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Spectral interpretation of zeros of zeta functions

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Spectral interpretation of zeros of zeta functions

Abstract
For every algebraic number field we construct an operator on a separable Hilbert space, whose eigenvalues are exactly the critical zeros of the Dedekind zeta function of the number field.

Keywords
Mathematics

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Spectral Interpretation of Zeros of Zeta Functions

BY

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M.S., University of Szeged (1998)

THESIS

Submitted to the University of New Hampshire in partial fulfillment of the requirements for the degree of

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Dedication

To Tünde
Acknowledgments

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ABSTRACT
Spectral Interpretation of Zeros of Zeta Functions
by
Tamás Waldhauser
University of New Hampshire, May, 2004

For every algebraic number field we construct an operator on a separable Hilbert space, whose eigenvalues are exactly the critical zeros of the Dedekind zeta function of the number field.
Chapter 1

Introduction

In this chapter we present some preliminaries from number theory. The notation introduced here will be used throughout the paper. We just state the facts that we will need in the sequel; the proofs can be found in any standard text on number theory (cf. [4],[5]). At the end of each section we illustrate the concepts on the quadratic field $\mathbb{Q}(\sqrt{2})$.

1.1 Algebraic Number Fields

An algebraic number field is a finite extension of the field of rational numbers. Let $K$ be such a field, and let $n$ be the degree of $K$ over $\mathbb{Q}$. The main object of study in algebraic number theory is $\mathcal{O}_K$, the ring of algebraic integers in $K$. As a $\mathbb{Z}$-module it is a free module of rank $n$, i.e. isomorphic to $\mathbb{Z}^n$. The multiplicative structure is more complicated: $\mathcal{O}_K$ is not a unique factorization domain in general, but it is a so-called Dedekind domain. This fact has several nice consequences: maximal ideals are the same as prime ideals; every ideal can be written uniquely as a product of prime ideals; $\mathcal{O}_K$ is a principal ideal domain if and only if it is a unique factorization domain.

Nonzero ideals form a semigroup under multiplication, which can be extended to a group $I_K$ by introducing fractional ideals. These are submodules of $K$, regarded as an $\mathcal{O}_K$-module. In other words, fractional ideals have all the defining properties
of an ideal (closed under subtraction, and multiplication by elements of $O_K$), except for that they are not necessarily subsets of $O_K$. When we want to emphasise that a fractional ideal is really an ideal of $O_K$, then we say that it is an integral ideal.

Principal (fractional) ideals form a subgroup $P_K$ of $I_K$; the corresponding factor group, $I_K/P_K$ is called the class group, and its cardinality, $h_K$ is the class number of $K$. It is a classical result in number theory that the class group is finite, hence $h_K$ is a natural number. Informally we can say that the class number measures how far $O_K$ is from being a principal ideal domain; $h_K$ is the number of different types of ideals 'modulo principal ideals'. We have $h_K = 1$ iff $O_K$ is a principal ideal domain.

There are $n = r_1 + 2r_2$ different field embeddings of $K$ into $\mathbb{C}$; here $r_1$ is the number of real embeddings (whose image lies in $\mathbb{R}$) and $2r_2$ is the number of complex embeddings (they appear in complex conjugate pairs, therefore the number of them is even). Let $\sigma_1, \ldots, \sigma_n$ denote these embeddings indexed in such a way that the first $r_1$ are the real embeddings, and $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ are complex ones such that none of them is the complex conjugate of the other. Most of the time we will work only with $\sigma_1, \ldots, \sigma_r$, where $r = r_1 + r_2$, because conjugate embeddings carry the same information about the field, so it is sufficient to take one member of each conjugate pair.

The norm of an element $\alpha \in K$ is

$$N(\alpha) = \sigma_1(\alpha) \cdots \sigma_n(\alpha).$$

This is the same as the determinant of the transformation $x \mapsto \alpha x$ of $K$ as a vector space over $\mathbb{Q}$. Therefore the norm is always a rational number, and it is multiplicative: $N(\alpha\beta) = N(\alpha)N(\beta)$. The norm of an algebraic integer is always a (rational) integer.

We define the norm of an ideal $\mathfrak{a} \triangleleft O_K$ to be $N(\mathfrak{a}) = |O_K/\mathfrak{a}|$, that is the number of cosets (residue classes) with respect to $\mathfrak{a}$. The norm of the principal ideal generated by $\alpha \in O_K$ is the same as the absolute value of the norm of $\alpha$: $N(\alpha O_K) = |N(\alpha)|$. 
Example 1 Let us consider the case $K = \mathbb{Q}(\sqrt{2})$. Every element $\alpha \in K$ can be written uniquely in the form $a + b\sqrt{2}$ ($a, b \in \mathbb{Q}$), and $\alpha \in \mathcal{O}_K$ if and only if $a, b \in \mathbb{Z}$. Thus $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank $n = 2$ : $\mathcal{O}_K = \mathbb{Z} [\sqrt{2}] = \mathbb{Z} \oplus \sqrt{2} \mathbb{Z} \cong \mathbb{Z}^2$. The class number is 1 for this field, thus $\mathcal{O}_K$ is a principal ideal domain. We have $r = r_1 = 2, r_2 = 0$, and the two real embeddings are $\sigma_1 : a + b\sqrt{2} \mapsto a + b\sqrt{2}$ and $\sigma_2 : a + b\sqrt{2} \mapsto a - b\sqrt{2}$. The norm of $\alpha = a + b\sqrt{2}$ is $N(\alpha) = (a + b\sqrt{2}) \cdot (a - b\sqrt{2}) = a^2 - 2b^2$. We can see that if $\alpha \in \mathcal{O}_K$, then $N(\alpha) \in \mathbb{Z}$.

1.2 Units

If $K \subseteq \mathbb{R}$, then it is a dense subset of the real line, otherwise it is dense in the complex plane, and in most cases even $\mathcal{O}_K$ is dense in $\mathbb{R}$ or $\mathbb{C}$. To get a nicer picture of $K$ (or at least of $\mathcal{O}_K$), we regard it as a subset of some bigger space. One such construction is $\mathbb{A}_K$, the ring of adeles, a complicated locally compact topological ring whose additive group contains $K$ as a discrete cocompact subgroup. In [1] the adele ring is used to obtain spectral interpretation for the zeros of zeta functions. Here we will use the following much simpler construction, which is also standard in number theory.

Let us embed $K$ into $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ via the map

$$\sigma : K \rightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}, \alpha \mapsto (\sigma_1(\alpha), \ldots, \sigma_r(\alpha)).$$

Algebraically we think of $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ as a ring (the direct product of $r_1$ copies of $\mathbb{R}$ and $r_2$ copies of $\mathbb{C}$), and $\sigma$ is clearly a ring homomorphism. If $u = (u_1, \ldots, u_r)$ and $v = (v_1, \ldots, v_r) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ are given, then we write $u \cdot v$ for $(u_1v_1, \ldots, u_rv_r)$ and $u/v$ for $(u_1/v_1, \ldots, u_r/v_r)$ if none of the $v_i$ equals zero, and let $\overline{u}$ denote $(\overline{u_1}, \ldots, \overline{u_r})$ (the complex conjugate is certainly unnecessary in the first $r_1$ coordinates).
Geometrically we identify \( C \) with \( \mathbb{R}^2 \), then \( \mathbb{R}^{r_1} \times C^{r_2} \) becomes \( \mathbb{R}^{r_1+2r_2} = \mathbb{R}^n \). The image of \( K \) might be still dense in this space, but \( \sigma(O_K) \), or more generally \( \sigma(\mathfrak{A}) \) for any ideal \( \mathfrak{A} \), is an \( n \)-dimensional lattice in \( \mathbb{R}^n \). We use the notation \( \langle u, v \rangle \) for the usual inner product in \( \mathbb{R}^n \):

\[
\langle u, v \rangle = \sum_{i=1}^{r_1} u_i v_i + \sum_{i=r_1+1}^{r} (\text{Re} \, u_i \text{Re} \, v_i + \text{Im} \, u_i \text{Im} \, v_i).
\]

Let \( \alpha \) be an element of \( K \), and \( x = (x_1, \ldots, x_r) = \sigma(\alpha) \). Then we have

\[
|N(\alpha)| = |x_1|^{l_1} \cdots |x_r|^{l_r},
\]

where \( l_1 = \cdots = l_{r_1} = 1 \) and \( l_{r_1+1} = \cdots = l_r = 2 \). Sometimes we will refer to \( |x_1|^{l_1} \cdots |x_r|^{l_r} \) as the 'norm' of \( x \) even if \( x \notin \sigma(K) \).

A nonzero element \( \varepsilon \) of \( O_K \) is called a unit if its reciprocal is also an algebraic integer. Units form a group \( U_K \) under multiplication, and two elements \( \alpha, \beta \in O_K \) generate the same principal ideal iff there is an \( \varepsilon \in U_K \) such that \( \alpha = \varepsilon \beta \). Thus principal ideals can be identified with the orbits of \( O_K \) under the action of \( U_K \); let \( [\alpha] \) denote the orbit of \( \alpha \in O_K \).

