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Multi-axial anisotropic material behavior using a collection of yield surfaces to model the stress-space coupling phenomena

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Multi-axial anisotropic material behavior using a collection of yield surfaces to model the stress-space coupling phenomena

Abstract
In the 1989 Loma Prieta earthquake in the San Francisco-Bay area and also the 1994 Northridge earthquake in Southern California, several buildings experienced significant damage because of inadequate member design. The existing codes of practice used for the structural design approximate the effects of nonlinear material behavior. This work formulates a nonlinear anisotropic material model for systems subjected to biaxial loading conditions such that the true nature of material degradation can be identified. This would aide structural designers in being able to predict accurate cyclic deformations under large earthquake events. The proposed model investigates anisotropic material behavior under bi-axial loading in principal stress space from a snapshot perspective using various material axes of anisotropy. Since the principal axes of stress are assumed to coincide with material axis of anisotropy, the shear stresses are zero on that given plane for the analysis. However, their effect is modeled so that, at each strain increment, a different set of principal axes is considered. Two experimentally verified uniaxial stress functions are coupled in this manner and used to describe the material anisotropy. The model is developed using a distortional energy approach and von Mises type of yielding surface, which is consistent with the snapshot assumption used for each set of material axes.

Keywords
Engineering, Civil
MULTI-AXIAL ANISOTROPIC MATERIAL BEHAVIOR USING A COLLECTION OF YIELD SURFACES TO MODEL THE STRESS-SPACE COUPLING PHENOMENA

BY

MIRJANA MARUSIC
BS, University Of Novi Sad, Serbia, 2004

THESIS

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ABSTRACT

MULTI-AXIAL ANISOTROPIC MATERIAL BEHAVIOR USING A COLLECTION OF YIELD SURFACES TO MODEL THE STRESS - SPACE COUPLING PHENOMENA

by

Mirjana Marusic

University of New Hampshire, September 2006

In the 1989 Loma Prieta earthquake in the San Francisco-bay area and also the 1994 Northridge earthquake in Southern California, several buildings experienced significant damage because of inadequate member design. The existing codes of practice used for the structural design approximate the effects of nonlinear material behavior. This work formulates a nonlinear anisotropic material model for systems subjected to biaxial loading conditions such that the true nature of material degradation can be identified. This would aide structural designers in being able to predict accurate cyclic deformations under large earthquake events. The proposed model investigates anisotropic material behavior under bi-axial loading in principal stress space from a snapshot perspective using various material axes of anisotropy. Since the principal axes of stress are assumed to coincide with material axis of anisotropy, the shear stresses are zero on that given plane for the analysis. However, their effect is modeled so that, at each strain increment, a different set of principal axes is considered. Two experimentally verified uniaxial stress functions are coupled in this manner and used to describe the material behavior.
anisotropy. The model is developed using a distortional energy approach and von Mises type of yielding surface, which is consistent with the snapshot assumption used for each set of material axes.
INTRODUCTION

Structural design has until recently been formulated mostly on the fundamentals of linear elastic theory. In these instances, building codes often accounted for inelastic behavior using various approaches including the application of reduction factors [1] so as to account for the ductility capability that various materials could provide during inelastic stages. On the theoretical elastic level, structural members were analyzed on the premise that its members could simply attain a maximum stress equal to the yielding stress in the material; factors were then used to account for the ductility and consequential inelastic demands. In acknowledging a member’s ductility capability, a material was actually able to reach its failing juncture well beyond the yielding stress where a large amount of strain hardening would occur. As such, these concepts of plasticity enabled engineers to design structures more economically by allowing for a greater material reserve through codified techniques ([1], [2], [3]). Ultimately, the application of plastic theory in design enables the carrying-load capacity of structures to be increased without causing total collapse.

The 1989 Loma Prieta earthquake in the San Francisco-bay area and also the 1994 Northridge earthquake in Southern California indicated that much of the incurred structural damage resulted from deficient design procedures and in particular targeted how the structural material had been assumed to behave. In fact, several buildings in the two earthquakes experienced significant damage because many connection elements exhibited an unexpected brittle fracture. This was only one instance of inadequate member design.
The prevailing thought is that there is still much work that needs to be done with respect to accurately quantifying nonlinear material behavior. In the case of the brittle failures, designers had assumed that adequate yielding of the material would take place. As such, a better understanding and a more accurate application of the nonlinear material mechanics at the fundamental level would enable the actual ductility level to be reached in the likelihood of a large seismic event. Thus, while existing specifications, such as those outlined in the Load and Resistance Factor Design (LRFD), conservatively approximate the effects of nonlinear material behavior, an objective of this research is to more accurately identify the true nature of material degradation. In doing so, this would ultimately aide structural designers in being able to predict accurate cyclic deformations under large earthquake events.

The general material model that accounts for the post-yield activity along the member length assuming anisotropic conditions is desired because of the complex nature of how today’s structures are loaded and analyzed using many of the available advanced computational tools and techniques. In particular, the current effort attempts to formulate a nonlinear anisotropic material model for systems subjected to biaxial loading conditions (although the three-dimensional case is considered in the accurate formulation of such a model).

The proposed material model is developed using the theory of yield surfaces, described in detail in Chapter 2, *Theoretical Formulation*. The chapter explains the importance of separating the total stress into the hydrostatic and deviatoric stresses in plasticity. The concept of a yield surface, its meaning and the ability to describe the material’s behavior once yielding has occurred is also presented.
Examples of basic uniaxial stress – strain curves used in practice are introduced in Chapter 3, *Uniaxial Stress-Strain Models*. Two nonlinear functions that accurately model uniaxial material behavior are presented and used in this work. Their coupling in order to capture material anisotropy sets a foundation for the remaining chapters.

Chapter 4, *Anisotropic Material Behavior*, is devoted to the previous work on the yielding criterion for anisotropic materials and the coupling phenomenon of two uniaxial stress-strain functions. The anisotropic material behavior under bi-axial loading in principal stress space from a snapshot perspective is presented.

Using the distortional strain energy density, $U_d$, the individual snapshot yield surfaces at each discretized plastic strain level is developed. The collection of all snapshots of yield surfaces models the anisotropic behavior in materials. Chapter 5, *Yield Surfaces for the Proposed Model*, explains how distortional strain energy density is used in this model. For each level of strain after yielding, the strain energy is separated into its “flow” (constant stress) and “hardening” (increasing stress) portions. The final form of the yield locus (ellipse) is obtained (again for each strain level). It is shown that the yield locus will both translate and expand in the stress plane during continuous straining, which proves that the strain hardening is a combination of both kinematic and isotropic hardening. A comprehensive computer program code, written in MATLAB, is developed for the entire formulation of this model.

The necessary adjustments for the anisotropic behavior are presented in Chapter 6. Since the obtained values of the stresses after plotting the yield loci for each snapshot did not match up with the uniaxial stresses for each strain increment, each ellipse was translated and rotated. This way, anisotropic behavior includes a rotational component in

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the plotting of the yield loci ([9], [10]). Each snapshot as well as the collection of all
snapshots of all yield surfaces is then plotted in stress space.

Chapter 7, *Backstress Evolutionary Function*, introduces the backstress function
from both the microscopic and macroscopic point of view. Its ability to determine the
stress-strain function of a material is presented. In the work of Armstrong and Frederick
[11], Chaboche [12], Voyiadjis and Sivakumar [13], the backstress evolution is predicted
by relations that expresses the back stress rate, $\dot{\alpha}$, in terms of the plastic strain rate, $\dot{\varepsilon}^p$,
accumulated plastic strain rate, $\dot{\varepsilon}$ (ref.), and/ or the stress rate, $\dot{\sigma}$. In this work, however,
the backstress function for each uniaxial snapshot is derived from the distortional energy
approach and obtained directly from the plots of the centers of the yield surfaces.

A part of this work was presented at the Ninth Pan-American Congress of
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Volume 11*, PACAM IX, Merida, Mexico].
CHAPTER 1

LIMIT DESIGN

In structures that are statically indeterminate, a reserve of strength exists in the material between the stages of when initial yielding takes place and when collapse of the structure occurs. In many current practices, limit design quantifies the nonlinear material behavior in members by using a perfectly-plastic stress distribution as shown in Figure 1.1.

