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A Beurling Theorem for Noncommutative Hardy Spaces Associated with a Semifinite von Neumann Algebra with Various Norms

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A BEURLING THEOREM FOR NONCOMMUTATIVE HARDY SPACES ASSOCIATED WITH A SEMIFINITE VON NEUMANN ALGEBRA WITH VARIOUS NORMS

BY

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Dedication

To Nathaniel, Mom and Dad. Thanks for everything.

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Soli Deo Gloria.

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ABSTRACT

A BEURLING THEOREM FOR NONCOMMUTATIVE HARDY SPACES ASSOCIATED WITH A SEMIFINITE VON NEUMANN ALGEBRA WITH VARIOUS NORMS

by

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We prove Beurling-type theorems for H^∞-invariant spaces in relation to a semifinite von Neumann algebra $\mathcal{M}$ with a semifinite, faithful, normal tracial weight τ, using an extension of Arveson's non-commutative Hardy space H^∞. First we prove a Beurling-Blecher-Labuschagne theorem for H^∞-invariant subspaces of $L^p(\mathcal{M}, \tau)$ when $0 < p \leq \infty$. We also prove a Beurling-Chen-Hadwin-Shen theorem for H^∞-invariant subspaces of $L^\alpha(\mathcal{M}, \tau)$ where α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. For a crossed product of a von Neumann algebra $\mathcal{M}$ by an action β, $\mathcal{M} \rtimes_\beta \mathbb{Z}$, we are able to completely characterize all H^∞-invariant subspaces of $L^\alpha(\mathcal{M} \rtimes_\beta \mathbb{Z}, \tau)$ using our results. As an example, we completely characterize all H^∞-invariant subspaces of the Schatten p-class, $S_p(\mathcal{H})$ $(0 < p \leq \infty)$, where H^∞ is the lower triangular subalgebra of $B(\mathcal{H})$. We also characterize the non-commutative Hardy space H^∞-invariant subspaces in a Banach function space $\mathcal{I}(\tau)$ on a semifinite von Neumann algebra $\mathcal{M}$.

Chapter 1

Preliminaries

1.1 Hilbert spaces

We begin by considering a complex vector space $\mathcal{X}$ with a norm $\|\cdot\|$. We define the *norm topology* on $\mathcal{X}$ for an element $x_0 \in \mathcal{X}$ and $\epsilon > 0$ by a family of neighborhoods $V(x_0, \|\cdot\|, \epsilon) = \{x \in \mathcal{X} : \|x - x_0\| < \epsilon\}$.

A complex vector space $\mathcal{X}$ with a norm $\|\cdot\|$, denoted $(\mathcal{X}, \|\cdot\|)$, is called a *normed space*.

Definition 1.1.1. *A complex vector space $(\mathcal{X}, \|\cdot\|)$ which is complete with respect to the norm topology on $\mathcal{X}$ is called a* Banach space*.*

We consider a complex vector space $\mathcal{H}$.

Definition 1.1.2. *A mapping $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by $(x, y) \to \langle x, y \rangle$ is called an* inner product *on a complex vector space $\mathcal{H}$ if:*

(i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for every $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{H}$;

(ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for every $x, y \in \mathcal{H}$;

(iii) $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{H}$.

If additionally, we have

(iv) $\langle x, x \rangle = 0$ if and only if $x = 0$,

then we call $\langle \cdot, \cdot \rangle$ a definite inner product*.*

If we combine (i) and (ii), we get an additional characteristic of an inner product:

(v) $\langle z, \alpha x + \beta y\rangle = \overline{\alpha}\langle z, x\rangle + \overline{\beta}\langle z, y\rangle$.

We call the pairing $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ an *inner product space*. We then use the inner product to define a seminorm on the complex vector space $\mathcal{H}$.

Proposition 1.1.3. *Suppose that $\langle\cdot,\cdot\rangle$ is an inner product on a complex vector space $\mathcal{H}$. Then for every $x \in \mathcal{H}$, the equation $\|x\| = \langle x, x\rangle^{1/2}$ defines a seminorm $\|\cdot\|$ on $\mathcal{H}$. If, in particular, $\langle\cdot,\cdot\rangle$ is a definite inner product, then $\|\cdot\|$ is a norm on $\mathcal{H}$.*

Proof. By Definition 1.1.2, it is clear that $\|x\| \geq 0$ for every $x \in \mathcal{H}$. Also,

$$\begin{aligned} \|\alpha x\| &= \langle \alpha x, \alpha x\rangle^{1/2} && \text{(for every } \alpha \in \mathbb{C} \text{ and } x \in \mathcal{H}) \\ &= (\alpha\overline{\alpha}\langle x, x\rangle)^{1/2} \\ &= |\alpha|\|x\| \end{aligned}$$

Additionally,

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle + 2Re(\langle x, y\rangle) + \langle y, y\rangle \\ &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && \text{(by the Cauchy-Schwarz inequality)} \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

So the triangle inequality is satisfied, and $\|\cdot\|$ is a seminorm on $\mathcal{H}$.

If additionally, $\langle\cdot,\cdot\rangle$ is a definite inner product on $\mathcal{H}$, then $\|x\| = \langle x, x\rangle^{1/2} = 0$ if and only if $\langle x, x\rangle = 0$. Therefore, if $\langle\cdot,\cdot\rangle$ is a definite inner product, we have that $\|\cdot\|$ is a norm. □

Definition 1.1.4. *A complex vector space $\mathcal{H}$ is said to be a* pre-Hilbert space *if the norm, $\|\cdot\|$, can be obtained from a definite inner product on $\mathcal{H}$. Namely, $\|x\| = \langle x, x\rangle^{1/2}$ for every $x \in \mathcal{H}$.*

We may now define a Hilbert space.

Definition 1.1.5. *A pre-Hilbert space $\mathcal{H}$ is called a* Hilbert space *if $\mathcal{H}$ is complete with respect to the norm $\|\cdot\|$ determined by a definite inner product on $\mathcal{H}$.*

We now discuss several examples of Hilbert spaces.

Example 1.1.6. *Consider the space $\mathbb{C}^n$, $n \in \mathbb{N}$, consisting of n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n \in \mathbb{C}$. We let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be elements of $\mathbb{C}^n$. We use the standard inner product on $\mathbb{C}^n$, $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$, and the associated norm $\|x\| = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{1/2}$. It may be verified that $\mathbb{C}^n$ is a Hilbert space.*

Example 1.1.7. *Suppose $\mathcal{A}$ is a set. We define $l^2(\mathcal{A}) = \{f | f : \mathcal{A} \to \mathbb{C}; \sum_{a\in\mathcal{A}} |f(a)|^2 < \infty\}$. We can see that, given $x, y \in l^2(\mathcal{A})$, then $\sum_{a\in\mathcal{A}} x(a)\overline{y}(a)$ converges, as $|x(a)\overline{y}(a)| \leq 1/2(|x(a)|^2 + |y(a)|^2)$, and $\sum_{a\in\mathcal{A}}(|x(a)|^2 + |y(a)|^2) < \infty$. We can define a definite inner product $\langle x, y\rangle = \sum_{a\in\mathcal{A}} x(a)\overline{y}(a)$ on $l^2(\mathcal{A})$. Then the norm $\|x\| = (\sum_{a\in\mathcal{A}} |x(a)|^2)^{1/2}$ is determined by the definite inner product. Therefore, $l^2(\mathcal{A})$ is a Hilbert space.*

Example 1.1.8. *Let $l^2_{(0)}(\mathbb{N})$ be defined to be the set of all complex valued functions on $\mathbb{N}$ taking non-zero values at only finitely many points of $\mathbb{N}$. Then $l^2_{(0)}(\mathbb{N}) \subseteq l^2(\mathbb{N})$, so $l^2_{(0)}(\mathbb{N})$ has a definite inner product and norm, inherited from $l^2(\mathbb{N})$. However, $l^2_{(0)}(\mathbb{N})$ is not complete with respect to $\|\cdot\|$, and is therefore a pre-Hilbert space, but not a Hilbert space.*

1.1.1 The adjoint operation

Consider a linear operator $T : X \to Y$, where X and Y are normed spaces. We say that T is bounded if there exists a $c \in \mathbb{R}$ such that $\|Tx\| \leq c\|x\|$ for every $x \in \mathcal{X}$.

Proposition 1.1.9. *T is continuous if and only if T is bounded.*

We define the norm of the linear operator T by $\|T\| = \sup_{x\in X;\|x\|\leq 1}\{\|Tx\|\}$. Then $B(X, Y) = \{T : X \to Y \mid \|T\| < \infty\}$.

Theorem 1.1.10. *Suppose that $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces. Suppose that $T \in B(\mathcal{H}, \mathcal{K})$. Then there exists a unique linear operator $T^* \in B(\mathcal{K}, \mathcal{H})$ such that $\langle T^*x, y\rangle = \langle x, Ty\rangle$ for $x \in \mathcal{K}$ and $y \in \mathcal{H}$. Moreover,*

(i) $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$ *for* $a, b \in \mathbb{C}$ *and* $S, T \in B(\mathcal{H}, \mathcal{K})$;

(ii) $(RS)^* = S^*R^*$ *for* $S \in B(\mathcal{H}, \mathcal{K})$ *and* $R \in B(\mathcal{K}, \mathcal{L})$;

(iii) $(T^*)^* = T$ *for every* $T \in B(\mathcal{H}, \mathcal{K})$;

(iv) $\|T^*T\| = \|T\|^2$ *for every* $T \in B(\mathcal{H}, \mathcal{K})$;

(v) $\|T^*\| = \|T\|$ *for every* $T \in B(\mathcal{H}, \mathcal{K})$.

Proof. See Theorem 2.4.1 in [22] for a proof. $\square$

Definition 1.1.11. *Given $T \in B(\mathcal{H}, \mathcal{K})$, T^*, as defined in Theorem 1.1.10, is called the* adjoint of T.

Remark 1.1.12. *For a Hilbert space $\mathcal{H}$, we may define the set of bounded linear operators $T : \mathcal{H} \to \mathcal{H}$, denoted $B(\mathcal{H}, \mathcal{H}) = B(\mathcal{H})$. Then, as in Theorem 1.1.10, for any $T \in B(\mathcal{H})$ there exists $T^* \in B(\mathcal{H})$, the adjoint of T.*

We can then classify the bounded linear operators on $\mathcal{H}$.

Definition 1.1.13. *A bounded linear operator $T \in B(\mathcal{H})$ is said to be:*

(i) self-adjoint *if* $T^* = T$;

(ii) normal *if* $TT^* = T^*T$;

(iii) unitary *if* $TT^* = T^*T = 1$;

(iv) positive *if* $\langle Tx, x\rangle \geq 0$ *for every* $x \in \mathcal{H}$.

1.2 C^*-algebras

We say that $\mathcal{A}$ is a *Banach algebra* if $\mathcal{A}$ is a Banach space with norm $\|\cdot\|$, and $\mathcal{A}$ has a bi-continuous multiplication $(A, B) \to AB$ such that $\|AB\| \leq \|A\| \cdot \|B\|$ for every $A, B \in \mathcal{A}$. We say that $\mathcal{A}$ is a unital Banach algebra if $\mathcal{A}$ contains a unit element I such that $\|I\| = 1$.

