotropy is compared with that which is observed. We show that the theory matches the observations remarkably well. We also calculate the proton temperature anisotropy and the power density spectrum for the waves generated by the reflected protons and the He\(^{2+}\) ions at the time when the process reaches a quasi-equilibrium. The predictions and the observations match each other remarkably well considering the accuracy of the measurements.

This Chapter is arranged as follows: Section 3.2 documents the observed wave spectrum and the ion distribution functions. Section 3.3 presents the theory for the evolution of the helium distribution function and its temperature anisotropy. Section 3.4
presents a calculation of the proton temperature anisotropy and the power density spectrum for the waves. Discussion and conclusions are presented in Section 3.5.

3.2 Observations

The four Cluster spacecraft traversed Earth’s bow shock several times on 31 March 2001 as they moved with increasing distance from Earth. The first crossing of the bow shock was at about 17:10 UT, followed by further inbound and outbound crossings relative to the shock until 20:00 UT as the bow shock moved inwards and outwards. The magnitude of the magnetic field, obtained by the Fluxgate Magnetometer (FGM) on Spacecraft 1 and shown in Figure 3-1(a), shows all the inbound and outbound crossings. The crossing we investigate in this chapter is an inbound crossing which occurred at 17:17:48 UT for Spacecraft 1. This event was chosen for study because the temperature anisotropy of He\textsuperscript{2+} exhibits a rapid increase just downstream of the shock before it decreases further downstream. The high-resolution magnetic field data, which has a resolution of about 0.04 s, is shown in Figure 3-1(b) for the time around shock passage. The characteristics of the upstream plasma, the detailed proton distribution functions, the He\textsuperscript{2+} distribution functions, and the wave spectra for this shock traversal are presented in this section.
Figure 3-1: Global magnetic field and detailed magnetic field around the shock we investigate
### 3.2.1 General Plasma Parameters

Table 3-1 lists the parameters characterizing the shock and the plasma upstream and downstream of the shock, including plasma flow velocity, plasma density, magnetic field.

<table>
<thead>
<tr>
<th></th>
<th>Upstream plasma</th>
<th>Shock parameters</th>
<th>Downstream plasma</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron density</td>
<td>$17 \text{ cm}^{-3}$</td>
<td>Mach number</td>
<td>3.3</td>
</tr>
<tr>
<td>Proton density</td>
<td>$14 \text{ cm}^{-3}$</td>
<td>Shock normal direction</td>
<td>[45, -8.0, 15]/48</td>
</tr>
<tr>
<td>He$^{2+}$ to proton ratio</td>
<td>0.12</td>
<td>Compression ratio $X$</td>
<td>2.7</td>
</tr>
<tr>
<td>Proton speed</td>
<td>$5.9 \times 10^2 \text{ km/s}$</td>
<td>Shock normal angle $\theta_{Bn}$</td>
<td>84°</td>
</tr>
<tr>
<td>Proton Temperature</td>
<td>$5.2 \times 10^4 \text{ K}$</td>
<td>Shock speed</td>
<td>48 km/s</td>
</tr>
<tr>
<td>Alfven speed</td>
<td>$1.5 \times 10^2 \text{ km/s}$</td>
<td>Flow speed</td>
<td>220 km/s; 145 km/s along shock normal</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Flow velocity</td>
<td>[-190, 10, 110] km/s</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Magnetic field</td>
<td>[17, -27, -86] nT $</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Electron density</td>
<td>47 cm$^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Proton thermal speed</td>
<td>$1.4 \times 10^2 \text{ km/s}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Proton gyrofrequency</td>
<td>1.5 Hz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Alfven speed</td>
<td>$2.9 \times 10^2 \text{ km/s}$</td>
</tr>
</tbody>
</table>

Table 3-1: The observed parameters for the shock and the plasma upstream and downstream of the shock.

**B**, Alfven Mach number, plasma-$\beta$, shock normal angle $\theta_{Bn}$, and compression ratio $X$. For cold upstream proton densities, some channels of the Composition and Distribution Function (CODIF) instrument are saturated. We therefore use earlier ACE data to
determine the upstream proton speed and temperature, and the ratio of \( \text{He}^{2+} \) density to proton density, with the appropriate time shift to account for solar wind propagation from ACE to Cluster as done by Alexandrova et al. [2004]. The appropriate time shift is 38 minutes for the spacecraft location and the solar wind speed during this interval. The electron density, as measured by the WHISPER instrument onboard Cluster Spacecraft 1, is obtained through private communication [Decreau, 2006]. We then combine the electron density and the helium-to-proton ratio to obtain the upstream densities of the helium and protons, ignoring other ion species. The Alfvén speed, the Alfvén Mach number, and the upstream plasma-\( \beta \) are calculated accordingly. The shock normal direction and velocity is calculated based on the shock traversal times of the four spacecraft [Maksimovic et al., 2003]. The shock is nearly perpendicular with a relatively low Mach number and plasma-\( \beta \). The parameters determined in this way should have an accuracy of about 10-15% considering the accuracy of the measurements. We note that the same event has been investigated by Walker et al. [2004] to determine the electric field lengthscale at a quasi-perpendicular shock. The parameters shown in Table 3-1 are consistent with the parameters listed by Walker et al. [2004] within this accuracy.

3.2.2 Ion Distribution Functions

The crossing of the shock ramp and the magnetic "overshoot" by Spacecraft 1 ended at 17:17:52 UT according to the high-resolution magnetic field data shown in Figure 3-1(b). Our analysis of the magnetic fluctuations and ion distribution functions downstream are based on the data obtained within several minutes after this time.
The CODIF instrument, working in its normal mode, measures a 3-dimensional distribution function for each ion species every 8 s. We take the proton distribution function measured during the time period 17:17:51-17:17:59 UT as the distribution

2001-03-31/17:17:51-17:17:59

Figure 3-2 Proton distribution function just downstream of the shock function of protons just downstream of the shock since Spacecraft 1 is in the overshoot for only ~1 s and in the downstream flow for ~7 s of this time period. A slice through the distribution function is shown in Figure 3-2, where $v_\perp$ ($v_z$) are velocity components
perpendicular (parallel) to B. It displays a high temperature anisotropy but does not reveal a separation between the reflected and transmitted proton components as anticipated. Current theory [Winske and Quest, 1988; Liu et al., 2005] and previous observations [Sckopke et al., 1983, 1990] show that reflected protons just downstream of the shock have a ring distribution separated in velocity-space from the transmitted core protons. One possible explanation for the absence of the separation in the Cluster observations is that the observed distribution function is an integration over a period of 8 s and the downstream plasma is turbulent. Protons have somewhat different distribution functions at different times; the time average obscures the distinction between the reflected and transmitted protons. Another possible explanation stems from the fact that the Alfvén Mach number of the shock is larger than those of the events documented by Sckopke et al. [1990], which are in the range 2-2.8. The shock ramp is therefore more turbulent so that the transmitted protons are heated more than in the events studied by Sckopke et al. [1990]; this again blurs the distinction between the reflected and transmitted protons. Though we cannot distinguish the reflected and core protons in the data, we shall estimate a number for the percentage of reflected protons. Further discussion of this issue will be presented in Section 3.5.