In this case, the plastic moment is computed as the product of the yield stress, \( \sigma_{\text{yield}} \), and the plastic modulus, \( Z \). As such, the distribution of stresses through the thickness of the cross section in the yielded region is considered constant. The plastic moment, \( M_p \), is then calculated as:

\[
M_p = T \frac{d}{2} = C \frac{d}{2} = (\sigma_{\text{yield}} \frac{bd}{2}) \frac{d}{2} = \sigma_{\text{yield}} \frac{bd^2}{4} = \sigma_{\text{yield}} Z
\]

Equation 1.1

In Equation 1.1, \( T \) and \( C \) are the internal tension and compression forces, respectively, and \( b \) and \( d \) are the dimensions of a rectangular cross-section.
In limit design, the redistribution of forces, which occurs when this plastic moment capacity is achieved, enables the structure to be re-analyzed at this current damaged state, wherein the critical section in question develops a full plastic hinge. If the load is further increased, an additional hinge will eventually form (unless the first hinge reverses, which is also a possibility). In this manner, the static determinacy of the structure keeps changing until a mechanism occurs where hinges form at a sufficient number of various locations. While the immediate effort of this research is to closely examine the mechanics with which plasticity develops, it is of great interest to underscore the overall applicability of such a detailed analysis in practice. As such, a typical example of a plastic collapse mechanism in the grand limit design scenario is illustrated using the fixed-fixed boundary condition beam shown in Figure 1.2. The uniform distributed load, \( w \), will first produce two hinges at the beam’s supports, and after the load is increased and a redistribution of forces occurs, a third plastic hinge will also develop at the center of the beam (ideally) thereby causing the collapsed state.

Figure 1.1 Distribution of the stresses through the thickness of the cross-section.
Using the method of virtual work, the magnitude of the nominal plastic moment can easily be calculated when sufficient number of hinges form to cause a collapse mechanism. In this manner a perfectly-plastic stress distribution is assumed, and the work due to the applied external loads over a small displacement, which occurs after the ultimate load is reached, is equated to the internal work that is absorbed by the hinges (Figure 1.3): 

\[ M_n(\theta + 2\theta + \theta) = wL(\frac{1}{2} \theta \frac{L}{2}) \Rightarrow M_n = \frac{wL^2}{16} \]

Equation 1.2
As another simple example, the pin-supported frame shown in Figure 1.4 (a) is statically indeterminate to the first degree. If the lateral load is increased to a yield level, the first plastic hinge will develop, and the frame will become statically determinate. If the load is then further increased, an additional plastic hinge forms, and a collapse mechanism will form. A third example is illustrated in Figure 1.4 (b) using a gable frame.
The development of the internal stresses through the cross-section is assumed to remain linear up until yielding is reached. In fact, the stresses are assumed to remain constant at this yield level ($\sigma_y$) even thereafter. However, in reality, the distribution after yielding has occurred is rather nonlinear especially during cyclic loading where the system responds with stages of unloading/reloading/reyielding during strain hardening. In a very precise manner, the gradual spread of plasticity through the cross section and along the member length should be considered in order to gain a better perspective of the development of a hinged point. In recent years, there has been a strong effort to accurately model nonlinear material behavior. Abbasania and Kassimali [4] use an idealized elastic-plastic material to model localized hinging. The modeling of ductile materials is also investigated using zero-length plastification models by Kim and Lee [5], which do not consider the spread of plasticity along the member's length.
CHAPTER 2

THEORETICAL FORMULATION

Hydrostatic and Deviatoric Stresses in Plasticity

In developing a general anisotropic model through plasticity theory, the first step is to separate the stress tensor components into their fundamental components: the hydrostatic and the deviatoric stresses. Experimental evidence has indicated that yielding is generally insensitive to hydrostatic (mean) pressure for certain materials. For materials composed of a metallic crystalline structure, only the deviatoric stress components are used to develop the theoretical flow of plasticity once systems begin to yield. The analysis of such systems in this manner presumes that these particular stress components are responsible for the change in the shape of materials. More specifically, the influence of the deviatoric stresses is manifested at the atomic level where the shearing of the atomic alignment results in an 'out of place' movement and structural re-alignment as shown in Figure 2.1. This crystalline slip, which is a typical characteristic for metals, requires a breaking of the inter-atomic bonds and a re-aligning of atoms. This process indicates that plastic slip is a shearing process which does not lead to the volume change. The difference at the elastic level of deformation, of course, is that only a non-permanent stretching of the bonds takes place where the system re-aligns itself in its original form.
Figure 2.1 Schematic representation of the slip in polycrystals.

However, hydrostatic stress does have a significant influence on the yielding criterion in nonmetal structures, such as rocks and soils, and cannot be neglected in those cases. A porous material may undergo plastic deformation under compression since the pores reduce in size. Thus, the volume changes and plastic deformation become dependent on the hydrostatic stress. As such, the potential function, which will be discussed in detail later, is more difficult to interpret and cannot be assumed to be equivalent to the yield function. Moreover, in the work of Christensen [19] it is indicated that the plastic potential for isotropic materials (not only ductile metals) is influenced by the distortional (shearing) effect, while the yielding function has been affected by both distortional and dilatational (hydrostatic) effects, which contribution is described by introduced parameters.
The hydrostatic stress tensor defined in indicial notation is given by the elements \( \sigma_m \delta_{ij} \):

\[
P_{ij} = \sigma_m \delta_{ij} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}
\]

Equation 2.1

The mean stress, \( \sigma_m \), is then defined as:

\[
\sigma_m = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})
\]

Equation 2.2

As previously mentioned, the deviatoric stresses, \( s_{ij} \), contribute to a material's ability to change shape. The tensor is quantified by subtracting the mean stress from the diagonal terms of the full stress tensor. In doing so, \( s_{ij} \) can be expressed in terms of the original stress tensor:

\[
s_{ij} = \sigma_{ij} - P_{ij} = \begin{bmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_m \end{bmatrix}
\]

Equation 2.3

In the theory of plasticity, the stresses and strains of a material are analyzed under loading conditions that strain the material beyond the elastic limit wherein plastic
yielding occurs. In order to better understand this phenomenon in the three-dimensional stress state, a widely accepted concept of yield surfaces is used.

**Yield Surface**

For the simple case of uniaxial loading, the yield stress, $\sigma_y$, represents a boundary point between the elastic and plastic regions of material deformation. In the more general three-dimensional stress space, there is an infinite number of yield combinations between the states that will cause the system to yield. This implies a generalization of the yield condition for uniaxial loading to the three-dimensional state of stress. The above mentioned points constitute the so-called yield surface that separates the stress space into the two prevailing elastic and plastic domains. Thus, this surface defines the boundary of the elastic limit at the points where yielding ensues under multi-axial loading and where plastic behavior will ultimately begin thereafter.

The yield surface is a convex surface in the stress space and has an arbitrary cross sectional shape — the convexity of which will be discussed later. It is defined mathematically as the surface given by $f(\sigma_y) = 0$ where the stress state (given as a stress pair, $\sigma$ or $\sigma_\tau$) is plotted inside the surface or on it. This implies the following:

- The material is said to behave elastically if $f(\sigma_y) < 0$
- The material is said to behave plastically if $f(\sigma_y) = 0$

The general yield surface in the principal stress space is shown in Figure 2.2. The vector OS is defined according to its coordinates $(\sigma_1, \sigma_2, \sigma_3)$ and lies on the yield surface, which implies, for this state of stress, that yielding occurs. This vector, as discussed, has the two aforementioned hydrostatic and deviatoric components. The first component is
represented by the vector OP which lies on the deviatoric plane given by equation \( \sigma_1 + \sigma_2 + \sigma_3 = 0 \). The hydrostatic stress vector, OH, has direction cosines \((1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\) and is perpendicular to the deviatoric plane. Since yielding is independent of the hydrostatic stress, the yield surface is a cylinder with generators perpendicular to the deviatoric plane. The object of interest is the intersection of the deviatoric plane \((\sigma_1 + \sigma_2 + \sigma_3 = 0)\) with the yield surface, known as yield locus (which is valid for a two dimensional stress state representation). For porous materials, such as rocks and soils, the cylinder is not parallel to the hydrostatic line \(\sigma_1=\sigma_2=\sigma_3\), because the hydrostatic stress can not be neglected in the plastic deformation.
In Figure 2.2, the coordinate axes represent the principal stresses. The yield locus can be plotted in the normal-shear plane as well [10].