Definition 1.2.1. *Suppose $\mathcal{A}$ is a Banach algebra. A mapping $* : \mathcal{A} \to \mathcal{A}$ taking $A \to A^*$ ($A \in \mathcal{A}$) is called an involution if the following conditions hold:*

(i) $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$ *for every $a, b \in \mathbb{C}$ and $S, T \in \mathcal{A}$;*

(ii) $(ST)^* = T^*S^*$ *for every $S, T \in \mathcal{A}$;*

(iii) $(T^*)^* = T$ *for every $T \in \mathcal{A}$.*

Definition 1.2.2. *A Banach algebra $\mathcal{A}$ with an involution satisfying:*

(iv) $\|TT^*\| = \|T\|^2$ *for every $T \in \mathcal{A}$*

is called a C^*-algebra.

Example 1.2.3. *Consider a Hilbert space $\mathcal{H}$. Recall that $B(\mathcal{H})$ is the set of all bounded linear operators from $\mathcal{H} \to \mathcal{H}$. Definition 1.1.11 defines the adjoint operator on $B(\mathcal{H})$ which satisfies the conditions given by Theorem 1.1.10. Therefore, the adjoint operator is an involution, and $B(\mathcal{H})$ is a C^*-algebra.*

1.2.1 Topologies on $B(\mathcal{H})$

Suppose that $\mathcal{H}$ is a Hilbert space. Recall that the C^*-algebra $B(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$.

Definition 1.2.4. *Suppose T_0 is an element of $B(\mathcal{H})$. The* strong operator topology *on $B(\mathcal{H})$ is given by the neighborhoods* $V(T_0; x_1, x_2, \ldots, x_m; \epsilon) = \{T \in B(\mathcal{H}) \mid \|(T - T_0)x_j\| < \epsilon$ *where* $j = 1, 2, \ldots, m;\ x_1, x_2, \ldots, x_m \in \mathcal{H};\ \epsilon > 0\}$.

Equivalently, a net $\{T_j\}$ in $B(\mathcal{H})$ converges to T_0 in the strong operator topology if and only if $\|(T_j - T_0)x\| \to 0$ for every $x \in \mathcal{H}$.

Definition 1.2.5. *Suppose T_0 is an element of $B(\mathcal{H})$. Define a linear functional $\omega_{x,y} : B(\mathcal{H}) \to \mathbb{C}$ by $\omega_{x,y}(A) = \langle Ax, y\rangle$ for $A \in B(\mathcal{H})$. The* weak operator topology *on $B(\mathcal{H})$ is given by the neighborhoods $\{T \in B(\mathcal{H}) \,|\, |\omega_{x,y}(T) - \omega_{x,y}(T_0)| < \epsilon\}$.*

Equivalently, a net $\{T_j\}$ in $B(\mathcal{H})$ conveges to T_0 in the weak operator topology if and only if $|\langle T_j x, y\rangle - \langle T_0 x, y\rangle| \to 0$ for every $x, y \in \mathcal{H}$.

Remark 1.2.6. *We have that $|\langle (T - T_0)x, y\rangle| < \epsilon$ for a given $\epsilon > 0$ when $\|(T - T_0)x\| < \frac{\epsilon}{1+\|y\|}$. Therefore, if a set is open in the weak operator topology, then it is open in strong operator topology. Hence, the weak operator topology is coarser than the strong operator topology.*

1.3 von Neumann algebras

Definition 1.3.1. *A von Neumann algebra is a C^*-algebra $\mathcal{M}$ acting on $\mathcal{H}$ which is weak operator topology closed and contains I.*

If the center of $\mathcal{M}$ is a subset of $\mathbb{C}I$, we say that $\mathcal{M}$ is a *factor*.

Example 1.3.2. *Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices with entries from the complex numbers, for $1 \leq n < \infty$. Then $M_n(\mathbb{C})$ is a von Neumann algebra.*

Example 1.3.3. *Consider the algebra of all bounded operators on a Hilbert space $\mathcal{H}$, which we denote by $B(\mathcal{H})$. It may be shown that $B(\mathcal{H})$ is a factor.*

1.3.1 Polar decompositions in von Neumann algebras

Definition 1.3.4. *An operator mapping a closed subspace $\mathcal{H}_1$ of a Hilbert space isometrically onto another closed subspace $\mathcal{H}_2$, which also annihilates the orthogonal compliment of $\mathcal{H}_1$ is called a* partial isometry*. The space $\mathcal{H}_1$ is called the* initial space *of the partial isometry and $\mathcal{H}_2$ is called* *the* final space. *The projections with ranges $\mathcal{H}_1$ and $\mathcal{H}_2$ are called the* initial *and* final projections, *respectively.*

Proposition 1.3.5. *If T is a bounded operator on a Hilbert space $\mathcal{H}$, then there exists a partial isometry V such that $T = V(T^*T)^{1/2} = (TT^*)^{1/2}V$. Also, if $T = WH$ where W is a partial isometry and H is positive, then $W = V$ and $H = (T^*T)^{1/2}$. We call such a decomposition the* polar decomposition of T.

Proof. It is easy to see that $\langle (T^*T)^{1/2}x, (T^*T)^{1/2}x\rangle = \langle (T^*T)x, x\rangle = \langle Tx, Tx\rangle$. Therefore, there exists a partial isometry V such that $T = V(T^*T)^{1/2}$. This implies that $T^* = (T^*T)^{1/2}V^*$, and $TT^* = VT^*TV^*$.

Hence $(TT^*)^{1/2} = V(T^*T)^{1/2}V^*$, and $T = V(T^*T)^{1/2} = V(T^*T)^{1/2}V^*V = (TT^*)^{1/2}V$.

It is clear that, if $T = WH$ where W is a partial isometry and H is positive, $W^*WH = H$. Thus, $T^*T = HW^*WH = H^2$. Hence, $(T^*T)^{1/2} = H$ and W must equal V. □

Proposition 1.3.6. *If T is a bounded operator in a von Neumann algebra M, and UH is the polar decomposition of T, then $U, H \in \mathcal{M}$.*

Proof. See Propositon 6.1.3 in [22]. □

1.3.2 Type decomposition of von Neumann algebras

Von Neumann algebras may be decomposed into five parts: type I_n, type I_∞, type II_1, type II_∞ and type III parts.

We need several definitions before we discuss the different types of von Neumann algebras.

Definition 1.3.7. *Suppose A is an operator in a von Neumann algebra $\mathcal{M}$. Then the* central carrier *of A is the projection $I - P$ such that P is the union of all projections P_a in the center of $\mathcal{M}$ which satisfy $P_aA = 0$.*

Definition 1.3.8. *Projections E and F in a von Neumann algebra $\mathcal{M}$ are* equivalent relative to $\mathcal{M}$ *if for some partial isometry V in $\mathcal{M}$, $V^*V = E$ and $VV^* = F$.*

Definition 1.3.9. *A projection E in a von Neumann algebra $\mathcal{M}$ is called an* abelian projection *if $E\mathcal{M}E$ is itself abelian.*

Definition 1.3.10. *Suppose E is a projection in a von Neumann algebra $\mathcal{M}$. If there is a projection E_0 such that E is equivalent to E_0, and $E_0 < E$, then E is called an* infinite projection *with respect to $\mathcal{M}$. If E is not infinite with respect to $\mathcal{M}$, then we say that E is a* finite projection*.*

Now, we may define the types of von Neumann algebras.

Definition 1.3.11. *A von Neumann algebra $\mathcal{M}$ is said to be of* type I *if it has an abelian projection with central carrier I. If I is the sum of n equivalent abelian projections ($n \in \mathbb{N}$), $\mathcal{M}$ is said to be of* type I_n*.*

If $\mathcal{M}$ has a finite projection with central carrier I, but no non-zero abelian projections, then $\mathcal{M}$ is said to be of type II*. If I is finite, then $\mathcal{M}$ is of* type II_1*, and if I is properly infinite $\mathcal{M}$ is of* type II_∞*.*

$\mathcal{M}$ is said to be of type III *if $\mathcal{M}$ has no non-zero finite projections.*

When $\mathcal{M}$ is a factor, the type definitions may be simplified.

Proposition 1.3.12. *Suppose $\mathcal{M}$ is a factor. Then $\mathcal{M}$ is either of type I_n, II_1, II_∞, or type III.*

The factor $\mathcal{M}$ is of type I is it has a minimal projection, and of type I_n if I can be written as the sum of n minimal projections ($n \in \mathbb{N}$).

If $\mathcal{M}$ has a finite projection but no minimal projection, then $\mathcal{M}$ is of type II. $\mathcal{M}$ is of type II_1 if I is finite, and type II_∞ if I is infinite.

If $\mathcal{M}$ has neither a non-zero finite projection nor a minimal projection, we say $\mathcal{M}$ is of type III.

Proof. See Corollary 6.3 in [22]. □

Proposition 1.3.13. *Suppose $\mathcal{M}$ is a type I_n factor. Then $\mathcal{M}$ is *-isomorphic to $B(\mathcal{H})$ where the dimension of the Hilbert space $\mathcal{H}$ is n.*

Proposition 1.3.14. *Suppose $\mathcal{M}$ is a countably decomposable type II_∞ von Neumann algebra. Then there exists a separable Hilbert space $\mathcal{H}$ such that $\mathcal{M} \cong B(\mathcal{H}) \otimes \mathcal{R}$ where $\mathcal{R}$ is a von Neumann algebra of type II_1.*

Example 1.3.15. *Suppose $\mathcal{A}$ is an abelian von Neumann algebra. We consider $M_n(\mathcal{A})$, the set of $n \times n$ matrices with entries from $\mathcal{A}$. $M_n(\mathcal{A})$ is a type I_n von Neumann algebra.*

Example 1.3.16. *Let $\mathcal{G}$ be a discrete infinite conjugacy class group. We have that $L(G)$ is a type II_1 factor, and for some separable Hilbert space $\mathcal{H}$, $B(\mathcal{H}) \otimes L(G)$ is a type II_∞ factor.*

Example 1.3.17. *Suppose that $\mathcal{M}$ is a type II_1 factor. Consider the $n \times n$ matrices with entries in $\mathcal{M}$, $M_n(\mathcal{M})$. Then when $1 \leq n < \infty$, $M_n(\mathcal{M})$ is a type II_1 factor. When $n = \infty$, $M_n(\mathcal{M})$ is a factor of type II_∞.*

It is a well known result of Murray and von Neumann in [29] that any von Neumann algebra $\mathcal{M}$ may be decomposed in the following way:

$$\mathcal{M} = \sum_{n \leq \dim \mathcal{H}} \mathcal{M}_{I_n} + \mathcal{M}_{I_\infty} + \mathcal{M}_{II_1} + \mathcal{M}_{II_\infty} + \mathcal{M}_{III}$$

where $\mathcal{M}_{I_n}$ is of type I_n, $\mathcal{M}_{I_\infty}$ is of type I_∞, $\mathcal{M}_{II_1}$ is of type II_1, $\mathcal{M}_{II_\infty}$ is of type II_∞ and $\mathcal{M}_{III}$ is of type III. Any of $\mathcal{M}_{I_n}$, $\mathcal{M}_{I_\infty}$, $\mathcal{M}_{II_1}$, $\mathcal{M}_{II_\infty}$, and $\mathcal{M}_{III}$ may be equal to zero.