The structure of the unit group is described by the following classical theorem.

**Theorem 2** (Dirichlet's Unit Theorem) \( U_K \cong W \times \mathbb{Z}^{r-1} \), where \( W \) is a finite cyclic group.

Let us briefly explain the construction used in the proof of this theorem. It is not hard to see that \( \varepsilon \in O_K \) is a unit if and only if \( N(\varepsilon) = \pm 1 \). Therefore the units (more precisely their images under \( \sigma \)) lie on the 'norm-one' surface

\[
S = \left\{ (x_1, \ldots, x_r) : |x_1|^{l_1} \cdots |x_r|^{l_r} = 1 \right\} \subset \mathbb{R}^n.
\]
If $x \in \mathbb{R}^* \times \mathbb{C}^r$, then let us simply write $\log(x)$ for \( (\ln|x_1|^{\|_1}, \ldots, \ln|x_r|^{\|_r}) \in \mathbb{R}^r \) and let $\ell$ denote the composition of $\sigma$ and this logarithmic function:

$$
\ell : K^* \to \mathbb{R}^r, \alpha \mapsto (\ln|\sigma_1(\alpha)|^{\|_1}, \ldots, \ln|\sigma_r(\alpha)|^{\|_r}).
$$

Clearly $\ell$ is a homomorphism from the multiplicative group $K^*$ of nonzero elements of $K$ to the additive group $\mathbb{R}^r$, and $\alpha \in \mathcal{O}_K$ is a unit iff $\ell(\alpha)$ lies on the 'trace-zero' hyperplane

$$
H = \{(y_1, \ldots, y_r) : y_1 + \ldots + y_r = 0\} \subseteq \mathbb{R}^r.
$$

(Notice the similarities and the differences between the definition of $S$ and $H$. Both are sets of $r$-tuples, but some of the $x_i$ are complex numbers, therefore $S$ is regarded as a subset of $\mathbb{R}^n$, while the $y_i$ are all real numbers, so $H$ is a subset of $\mathbb{R}^r$.)

Dirichlet's theorem states that $\ell(U_K)$ is a lattice in $H$ (therefore isomorphic to $\mathbb{Z}^{r-1}$), and the kernel of $\ell$ is a finite group $W$, namely the group of the roots of unity belonging to $K$. A system of units $\varepsilon_1, \ldots, \varepsilon_{r-1}$ is called a fundamental system of units, if $\ell(\varepsilon_1), \ldots, \ell(\varepsilon_{r-1})$ is a basis for the lattice $\ell(U_K)$, that is $\ell(U_K) = \ell(\varepsilon_1) \mathbb{Z} \oplus \ldots \oplus \ell(\varepsilon_{r-1}) \mathbb{Z}$.

**Example 3** Both $K = \mathbb{Q}(\sqrt{2})$ and $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ are dense in $\mathbb{R}$, but the image of $\mathcal{O}_K$ under the map $\sigma : a + b\sqrt{2} \mapsto (a + b\sqrt{2}, a - b\sqrt{2})$ is the two-dimensional lattice $(1, 1)\mathbb{Z} \oplus (\sqrt{2}, -\sqrt{2})\mathbb{Z}$ in $\mathbb{R}^2$. Figure 1-1 shows this lattice with the basis vectors $\sigma(1) = (1, 1)$ and $\sigma(\sqrt{2}) = (\sqrt{2}, -\sqrt{2})$.

We have $t_1 = t_2 = 1$, and $\ell(a + b\sqrt{2}) = (\ln|a + b\sqrt{2}|, \ln|a - b\sqrt{2}|)$. The 'norm-one' surface $S$ is defined by $|x_1x_2| = 1$, so it is a union of two hyperbolas. On Figure 1-2, we can see this surface (black hyperbolas), and the 'norm-two', 'norm-three', etc. surfaces (gray hyperbolas) as well. The ten bigger dots represent units (more precisely, their images under $\sigma$).
Figure 1-1: The lattice $\sigma (\mathcal{O}_K)$

Figure 1-2: The norm-one surface $S$ and $\sigma (U_K)$
Figure 1-3: Construction of the trace-zero hyperplane $H$ and the lattice $\ell(U_K)$

If we take absolute values in both coordinates we get the hyperbola-branch shown on the left side of Figure 1-3. Note that each of the five dots are covered by two of the ten dots of Figure 1-2. This fact shows that the kernel of $\ell$ is a two-element group, namely $W = \{1, -1\}$. Taking logarithms we obtain $\ell(U_K)$, a one-dimensional lattice in the hyperplane $H = \{(y_1, y_2) : y_1 + y_2 = 0\} \subseteq \mathbb{R}^2$ (the right side of Figure 1-3). A generating vector for this lattice is $\ell(1 + \sqrt{2}) = (\ln|1 + \sqrt{2}|, \ln|1 - \sqrt{2}|) = (\ln(1 + \sqrt{2}), -\ln(1 + \sqrt{2}))$, thus $\varepsilon = 1 + \sqrt{2}$ is a fundamental unit. (In this example $n = r = 2$, so both $S$ and $H$ lie in $\mathbb{R}^2$, but this is just a coincidence; usually $S$ is in a higher dimensional space than $H$.) Finally, we obtain that the group of units is $U_K = \{\pm (1 + \sqrt{2})^k : k \in \mathbb{Z}\} \cong \{\pm 1\} \times \mathbb{Z}$.

1.3 Zeta Functions

The Dedekind zeta function of $K$ is defined by the Dirichlet series

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

(1.2)

the sum is taken over all nonzero integral ideals of $\mathcal{O}_K$, and $a_n$ is the number of integral ideals with norm $n$. This series converges absolutely in the half-plane $\text{Re}\,(s) > 1$, and
has an analytic continuation to the whole complex plane with a simple pole at \( s = 1 \) as the only singularity.

The zeros of this function are some negative integers (these are the trivial zeros), and some numbers in the critical strip defined by \( 0 < \text{Re}(s) < 1 \); we call these critical zeros. The location of these zeros has deep connections with the distribution of the prime ideals of \( \mathcal{O}_K \). The Generalized Riemann Hypothesis states that all critical zeros lie on the critical line, given by \( \text{Re}(s) = 1/2 \).

We can break (1.2) into the sum of the \( h_K \) partial zeta functions that correspond to summations on the ideal classes. If \( C \) is an ideal class (i.e. \( C \in \mathcal{I}_K/P_K \)), then let

\[
\zeta_K(s, C) = \sum_{\mathfrak{a} \in C} \frac{1}{N(\mathfrak{a})^s};
\]

the summation is taken over all nonzero integral ideals belonging to the class \( C \). Just like (1.2), this series converges for \( \text{Re}(s) > 1 \), and has an analytic continuation with a simple pole at \( s = 1 \) as the only singularity. Clearly the Dedekind zeta function is the sum of these partial zeta functions:

\[
\zeta_K(s) = \sum_{C \in \mathcal{I}_K/P_K} \zeta_K(s, C).
\]

Let us fix an ideal class \( C \), and let \( \mathfrak{B} \) be an integral ideal in \( C^{-1} \). Then for every \( \alpha \in \mathfrak{B} \) the ideal \( \alpha \mathcal{O}_K \cdot \mathfrak{B}^{-1} \) is integral, and belongs to \( C \), and conversely, every integral ideal \( \mathfrak{a} \in C \) can be written in this form. The principal ideal \( \alpha \mathcal{O}_K \) is uniquely determined, i.e. \( \alpha \) is unique up to a unit factor. Therefore we can write the partial zeta function in the following form (cf. [6]).

\[
\zeta_K(s, C) = \sum_{[\alpha] \in \mathfrak{B}^*/U_K} \frac{1}{N(\alpha \mathcal{O}_K \cdot \mathfrak{B}^{-1})^s} = \sum_{[\alpha] \in \mathfrak{B}^*/U_K} \frac{1}{|N(\alpha)/N(\mathfrak{B})|^s} = N(\mathfrak{B})^s \sum_{[\alpha] \in \mathfrak{B}^*/U_K} \frac{1}{|N(\alpha)|^s}. \tag{1.3}
\]

Here \( \mathfrak{B}^*/U_K \) denotes the set of orbits of the (multiplicative) action of \( U_K \) on \( \mathfrak{B}' = \mathfrak{B} \setminus \{0\} \), and \([\alpha]\) is the orbit of \( \alpha \).
Example 4 For the field $K = \mathbb{Q}(\sqrt{2})$ we have just one partial zeta function, and it is equal to the Dedekind zeta function $\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where $a_n$ is the number of inequivalent solutions of the Pell equation $x^2 - 2y^2 = n$. Here two solutions $(x_1, y_1)$ and $(x_2, y_2)$ are considered equivalent if there is an integer $k$ such that $x_2 + y_2\sqrt{2} = \pm (x_1 + y_1\sqrt{2})^k$. The first few terms of this Dirichlet series are the following: $\zeta_K(s) = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{2}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{2}{14^s} + \frac{1}{16^s} + \frac{2}{17^s} + \frac{1}{18^s} + \cdots$. 