When the state of stress is on the surface, three cases of material behavior can result. The first one is the loading condition shown in Figure 2.3 and is represented by:

\[ df = \frac{\partial f}{\partial \sigma_y} \sigma_y > 0 \]
This signifies that the stress state moves outward from the yield surface. In this case, the material is said to begin a hardening process. In the case of neutral loading, the stress state moves along the yield surface (Figure 2.4) where the material has simply experiencing a plastic flow and is not yet experiencing a plastic hardening. This is represented by the following:

\[ df = \frac{\partial f}{\partial \sigma_y} \sigma_y = 0 \]

Finally, during the unloading process, the stress rate decreases and tends back towards the inside of the surface. This is given in Figure 2.5 by:

\[ df = \frac{\partial f}{\partial \sigma_y} \sigma_y < 0 \]
Figure 2.3 Loading Condition.
Figure 2.4 Neutral Loading Condition.
Yield Criterion

As previously mentioned, the yield surface itself is convex, but its exact shape is a very complex problem in itself. The convex nature of the surface is a direct result of Drucker's postulate [6]. The yield criterion that is assumed actually provides an answer to the question as to when (and for which state of stress) yielding will initiate and also what precise shape the yield surface should have so as to model the infinite stress combinations resulting from multi-directional loading. There are many proposed yield criteria, among which the two most commonly used are Tresca and von Mises. In the former, yielding occurs if one half the largest difference between the principal stresses reaches the value $k_t$, which depends on the properties of the material:
Equation 2.4

\[ k_t = \frac{1}{2} (\sigma_{\text{max}} - \sigma_{\text{min}}) \]

where \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) represent the maximum and minimum principal stress respectively. Experimentally, the value \( k_t \) is obtained from a simple tensile test. When a material reaches a yielding point, the maximum principal stress is the yielding stress, \( \sigma_{\text{yield}} \), of the material, and the minimum stress is zero, therefore:

\[ k_t = \frac{\sigma_{\text{yield}}}{2} \]

Equation 2.5

According to the Von Mises criterion, yielding will occur when the second invariant of the stress deviator tensor equals the value \( k_m^2 \) where \( k_m \) depends only on the material. This is indicated by the relation below:

\[ J_2 = \frac{1}{2} s_{ij} s_{ij} = k_m^2 \]

Equation 2.6

The deviatoric stress invariant, \( J_2 \), can be expressed in different forms (using either deviatoric stresses, \( s_{ij} \), or total stresses \( \sigma_{ij} \)).
\[ J_2 = \frac{1}{6} \left[ (s_{11} - s_{22})^2 + (s_{22} - s_{33})^2 + (s_{33} - s_{11})^2 \right] + s_{12}^2 + s_{23}^2 + s_{31}^2 \]

\[ J_2 = \frac{1}{6} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \]

Equations 2.7

or in terms of just principal stresses as:

\[ J_2 = \frac{1}{6} \left[ (s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2 \right] \]

\[ J_2 = \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

Equations 2.8

The value \( k_m \) is related to the uniaxial tensile stress when the material yields and

is given as:

\[ k_m = \frac{\sigma_{\text{yield}}}{\sqrt{3}} \]

Equation 2.9

Now, Equation 2.6 can finally be expressed as:

\[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{\text{yield}}^2 \]

Equation 2.10

or for the plane state of stress as:
\[ \sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_{\text{yield}}^2 \]

Equation 2.11

This represents a second order function, which is an ellipse rotated at 45 degrees about \( \sigma_3 \) principal coordinate axis. This is shown in Figure 2.6.

Figure 2.6 Von Mises yield locus.

Another approach for deriving the von Mises criterion is to use the maximum distortion strain-energy theory [6]. According to this theory, yielding will occur when the distortional strain energy density, \( U_d \), equals or exceeds a value \( k \) that depends on the material. \( U_d \) is the difference between the total elastic strain energy density, \( U_e \), and the
hydrostatic strain energy density, $U_h$. The total elastic strain energy density is then given by:

$$U_t = \int (\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z)$$

Equation 2.12

which represents the area under the stress-strain curves (see Figure 2.7).

![Figure 2.7](image)

Figure 2.7 Total elastic strain energy density.

After substituting in the strain increments in terms of the stress increments using Hook's law and after integration, Equation 2.12 can be expressed as:

$$U_t = \frac{1}{2E} \left[ (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - 2\nu(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) \right]$$

Equation 2.13
Finally, after subtracting mean stress from the stresses in Equation 2.13, the expression for the distortional strain energy density can be obtained as:

\[
U_d = \frac{1}{6E} (1 + \nu)[(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2]
\]

Equation 2.14

As stated, the value of k can be determined from the uniaxial tensile test. For the yielding state of stress, \(\sigma_x\) will be equal to \(\sigma_{\text{yield}}\) and \(\sigma_y, \sigma_z\) will be zero. Therefore, from Equation 2.14, \(U_{d,\text{tension}}\) can be expressed as:

\[
U_{d,\text{tension}} = \frac{1}{3E} (1 + \nu)\sigma_{\text{yield}}^2 = k
\]

Equation 2.15

Setting k equal to \(U_d\) so as to define the onset of yielding, and after performing some algebra, the von Mises criterion for the three dimensional state of stress can be readily expressed as:

\[
\frac{\sqrt{2}}{2} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 \right]^{\frac{1}{2}} = \sigma_{\text{yield}}
\]

Equation 2.16

In the two dimensional state of stress, this reduces to:
\[ \sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_{yield}^2 \]

Equation 2.17

where \( \sigma_1 = \sigma_x \) and \( \sigma_2 = \sigma_y \). This equation is the same as Equation 2.11.

In the three-dimensional stress space, the von Mises yield surface is a circular cylinder with a cross section defined by radius \( \sqrt{2/3 \sigma_{yield}} \) as shown in the following figure:

![Von Mises yield surface](image)

Figure 2.8 Von Mises yield surface.
Von Mises ellipse (Equation 2.17) is plotted in Figure 2.9:

Locus of points on the ellipse represents the yield locus

Figure 2.9 Von Mises yield locus.
Material Hardening

There are two general types of hardening that the material may generally experience once yielding has occurred. These initially defined independently as isotropic and kinematic (the combination of the two notwithstanding).

The isotropic hardening concept assumes a uniform expansion of the yield surface about the origin during loading while maintaining the initial shape of the surface (see Figure 2.10). In this case, the yield stress in tension is equal to the yield stress in compression.

Figure 2.10 Expansion of yield surfaces (loci) for the isotropic hardening.
In defining the kinematic hardening model, the assumption is that the yield surfaces are able to translate in the deviatoric stress space without experiencing a change in shape or size (see Figure 2.11). This accounts for the Bauschinger effect phenomenon where the initial side of yielding (either tension of compression) is larger than yield stresses on the opposite side. This is actually due to an annihilation of atomic bonds at the post-yield level and is representative of an anisotropic type of behavior that materials generally experience once yielding occurs.

Figure 2.11 Kinematic Hardening with backstress evolutionary function, $\alpha$. 

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In the present research effort, a combined isotropic-kinematic rule of hardening is assumed, where both the growth and translation of the yield surfaces are considered. Experimentally, this has been proven over recent years to be a well-encompassing basis model for predicting nonlinear behavior.

Flow Rule

In order to describe a material's plastic behavior, a type of flow rule needs to be established that assess how a material will flow upon yielding. As such, the flow rule specifies the increment in plastic strain once the material has yielded.

Saint-Venant (1870) was the first to formulate the stress-strain relations for plastic deformation, where the principal axis of strain increment (or strain rate) was assumed to coincide with the axis of principal stresses. The elastic strain was neglected in this case. It can be shown that that the principal stress axes coincided with the axis of the deviatoric axis. Consequently, the strain increment, $\Delta e_{ij}$ (or strain rate, $\varepsilon_{ij}$) is coaxial with the deviatoric stress, $S_{ij}$. In indicial notation, this is given as:

$$
\dot{\varepsilon}_{ij} = \dot{\Lambda} S_{ij}
$$

Equation 2.18

or in the Cartesian coordinate system:
where $\dot{\Lambda}$ is a proportional positive scalar factor and can be determined according to the assumed yield criterion. Equation 2.19 are called Levy-Mises equations. In order to consider elastic, perfectly plastic materials, Prandtl and Reuss suggested a modification of the Equation 2.19 such that plastic strain rates are considered. In this light, the plastic strain rates depend only on the current deviatoric stresses.

Prandtl-Reuss equations were empirically postulated using the results from experimental observations in metals for perfectly plastic behavior. However, a general mathematical treatment was needed in order to describe plastic deformation.