1.3.3 Semifinite von Neumann algebras

When $\mathcal{M}_{III} = 0$ in Murray and von Neumann's result, we call $\mathcal{M}$ a *semifinite* von Neumann algebra. However, we will use an alternate definition of a semifinite von Neumann algebra.

Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{M}^+$ be the positive part of $\mathcal{M}$.

Definition 1.3.18. *A mapping $\tau : \mathcal{M}^+ \to [0, \infty]$ is a* tracial weight *on $\mathcal{M}$ if*

1. *$\tau(x + y) = \tau(x) + \tau(y)$ for $x, y \in \mathcal{M}^+$;*

2. *$\tau(ax) = a\tau(x)$ for $x \in \mathcal{M}^+$ and $a \in [0, \infty]$;*

3. $\tau(xx^*) = \tau(x^*x)$ *for every* $x \in \mathcal{M}$.

A tracial weight τ is called *normal* if $\tau : \mathcal{M}^+ \to \mathbb{C}$ is continuous with respect to the weak $*$-topology. τ is *faithful* if for every $a \in \mathcal{M}^+$, $\tau(a^*a) = 0$ implies $a = 0$. τ is said to be *finite* if $\tau(I) < \infty$, and *semifinite* if for any nonzero $x \in \mathcal{M}^+$, there is a nonzero $y \in \mathcal{M}^+$ such that $\tau(y) < \infty$ and $y \leq x$. A von Neumann algebra $\mathcal{M}$ is called a *semifinite* von Neumann algebra if a faithful, normal semifinite tracial weight τ exists.

1.3.4 The predual of a von Neumann algebra

A third topology on a von Neumann algebra is the weak *-topology, for which we need a predual space.

Definition 1.3.19. *Suppose that* $\mathcal{M}$ *is a von Neumann algebra over a Hilbert space* $\mathcal{H}$. *Denote by* $\mathcal{M}_\#$ *the linear space of linear functionals on* $\mathcal{M}$ *which are weak operator topology continuous on the unit ball of* $\mathcal{M}$. *The space* $\mathcal{M}_\#$ *is called the* predual *of* $\mathcal{M}$.

Definition 1.3.20. *Suppose* $\mathcal{M}$ *is a von Neumann algebra with predual* $\mathcal{M}_\#$. *The* weak *-topology *on* $\mathcal{M}$, $\phi(\mathcal{M}_\#, \mathcal{M})$, *is given by the neighborhoods* $\{\rho \in \mathcal{M}_\# : |\rho(x_j) - \rho_0(x_j)| < \epsilon \ (j = 1, 2, \ldots, m)\}$ *of* ρ_0 *where* $x_1, x_2, \ldots, x_m \in \mathcal{M}_\#$ *and* $\epsilon > 0$.

Equivalently, a net $\{\rho_\lambda\}_{\lambda \in \Lambda}$ conveges to ρ_0 in the weak *-topology if and only if $|\rho_\lambda(x) - \rho_0(x)| \to 0$. The weak *-topology on $\mathcal{M}$ is induced by the predual $\mathcal{M}_\#$.

The following lemma is known (see, for example Theorem 1.7.8 in [35]), but useful when dealing with the weak *-topology on a von Neumann algebra $\mathcal{M}$.

Lemma 1.3.21. *Let* $\mathcal{M}$ *be a von Neumann algebra. If* $\{e_\lambda\}_{\lambda \in \Lambda}$ *is a net of projections in* $\mathcal{M}$ *such that* $e_\lambda \to I$ *in the weak* $*$*-topology, then* $e_\lambda x \to x$, $xe_\lambda \to x$ *and* $e_\lambda x e_\lambda \to x$ *in weak* $*$*-topology for all* x *in* $\mathcal{M}$.

Lemma 1.3.22. *Let* $\mathcal{M}$ *be a von Neumann algebra with a semifinite, faithful, normal, tracial weight* τ. *Then the following are true.*

1. *There exists a family $\{e_j\}_{j\in J}$ of orthogonal projections in $\mathcal{M}$ such that (i) $\sum_j e_j$ converges to I in weak $*$-topology and (ii) $\tau(e_j) < \infty$ for each $j \in J$.*

2. *There exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{M}$ such that (i) $e_\lambda \to I$ in weak $*$-topology and (ii) $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$.*

Proof. It is not hard to see that (2) follows from (1). For the purpose of completeness, we sketch the proof of (1) here. Actually, we need only to show that every nonzero projection e in $\mathcal{M}$ contains a nonzero subprojection $\tilde{e}$ such that $\tau(\tilde{e}) < \infty$. Then the rest follows directly from Zorn's lemma.

Let e be a nonzero projection in $\mathcal{M}$. Since τ is semifinite, there is a $y \in \mathcal{M}^+$, $y \neq 0$ such that $\tau(y) < \infty$ and $y \leq f$. Therefore, there exist a positive number $\lambda > 0$ and a nonzero spectral projection $\tilde{e}$ of y in $\mathcal{M}$ such that $\lambda\tilde{e} \leq y$. Hence $\tilde{e}$ is a non-zero subprojection of e such that $\tau(\tilde{e}) < \infty$. The rest of the proof follows. $\square$

1.4 L^α spaces for a semifinite von Neumann algebra

Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ. Let $\mathcal{I} = span\{\mathcal{M}e\mathcal{M} : e = e^* = e^2 \in \mathcal{M}$ with $\tau(e) < \infty\}$ be the set of elementary operators in $\mathcal{M}$. (For more information, see the quasi-simple operators in Remark 2.3 of [42].) It may be shown that $\mathcal{I}$ is a two-sided ideal of $\mathcal{M}$.

We define a family of norms on $\mathcal{I}$.

Definition 1.4.1. *We call a norm $\alpha : \mathcal{I} \to [0,\infty)$ a* unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ *on $\mathcal{I}$ if it satisfies the following characteristics:*

1. *α is* unitarily invariant *if for all unitaries u, v in $\mathcal{M}$ and every x in $\mathcal{I}$, $\alpha(uxv) = \alpha(x)$;*

2. *α is* locally $\|\cdot\|_1$-dominating *if for every projection e in $\mathcal{M}$ with $\tau(e) < \infty$, there exists $0 < c(e) < \infty$ such that $\alpha(exe) \geq c(e)\|exe\|_1$ for every $x \in \mathcal{I}$;*

3. *α is* mutually continuous with respect to τ*; namely*

(a) *If $\{e_\lambda\}$ is an increasing net of projections in $\mathcal{I}$ such that $\tau(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$. Or, equivalently, if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $e_\lambda \to I$ in the weak* topology, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$.*

(b) *If $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$.*

Remark 1.4.2. *Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. We may define a mapping $\|\cdot\|_p : \mathcal{I} \to [0, \infty)$ for $0 < p < \infty$ by*

$$\|x\|_p = (\tau(|x|^p))^{1/p} \quad \textit{for every } x \in \mathcal{I}.$$

When $1 \le p < \infty$, it may be shown that $\|\cdot\|_p$ is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ.

Definition 1.4.3. *Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ, and suppose that $\mathcal{I} = span\{\mathcal{M}e\mathcal{M} : e = e^* = e^2 \in \mathcal{M}$ with $\tau(e) < \infty\}$ is the set of elementary operators in $\mathcal{M}$. Define $L^\alpha(\mathcal{M}, \tau)$ for a norm α on $\mathcal{I}$ to be the completion of $\mathcal{I}$ under α, namely*

$$L^\alpha(\mathcal{M}, \tau) = \overline{\mathcal{I}}^\alpha.$$

We denote $\overline{\mathcal{I}}^{\|\cdot\|_p}$ by $L^p(\mathcal{M}, \tau)$.

Notation 1.4.4. *If $S \subseteq L^\alpha(\mathcal{M}, \tau)$, then we denote the closure of S in $L^\alpha(\mathcal{M}, \tau)$ by $[S]_\alpha$.*

1.5 Arveson's non-commutative Hardy space

In this subsection, we will recall Arveson's definition of non-commutative Hardy spaces, and the expansion of Arveson's definition to $L^\alpha(\mathcal{M}, \tau)$. Assume that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Let $\mathcal{A} \subseteq \mathcal{M}$ be a weak*-closed unital subalgebra of $\mathcal{M}$, and let $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Assume that $\Phi : \mathcal{M} \to \mathcal{D}$ is faithful, normal conditional expectation from $\mathcal{M}$ onto $\mathcal{D}$.

Definition 1.5.1. *$\mathcal{A}$ is a called a semifinite subdiagonal subalgebra, or a semifinite non-commutative Hardy space, with respect to $(\mathcal{M}, \Phi)$ if*

1. *The restriction of τ on $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ is semifinite;*
2. *$\Phi(xy) = \Phi(x)\Phi(y)$ for every $x, y \in \mathcal{A}$;*
3. *$\mathcal{A} + \mathcal{A}^*$ is weak* dense in $\mathcal{M}$;*
4. *$\tau(\Phi(x)) = \tau(x)$ for every positive operator x in $\mathcal{M}$.*

In this case, $\mathcal{A}$ will also be denoted by H^∞. Furthermore, we denote $[\mathcal{A} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$, the closure in α-norm, by H^α.

Remark 1.5.2. *It was shown in [47], [13] and [24] that such a subalgebra H^∞ with respect to $(\mathcal{M}, \Phi)$ is maximal among semifinite subdiagonal subalgebras satisfying (1), (2), (3) and (4). From this fact, it follows that*

$$H^\infty = \{a \in \mathcal{M} \,:\, \Phi(xay) = 0,\ \forall\, x \in H^\infty, y \in H^\infty \cap \ker(\Phi)\}.$$

Remark 1.5.3. *Following notation from Definition 1.5.1, we know that the conditional expectation $\Phi : \mathcal{M} \to \mathcal{D}$ can be extended to a projection from $L^p(\mathcal{M}, \tau)$ onto $L^p(\mathcal{D}, \tau)$ with the norm $\|\cdot\|_p$ $(1 \le p < \infty)$ (see Proposition 2.3 in [47] or [2]). Such an extended projection will still be denoted by Φ. Moreover,*

$$\Phi(axb) = a\Phi(x)b, \qquad \forall\ a, b \in \mathcal{D},\ x \in L^p(\mathcal{M}, \tau)\ \ (1 \le p < \infty).$$

Notation 1.5.4. *We will let $H_0^\infty = \ker(\Phi) \cap H^\infty$, and $H_0^\alpha = \ker(\Phi) \cap H^\alpha$.*

The next result follows directly from Definition 1.5.1 and can be found in Lemma 3.1 of [2].

