Planar algebras: A category theoretic point of view

Shamindra Kumar Ghosh
University of New Hampshire, Durham

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PLANAR ALGEBRAS: A CATEGORY THEORETIC POINT OF VIEW

BY

SHAMINDRA KUMAR GHOSH
B.S, Calcutta University, 1996
M.S, Indian Statistical Institute, 1998
Ph.D, Indian Statistical Institute, 2004

DISSERTATION

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in partial fulfillment of
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in
Mathematics

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This dissertation has been examined and approved.

Dissertation Director Dmitri Nikshych
Associate Professor of Mathematics Department

Eric Grinberg
Professor of Mathematics Department

Don Hadwin
Professor of Mathematics Department

Maria Basterra
Assistant Professor of Mathematics Department

David Feldman
Associate Professor of Mathematics Department

07/19/2006
Date
DEDICATION

To Mom and Dad
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At first, I would like to thank my advisor Professor Dmitri Nikshych for systematically teaching me the subject of Fusion categories and inspiring me to do research. I am grateful to him for transmitting his endless zest for doing mathematics into me.

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FOREWORD

In [Jon1], Vaughan Jones introduced the notion of index for type $II_1$ subfactors and also examined the canonical tower,

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

of basic construction of $N \subset M$. For a finite index inclusion of type $II_1$ factors, $N \subset M$, the grid of finite dimensional algebras of relative commutants,

$$N \cap N \subset N' \cap M \subset N' \cap M_1 \subset N' \cap M_2 \subset \cdots$$

$$M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \cdots$$

known as the standard invariant, became an important invariant for $N \subset M$ (see [GHJ], [JS], [Pop1], [Pop2]).

Sorin Popa in [Pop2] studied the question of which families $\{A_{ij} : -1 \leq i \leq j < \infty\}$ of finite-dimensional $C^*$-algebras could arise as the tower of relative commutants of an extremal finite-index subfactor, that is, when does there exist such a subfactor $M_{-1} \subset M_0$ such that $A_{ij} = M_i \cap M_j$; and he obtained a beautiful algebraic axiomatization of such families, which he called $\lambda$-lattices. Subsequently, Jones used this characterisation of $\lambda$-lattices to obtain an algebraic and geometric reformulation of the standard invariant, which he called Planar Algebras (see [Jon2]). Jones then introduced the notion of 'modules over a planar algebra' in [Jon3] where he explicitly found the irreducible modules over the Temperley–Lieb planar algebras for index greater than 4. This notion became a powerful tool to construct subfactors of index less than 4, namely, the subfactors with principal graph, $E_6$ and $E_8$. Shamindra Ghosh (see [Gho]) established a one-to-one correspondence of all modules over the group planar algebra, that is, the planar algebra associated to the subfactor arising from the action of a finite group, and the representations of a non-trivial quotient of the quantum double of the group over a certain ideal; the reason for the appearance of a quotient of the quantum double instead of just the quantum double was allowing rotation of internal discs in the definition of the modules over a planar algebra. Ghosh also answered the Jones’s question in affirmative in the case of group planar algebras whether the radius of convergence of the dimension of a module is at least as big the inverse-square of the modulus.

In this thesis, we mainly address two problems.
Problem 1: To every subfactor $N \subset M$, one can associate the bicategory of $N - N$, $N - M$, $M - N$, $M - M$ bimodules. It is natural to expect a correspondence between the bicategory and the planar algebra associated to the subfactor. Is it possible to construct a planar algebra directly from a bicategory?

Problem 2: From [Gho], it seems that if the modules over a planar algebra are defined with rigid internal disc then they are more interesting because of the connection with quantum double in the case of group planar algebras. How can one find such modules (called affine representations) over a finite depth planar algebra and can one answer Jones's question on radius of convergence of the dimension of the affine representation of such planar algebras?

We answer Problem 1 by constructing a planar algebra starting with a bicategory with the extra assumption of pivotality. In problem 2, we shed some light on the affine representations of a finite depth planar algebra by proving some finiteness results and also answer Jones's question in affirmative.

Next, we give a chapter by chapter summary of the thesis.

In the first chapter, we discuss the preliminaries from basic category theory. The first section recalls the definition of multicategories and maps between them from [Lei]. We introduce the notion of the structure of empty objects in a multicategory and the common examples, namely multicategory of sets or vector spaces, possesses structure of empty objects. In the second section, we discuss basics of bicategory theory and several structures related to a bicategory, namely functors, transformation between functors and rigidity.

We construct a new example of multicategory with the structure of empty objects which we call Planar Tangle Multicategory in the second chapter. We re-define Jones's planar algebra simply as a map of multicategories from the Planar Tangle Multicategory to the multicategory of vector spaces. We think that this is probably the neatest and approachable way of defining a planar algebra. In the end, we discuss more structures (modulus, connectedness, local finiteness, $C^*$-structure, etc.) on a planar algebra and discuss the most basic example of the Temperley Planar Algebra.

In the first section of the third chapter, we start with fixing an 1-cell in a pivotal strict 2-category and construct a planar algebra. The construction is very similar to Jones's construction of a planar algebra from a subfactor. However, we would like to mention that this construction is totally algebraic and heavily depends on the graphical calculus of the 2-cells and pivotal structure plays a key role here. In the second section, we briefly discuss the example of group planar algebras.

Motivated with the connection of annular representation of the group planar algebra with the representations of a certain quotient of the quantum double of the group, we introduce the concept of affine representations of a planar algebra in the fourth chapter. This is a generalization of Hilbert space representation of annular Temperley-Lieb by Jones and Reznikoff (see [JR]). In the end, we discuss some general theory of such representations.
In the fifth and the final chapter, we discuss affine representations of a planar algebra associated to a finite depth subfactor. We find a bound on the weights of these representations which is dependent on the depth of the planar algebra. We also proved that at each weight there could be only finitely many irreducible affine representations. We answers Jones's question on radius of convergence of dimension of affine representations.
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ABSTRACT

PLANAR ALGEBRAS: A CATEGORY THEORETIC POINT OF VIEW

by

SHAMINDRA KUMAR GHOSH
University of New Hampshire, September, 2006

We define Jones's planar algebra as a map of multicategories and construct a planar algebra starting from a 1-cell in a pivotal strict 2-category. We introduce the concept of affine representations of a planar algebra and prove some finiteness results for the affine representations of finite depth planar algebras. We also show that the radius of convergence of the dimension of an affine representation of the planar algebra associated to a finite depth subfactor is at least as big as the inverse-square of the modulus.
CHAPTER 1

PRELIMINARIES

1.1 Multicategories

In this section, we revisit the definition of multicategory and an algebra for a multicategory (introduced in [Lei]). We introduce the structure of empty object in a multicategory which will be useful in the next chapters.

Definition 1 A multicategory $C$ consists of:

- a class $C_0$ whose elements are called objects of $C$,
- for all $n \in \mathbb{N}$, for all $a = (a_1, a_2, \cdots, a_n) \in (C_0)^n$ and for all $a \in C_0$, there exists a class $C(a; a)$ called morphism space from $a$ to $a$, whose elements are called arrows,
- Composition maps: for all $n$, $k_1, k_2, \cdots, k_n \in \mathbb{N}$ and for all $a = (a_1, a_2, \cdots, a_n) \in (C_0)^n$, $a \in C_0$, there exists a composition map $\circ$, denoted in the following way:

$$
C(\theta; a) \times C(a_1; a) \times C(a_2; a) \times \cdots \times C(a_n; a) \ni (\theta, \theta_1, \theta_2, \cdots, \theta_n) \mapsto \theta \circ (\theta_1, \theta_2, \cdots, \theta_n)
$$

where $(a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, a_n) \in (C_0)^n$ satisfies the following conditions:

- Associativity: $\theta \circ (\theta_1 \circ (\theta_1', \theta_2', \cdots, \theta_{n-1}') \circ (\theta_1'' \circ (\cdots \circ (\theta_1''', \theta_2'''', \cdots, \theta_{n-1}''')) \circ \cdots \circ (\theta_1, \theta_2, \cdots, \theta_n))) = (\theta \circ (\theta_1, \theta_2, \cdots, \theta_n)) \circ (\theta_1', \theta_2', \cdots, \theta_{n-1}') \circ (\theta_1'', \theta_2'', \cdots, \theta_{n-1}'')$ whenever the composites make sense,

- Identity: for each $a \in C_0$, there exists an arrow $1_a \in C(a; a)$ (called the identity on $a$) such that:

$$
\theta \circ (1_a, 1_a, \cdots, 1_a) = \theta = 1_a \circ \theta \text{ for all } \theta \in C((a_1, a_2, \cdots, a_n); a).
$$
Remark 2  The associativity and identity axioms are easier to understand with pictorial notation of arrows (see [Lei]).

Example 3  The collection of sets $\mathcal{MSet}$ (resp. vector spaces $\mathcal{MVec}$) forms a multicategory where arrows are given by maps from cartesian product of finite collection of sets to another set (resp. multilinear maps from a finite collection of vector spaces to another vector space).

Example 4  Any tensor category $\mathcal{C}$ has an inbuilt multicategory structure in the obvious way by setting $\mathcal{C}_0 = \text{ob}(\mathcal{C}) = \text{set of objects of } \mathcal{C}$ and $\mathcal{C}((a_1, a_2, \cdots, a_n); a) = \text{Mor}_C((a_1 \otimes a_2) \otimes \cdots \otimes a_{n-1} \otimes a_n, a)$.

Definition 5  Let $\mathcal{C}$ and $\mathcal{C}'$ be multicategories. A map of multicategories $f : \mathcal{C} \to \mathcal{C}'$ consists of a map $f : \mathcal{C}_0 \to \mathcal{C}'_0$ together with another map

$$f : \mathcal{C}(a_1, a_2, \cdots, a_n; a) \to \mathcal{C}'(f(a_1), f(a_2), \cdots, f(a_n); f(a))$$

such that composition of arrows and identities are preserved. (If $\mathcal{C}$ and $\mathcal{C}'$ are multicategories with each morphism space being vector space and composition being multilinear, then we will assume that the map of multicategories is linear between the morphism spaces.)

Definition 6  Let $\mathcal{C}$ be a multicategory. A $\mathcal{C}$-algebra is simply a map of multicategories from $\mathcal{C}$ to $\mathcal{MSet}$. (If $\mathcal{C}$ is a multicategory with each morphism space being vector space and composition being multilinear, then we will consider a $\mathcal{C}$-algebra to be a map of multicategories from $\mathcal{C}$ to $\mathcal{MVec}$.)