In the theory of elasticity, the strain tensor is related to the stress tensor through an elastic potential function (complementary strain energy $U_c$, see Figure 2.12) in the following manner:

$$
\dot{\varepsilon} = \frac{\partial U_c}{\partial \sigma}
$$

Equation 2.20

where for the general principal state of stress, $U_c$ is expressed as:
\[ U_c = \int \varepsilon_1 d\sigma_1 + \varepsilon_2 d\sigma_2 + \varepsilon_3 d\sigma_3 \]

Equation 2.21

Figure 2.12 Complementary and strain energy density.

Von Mises applied this idea to plasticity theory, where the existence of a plastic potential function, \(Q(\sigma_\beta)\), was proposed, such that:

\[ \dot{\varepsilon}_\beta^p = \lambda \frac{\partial Q(\sigma_\beta)}{\partial \sigma_\beta} \]

Equation 2.22

The plastic potential and the plastic strain rate represent a surface and a vector in stress space, respectively, such that the vector is perpendicular to the surface (predicated on the normality rule of plastic theory, see Figure 2.13). If a state of incompressibility is assumed, then the plastic potential function can be represented as a cylinder in which case \(\sigma_{11} = \sigma_{22} = \sigma_{33}\), or in the principal state of stress \(\sigma_1 = \sigma_2 = \sigma_3\). For materials whose
plastic deformation is not extenuated by the experienced volume change, but only by a shape change, the hydrostatic axis, $\sigma_{11} = \sigma_{22} = \sigma_{33}$, is parallel to the said plastic potential surface. Plastic deformation of materials such as concrete, rock, and soil are significantly impacted by volume change as well, and thus, the hydrostatic axis is not parallel to the plastic potential surface.

The problem at this point is in defining an appropriate form of $Q(\sigma_{ij})$. To date, the plastic potential remains unspecified exactly. However, if the assumption is made where the plastic potential is equal to the yield surface, which is a most common approach in the plasticity theory as has been indicated experimentally, then the associated flow rule [16] is defined such that plastic flow is directly associated with the yielding criterion as shown, which the Levy-Mises rule does not indicate:

$$\dot{\varepsilon}_{ij} = \Lambda \frac{\partial f}{\partial \sigma_{ij}}$$

Equation 2.23

If $Q(\sigma_{ij}) \neq f(\sigma_{ij})$, then this would in the general scope define a nonassociated flow rule [16]. While this more generally describes the plastic deformation of porous materials such as concrete, soil, and rock, Drucker’s postulate is not applicable in this case.

In order to better understand the role of the scalar $\Lambda$, first consider Hooke’s law in its basic form:
\[ \varepsilon_1 = \frac{1}{E} [\sigma_1 - \nu(\sigma_2 + \sigma_3)] \]

\[ \varepsilon_2 = \frac{1}{E} [\sigma_2 - \nu(\sigma_1 + \sigma_3)] \]

\[ \varepsilon_3 = \frac{1}{E} [\sigma_3 - \nu(\sigma_1 + \sigma_2)] \]

Equations 2.24

For the plastic strain, using analogy, we have:

\[ d\varepsilon_1^p = d\Lambda [\sigma_1 - \nu(\sigma_2 + \sigma_3)] \]

\[ d\varepsilon_2^p = d\Lambda [\sigma_2 - \nu(\sigma_1 + \sigma_3)] \]

\[ d\varepsilon_3^p = d\Lambda [\sigma_3 - \nu(\sigma_1 + \sigma_2)] \]

Equations 2.25

Poisson’s ratio assumed the value 0.5 if incompressibility (constant volume plastic deformation) is considered. It can be observed from the above equations that \( d\Lambda \) has replaced \( 1/E \) in the Hooke’s law although as literature widely validates, the parameter \( d\Lambda \) is not a material constant as is \( E \) and actually represents a positive scalar.
Figure 2.13 Normality of the plastic strain increment vector at the point of yielding.
CHAPTER 3

UNIAXIAL STRESS-STRAIN MODEL

Uniaxial stress-strain models are defined when applied loading results in straining in one direction. There are various uniaxial stress-strain models that have been used in several analyses and applied in various design codes of practice. Many of these models, however, idealize the true stress-strain behavior of the material, which they are representing; some of these are shown in Figure 3.1.
Figure 3.1 Idealized stress-strain diagrams (a) Rigid (b) Linear elastic (c) Rigid perfectly plastic (d) Rigid plastic with linear strain hardening (e) Linear elastic perfectly plastic (f) Linear elastic, plastic with linear strain hardening.
Although design codes of practice, such as the UBC [1], do not explicitly utilize a specific post-yield model for predicting inelastic deformation levels under cyclic conditions, they do enable engineers to consider such member nonlinearities by utilizing reduction factors applied to the elastic response levels of a system. In this way, the advantages that ductility provides to a system's ability to remain functional can be utilized. However, the results attained from these procedures are often on the conservative side and can actually result in very inaccurate response predictions if cyclic loading and/or dynamic loading are considered. Therefore, in order to more accurately describe the overall plastic behavior of systems, a highly-nonlinear strain hardening definition should be stipulated (see for example Figure 3.2).

Figure 3.2 Stress-strain diagram (monotonic loading). Linear elastic, plastic with nonlinear strain hardening.

In this work, individual uniaxial nonlinear stress-strain relations, which have been verified experimentally, are used in combination to develop a general biaxial plasticity model for anisotropic materials. The general model is actually developed initially by
considering the three-dimensional state so as to preserve the accuracy of the mathematical derivations that follow.

One of the models that is used is a constitutive relationship [8] that is represented by a continuous second-order polynomial function defined by a hardening index parameter ($\alpha$) and a post-yield strain coefficient ($\Delta_e$). These parameters enable the constitutive mathematics of the stress-strain relationship to be accurately developed so that for bending stress:

$$
\sigma_x = \sigma_{yield} - \alpha \sigma_{yield} \left( 2 + \frac{1}{\Delta_e} \right) + \frac{2\alpha \sigma_{yield}}{e} \left( 1 + \frac{1}{\Delta_e} \right) - \frac{\alpha \sigma_{yield} y^2}{\Delta_e e^2}
$$

Equation 3.1

In Equation 3.1, $\sigma_x$ is defined as the post-yield stress at some distance $y$ away from the neutral axis of the member’s cross section. The depth of linear elastic activity through the section depth is defined by the value $e$ (see Figure 3.3, Figure 3.4 and Figure 3.5), and $\Delta \sigma_p$ is the post-yield stress measured at the top fiber. The value of $e$ actually decreases starting from $h/2$ where $e=h/2$ can be found at the tip of the member (for shear-frames) where the section has ‘just-yielded;’ the section depth is given as $h$. The value of $e$ can attain a theoretical minimum depth of $e = 0$, wherein the section has fully plastified. As such, the distance along the member that has achieved at least some level of yielding can then be computed. The underlying assumption in applying this model to calculate the finite-element member displacements is that the cross-section geometry is symmetric about both of its in-plane axes. The uniaxial stress-strain relationship is shown in Figure
3.6 where different stress states representing the various degrees of through-thickness plasticity levels are shown (in terms of e).

The hardening index parameter, $\alpha$, guides the stress in the post-yield material stress range. It defines the average modulus degradation between any two states (the yield and ultimate for example) and helps create a continuous post-yield distribution. In the case where $\alpha = 0$, the elastic-perfectly plastic case is represented. Equation 3.1 is plotted in Figure 3.7 for $\alpha=0.18$ and $\Delta_e=14$.

![Through-thickness stress distribution](image)

Figure 3.3 Through-thickness stress distribution.
Figure 3.4 Through-thickness strain distribution.
Figure 3.5 Shear frame member with post-yield state distribution e.
Figure 3.6 Stress-strain model with different values of $e$. 