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Chapter 2

The Pólya-Hilbert Space

In this chapter our goal is to define a Hilbert space, and in Chapter 3 we will define an operator on this space, whose eigenvalues are exactly the critical zeros of ζ_K. We will follow the ideas described in [3], where the same work has been done for the case K = Q. Our operator will be the same, and the Hilbert space where it acts will be defined in an analogous way.

2.1 The Setup

The Pólya-Hilbert space will appear as the orthogonal complement of a subspace of L^2(\R^+, \mu(x) d^x), where \mu(x) = \frac{1}{1+x}, and d^x = \frac{dx}{x} is the multiplicative Haar-measure on the positive real numbers. The weight \mu(x) is just a technical detail: we need it to make some functions of the form x^* (the eigenfunctions) integrable. We will denote the norm and the inner product in L^2(\R^+, \mu(x) d^x) by ||||_\mu and \langle \cdot, \cdot \rangle_\mu respectively. Let S(\R^+) be the space of Schwartz functions on the positive half-line, i.e. the set of smooth functions that vanish rapidly at 0 and at \infty along with all their derivatives. The space of those Schwartz functions whose integral is 0 (w.r.t. Lebesgue measure) is denoted by S(\R^+)_0.

For a function f \in S(\R^+) let us define Ef by the following formula:

\[ Ef(x) = \sum_{\Re \alpha \in \Omega_K} f(N(\alpha)x) = \sum_{n=1}^{\infty} a_n f(nx). \]
For any given $m > 1$ there is a constant $C > 0$ such that $|f(x)| \leq C/x^m$, because $f$ is a Schwartz function. Then $\sum_{n=1}^{\infty} |a_n f(nx)| \leq \sum_{n=1}^{\infty} a_n C/(nx)^m = C\zeta_K(m)/x^m$, hence the sum that defines $Ef(x)$ is absolutely convergent, and $Ef$ vanishes rapidly at infinity. The convergence is uniform on any interval of the form $[\delta, \infty)$ (with $\delta > 0$), so we can differentiate (2.1) term by term. Differentiating $k$ times we obtain (for $x > 0$):

$$\frac{d^k}{dx^k} Ef(x) = \sum_{n=1}^{\infty} a_n \frac{d^k}{dx^k} f(nx) = \sum_{n=1}^{\infty} a_n n^k f^{(k)}(nx) = \frac{1}{x^k} \sum_{n=1}^{\infty} a_n (nx)^k f^{(k)}(nx).$$

Here $f^{(k)}$ denotes the $k$-th derivative of $f$, and we can recognize $E(x^k f^{(k)}(x))$ on the right hand side, thus we have:

$$\frac{d^k}{dx^k} Ef(x) = \frac{1}{x^k} E(x^k f^{(k)}(x)).$$

Clearly $x^k f^{(k)}(x) \in \mathcal{S}(\mathbb{R}^+)$, therefore we can conclude that all the derivatives of $Ef(x)$ vanish rapidly at infinity.

In general $Ef$ does not vanish at zero; for this we need the integral of $f$ to be 0, so we will only consider functions from $\mathcal{S}(\mathbb{R}^+)_0$. The major part of this section is devoted to prove that $E$ maps this space into $\mathcal{S}(\mathbb{R}^+)$ (Theorem 13). Suppose for the moment that this has already been established, and let us finish the definition of the Pólya-Hilbert space.

To handle the factor $\frac{1}{1+x^2}$ in the definition of the measure we use the following map $\Phi$ instead of $E$:

$$\Phi: \mathcal{S}(\mathbb{R}^+)_0 \rightarrow \mathcal{S}(\mathbb{R}^+), f(x) \mapsto (1 + x^2) Ef(x).$$

Obviously $\mathcal{S}(\mathbb{R}^+) \subseteq L^2(\mathbb{R}^+,\mu(x)\,d^*x)$, so we can regard $\Phi(\mathcal{S}(\mathbb{R}^+)_0)$ as a linear subspace of $L^2(\mathbb{R}^+,\mu(x)\,d^*x)$. Finally, the Pólya-Hilbert space $\mathcal{H}$ that we consider is the orthogonal complement of this subspace:

$$\mathcal{H} = \Phi(\mathcal{S}(\mathbb{R}^+)_0)^\perp \subseteq L^2(\mathbb{R}^+,\mu(x)\,d^*x).$$

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Example 5 For $K = \mathbb{Q}(\sqrt{2})$ we have $Ef(x) = \sum_{n=1}^{\infty} a_n f(nx) = f(x) + f(2x) + f(4x) + 2f(7x) + f(8x) + f(9x) + 2f(14x) + f(16x) + 2f(17x) + f(18x) + \cdots$

2.2 The Auxiliary Function $\Omega_K$

Let us break up (2.1) into the sums on the ideal classes, just as we did with the Dedekind zeta function:

$$Ef(x) = \sum_{\mathfrak{a} \in \mathcal{O}_K} f(N(\mathfrak{a})x) = \sum_{\mathcal{C} \in \mathcal{C}/\mathcal{P}_K} \sum_{\mathfrak{a} \in \mathcal{C}} f(N(\mathfrak{a})x). \quad (2.3)$$

This is a sum of $h_K$ many functions, so it is enough to show that all of these are Schwartz functions. Let $\mathcal{C}$ be an ideal class, and $\mathfrak{B}$ an arbitrary (but fixed) integral ideal from $\mathcal{C}^{-1}$. Proceeding the same way as in (1.3) we can express the summand of (2.3) corresponding to this class in the form

$$\sum_{\mathfrak{a} \in \mathfrak{B}^*/\mathfrak{U}_K} f \left( \left[ N(\mathfrak{a}) \right]/N(\mathfrak{B}) \right) x. \quad (2.4)$$

Introducing the notation

$$E_{\mathcal{C}}f(x) = \sum_{\mathfrak{a} \in \mathfrak{B}^*/\mathfrak{U}_K} f \left( \left| N(\mathfrak{a}) \right| x \right)$$

we can write (2.4) as $E_{\mathcal{C}}f(x/N(\mathfrak{B}))$, so it suffices to prove that $E_{\mathcal{C}}(S(R^+)_0) \subseteq S(R^+)$. In order to prove this we will construct a function on $\mathbb{R}^1 \times \mathbb{C}^2$, such that $E_{\mathcal{C}}f(x)$ can be expressed as a sum of this function on the lattice $\Gamma = \sigma(\mathfrak{B})$, and then we will use the Poisson Summation Formula just as in [3]. The main difficulty compared to the case $K = \mathbb{Q}$ discussed there is that in an arbitrary algebraic number field the group of units can be infinite. (From Dirichlet's Unit Theorem one sees that this is the case whenever $K$ is neither an imaginary quadratic field, nor $\mathbb{Q}$.) To handle this difficulty we will define a smooth weight function $\Omega_K$ on $\mathbb{R}^1 \times \mathbb{C}^2$ such that for every $x \in \mathbb{R}^1 \times \mathbb{C}^2$ the following property holds:

$$\sum_{c \in U_K} \Omega_K (c \cdot \sigma(e)) = 1. \quad (2.5)$$

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Informally it means that if we let \( U_K \) act on \( \mathbb{R}^r \times \mathbb{C}^2 \) by

\[
x = (x_1, \ldots, x_r) \mapsto (x_1 \sigma_1(\varepsilon), \ldots, x_r \sigma_r(\varepsilon)) = x \cdot \sigma(\varepsilon),
\]

then the sum of \( \Omega_K \) on each orbit is 1.