The He^{2+} ions are decelerated less compared with the transmitted protons when they traverse the shock potential between the upstream and downstream plasma because they have a larger mass per charge. These ions gyrate downstream of the shock before wave-ion interactions change their distribution function [Fuselier and Schmidt, 1994, 1997; Liu et al., 2005]. The first downstream helium distribution function, averaged over the time period from 17:17:55 UT to 17:18:03 UT, is shown in Figure 3-3 with the same
format as Figure 3-2. It has a ring-like distribution with \( v_L=200\pm50 \) km/s. Here the specified variation is the range of \( v_L \) for the red area in the observed distribution function. According to Liu et al. [2005], the shock potential energy \( e\phi \) is approximately equal to the kinetic energy difference between the incident and transmitted core protons; the gyration speed of helium is therefore \( V[(1/2+1/X^2)^{1/2}-1/X] \). For the parameters of this event, the \( \text{He}^{2+} \) gyration speed yields 230 km/s, which matches the observed gyration speed very well.

\[
\text{Figure 3-3 \ He}^{2+} \text{ ions distribution function downstream}
\]
Figure 3-4 Time evolution of temperature and anisotropy for both He\textsuperscript{2+} ions and protons with the observed magnetic field
Figure 3-4 shows the downstream magnetic field magnitude along with time profiles of the He$^{2+}$ perpendicular temperature, He$^{2+}$ parallel temperature, and the temperature anisotropy of both helium and protons downstream of the shock. The temperature anisotropy as defined before is $T_\perp/T_\parallel$, where $T_\perp$ and $T_\parallel$ are calculated directly from the distribution function based on the average magnetic field direction during the specific time interval. The proton temperature anisotropy decreases monotonically from 3.4 to 1.6 as the spacecraft measures further downstream. In the spacecraft frame it takes the protons $t_s$ (~ 1 min) to reach a saturated value of the temperature anisotropy. The saturation time following a parcel of downstream plasma is

$$t'_s = v_s t_s/v_p$$

(3.1)

where $v_s$ ($= 48 \text{ km/s}$) is the speed of the spacecraft relative to the shock and $v_p$ ($= 1.5 \times 10^5 \text{ km/s}$) is the normal component of the downstream plasma velocity. We obtain $t'_s$ ~ 20 s. The gyrofrequency of the downstream protons is 1.5 Hz, so that the interaction between protons and waves saturates after about 30 gyroperiods. This timescale will be compared with the predicted timescale in Section 3.3.

For helium both $T_\perp$ and $T_\parallel$ increase downstream of the bow shock. Just downstream of the shock $T_\perp$ increases more rapidly than $T_\parallel$, so that the temperature anisotropy increases; further downstream, $T_\parallel$ increases more rapidly, so that the temperature anisotropy decreases. The helium anisotropy first increases from 3.0 to 4.1 in about 8 s. It then decreases to a “residual anisotropy” of about 2 in ~ 1 minute, which is approximately the same relaxation time as for the protons. If we follow a parcel of plasma traveling downstream, it takes ~ 2 - 3 s for the He$^{2+}$ temperature anisotropy to increase and then ~ 20 s to decrease. Obviously some of the He$^{2+}$ ions are accelerated in
the direction perpendicular to the magnetic field. The "residual temperature anisotropy" of the He\(^{2+}\) ions (\(~ 2\)) is larger than that for the protons (\(~ 1.7\)).

3.2.3 Magnetic Fluctuations Downstream of the Shock

The magnetic field was measured with a high time resolution of about 25 Hz [Balogh et al., 2001]. During each 16 s interval starting at 17:17:52 UT, the average magnetic field is calculated and taken to be the ambient magnetic field. In each interval the magnetic fluctuations are projected onto directions parallel and perpendicular to the ambient magnetic field. The perpendicular fluctuations are further separated into left-hand and right-hand circularly polarized waves. With the reasonable assumption that these waves propagate along the magnetic field [Gary and Winske, 1986], the Doppler shift is negligible since the magnetic field direction is nearly perpendicular to the flow velocity obtained by the Hot Ion Analyzer (HIA) instrument. The frequencies and polarization measured in the spacecraft frame are taken to be the frequencies and polarization in the plasma frame. The time interval of 16 s is chosen because this interval is large enough to include the dominant frequencies of the low frequency waves, and yet small enough to investigate spectral evolution with time.

Figure 3-5 shows the power density spectra of the magnetic fluctuations \(K(\nu)\) which is defined by

\[
\langle \mathbf{B} \cdot \mathbf{B} \rangle = \sum_{\chi=L,R,P} \int_{0}^{\infty} K_{\chi}(\nu) d\nu
\]

where \(\nu\) is the wave frequency in Hz, L and R represent left-hand and right-hand circularly-polarized waves, respectively, and P represents fluctuations parallel to the ambient magnetic field. Similar to the events documented by Sckopke et al. [1990], the left-hand circularly polarized wave enhancements have much larger amplitudes than the
right-hand modes and parallel fluctuations. Most of the energy resides at frequencies lower than the proton gyrofrequency. We focus on the left-hand circularly polarized wave spectra within this range. The last two wave spectra downstream display a double-hump structure within $0 < v < v_g$ with a dip between the two humps as shown in Panels (b) and (c). One hump is located at a lower-frequency $\sim 0.2 v_g$. The other hump is located at a higher frequency $\sim 0.6 v_g$ ($\sim 0.9$ Hz). We identify the hump at lower frequency as a result of the helium instability and the other hump at higher frequency as the proton instability [Gary et al., 1993]. The detailed analysis for the waves excited in these different frequency ranges and resonant with the different ion species is presented in Section 3.4.2.

The wave enhancement hump at lower frequency just downstream of the shock is large. It peaks 16 s after shock passage before it starts to decay. The wave enhancement hump at higher frequency does not appear in the spectrum of the magnetic fluctuations in the first period of 16 s after shock passage. The second and third spectra in Figure 3-5(b) and 3-5(c) show this wave enhancement and its evolution during this time.