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Another uniaxial stress-strain relationship is formulated on the Ramberg-Osgood model [9], which is given as:

\[ \sigma_y = A \varepsilon^n \]

Equation 3.2

where A and n are material constants determined experimentally. This equation is developed from a dislocation theory stating that the stress is proportional to the square root of the density of dislocations, \( \rho_d \):
\[ \sigma = \kappa b \rho_d^{1/2} \]  

Equation 3.3

If the dislocation density, \( \rho_{do} \), that corresponds to the elastic limit, \( \sigma_{yield} \), is included, then the equation becomes

\[ \sigma = \sigma_{yield} + \kappa b (\rho_d - \rho_{do})^{1/2} \]  

Equation 3.4

where \( \kappa \) and \( b \) are material constants. Analogous to the previous equation and considering strains on a macroscopic level, the following is true:

\[ \sigma = \sigma_{yield} + K_Y \varepsilon^{1/M_Y} \]  

Equation 3.5

where \( K_Y \) is the coefficient of plastic resistance, and \( M_Y \) is the hardening exponent. Equation 3.2 is plotted in Figure 3.8 for \( A=715.15 \) and \( n=0.2070 \). These parameters are optimally computed from an uniaxial experimental test on stainless steel 316 under low-rate monotonic loading [15].
Figure 3.8 Uniaxial stress-post-yield strain relationship (Equation 3.2).
CHAPTER 4

ANISOTROPIC MATERIAL BEHAVIOR

The material properties of isotropic materials are the same in any direction. This means that only the magnitudes of the principal stresses are needed to describe the failure behavior of the material since the shear stresses can be assumed as zero along these principal planes. On the other hand, material properties for anisotropic materials vary according to the orientation of the tested sample, which means that both the magnitudes of the principal stresses and their orientation are needed in describing the responses. Also, the shear stresses can play an overall significant role in the development of the yield surfaces for the anisotropic models. This makes anisotropic material behavior much more difficult to model.

Table 4.1 summarizes some of the proposed yield functions for anisotropic materials to date (including Tresca and von Mises for the isotropic materials):
### Table 4.1 Yield criteria for anisotropic materials.

<table>
<thead>
<tr>
<th>Yield Criterion</th>
<th>Type</th>
<th>Shear stress included</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tresca</td>
<td>Isotropy</td>
<td>-</td>
</tr>
<tr>
<td>Von Mises</td>
<td>Isotropy</td>
<td>-</td>
</tr>
<tr>
<td>Hill (1948)</td>
<td>Cross Anisotropy</td>
<td>+</td>
</tr>
<tr>
<td>Hill (1979)</td>
<td>Cross Anisotropy</td>
<td>-</td>
</tr>
<tr>
<td>Hosford (1979)</td>
<td>Cross Anisotropy</td>
<td>-</td>
</tr>
<tr>
<td>Barlat and Lian (1989)</td>
<td>Cross Anisotropy</td>
<td>+</td>
</tr>
<tr>
<td>Barlat (1991)</td>
<td>Anisotropy</td>
<td>+</td>
</tr>
</tbody>
</table>

In 1948 Hill proposed the following constitutive relation for the yielding function of anisotropic metals [7]:

\[
2f = F(\sigma_{yy} - \sigma_{zz})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2L\sigma_{yx}^2 + 2M\sigma_{zx}^2 + 2N\sigma_{xy}^2 = 1
\]

Equation 4.1

Hill's criterion is based on the von Mises criterion for isotropic materials and includes six material constants (F, G, H, L, M, N) that describe the current state of anisotropic yielding. Hill assumed that the material is orthotropic, which means that there are three mutually orthogonal planes of symmetry at each material point. Hill also assumed that there is no Bauschinger effect and that hydrostatic stress does not affect yielding. This criterion includes shear stresses as well. Unfortunately, however, the
obtained shape of the yield surface using Hill's formulation was not consistently accurate with experimental validation.

Hill's criterion proposed in 1979 [17] is given by:

\[
\begin{align*}
&f \left| \sigma_2 - \sigma_3 \right|^m + g \left| \sigma_3 - \sigma_1 \right|^m + h \left| \sigma_1 - \sigma_2 \right|^m + a \left| 2\sigma_1 - \sigma_2 - \sigma_3 \right|^m + \\
b \left| 2\sigma_2 - \sigma_3 - \sigma_1 \right|^m + c \left| 2\sigma_3 - \sigma_1 - \sigma_2 \right|^m = e
\end{align*}
\]

Equation 4.2

where \( f, g, h, a, b, \) and \( c \) are material constants and \( \sigma_1, \sigma_2, \sigma_3 \) are principal stresses.

This yield function does not include shear stresses, which assumes that the principal axes coincide with the axis of material symmetry; this becomes problematic when trying to assess anisotropic material behavior.

Hosford (1979) developed a model for the plane state of stress without shear stresses [18]:

\[
\left| \sigma_1 \right|^M + \left| \sigma_2 \right|^M + r \left| \sigma_1 - \sigma_2 \right|^M = (r + 1)Y^M
\]

Equation 4.3

where the parameter \( M \) determines the shape of the yield function, and \( r \) is the ratio of the yield stress in biaxial tension to the average yield stress in uniaxial tension. Both of these parameters can be determined experimentally, where calibration of the former requires more complicated testing procedures. \( Y \) is the yield stress of a bar in uniaxial tension.
Bi-axial loading

A typical frame subjected to bi-axial loading is presented in the following figure:

![Bi-axial loading of a frame element](image)

Figure 4.1 Bi-axial loading of a frame element.

In Figure 4.1, the axial force P and the moment M will generate the principal stress distribution $\sigma_x$, and any lateral loading along the member length will generate the $\sigma_y$ stresses as indicated on the element in the same figure.

The proposed model for anisotropic material behavior uses the results of the uniaxial tests to describe the coupled stress effects under such a biaxial loading condition.
Proposed model

The proposed model investigates anisotropic material behavior under bi-axial loading in principal stress space from a snapshot perspective. The yield surfaces are generated by coupling the two uniaxial models given by Equation 3.1 and Equation 3.2. Each equation is projected onto a set of material axes of assumed anisotropy. This is defined as a snapshot.

Since the principal axes of stress can be assumed to coincide with the material axes of anisotropy at any given instant, the shear stresses are taken as zero on that given plane (for that particular snapshot). As such, a different set of principal axes is considered at each independent snapshot at each strain increment. The more snapshot sets of principal axis that are selected at each strain increment, the more accurate the shear stress effect is proposed to be taken into account. Theoretically, an $n$ number of sets can be chosen. In this work, ten are considered in all. Figure 4.2 shows two of the ten snapshot stress-strain relationships for the biaxial state of stress at some distance $x$ along the length of a wide-flange section.

Each snapshot results from a different combination of $\sigma_x$ and $\sigma_y$ stresses. The basic $\sigma_x$ and $\sigma_y$ functions (Equation 3.1 and Equation 3.2) are shown in Figure 4.3. Other stress-strain relationships ($\sigma_j$ for the $j^{th}$ snapshot) are determined by combining the two functions through an ellipsoid. Figure 4.4 shows two of such combinations for the snapshots $j=1$ and $j=2$. 

50

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Figure 4.2 Two representative snapshots of material anisotropy at distance $x$. 

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Figure 4.3 Ellipsoid model of various stress-strain snapshots (j=1 to 10).
Figure 4.4 Biaxial stress state using snapshots $j=1$ and $j=2$.

For each snapshot, a percentage of one curve in Figure 4.3 is utilized in combination with a percentage of the other curve. This combination is formulated on the basis of an assumed elliptical connection between the two uniaxial functions $\sigma_x$ and $\sigma_y$ in Figure 4.3. This 'ellipsoid combination' is an assumption undertaken as part of this study in trying to compute the uniaxial yield levels from the distortional strain energy approach as discussed earlier. However, the use of the ellipsoid is logically predicated on the fact that the yield surfaces will be elliptical functions in this snapshot perspective (where each snapshot is again an independent isotropic condition). Since the distortional strain energy (which will be discussed later) is equal to the strain energy density under uniaxial...
conditions, the strain energy condition for the uniaxial yield case is obtained using the above mentioned elliptical connection. In other words, the $\sigma_x$ and $\sigma_y$ curves are joined elliptically so as to define the uniaxial yield curve for each snapshot predicated on the presumed axis orientation of Figure 4.2.
CHAPTER 5

YIELD SURFACES FOR THE PROPOSED MODEL

The distortional strain energy density, $U_d$, is used to develop a collection of yield surfaces using the individual snapshot yield surfaces at each discretized plastic strain level; in this way, the entire collection of yield surface models the anisotropic behavior in the material. As previously mentioned, the distortional strain energy density is often used to model isotropic behavior and will thus be used in this regard to model each snapshot obtained by integrating the stress-strain diagram (see Figure 4.3). As such, each snapshot of yield surfaces essentially acts as a model for isotropic behavior; it’s the collection that then models anisotropic behavior.