First of all, we need a smooth function \( \eta(x) \) defined on the interval \([0, 1/2]\) such that \( \eta(0) = 0, \eta(1/2) = 1/2 \), and all the derivatives of \( \eta \) vanish at 0 and at 1/2. For example \( \eta(x) = \frac{1}{2} e^{-e^{2\pi i x}} \) is such a function (Figure 2.1 shows \( \eta(x) = \frac{1}{2} e^{-0.3 e^{2\pi i x}} \)). Now we paste four pieces of \( \eta \) together to get the function \( \omega(x) \) (see Figure 2.2):

\[
\omega(x) = \begin{cases} 
1 - \eta(|x|), & \text{if } |x| \leq 1/2, \\
\eta(1 - |x|), & \text{if } 1/2 \leq |x| \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]

This function is smooth and compactly supported, therefore it is a Schwartz function on \( \mathbb{R} \). Moreover, thanks to its symmetries, for every real number \( x \) we have \( \sum_{k=-\infty}^{\infty} \omega(x + k) = 1 \). Now, if we define \( \omega^{(m)}(x_1, \ldots, x_m) = \omega(x_1) \cdots \omega(x_m) \) for any natural number \( m \), then \( \omega^{(m)} \) is a smooth function on \( \mathbb{R}^m \), supported on \([-1, 1]^m\), and for all \( x \in \mathbb{R}^m \) we have

\[
\sum_{k \in \mathbb{Z}^m} \omega^{(m)}(x + k) = 1. \tag{2.7}
\]

Now let us apply \( \omega^{(r-1)} \) to the 'trace-zero' hyperplane \( H \) distorted in such a way that the lattice \( \ell(U_K) \) takes the place of \( \mathbb{Z}^m \) in (2.7). To do this let us fix a fundamental system of units \( \varepsilon_1, \ldots, \varepsilon_{r-1} \) for \( K \). Then \( \ell(\varepsilon_1), \ldots, \ell(\varepsilon_{r-1}) \) is a basis for \( H \), and appending the vector \( \mathbf{1} = (1, \ldots, 1) \) we get a basis for \( \mathbb{R}^r \). We define the function \( \omega_K \) on \( \mathbb{R}^r \) by the formula

\[
\omega_K(c_1 \ell(\varepsilon_1) + \cdots + c_{r-1} \ell(\varepsilon_{r-1}) + c_r \mathbf{1}) = \omega^{(r-1)}(c_1, \ldots, c_{r-1}).
\]

(Thus in order to get \( \omega_K(y) = \omega_K(y_1, \ldots, y_r) \) we need to compute the coordinates of \( y \) in the above mentioned basis, drop the last coordinate, and plug the rest into \( \omega^{(r-1)} \).)
Figure 2-1: Graph of $\eta(x)$

Figure 2-2: Graph of $\omega(x)$
On every translate of $H$ this function looks like $\omega^{(r-1)}$ (distorted suitably), therefore the following version of (2.7) holds (for every $y \in \mathbb{R}^r$):

$$\sum_{\varepsilon \in U_K} \omega_K(y + \ell(\varepsilon)) = |W|.$$  \hspace{1cm} (2.8)

(Each element of $\ell(U_K)$ appears $|W|$ times in this sum, because the kernel of $\ell$ is $W$.)

Finally, we take the composition of $\omega_K$ and the logarithmic function on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Given any $x = (x_1, \ldots, x_r) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ with $x_i \neq 0$, let $c_1, \ldots, c_r$ be the coordinates of $\log(x) = \left(\ln|x_1|^{i_1}, \ldots, \ln|x_r|^{i_r}\right)$ in the basis $\ell(\varepsilon_1), \ldots, \ell(\varepsilon_{r-1}), 1$. Let us define the function $\Omega_K$ in the following way:

$$\Omega_K(x) = \frac{1}{|W|} \omega_K(\log(x)) = \frac{1}{|W|} \omega^{(r-1)}(c_1, \ldots, c_{r-1}).$$

Using (2.8) we can check that property (2.5) holds (with $y = \log x$):

$$\sum_{\varepsilon \in U_K} \Omega_K(x \cdot \sigma(\varepsilon)) = \sum_{\varepsilon \in U_K} \frac{1}{|W|} \omega_K(\log(x) \cdot \sigma(\varepsilon)))
= \sum_{\varepsilon \in U_K} \frac{1}{|W|} \omega_K(\log(x) + \log(\sigma(\varepsilon)))
= \frac{1}{|W|} \sum_{\varepsilon \in U_K} \omega_K(y + \ell(\varepsilon)) = 1.$$

Note that $\Omega_K$ is not yet defined at those points of $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, where some of the coordinates are zero, but it is smooth where it is defined. The limit of $\Omega_K$ at the origin does not exist, hence it is impossible to define it there to make it continuous. If we define $\Omega_K$ to be zero at those points where some, but not all of the coordinates are zero, then our function will be locally constant, hence smooth at these points. This follows from the next lemma, where we prove that if $(x_1, \ldots, x_r)$ belongs to the support of $\Omega_K$, then the numbers $|x_i|^{i_1}$ are close to each other in the sense that none of them is too far from their geometric mean.
Lemma 6 There are constants $0 < a < 1 < A$ such that if $\Omega_K(x_1, \ldots, x_r) \neq 0$, then for every $i = 1, \ldots, r$ we have

$$a \sqrt{|x_1|^{i_1} \cdots |x_r|^{i_r}} \leq |x_i|^{i_i} \leq A \sqrt{|x_1|^{i_1} \cdots |x_r|^{i_r}}.$$  

Proof. Let us choose $0 \neq x = (x_1, \ldots, x_r) \in \mathbb{R}^r$ such that $\Omega_K(x) \neq 0$. Let $c_1, \ldots, c_r$ denote the coordinates of $y = \log(x)$ in the basis $\ell(e_1), \ldots, \ell(e_{r-1}), 1$. Since the support of $\omega^{(r-1)}$ is the cube $[-1,1]^{r-1}$, we have $c_i \in [-1,1]$ for $i = 1, \ldots, r-1$. Thus the intersection of $H$ and the support of $\omega_K$ is the union of the $2^{r-1}$ fundamental parallelepipeds of the lattice $\ell(U_K)$ that meet at the origin. This is a compact set, hence contained in the cube $[-M,M]^r$ for some $M > 0$. Since $\Omega_K(x) = \frac{1}{|M|^r} \omega_K(y) \neq 0$, the component of $y$ lying in $H$ has to belong to this cube. This component is $y - c_r 1$, and $c_r = (y, 1) / \|1\|^2 = (y_1 + \cdots + y_r)/r$, therefore we have (for $i = 1, \ldots, r$):

$$-M \leq y_i - \frac{y_1 + \cdots + y_r}{r} \leq M.$$  

Taking exponentials and recalling that $y_i = \ln |x_i|^{i_i}$ we obtain

$$e^{-M} \leq \frac{|x_i|^{i_i}}{\sqrt{|x_1|^{i_1} \cdots |x_r|^{i_r}}} \leq e^M,$$

which shows that the statement of the lemma holds with $a = e^{-M}$ and $A = e^M$. ■

Example 7 Let $\varepsilon = 1 + \sqrt{2}$ (a fundamental unit of $K = \mathbb{Q}(\sqrt{2})$), then the vectors $\ell(\varepsilon) = (\ln (1 + \sqrt{2}), -\ln (1 + \sqrt{2}))$ and $1 = (1,1)$ form a basis for $\mathbb{R}^2$. The coordinates of $(y_1, y_2)$ in this basis are $c_1 = (y_1 - y_2) / (2 \ln (1 + \sqrt{2}))$ and $c_2 = (y_1 + y_2) / 2$, thus

$$\omega_K(y_1, y_2) = \omega^{(i)}(c_1) = \omega \left( \frac{y_1 - y_2}{2 \ln (1 + \sqrt{2})} \right).$$

Let us see the construction of this function step by step. First we put (the graph of) $\omega$ onto $H$ streched horizontally so that the support is the segment between $\ell(\varepsilon)$ and $-\ell(\varepsilon)$ (the dots next to the origin on the left side of Figure 2-3).
Figure 2-3: Construction of the function $\omega_K (y_1, y_2)$

Figure 2-4: Graph of $\omega_K (y_1, y_2)$
Figure 2-5: Graph of $\Omega_K(x_1, x_2)$

Figure 2-6: Density plot of $\Omega_K(x_1, x_2)$
Now we translate this curve in the direction given by the vector $\mathbf{1} = (1,1)$ (see the right side of Figure 2-3) to get the graph of $\omega_K$ (Figure 2-4). The support of this function is the strip between the lines $y_2 = y_1 + 2 \ln (1 + \sqrt{2})$ and $y_2 = y_1 - 2 \ln (1 + \sqrt{2})$.

To get $\Omega_K$ we compose $\omega_K$ with the function $(y_1, y_2) = (\ln |x_1|, \ln |x_2|)$:

$$\Omega_K(x_1, x_2) = \frac{1}{2} \omega \left( \frac{\ln |x_1| - \ln |x_2|}{2 \ln (1 + \sqrt{2})} \right) (x_1, x_2 \neq 0).$$

Figures 2-5 and 2-6 show the graph and the density plot of this function. We can see that the support of $\Omega_K$ is a union of two cones (bounded by the four lines $x_2 = \pm (1 + \sqrt{2}) \pm ^2 x_1$). It is clear that if $x_1 \neq 0$ and $x_2 = 0$ (or conversely), then there is a neighborhood of $(x_1, x_2)$ where $\Omega_K$ is constant zero (Figure 2-6 shows a such a neighborhood of the point $(3, 0)$). However, the limit at the origin does not exist; for example $\lim_{x \to 0} \Omega_K(x, x) = 1$ and $\lim_{x \to 0} \Omega_K(x, 0) = 0$.