Lemma 1.5.5. *If e is a projection in $\mathcal{D} = H^\infty \cap (H^\infty)^*$ with $0 < \tau(e) < \infty$, then $eH^\infty e$ (denoted H_e^∞) is a finite subdiagonal subalgebra of $e\mathcal{M}e$ (denoted $\mathcal{M}_e$), and $[eH^\infty e]_\alpha = eH^\alpha e$, which we denote by H_e^α.*

We will need the following technical lemma in the later chapters.

Lemma 1.5.6. *Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ. Let H^∞ be a semifinite, subdiagonal subalgebra in $\mathcal{M}$ in the sense of Definition 1.5.1 (namely, the restriction of τ on $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is semifinite). Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mututally continuous norm with respect to τ.*

Then for every $x \in L^\alpha(\mathcal{M}, \tau)$ with $0 < p < \infty$ and for every $e \in \mathcal{D}$ with $0 < \tau(e) < \infty$, there exist an $h_1, h_3 \in eH^\infty e$ and an $h_2, h_4 \in eH^\alpha e$ such that:

(i) $h_1 h_2 = e = h_2 h_1$ and $h_3 h_4 = h_4 h_3 = e$

(ii) $h_1 ex$ and xeh_3 are in $\mathcal{M}$.

Proof. Let $ex = \sqrt{exx^*e}u = |x^*e|u$ be the polar decomposition of $(ex)^*$ in $L^\alpha(\mathcal{M}, \tau)$ where u is a partial isometry in $\mathcal{M}$ and $|x^*e|$ is a positive operator in $L^\alpha(\mathcal{M}, \tau)$. Note that $|x^*e|$ is in $eL^\alpha(\mathcal{M}, \tau)e = L^\alpha(\mathcal{M}_e, \tau)$. Since $0 < \tau(e) < \infty$, we know that $\mathcal{M}_e$ is a finite von Neumann algebra with a faithful, normal tracial state $\frac{1}{\tau(e)}\tau$. By Lemma 1.5.5, we have that H_e^∞ is a finite subdiagonal subalgebra of $\mathcal{M}_e$ with $[H_e^\infty]_\alpha = H_e^\alpha$.

We have that $|x^*e| \in L^\alpha(\mathcal{M}_e, \frac{1}{\tau(e)}\tau)$, and $0 < \tau(e) < \infty$. Then $w = (e + |x^*e|)^{-1}$ is an invertible operator in $\mathcal{M}_e$ with $w^{-1} \in L^\alpha(\mathcal{M}_e, \frac{1}{\tau(e)}\tau)$. We know that $\mathcal{M}_e$ is a finite von Neumann algebra with faithful, normal tracial state $\frac{1}{\tau(e)}\tau$, and α_e on $\mathcal{M}_e$ is a unitarily invariant, ϵ-$\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}_e$. Therefore, from Proposition 5.2 in [8], there exists a unitary v in $\mathcal{M}_e$, $h_1 \in H_e^\infty$, and $h_2 \in H_e^\alpha$ such that

(i) $h_1 h_2 = e = h_2 h_1$; and

(ii$_a$) $w = vh_1$.

By (ii$_a$), we get (ii$_b$) $h_1|x^*e| = v^*w|x^*e| = v^*(e + |x^*e|)^{-1}|x^*e| \in \mathcal{M}_e \subseteq \mathcal{M}$. Since u_1 is a partial isometry in $\mathcal{M}$, $h_1 ex = h_1|x^*e|u_1 \in \mathcal{M}$. Therefore, (ii) holds.

The proof for h_3 and h_4 is similar. $\square$

The following lemma is also useful.

Lemma 1.5.7. *Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ. Let H^∞ be a semifinite subdiagonal subalgebra with respect to $(\mathcal{M}, \Phi)$, where Φ is a faithful, normal conditional expectation from $\mathcal{M}$ onto $\mathcal{D} = H^\infty \cap (H^\infty)^*$.*

Then there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that such that

(i) $e_\lambda \to I$ in the weak $*$-topology of $\mathcal{M}$ and $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$.

(ii) We have, for every $x \in L^\alpha(\mathcal{M}, \tau)$ with $0 < p < \infty$,

$$\lim_\lambda \alpha(e_\lambda x - x) = 0; \;\; \lim_\lambda \alpha(x e_\lambda - x) = 0; \quad \textit{and} \quad \lim_\lambda \alpha(e_\lambda x e_\lambda - x) = 0.$$

Proof. We know that H^∞ is a semifinite subdiagonal subalgebra of $\mathcal{M}$, therefore the restriction of τ to $\mathcal{D}$ is semifinite. From Lemma 1.3.22, there exists a net of projections $\{e_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{D}$ such that $e_\lambda \to I$ in the weak* topology on $\mathcal{D}$, and $\tau(e_\lambda) < \infty$ for all $\lambda \in \Lambda$. Therefore,

$$\lim_\lambda |\tau(e_\lambda z - z)| = 0 \text{ for every } z \in L^1(\mathcal{D}, \tau).$$

Also, for each y in $L^1(\mathcal{M}, \tau)$, we have that

$$\lim_\lambda |\tau(e_\lambda y - y)| = \lim_\lambda |\tau(\Phi(e_\lambda y - y))| = \lim_\lambda |\tau(e_\lambda \Phi(y) - \Phi(y))| = 0.$$

Namely, $e_\lambda \to I$ in the weak* topology on $\mathcal{M}$, and $\tau(e_\lambda) < \infty$ for every $\lambda \in \Lambda$. (i) is satisfied.

Then from (i) and Definition 1.4.1, we may conclude that (ii) holds. Namely, for every $x \in L^\alpha(\mathcal{M}, \tau)$,

$$\lim_\lambda \alpha(e_\lambda x - x) = 0; \quad \lim_\lambda \alpha(x e_\lambda - x) = 0; \text{ and } \lim_\lambda \alpha(e_\lambda x e_\lambda - x) = 0.$$

Therefore, the lemma is proven. $\square$

1.6 Row sums of von Neumann algebras

Now we recall the following definition for the row sum of subspaces in $L^p(\mathcal{M}, \tau)$ for $0 < p \le \infty$ as follows.

Definition 1.6.1. *Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, normal faithful, tracial weight τ and $0 < p < \infty$. Let X be a closed subspace of $L^p(\mathcal{M}, \tau)$. Then X is called an internal row sum of closed subspaces $\{X_i\}_{i\in\mathcal{I}}$ of $L^p(\mathcal{M}, \tau)$, denoted by $X = \bigoplus_{i\in\mathcal{I}}^{row} X_i$, if*

1. *$X_j X_i^* = \{0\}$ for all distinct $i, j \in \mathcal{I}$; and*
2. *the linear span of $\{X_i : i \in \mathcal{I}\}$ is dense in X, i.e. $X = [span\{X_i : i \in \mathcal{I}\}]_p$. We will denote $span\{X_i : i \in \mathcal{I}\}$ by $\sum_{i\in\mathcal{I}} X_i$.*

Definition 1.6.2. *Let $\mathcal{M}$ be a von Neumann algebra. Let X be a weak $*$-closed subspace of $\mathcal{M}$. Then X is called an internal row sum of a family of weak*-closed subspaces $\{X_i\}_{i\in\mathcal{I}}$ of $\mathcal{M}$, denoted by $X = \bigoplus_{i\in\mathcal{I}}^{row} X_i$, if*

1. *$X_j X_i^* = \{0\}$ for all distinct $i, j \in \mathcal{I}$; and*
2. *the linear span of $\{X_i : i \in \mathcal{I}\}$ is weak*-dense in X, i.e. $X = \overline{span\{X_i : i \in \mathcal{I}\}}^{w*}$. We will denote $span\{X_i : i \in \mathcal{I}\}$ by $\sum_{i\in\mathcal{I}} X_i$.*

1.7 The Beurling theorem

In 1949, A. Beurling proved his classical theorem for invariant subspaces (see [5]). We recall his Theorem now. Suppose that $\mathbb{T}$ is the unit circle, and let μ be the measure on $\mathbb{T}$ such that $d\mu = \frac{1}{2\pi} d\theta$. We let $L^\infty(\mathbb{T}, \mu)$ be the commutative von Neumann algebra on $\mathbb{T}$ and define $L^2(\mathbb{T}, \mu)$ to be the closure of $L^\infty(\mathbb{T}, \mu)$ under the $\|\cdot\|_2$-norm. Let $H^2 = \overline{span(\{z^n : n \geq 0\}}^{\|\cdot\|_2}$, as subspace of $L^2(\mathbb{T}, \mu)$, and let $H^\infty = H^2 \cap L^\infty(\mathbb{T}, \mu)$. Define $M_\phi(f) = \phi(f)$ for every $f \in L^2(\mathbb{T}, \mu)$. It may be shown that $L^\infty(\mathbb{T}, \mu)$ has a representation onto $\mathcal{B}(L^2(\mathbb{T}, \mu))$ via the map $\phi \to M_\phi$. Hence, $L^\infty(\mathbb{T}, \mu)$ and H^∞ may be assumed to act naturally on $L^2(\mathbb{T}, \mu)$ via left or right multiplication.

The classical Beurling theorem (from [5]) may be stated as follows: *If $\mathcal{W}$ is a nonzero, closed, H^∞-left-invariant subspace of H^2 (equivalently, $z\mathcal{W} \subseteq \mathcal{W}$ for all $z \in H^\infty$), then $\mathcal{W} = \phi H^2$ for some ϕ in H^∞ such that $|\phi| = 1$ a.e.(μ).*

When we define $L^p(\mathbb{T},\mu) = \overline{L^\infty(\mathbb{T},\mu)}^{\|\cdot\|_p}$, and $H^p = \{f \in L^p(\mathbb{T},\mu) \,:\, \int_{\mathbb{T}} f(e^{i\theta})e^{in\theta}d\mu(\theta) \forall n \in \mathbb{N}\}$ for $1 \leq p < \infty$, the Beurling theorem has been extended to the H^∞-left-invariant subspace on the Hardy spaces H^p for $1 \leq p \leq \infty$. (For example, see [7], [17], [18], [19], [21], [44], and others).

The Beurling theorem has been extended in other ways as well. Our goal is to extend it in several new cases.

Chapter 2

Invariant subspaces of L^p-Spaces

Let $\mathcal{H}$ be an infinite dimensional Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$, and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Let $\tau = Tr$ be the usual trace on $B(\mathcal{H})$, i.e.

$$\tau(x) = \sum_{m\in\mathbb{Z}} \langle xe_m, e_m\rangle, \qquad \text{for all positive } x \text{ in } B(\mathcal{H}).$$

For each $0 < p < \infty$, the Schatten p-class $S^p(\mathcal{H})$ consists all the elements x in $B(\mathcal{H})$ such that $\tau(|x|^p) < \infty$. It is well-known (for example, see [9]) that $S^p(\mathcal{H})$ is a complete metric space (a Banach space when $p \geq 1$ and a Hilbert space when $p = 2$). Moreover, $S^p(\mathcal{H})$ is a two sided ideal of $B(\mathcal{H})$.

Let

$$\mathcal{A} = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0, \quad \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$. In this chapter, we are interested in answering the following question, which is implicitly asked by McAsey, Muhly and Saito in Example 2.6 of [28].