In order to analyze what happens after the yield stress is reached, the area under the stress-strain curve is discretized at various levels of plastic strains. In this manner, the discretization allows Hook’s law to be applied only on the linearized hardening portions of the stress-strain curve; the linearization is a result of the discretization of the nonlinear stress-strain function. For each level of strain after yielding, the strain energy is separated into its “flow” (constant stress) and “hardening” (increasing stress) portions, as shown in Figure 5.1. A constant strain increment, $\Delta \varepsilon$, is considered in this sense. In this way, the stress and strain relations during plastic flow are assumed after the hardening portions are first considered using Hooke’s Law, which is again only applicable in those hardening regions; as such, Hooke’s Law is not used in the flow areas as it is not applicable.
Therefore, the proportionality scaling factor, $\Lambda$, can be solved afterwards by equating these results to those equations shown in Chapter 2 (Equations 2.25) so as to be consistent with literature, where $\Lambda$ is often desired. Also, as Figure 5.1 below shows, the strain increment ($\varepsilon_{x2} - \varepsilon_{x1}$) is defined by the boundary of parallel lines to the $\sigma$ axis. This alleviates the need to compute a true plastic strain, which would be determined by hypothetically unloading the stress – strain curve at each stress level (e.g., at $\sigma_{x2}$) and computing where it crosses the strain axis. In that case, the calculation of $\Lambda$ would become necessary. In the proposed approach, the methodology appears to be much more straight-forward.

\[
\begin{align*}
\sigma & \\
\sigma_{x2} & \\
\sigma_{x1} & \\
\varepsilon_{x1} & \\
\varepsilon_{x2} & \\
\varepsilon
\end{align*}
\]

**Figure 5.1 Hardening and Flow components.**

Since the mechanical characteristics of anisotropic materials vary with the orientation of the tested sample, different values of Young’s Moduli (E) and Poisson’s
ratios (ν) were also determined depending on the specific snapshot orientation that is under consideration at that time. The state of principal stresses in three dimensions predicates that all three moduli (E_x, E_y, E_z) and Poisson ratios (ν_x, ν_y, ν_z) be calculated for each snapshot (j) even though only bi-axial loading is being considered; this was necessary in order to preserve the mathematical integrity of the approach. This was alluded to earlier. As such, the following simple commonly-used equations for elastic (isotropic) materials are utilized for each snapshot:

\[
\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_y}{E_y} \sigma_y - \frac{\nu_z}{E_z} \sigma_z \quad (j^{th} \text{ snapshot})
\]

\[
\varepsilon_y = \frac{\sigma_y}{E_y} - \frac{\nu_x}{E_x} \sigma_x - \frac{\nu_z}{E_z} \sigma_z \quad (j^{th} \text{ snapshot})
\]

\[
\varepsilon_z = \frac{\sigma_z}{E_z} - \frac{\nu_x}{E_x} \sigma_x - \frac{\nu_y}{E_y} \sigma_y \quad (j^{th} \text{ snapshot})
\]

Equation 5.1

Total strain energy density for the hardening (see Figure 5.2) on the j^{th} snapshot and i^{th} state of strain is then given by:

\[
U_{i,j} = \frac{1}{2} \Delta\sigma_{x,i} \Delta\varepsilon_{x,j} + \frac{1}{2} \Delta\sigma_{y,i} \Delta\varepsilon_{y,j} + \frac{1}{2} \Delta\sigma_{z,i} \Delta\varepsilon_{z,j} \quad (j^{th} \text{ snapshot})
\]

Equation 5.2

where \(\Delta\sigma_{x,i} = \sigma_{x,i} - \sigma_{x,i-1}\) and \(\Delta\varepsilon_{x,i} = \varepsilon_{x,i} - \varepsilon_{x,i-1}\) = const.
The total strain energy density for the flow portion of the discretized curve can then be calculated as:

\[ U_{ij} = \sigma_{x,i-1} \Delta \varepsilon_{x,i} + \sigma_{y,i-1} \Delta \varepsilon_{y,j} + \sigma_{z,i-1} \Delta \varepsilon_{z,j} \]  

\text{(j^{th} snapshot)}

\text{Equation 5.3}

This is graphically represented for the x-principal direction in Figure 5.3 below where the increment \( \Delta \varepsilon_{x,i} = \varepsilon_i - \varepsilon_{i-1} = \text{constant} \).
Figure 5.3 Total strain energy density for the flow.

After substituting for the strains using Equation 5.1 in the equations for the total strain energy (hardening and flow) and after subtracting the hydrostatic stresses, the following results. As a note, in order to attain correct expression for the distortional strain energy density for the plane state of stress, one must start from the three-dimensional state of stress and then consider $\sigma_3=0$:

$$U_{di, hardening} = a_i(\Delta \sigma_{x,i} + \Delta \sigma_{y,i})^2 + b_i \Delta \sigma_{x,i} \Delta \sigma_{y,i} + c_i \Delta \sigma_{x,i}^2 + s_i \Delta \sigma_{y,i}^2 + e_i \Delta \sigma_{x,i} (\Delta \sigma_{x,i} + \Delta \sigma_{y,i}) + f_i \Delta \sigma_{y,i} (\Delta \sigma_{x,i} + \Delta \sigma_{y,i})$$

Equation 5.4

$$U_{di, flow} = g_i \Delta \sigma_{x,i} + h_i \Delta \sigma_{y,i} + m_i (\Delta \sigma_{x,i} + \Delta \sigma_{y,i}) + n_i (2\Delta \sigma_{x,i} - \Delta \sigma_{y,i}) + k_i (2\Delta \sigma_{y,i} - \Delta \sigma_{x,i})$$

Equation 5.5
where:

\[
\begin{align*}
a_i &= \frac{0.5}{9} \left\{ \frac{1}{E_{x,i}} (1-2\nu_{x,i}) + \frac{1}{E_{y,i}} (1-2\nu_{y,i}) + \frac{1}{E_{z,i}} (1-2\nu_{z,i}) \right\} \\
b_i &= -0.5 \left\{ \frac{2}{3} \left( \frac{1}{E_{x,i}} + \frac{1}{E_{y,i}} \right) + \frac{v_{x,i}}{E_{x,i}} + \frac{v_{y,i}}{E_{y,i}} \right\} \\
c_i &= 0.5 \frac{1}{3E_{y,i}} \\
s_i &= 0.5 \frac{1}{3E_{y,i}} \\
e_i &= 0.5 \left\{ \frac{1}{3} \left( \frac{2v_{x,i}}{E_{x,i}} + \frac{v_{y,i}}{E_{y,i}} + \frac{v_{z,i}}{E_{z,i}} \right) \right\} \\
f_i &= 0.5 \left\{ \frac{1}{3} \left( \frac{v_{x,i}}{E_{x,i}} + \frac{2v_{y,i}}{E_{y,i}} + \frac{v_{z,i}}{E_{z,i}} \right) \right\} \\
g_i &= \frac{2\sigma_{x,i-1} - \sigma_{y,i-1}}{3E_{y,i}} \\
h_i &= \frac{2\sigma_{y,i-1} - \sigma_{x,i-1}}{3E_{y,i}} \\
m_i &= \frac{1}{9} \left\{ \left( \frac{1}{E_{x,i}} + \frac{1}{E_{y,i}} + \frac{1}{E_{z,i}} \right) (\sigma_{x,i-1} + \sigma_{y,i-1}) - 3 \left( \frac{\sigma_{x,i-1}}{E_{x,i}} + \frac{\sigma_{y,i-1}}{E_{y,i}} \right) \frac{v_{z,i}}{E_{z,i}} (\sigma_{x,i-1} + \sigma_{y,i-1}) \right\} \\
n_i &= \frac{v_{x,i}}{9E_{x,i}} (2\sigma_{x,i-1} - \sigma_{y,i-1}) \\
k_i &= \frac{v_{y,i}}{9E_{y,i}} (2\sigma_{y,i-1} - \sigma_{x,i-1})
\end{align*}
\]

Equations 5.6

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In Equations 5.6, coefficients $a_i$, $b_i$, $c_i$, $s_i$, $e_i$, $f_i$ affect material hardening, while coefficients $g_i$, $h_i$, $m_i$, $n_i$, and $k_i$ influence the material flow. Combining Equation 5.4 and Equation 5.5, the distortional strain energy density for each segment under the stress-strain curves can be computed. Note that Young’s moduli and Poisson’s ratios are different for each segment and can in fact be negative as literature has shown [20]. The usual relationship between the ratios of Young’s moduli and Poisson’s ratios for each post-yield strain state for the anisotropic materials [10] is as follows:

$$\frac{E_{x,i}}{E_{y,i}} = \frac{\nu_{x,i}}{\nu_{y,i}}$$

Equation 5.7

If the distortional strain energy density is then set equal to the distortional strain energy density for uniaxial loading, the yield locus for multi-axial loading using the Von Mises approach will be conceived. In this way, the final form is that of an ellipse in the principal $\sigma_x$-$\sigma_y$, rotated at 45° about the $\sigma_z$ axis (where $\sigma_z$ is now 0). As mentioned in Chapter 2 (Figure 2.9), this angle is a result of considering isotropic material for which principal uniaxial stresses have the same value. Since the proposed model accounts for anisotropic materials, necessary rotations of the yield surfaces took place as will be explained in the following chapter. After some algebraic reduction using the parameters from Equations 5.6, the equation of the ellipse for every snapshot and $i^{th}$ strain increment will take on the following form:
\[ A_i x_i^2 - B_i x_i y_i + C_i y_i^2 + D_i x_i + E_i y_i = d_i \]