### 2.3 Poisson Summation

The goal of this section is to prove several lemmas that prepare the proof of $E_C \left( S(\mathbb{R}^+) \right) \subseteq S(\mathbb{R}^+)$. For any function $f \in S(\mathbb{R}^+)$ we construct a function $g \in S(\mathbb{R}^n \times \mathbb{C}^2)$ as a composition of $f$ and the 'norm function' on $\mathbb{R}^n \times \mathbb{C}^2$ multiplied with the weight function $\Omega_K$:

$$g(x_1, \ldots, x_r) = f \left( |x_1|^{i_1} \ldots |x_r|^{i_r} \right) \cdot \Omega_K(x_1, \ldots, x_r). \quad (2.9)$$

We need the $\Omega_K$ factor for two reasons. The first one is that $f \left( |x_1|^{i_1} \ldots |x_r|^{i_r} \right)$ does not vanish at infinity. The next example illustrates this, and gives an idea why $g$ vanishes at infinity. After the example we prove that $g$ is indeed a Schwartz function on $\mathbb{R}^n \times \mathbb{C}^2 \cong \mathbb{R}^n$. 

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Example 8 Let us consider the case $K = \mathbb{Q}(\sqrt{2})$ and the function $f \in S(\mathbb{R}^+) \cap C^\infty$ shown on Figure 2.7. Then $g$ is defined on $\mathbb{R}^1 \times \mathbb{R}^2 = \mathbb{R}^2$, and we have $g(x_1, x_2) = f(|x_1| \cdot |x_2|) \cdot \Omega_K(x_1, x_2)$. The function $f(|x_1| \cdot |x_2|)$ is smooth, but does not vanish at infinity, because it is constant on the norm surfaces (hyperbolas) as we can see on Figures 2-8 and 2-9. The factor $\Omega_K(x_1, x_2)$ is zero on the distant parts of these hyperbolas, therefore $g$ vanishes at infinity (see Figures 2-10 and 2-11).

Lemma 9 The function $g$ defined by (2.9) is a Schwartz function on $\mathbb{R}^n$ if $f \in S(\mathbb{R}^+)$. 

Proof. We need to show that $g$ is a smooth function, and all of its partial derivatives (of arbitrary order) vanish rapidly at infinity. The smoothness is clear, because the only singularity of $\Omega_K$ is at the origin, and $f(|x_1|^{\frac{1}{r}} \cdots |x_r|^{\frac{1}{r}})$ vanishes quickly here.
Figure 2-8: Graph of $f(|x_1| \cdot |x_2|)$

Figure 2-9: Density plot of $f(|x_1| \cdot |x_2|)$
Figure 2-10: Graph of $g(x_1, x_2)$

Figure 2-11: Density plot of $g(x_1, x_2)$
Now let $x = (x_1, \ldots, x_r)$ tend to infinity in such a way that it never leaves the support of $\Omega_K$. (We can suppose this, because $g$ is zero outside this set.) Any partial derivative of $g$ is a linear combination of products, where each factor of each product is either of the form $f^{(k)} \left( |x_1|^{i_1} \cdots |x_r|^{i_r} \right)$, or a partial derivative of $|x_1|^{i_1} \cdots |x_r|^{i_r}$ or of $\Omega_K (x_1, \ldots, x_r)$ (here $f^{(k)}$ denotes the $k$-th derivative of $f$). The latter two terms grow at most polynomially (in fact the derivatives of $\Omega_K$ are bounded on any set that excludes a neighborhood of the origin), so it is enough to show that $f^{(k)} \left( |x_1|^{i_1} \cdots |x_r|^{i_r} \right)$ converges to zero rapidly.

Since $x \to \infty$, at least one of its coordinates, say $x_{i_0}$ tends to infinity. Applying Lemma 6 to this coordinate we get $\sqrt{|x_1|^{i_1} \cdots |x_r|^{i_r}} \geq A^{-1} |x_{i_0}|^{i_{i_0}}$, and applying the lemma now to any coordinate we obtain $|x_i|^{i_i} \geq aA^{-1} |x_{i_0}|^{i_{i_0}}$ for $i = 1, \ldots, r$. This estimate together with the fact that $f^{(k)}$ is a Schwartz function on $\mathbb{R}^+$ proves the rapid decay of $f^{(k)} \left( |x_1|^{i_1} \cdots |x_r|^{i_r} \right)$. ■

The other reason for the $\Omega_K$ factor in the definition of $g$ is that it helps to rewrite $E_c f$ with a summation over the lattice $\Gamma = \sigma (\mathcal{B})$ as explained at the beginning of Section 2.2. We make use of the action (2.6) again, but this time we let $\mathcal{B}^*$ act on $\mathbb{R}^r \times \mathbb{C}^2 \cong \mathbb{R}^n$, not just $U_K$. This is the analogue of the action of $\mathbb{N}^*$ on $\mathbb{R}$ in [3].

**Lemma 10** If $x \in \mathbb{R}^+$ and $x = (x_1, \ldots, x_r) \in \mathbb{R}^r \times \mathbb{C}^2$, such that $x = |x_1|^{i_1} \cdots |x_r|^{i_r}$, then $E_c f (x) = \sum_{\alpha \in \mathcal{B}} g (x \cdot \sigma (\alpha))$.

**Proof.** The series on the right side converges absolutely, because $g \in \mathcal{S}(\mathbb{R}^n)$, so we can rearrange it (note that we did not exclude $\alpha = 0$, but this does not matter, since $g (0) = 0$).

Let us split $\mathcal{B}$ into orbits with respect to the action of $U_K$ as we did in (1.3), and first sum on each orbit:

$$\sum_{[\alpha] \in \mathcal{B}/U_K} \sum_{U \in U_K} g (x \cdot \sigma (\alpha \omega)) = \sum_{[\alpha] \in \mathcal{B}/U_K} \sum_{U \in U_K} g (x_1 \sigma_1 (\alpha \omega), \ldots, x_r \sigma_r (\alpha \omega)).$$
Using the definition of $g$ and formula (1.1) for the norm we get
\[
\sum_{[\alpha] \in \mathbb{H}/U_K} \sum_{\varepsilon \in U_K} f \left( |x_1|^\varepsilon \cdots |x_r|^\varepsilon \cdot |N(\alpha\varepsilon)| \right) \cdot \Omega_K \left( x_1 \sigma_1(\alpha\varepsilon), \ldots, x_r \sigma_r(\alpha\varepsilon) \right).
\]

The argument of $f$ is just $x \cdot |N(\alpha)|$, because $\varepsilon$ is a unit, and therefore $|N(\varepsilon)| = 1$. This term does not depend on $\varepsilon$, so we can factor this out from the inner summation:
\[
\sum_{[\alpha] \in \mathbb{H}/U_K} f(x \cdot |N(\alpha)|) \sum_{\varepsilon \in U_K} \Omega_K((x \cdot \sigma(\alpha)) \cdot \sigma(\varepsilon)).
\]

Writing $x \cdot \sigma(\alpha)$ in the place of $x$ in (2.5) we see that the value of the inner sum is 1, and what remains is just $E_c f(x)$. \qed

In the next lemma we show that the rapid decay of $E_c f$ at 0 depends on the the value of the Fourier transform of $g$ at zero.

**Lemma 11** Let $f \in S(\mathbb{R}^r)$ and let $g$ be the function defined by (2.9). Then $E_c f$ vanishes rapidly at zero if and only if $\hat{g}(0) = 0$.

**Proof.** We will proceed the same way as in Lemma 1 of [3]. Let us fix $x > 0$ and $(x_1, \ldots, x_r) \in \mathbb{R}^r \times \mathbb{C}^r$, such that $x = |x_1|^\varepsilon \cdots |x_r|^\varepsilon$. We define a function $G$ on $\mathbb{R}^r \times \mathbb{C}^r$ by
\[
G(u) = G(u_1, \ldots, u_r) = g(u_1 x_1, \ldots, u_r x_r) = G(u \cdot x).
\]

Then $G$ is a Schwartz function on $\mathbb{R}^n$, and the previous lemma means that
\[
E_c f(x) = \sum_{\alpha \in \mathcal{B}} G(\sigma(\alpha)) = \sum_{\gamma \in \Gamma} G(\gamma),
\]
where $\Gamma$ is the lattice $\sigma(\mathcal{B})$.

This is the desired form of $E_c f(x)$. Now we can apply Poisson's Summation Formula:
\[
E_c f(x) = \sum_{\gamma \in \Gamma} G(\gamma) = \frac{1}{V} \sum_{\beta \in \Gamma^1} \hat{G}(\beta).
\] (2.10)
The constant on the right side is the volume of the fundamental parallelepiped of \( \Gamma \); its exact value is \( V = \sqrt{|d_K|}/2^n \), where \( d_K \) is the discriminant of the field \( K \).