We claim that these waves are ion-cyclotron waves excited in the plasma downstream of the shock, and that they are the source of the He$^{2+}$-ion acceleration in the direction perpendicular to the magnetic field. The result is the observed spike in the temperature anisotropy of He$^{2+}$ ions evident in Figure 3-4. The analysis supporting this claim is presented in the next section.
Figure 3-5 Power spectra density of the observed magnetic fluctuations downstream of the shock
3.3 Theory of the Ion-Wave Interaction

The ion-cyclotron waves exist downstream of the shock as shown in the previous section. The condition for cyclotron resonance between a He$^{2+}$ ion with velocity component $v_z$ parallel to the ambient magnetic field and a parallel-propagating wave with angular frequency $\omega$ ($2\pi v = |\omega|$) and wavenumber $k$ is [Stix, 1992]

$$\omega(k) = kv_z - \Omega_a,$$

(3.2)

where $\Omega_a (=2\pi v_{ga})$ is the downstream He$^{2+}$ angular gyrofrequency. The resonance condition is represented by a straight line in the $\omega$ - $k$ diagram shown in Figure 3-6 with a slope equal to $v_z$ and an $\omega$ - intercept equal to $-\Omega_a = -\Omega/2$, where $\Omega (=2\pi v_g)$ is the downstream proton angular gyrofrequency. The function $\omega(k)$ in equation (3.2) is the dispersion relation for left-hand polarized ion-cyclotron waves propagating parallel to the ambient magnetic field, which is determined by the distribution functions of the ion and electron components of the plasma. For a plasma comprised of bimaxwellian protons, electrons and He$^{2+}$ ions, the dispersion relation is shown by the two curves passing through the origin in Figure 3-6. The plasma parameters are chosen to simulate the plasma downstream of the shock crossing we investigate so that the parallel and perpendicular temperatures, and the number densities for both helium and protons, are those measured just downstream of the shock. The electron temperature is taken to equal the parallel temperature of the protons and their temperature anisotropy is taken to be unity. The intersection(s) of the resonance line and the dispersion relations for parallel and antiparallel wave propagation gives the resonant wave frequency $\omega$ and wavenumber $k$. The flatter solid straight line is chosen for $|v_z| = v_{z0} \approx 0.05V_A$, which has two
intersections with the dispersion relation curve. As shown by a dashed resonance line in Figure 3-6, for a He\textsuperscript{2+} ion moving with $|v_z| < v_{z0}$, the resonance line intersects with the wave dispersion relation at three different locations. The crossing at large $|k|$ is neglected in our analysis. This interaction is weak in general and the wave intensity at large $|k|$ is generally small, as

![Figure 3-6 Dispersion relation and resonance condition](image)

we will show in more detail in Section 3.1. We also note that the dispersion relation of the waves which have a single resonance with He\textsuperscript{2+} ions is very close to straight lines $\omega = \pm kV_p$ as shown by the dotted lines in Figure 3-6. We find that $V_p \equiv 0.85V_A$. We therefore
assume that these waves have a constant phase speed of $V_P = 0.85V_A$ when we consider the scattering of He$^{2+}$ ions by these waves.

Previous studies have shown that ions are scattered along a bispherical shell or bispherical shells when interacting with unstable waves [Galeev and Sagdeev, 1988; Isenberg and Lee, 1996; Isenberg, 2001]. Because the particles conserve energy in the resonant unstable wave frame, they are scattered on spheres (or segments of spheres) centered on the wave phase speed. These shells for He$^{2+}$ ions are shown schematically in Figure 3-7 where we assume that the phase velocity $V_P$ is constant so that the curves are arcs of circles centered on $\pm V_P$. Note here that we also neglect the dispersion of the waves which have a double resonance with He$^{2+}$ ions and assume that they have the same phase velocity $V_P$ as the lower frequency waves. The doubly resonant waves are in a narrow angular frequency range of width $\sim 0.1 \Omega$. Although they are actually dispersive in this range, our assumption should be adequate for such a narrow range. The shells overlap in the region $|\nu_\parallel| < \nu_{\parallel 0} \equiv 0.05 V_A$, where He$^{2+}$ ions have a double resonance. These overlapping shells provide a path for some He$^{2+}$ ions to diffuse in $\nu_\perp$ as shown by the thick line with an arrow in Figure 3-7. This process is analogous to second-order Fermi acceleration as investigated by Terasawa [1989] and many others. Just downstream of the shock, most He$^{2+}$ ions, which have $|\nu_\parallel| < \nu_{\parallel 0}$, are accelerated through this stochastic process and increase $T_\perp$ due to the double resonance with the ion-cyclotron waves; a smaller number of the He$^{2+}$ ions which have $|\nu_\parallel| > \nu_{\parallel 0}$ are scattered to larger $\nu_\parallel$ and thus increase $T_\parallel$. $T_\parallel$ increases more slowly than $T_\perp$ because the number of ions scattering to larger $\nu_\parallel$ is smaller initially. Therefore, the temperature anisotropy just downstream increases. After a short period of time, enough He$^{2+}$ ions are scattered to $|\nu_\parallel| > \nu_{\parallel 0}$ that
scattering to larger $|\nu_2|$ dominates and the temperature anisotropy decreases. We suspect that this is the explanation for why the observed temperature anisotropy of helium has a spike just downstream of the shock before it starts to decrease more gradually. On the other hand, since this process happens in a very short period of time, it is difficult for the spacecraft to catch it in the right phase. This may be the reason why the spike of the helium temperature anisotropy is not present in other shock crossings during the same day. The evolution of the He$^{2+}$ distribution function and the decay of the temperature anisotropy after it peaks are calculated in Section 3.1.

Figure 3-7 A schematic illustration for He$^{2+}$ ions accelerated to higher $\nu_\perp$. 

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3.3.1 An Analytical Theory for the Relaxation of the He$^{2+}$ Temperature Anisotropy

The calculations presented in this section are based on the following assumptions: (1) the wave intensity of the magnetic fluctuations downstream of the shock is small compared with $B_0^2$, where $B_0$ is the ambient magnetic field magnitude, so that quasilinear theory is expected to be valid; (2) the wave dispersion relation $\alpha(k)$ does not change when the He$^{2+}$ and proton distribution functions change as a result of ion-wave interactions; (3) the double resonance range of $v_z$ is small ($v_{z0} \ll V_A$) so that scattering along a certain shell is much faster than the stochastic process which eventually accelerates some He$^{2+}$ ions to higher $v_L$ (here we omit the early stage when most He$^{2+}$ ions are in the double resonance range and the temperature anisotropy is increasing, and concentrate on the relaxation of the temperature anisotropy); (4) since ions in Regions I and III of Figure 3-7 scatter rapidly on a shell for $|v_z| > v_{z0}$, we make the assumption that the He$^{2+}$ ions are evenly distributed along the shell for $|v_z| > v_{z0}$ when we consider the slower process of stochastic acceleration; (5) the wave dispersion relation is given by $\omega = \pm kV_p$ so that each shell is a spherical cap as shown in Figure 3-7; (6) we neglect the dependence of the wave intensity on $k$ for those waves resonant with doubly resonant He$^{2+}$ ions; (7) the distribution function of the He$^{2+}$ ions averaged over a convected gyroradius downstream is gyrotropic. Based on these assumptions we now calculate the evolution of the distribution function with time appropriate to the decay phase of the temperature anisotropy.