Definition 7  A multicategory $\mathcal{C}$ is said to be symmetric if the following conditions hold:

1. for all $n \in \mathbb{N}$, $a \in (\mathcal{C}_0)^n$, $a \in \mathcal{C}_0$, $\sigma \in S_n$, there exists a map $- \cdot \sigma : \mathcal{C}(a; a) \to \mathcal{C}(a \cdot \sigma; a)$ (where $(a_1, a_2, \cdots, a_n) \cdot \sigma = (a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(n)})$) satisfying:

2. $(\theta \cdot \sigma) \cdot \rho = \theta \cdot (\sigma \cdot \rho)$ and $\theta = \theta \cdot 1_{S_n}$ for all $n \in \mathbb{N}$, $a \in (\mathcal{C}_0)^n$, $a \in \mathcal{C}_0$, $\sigma, \rho \in S_n$, $\theta \in \mathcal{C}(a; a)$,

3. $(\theta \cdot \sigma) \circ (\theta_{\sigma(1)} \cdot \pi_{\sigma(1)}, \theta_{\sigma(2)} \cdot \pi_{\sigma(2)}, \cdots, \theta_{\sigma(n)} \cdot \pi_{\sigma(n)})$

   $$= (\theta \circ (\theta_1, \theta_2, \cdots, \theta_n)) \cdot (\sigma \cdot (\pi_{\sigma(1)}, \pi_{\sigma(2)}, \cdots, \pi_{\sigma(n)}))$$
for all \( n, k_i \in \mathbb{N}, a \in C_0, a = (a_1, a_2, \ldots, a_n) \in (C_0)^n, a_i \in (C_0)^{k_i}, \theta \in C(a; a), \theta_i \in C(a_i; a_i), \sigma \in S_n, \pi_i \in S_{k_i}, \) for \( 1 \leq i \leq n, \) where \( \tilde{\sigma} \) and \( (\pi_{\sigma(1)}, \pi_{\sigma(2)}, \ldots, \pi_{\sigma(n)}) \) are permutations in \( S_{k_1 + k_2 + \ldots + k_n} \) defined by:

\[
\tilde{\sigma} \left( j + \sum_{i=0}^{i-1} k_{\sigma(i)} \right) = j + \sum_{i=0}^{\sigma(i)-1} k_i \\
(\pi_{\sigma(1)}, \pi_{\sigma(2)}, \ldots, \pi_{\sigma(n)})(j + \sum_{i=0}^{i-1} k_{\sigma(i)}) = (\pi_{\sigma(1)}(j) + \sum_{i=0}^{i-1} k_{\sigma(i)})
\]

for all \( 1 \leq i \leq n, 1 \leq j \leq k_{\sigma(i)} \) assuming \( \sigma(0) = 0 = k_0. \)

It will be easier to understand the axioms of symmetry in pictorial notation as in [Lei].

**Remark 8** Clearly, the multicategories \( M\text{Set}, M\text{Vec} \) and the one arising from a symmetric tensor category are symmetric.

**Definition 9** A multicategory \( C \) is said to have the structure of empty object if for all \( a \in C_0, \) there exists a class \( C(0; a) \) such that the composition in \( C \) extends in the following way:

\[
\text{for all } n \in \mathbb{N}, 1 \leq s \leq n, a \in C_0, a = (a_1, a_2, \ldots, a_n) \in (C_0)^n, \theta \in C(0; a), \theta_s \in C(0; a_s),
\]

\[
\text{for all } k_i \in \mathbb{N}, a_i \in (C_0)^{k_i}, \theta_i \in C(a_i; a_i) \text{ where } s \neq i \in \{1, 2, \ldots, n\},
\]

\[
C(a; a) \times C(a_1; a_1) \times \cdots \times C(0; a_s) \times \cdots \times C(a_n; a_n) \ni (\theta, \theta_1, \ldots, \theta_s, \ldots, \theta_n)
\]

\[
\xrightarrow{\circ} \theta \circ (\theta_1, \ldots, \theta_s, \ldots, \theta_n) \in C \left( \left( a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_n \right); a \right)
\]

such that this composition map is associative.

(We demand that a map of multicategories both having the structure of empty object, should preserve this structure.)

### 1.2 Bicategories

In this section, we will recall the definition of bicategories and various other notions related to bicategories which will be useful in Chapter 3. Most of the materials in this section can be found in any standard textbook on bicategories.

**Definition 10** A bicategory \( B \) consists of:

---

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• a class $B_0$ whose elements are called objects or 0-cells,

• for each $A, B \in B_0$, there exists a category $B(A, B)$ whose objects $f$ are called 1-cells of $B$ and denoted by $A \xrightarrow{f} B$ and whose morphisms $\gamma$ are called 2-cells of $B$ and denoted by

\[
\begin{array}{ccc}
A & \xrightarrow{\gamma} & B \\
\downarrow & & \downarrow \\
\end{array}
\]

• for each $A, B, C \in B_0$, there exists a functor $\otimes : B(B, C) \times B(A, B) \to B(A, C)$,

• Identity Objects: for each $A \in B_0$, there exists an object $1_A \in \text{ob}(B(A, A))$ (the identity on $A$),

• Associativity Constraints: for each triple $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$ of 1-cells, there exists an isomorphism $A \xrightarrow{\alpha_{h,g,f}} D \in \text{Mor}(B(A, D))$,

• Unit Constraints: for each 1-cell $A \xrightarrow{f} B$, there exist isomorphisms

\[
\begin{array}{ccc}
1_B \otimes f & & f \otimes 1_A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\lambda_f} & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{\rho_f} & B \in \text{Mor}(B(A, B))
\end{array}
\]

such that $\alpha_{h,g,f}$, $\lambda_f$, and $\rho_f$ are natural in $h$, $g$, $f$, and satisfy the pentagon and the triangle axioms (which are exactly similar to the ones in the definition of a tensor category).

A bicategory will be called a strict 2-category if the associativity and the unit constraints are identities.

An abelian (resp. semisimple) bicategory $B$ is a bicategory such that $B(A, B)$ is an abelian (resp. semisimple) category for every $A, B \in B_0$ and the functor $\otimes$ is additive.

Remark 11. $B(A, A)$ is a tensor category and $B(A, B)$ is a $(B(B, B), B(A, A))$-bimodule category for 0-cells $A, B$. (See [ENO], [Ost] for definition of module category.)
Example 12 A bicategory with only one 0-cell is simply a tensor category.

Example 13 A bicategory can be obtained by taking rings as 0-cells, 1-cells \( A \to B \) being \((B, A)\)-bimodules and 2-cells being bimodule maps. The tensor functor is given by the obvious tensor product over a ring.

Definition 14 Let \( B, B' \) be bicategories. A weak functor \( F = (F, \varphi) : B \to B' \) consists of:

- a function \( F : B_0 \to B'_0 \),
- for all \( A, B \in B_0 \), there exists a functor \( F^{A,B} : B(A, B) \to B'(F(A), F(B)) \) written simply as \( F \),
- for all \( A, B, C \in B_0 \), there exists a natural isomorphism \( \varphi^{A,B,C} : \otimes' \circ (F^{A,B} \times F^{A,B'}) \rightarrow F^{A,C} \circ \otimes' \) written simply as \( \varphi \) (where \( \otimes \) and \( \otimes' \) are the tensor products of \( B \) and \( B' \) respectively),
- for all \( A \in B_0 \), there exists an invertible (with respect to composition) 2-cell \( \varphi_A : 1_{F(A)} \to F(1_A) \),

satisfying commutativity of certain diagrams (consisting of 2-cells) which are analogous to the hexagonal and rectangular diagrams appearing in the definition of a tensor functor.

A coweak functor \( F : B \to B' \) satisfies all conditions of a weak functor except \( F^{A,B} \) is replaced by a contravariant functor \( F^{A,B} : B(A, B) \to B'(F(A), F(B)) \) and \( \varphi^{A,B,C} \) is replaced by a natural isomorphism \( \varphi^{A,B,C} : \otimes' \circ (F^{A,B} \times F^{B,C}) \rightarrow F^{A,C} \circ \otimes' \circ \text{(flip)} \) where \( \text{(flip)} \) is the functor obtained by interchanging the coordinates.

Definition 15 Let \( F = (F, \varphi), G = (G, \psi) : B \to B' \) be weak functors. A weak transformation \( \sigma : F \to G \) consists of:

- for all \( A \in B_0 \), there exists a 1-cell \( \sigma_A \in \text{ob}(B(F(A), G(A))) \),
- for all \( A, B \in B_0 \), there exists a natural transformation \( \sigma^{A,B} : (\sigma_B \otimes' F^{A,B}) \to G^{A,B} \otimes' \sigma_A \) written simply as \( \sigma \) (where \( \sigma_B \otimes' F^{A,B} \), \( G^{A,B} \otimes' \sigma_A : B(A, B) \to B(F(A), G(B)) \) are functors defined in the obvious way), satisfying the following:
• for all \( x \in \text{ob}(B(B, C)) \), \( y \in \text{ob}(B(A, B)) \) where \( A, B, C \in B_0 \), the following diagrams commute:

\[
\begin{align*}
\sigma_C \otimes' F(x) \otimes' F(y) &\xrightarrow{\sigma_C \otimes' 1_{F(y)}} G(x) \otimes' \sigma_B \otimes' F(y) \xrightarrow{1_{G(x)} \otimes' \psi} G(x) \otimes' G(y) \otimes' \sigma_A \\
1_{\sigma_C} \otimes' \psi_{x,y} &\xrightarrow{} G(\sigma_C \otimes' F(x) \otimes' y) \\
\sigma_C \otimes' F(x \otimes' y) &\xrightarrow{\sigma_C \otimes' \psi} G(x \otimes' y) \otimes' \sigma_A \\
1_{\sigma_C} \otimes' \psi_{x,y} &\xrightarrow{} G(x \otimes' y) \otimes' \sigma_A
\end{align*}
\]

where \( \lambda', \rho' \) are the left and right unit constraints of \( B' \).

**Remark 16** Composition of weak or coweak functors and weak transformations follows exactly from composition of functors and natural transformations in categories; moreover composition of two coweak functors is a weak functor. One can also extend the notion of natural isomorphisms in categories to weak isomorphism in bicategories.

**Theorem 17** *(Coherence Theorem for Bicategories)* Let \( B \) be a bicategory. Then there exists a strict 2-category \( B' \) and functors \( F : B \to B' \), \( G : B' \to B \) such that \( \text{id}_B \) (resp. \( \text{id}_{B'} \)) is weakly isomorphic to \( G \circ F \) (resp. \( F \circ G \)).