Problem 2.0.1. *Given a closed subspace $\mathcal{K}$ of the Schatten p-class $S^p(\mathcal{H})$ where $0 < p < \infty$, such that $\mathcal{K}$ satisfies $\mathcal{A}\mathcal{K} \subseteq \mathcal{K}$, how can we characterize the subspace $\mathcal{K}$?*

The answer to Problem 2.0.1 is closely related to our generalization of the classical Beurling theorem for a Hardy space.

One extension of the Beurling theorem comes from the work of D. Blecher and L. Labuschagne in [6]. We recall the construction of $L^p(\mathcal{M}, \tau)$. Let $\mathcal{M}$ be a semifinite von Neumann algebra, and let τ be a faithful, normal tracial weight on $\mathcal{M}$ (when $\tau(I) < \infty$, $\mathcal{M}$ is finite). Let $\mathcal{I}$ be the set of elementary operators in $\mathcal{M}$ (when $\mathcal{M}$ is finite, $\mathcal{I} = \mathcal{M}$). Then define a mapping from $\mathcal{I}$ to $[0, \infty)$ by $\|x\|_p = (\tau(|x|^p))^{1/p}$ for every $x \in \mathcal{I}$, and where $|x| = \sqrt{x^*x}$. It is nontrivial to prove that when $1 \leq p < \infty$, $\|\cdot\|_p$ defines a norm on $\mathcal{I}$, which we call the *L_p-norm*. We may then define $L^p(\mathcal{M}, \tau) = \overline{\mathcal{I}}^{\|\cdot\|_p}$. We let $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$, and this space acts naturally on $L^p(\mathcal{M}, \tau)$ by left (or right) multiplication.

We then recall the definition of the semifinite extension of Arveson's non-commutative Hardy space from [1]. If $\mathcal{M}$ is a von Neumann algebra, with faithful, normal, semifinite tracial weight τ, let $\mathcal{A} \subseteq \mathcal{M}$ be a weak* closed unital subalgebra. Then let $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ be a von Neumann subalgebra of $\mathcal{M}$, such that $\tau|_{\mathcal{D}}$ is semifinite. There exists $\Phi : \mathcal{M} \to \mathcal{D}$, a faithful, normal conditional expectation, which can be extended to $\Phi : L^1(\mathcal{M}, \tau) \to L^1(\mathcal{D}, \tau)$. Then $\mathcal{A}$ is called a non-commutative Hardy space if (1) $\Phi(xy) = \Phi(x)\Phi(y)$ for every $x, y \in \mathcal{A}$; (2) $\mathcal{A} + \mathcal{A}^*$ is weak* dense in $\mathcal{M}$; (3) $\tau(\Phi(x)) = \tau(x)$ for every positive element $x \in \mathcal{M}$.

Blecher and Labuschagne proved the following theorem for finite von Neumann algebras in [6]. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, tracial, normal state τ, and H^∞ be a maximal subdiagonal subalgebra of $\mathcal{M}$ with $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Suppose that $\mathcal{K}$ is a closed H^∞-right-invariant subspace of $L^p(\mathcal{M}, \tau)$, for some $1 \leq p \leq \infty$. (For $p = \infty$ it is assumed that $\mathcal{K}$ is weak* closed.) Then $\mathcal{K}$ may be written as a column L^p-sum $\mathcal{K} = \mathcal{Z} \oplus^{col} (\oplus_i^{col} u_i H^p)$, where $\mathcal{Z}$ is a closed (indeed, weak*closed if $p = \infty$) subspace of $L^p(\mathcal{M}, \tau)$ such that $\mathcal{Z} = [\mathcal{Z} H_0^\infty]_p$, and where u_i are partial isometries in $\mathcal{M} \cap \mathcal{K}$ satisfying certain conditions.* (For more details, see [6].) Here $\oplus_i^{col} u_i H^p$ and $\mathcal{Z} = [\mathcal{Z} H_0^\infty]_p$ are of type 1, and type 2 respectively (also see [6] for definitions of invariant subspaces of different types).

Examples of finite von Neumann algebras include the spaces $M_n(\mathbb{C})$ of all $n \times n$ matrices with complex entries when $1 \leq n < \infty$. However, if $\mathcal{H}$ is an infinite dimensional separable Hilbert space and we view $B(\mathcal{H})$ as $M_\infty(\mathbb{C})$, the set of all (bounded) $\infty \times \infty$ matrices with complex entries, then $B(\mathcal{H})$ is a semifinite von Neumann algebra, and no longer satisfies the hypothesis of the Beurling-Blecher-Labuschagne theorem.

In this paper, we therefore consider a version of Blecher and Labuschagne's Beurling theorem for semifinite von Neumann algebras. We seek to characterize H^∞-invariant spaces of $L^p(\mathcal{M},\tau)$ spaces. Adapting Blecher and Labuschagne's theorem to the semifinite case, we prove the following results:

Theorem 2.3.5. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$ (see Definition 1.5.1). Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K} \subseteq \mathcal{M}$ is weak $*$-closed subspace such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then there exist a weak closed subspace Y of $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that:*

(i) $u_\lambda Y^* = 0$ *for all* $\lambda \in \Lambda$.

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$.

(iii) $Y = \overline{H_0^\infty Y}^{w^*}$.

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\infty u_\lambda)$

Here $\oplus^{row}$ is the row sum of subspaces defined in Definition 1.6.2.

Theorem 2.3.6. *Let $1 \leq p < \infty$. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$ (see Definition 1.5.1). Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M},\tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then there exist a closed subspace Y of $L^p(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that:

(i) $u_\lambda Y^* = 0$ *for all* $\lambda \in \Lambda$.

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$.

(iii) $Y = [H_0^\infty Y]_p$.

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda)$

Here $\oplus^{row}$ is the row sum of subspaces defined in Definition 1.6.1.

However, many of the methods used by Blecher and Labuschagne do not apply directly when $\mathcal{M}$ is a semifinite von Neumann algebra. Thus, we prove a density theorem for semifinite von Neumann algebras through a series of lemmas and propositions.

Proposition 2.3.1. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let $1 \le p < \infty$. Assume that $\mathcal{K}$ is a closed subspace in $L^p(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then the following statements are true.*

(i) $\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M},\tau)$.

(ii) $\mathcal{K} = [\mathcal{K} \cap \mathcal{M}]_p$.

Proposition 2.3.2. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Assume that $\mathcal{K} \subseteq \mathcal{M}$ is weak*-closed subspace such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then*

$$\mathcal{K} = \overline{[\mathcal{K} \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w^*}, \quad \forall\ 1 \le p < \infty.$$

Lemma 2.3.3. *If u is a partial isometry in $\mathcal{M}$ such that $uu^* \in \mathcal{D}$, then*

(i) $[(H^\infty u) \cap L^p(\mathcal{M},\tau)]_p = H^p u$ *for all* $1 \le p < \infty$, *and*

(ii) $H^\infty u = \overline{H^p u \cap \mathcal{M}}^{w^*}$ *for all* $1 \le p < \infty$.

Proposition 2.3.4. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Assume that $S \subseteq \mathcal{M}$ is a subspace such that $H^\infty S \subseteq S$. Then*

$$[S \cap L^p(\mathcal{M},\tau)]_p = [\overline{S}^{w^*} \cap L^p(\mathcal{M},\tau)]_p, \quad \forall\ 1 \le p < \infty.$$

Subsequently, we are able to prove a noncommutative Beurling-Blecher-Labuschagne theorem for the semifinite case when $0 < p < 1$.

Theorem 2.4.4. *Let $0 < p < 1$. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$ (see Definition 1.5.1). Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then there exist a closed subspace Y of $L^p(\mathcal{M}, \tau)$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M}$ such that:

(i) $u_\lambda Y^* = 0$ *for all* $\lambda \in \Lambda$.

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$.

(iii) $Y = [H_0^\infty Y]_p$.

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda \in \Lambda}^{row} H^p u_\lambda)$

Here $\oplus^{row}$ is the row sum of subspaces defined in Definition 1.6.1.

Here, we use similar methods to our proof for $1 \leq p < \infty$, including proving a similar density theorem (see Proposition 2.4.1, Proposition 2.4.2).

We also prove a corollary for the case when $0 < p < \infty$.

Corollary 2.4.5. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ.*

(i) *Let $0 < p < \infty$. If $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M}, \tau)$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$, then there exists a projection $q \in \mathcal{M}$ such that $\mathcal{K} = L^p(\mathcal{M}, \tau)q$.*

(ii) *If $\mathcal{K}$ is a weak*-closed subspace of $\mathcal{M}$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$, then there exists a projection $q \in \mathcal{M}$ such that $\mathcal{K} = \mathcal{M}q$.*

2.1 L^p-spaces of semifinite von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ. We let

$$\mathcal{I} = span\{\mathcal{M}e\mathcal{M} : e = e^* = e^2 \in \mathcal{M} \text{ with } \tau(e) < \infty\}$$

be the set of elementary operators in $\mathcal{M}$ (see quasi-simple operators in Remark 2.3 in [42]). Then $\mathcal{I}$ is a two-sided ideal of $\mathcal{M}$. We recall the construction of the space $L^p(\mathcal{M},\tau)$.

For each $0 < p < \infty$, we define a mapping $\|\cdot\|_p : \mathcal{I} \to [0,\infty)$ as follows

$$\|x\|_p = (\tau(|x|)^p)^{\frac{1}{p}} \qquad \text{for every } x \in \mathcal{I}.$$

It is a well-known fact that $\|\cdot\|_p$ is a norm on $\mathcal{I}$ for $1 \le p < \infty$, and a p-norm on $\mathcal{I}$ for $0 < p < 1$. (see Theorem 4.9 in [14])

Recall the following from Defintion 1.4.3. We define $L^p(\mathcal{M},\tau)$, for $0 < p < \infty$, to be the completion of $\mathcal{I}$ under $\|\cdot\|_p$, i.e.

$$L^p(\mathcal{M},\tau) = \overline{\mathcal{I}}^{\|\cdot\|_p}.$$

As usual, we let $L^\infty(\mathcal{M},\tau)$ be $\mathcal{M}$.

Notation 2.1.1. *If S is a subset of $L^p(\mathcal{M},\tau)$ with $0 < p < \infty$, we will denote by $[S]_p$ the closure of S in $L^p(\mathcal{M},\tau)$. If S is a subset of $\mathcal{M}$, we will denote by $\overline{S}^{w*}$ the closure of S in $\mathcal{M}$ under the weak $*$-topology.*

The following two lemmas are well known.

Lemma 2.1.2. *Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ. The following are true.*

1. *(Hölder's Inequality) For $0 < p, q, r \le \infty$ with $1/p + 1/q = 1/r$, we have $xy \in L^r(\mathcal{M},\tau)$ and*

$$\|xy\|_r \le \|x\|_p\|y\|_q \qquad \textit{for all } x \in L^p(\mathcal{M},\tau) \textit{ and } y \in L^q(\mathcal{M},\tau).$$

2. *For each $0 < r \le \infty$, we have $\|axb\|_r \le \|a\|\|x\|_r\|b\|$ for $x \in L^r(\mathcal{M},\tau)$ and $a, b \in \mathcal{M}$. Therefore, $L^r(\mathcal{M},\tau)$ is an $\mathcal{M}$ bi-module for each $0 < r \le \infty$.*

3. *(Duality) For any $1 \le p < \infty$ and $1 < q \le \infty$ with $1/p + 1/q = 1$, we have*

$$(L^p(\mathcal{M},\tau))^\sharp = L^q(\mathcal{M},\tau) \qquad (\textit{isometrically}),$$

where the duality between $L^p(\mathcal{M},\tau)$ and $L^q(\mathcal{M},\tau)$ is given by $\langle x,y\rangle = \tau(xy)$. Thus, $L^1(\mathcal{M},\tau)$ is the predual of $\mathcal{M}$.