The quasilinear evolution of helium ions with a gyrotropic distribution function $F(v_z, v_L, t)$, interacting with transverse parallel-propagating waves, is governed by [Lee and Ip, 1987; Bogdan et al., 1991; Isenberg, 2005]:

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\[
\frac{\mathcal{F}}{\alpha} = \frac{C}{v_\perp} \int k dk \sum_{\rho = \pm} \left( \frac{\omega_\rho}{k} \right)^2 G^\rho \left[ v_\perp \delta (\omega_\rho - kv_\perp + \Omega_\alpha) I_\rho (k) G^\rho F \right]
\]  

(3.3)

where \(C = \pi (q/mc)^2/2\), \(q\) and \(m\) are the charge and mass of a He\(^{2+}\) ion, and

\[
G^\rho = \left( 1 - \frac{k v_\perp}{\omega_\rho} \right) \frac{\partial}{\partial v_\perp} + \frac{k v_\perp}{\omega_\rho} \frac{\partial}{\partial v_z}.
\]  

(3.4)

Here \(\rho = \pm\) corresponds to waves propagating in the (+) or (−) direction. The wave power intensity spectrum \(I(k)\) is defined as

\[
\langle \delta \mathbf{B} \cdot \delta \mathbf{B} \rangle = \sum_{\rho = \pm} \int_0^\infty I_\rho (k) dk
\]

and it relates to \(K(\nu)\) through

\[
\sum_{\nu = L,R,P} \int_{\nu = L,R} K_\nu (\nu) d\nu = \sum_{\rho = \pm} (I_\rho (k) + I_\rho (-k)) dk.
\]

The function \(v(k)\) is determined by the dispersion relation \(|\omega| = |\omega(k)| = 2\pi v\). The \(G^\rho\) operator is proportional to the velocity-space gradient along a bispherical shell. If we neglect wave dispersion, \(G\) can be simplified as

\[
G^\rho = \frac{v_\perp}{V_\rho^2 v_\rho} \frac{\partial}{\partial \mu_\rho}
\]

where \(V_\rho^2 = \pm V_\rho\) is the wave phase speed and \(v_\rho\) and \(\mu_\rho\) are the ion speed and cosine of the pitch angle in the frame of the wave propagating with speed \(V_\rho^\rho\). In Regions I or III, where the He\(^{2+}\) ions are resonant only with waves propagating in one direction, equation (3.3) becomes

\[
\frac{\mathcal{F}}{\alpha} = C \frac{\partial}{\partial \mu_\rho} \left[ \frac{1 - \mu_\rho^2}{v_\rho^2 \mu_\rho} I_\rho (k_\parallel) \frac{\partial \mathcal{F}}{\partial \mu_\rho} \right],
\]  

(3.5)

where \(k_\parallel\) is the resonant wave number, \(\rho = -\) for Region I, and \(\rho = +\) for Region III.
As we have mentioned previously, He$^{2+}$ ions in Region II have three resonances with ion-cyclotron waves. The scattering process is inversely proportional to $k^2$ from equation (3.3). With the recognition that the variation of the resonant frequency is small, the interaction of waves and ions at the large resonant wavenumber is weak and can be neglected. We only consider the two resonances at small $k$ in the following calculation.

Now consider spherical coordinates $v$ and $\mu$ in the plasma frame, where $v = (v_x^2 + v_z^2)^{1/2}$ and $\mu = u_y/v$; the $G^\rho$ operator in equation (3.4) becomes

$$G^\rho = \frac{v_\perp}{v} \left[ \frac{\partial}{\partial v} + \left( \pm \frac{1}{V_p} \frac{\mu}{v} \frac{\partial}{\partial \mu} \right) \right].$$

(3.6)

For the domain of $v_z$ in the vicinity of $v_z = 0$ for which ions have a double resonance, we can neglect the small term $\mu/v$, then the operator $G^\perp$ becomes

$$G^\perp = \frac{v_\perp}{v} \left( \frac{\partial}{\partial v} + \frac{1}{V_p} \frac{\partial}{\partial \mu} \right).$$

(3.7)

Equation (3.3) in the domain $|v_z| < v_{z0}$ then becomes

$$\frac{\partial \mathcal{F}}{\partial \varepsilon} = \frac{CV_p^2}{v} \left\{ \left( \frac{\partial}{\partial v} + \frac{1}{V_p} \frac{\partial}{\partial \mu} \right) \left[ I_+(k_z) v \left( \frac{\partial \mathcal{F}}{\partial v} + \frac{1}{V_p} \frac{\partial \mathcal{F}}{\partial \mu} \right) \right] + \right.$$  

$$\left. \left( \frac{\partial}{\partial v} - \frac{1}{V_p} \frac{\partial}{\partial \mu} \right) \left[ I_-(k_z) v \left( \frac{\partial \mathcal{F}}{\partial v} - \frac{1}{V_p} \frac{\partial \mathcal{F}}{\partial \mu} \right) \right] \right\}$$

(3.8)

Since we neglect the dependence of the wave intensity $I$ on $k$ for the waves within the double resonance range,

$$I_+(k_z) = I_-(k_z) = I,$$

(3.9)

and equation (3.8) now simplifies to
Now consider an incremental volume $\Gamma$ across Regions I, II and III as shown in Figure 3-8. The volume in Region I (III) is part of a spherical shell centered at $v_1 = 0, v_z = -(+)V_p$ with radius $v_1(v_2)$ and thickness $\delta v_1(\delta v_2)$; the volume in Region II is part of a spherical shell centered at the origin with radius $v$ and thickness $\delta v$. The radii and thicknesses satisfy

$$v_+ = v_- = \sqrt{v^2 + V_p^2}$$

(3.11a)

$$\delta v_+ = \delta v_- = \sqrt{1 - \mu_0^2} \delta v,$$

(3.11b)

where

$$\mu_0 = \frac{V_p}{v_+}.$$

(3.12)