Let \( A \xrightarrow{f} B \) be a 1-cell in a bicategory \( B \). A right (resp. left) dual of \( f \) is an 1-cell \( B \xrightarrow{\overline{f}} A \) (resp. \( A \xrightarrow{\underline{f}} B \)) such that there exists 2-cells

\[
\begin{align*}
& \frac{f \otimes f}{A} & & \frac{1_B}{A} \\
& \frac{\varepsilon_f}{B} & & \frac{\eta_f}{B} \\
& \frac{1_A}{B} \quad \text{and} \quad \frac{\alpha_f}{A} & & \frac{\beta_f}{A} \\
& \frac{f \otimes f \otimes \alpha_f}{(\text{resp. } B \xrightarrow{f} B \text{ and } A \xrightarrow{f} A)} & & \frac{1_A}{A}
\end{align*}
\]
such that the following identities (ignoring the associativity and unit constraints) are satisfied:

\[
\begin{align*}
(1_f \otimes e_f) \circ (c_f \otimes 1_f) &= 1_f \\
(e_f \otimes 1_f) \circ (1_f \otimes c_f) &= 1_f.
\end{align*}
\]

(resp. \((1_f \otimes e_f) \circ (c_f \otimes 1_f) = 1_f\).

\(e_f \otimes 1_f = 1_f\)

(Here \(e\) stands for evaluation and \(c\) stands for coevaluation.) One can show that two right (resp. left) duals are isomorphic via an isomorphism which is compatible with the evaluation and coevaluation maps.

A bicategory is said to be rigid if right and left dual exists for every 1-cell. Further, in a rigid bicategory \(\mathcal{B}\), one can consider right dual as a coweak functor \(* = (\ast, \varphi) : \mathcal{B} \to \mathcal{B}\) in the following way:

- for each 1-cell \(f\), we fix a triple \((f^*, e_f, c_f)\) so that when \(f = 1_A\) where \(A \in \mathcal{B}_0\), then \(f^* = 1_A, e_f = \lambda_{1_A}\) \((= \rho_{1_A}\), see [Kas] for proof\), \(c_f = \lambda_{1_A}^{-1} = \rho_{1_A}^{-1}\),

- \(*\) induces identity map on \(\mathcal{B}_0\),

- for all \(A, B \in \mathcal{B}_0\), \(f, g \in \text{ob}(\mathcal{B}(A, B))\) and 2-cell \(\gamma : f \to g\), define the contravariant functor \(* : \mathcal{B}(A, B) \to \mathcal{B}(B, A)\) by \(*f = f^*\) and \(*\gamma\) denoted by \(*\gamma\), is given by the composition of the following 2-cells

\[
\begin{array}{c}
g^* \circ \rho_{1_A}^{-1} \circ g^* \otimes 1_A \circ 1_{f^*} \circ f^* \circ 1_f \circ g \circ f^* \circ (e_{1_f} \otimes 1_f) \circ 1_f \circ 1_f \\
= 1_B \otimes f^* \circ f \circ f^*.
\end{array}
\]

- for all \(A, B, C \in \mathcal{B}_0\), the natural isomorphism \(\varphi_{A,B,C} : \otimes \circ (\ast_{\mathcal{A}^B \times \ast_{\mathcal{B}^C}}) \to \ast_{\mathcal{A}^C} \circ \otimes \circ (\text{flip})\) is defined by:

for \(f \in \text{ob}(\mathcal{B}(A, B)), g \in \text{ob}(\mathcal{B}(B, C))\), the invertible 2-cell \(\varphi_{f,g}\) is given by the composition of the following 2-cells

\[
\begin{array}{c}
f^* \otimes g \\
\downarrow (f^* \otimes g^*) \otimes c_{(g^* f)}
\end{array}
\]

ignoring the associativity and the unit constraints necessary to make sense of the composition,

- for all \(A \in \mathcal{B}_0\), the invertible 2-cell \(\varphi_A : 1_A \to 1_A\) is given by identity morphism on \(1_A\).

Similarly, one can define a left dual coweak functor in a rigid bicategory.
CHAPTER 2

PLANAR ALGEBRAS

In this chapter, we will introduce a new example of a symmetric multicategory, namely, the Planar Tangle Multicategory ($\mathcal{P}$) which possesses the additional structure of empty object. Any planar algebra, in the sense of [Jon2], turns out to be a $\mathcal{P}$-algebra. In the end, we also exhibit some examples and define more structures on a planar algebra.

Let us first define planar tangles which are the building blocks of the planar tangle multicategory. Fix $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\varepsilon \in \{+, -\}$.

Definition 18 A $(k, \varepsilon)$-planar tangle is an isotopy class of pictures containing:

- an external disc $D$ on the Euclidean plane $\mathbb{R}^2$ with $2k$ distinct points on the boundary numbered clockwise,

- finitely many (possibly zero) non-intersecting internal discs $D_1, D_2, \ldots, D_n$, lying in the interior of $D$ with $2k_i$ distinct points on the boundary of $D_i$ numbered clockwise where $k_i \in \mathbb{N}_0$ for $1 \leq i \leq n$,

- a collection $\mathcal{S}$ of non-intersecting oriented curves (called strings) on $D \setminus \left( \bigcup_{i=1}^{n} D_i \right)$ such that:
  
  (a) each marked point on the boundaries of $D, D_1, D_2, \ldots, D_n$ is connected to exactly one string,

  (b) each string either has no end-points or has exactly two end-points on the marked points,

  (c) the orientations induced on each connected component of $D \setminus \left( \bigcup_{i=1}^{n} D_i \right)$ by different bounding strings should be same,

- the orientation induced in the connected component of $D \setminus \left( \bigcup_{i=1}^{n} D_i \right)$, adjacent to the first and the last marked point on the boundary of $D$, should have orientation positive (anti-clockwise) or negative (clockwise) according to the sign of $\varepsilon$. 

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Figure 2-1. Example of a $(4, +)$-planar tangle with two internal discs $D_1, D_2$ with colours $(2, +), (3, -)$ respectively.

Remark 19 For each $i \in \{1, 2, \cdots, n\}$, we can assign $\varepsilon_i \in \{+, -\}$ to the internal disc $D_i$ depending on the orientation of the connected component of $D^o \setminus \left( \bigcup_{i=1}^{n} D_i \right) \cup S$, adjacent to the first and the last marked points on the boundary of $D_i$. $(k_i, \varepsilon_i)$ will be called the colour of $D_i$ and $(k, \varepsilon)$ will be the colour of $D$.

Sometimes, instead of numbering each marked point on the boundary of a disc with colour $(k, \varepsilon)$, we will write $\varepsilon$ very close to the boundary of the disc and in the connected component adjacent to the first and the last points.

Let $T((k_1, \varepsilon_1), (k_2, \varepsilon_2), \cdots, (k_n, \varepsilon_n); (k, \varepsilon))$ be the set of $(k, \varepsilon)$-planar tangles with $n$ internal discs $D_1, D_2, \cdots, D_n$ with colours $(k_1, \varepsilon_1), (k_2, \varepsilon_2), \cdots, (k_n, \varepsilon_n)$ respectively, $T(\emptyset; (k, \varepsilon))$ be the set of $(k, \varepsilon)$-planar tangles with no internal disc and $T_{(k, \varepsilon)}$ be the set of all $(k, \varepsilon)$-planar tangles. The composition of two tangles $T \in T((k_1, \varepsilon_1), (k_2, \varepsilon_2), \cdots, (k_n, \varepsilon_n); (k, \varepsilon))$ and $S \in T((l_1, \delta_1), (l_2, \delta_2), \cdots, (l_m, \delta_m); (k_i, \varepsilon_i))$ (resp. $S \in T(\emptyset; (k_1, \varepsilon_1)))$, denoted by
The Planar Tangle Multicategory, denoted by \( \mathcal{P} \), is defined as:

- \( \mathcal{P}_0 = \{ (k, \varepsilon) : k \in \mathbb{N}_0, \varepsilon \in \{ +, - \} \} \),
- \( \mathcal{P}((k_1, \varepsilon_1), (k_2, \varepsilon_2), \ldots, (k_n, \varepsilon_n); (k, \varepsilon)) \) (resp. \( \mathcal{P}(\emptyset; (k, \varepsilon)) \)) is the vector space with \( \mathcal{T}((k_1, \varepsilon_1), (k_2, \varepsilon_2), \ldots, (k_n, \varepsilon_n); (k, \varepsilon)) \) (resp. \( \mathcal{T}(\emptyset; (k, \varepsilon)) \)) as a basis,
- composition is a multilinear map induced by the composition of tangles as described above,
- the identity morphism \( 1_{(k, \varepsilon)} \in \mathcal{T}((k, \varepsilon); (k, \varepsilon)) \subset \mathcal{P}((k, \varepsilon); (k, \varepsilon)) \) is given by the \( (k, \varepsilon) \)-planar tangle with exactly one internal disc with colour \( (k, \varepsilon) \), containing precisely \( 2k \) strings such that \( i^{th} \) point on the internal disc is connected to the \( i^{th} \) point on the external disc by a string for \( 1 \leq i < 2k \).
We leave the checking of associativity and identity axioms to the reader. A moment’s observation also reveals that \( \mathcal{P} \) is symmetric and has the structure of empty object.

**Definition 20** A planar algebra \( \mathcal{P} \) is simply a \( \mathcal{P} \)-algebra, that is, a map of multicategories from \( \mathcal{P} \) to \( \mathcal{MVec} \).

**Remark 21** The above definition of planar algebra is equivalent to that of general planar algebra in [Jon2].

**Remark 22** One can define a planar algebra by sending \((k,\varepsilon)\) to \( \mathcal{P}(k,\varepsilon) \) which is defined as the vector space with \( T(k,\varepsilon) \) as a basis. This is called the Universal Planar Algebra in [Jon2].

**Remark 23** For a planar algebra \( \mathcal{P} \), the collection of vector spaces \( \{\mathcal{P}(k,\varepsilon)\}_{k \in \mathbb{N}_0} \) forms a unital filtered algebra where \( \varepsilon \in \{+,-\} \). The multiplication of \( \mathcal{P}(k,\varepsilon) \), inclusion of \( \mathcal{P}(k,\varepsilon) \) inside \( \mathcal{P}(k+1,\varepsilon) \) and identity of \( \mathcal{P}(k,\varepsilon) \) are induced by the following tangles:

![Tangles](image)

and respectively.

**Example 24** Temperley-Lieb Planar Algebra with modulus \((\delta_+,\delta_-)\)

Let \( \delta_+, \delta_- \in \mathbb{C} \). Consider the subspace \( \mathcal{W}_{(k,\varepsilon)} \) of \( \mathcal{P}(\emptyset; (k,\varepsilon)) \) generated by elements of the form \( T_1 - \delta_+^{m_1} \delta_-^{n_1} T_2 \) where \( T_1, T_2 \in T(\emptyset; (k,\varepsilon)) \) such that \( T_1 \) can be isotopically obtained from \( T_2 \) by attaching \( m \) many loops oriented clockwise and \( n \) many loops oriented anti-clockwise. Set \((T_L(\delta_+,\delta_-))(k,\varepsilon))_{(T_L(\delta_+,\delta_-))(k,\varepsilon)} = \frac{\mathcal{P}(\emptyset; (k,\varepsilon))}{\mathcal{W}_{(k,\varepsilon)}} \).