Proof. See [14]. □

We have the following as a consequence of Lemma 1.5.7

Lemma 2.1.3. *Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ and $0 < p < \infty$. If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a net of projections in $\mathcal{M}$ such that such that $e_\lambda \to I$ in the weak $*$-topology, then for every $x \in L^p(\mathcal{M},\tau)$*

$$\lim_\lambda \|e_\lambda x - x\|_p = 0; \quad \lim_\lambda \|x e_\lambda - x\|_p = 0; \quad \text{and} \quad \lim_\lambda \|e_\lambda x e_\lambda - x\|_p = 0.$$

2.2 Beurling-Blecher-Labuschagne theorem for semifinite Hardy spaces, p=2

In this section, we will prove a Beurling-Blecher-Labuschagne type theorem for semifinite noncommutative Hardy spaces.

Theorem 2.2.1. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a weak*-closed subalgebra of $\mathcal{M}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$ be a von Neumann subalgebra of $\mathcal{M}$, and $\Phi : \mathcal{M} \to \mathcal{D}$ be a faithful normal condition expectation.*

Assume that H^∞ is a semifinite subdiagonal subalgebra with respect to $(\mathcal{M},\Phi)$ (see Definition 1.5.1). Let $\mathcal{K}$ be a closed subspace of $L^2(\mathcal{M},\tau)$ satisfying $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then there exist a closed subspace Y of $L^2(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$, satisfying

(i) $u_\lambda Y^* = 0$ *for all* $\lambda \in \Lambda$.

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda,\mu \in \Lambda$ *with* $\lambda \neq \mu$.

(iii) $Y = [H_0^\infty Y]_2$, *where* $H_0^\infty = H^\infty \cap \ker(\Phi)$.

(iv) $\mathcal{K} = Y \oplus \left(\oplus_{\lambda\in\Lambda} H^2 u_\lambda\right)$

The proof of this result uses a similar idea as the one in [6] for finite von Neumann algebras. We will modify the argument in [6] to prove the preceding result for the case of semifinite von Neumann algebras. First, we present a series of technical lemmas.

2.2.1 Some lemmas

Following the notation above, we let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$ be a von Neumann subalgebra of $\mathcal{M}$ and $\Phi : \mathcal{M} \to \mathcal{D}$ a faithful normal conditional expectation. From Remark 1.5.3, we know that Φ can be extended to a positive contraction from $L^p(\mathcal{M}, \tau)$ onto $L^p(\mathcal{D}, \tau)$ for each $1 \le p < \infty$, such that

$$\Phi(axb) = a\Phi(x)b, \qquad \forall\ a, b \in \mathcal{D},\ x \in L^p(\mathcal{M}, \tau),\ 1 \le p < \infty.$$

We find the following observation useful. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. If x in $L^1(\mathcal{M}, \tau)$ satisfies

$$\tau(xz) = 0 \qquad \text{for all } z \in H^\infty + (H^\infty)^*,$$

then $x = 0$. This follows from the weak*-density of $H^\infty + (H^\infty)^*$.

Lemma 2.2.2. *Let $\mathcal{K}$ be a closed subspace of $L^2(\mathcal{M}, \tau)$ satisfying $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Let*

$$X = \mathcal{K} \ominus [H_0^\infty \mathcal{K}]_2 \subseteq \mathcal{K} \subseteq L^2(\mathcal{M}, \tau).$$

Then the following are true.

(i) $XX^* \subseteq L^1(\mathcal{D}, \tau)$.

(ii) *X is a left $\mathcal{D}$-module, i.e. for every $d \in \mathcal{D}$ and $x \in X$, we have $dx \in X$.*

(iii) *Let x be an element in X and $x = hu$ where u^*h is the polar decomposition of x^* in $L^2(\mathcal{M}, \tau)$, where u is a partial isometry in $\mathcal{M}$ and $h = |x^*| \in L^2(\mathcal{M}, \tau)$. Then*

(a) $h \in L^2(\mathcal{D}, \tau)$ *and* $uu^* \in \mathcal{D}$;

(b) $[\mathcal{D}x]_2 = L^2(\mathcal{D}, \tau)u$;

(c) $[H^\infty x]_2 = H^2 u$. *In particular,* $H^2 u \subseteq X$.

(iv) *There exists a family* $\{u_\lambda\}_{\lambda \in \Lambda}$ *of partial isometries in* $\mathcal{M}$ *such that*

(a) $X = \oplus_{\lambda \in \Lambda} H^2 u_\lambda$;

(b) $u_\lambda u_\lambda^*$ *is a projection in* $\mathcal{D}$; *and*

(c) $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$.

Proof. (i): Proving (i) is equivalent to showing that for every $x, y \in X$, $yx^* \in L^1(\mathcal{D}, \tau)$.

Assume that $x, y \in X \subseteq L^2(\mathcal{M}, \tau)$. Thus $yx^* \in L^1(\mathcal{M}, \tau)$. Recall $\Phi : L^1(\mathcal{M}, \tau) \to L^1(\mathcal{D}, \tau)$ is a positive contraction such that

$$\Phi(d_1 a d_2) = d_1 \Phi(a) d_2, \qquad \forall\ d_1, d_2 \in \mathcal{D} \ \text{ and } \ a \in L^1(\mathcal{M}, \tau),$$

and thus

$$\Phi(da) = d\Phi(a), \qquad \forall\ d \in L^1(\mathcal{D}, \tau) \text{ and } \ a \in \mathcal{M}. \tag{2.1}$$

Thus, to prove that $yx^* \in L^1(\mathcal{D}, \tau)$, it is enough to show that $yx^* - \Phi(yx^*) = 0$. By the observation preceeding the lemma and the fact that $yx^* - \Phi(yx^*) \in L^1(\mathcal{M}, \tau)$, we need only to prove that

$$\tau([yx^* - \Phi(yx^*)]z) = 0 \quad \text{for every } z \in H^\infty + (H^\infty)^*.$$

We will proceed with the proof according to the cases (1) $z \in H_0^\infty$, (2) $z \in \mathcal{D}$, and (3) $z \in (H_0^\infty)^*$. Case (1): Let $z \in H_0^\infty$. Then

$$\begin{aligned}
\tau([yx^* - \Phi(yx^*)]z) &= \tau(yx^* z) - \tau(\Phi(yx^*)z) \\
&= \tau(yx^* z) - \tau(\Phi(\Phi(yx^*)z)) && (\Phi \text{ is trace preserving}) \\
&= \tau(zyx^*) - \tau(\Phi(yx^*)\Phi(z)) && (\text{by Equation 2.1}) \\
&= 0 && (\text{as } x, y \text{ are in } X \text{ and } z \text{ is in } H_0^\infty)
\end{aligned}$$

Case (2): Let $z \in \mathcal{D}$. Then

$$\begin{aligned}
\tau([yx^* - \Phi(yx^*)]z) &= \tau(\Phi([yx^* - \Phi(yx^*)]z)) && (\Phi \text{ is trace preserving}) \\
&= \tau([\Phi(yx^*) - \Phi(yx^*)]z) \\
&= 0. && (\text{as } x, y \text{ are in } X \text{ and } z \text{ is in } H_0^\infty)
\end{aligned}$$

Case (3): Let $z \in (H_0^\infty)^*$. Then

$$\begin{aligned}
\tau([(yx^*) - \Phi(yx^*)]z) &= \tau(yx^*z) - \tau(\Phi(yx^*)z) \\
&= \tau(y(z^*x)^*) - \tau(\Phi(\Phi(yx^*)z)) && (\Phi \text{ is trace preserving}) \\
&= \tau(y(z^*x)^*) - \tau(\Phi(yx^*)\Phi(z)) && (\text{by equation 2.1}) \\
&= 0 && (\text{as } x, y \text{ are in } X \text{ and } z \text{ is in } H_0^\infty)
\end{aligned}$$

This ends the proof of part (i).

(ii): Let $d \in \mathcal{D}$ and $x \in X \subseteq \mathcal{K}$. Since $H^\infty \mathcal{K} \subseteq \mathcal{K}$, we have $dx \in \mathcal{K}$. Now, for $h_0 \in H_0^\infty$ and $k \in K$,

$$\tau(h_0 k(dx)^*) = \tau(h_0 k x^* d^*) = \tau(d^* h_0 k x^*) = 0,$$

as $d^* h_0 \in H_0^\infty$, and $x \in X = \mathcal{K} \ominus [H_0^\infty \mathcal{K}]_2$. Hence $dx \perp [H_0^\infty K]_2$. Thus $dx \in X$ and X is a left $\mathcal{D}$-module.

(iii): Assume x is an element in X. Let $x = hu$ where u^*h is the polar decomposition of x^* in $L^2(\mathcal{M}, \tau)$, where u is a partial isometry in $\mathcal{M}$ and $h = |x^*| \in L^2(\mathcal{M}, \tau)$. From the result in (i), we know that h is in $L^2(\mathcal{D}, \tau)$. Therefore uu^*, as the range projection of h, is in $\mathcal{D}$. This shows that (a) is true.

From (a), it follows that $[L^2(\mathcal{D}, \tau)uu^*]_2 = L^2(\mathcal{D}, \tau)(uu^*)$. Observe that uu^* is the range projection of h. Therefore, we have $[\mathcal{D}h]_2 = L^2(\mathcal{D}, \tau)(uu^*)$, whence

$$[\mathcal{D}h]_2 u = L^2(\mathcal{D}, \tau)(uu^*u) = L^2(\mathcal{D}, \tau)u. \tag{2.2}$$

We claim that

$$[\mathcal{D}x]_2 = [\mathcal{D}h]_2 u.$$

In order to prove this claim, we observe that for any sequence $\{d_n\}$ in $\mathcal{D}$, d_n converges in $\|\cdot\|_2$-norm to some element b in $[\mathcal{D}x]_2$ if and only if $d_n x u^* = d_n h$ is $\|\cdot\|_2$-norm convergent to bu^* in $[\mathcal{D}x]_2$. Uniqueness of limits ensures that the reverse part of the implication holds.

From this observation and equation (2.2), we conclude that

$$[\mathcal{D}x]_2 = [\mathcal{D}h]_2 u = L^2(\mathcal{D}, \tau)u.$$

This ends the proof of part (b). The proof of (c) is similar to (b).

(iv) We may assume that $X \neq 0$. From the result in (iii) and Zorn's lemma, we may assume that there exists a maximal family $\{u_\lambda\}_{\lambda\in\Lambda}$ of nonzero partial isometries in $\mathcal{M}$ with respect to which

(a$_1$) $H^2 u_\lambda \subseteq X$ for each $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^*$ is a projection in $\mathcal{D}$; and

(c) $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$.