Figure 3-8 The shell bounded the bispherical shells in region I and III and spherical shells in region II.
The total number of particles $\delta N$ in this volume $\Gamma$ is

$$\delta N = 4\pi \delta v, \nu^2 (1 - \mu_0) F(v) \quad (3.13)$$

where we neglect the domain $|v_2| < v_20$ since $v_20$ is assumed to be small compared with $V_A$ and characteristic values of $|v_2|$. Particles in Regions I and III are scattered along the shell, but this does not change the total number $\delta N$. The number $\delta N$ changes by particle scattering across the shell shown in Figure 3-8. This occurs for particles in Region II. We integrate the first term on the right hand side of equation (3.10) to obtain the total change in $\delta N$. Explicitly, we have

$$\frac{d\delta N}{dt} = 2CV_p I \delta v \nu^2 2\pi 2\varepsilon \frac{1}{\nu} \frac{\partial}{\partial \nu} \left( \nu \frac{\partial F}{\partial \nu} \right) \quad (3.14)$$

where $\varepsilon = v_20/v$. Combining equations (3.11), (3.13), and (3.14), we obtain

$$\frac{\partial F}{\partial \nu} = 2CV_p \nu_20 I \frac{\nu_20^2}{\nu \left( \sqrt{\nu^2 + V_p^2} - V_p \right)} \frac{\partial}{\partial \nu} \left( \nu \frac{\partial F}{\partial \nu} \right) \quad (3.15)$$

In the limit of $\nu >> V_p$, equation (3.15) becomes

$$\frac{\partial F}{\partial \nu} = 2CV_p \frac{\nu_20^2}{\nu^2} \frac{\partial}{\partial \nu} \left( \nu \frac{\partial F}{\partial \nu} \right) \quad (3.16)$$

This equation can be simplified as

$$\frac{\partial F'}{\partial \nu'} = \frac{1}{\nu^{2}} \frac{\partial}{\partial \nu'} \left( \nu' \frac{\partial F'}{\partial \nu'} \right) \quad (3.17)$$

where $\nu' = \nu/V_p, \nu' = \sigma t$ where

$$\sigma = \frac{2CI\nu_20}{V_p^2} \quad (3.18)$$

The solution of (3.17) with initial condition

$$F(\nu', t' = 0) = \delta(\nu' - \nu_0) \quad (3.19)$$
where \( I_0(x) \) is the standard modified Bessel function. This solution is shown in Figure 3-9 for \( t' = 0.1, 0.2 \) and 0.5 for \( u_0 = 2 \). Note that \( F(\nu) \) is the velocity-space density throughout the volume \( \Gamma(\nu) \) specified by the particle speed at \( \mu = 0 \). The results show that He\(^{2+}\) ions, starting with a uniform distribution in the volume \( \Gamma(2 < \nu' < 2+\delta\nu') \), then diffuse to fill the phase space of \( \nu < u_0 \) faster than they are accelerated to higher \( \nu \).

Scattering across the shells is faster at lower \( \nu \) where the intersecting shells cross each other at a smaller angle than at larger \( \nu \). [The crossing angle at the lower (upper) position of Figure 3-7 is slightly less (greater) than 90°]. Even though particle diffusion fills the shells of smaller \( \nu \) faster, the total energy of the ions still increases due to the combined effect of the diffusion coefficient in equation (3.17) and the phase space factor \( \nu^5 \) in \( \nu \) integration for the total energy.

On the other hand, if \( \nu \ll V_p \), equation (3.15) becomes

\[
\frac{\partial F}{\partial t} = 4C V_p^2 \frac{u_0 I}{\nu^3} \frac{\partial}{\partial \nu} \left( \nu F \frac{\partial}{\partial \nu} \right)
\]  

(3.21)

In dimensionless form equation (3.21) becomes

\[
\frac{\partial F}{\partial t'} = \frac{1}{\nu' ^ 3} \frac{\partial}{\partial \nu'} \left( \nu' \frac{\partial F}{\partial \nu'} \right),
\]  

(3.22)

where \( t'' = 2t' \). The solution of equation (3.22) with the initial condition (3.19) is

\[
F(\nu',t'') = \frac{u_0 ^ 3}{2t''} I_0 \left[ \frac{(\nu' \nu_0 ^ 2)^2}{8t''} \right] \exp \left( -\frac{u_0 ^ 4 + \nu' ^ 4}{16t''} \right)
\]  

(3.23)
3.3.2 Comparison with Observations

The initial gyration speed of He$^{2+}$ ions calculated in Section 3.2.2 is $\sim 230$ km/s and the Alfvén speed downstream is $\sim 290$ km/s. The dimensionless gyration speed $v_0 = 0.8$ is neither much larger nor much smaller than unity. We need to solve equation (3.15) numerically to obtain the evolution of the distribution function for He$^{2+}$ ions in this event.

The dimensionless form of equation (3.15) is

$$\frac{\partial F}{\partial t'} = \frac{1}{v'(\sqrt{v'^2 + 1} - 1)} \frac{\partial}{\partial v'} \left( v' \frac{\partial F}{\partial v'} \right).$$

We solve equation (3.24) numerically and calculate the temperature anisotropy based on the distribution function obtained. The resonant wave intensity $I = V_p K(v_0)/4\pi$, where $K(v_0)$ is taken to be 300 nT$^2$/Hz according to the power spectrum of magnetic fluctuations presented in the previous section. Here we assume that all fluctuations are propagating at
the same phase speed $V_p$ and that the wave intensity is obtained from the peak value of the left-hand polarized wave spectrum in Figure 3-5(c) near frequency $\sim 0.5 \nu_g$. The range of the double resonance is $\nu_{50} = 0.05 V_A$. Using these values we obtain $\nu \approx 0.01 \text{ s}^{-1}$.