Equation 5.8

where \( x_i \) represents \( \Delta \sigma_{x,i} \), \( y_i \) represents \( \Delta \sigma_{y,i} \) and:

\[
A_i = a_i + c_i + e_i \\
B_i = 2a_i + b_i + e_i + f_i \\
C_i = a_i + s_i + f_i \\
D_i = g_i + m_i + 2n_i - k_i \\
E_i = h_i + m_i - n_i + 2k_i
\]

Equations 5.9

The value on the right side of Equation 5.8 (which is shown as the constant \( d_i \)) is simply determined from either

\[
d_i = \frac{\Delta \sigma_{x,i}^2}{18} + \frac{\sigma_{x,i} \Delta \sigma_{x,i}}{9} \left\{ \frac{4}{E_{x,i}} (1 + \nu_{x,i}) + \frac{1}{E_{y,i}} (1 + \nu_{y,i}) + \frac{1}{E_{z,i}} (1 + \nu_{z,i}) \right\}
\]
or

\[
d_i = \frac{\Delta \sigma_{y,i}^2}{18} + \frac{\sigma_{y,i} \Delta \sigma_{y,i}}{9} \left\{ \frac{1}{E_{x,i}} (1 + \nu_{x,i}) + \frac{4}{E_{y,i}} (1 + \nu_{y,i}) + \frac{1}{E_{z,i}} (1 + \nu_{z,i}) \right\}
\]

Equation 5.10
depending on the assumed uniaxial loading direction according to the Mises criterion

where either \( \sigma_{y,j} = \Delta \sigma_{y,j} = 0 \) or \( \sigma_{x,j} = \Delta \sigma_{x,j} = 0 \).

After dividing through by \( A_i \), Equation 5.8 finally takes on the following form:

\[
x_i^2 - x_i y_i + y_i^2 + M_i x_i + N_i y_i = F_i
\]

Equation 5.11

where \( M_i = \frac{D_i}{A_i} \) and \( N_i = \frac{E_i}{A_i} \).

Equation 5.11 is then expressed in the local \((X_i, Y_i)\) rotated coordinate system as:

\[
\frac{[X_i + \sqrt{2} (M_i + N_i)]^2}{2F_i + \frac{1}{2} (M_i + N_i)^2 + \frac{1}{6} (-M_i + N_i)^2} + \frac{[Y_i + \sqrt{2} (-M_i + N_i)]^2}{\frac{1}{3} [2F_i + \frac{1}{2} (M_i + N_i)^2 + \frac{1}{6} (-M_i + N_i)^2]} = 1
\]

Equation 5.12

where \( X_i = \frac{\sqrt{2}}{2} (x_i + y_i) \) and \( Y_i = \frac{\sqrt{2}}{2} (-x_i + y_i) \).

Analyzing Equation 5.12, it can be noticed that both coefficients \( M_i \) and

\( N_i \) influence the position as well as the size of the yield surface.

Standard form of the von Mises yield criterion for isotropic metals widely used in

the published literature ([6], [10], [12], [15]) is:
\[
\frac{1}{2\sqrt{3}} \left[ ((\sigma_{ij} - \alpha_y)(\sigma_{ij} - \alpha_y)) \right]^{\frac{1}{2}} - \sigma_{\text{yield}} - R = 0
\]

Equation 5.13

where \( \alpha_y \) is the tensor which defines the center of the yield surface, \( \sigma_{ij} \) are the deviatoric components of the stress tensor and \( R \) is the isotropic hardening variable (accounts for the change in size or expansion of the yield surface). It can be noticed that von Mises yield criterion separates the so-called kinematic variable, \( \alpha_y \), from the isotropic variable, \( R \) and their evolution has been a subject of intensive research. In this work, however, the distortional strain energy density accumulation leads to the form in Equation 5.11 or Equation 5.12 and the above mentioned variables are not treated separately. Thus, the proposed \( \alpha_y \) is determined from the natural behavior of the yield surfaces, their translation and rotation as will be explained in the following chapter.
CHAPTER 6

ADJUSTMENTS FOR ANISOTROPIC BEHAVIOR

As was previously shown in Figure 2.9, the values of the uniaxial yield stresses on each principal axis are the same for an isotropic material. While the entire formulation to this point was developed in an extensive MatLab program, it was desired to make suitable adjustments for the anisotropic nature of the material (as alluded earlier). After the yield loci for each snapshot were plotted, the material had essentially been treated in an isotropic sense with respect to each snapshot. As such, because the obtained values of the uniaxial stresses did not match up with those stresses plotted in the stress-stress space, an adjustment was made on each set yield loci for each snapshot and for the entire collection of surfaces. Therefore, each ellipse was first rotated and then translated in the principal stress plane as shown in Figure 6.1 below so that the uniaxial stresses matched the individual $\sigma_x$ and $\sigma_y$ principal stresses as determined from the discretized curves (Figure 4.3). In performing this extensive task, the final location of the center of each yield surface (locus) was determined. Also, this was consistent with the fact that anisotropic behavior includes a rotational component in the plotting of the yield loci ([9], [10]). For example, the ellipse in Figure 6.1 below was translated from its original position to the new position centered at the origin, or point 0. The ellipse is then translated until $\sigma_y$ matches the value of $\sigma_y$ corresponding to Figure 4.3 at the $i^{th}$ state and $j^{th}$ snapshot (point 2 in Figure 6.1). The obtained ellipse is then rotated about point 2 until
σ_x matches the value of σ_x corresponding to Figure 4.3 and providing the form of the ‘translated and rotated’ ellipse as shown in Figure 6.1. The center of the obtained ellipse is translated to the final position by vector PG, which is parallel and equal in magnitude to the vector BR. This continued for each ellipse on each snapshot and for all the snapshots; altogether, this was performed for 1700 ellipses where each ellipse was discretized into 80 points.

![Figure 6.1 Rotation and translation of the yield locus.](image)

Figure 6.1 Rotation and translation of the yield locus.
Collection of yield surfaces

The final set of the yield surfaces for each snapshot (j = 1 to 10) is shown in the following figures along with the unrotated counterparts. Therefore, it is quite evident that the rotational adjustments made are quite significant to the overall interpretation of the final yield surfaces. The percentage combinations of the $\sigma_x$ and $\sigma_y$ stresses are shown below (again conforming to the ellipsoid that combined each uniaxial snapshot curve as discussed earlier):

<table>
<thead>
<tr>
<th>Stress combination</th>
<th>$\sigma_x$ (%)</th>
<th>$\sigma_y$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>43.6</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>70</td>
<td>71.4</td>
</tr>
<tr>
<td>5</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>6</td>
<td>50</td>
<td>86.6</td>
</tr>
<tr>
<td>7</td>
<td>40</td>
<td>91.7</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>95.4</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>98</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>99.5</td>
</tr>
</tbody>
</table>

Table 6.1 Combination of stresses for each snapshot (j=1 to 10).
Figure 6.2 Yield surfaces for snapshots $j=1$ and $j=2$. 

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Figure 6.3 Yield surfaces for snapshots $j=3$ and $j=4$. 

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Figure 6.4 Yield surfaces for snapshots $j=5$ and $j=6$. 

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Figure 6.5 Yield surfaces for snapshots j=7 and j=8.
Figure 6.6 Yield surfaces for snapshots $j=9$ and $j=10$. 

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It can be noticed that from $j=1$ to $j=7$ the 'rotated' surfaces at the higher plastic level tend to gradually pull away from the other surfaces in a counter-clockwise manner. Starting at around $j=8$, the 'rotated' yield surfaces at the smaller post-yield strains start to catch-up to the other surfaces where all the surfaces start to rotate together in a clockwise manner. There is no distinguishing rotation among the 'unrotated' surfaces. The collection of all snapshot yield surfaces is shown in Figure 6.7.