Our next task is to compute the Fourier transform of \( G \):

\[
\hat{G}(u) = \int_{\mathbb{R}^n} G(v) e^{2\pi i (v \cdot u)} dv = \int_{\mathbb{R}^n} g(v \cdot x) e^{2\pi i (v \cdot u)} dv
\]

\[
= \int_{\mathbb{R}^n} g(w) e^{2\pi i (w \cdot u/x)} \frac{dw}{|x_1| \cdots |x_r|}
\]

\[
= \int_{\mathbb{R}^n} g(w) e^{2\pi i (w \cdot u/x)} \frac{dw}{x^r} = \frac{1}{x^r} \hat{g}(u/x).
\]

Substituting the above expression for \( \hat{G} \) into (2.10) we get the following formula for \( E_\mathcal{C} f \):

\[
E_\mathcal{C} f(x) = \frac{1}{V} \frac{1}{x} \sum_{\beta \in \Gamma^\perp} \hat{g}(\beta/x) = \frac{1}{V} \frac{1}{x} \hat{g}(0) + \frac{1}{V} \frac{1}{x} \sum_{\beta \in \Gamma^\perp \setminus \{0\}} \hat{g}(\beta/x).
\]

Now let \( x \to 0 \), and let us choose the corresponding \( x \) so that each of its coordinates converges to 0 as well. Then the lattice \( \Gamma^\perp / x = \{ \beta/x : \beta \in \Gamma^\perp \} \) gets sparser and sparser, so the second term on the right hand side vanishes quickly (take into account, that \( \hat{g} \) is a Schwartz function). Therefore the behaviour of \( E_\mathcal{C} f \) at zero depends only on the value of \( \hat{g}(0) \): if \( \hat{g}(0) = 0 \), then \( E_\mathcal{C} f \) vanishes rapidly at the origin, otherwise \( \lim_{x \to 0} E_\mathcal{C} f(x) = \pm \infty \) (and the speed of the divergence is the same as that of \( 1/x \)).

\[\blacksquare\]

**Example 12** The curve approaching the origin on Figure 2-12 is the graph of \( E f(x) = \sum_{n=1}^{\infty} f(nx) \) (this is the operator \( E \) corresponding to the case \( K = \mathbb{Q} \)) for the function \( f \) considered in Example 8 (the integral of this function is zero). The other curve is the graph of \( E f_1 \), where \( f_1 \) is obtained from \( f \) by a very small perturbation that makes its integral nonzero. We can see that \( E f_1 \) behaves like \( 1/x \) around the origin.
2.4 Computation of $\widehat{g}(0)$

In this section we finally prove the result promised in Section 2.1.

**Theorem 13** If $f \in S(\mathbb{R}^+)_{0}$, then $Ef \in S(\mathbb{R}^+)_{0}$.

**Proof.** We have already seen that if $f \in S(\mathbb{R}^+)_{0}$, then $Ef$ is a smooth function, and all of its derivatives vanish rapidly at infinity, so we only need to show that the derivatives of $Ef$ vanish quickly at zero as well. If $f$ belongs to $S(\mathbb{R}^+)_{0}$, then so does $x^k f^{(k)}$ (we can check this integrating by parts $k$ times), therefore (2.2) shows that it is enough to prove that $Ef$ vanishes rapidly for every $f \in S(\mathbb{R}^+)_{0}$.

Let us take an arbitrary function $f \in S(\mathbb{R}^+)_{0}$, and consider the function $g$ given by (2.9). We need to prove that $E_C f$ vanishes rapidly at zero for every ideal class $C$. According to Lemma 11 we just have to show that $\widehat{g}(0) = 0$, so let us compute the
Here \( dx \) indicates the Lebesgue measure on \( \mathbb{R}^n \) and \( dx_i \) the Lebesgue measure on \( \mathbb{R} \) (for \( i = 1, \ldots, r_1 \)) or on \( \mathbb{R}^2 \) (for \( i = r_1 + 1, \ldots, r \)). Recalling the definitions of \( g \) and \( \Omega_K \), one can see that the value of \( g(x_1, \ldots, x_r) \) depends only on the absolute values of the \( x_i \)'s. Therefore, instead of (2.11) it suffices to compute the following integral (its value is \( \tilde{g}(0)/2^r \)):

\[
\int_{r_1} \cdots \int_{r_1} g(x_1, \ldots, x_r) \, dx_1 \cdots dx_r. \tag{2.12}
\]

Now we change to polar coordinates on each of the \( r_2 \) planes: let \( \rho_i = |x_i| \) and \( \theta_i = \text{arg} \, x_i \) \((i = r_1 + 1, \ldots, r)\). We will use \( \rho_i \) instead of \( x_i \) for the real coordinates too (\( \rho_i = |x_i| \) holds, because \( x_i > 0 \) for \( i = 1, \ldots, r_1 \)).

Then (2.12) takes the following form:

\[
\int_{r_2} \cdots \int_{r_2} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} f(\rho_1^i, \ldots, \rho_r^i) \Omega_K(x_1, \ldots, x_r) \rho_{r_1+1} \cdots \rho_r \rho_1 \cdots \rho_r d\theta_{r_1+1} \cdots d\theta_r. \tag{2.13}
\]

By definition, \( \Omega_K(x_1, \ldots, x_r) = \frac{1}{|W|} \omega_K (\ln \rho_1^i, \ldots, \ln \rho_r^i) \), thus the integrand does not depend on the \( \theta_i \), so we have just a constant (exactly: \( (2\pi)^r / |W| \)) times the integral

\[
\int_{r_2} \cdots \int_{r_2} f(\rho_1^i, \ldots, \rho_r^i) \omega_K (\ln \rho_1^i, \ldots, \ln \rho_r^i) \rho_{r_1+1} \cdots \rho_r \rho_1 \cdots \rho_r. \tag{2.13}
\]
Now let us recall the definition of $\omega_K$. First we have to express the vector $a = (\ln \rho_1^i, \ldots, \ln \rho_r^i)$ in some basis of $\mathbb{R}^r$ (namely $\ell(\varepsilon_1), \ldots, \ell(\varepsilon_{r-1}), \mathbf{1}$ given by a fundamental system of units), and then we have to apply $\omega^{r-1}$ to the component lying in the hyperplane $H$ (i.e., perpendicular to $\mathbf{1}$). Since this basis is not given explicitly (in fact, it is not an easy task to find a fundamental system of units), let us take a concrete one instead:

$$
\begin{align*}
\mathbf{b}_1 &= (r-1, -1, -1, \cdots, -1, -1, -1), \\
\mathbf{b}_2 &= (0, r-2, -1, \cdots, -1, -1, -1), \\
\vdots \\
\mathbf{b}_{r-2} &= (0, 0, 0, \cdots, 2, -1, -1, -1), \\
\mathbf{b}_{r-1} &= (0, 0, 0, \cdots, 0, 1, -1, -1), \\
\mathbf{b}_r &= (1, 1, 1, \cdots, 1, 1, 1).
\end{align*}
$$

This is an orthogonal basis for $\mathbb{R}^r$, and the first $r-1$ vectors form a basis of $H$ (the last vector is the same vector $\mathbf{1}$ that we used before). The coordinates of $a$ in this basis are $(a, \mathbf{b}_i) / \|\mathbf{b}_i\|^2$. Let us introduce these as new variables (we drop the constants $1/\|\mathbf{b}_i\|^2$):

$$
\begin{align*}
u_1 &= (r-1) \ln \rho_1^i - \ln \rho_2^i - \cdots - \ln \rho_{r-1}^i - \ln \rho_r^i, \\
u_2 &= (r-2) \ln \rho_2^i - \ln \rho_3^i - \cdots - \ln \rho_{r-1}^i - \ln \rho_r^i, \\
\vdots \\
u_{r-1} &= \ln \rho_{r-1}^i - \ln \rho_r^i, \\
u_r &= \ln \rho_1^i + \ln \rho_2^i + \cdots + \ln \rho_{r-1}^i + \ln \rho_r^i.
\end{align*}
$$
The Jacobian of this change of variables is \( \frac{\rho_1 \cdots \rho_r}{t_1 \cdots t_r} = \frac{1}{2^{r^2} r!} \rho_1 \cdots \rho_r \), so our integral (2.13) becomes

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(e^{\omega t}) \omega(t+1) (\cdots) \rho_{t_1+1} \cdots \rho_{t_r} \frac{1}{2^{r^2} \cdot r!} \rho_1 \cdots \rho_r du_1 \cdots du_r
\]

\[
= \frac{1}{2^{r^2} \cdot r!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(e^{\omega t}) \omega(t+1) (\cdots) \rho_1 \cdots \rho_r du_1 \cdots du_r
\]

\[
= \frac{1}{2^{r^2} \cdot r!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(e^{\omega t}) e^{\omega u} du_1 \cdots du_r.
\]

By the definition of \( \omega_K \), the arguments of \( \omega(t+1) \) depend only on the component of \( t \) lying in \( H \), and this is expressible in terms of \( u_1, \ldots, u_{r-1} \). Therefore, this part of the integrand does not depend on \( u_r \), while the rest depends only on this variable.