We will show that

(a) $X = \oplus_{\lambda\in\Lambda} H^2 u_\lambda$.

In fact, from (a$_1$), we know that each $H^2 u_\lambda \subseteq X$. Combining with (c), we conclude that $\{H^2 u_\lambda\}_{\lambda\in\Lambda}$ is a family of orthogonal subspaces of X, whence $\oplus_{\lambda\in\Lambda} H^2 u_\lambda$ is a subspace of X.

Now assume that $X \ominus (\oplus_{\lambda\in\Lambda} H^2 u_\lambda)$ is not equal to 0. Pick a nonzero x in $X \ominus (\oplus_{\lambda\in\Lambda} H^2 u_\lambda)$ and assume that $x = hu$ is the polar decomposition of x^* in $L^2(\mathcal{M}, \tau)$, where u is a nonzero partial isometry in $\mathcal{M}$ and $h = |x^*| \in L^2(\mathcal{M}, \tau)$. It follows from the result proved in (iii) that $H^2 u \subseteq X$ and uu^* is in $\mathcal{D}$.

By Lemma 1.5.7, there exists a net $\{e_j\}_{j\in J}$ of projections in $\mathcal{D}$ such that such that $e_j \to I$ in the weak $*$-topology and $\tau(e_j) < \infty$ for each $j \in J$.

Let $j \in J$. Then by the choice of x, we get that $H^2 u_\lambda$ and x are orthogonal. So,

$$\tau(d e_j u_\lambda x^*) = 0, \qquad \forall\ d \in \mathcal{D}.$$

From (i), $e_j u_\lambda x^*$ is in $L^1(\mathcal{D}, \tau)$. By the observation preceeding Lemma 2.2.2 we conclude that $e_j u_\lambda x^* = 0$ for each $j \in J$.

As $u_\lambda x^* \in L^2(\mathcal{M}, \tau)$, $\lim_j \|e_j u_\lambda x^* - u_\lambda x^*\|_2 = 0$ by Lemma 2.1.3. Thus we have that $u_\lambda x^* = u_\lambda u^* h = 0$. The fact that the initial projection of u^* is the range projection of h induces that $u_\lambda u^* = 0$. Therefore, u is a nonzero partial isometry in $\mathcal{M}$ such that $H^2 u \subseteq X$, $uu^* \in \mathcal{D}$, and $u_\lambda u^* = 0$ for each $\lambda \in \Lambda$. This contradicts the assumption that the family $\{u_\lambda\}_{\lambda \in \Lambda}$ is maximal with respect to (a$_1$), (b) and (c). Therefore, $X = \oplus_{\lambda \in \Lambda} H^2 u_\lambda$. This concludes the proof of part (iv). $\square$

Lemma 2.2.3. *Let $\mathcal{K}$ be a closed subspace of $L^2(\mathcal{M}, \tau)$ satisfying $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Let*

$$X = \mathcal{K} \ominus [H_0^\infty \mathcal{K}]_2 \quad \text{and} \quad Y = \mathcal{K} \ominus [H^\infty X]_2.$$

Then the following are true.

(i) $YX^* = 0$, *or equivalently* $XY^* = 0$.

(ii) $Y = [H_0^\infty Y]_2$

Proof. (i) We will show that $yx^* = 0$ for every $y \in Y$ and $x \in X$.

Note that $Y \subseteq \mathcal{K} \subseteq L^2(\mathcal{M}, \tau)$ and $X \subseteq \mathcal{K} \subseteq L^2(\mathcal{M}, \tau)$. We have that $YX^* \subseteq L^1(\mathcal{M}, \tau)$. Assume $y \in Y$ and $x \in X$. Then by the observation preceeding Lemma 2.2.2, it suffices to show that

$$\tau(yx^* z) = 0 \qquad \text{for every } z \in H^\infty + (H^\infty)^*.$$

We will proceed with the proof according to the cases (1) $z \in H_0^\infty$, (2) $z \in \mathcal{D}$, and (3) $z \in (H_0^\infty)^*$.
Case (1): Let $z \in H_0^\infty$. Then

$$\tau(yx^* z) = \tau(zyx^*) = 0,$$

since $x \in X$, $zy \in H_0^\infty K$, and $X \perp [H_0^\infty K]_2$.

Case (2): Let $z \in \mathcal{D}$. Then

$$\tau(yx^* z) = \tau(y(z^* x)^*) = 0,$$

as $y \in Y$, $z^*x \in H^\infty X$, and $Y \perp H^\infty X$.

Case (3): Let $z \in (H_0^\infty)^*$. Then

$$\tau(yx^*z) = \tau(y(z^*x)^*) = 0,$$

as $y \in Y$, and $z^*x \in H_0^\infty X$.

Therefore, $YX^* = 0$, which ends the proof of (i).

(ii) From part (i), we know that $YX^* = 0$, whence $H_0^\infty YX^* = 0$. Recall $Y = \mathcal{K} \ominus [H^\infty X]_2$. It follows that $[H_0^\infty Y]_2 \subseteq Y$. Let $Z = Y \ominus [H_0^\infty Y]_2 = 0$. To prove (ii), it suffices to show that $ZZ^* = 0$. Because $Z \subseteq Y$, we have that $Z \perp [H^\infty X]_2$, whence $Z \perp [H_0^\infty(Y \oplus [H^\infty X]_2)]_2$. This implies that $Z \perp [H_0^\infty \mathcal{K}]_2$. Note that $X = \mathcal{K} \ominus [H_0^\infty \mathcal{K}]_2$. We conclude that $Z \subseteq X$. Note that $YX^* = 0$. Since $Z \subseteq X$ and $Z \subseteq Y$, we have that $ZZ^* \subseteq YX^* = 0$. This ends the proof of (ii). $\square$

2.2.2 Proof of Theorem 2.2.1

We are ready to prove the main result in this section.

Proof. Recall that $\mathcal{K}$ is a closed subspace of $L^2(\mathcal{M}, \tau)$ satisfying $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Let

$$X = \mathcal{K} \ominus [H_0^\infty \mathcal{K}]_2 \quad \text{and} \quad Y = \mathcal{K} \ominus [H^\infty X]_2.$$

By Lemma 2.2.2, there exists a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M}$ such that

$$X = \oplus_{\lambda \in \Lambda} H^2 u_\lambda;$$

and

$$u_\lambda u_\lambda^* \text{ is a projection in } \mathcal{D}, \text{ and } u_\lambda u_\mu^* = 0 \text{ for all } \lambda, \mu \in \Lambda \text{ with } \lambda \neq \mu. \tag{ii}$$

By the choice of Y, we have

$$\mathcal{K} = Y \oplus X = Y \oplus (\oplus_{\lambda \in \Lambda} H^2 u_\lambda). \tag{iv}$$

Moreover, from Lemma 2.2.3, we know that

$$e_i u_\lambda Y^* = 0$$

for all $\lambda \in \Lambda$, and a net of projections $\{e_i\}$ such that $\tau(e_i) < \infty$, and $e_i \to I$ in the weak*-topology. Therefore

$$u_\lambda Y^* = 0. \tag{i}$$

Also,

$$Y = [H_0^\infty Y]_2. \tag{iii}$$

This ends the proof of Theorem 2.2.1. □

2.3 Beurling-Blecher-Labuschagne theorem for semifinite Hardy spaces, $1 \leq p \leq \infty$

2.3.1 Dense subspaces

Proposition 2.3.1. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let $1 \leq p < \infty$. Assume that $\mathcal{K}$ is a closed subspace in $L^p(\mathcal{M}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then the following statements are true.*

(i) $\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M}, \tau)$.

(ii) $\mathcal{K} = [\mathcal{K} \cap \mathcal{M}]_p$.

Proof. (i) It is easily observed that

$$\mathcal{K} \cap \mathcal{M} \subseteq \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M}, \tau).$$

We will show that

$$\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M}, \tau).$$

Assume, to the contrary, that $\mathcal{K} \cap \mathcal{M} \subsetneqq \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M}, \tau)$. Then there exists an $x \in \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M}, \tau)$ such that $x \notin \mathcal{K} \cap \mathcal{M}$. Clearly, $\overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \subseteq \mathcal{M}$, so $x \notin \mathcal{K} \cap \mathcal{M}$ implies that

$x \notin \mathcal{K}$. By the Hahn-Banach theorem, there exists $\varphi \in L^p(\mathcal{M},\tau)^\# = L^q(\mathcal{M},\tau)$ (where $\frac{1}{p}+\frac{1}{q}=1$) such that $\varphi(x) \neq 0$ and $\varphi(y) = 0$ for every $y \in \mathcal{K}$. Pick ξ in $L^q(\mathcal{M},\tau)$ such that $\tau(\xi x) \neq 0$, and $\tau(\mathcal{K}) = 0$.

By Lemma 1.5.7, there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$, and $\lim_\lambda \tau(e_\lambda x\xi) = \tau(x\xi)$. So, we can always assume that there exists a projection e in $\mathcal{D}$ with $0 < \tau(e) < \infty$ such that $\tau(ex\xi) \neq 0$ and $\tau(ey\xi) = 0$ for every $y \in \mathcal{K}$ (as $\mathcal{K}$ is H^∞-invariant and $e \in \mathcal{D} \subseteq H^\infty$).

Now we claim that $\xi e \in L^1(\mathcal{M},\tau)$, as $||\xi e||_1 \leq ||\xi||_q||e||_p < \infty$.

Since $x \in \overline{\mathcal{K}\cap\mathcal{M}}^{w^*} \cap L^p(\mathcal{M},\tau)$, we can find a net $\{y_i\}_{i\in I}$ in $\mathcal{K}\cap\mathcal{M}$, such that $y_i \to x$ in the weak*-topology. Combining this with the fact that $\xi e \in L^1(\mathcal{M},\tau)$, we have

$$\tau(ex\xi) = \tau(x\xi e) = \lim_i \tau(y_i\xi e) = \lim_i \tau(ey_i\xi) = 0,$$

which contradicts the fact that $\tau(ex\xi) \neq 0$. This ends the proof of part (i).

(ii) Suppose, to the contrary, that $[\mathcal{K}\cap\mathcal{M}]_p \subsetneqq \mathcal{K}$. Then there exists an $x \in \mathcal{K}$ such that $x \notin [\mathcal{K}\cap\mathcal{M}]_p$. Then, by the Hahn-Banach theorem, there exists $\varphi \in L^p(\mathcal{M},\tau)^\# = L^q(\mathcal{M},\tau)$ (where $\frac{1}{p}+\frac{1}{q}=1$), such that $\varphi(x) \neq 0$ and $\varphi(y) = 0$ for every $y \in [\mathcal{K}\cap\mathcal{M}]_p$. This occurs if and only if there exists a $\xi \in L^q(\mathcal{M},\tau)$ such that $\tau(x\xi) \neq 0$ and $\tau(y\xi) = 0$ for every $y \in [\mathcal{K}\cap\mathcal{M}]_p$.