Define the map of multicategories \( T_L(\delta_+,\delta_-) : \mathcal{P} \to \mathcal{MVec} \) by

\[
T_L(\delta_+,\delta_-)(k,\varepsilon) = (T_L(\delta_+,\delta_-))(k,\varepsilon),
\]
\[ TL(\delta_+, \delta_-)(X)([Y_1], [Y_2], \ldots, [Y_n]) = [X \circ (Y_1, Y_2, \ldots, Y_n)] \]

where \( X \in P((k_1, \varepsilon_1), (k_2, \varepsilon_2), \ldots, (k_n, \varepsilon_n); (k, \varepsilon)) \), \( Y_i \in P(0; (k_i, \varepsilon_i)) \) for \( 1 \leq i \leq n \). We leave the checking of preserving associativity, identity and structure of empty object to the reader. This is the first basic and non-trivial example of a planar algebra.

We will now define more structures on a planar algebra. A planar algebra \( P \) is said to be connected (resp. locally finite) if \( \dim(P(0, +)) = 1 = \dim(P(0, -)) \) (resp. \( \dim(P(k, \varepsilon)) < \infty \) for all \( (k, \varepsilon) \)). A planar algebra \( P \) is said to have modulus \((\delta_+, \delta_-)\) if \( P(T) = \delta_+ P(T_1) \) (resp. \( P(T) = \delta_- P(T_1) \)) where \( T \) is a planar tangle with a contractible loop oriented clockwise (resp. anti-clockwise) and \( T_1 \) is the tangle \( T \) with the loop removed. A connected planar algebra \( P \) is called spherical if two tangles \( T_1 \in T(0, e) \) and \( T_2 \in T(0, \eta) \) induce the same multilinear functional by expressing the images of \( P(T_1) \) and \( P(T_2) \) as scalar multiples of the identities of \( P(0, e) \) and \( P(0, \eta) \) respectively whenever one can obtain \( T_1 \) from \( T_2 \) after embedding them on the unit sphere and using spherical isotopy.

If \( T \in T((k_1, \varepsilon_1), (k_2, \varepsilon_2), \ldots, (k_n, \varepsilon_n); (k, \varepsilon)) \) (resp. \( T \in T(0; (k, \varepsilon))) \), then \( T^* \in T((k_1, \varepsilon_1), (k_2, \varepsilon_2), \ldots, (k_n, \varepsilon_n); (k, \varepsilon)) \) (resp. \( T^* \in T(0; (k, \varepsilon))) \)) is defined as the tangle obtained by reflecting \( T \) about any straight line not intersecting \( T \), and the first point of an internal (resp. external) disc of the reflected \( T \) is taken to be the reflected point of the last point of the corresponding internal (resp. external) disc in \( T \) such that the reflection preserves the colour of each disc. For example, the * of the tangle in Figure 2-1 is given by:
(where we reflect the Figure 2-1 about a vertical line). We extend the map $T_{(k,e)} \ni T \mapsto T^* \in T_{(k,e)}$ conjugate linearly to $*: P_{(k,e)} \to P_{(k,e)}$. It is clear that $*$ is an involution. This makes $\{P_{(k,e)}\}_{k \in \mathbb{N}_0}$ into a unital filtered $*$-algebra for $e \in \{+, -\}$. $P$ is said to be a $*$-planar algebra (resp. $C^*$-planar algebra) if $P$ is a planar algebra, $P(k,e)$ is a $*$-algebra (resp. $C^*$-algebra) for each colour $(k,e)$ and the map $P$ is $*$ preserving in the sense:

if $0 \in P((k_1,e_1),(k_2,e_2),\ldots,(k_n,e_n);(k,e))$ (resp. $P(0;(k,e))$) and $f_i \in P(k_i,e_i)$ for $1 \leq i \leq n$, then

\[ P(\theta^*(f_1^*,\ldots,f_n^*)) = (P(\theta)(f_1^*,\ldots,f_n))^*. \]

A locally finite spherical $C^*$-planar algebra is called subfactor-planar algebra.

**Theorem 25** (Jones) Any extremal subfactor with finite index gives rise to a subfactor-planar algebra. Conversely, any subfactor planar algebra gives rise to an extremal subfactor with finite index.

We will not attempt to give a proof here (see [Jon2]). However, we would like to comment that the proof of the converse uses Popa’s $\lambda$-lattices ([Pop2]). The proof of the forward direction is very similar to the construction of a planar algebra from a bicategory which we will discuss in the next chapter.
CHAPTER 3

PLANAR ALGEBRA ARISING FROM A BICATEGORY

In this chapter, we will show how one can construct a planar algebra from a 1-cell of an abelian ‘pivotal’ strict 2-category with exactly two 0-cells. However this construction can be extended for any abelian pivotal bicategory. The prescription is very similar to the planar algebra associated to an extremal finite index subfactor (see [Jon2]). In the end, we discuss the planar algebra associated to the subfactor arising from the outer action of a finite group.

3.1 Construction of the Planar Algebra

Before we proceed towards the construction, we will first state or deduce some useful results and set up some notations.

Definition 26 A bicategory $B$ is called pivotal if $B$ is rigid and there exists a weak transformation $\alpha : \text{id}_B \to \ast$ such that $\alpha_s = 1_s \in \text{ob}(B(s,s))$ for all $s \in B_0$, where $\ast = (\ast, K)$ is the right dual coweak functor and $\ast \ast = (\ast \ast, J)$ is the weak functor $\ast \circ \ast$.

From now on, we will consider only strict 2-category instead of general bicategories unless otherwise mentioned; however all results modified with appropriate associativity and unit constraints, will hold even in the absence of the ‘strict’ assumption by the coherence theorem for bicategories. We next set up some pictorial notation to denote 2-cells which is analogous to the graphical calculus of morphism in a tensor category (see [Kas], [BK]). Let $B$ be a pivotal strict 2-category as defined above. We denote a 2-cell $f : Y \to Z$ by a rectangle labelled with $f$, placed on $\mathbb{R}^2$ so that one of the sides is parallel to the $X$-axis and a vertical line segment labelled with $Y$ (resp. $Z$) is attached to the top (resp. bottom) side of the rectangle.
\[ f = \begin{array}{c}
\text{rectangle with } f \\
Y \\
\downarrow Z
\end{array} \]

Sometimes we will not label the strings attached to a rectangle labelled with a 2-cell; the 2-cell itself will induce the obvious labelling to the strings. We list below pictorial notations of several other 2-cells which will be the main constituents of the construction without describing them meticulously in words like the way we described \( f \) above.

\[ Y = \begin{array}{c}
1_Y \\
Y \\
\downarrow Y
\end{array} ; \quad Y \otimes Z = \begin{array}{c}
Y \\
\downarrow Z
\end{array} ; \quad f \circ g = \begin{array}{c}
f \\
\downarrow f \\
Y \\
\downarrow g
\end{array} \]

\[ Y^* \otimes Y \xrightarrow{e_Y} 1_Y = \bigcup Y; \quad 1_Y \xrightarrow{e_Y} Y \otimes Y^* = \bigcup Y; \]

\[ \begin{array}{c}
Y^* \\
\downarrow Y^*
\end{array} = \begin{array}{c}
y_f \\
\downarrow Y^*
\end{array} ; \quad \bigcup Y = \begin{array}{c}
Y \\
\downarrow Y
\end{array} \]

where \( Y, Z \) are 1-cells and \( f, g \) are 2-cells. We will next exhibit some easy consequences in terms of the pictorial notation; detailed proof are not given.
Lemma 27

(i) \[ Y = \begin{array}{c} Y \\ Y \end{array} = \begin{array}{c} Y \\ Y \end{array}; \quad Y^* = \begin{array}{c} Y \\ Y \end{array} = \begin{array}{c} Y \\ Y \end{array}; \]

where \( Y \) is any 1-cell,

(ii) for any 2-cell \( f : Y \to Z \),

\[ f^* = \begin{array}{c} Y \\ Z \end{array} = \begin{array}{c} f \\ Z \end{array} \quad \text{and} \quad f^{**} = \begin{array}{c} Y^* \\ Z \end{array} = \begin{array}{c} f \\ Z \end{array} \]

(iii) \[ K_{Y,Z} = \begin{array}{c} Y \times Z \end{array} \]

and \( K_{Y,Z}^{-1} = \begin{array}{c} Y \times Z \end{array} \]

(iv) \[ J_{Y,Z} = \begin{array}{c} Y \times Z \end{array} \]

\[ (Y \times Z)^* \]
and $J_{Y,Z} = \begin{array}{c}
(Y \otimes Z)^{*} \\
Z' \\
Y \\
Y \otimes Z \\
Z \\
(\otimes Z)^{*}
\end{array}$

(v) $J_{Y,Z} \circ (a_Y \otimes a_Z) = a_{Y \otimes Z}$ for all 1-cells $Y$ and $Z$.

**Proof.** (i) follows from the definition of $Y^*$ being the right dual of $Y$ and $a$ being invertible.

First part of (ii) follows from the definition of $f^*$ and naturality of $a$ and the second part easily follows from the first one.

(iii) (resp. (iv)) follows from the way the coweak (resp. weak) functor $*$ (resp. **) is defined.

Definition of the pivotal structure $a$ implies (v). ■

**Remark 28** Parts (iii) and (iv) of the above lemma do not use the pivotal structure $a$ at all. However, with the help of pivotal structure, especially part (v) of the above lemma, one may also prove the following:

$K_{Y,Z} = \begin{array}{c}
Y \\
Z \\
Y \otimes Z \\
Z \\
(\otimes Z)^{*}
\end{array}$

$J_{Y,Z} = \begin{array}{c}
Y' \\
Y \otimes Z \\
Y \\
Z \\
(\otimes Z)^{*}
\end{array}$

**Proposition 29** For any 2-cell $f : Y_1 \otimes \cdots \otimes Y_n \to Z_1 \otimes \cdots \otimes Z_m$, the following identities hold:
Proof. It is enough to show one of the identities (because applying the reverse rotation and using Lemma 27 (i), one can deduce the other identity). We will sketch the proof of the first identity.

For the case $m = n = 1$, the result follows trivially from the naturality of $a$.

Suppose $n = 2$. Then LHS of the first identity

Using Lemma 27 (v))

For $n > 2$, an analogous result can be deduced but calculations will be more involved.