The predicted and observed evolution of the temperature anisotropy are shown in Figure 3-10. We skip the early stage of the evolution when the ions are not uniformly distributed on the shells and the equations of Section 3.3.1 are not valid, so that the initial time at $t = 0$ is taken to be the time when the temperature anisotropy of the He$^{2+}$ ions has its maximum value. The time assigned to the observed temperature anisotropy is the time in the plasma frame based on the speed of the shock relative to the spacecraft and the speed of the downstream plasma according to equation (3.1). The observed relaxation of the temperature anisotropy matches the predicted temperature anisotropy very well in general shape and in timescale. Most of the observed values of the temperature anisotropy are slightly smaller than those predicted.
Figure 3-10 The predicted (solid line) and observed (*) time evolution of temperature anisotropy
3.3.3 Timescale for Proton Relaxation

The downstream reflected protons have a ring distribution with $v_r = v_{p0}$, this speed is calculated according to the trajectory of the reflected protons and the velocity of the plasma downstream and upstream of the shock. The details of this calculation have been presented in Liu et al. [2005]. With the parameters given in Table 3-1, we obtain $v_{p0} \sim 4 V_A$. The quasilinear timescale for the reflected protons to relax to a shell distribution is [Liu et al., 2005]

$$\tau \sim D_{\mu\nu}^{-1}$$

(3.25)

where the pitch–angle diffusion coefficient is given by

$$D_{\mu\nu} = \frac{\pi e^2}{2m_p^2c^2|\mu|\nu} I(k_s)$$

(3.26)

and where $m_p$ is the mass of a proton, $v_r$ is the speed of the reflected protons in the downstream plasma frame and $I(k_s)$ is the resonant wave intensity. Since we do not know the exact number density of the reflected protons, we use equation (3.26) instead of equation (3.22) in Liu et al. [2005] to estimate the timescale. The observed wave intensity $K(\nu)$ is 200 nT$^2$/Hz which is the enhanced intensity within the frequency range 0.8 Hz ~ 1.2 Hz in the time period 17:18:24 UT to 17:18:40 UT. The frequency range is chosen for waves resonant with protons moving with small $v_z$ representative of the protons just downstream of the shock. The characteristic speed of the reflected protons is $v_r \sim 4 V_A$. The predicted timescale is $\tau \sim 21$ s. This value corresponds to $\sim 63$ s in the spacecraft frame which is in good agreement with the observed saturation time of $\sim 60$ s.
3.4. Wave Spectrum and Quasilinear Theory

The wave power spectrum generated downstream includes contributions from core protons, reflected protons, and He$^{2+}$ ions. This power spectrum was calculated by Liu et al. [2005] for low Mach number events, while ignoring the stochastic acceleration process for He$^{2+}$ ions which are transported to higher and lower $v_\perp$ due to double resonances with the ion-cyclotron waves. In this event the amount of He$^{2+}$ ions is relatively large and their transport to higher $v_\perp$ is apparent in the observed temperature; thus their contribution cannot be neglected. Using the He$^{2+}$ distribution function we obtained in Section 3.2, we estimate the contribution to the power spectrum by He$^{2+}$ ions. Since some of the He$^{2+}$ ions are scattered to larger $v_\perp$ and gain energy from the waves, this process is not self-sustaining unless there is a source of wave energy at the same time. We will find that the interactions between the protons and waves can provide sufficient energy for the waves to energize the He$^{2+}$ ions.

3.4.1 Proton Contribution

We assume that the percentage of protons which are reflected protons is ~10%, which is chosen to be intermediate between the 3 ~ 5% characteristic of very low Mach number shocks ($M_A = 1.9 \sim 2.5$) and the 20% for a high Mach number ($M_A = 4.9$) shock documented by Sckopke et al. [1990]. The effect of this parameter on the wave spectrum will be discussed later.

These downstream reflected protons are scattered by ion-cyclotron waves on a dispersive shell, the shape of which has been discussed in detail by Liu et al. [2005]. Every small area on the shell is part of a sphere centered on the phase velocity of the
resonant wave. The cyclotron resonance condition for protons with velocity component \( v_z \) and waves of wavenumber \( k \) and angular frequency \( \omega \) is

\[
\omega = k v_z - \Omega .
\]  

(3.27)

The dispersion relation we use here is

\[
\omega = \pm k V_A \sqrt{1 - \alpha \frac{\omega}{\Omega}},
\]  

(3.28)

which is an approximation for the dispersion relation shown in Figure 3-6 for the warm plasma downstream of Earth's bow shock. Here \( \alpha \) is chosen to be 1.3. Figure 3-11 shows a comparison between the warm plasma dispersion relation (solid line) and this approximate dispersion relation (dashed line). Thus we use equation (3.28) as an approximation of the dispersion relation for mathematical convenience.

![Dispersion Relation Graph](image)

Figure 3-11 Warm plasma wave dispersion relation (solid line) and the approximate form for it (Dashed line).
The protons evolve from a ring distribution to a uniform distribution on the shell; the number of protons transported through the plane \( v_z = v_{zs} > 0 \) in velocity space is

\[
N_{ps}(v_{zs}) = n_r \frac{v_{zf} - v_{zs}}{v_{zf}} \tag{3.29}
\]

where \( v_{zf} \) is the largest \( v_z \) on the shell. Here we presume that all protons are scattered to \( v_z > 0 \) because protons scattered to \( v_z < 0 \) are symmetric. These protons lose energy to the resonant waves. The energy change \( dE \) corresponding to a small velocity change \( dv_{zs} \) is [Isenberg and Lee, 1996; Liu et al., 2005]

\[
dE = N_{ps} m V_p dv_{zs}, \tag{3.30}
\]

where \( V_p \) is the phase velocity of the waves resonant with particles moving with \( v_{zs} \). This energy will go into waves in the small band of resonant frequencies \( (\omega, \omega + d\omega) \), where \( \omega \) and \( \omega + d\omega \) are the frequencies of the waves resonant with protons moving with \( v_{zs} \) and \( v_{zs} + dv_{zs} \), respectively. The wave power intensity as a function of \( \omega \) is

\[
E(\omega) = N_{ps} m V_p \left| \frac{dv'}{d\omega} \right|, \tag{3.31}
\]

Since we are using an approximate dispersion relation to estimate the wave power spectrum we simply assume that the power of the magnetic fluctuations is half of the total wave power. The power density of magnetic fluctuations \( K_L(\nu) \) for left-hand polarized waves is

\[
K_L(\nu) = 8\pi^2 E(-2\pi\nu) \tag{3.32}
\]

The red line in Figure 3-12 shows the predicted wave spectrum \( K_L(\nu) \) excited by protons. It occupies the frequency range \((0.2 \, v_g, 0.7 \, v_g)\) and must be added to the \( \text{He}^{2+} \) contribution to obtain the total wave magnetic fluctuation spectrum.
Figure 3-12 Power spectrum excited by protons (red line) and He$^{2+}$ ions (green line) and the sum of them adding the waves damped to accelerate He$^{2+}$ ions (blue line).

3.4.2 He$^{2+}$ Contribution

He$^{2+}$ ions are scattered from a ring distribution function to a multi-shell distribution function as calculated in Section 3.3. Consider a particle transported from ($v_z = 0$, $v_\perp = v_0$) to ($v_z = v_{zo}$, $v_\perp = v_{z0}$) on the shell going through ($v_z = 0$, $v_\perp = \eta$) which we label as shell $\eta$. It is first scattered along the $v_\perp$ axis to the top of the shell ($v_z = 0$, $v_\perp = \eta$), and then scattered along the shell to ($v_z = v_{zo}$, $v_\perp = v_{z0}$). The energy gained or lost by all ions is calculated in two corresponding parts. The first part is to measure the energy gained or lost by all the ions as they are transported along the $v_\perp$ axis to the top of the shell on which they finally reside; the second part is to measure the energy lost to the
waves for all the ions as they are transported from the top of the shell to form a uniform distribution on each shell. This energy is transferred to or from the resonant waves, resulting in the excited wave spectrum.