Thus we can split the integral into the product of these two integrals:

\[
\frac{1}{2^{r^2} \cdot r!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \omega(t+1) (\cdots) du_1 \cdots du_{r-1} \cdot \int_{-\infty}^{\infty} f(e^{\omega t}) e^{\omega u} du_r.
\]

Changing variables once more again \( t = e^{\omega u} \), the second part transforms to \( \int_{0}^{\infty} f(t) dt \) which is 0, because \( f \) belongs to \( \mathcal{S}(\mathbb{R}^+) \). This completes the proof of Theorem 13. \( \blacksquare \)
Chapter 3

The Operator

In this chapter we define an unbounded linear operator $D$ on the Pólya-Hilbert space constructed in the previous chapter, such that the point spectrum of $D$ is the set of critical zeros of the function $\zeta_K$.

3.1 Definition of $D$

We will consider the same operator $D$ on $L^2(\mathbb{R}^+, \mu(x) dx)$ as in [3], and we will restrict it to the closed subspace $\mathcal{H}$ defined in Section 2.1. It is only this subspace whose definition involves the field $K$, and it is defined in such a way that the eigenvalues of $D|_{\mathcal{H}}$ are exactly the critical zeros of $\zeta_K$. Let us mention that it is possible to define $E$ and the corresponding Hilbert space $\mathcal{H}$ for any Dirichlet series $\sum a_n$, not just zeta functions of fields. The validity of the results in this section depend only on Theorem 13. We used very heavily in the proof of this theorem that the coefficients $a_n$ came from an algebraic number field, but the analytic continuation and the spectral interpretation of critical zeros given in Theorem 17 remain valid in the general case, provided $E(S(\mathbb{R}^+)_0) \subseteq S(\mathbb{R}^+)$ holds.

Following the philosophy outlined in [3], we define the operators $V_t$ on the Hilbert space $L^2(\mathbb{R}^+, \mu(x) dx)$ by $V_t f(x) = f(tx)$ in order to study the action of the multiplicative semigroup $\mathbb{N}^*$ on the real line.
Lemma 14 The family \( \{V_t : t \geq 1\} \) is a strongly continuous one-parameter semi-
group of bounded operators on \( L^2(\mathbb{R}^+, \mu(x) \, dx) \).

**Proof.** The identity \( V_t V_s = V_{ts} \) shows that we have indeed a one-parameter semi-
group. (Actually we should consider \( \{V_\varepsilon : t \geq 0\} \), because in the usual definition of a
semigroup of operators the parameter semigroup is additive: \( V_\varepsilon V_\kappa = V_{\varepsilon + \kappa} \). We will
omit this exponential reparametrization for convenience. The only difference com-
pared to the additive case is that we will have to consider limits as \( t \to 1 \) instead
of \( t \to 0 \) in the definition of the infinitesimal generator.) The boundedness of these
operators follows from the following estimate.

\[
\|V_t f\|_\mu^2 = \int_0^\infty |f(tx)|^2 \mu(x) \, dx = \int_0^\infty |f(x)|^2 \mu\left(\frac{x}{t}\right) \, dx \\
\leq \int_0^\infty |f(x)|^2 (1 + t^2) \mu(x) \, dx = (1 + t^2) \|f\|_\mu^2
\]

Strong continuity of the semigroup means that the map \( t \mapsto V_t \) is continuous
in the strong operator topology. Clearly it is enough to check the continuity at
\( t = 1 \), and since \( \|V_t\| \leq \sqrt{1 + t^2} \) is bounded in a neighborhood of 1, it suffices to
prove \( \lim_{t \to 1} V_t f = f \) for \( f \) in a dense subset of \( L^2(\mathbb{R}^+, \mu(x) \, dx) \) (cf. Proposition
5.3 in [2]). For example, we can suppose that \( f \) is a compactly supported continuous
function, whose support does not contain 0. Then \( f \) is uniformly continuous, therefore
the integrand of \( \|V_t f - f\|_\mu^2 = \int_0^\infty |f(tx) - f(x)|^2 \mu(x) \, dx \) can be made arbitrarily
small, if \( t \) is sufficiently close to 1, and the interval of integration can be replaced with
a finite interval bounded away from 0, so the integral converges to 0 as \( t \to 1 \). □

Now let us define the bounded operators \( D_\varepsilon = \frac{V_{1+\varepsilon} - V_1}{\varepsilon} \), and let \( D \) be the strong
limit of \( D_\varepsilon \) as \( \varepsilon \to 0 \):

\[
D = \lim_{\varepsilon \to 0} \frac{V_{1+\varepsilon} - V_1}{\varepsilon}.
\]
Then $D$ is called the \textit{infinitesimal generator} of the semigroup $\{V_t : t \geq 1\}$, and it is a densely defined closed unbounded operator on $L^2(\mathbb{R}^+ , \mu (x) \, d^x x)$. For $f$ in the domain of $D$ we have $Df (x) = \lim_{\epsilon \to 0} \frac{f((1+\epsilon)x) - f(x)}{\epsilon}$, where the limit is taken with respect to the norm $\| \cdot \|_{\mu}$. If we considered it as a pointwise limit, then we would have $Df (x) = xf'(x)$. In the next lemma we will show, that this is actually true, because every function in the domain of $D$ is absolutely continuous (hence differentiable almost everywhere).

**Lemma 15** If $f$ is in the domain of $D$, then $f$ is absolutely continuous, and $Df (x) = xf'(x)$ holds outside a set of measure zero.

**Proof.** Let us fix arbitrary numbers $0 < a < b$, and consider the limit

$$\lim_{\epsilon \to 0} \int_a^b D\epsilon f (x) \mu (x) \, d^x x. \tag{3.1}$$

This limit is $\int_a^b Df (x) \mu (x) \, d^x x$; to prove this we make use of the Cauchy-Schwartz inequality in the Hilbert space $L^2([a,b], \mu (x) \, d^x x)$:

$$\left| \int_a^b (D\epsilon f (x) - Df (x)) \mu (x) \, d^x x \right| \leq \sqrt{\int_a^b (D\epsilon f (x) - Df (x))^2 \mu (x) \, d^x x} \cdot \sqrt{\int_a^b \mu (x) \, d^x x}.$$ 

The right hand side is dominated by a constant times $\|D\epsilon f - Df\|_{\mu}$, so it converges to zero according to the definition of $Df$.

Now let us compute (3.1) in a different way:

$$\int_a^b \frac{f((1+\epsilon)x) - f(x)}{\epsilon} \mu (x) \, d^x x = \int_a^b \frac{1}{\epsilon} \frac{f(x) \mu \left( \frac{x}{1+\epsilon} \right) \, d^x x - \frac{1}{\epsilon} \int_a^b f(x) \mu (x) \, d^x x}{\frac{b+a}{b+\epsilon}}.$$ 

$$= \int_a^b f(x) \frac{\mu \left( \frac{x}{1+\epsilon} \right) - \mu (x)}{\epsilon x} \, dx + \frac{1}{\epsilon} \int_a^b f(x) \left( \mu \left( \frac{x}{1+\epsilon} \right) - \mu (x) \right) \, d^x x - \frac{1}{\epsilon} \int_a^b f(x) \left( \mu \left( \frac{x}{1+\epsilon} \right) - \mu (x) \right) \, d^x x.$$ 

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The integrand of the first term is dominated by a constant multiple of \( f(x)(-\mu'(x)) \), therefore by the Lebesgue Dominated Convergence Theorem, its limit is \(-\int_a^b f(x)\mu'(x)\,dx\). The limit of the second term is \( b \cdot F'(b) \), where \( F(x) = \int_1^x f(x)\mu(x)\frac{1}{x}\,dx \). This function is differentiable almost everywhere, and \( F'(x) = f(x)\mu(x)\frac{1}{x} \), therefore \( b \cdot F'(b) = f(b)\mu(b) \) for almost every \( b \). Similarly, the third term converges to \( f(a)\mu(a) \) for almost every \( a \). The last two terms tend to zero, because \( \mu \) is uniformly continuous.

Comparing the two expressions that we have obtained for (3.1) we get for almost all \( a, b \in \mathbb{R}^+ \)

\[
-\int_a^b f(x)\mu'(x)\,dx + f(b)\mu(b) - f(a)\mu(a) = \int_a^b Df(x)\mu(x)\frac{dx}{x}.
\]

Let us fix an \( a \) for which this holds, then we have for almost all \( b \)

\[
f(b)\mu(b) = \int_a^b \left( f(x)\mu'(x) + Df(x)\mu(x)\frac{1}{x} \right)\,dx + f(a)\mu(a),
\]

which means that \( f(b) \) is absolutely continuous (or can be made absolutely continuous by redefining on a set of measure zero). Differentiating with respect to \( b \), we get

\[
f'(b)\mu(b) + f(b)\mu'(b) = f(b)\mu'(b) + Df(b)\mu(b)\frac{1}{b},
\]

therefore \( bf'(b) = Df(b) \).