After working on the rest of the curves (emanating from the top of the rectangle labelled with $f$) in the same way as above, the LHS of the first identity

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We now construct a planar algebra from a bicategory. Let $\mathcal{B}$ be a pivotal abelian strict 2-category with $\{+, -\}$ as the set of 0-cells and fix $X \in \text{ob}(\mathcal{B}(+, -))$. For each colour $(k, \varepsilon)$, set

$$X_{(k, \varepsilon)} = \begin{cases} X \otimes X^* \otimes X \otimes X^* \otimes \cdots \otimes k \text{ many tensor factors if } \varepsilon = +, \\ X^* \otimes X \otimes X^* \otimes X \otimes X^* \otimes \cdots \otimes k \text{ many tensor factors if } \varepsilon = -\end{cases}$$

if $k \geq 1$ and $X_{(0, \varepsilon)} = 1_{\varepsilon} \in \text{ob}(\mathcal{B}(\varepsilon, \varepsilon))$. Define $P_{(k, \varepsilon)} = \text{End}(X_{(k, \varepsilon)})$.

Now, for a $(k, \varepsilon)$-planar tangle $\theta \in T((k_1, \varepsilon_1), (k_2, \varepsilon_2), \cdots, (k_n, \varepsilon_n); (k, \varepsilon))$, we wish to define a multilinear map $P(\theta) : P_{(k_1, \varepsilon_1)} \times \cdots \times P_{(k_n, \varepsilon_n)} \to P_{(k, \varepsilon)}$. For this we extensively use the graphical calculus of the 2-cells of $\mathcal{B}$.

For the ease of dealing with 2-cells replaced by labelled rectangles, we will consider the planar tangle $\theta$ as an isotopy class of pictures where each disc (internal or external) is replaced by a rectangle with first half of the strings being attached to one of the side (called the top side) and the remaining half of the strings attached to the opposite side (called the bottom side). Next, in the isotopy class of $\theta$, we fix a picture $\Theta$ placed on $\mathbb{R}^2$ with the bottom side of the external rectangle being parallel to the $X$-axis, satisfying the following properties:

- the collection of strings in $\Theta$ must have finitely many local maxima and minima,

- each internal rectangle is aligned in such a way that the top side of the external rectangle is parallel and also nearer to the top side of the internal rectangle than its bottom side,
the projections of the maximas, minimas and one of the vertical sides of each internal rectangle (that is, the sides other than the top and bottom ones) on the vertical side of the external rectangle of $\Theta$ are disjoint.

We will say that an element $\Theta$ in the isotopy class of $\theta$ is in *standard form* if $\Theta$ satisfies the above conditions. For example, is a picture in standard form coming from the example of the tangle in Figure 2-1.

Let $\Theta$ be an element in standard form of the isotopy class of $\theta$. We now cut $\Theta$ into horizontal stripes so that every stripe should have at most one local maxima, minima or internal rectangle. Each component of every string in a horizontal stripe is labelled with $X$ or $X^*$ according as the orientation of the string is from the bottom side to the top side of the horizontal stripe or reverse respectively; each local maxima or minima is labelled with $X$ and the orientation is induced by the orientation of the actual string in $\Theta$. For example,

will be replaced by

To define $P(\theta) : P_{(k, r_1)} \times \cdots \times P_{(k, r_n)} \to P_{(k, \varepsilon)}$, we fix $2$-cells $f_i \in P_{(k, r_i)}$ for $1 \leq i \leq n$. We label the $i^{th}$ internal rectangle (contained in some horizontal stripe) with $f_i$. Now, each horizontal stripe makes sense as a $2$-cell according to the notation already set up. We define $P(\theta)(f_1, f_2, \ldots, f_n)$ as the composition of these.
2-cells. It is easy to easy that $P(\theta)$ is a multilinear map from $P(k_1, e_1) \times \cdots \times P(k_n, e_n)$ to $P(k, e)$. Natural question to ask will be why $P(\theta)(f_1, f_2, \cdots, f_n)$ is independent of the choice of $\Theta$ in the isotopy class of $\theta$.

We will attempt to show the independence by proving two labelled pictures $\Theta_1$ and $\Theta_2$ both in standard form, arising from $\theta$ and $\{f_1, f_2, \cdots, f_n\}$, differ by the following set of moves and the moves are preserved when they are translated in terms of 2-cells in $B$.

- **Sliding Move:**

  ![Sliding Move Diagram]

- **Rotation Move:**

  ![Rotation Move Diagram]

- **Wiggling Move:**

  ![Wiggling Move Diagram]

Sliding moves hold when translated into 2-cells from the functoriality of $\otimes$; rotation moves follow from Proposition 29; finally, wiggling moves are justified by Lemma 27 (i).

Now, what remains to show is that if $\Theta_1$ and $\Theta_2$ are elements in standard form in the isotopy class of $\theta$, then one can use the above three moves to get one from the other. This part will be heavily dependent on
the arguments used by Jones in the proof of Theorem 4.2.1 in [Jon2]. We describe below parts of the proof which is relevant to our context.

Starting from a planar \((k, e)\)-tangle \(\theta\), Jones fixed a picture \(\Theta\) in the isotopy class of \(\theta\), and viewed it in the following way:

- the boundary of the external disc of \(\Theta\) is made into a straight horizontal line by cutting at some point between the first and the last points; the \(2k\) marked points on the boundary of the external disc are marked on the horizontal line from left to right;

- each internal discs of \(\Theta\) is shrunk to a point lying strictly below the horizontal line;

- the strings in \(\Theta\) appear below the horizontal line in such a way that:
  
  (i) they meet the horizontal line transversely,

  (ii) the strings attached to an internal disc \(D_i\) in \(\Theta\) appear as a cusp \(C_i\) at the point to which \(D_i\) is shrunk, so that the string attached to the first point of \(D_i\) becomes the left most string at the cusp \(C_i\),

  (iii) planarity of the strings are preserved,

  (iv) the \(y\)-coordinates of the cusps, local maximas and minimas are all distinct.

Jones named such a picture arising from \(\Theta\) as a **standard picture**. For example,

![Diagram](image)

is a standard picture of the tangle in Figure 2.1. Thereafter, he introduced the three kinds of moves, namely, *sliding*, *rotation* and *wiggling* which are analogous to the moves we defined above. Finally, he argued that two standard pictures \(\Theta_1\) and \(\Theta_2\) which are isotopically equivalent can be obtained from one another by using the three moves.
Getting back to our context, if $\Theta_1, \Theta_2$ are in standard form in the isotopy class of $\theta$, then we get two standard pictures (in the sense of Jones) $J_1$ and $J_2$ of $\theta$ respectively via the following prescription:

Again, from $J_1$ and $J_2$, we get $\Theta'_1$ and $\Theta'_2$ in standard form in the isotopy class of $\theta$ respectively via the reverse prescription:

Observe that:

(i) $\Theta_i$ can be obtained from $\Theta'_i$ using the sliding and wiggling moves where $i \in \{1, 2\}$,
(ii) $J_1$ can be obtained from $J_2$ using the three moves by Jones,
(iii) one can show that $\Theta'_1$ can be obtained from $\Theta'_2$ using the moves induced by the ones used for obtaining $J_1$ from $J_2$ via the second prescription.

The above three observations clearly implies that $\Theta_1$ can be obtained from $\Theta_2$ using the three moves. Thus, we have a well-defined map $P : T((k_1, \xi_1), \cdots, (k_n, \xi_n); (k, \xi)) \to MVec((P_{(k_1, \xi_1)}, \cdots, P_{(k_n, \xi_n)}; P_{(k, \xi)}))$

(resp. $P : T(\emptyset; (k, \xi)) \to MVec(\emptyset; P_{(k, \xi)})$).

Finally, define the planar algebra $P : P \to MVec$ by:

- $P(k, \xi) = P_{(k, \xi)}$,
• the linear map \( P : \mathcal{P}(\{(k_1, \varepsilon_1), \ldots, (k_n, \varepsilon_n); (k, \varepsilon)\}) \to \mathcal{MVec}(P(\{k_1, \varepsilon_1\}), \ldots, P(\{k_n, \varepsilon_n\}); P(k, \varepsilon)) \) (resp. \( P : \mathcal{P}(\emptyset; (k, \varepsilon)) \to \mathcal{MVec}(\emptyset; P(\{k, \varepsilon\})) \)) is defined by extending the map \( P : T(\{(k_1, \varepsilon_1), \ldots, (k_n, \varepsilon_n); (k, \varepsilon)\}) \to \mathcal{MVec}(P(\{k_1, \varepsilon_1\}), \ldots, P(\{k_n, \varepsilon_n\}); P(k, \varepsilon)) \) (resp. \( P : T(\emptyset; (k, \varepsilon)) \to \mathcal{MVec}(\emptyset; P(\{k, \varepsilon\})) \)) linearly.

It is not hard to check that \( P \) preserves composition and identities. This completes the construction of the planar algebra.

### 3.2 Example: Group Planar Algebra

The planar algebra associated to the subfactor arising from the outer action of a finite group is discussed extensively and explicitly by Zeph Landau in [Lan] using operator algebraic techniques. One can also obtain this planar algebra called Group Planar Algebra from the above construction. For this, we need to consider the bicategory associated to the subfactor. We will first describe this bicategory and then establish a connection between the planar algebra arising from the bicategory and the group planar algebra in terms of 'generators and relations'.

Let \( G \) be a finite group. Consider the fusion category \( \text{Vec}(G) \) (= category of finite dimensional \( G \)-graded vector spaces with trivial associator). Let \( A = (A_g) \) be the algebra in \( \text{Vec}(G) \) where \( A_g = \mathbb{C} \) for all \( g \in G \) and the multiplication is given by:

\[
(A \otimes A)_g = \bigoplus_h (A_h \otimes A_{h^{-1}g}) \ni \bigoplus_h a_h \mapsto \sum_h a_h \in A_g
\]

and identity is given by the obvious inclusion of \( \mathbb{C}_1 \) (the object of \( \text{Vec}(G) \) with \( \mathbb{C} \) at 1 and zero vector spaces at all other elements of \( G \)) inside \( A \).

Define

\( B(+,+) = \text{category of } A - A \)-bimodules in \( \text{Vec}(G) \),

\( B(+,-) = \text{category of left } A \)-modules in \( \text{Vec}(G) \),

\( B(-,+)= \text{category of right } A \)-modules in \( \text{Vec}(G) \),

\( B(-,-)= \text{Vec}(G) \).
For definitions of an algebra and modules over an algebra in a fusion category, see [Ost]. They are basically categorical analogues of usual algebras and modules. We will mention several results about Vec(G) in the rest of this section without proofs; the proofs are not at all obvious but completely routine and depends on simple linear algebra techniques. Some of the proofs can be found in [Ost], [ENO].