Initially the kinetic energy density of the He\textsuperscript{2+} ions is

\[ E_{a1} = \frac{1}{2} n_\alpha m v_0^2. \]  

(3.33)

The saturated distribution function for He\textsuperscript{2+} ions is \( F(v) \), as calculated in the previous section. When they are at the tops of their shells, their energy is

\[ K_\alpha = \left( \frac{334}{334} \right) \]

where \( v_\alpha \) corresponds to the ion speed in the wave frame. Numerical evaluation shows that \( E_{a2} > E_{a1} \) so that the ions gain energy from the waves during the stochastic acceleration. Now we assume that the energy is evenly distributed amongst the waves within the frequency range \((v_{31} \approx 0.44 v_g, v_{32} \approx 0.55 v_g)\) resonant with He\textsuperscript{2+} with velocity component \( v_z \) in the range \((-v_{30} < v_z < v_{30})\). Then the absorbed wave power density spectrum \( K \sim 4\pi(E_{a2} - E_{a1})/(v_{32} - v_{31}) \), for the chosen parameters turns out to be \( \sim 2B_0^2/v_g \).

Now we consider the energy lost to the waves when the ions are scattered from the top of the shell and become evenly distributed on each shell. Taking one shell with thickness \( d\eta \), and the corresponding speed in the wave frame \( \xi \), the number of ions crossing the plane \( v_z = v_{32} \) is

\[ \delta N_{a1}(\xi, v_{32}) = 4\pi \xi^2 F(\sqrt{\xi^2 - V_p^2}) d\xi \left( 1 - \frac{v_{32} + V_p}{\xi} \right) \]  

(3.35)
The total number of He\(^{2+}\) ions scattered across the plane is the integration of equation (3.35) for all possible values of \(\xi\):

\[
N_{as}(v_{zs}) = \int_{v_{zs} + V_p}^{\infty} \partial N_{as}(\xi, v_{zs}) d\xi
\]  

(3.36)

The wave energy spectrum excited in this process is

\[
E(\omega) = N_{as} m V_p \left| \frac{d v_{zs}}{d\omega} \right|
\]  

(3.37)

According to equation (3.2) and the approximate dispersion relation \(\omega = \pm k V_p\), He\(^{2+}\) ions with \(v_z = v_{zs}\) resonate with waves of frequency

\[
\omega = \frac{V_p}{v_{zs} + V_p} \Omega_a.
\]  

(3.38)

Altogether, the estimated wave power density contributed by He\(^{2+}\) ions is:

\[
E(\omega) = N_{as} m V_p^2 \frac{\Omega_a}{\omega^2}.
\]  

(3.39)

Combining equations (3.39) and (3.32), and including the wave absorption due to the stochastic acceleration of helium, we obtain the power density spectrum of magnetic fluctuations \(K_f(v)\), which is shown by the green line in Figure 3-12. The precipitous decay of the predicted wave spectrum at \(v = 0.5 v_g\) is due to ion scattering along the shell to larger \(v_z\) and losing energy to waves are resonant with waves of frequencies \(v \leq 0.5 v_g\).

3.4.3 Comparison with Observations

The estimated power spectrum of the magnetic fluctuations, obtained by summing the contributions of both protons and He\(^{2+}\), is shown by the blue line in Figure 3-12. It has a double hump structure with one hump at frequency \(~0.3 v_g\) (hump I), and another hump at frequency \(~0.6 v_g\) (hump II). The slot between these two peaks corresponds to the waves damped by the acceleration of He\(^{2+}\) ions. Due to the existence of this slot, the
general shape of the predicted spectrum should not change if we chose a different value for the percentage of the reflected protons. The peak values of the humps I and II for the magnetic fluctuations are $1.6 \frac{B_0^2}{\nu_g} \sim 10^3 \text{nT}^2/\text{Hz}$ and $0.3 \frac{B_0^2}{\nu_g} \sim 10^2 \text{nT}^2/\text{Hz}$, respectively. Both values match the order-of-magnitude values for the observed peaks of the magnetic fluctuation power spectrum shown in Figure 3-5(c). We also note that the predicted orders of magnitude for the peak values are also in reasonable agreement with those observed even if we choose a different number between 5% and 20% for the percentage of the reflected protons.

It is obvious that the protons generate hump II; we designate this hump as the “proton instability”. Hump I includes waves generated by both protons and He$^{2+}$ ions; we designate this hump as the “helium-proton instability”. This conclusion is different from that of Gary et al. [1993], who conclude that these two humps result from the proton instability and the helium instability separately. We suggest that it is more appropriate to predict the wave spectrum using quasilinear theory rather than linear theory based on growth rate calculations.

The predicted frequency ranges for the humps and the slot do not match the observations perfectly. The observed slot has a width of about $0.2 \nu_g$ while the predicted width is only about $0.1 \nu_g$. More importantly, the absorbed wave intensity is larger than the sum of the wave intensity excited by helium and by protons in the frequency range of the slot, which means that the energy provided by the protons is not large enough to accelerate He$^{2+}$ ions to the shells we predicted. This discrepancy may be explained by resonance broadening [Karamabadi, 1999] so that waves close to the frequencies specified by equation (3.2) are also damped and provide energy to accelerate
the He$^{2+}$ ions. Another interesting feature of the observed wave spectrum is that hump II is not present just downstream of the shock as shown in Panel (a) of Figure 3-6. The evolution of the power density spectrum is not considered in our application of quasilinear theory. It could be that the waves which have double resonances with He$^{2+}$ ions are damped to such a degree that it slows down the proton-wave interaction. Therefore, the second peak generated exclusively by protons cannot be seen in the early stage. At the same time, damping of these waves does not slow down the singly-resonant helium-wave interactions which generate the hump I.