![Figure 6.7 Collection of the snapshot yield surfaces.](image)

The distribution of the normal stresses for each strain increment can be obtained by intersection of the loading curve on the $f_j$ snapshot yield surfaces. The appropriate
combination of the other snapshot stresses generates the shear state of stress, although this is still under investigation.
CHAPTER 7

THE BACKSTRESS EVOLUTIONARY FUNCTION

Due to the plastic strain hardening of the material, the yield surface by definition translates in the stress space. This translation not only describes how the material yields during loading but also defines the unloading and reloading phases of the material as well. The function that characterizes this behavior is termed the backstress evolutionary function as mentioned in the previous chapter where the evolutionary effect is present because of the particular state (level of plasticity) of the material during unloading.

According to the dislocation theory [6], strain hardening occurs as dislocations pile up at the surface of the material as it is yielding. This in effect prevents atomic slip from occurring, which thus prevents a free flow of the material. This free flow would otherwise be defined by an increase in the strain at constant stress. In order to break this pile-up of dislocations, the material experiences a larger quantitative stress. The terminology of ‘back stress’ thus exists because the process opposes and requires larger applied stresses during this back up of dislocations. This phenomenon is what actually defines hardening. Thus, the concept of hardening would not exist if dislocations did not pile up at the material’s surface during yielding. In that case, the material would simply flow and would define a perfectly plastic condition shown previously where there would be no prevention of surface atoms to slip.
Macroscopically, the backstress function (usually denoted as $\alpha$ in literature) is a mathematical representation of the stress-plastic strain relation. Using the derived theory of yield surface formulations, the backstress is described herein by the position of the centers of the yield surfaces. Thus, by knowing how the yield surfaces move in the stress space or, more precisely, how the yield loci move in the stress plane for a biaxial state of stress, and by using the obtained centers of the yield surfaces, the stress-strain function of a material can easily be calculated.

In the work of Prager [14], Armstrong and Frederick [11], Chaboche [12], Voyiadjis and Sivakumar [13] backstress evolution is predicted by relations that expresses the backstress rate, $\dot{\alpha}$, in terms of the plastic strain rate, $\dot{\varepsilon}^p$, accumulated plastic strain rate, $p$ [6], or/and stress rate, $\dot{\sigma}$. One of the developed models for the back stress function [15] is given in the following form:

$$\dot{\alpha}_{ij} = \frac{2}{3} C \dot{\varepsilon}_{ij} - \gamma \alpha_{ij} \dot{p} + \beta \sigma_{ij} \dot{p}$$

Equation 7.1

where

$$p = \sqrt{\frac{2}{3} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}}$$

After integration, the following relationship is determined:
\[ \alpha = \mu + (\alpha_0 - \mu)e^{-\frac{\gamma(\varepsilon_p - \varepsilon_{p0})}{1-\beta}} \]

Equation 7.2

where

\[ \mu = \frac{3}{\gamma} \frac{2C + \beta b}{C + \beta b} \]

The state \((\varepsilon_{p0}, \alpha_0)\) results from the previous flow, and \(C, \lambda, \beta\) are material constants. Thus, the backstress function is actually a function of the plastic strain as shown schematically in Figure 7.1 and Figure 7.2. The third term in Equation 7.1 is added in order to account for the experimental observations, which showed that the direction of the movement of the center of the yield surface occurred between the stress rate tensor and plastic strain rate tensor. The constants \(C, \lambda, \beta\) can be determined using the stress-strain data obtained from the first half cycle of a uniaxial tension or compression experiment (Figure 7.3).
Figure 7.1 Schematic representation of the plastic strain, $\varepsilon^p$.

Figure 7.2 Schematic representation of the plastic strain increment, $d\varepsilon^p$. 

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Figure 7.3 Half cycle of stress-strain data.

For each test data point \((\sigma_i, \varepsilon_i^p)\) a value of \(\alpha_i\) is obtained as \(\alpha_i = \sigma_i - \sigma_i^0\), where \(\sigma_i^0\) is the user-defined size of the yield surface at the corresponding plastic strain for the isotropic hardening component or the initial yield stress if the isotropic hardening component is not defined.

In this work, however, the backstress function is derived from the distortional energy approach. As explained in the previous chapter, the stress-strain curve is discretized and the yield surface for each segment is obtained by adding the energy of its previous segment of the stress-strain curve. This work models anisotropic material behavior by generating ten snapshots (Figure 4.3) as uniaxial stress-strain functions for isotropic materials and thereby considering these functions as a collection in order to capture the anisotropic behavior. Since the centers of the yielding surfaces generate the backstress function for an isotropic material, ten backstress functions are obtained in this work and represented in the following figure:
Figure 7.4 Yield surface center distribution of each snapshot.

Each of these ten backstress functions is plotted in the stress-plastic strain plane and compared with the corresponding uniaxial stress-plastic strain function. The difference in the stress values for each plastic strain was small as Figure 7.5 through Figure 7.7 for snapshots j=5, j=7, j=9, respectively indicates, which validates the derivations.
Figure 7.5 Backstress and uniaxial stress for the snapshot $j=5$ (40% of $\sigma_x$).

Figure 7.6 Backstress and uniaxial stress for the snapshot $j=7$ (60% of $\sigma_x$).
Figure 7.7 Backstress and uniaxial stress for the snapshot j=9 (80% of $\sigma_x$).

After computing the backstress functions for each snapshot, the incremental evolution of the general backstress function using all snapshots is obtained:

$$d\alpha_{m,ij} = 2U_j e_{m,ij} d\varepsilon_{m,ij} + V_j d\varepsilon_{m,ij}$$

Equation 7.3

where $U_j$ and $V_j$ depend on the unloading state on any one snapshot. The 'x' and 'y' components of the centers ($\alpha_{x,ij}$, $\alpha_{y,ij}$) of the yield surfaces are distinguished by the subscript m. The post-yield strain is defined as $\varepsilon_{m,ij}$. The following figures show the distribution of $\alpha_{x,ij}$ and $\alpha_{y,ij}$ for all the snapshots, respectively.
Figure 7.8 Snapshot distribution of the backstress evolution $\alpha_{m,ij}$ projected into $\alpha_{x,ij}$ component.

Figure 7.9 Snapshot distribution of the backstress evolution $\alpha_{m,ij}$ projected into $\alpha_{y,ij}$ component.
The obtained backstress functions can be used to determine the stress-strain relationship of the anisotropic material with respect to the direction of loading. Each snapshot is related to one particular loading direction. Ten of them are considered as explained in Chapter 4. By knowing how material behaves during loading, unloading and reloading, structural designers can be able to accurately predict cyclic deformations under large earthquake events.
CHAPTER 8

CONCLUSIONS AND FUTURE WORK

In this work, a new general plasticity model for anisotropic material behavior for systems subjected to biaxial loading conditions has been proposed.

The widely accepted yield surface approach in the theory of plasticity has been used in order to determine the behavior of anisotropic materials in the post-yield stress-strain material range. The uniqueness of the model is the coupling effect of two uniaxial stress-strain functions, thus the ability to use the experimental results from uniaxial instead of biaxial tests. The coupling phenomenon is achieved by assuming the elliptical connection between the two uniaxial functions. Ten individual isotropic snapshots of yield surfaces are developed with that regard using the distortional strain energy density. Stress-strain function was discretized and for each strain segment distortional energy was separated into its hardening and flow portion. Performing this task, Young's modules and Poisson's ratios are different for each segment. Some Poisson's ratios have a negative value, which was also shown by Guo and Wheeler [20]. The obtained yield surface is of von Mises type. Each set of ellipses is translated and rotated so as to match the uniaxial yield stresses to each corresponding snapshot stress-strain relationship, which is a result of the assumed material anisotropy. The collection of all snapshots of yield surfaces models the overall anisotropic material behavior. Using the yield surfaces, a backstress
evolutionary function is computed and compared to the individual uniaxial snapshot stress-strain relationships. The results match very closely for all snapshots.

Using the proposed model, an accurate time-history response of a multi-storey frame under the earthquake event can be predicted. The control of those responses (deformations, velocities and accelerations) using magnetorheological (MR) dampers to meet specific performance objectives remains to be the future task and the final goal. MR dampers appear to be quite promising for seismic response reduction. They have the ability to dynamically modify their rheological properties which enables them to generate optimal earthquake resisting forces to meet specific performance demands in structures. The algorithm will be developed and written in Visual Basic and will have the capability of calculating optimal resisting forces generated by MR dampers, which will be integrated within the frame.
LIST OF REFERENCES