The fusion category $B(\pm, \pm)$ is equivalent to the fusion category of finite dimensional representation of $G$ and each of $B(\pm, -)$ and $B(-, \pm)$ is equivalent to the abelian category $Vec$, the category of finite dimensional vector spaces preserving the module category structures. The tensor product bifunctor $\otimes : B(\varepsilon, \pm) \times B(\pm, \eta) \to B(\varepsilon, \eta)$ is induced by the usual tensor in $Vec(G)$ whereas the bifunctor $\otimes : B(\varepsilon, \pm) \times B(\pm, \eta) \to B(\varepsilon, \eta)$ is given by tensor product over the algebra $A$, denoted by, $\otimes$ where $\varepsilon, \eta \in \{\pm, -\}$. This makes $B$ into a bicategory. Since the element $A_A$ (resp. $A^A$) of $ob(B(-, \pm))$ (resp. $ob(B(+, -))$) corresponds to the one dimensional vector spaces in the equivalence with $Vec$. The algebra $A$ is self dual and the dual of $A^A$ is $A_A$.

We now consider the planar algebra arising from the object $A_A$ as described in the previous section and denote it by $P$. One can describe a planar algebra $P$ in terms of generators and relations (see [Jon2]) where generators are given by a certain set of elements in the two filtered algebras given by $P$ and relations are given by a set of linear combinations of tangles whose internals discs, if any, are labelled by the generators.

Let $s^h = (s^h_g) \in End(A)$ be defined by $s^h_g = \delta_{h,g} \text{id}_{A_A}$. The generators of $P$ are given by:

![Diagram](image)

$P \in End(A) \otimes A_A)$. The equivalence between $B(\pm, -)$ and $Rep(G)$ takes $A_A \otimes A_A$ to the regular representation ($CG$ as a left $G$-module) and the $A - A$ linear endomorphism $t^g$ corresponds to right multiplication by $g^{-1}$.

The relations are given by:
These relations can be proved using simple linear algebra and we leave it to the reader. However, we would like to comment that the bicategory which we are considering here is not strict and hence one has to take care of the appropriate associativity and unit constraints.
CHAPTER 4

AFFINE REPRESENTATIONS OF A PLANAR ALGEBRA

In this chapter, we will introduce the notion of an affine representation of a planar algebra which is a generalization of the concept of the Hilbert space representation of annular Temperley-Lieb by Vaughan Jones and Sarah Reznikoff ([JR]); one can also treat this as an annular representations of a planar algebra with rigid boundaries. We then discuss some general theory of these representations.

Before going into the definition of affine representations, we will first introduce the affine category over a planar algebra.

Definition 30 An \((m, \eta), (n, \epsilon)\)-affine tangle is an isotopy class of pictures consisting of:

- the annulus \( A = \{ z \in \mathbb{C} : 1 < z < 2 \} \),
- the set of points \( \{ 2e^{\frac{2\pi ik}{m}} : 0 \leq k \leq 2m-1 \} \) (resp. \( \{ e^{\frac{2\pi ik}{n}} : 0 \leq k \leq 2n-1 \} \)) are numbered clockwise starting from 2 (resp. 1) as the first points,
- \( A \) consists of internal discs \( D_1, D_2, \ldots, D_t \) with colour \( (k_1, \epsilon_1), (k_2, \epsilon_2), \ldots, (k_t, \epsilon_t) \) respectively and non-intersecting oriented strings (just like in an ordinary planar tangle described in Definition 18) so that the inner (resp. outer) boundary of \( A \) gets the colour \( (n, \epsilon) \) (resp. \( (m, \eta) \)),
- any isotopy should keep the boundary of \( A \) fixed.

Let \( P \) be a planar algebra. An \((m, \eta), (n, \epsilon)\)-affine tangle is said to be \( P \)-labelled if, to each internal disc \( D \) of \( A \) with colour \( (k, \epsilon') \), an element of \( P_{(k, \epsilon')} \) is assigned. Let \( A_{(n, \epsilon)}^{(m, \eta)} \) denote the set of all \((m, \eta), (n, \epsilon)\)-affine tangles and \( A_{(n, \epsilon)}^{(m, \eta)}(P) \) denote the set of all \( P \)-labelled \((m, \eta), (n, \epsilon)\)-affine tangles. If \( A \in A_{(n, \epsilon)}^{(m, \eta)} \) and \( B \in A_{(t, \delta)}^{(n, \epsilon)} \), then we can define \( A \circ B \in A_{(t, \delta)}^{(m, \eta)} \) as the affine tangle obtained by considering the picture \( \frac{1}{2}(2A \cup B) \). We might have to smoothen out the strings which are attached with the inner boundary of \( 2A \)
Figure 4-1. Example of a \(((2, -), (1, +))\)-affine tangle

and outer boundary of \(B\); this can also be avoided by requiring the strings to meet the inner and the outer boundaries radially in the definition of an affine tangle.

We now set up a convenient way of sketching an affine tangle; instead of marking the points on the inner (resp. outer) boundary at the roots of unity (resp. twice the roots of unity), we will mark them close to each other on the top with 1 as the leftmost point. Further, with the help of isotopy, every \(A \in \mathcal{A}_{(m, n)}^{(m, n)}\) can be expressed as:

for some \(T \in \mathcal{T}_{(m+n+2l, n)}\). Note that \(T\) and \(l\) are not unique. For example, the affine tangle in Figure 4-1 can be expressed as the above annular tangle for \(m = 2\), \(n = 1\), \(l = 1\) and
Let \((F.AP)_{(m,n)}(P)\) be the vector space with \(A_{(m,n)}(P)\) as a basis, \(T_{(k,e)}(P)\) be the set of all \(P\)-labelled \((k,e)\)-planar tangles, \(P_{(k,e)}(P)\) (resp. \(T_{(k,e)}(P)\)) be the vector space with \(T_{(k,e)}(P)\) (resp. \(T_{(k,e)}(P)\)) as a basis and \(\Psi_{(m,n),(n,e)}\) be the the following annular tangle:

\[
\begin{align*}
T = & \hspace{1cm} \begin{array}{c}
\includegraphics[width=0.5\textwidth]{tangle.png}
\end{array}
\end{align*}
\]

Observe that induces a linear map \(\psi_{(m,n),(n,e)} : P_{(m+n+l,e)}(P) \rightarrow (F.AP)_{(m,n)}(P)\) which also lifts to the level of the \(P\)-labelled ones. Moreover, for any \(A \in (F.AP)_{(m,n)}(P)\), there exists \(l \in \mathbb{N}_0\) and \(T \in P_{(m+n+l,e)}(P)\) such that \(A = \psi_{(m,n),(n,e)}(T)\). Set

\[
W_{(m,n)} = \left\{ A \in (F.AP)_{(m,n)}(P) \left| \begin{array}{l}
A = \psi_{(m,n),(n,e)}(T) \text{ for some } l \in \mathbb{N}_0, \\
T \in P_{(m+n+l,e)}(P) \text{ and } P(T) = 0
\end{array} \right. \right\}.
\]

Note that the map of multicategories \(P\) induces an obvious linear map \(P : P_{(k,e)}(P) \rightarrow P_{(k,e)}\). It is a fact that \(W_{(m,n)}\) is a vector subspace of \((F.AP)_{(m,n)}(P)\). For instance, if \(A_i \in W_{(m,n)}\) and \(l_i \in \mathbb{N}_0\), \(T_i \in P_{(m+n+i,e)}(P)\) such that \(A_i = \psi_{(m,n),(n,e)}(T_i), P(T_i) = 0\) for \(i = 1, 2\) and \(l_1 \leq l_2\), then one can obtain \(\overline{T}_1 \in P_{(m+n+2,e)}(P)\) such that \(A_1 = \psi_{(m,n),(n,e)}(\overline{T}_1)\) by wiggling back and forth a string emanating from either of the vertical sides of \(T_1\) around the inner disc of \(A_1\) until the total number of strings around the inner disc of \(A_1\) increases from \(l_1\) to \(l_2\); finally, \(A_1 + A_2 = \psi_{(m,n),(n,e)}(\overline{T}_1 + T_2)\).

Define the category \(AffP\) by:
\[ \text{ob}(\text{AffP}) = \{(k, e) : k \in \mathbb{N}_0, e \in \{+, -\} \} \]

- \[ \text{Hom}(\text{AffP})((n, s), (m, \eta)) = \frac{(\text{FAp})^{(m, \eta)}}{\mathcal{W}^{(n, s)}} = \text{the quotient vector space of} \ (\text{FAp})^{(m, \eta)} \text{over} \ \mathcal{W}^{(n, s)} \] (also denoted by \((\text{AffP})^{(m, \eta)}\)),

- the composition of affine tangles is linearly extended for \((\text{FAp})^{(m, \eta)}\); one can easily verify that \(A \circ B \in \mathcal{W}^{(m, \eta)}\) whenever \(A \in \mathcal{W}^{(m, \eta)}\) and \(B \in (\text{FAp})^{(n, s)}\), or \(A \in (\text{FAp})^{(n, s)}\) and \(B \in \mathcal{W}^{(m, \eta)}\); this implies the composition is induced in the level of quotient vector spaces as well,

- the identity of \((k, e)\) denoted by \(1_{(k, e)}\), is given by an \((k, e), (k, s))\)-affine tangle obtained by joining the \(i\)th point of the inner boundary with the \(i\)th point of the outer boundary by a straight string.

We call the category \text{AffP} the affine category over \(P\).

**Definition 31** An additive functor \(F : \text{AffP} \to \text{Vec}\) is said to be an affine representation of \(P\).

**Remark 32** The functor induced by \(P\) itself gives an affine representation of \(P\); this is called the 'trivial' affine representation.

**Lemma 33** If \(F\) is an affine representation of \(P\), then

\[ F(k, e) \rightarrow F(k + 1, e), \]

\[ F(k, e) \text{ is isomorphic to } F(k, -e) \]

for all colours \((k, e)\).

**Proof.** The inclusion in part (a) is given by considering the \(F\)-image of the inclusion tangle.

For part (b), consider the rotation tangle \(R_{(k, e)} \in \mathcal{A}^{(k, -e)}\) obtained by joining the points \(e^{-\frac{i\pi}{2}}\) and \(2e^{-\frac{i\pi}{4}}\) on the boundary of \(A\) by a string which does not make a full round about the inner disc. \(F(R_{(k, e)})\) gives the desired isomorphism in (b).

**Remark 34** It may seem so that \((R_{(k, e)})^{2k}\) is the identity \(((k, e), (k, e))\)-affine tangle (that is, the tangle obtained by joining the points \(e^{-\frac{i\pi}{2}}\) and \(2e^{-\frac{i\pi}{4}}\) by a straight line), but this is not true because of the restriction of the isotopy being identity on boundary of \(A\). This is the main difference between the annular representations of \(P\) (in [Jon3] and [Gho]) and the affine representations.
The weight of an affine representation \( F \) denoted by \( wt(F) \), is given by the smallest integer \( k \) such that \( \dim(F(k, e)) \neq 0 \). The \( wt(F) \) is well-defined by Lemma 33.