3.5 Discussion and Conclusions

We have presented a synopsis of the observed ion temperatures, ion temperature anisotropies, and magnetic fluctuation spectra downstream of Earth’s bow shock as observed by Cluster in the time interval 17:17:00-17:19:00 UT on 31 March 2001. At this time and location the shock was of moderate strength and nearly perpendicular. We have also presented a quasilinear theory to predict the evolution of the helium distribution function under the effect of double resonance with ion-cyclotron waves, and to estimate the wave power spectrum when interacting with an anisotropic distribution of both protons and helium ions. The predictions and the observations match remarkably well, considering the accuracy of the measurements. We draw the following conclusions:

1. Due to double resonances with the ion-cyclotron waves, some of the He$^{2+}$ ions are scattered to larger or smaller $v_\perp$; the waves resonant with those ions which gain energy are damped, as manifest in the observed wave spectra by a slot between two humps.
2. Quasilinear theory works remarkably well to predict (i) the relaxation of the temperature anisotropy of He$^{2+}$ ions scattered by ion-cyclotron waves downstream of Earth's bow shock, (ii) the timescale for proton relaxation and (iii) the excited wave spectrum generated by the ion-wave interaction;

3. The double-hump structure of the observed power spectrum can be explained as a result of the proton instability at higher frequencies and a joint helium-proton instability at lower frequencies.

Liu et al. [2005] explained the double hump structure of the downstream wave spectra documented by Sckopke et al. [1990] as a result of core and reflected proton instabilities. We account for the observed power spectra presented in this chapter with proton and helium-proton instabilities. The two humps addressed in this chapter are distributed in different frequency ranges than those documented by Sckopke et al. [1990], and the theory based on reflected proton and helium excitation matches the observations very well. There are still unanswered questions which need to be addressed in future studies.

First of all, we do not treat the evolution of the distribution function of He$^{2+}$ and the wave intensity spectrum self-consistently. We predicted the evolution of the He$^{2+}$ temperature anisotropy with an effective wave intensity; we did not calculate how the wave power spectrum evolves in time and whether this variation affects substantially the evolution of the He$^{2+}$ distribution function and its temperature anisotropy. Some simulation work needs to be done to solve this problem self-consistently.

Secondly, we assumed that the double resonance range is narrow and that scattering on one shell is rapid. Therefore, we cannot account for the initial stage of the
stochastic acceleration of He\(^{2+}\) when its temperature anisotropy increases. Numerical work needs to be done to solve the diffusion equation in (\(u_\perp, u_\parallel\))-space with a finite double resonance range and a finite pitch-angle scattering rate to account for the spike in the temperature anisotropy observed just downstream of the shock.

Lastly, there may be nonlinear processes, like wave ripples recently observed on the surface of Earth's bow shock or resonance broadening, which affect ion-wave interactions downstream of the shock [Mouillard et al., 2006]. They may contribute to the wave spectrum as we discussed in Section 3.4. More investigation needs to be done to determine their contributions quantitatively.
CHAPTER 4

CONCLUSIONS

We have developed a quasilinear theory to describe the ion thermalization and wave excitation downstream of Earth's bow shock. The theory predicts that the downstream reflected protons relax to a bispherical shell distribution and excite ion-cyclotron waves. The wave power spectra and the residual temperature anisotropy of the protons match the AMPTE/IRM observations documented by Sckopke et al. [1990] well if wave dispersion and core-proton contributions are included in the calculation. The transmitted He$^{2+}$ ions can be accelerated to higher $v_\perp$ due to their double resonances with the ion-cyclotron waves. In the initial treatment of these ions presented in Chapter 2, we neglected the stochastic acceleration which accelerates He$^{2+}$ ions to higher $v_\perp$ and simply assumed that they are scattered to a shell distribution like the reflected protons. Later, as presented in Chapter 3, we extended the theory to calculate the time evolution of the helium distribution function including the stochastic processes, and predicted the wave spectrum excited by the protons and helium ions.

In Chapter 3, we presented Cluster observations of the ion distribution functions downstream of Earth's bow shock following one particular nearly-perpendicular shock crossing. The data clearly demonstrate that He$^{2+}$ ions just downstream of the shock have a ring distribution which implies that the helium ions gyrate about the transmitted protons. The observed perpendicular temperature of the He$^{2+}$ ions increases further away from the
shock. This observation shows that the He$^{2+}$ ions are accelerated along the directions perpendicular to the ambient magnetic field. The observed temperature anisotropy as a function of time matches that predicted by our theory remarkably well.

Some the He$^{2+}$ ions are accelerated and damp the resonant waves. The observed magnetic power density spectrum has a slot between two humps. The theory shows that the slot is due to the energy absorbed by the accelerated He$^{2+}$ ions. The general shape and the predicted wave spectrum match the observations. The predicted order magnitude of the peak values for the humps match those of observed humps very well.

With four Cluster spacecraft, we can determine the time scale of the relaxation process by calculating the distance from the spacecraft to the shock. The distance is calculated/estimated from the inferred speed of the shock sweeping past the four spacecraft. The observed timescale matched the predicted timescale very well for the event we investigated.

In general, the theory we developed matches the observations, but the detailed features need further work, especially, for predicting the wave spectrum with He$^{2+}$ ions involved. To determine the detailed evolution of the helium distribution functions, the scattering along the spherical shells and the stochastic acceleration need to be considered simultaneously.

Solving for the shell scattering and the stochastic acceleration simultaneously would probably reveal the temperature anisotropy "spike" just downstream of the shock. To do this, we would need to solve equation (3.3) in Chapter 3 numerically. Similar work to predict heavy ion acceleration in the solar corona has been done by Isenberg and Vasquez [2006]. They predicted the temperature anisotropy based on a pre-determined
wave spectrum and dispersion relation. One challenge to solve the shell scattering together with the stochastic acceleration downstream of the shock is that the evolution of the ion distribution may influence the wave dispersion relation and therefore the wave spectrum. These effects are usually negligible in the solar corona plasma since the heavy ions have small number densities and can be treated as test particles. For the plasma downstream of Earth's bow shock, during events with high helium concentration, however, these effects may be much larger. To incorporate these effects into the theory, we need to combine equation (3.3) in Chapter 3 and the dispersion relation, and solve them together numerically.

Another effect which could be considered in a stricter treatment is resonance broadening, which may have influenced the ion-wave interaction in the event we presented in Chapter 3.

Recently, ripples on the bow shock surface were observed by Cluster [Moullard et al., 2006]. How do the ripples affect the reflected protons? How do ion-wave interactions influence the shock structure? Do the waves generated by downstream protons affect the large scale shape of the shock and the magnetopause, and space weather predictions? All these questions need to be considered in future studies.

Through the comparison of theory and observations, we conclude that the quasilinear theory works remarkably well in describing the ion thermalization and wave excitation downstream of Earth's bow shock for marginally critical conditions. We also conclude that a multiple spacecraft mission is a power tool to investigate space physics processes, and we look forward to the Magnetospheric Multi-Scale Mission.
LIST OF REFERENCES


Hollweg, J. V. (2004), Private communication.


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