In the next lemma we prove that \( \mathcal{H} \) is an invariant subspace for each of the operators \( V_t \), thus \( \{V_t|_{\mathcal{H}}: t \geq 1 \} \) is a strongly continuous one-parameter semigroup of operators on this Hilbert space, and its infinitesimal generator, \( D|_{\mathcal{H}} \), is a densely defined closed unbounded operator on \( \mathcal{H} \). The statement of the previous lemma certainly remains valid for this operator.
Lemma 16 For every \( t \geq 1 \), \( V_t \) preserves the subspace \( \mathcal{H} \) of \( L^2(\mathbb{R}^+, \mu(x)\,d^*x) \).

Proof. Let \( h \in \mathcal{H} \), and \( f \in S(\mathbb{R}^+) \). We want to show that \( \langle V_t h, \Phi f \rangle_\mu = 0 \). It is clear that all the \( V_t \)'s preserve \( S(\mathbb{R}^+) \), and they commute with \( E \) on this space. This allows us to compute the following way:

\[
\langle V_t h, \Phi f \rangle_\mu = \int_0^\infty V_t h(x) (1 + x^2) \frac{E f(x)\mu(x)}{x} \,d^*x = \int_0^\infty h(tx) \frac{E f(x)\mu(x)}{x} \,d^*x
\]
\[
= \int_0^\infty h(y) \frac{E f(t^{-1}y)}{y} \,d^*y = \int_0^\infty h(y) \frac{E (V_{t^{-1}} f)}{y} \,d^*y
\]
\[
= \int_0^\infty h(y) (1 + y^2) \frac{E (V_{t^{-1}} f)}{y} \,d^*y = \langle h, \Phi (V_{t^{-1}} f) \rangle_\mu.
\]

Since \( h \) belongs to \( \mathcal{H} \), it is orthogonal to \( \Phi (V_{t^{-1}} f) \), because \( V_{t^{-1}} f \) is in \( S(\mathbb{R}^+) \). Thus \( \langle V_t h, \Phi f \rangle_\mu = \langle h, \Phi (V_{t^{-1}} f) \rangle_\mu = 0 \). \( \blacksquare \)

3.2 The Spectrum

It is easy to check that the solutions of the differential equation \( x f'(x) = \lambda f(x) \) are the functions of the form \( cx^\lambda \). Such a function belongs to \( L^2(\mathbb{R}^+, \mu(x)\,d^*x) \) iff \( 0 < \text{Re}(\lambda) < 1 \) (this is the reason for the weight \( \mu(x) \)). Thus the set of eigenvalues of \( D \) is exactly the critical strip. The subspace \( \mathcal{H} \) was constructed in such a way that the restriction of \( D \) to \( \mathcal{H} \) has only the zeros of \( \zeta_K \) from the critical strip as eigenvalues.

Theorem 17 The Dirichlet series (1.2) has an analytic continuation to the complex plane with a simple pole at \( s = 1 \) as the only singularity, and the point spectrum of \( D|_\mathcal{H} \) is the set of critical zeros of \( \zeta_K \).

Proof. Let us fix a \( \lambda \) in the critical strip, and a function \( f \in S(\mathbb{R}^+) \). We will compute the inner product \( \langle x^\lambda, \Phi f(x) \rangle_\mu \) to see if \( x^\lambda \) belongs to \( \mathcal{H} \) or not.
First let us calculate \( \langle x^s, \Phi f(x) \rangle_\mu \) for \( \text{Re}(s) > 1 \). Though \( x^s \) is not square integrable with respect to \( \mu(x)dz \), the integral that defines this inner product is absolutely convergent, because \( \Phi f \) is a Schwartz function.

\[
\langle x^s, \Phi f(x) \rangle_\mu = \int_0^\infty x^s (1 + x^2) \overline{E f(x)} \mu(x) dz = \int_0^\infty x^s \sum_{n=1}^{\infty} a_n \overline{\mathcal{F}(nx)} dz
\]

\[
= \sum_{n=1}^{\infty} \int_0^\infty x^s a_n \overline{\mathcal{F}(nx)} dz = \sum_{n=1}^{\infty} \int_0^\infty \left( \frac{y}{n} \right)^s \overline{f(y)} dy
\]

\[
= \sum_{n=1}^{\infty} a_n \int_0^\infty y^s \overline{f(y)} dy = \zeta_K(s) \cdot \int_0^\infty y^s \overline{f(y)} \frac{dy}{y}
\]

Thus we have

\[
\langle x^s, \Phi f(x) \rangle_\mu = \zeta_K(s) \cdot \langle x^{s-1}, f(x) \rangle,
\]

where the second inner product is taken with respect to Lebesgue measure. The left hand side of this equality defines an entire function of \( s \). The right hand side is also analytic (and equal to the left hand side) in the half-plane \( \text{Re}(s) > 1 \). Thus, we can use the left side as the analytic continuation of the right side. The factor \( \langle x^{s-1}, f(x) \rangle \) is analytic on the whole complex plane, so the analytic continuation of \( \zeta_K \) can be given by

\[
\zeta_K(s) = \frac{\langle x^s, \Phi f(x) \rangle_\mu}{\langle x^{s-1}, f(x) \rangle} = \frac{\langle x^{s-1}, Ef(x) \rangle_\mu}{\langle x^{s-1}, f(x) \rangle}.
\]

Note that in the case \( s = 1 \) the denominator is just the integral of \( f \), which is zero for every \( f \in \mathcal{S}(\mathbb{R}^+) \), showing that \( \zeta_K \) has a pole at \( s = 1 \), and this is the only singularity, because we can choose \( f \) so that \( \langle x^{s-1}, f(x) \rangle = 0 \) iff \( s = 1 \). In order to prove that this pole is simple, we need to show that the limit

\[
\lim_{s \to 1} (s - 1) \cdot \frac{\langle x^{s-1}, Ef(x) \rangle_\mu}{\langle x^{s-1}, f(x) \rangle} = \lim_{s \to 1} \frac{\langle x^{s-1}, Ef(x) \rangle}{\langle x^{s-1}, f(x) \rangle}
\]

is finite, and for this it suffices to prove that the limit of the denominator is nonzero.
Since the integral of $f$ is 0, we can rewrite the denominator the following way:

$$\left\langle \frac{x^{s-1}}{s-1}, f(x) \right\rangle = \int_0^\infty \frac{x^{s-1}}{s-1} f(x) dx = \int_0^\infty \frac{x^{s-1} - 1}{s-1} f(x) dx.$$ 

If we choose $f$ such that its support is a finite interval $[a, b]$ with $a > 0$, then the integrand is bounded, and the interval of integration can be replaced with $[a, b]$, hence we can change the order of the limit and the integration by the Lebesgue Dominated Convergence Theorem. The limit of the integrand is $\ln(x) f(x)$, so the value of the limit (3.3) is $\int_a^b \ln(x) f(x) dx$, which can be made nonzero by choosing a suitable function $f$.

To finish the proof, let us suppose first that $\lambda$ is an eigenvalue of $D|_H$, that is, $x^\lambda$ belongs to $H$. Then $\langle x^\lambda, \Phi f(x) \rangle_\mu = 0$ for all $f \in S(\mathbb{R}^+)_0$, and choosing one for which $\langle x^{\lambda-1}, f(x) \rangle \neq 0$, we can conclude $\zeta_K(s) = 0$ by (3.2).

Conversely, suppose that $\zeta_K(s) = 0$, and let $f \in S(\mathbb{R}^+)_0$ be an arbitrary function. Using (3.2) again, we have $\langle x^\lambda, \Phi f(x) \rangle_\mu = \zeta_K(s) \cdot \langle x^{\lambda-1}, f (x) \rangle = 0$, so $x^\lambda \in H$, and therefore $\lambda$ is an eigenvalue of $D|_H$. □

**Remark 18** It might seem that we have obtained a new proof for the analytic continuation of zeta functions of algebraic number fields, e.g. the Riemann zeta function. Let us apply (3.2) with $f(x) = e^{-\pi x^2}$ (which is not in $S(\mathbb{R}^+)_0$). The right hand side is

$$\zeta(s) \cdot \int_0^\infty x^{s-1} e^{-\pi x^2} dx = \zeta(s) \cdot \frac{1}{2} \pi^{-s/2} \Gamma \left( \frac{s}{2} \right).$$

(3.4)

This is the Riemann zeta function completed with the gamma factor.

For $E f(x)$ we have $\sum_{n=1}^\infty e^{-\pi n^2 x^2} = \psi(x^2)$, where $\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$. The left hand side of (3.2) becomes (introducing $y = x^2$ as a new variable)

$$\int_0^\infty x^s \psi(x^2) \frac{dx}{x} = \frac{1}{2} \int_0^\infty y^{s/2-1} \psi(y) dy.$$ (3.5)
Comparing (3.4) and (3.5) we obtain

$$\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \int_0^\infty x^{s/2-1} \psi(x) \, dx,$$

which is the first step in one of the proofs of the functional equation and the analytic continuation of the Riemann zeta function.
Bibliography