An affine representation \( F \) will be called locally finite if \( F(k, e) \) is finite dimensional for all colours \( (k, e) \).

The dimension of an affine representation \( F \) is defined as a pair of formal power series \( (\Phi_F^+, \Phi_F^-) \) where

\[
\Phi_F^\varepsilon(k) = \sum_{k=0}^{\infty} \dim(F(k, e)) z^k \quad \text{for } \varepsilon \in \{+, -\}.
\]

**Question (Jones):** If \( P \) is a planar algebra with modulus \( (\delta, \delta) \), is the radius of convergence of the dimension of an affine representation greater than or equal to \( \delta^{-2} \)?

The above question was asked by Vaughan Jones for annular representations of a planar algebra (see [Jon3]). The question for annular representations was answered in affirmative for the Temperley-Lieb planar algebras by Vaughan Jones himself (in [Jon3]) and for the Group Planar Algebras by Shamindra Ghosh (in [Gho]).

Let \( P \) be a \(*\)- or a \( C^*\)-planar algebra. Then, \( P(k, e)(P) \) becomes a \(*\)-algebra by \(*\) of a labelled tangle being \(*\) of the unlabelled tangle labelled with \(*\) of the labels. One can define \(*\) of an affine tangle by reflecting it around a circle concentric to inner or outer boundary and then isotopically stretch or shrink to fit into the annulus \( A \) such that the first point of inner or outer boundary after reflection remains the same whereas the first point of any internal disc after reflection is given by the reflection of the last point and colours of all discs are preserved; this can be induced in the \( P\)-labelled ones by labelling the internals discs of the reflected tangle with \(*\) of the labels. Note that \(*\) is an involution. Extending \(*\) conjugate linearly, we can define the map \( * : (FA P)_{(m,\eta)}^{(n,\xi)} \to (FA P)_{(m,\eta)}^{(n,\xi)} \) for all colours \( (m, \eta), (n, \xi) \). It is not hard to check that \( * (\gamma_{(m,\eta)}^{(n,\xi)}) = \gamma_{(m,\eta)}^{(n,\xi)} \). This makes the category \( \text{Aff} P \) a \(*\)-category. An additive functor \( F : \text{Aff} P \to \text{Hil} \) is said to be an affine \(*\)-representation if \( F \) is \(*\) preserving, that is, \( F(A^*) = (F(A))^* \) for all \( A \in \text{Mor}_{\text{Aff} P} \) where \( \text{Hil} \) denotes the category of Hilbert spaces.

**Remark 35** Note that if \( F \) is an affine \(*\)-representation, then

\[
\langle F(A)(v), w \rangle = \langle v, F(A^*)(w) \rangle
\]

31
for all $A \in \left( \text{Aff} P \right)_{(m,\eta)}^{(n,\epsilon)}$, $v \in F(n,\epsilon)$, $w \in F(m,\eta)$.

The category of affine representations of a planar algebra $P$ with natural transformations as morphism space, forms an abelian category and the dimension is additive with respect to direct sum. One can further talk about irreducibility and indecomposability of an affine representation. For example, the trivial affine representation of $P$ is irreducible. However, if we restrict ourselves to the case of a locally finite, non-degenerate $C^*$-planar algebra $P$ and the category of locally finite affine $*$-representations, the notions of irreducibility and indecomposability coincide. In this case, one can also talk about orthogonality of affine representations. These treatments for annular representations can be found in more details in [Jon3].

Vaughan Jones indicated a procedure of finding annular representations of a locally finite $C^*$-planar algebra $P$ with modulus $(\delta,\delta)$ in [Jon3]; the same works for the affine ones as well. For this we need to consider a subspace of the morphism space $\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)}$, namely,

$$\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)} = \left\{ \begin{array}{l}
A \in \left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)} \quad \text{A is a linear combination of} \\
\quad \text{elements of the form } B \circ C \text{ where} \\
\quad \quad B \in \left( \text{Aff} P \right)_{(n,\eta)}^{(k,\epsilon)}, \quad C \in \left( \text{Aff} P \right)_{(k,\epsilon)}^{(n,\eta)} \\
\quad \text{for some colour } (n,\eta) \text{ such that } n < k
\end{array} \right\}.
$$

It is easy to see that $\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)}$ is an ideal in $\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)}$. We list some common properties shared by affine $*$-representations and annular $*$-representations of $P$; the proofs can be found in [Jon3].

(i) An affine representation $F$ is irreducible iff $F(k,\epsilon)$ is irreducible as an $\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)}$-module for all colours $(k,\epsilon)$.

(ii) If $W$ is an irreducible $\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)}$-submodule of $F(k,\epsilon)$ for some colour $(k,\epsilon)$, then $W$ generates an irreducible subrepresentation of $F$.

(iii) Orthogonal $\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)}$-submodules of $F(k,\epsilon)$ for some colour $(k,\epsilon)$, generate orthogonal subrepresentations of $F$.

(iv) If $F$ and $G$ are representations with $F$ being irreducible and if $\theta : F(k,\epsilon) \to G(k,\epsilon)$ is a non-zero $\left( \text{Aff} P \right)_{(k,\epsilon)}^{(k,\epsilon)}$-linear homomorphism for some colour $(k,\epsilon)$, then $\theta$ extends to an injective homomorphism from $F$ to $G$, that is, an injective natural transformation from $F$ to $G$.
(v) If $W_{(k,e)} = \text{span}\{F(A)(v) : A \in (AffP)_{(k,e)}, v \in F(n,\eta), k > n \geq 0\}$, then

$$(W_{(k,e)})^\perp = \bigcap_{A \in (AffP)_{(k,e)}} \text{kernel}(F(A)).$$

From (v), we can conclude that for an affine $\ast$-representation $F$ with weight $k$, we have

$$F(k,e) = \bigcap_{A \in (AffP)_{(k,e)}} \text{kernel}(F(A))$$

since $W_{(k,e)}$ turns out to be zero and hence $F(k,e)$ forms a module over the quotient $(AffP)_{(k,e)}/F(A)$. We denote this quotient algebra by $(LW_P)_{(k,e)}$ (Lowest Weight algebra at $(k,e)$).

By (i), if $F$ is an irreducible affine $\ast$-representation with weight $k$, then $F(k,e)$ is an irreducible module over $(LW_P)_{(k,e)}$. In order to find the irreducible affine $\ast$-representations of $P$, it suffices to do the following:

(i) find the irreducible representations of $(LW_P)_{(k,e)}$,

(ii) find which irreducible representation of $(LW_P)_{(k,e)}$ gives rise to an irreducible affine $\ast$-representation of the planar algebra.

We will use this method to deduce some results of the irreducible affine $\ast$-representations of a finite depth planar algebra in the next chapter.
In this chapter, we will define the notion of the depth of a planar algebra which is motivated from the depth of a finite index subfactor in the context of Operator Algebras. We then prove some finiteness results for the category of affine representation in the case of some special class of planar algebras called subfactor-planar algebra. Finally, we answer the question posed by Vaughan Jones for this special class with extra assumption of finite depth.

Let $P$ be a planar algebra with modulus $(\delta_+, \delta_-)$. We first define below a tangle called Jones projections.

$E(k, \varepsilon) = P_{k+1, \varepsilon} \quad \in P_{k+1, \varepsilon}$ where $k \in \mathbb{N}$ and $\varepsilon \in \{+,-\}$. Note that

$E(k, \varepsilon)^2 = \begin{cases} \delta_+ E(k, \varepsilon), & \text{if } k \text{ is odd}, \\ \delta_- E(k, \varepsilon), & \text{if } k \text{ is even}. \end{cases}$

From now on, we will work with the case $\delta_+ = \delta_- = \delta$. In this case, $E(k, \varepsilon) = \frac{1}{\delta} E(k, \varepsilon)$ becomes an idempotent. Two more immediate consequences are:

(i) $E(k, \varepsilon) \cdot E(k+1, \varepsilon) \cdot E(k, \varepsilon) = E(k, \varepsilon)$,

(ii) $E(k, \varepsilon) \cdot E(l, \varepsilon) = E(l, \varepsilon) \cdot E(k, \varepsilon)$ whenever $|k - l| \geq 2$

where $\cdot$ denotes the multiplication in the planar algebra $P$.

**Lemma 36** The subspace $I_{(k, \varepsilon)} = P_{(k, \varepsilon)} e_{(k, \varepsilon)} P_{(k, \varepsilon)} = \text{span} \{ x \cdot e_{(k, \varepsilon)} : x, y \in P_{(k, \varepsilon)} \}$ is a two-sided ideal of $P_{(k+1, e)}$. 

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**Proof.** The proof of being right ideal easily follows by considering the tangle

![Diagram](image)

and being left ideal follows from the same with upside down. ■

**Lemma 37** If \( I_{(k,e)} = P_{(k+1,e)} \), then \( I_{(k+1,e)} = P_{(k+2,e)} \).

**Proof.** By Lemma 36, \( I_{(k+1,e)} \) is an ideal in \( P_{(k+2,e)} \). So, it is enough to show \( 1 \in I_{(k+1,e)} \). Now, implies \( 1 \in P_{(k+1,e)} = I_{(k,e)} = P_{(k,e)}E_{(k,e)}P_{(k,e)} = P_{(k,e)}E_{(k,e)}E_{(k+1,e)}E_{(k,e)}P_{(k,e)} \subset P_{(k+1,e)}E_{(k,e)}P_{(k+1,e)} = I_{(k+1,e)} \). ■

**Lemma 38** If \( I_{(k,e)} = P_{(k+1,e)} \), then

\[
\begin{align*}
I_{(k+1,e)} &= P_{(k+1,e)} 	ext{ if } k \text{ is even}, \\
I_{(k+1,e)} &= P_{(k+2,e)} 	ext{ if } k \text{ is odd}.
\end{align*}
\]

**Proof.** Define the tangle

![Diagram](image)

\( S_{(k,e)} = \in T((k,e),(k+1,e)) \).

Note that \( I_{(k,e)} = P_{(k+1,e)} \) if and only if \( P_{(k+1,e)} = \text{Range} \left( P\left(S_{(k,e)}\right)\right) \) (span of the image of \( P\left(S_{(k,e)}\right)\)).

We first consider the case \( k \) being even. Then, isotopically

\[
(R_{(k+1,e)})^{k+1} \circ S_{(k,e)} \circ (R_{(k,e)})^{k+1} = S_{(k,-e)}.
\]

Hitting both sides with \( P \) and using the invertibility of the rotation tangles, we get the desired equality.

If \( k \) is odd, then by Lemma 36, \( I_{(k+1,e)} = P_{(k+2,e)} \) where \( k + 1 \) is even and thus we are through by the first case. ■
Remark 39 Note that in the above three lemmas, the modulus of $P$ is not used at all.

Definition 40 A planar algebra $P$ is said to have finite depth if $I_{(i_e)} = P_{(l_e+1,e)}$ for some $l \in \mathbb{N}$, $e \in \{+, -\}$ and in that case, the depth of $P$ will be a pair of natural numbers $(l_+, l_-)$ such that $l_e$ is the smallest natural number such that $I_{(i_e, e)} = P_{(l_e+1,e)}$.

Remark 41 From the lemmas preceding the above definition, one can deduce that if $(l_+, l_-)$ denotes the depth of $P$ and $l_+$ is even (resp. odd), then $l_- \in \{l_e - 1, l_e\}$ (resp. $l_- \in \{l_e, l_e + 1\}$), that is, either both $l_+$ and $l_-$ are same or they are consecutive natural numbers with the larger one being even.

Let $S_{(k,e)}^n$ denote the tangle $S_{(k,e)}^n = S_{(k,+)}$ and $S_{(k,e)}^2 = S_{(k,e)}$ defined in the proof of the Lemma 38.

Lemma 42 If $P$ has finite depth with depth $(l_+, l_-)$, then

$$\text{Range } \left\{ P \left( S_{(k,e)}^n \right) \right\} = P(k + m - 1, e)$$

whenever $k \geq l_e$ is the $e$-depth of $P$ and $m \in \mathbb{N}$.

Proof. The case $m = 1$ is trivial and $m = 2$ follows from the proof of Lemma 38.

Suppose the statement of the lemma is true for all $m \leq n$. To show the same for $m = (n + 1)$, consider the tangle $S_{(k+n-1,e)}^2 \circ \left( S_{(k,e)}^n, S_{(k,e)}^n \right)$. Clearly,

$$\text{Range } \left( P \left( S_{(k+n-1,e)}^2 \circ \left( S_{(k,e)}^n, S_{(k,e)}^n \right) \right) \right) = P(k + n, e)$$

since $k \geq l_e$. Again, for even $n (= 2p$ say), the tangle $S_{(k+n-1,e)}^2 \circ \left( S_{(k,e)}^n, S_{(k,e)}^n \right)$ isotopically looks like:
Note that the tangle bounded by the dotted line, denoted by $T$, is a $(k, \varepsilon)$-planar tangle and thus $S^2_{(k+n-1, \varepsilon)} \circ \left(S^n_{(k, \varepsilon)} \circ S^n_{(k, \varepsilon)}\right) = S^{n+1}_{(k, \varepsilon)} \circ (1_{(k, \varepsilon)}, \cdots, 1_{(k, \varepsilon)}, T, 1_{(k, \varepsilon)}, \cdots, 1_{(k, \varepsilon)})$ where $T$ sits in the $(p + 1)^{th}$ position; this further implies

$$P(k + n, \varepsilon) = \text{Range} \left[P \left(S^2_{(k+n-1, \varepsilon)} \circ \left(S^n_{(k, \varepsilon)} \circ S^n_{(k, \varepsilon)}\right)\right]\right] \subset P(k + n, \varepsilon).$$

Similar arguments can be used to prove the same for odd $n$. Hence

$$\text{Range} \left(P \left(S^{n+1}_{(k, \varepsilon)}\right)\right) = \text{Range} \left(P \left(S^2_{(k+n-1, \varepsilon)} \circ \left(S^n_{(k, \varepsilon)} \circ S^n_{(k, \varepsilon)}\right)\right)\right) = P(k + n, \varepsilon).$$

**Proposition 43** If $P$ is a finite depth planar algebra with $(I_+, I_-)$ as its depth, then

$$(\text{Aff} P)_{(p, \varepsilon)}^{(q, \eta)} = \text{span} \left\{ (\text{Aff} P)_{(p, \varepsilon)}^{(s, \nu)} \circ (\text{Aff} P)_{(q, \eta)}^{(s, \nu)} \right\}$$

for all colours $(p, \varepsilon), (q, \eta), \nu \in \{+, -\}$ where $s = \left\lfloor \frac{1}{2} \min \{I_+, I_-\} \right\rfloor$. ($\lfloor \cdot \rfloor$ denotes the greatest integer function.)
Proof. If either of \( p \) and \( q \) is less than or equal to \( s \), then the equality can easily be established by wiggling a string sufficiently and then decomposing the affine tangle. One can also assume \( \varepsilon = \eta \) because the case when they are different can be deduced using rotation tangles. Without loss of generality, let \( l = l_+ \leq l_- \), \( p, q \geq s + 1 \) and \( \eta = \varepsilon = +. \) Let \( A \in (\text{AffP})^{(p,\eta)}_{(q,\eta)} \). Then, \( A \) can be expressed as the equivalence class of the affine tangle \( \Psi^{(p,\eta)}_{(p,\eta), (q,\eta)} \) such that the internal rectangle is labelled with an element of \( P_{(p+q+r,\eta)} \) where \( r \) can be chosen to exceed \( l \) (using wiggling around the inner disc). By Lemma 42, \( A \) is a linear combination (l.c.) of equivalence class (eq. cl.) of labelled strings. Now we consider two cases.

Case 1: \( l \) is odd, that is, \( l - 1 = 2s \). We can isotopically move the internal rectangles attached to left side of the above tangle around the inner disc and bring them to the to the right side. In this way, we express \( A \) as:
Identifying the two vertical sides of the last tangle we get an affine tangle which we cut along the dotted line; this cutting induces a decomposition of $A$. Note that the dotted line intersects exactly $2s$ strings. Thus $A \in \text{span} \left\{ (\text{Aff} P)^{(p,+)}_{(q,+)} \circ (\text{Aff} P)^{(s,+)}_{(q,+)} \right\}$.

Case 2: $l$ is even, that is, $l = 2s$. Using similar arguments, we can say that $A$ is a l.c. of eq. cl. of $\psi_{(p,+),(q,+)}^{2s}$. Following the same steps as in Case 1, we get $A \in \text{span} \left\{ (\text{Aff} P)^{(p,+)}_{(s,+)} \circ (\text{Aff} P)^{(s,+)}_{(q,+)} \right\}$. ■

**Corollary 44** If $P$ is a finite depth planar algebra with $(l_+, l_-)$ being its depth, then $(\text{Aff} P)_{(k,e)} = (\text{Aff} P)^{(k,e)}_{(l_+,l_-)}$ for all colours $(k,e)$ such that $k > s = \left\lfloor \frac{1}{2} \min \{l_+, l_-\} \right\rfloor$.

**Proof.** Follows immediately from the Proposition and definition of $(\text{Aff} P)_{(k,e)}$. ■

**Theorem 45** If $P$ is a finite depth subfactor-planar algebra with $(l_+, l_-)$ as its depth, then the affine $*$-representations of $P$ can have weight at most $s = \left\lfloor \frac{1}{2} \min \{l_+, l_-\} \right\rfloor$.

**Proof.** Corollary 44 implies that the lowest weight algebra $(LWP)_{(k,e)} = \{0\}$ whenever $k > s$. Thus, from the discussion of finding irreducible affine representations in Chapter 4, all irreducible affine representations have weight at most $s$. To prove the same for non-irreducible ones, note that taking direct sums never increases the weight. ■
Theorem 46 If $P$ is a finite depth subfactor-planar algebra with modulus $(\delta, \delta)$, then every irreducible affine $*$-representation of $P$ is locally finite and the radius of convergence of its dimension is at most $\frac{1}{2\delta}$. Moreover, the number of irreducibles at each weight is finite.

Proof. Let $F$ be an irreducible affine $*$-representation with weight $k$. So, $F(k, \varepsilon)$ is an irreducible module of $(LWP)^{k,\varepsilon}$. Irreducibility of $F$ says that $F$ induces a surjective linear map from $(AffP)^{(p,\eta)}_{(k,\varepsilon)} \otimes F(k, \varepsilon)$ to $F(p, \eta)$. Therefore, we have $\dim(F(p, \eta)) \leq \dim((AffP)^{(p,\eta)}_{(k,\varepsilon)}) \dim(F(k, \varepsilon))$. We look back once again into the two cases in the proof of Proposition 43. Let $l$ and $s$ be as in Proposition 43 for the rest of the proof.

A careful observation on the two cases will say that there exists a surjective linear map from $P(l, +)^{(p+k)}$ (resp. $P(l, +)^{(p+k+1)}$) to $(AffP)^{(p,\eta)}_{(k,\varepsilon)}$ when $l$ is odd (resp. even). Therefore,

$$\dim((AffP)^{(p,\eta)}_{(k,s)}) = \dim((AffP)^{(p,\eta)}_{(k,s)}) \leq (p + k + 1) \dim(P(l, +)) \leq \infty$$

since $P$ is locally finite. The lowest weight algebras become finite dimensional and hence there are finitely many irreducibles at each weight. This also implies $F(k, \varepsilon)$ has finite dimension. Thus $F$ is locally finite.

Next consider the labelled affine tangle obtained by the action of $\psi^{-1}_{(p,\eta),(k,\varepsilon)}$ (resp. $\psi^1_{(p,\eta),(k,\varepsilon)}$) on the tangle if $l$ is odd (resp. even). By Lemma 42 and proof of Proposition 43, eq. cl. of such labelled tangles generate $(AffP)^{(p,\eta)}_{(k,\varepsilon)}$. Therefore,

$$\dim((AffP)^{(p,\eta)}_{(k,s)}) = \dim((AffP)^{(p,\eta)}_{(k,s)}) \leq (k + l) \dim(P(l, +)) \dim(P(p, +)).$$

So, $\dim(F(p, \eta)) \leq (k + l) \dim(P(l, +)) \dim(P(p, +))$. Now, we try to find the limit of $((k + l) \dim(P(l, +)) \dim(P(p, +)))^{\frac{1}{2}}$ as $p \to \infty$. Note that $(k + l) \dim(P(l, +))$ is constant. Next,
\[ \lim_{p \to \infty} (\dim (P(p,+)))^{\frac{1}{2}} = \text{norm of the principal graph} = \text{the index of the finite depth subfactor corresponding to the planar algebra}. \]

By Jones’ theorem, index of the subfactor is square of the modulus. Hence,

\[ \lim \sup_{p \to \infty} (\dim (F(p, q)))^{\frac{1}{2}} \leq \lim \sup_{p \to \infty} ((k + l) \dim (P(l,+)) \dim (P(p,+)))^{\frac{1}{2}} = \delta^{2} \]

which implies radius of convergence of \( \Phi_{k}^{q} \) is at least \( \frac{1}{\delta^{2}} \). This ends the proof. ■
BIBLIOGRAPHY