By Lemma 1.5.7, there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$, and $\lim_\lambda \tau(e_\lambda x\xi) = \tau(x\xi)$. So, we may always assume that there exists a projection e in $\mathcal{D}$ with $0 < \tau(e) < \infty$ such that

(a) $\tau(ex\xi) \neq 0$; and

(b) $\tau(ey\xi) = 0$ for every $y \in \mathcal{K}\cap\mathcal{M}$ (as $\mathcal{K}$ is H^∞-invariant, and $e \in \mathcal{D} \subseteq H^\infty$).

Since $x \in L^p(\mathcal{M},\tau)$ and e is a projection in $\mathcal{D}$ such that $\tau(e) < \infty$, by Lemma 1.5.6, there exists a $h_1 \in eH^\infty e$, and $h_2 \in eH^pe$ such that $h_1ex \in \mathcal{M}$ and $h_1h_2 = h_2h_1 = e$. From the fact that

$h_2 \in eH^pe$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $eH^\infty e$ such that $\lim_{n\to\infty}\|a_n - h_2\|_p = 0$. Therefore

$$\begin{aligned}\lim_{n\to\infty}|\tau(a_nh_1ex\xi) - \tau(ex\xi)| &= \lim_{n\to\infty}|\tau(a_nh_1ex\xi) - \tau(h_2h_1ex\xi)|\\ &\leq \lim_{n\to\infty}\|a_n - h_2\|_p\|h_1ex\|\|\xi\|_q\\ &= 0.\end{aligned}$$

On the other hand, since a_n, h_1 and e are in H^∞ and $h_1ex \in \mathcal{M}$, we know that $a_nh_1ex \in \mathcal{K}\cap\mathcal{M}$. From assumption (b), it follows that $\tau(a_nh_1ex\xi) = 0$ for all $n \geq 1$. Therefore $\tau(ex\xi) = 0$, which contradicts the assumption (a) that $\tau(x\xi e) \neq 0$. This ends the proof of part (ii). $\square$

Proposition 2.3.2. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Assume that $\mathcal{K} \subseteq \mathcal{M}$ is a weak*-closed subspace such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then*

$$\mathcal{K} = \overline{[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w^*}, \quad \forall\ 1 \leq p < \infty.$$

Proof. First, we show that

$$\mathcal{K} \subseteq \overline{[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w*}.$$

Let x be an element in $\mathcal{K} \subseteq \mathcal{M}$. By Lemma 1.5.7, there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that such that $e_\lambda \to I$ in the weak* topology and $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$. By Lemma 1.3.21, $e_\lambda x \to x$ in the weak* topology. To show that $x \in \overline{[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w*}$, it suffices to show that $e_\lambda x \in \overline{[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w*}$ for each $\lambda \in \Lambda$.

Since $\mathcal{K} \subseteq \mathcal{M}$ is left H^∞-invariant and $x \in \mathcal{K}$, we have $e_\lambda x \in \mathcal{K}$. Moreover, $||e_\lambda x||_p \leq ||e_\lambda||_p||x||_\infty < \infty$, so $e_\lambda x \in L^p(\mathcal{M},\tau)$. It follows that $e_\lambda x \in \mathcal{K}\cap L^p(\mathcal{M},\tau)$ for each $\lambda \in \Lambda$. As $e_\lambda x \to x$ in the weak* topology, $x \in \overline{[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w*}$. We obtain $\mathcal{K} \subseteq \overline{[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w*}$.

Next, we will show that

$$\overline{[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w*} \subseteq \mathcal{K}.$$

Since $\mathcal{K}$ is weak*-closed, it suffices to show that

$$[\mathcal{K}\cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M} \subseteq \mathcal{K}.$$

Assume, to the contrary, that x is an element in $[\mathcal{K} \cap L^p(\mathcal{M}, \tau)]_p \cap \mathcal{M}$, but $x \notin \mathcal{K}$. Thus, by the Hahn-Banach theorem, there exists a weak* continuous linear functional φ on $\mathcal{M}$ such that $\varphi(x) \neq 0$ and $\varphi(y) = 0$ for every $y \in \mathcal{K}$. Equivalently, there exists a $\xi \in L^1(\mathcal{M}, \tau)$ such that

(a) $\tau(x\xi) \neq 0$; and

(b) $\tau(y\xi) = 0$ for every $y \in \mathcal{K}$.

By Lemma 1.5.7, there exists a net $\{e_\lambda\}_{\lambda \in \Lambda}$ of projections in $\mathcal{D}$ such that $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$ and $\lim_\lambda \tau(e_\lambda x \xi) = \tau(x\xi)$. So we may always assume that there exists a projection e in $\mathcal{D}$ with $0 < \tau(e) < \infty$ such that

(a_1) $\tau(ex\xi) \neq 0$; and

(b_1) $\tau(ey\xi) = 0$ for every $y \in \mathcal{K}$ (as $\mathcal{K}$ is H^∞-invariant and $e \in \mathcal{D} \subseteq H^\infty$).

We claim there exists a $z = ze \in \mathcal{M}e$ such that

(a_2) $\tau(xz) \neq 0$; and

(b_2) $\tau(yz) = 0$ for every $y \in \mathcal{K}$.

Observe that ξ is in $L^1(\mathcal{M}, \tau)$, and e is a projection in $\mathcal{D}$ such that $\tau(e) < \infty$. From Lemma 1.5.6, there exist $h_3 \in eH^\infty e$ and $h_4 \in eH^1 e$ such that $\xi e h_3 \in e\mathcal{M}e$ and $h_3 h_4 = e$. Thus there exists a sequence $\{k_n\}_{n \in \mathbb{N}}$ of elements in $eH^\infty e$ such that $\lim_{n\to\infty} \|k_n - h_4\|_1 = 0$. It follows that

$$\begin{aligned}\lim_{n\to\infty} |\tau(ex\xi) - \tau(x\xi e h_3 k_n)| &= \lim_{n\to\infty} |\tau(x\xi e h_3 h_4) - \tau(x\xi e h_3 k_n)| \\ &\leq \lim_{n\to\infty} \|x\| \|\xi e h_3\| \|h_4 - k_n\|_1 = 0.\end{aligned}$$

Combining this with (a_1), we know that there exists an $N \in \mathbb{N}$ such that $\tau(x\xi e h_3 k_N) \neq 0$. Let $z = (\xi e h_3) k_N$ be in $\mathcal{M}$. Then $z = ze \in \mathcal{M}e$ satisfies

(a_2) $\tau(xz) = \tau(x\xi e h_3 k_N) \neq 0$; and

(b_2) $\tau(yz) = \tau(y\xi e h_3 k_N) = \tau((e h_3 k_N) y \xi) = 0$ for every $y \in \mathcal{K}$.

Note that $x \in [\mathcal{K} \cap L^p(\mathcal{M}, \tau)]_p \cap \mathcal{M}$. There exists a sequence $\{x_n\}_{n\in\mathcal{N}}$ in $\mathcal{K} \cap L^p(\mathcal{M}, \tau)$ such that $\lim_{n\to\infty} \|x_n - x\|_p = 0$. Thus we have

$$|\tau(xz - x_n z)| = |\tau((x - x_n)ze)| \leq \|x_n - x\|_p \|z\| \|e\|_q \to 0, \tag{2.3}$$

where q satisfies $1/p + 1/q = 1$. On the other hand, since $\{x_n\}_{n\in\mathcal{N}}$ is in $\mathcal{K} \cap L^p(\mathcal{M}, \tau)$, by ($b_2$) we have

$$\tau(x_n z) = 0, \qquad \forall\ n \in \mathbb{N}.$$

Combining with inequality (2.3), we have

$$\tau(xz) = 0.$$

This contradicts the assumption in (a_2) that $\tau(xz) \neq 0$. Therefore,

$$\overline{[\mathcal{K} \cap L^p(\mathcal{M}, \tau)]_p \cap \mathcal{M}}^{w*} \subseteq \mathcal{K}.$$

Hence

$$\mathcal{K} = \overline{[\mathcal{K} \cap L^p(\mathcal{M}, \tau)]_p \cap \mathcal{M}}^{w*}.$$

□

Lemma 2.3.3. *If u is a partial isometry in $\mathcal{M}$ such that $uu^* \in \mathcal{D}$, then*

(i) $[(H^\infty u) \cap L^p(\mathcal{M}, \tau)]_p = H^p u$ *for all* $1 \leq p < \infty$*, and*

(ii) $H^\infty u = \overline{H^p u \cap \mathcal{M}}^{w^*}$ *for all* $1 \leq p < \infty$.

Proof. (i) can be verified directly. (ii) follows from Proposition 2.3.2 and (i). □

Proposition 2.3.4. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ and let H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Assume that $S \subseteq \mathcal{M}$ is a subspace such that $H^\infty S \subseteq S$. Then*

$$[S \cap L^p(\mathcal{M}, \tau)]_p = [\overline{S}^{w^*} \cap L^p(\mathcal{M}, \tau)]_p, \quad \forall\ 1 \leq p < \infty.$$

Proof. It suffices to show that

$$\overline{S}^{w^*} \cap L^p(\mathcal{M},\tau) \subseteq [S \cap L^p(\mathcal{M},\tau)]_p.$$

Let $x \in \overline{S}^{w^*} \cap L^p(\mathcal{M},\tau)$. By Lemma 1.5.7, there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that $e_\lambda \to I$ in the weak* topology and $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$. By Lemma 2.1.3, $\lim_\lambda \|e_\lambda x - x\|_p = 0$. To show that $x \in [S \cap L^p(\mathcal{M},\tau)]_p$, it is enough to show that $e_\lambda x \in [S \cap L^p(\mathcal{M},\tau)]_p$ for each $\lambda \in \Lambda$.

By Proposition 2.3.1, we have

$$[S \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M} = \overline{[S \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M},\tau).$$

Since $x \in \overline{S}^{w^*} \cap L^p(\mathcal{M},\tau)$, there exists a net $\{x_j\}_{j\in J}$ in S such that $x_j \to x$ in the weak* topology. By Lemma 1.3.21, $e_\lambda x_j \to e_\lambda x$ in the weak* topology for each λ. Note that $\|e_\lambda x_j\|_p \le \|e_\lambda\|_p \|x_j\|$ and $H^\infty S \subseteq S$. We therefore know that $e_\lambda x_j \in S \cap L^p(\mathcal{M},\tau)$. So $e_\lambda x$ is in $\overline{[S \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w^*}$. It is trivial to see that $e_\lambda x \in L^p(\mathcal{M},\tau)$. Hence,

$$e_\lambda x \in \overline{[S \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w^*} \cap L^p(\mathcal{M},\tau) = [S \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}.$$

So

$$x \in [S \cap L^p(\mathcal{M},\tau)]_p.$$

Thus

$$\overline{S}^{w^*} \cap L^p(\mathcal{M},\tau) \subseteq [S \cap L^p(\mathcal{M},\tau)]_p.$$

Hence

$$[S \cap L^p(\mathcal{M},\tau)]_p = [\overline{S}^{w^*} \cap L^p(\mathcal{M},\tau)]_p, \quad \forall\, 1 \le p < \infty.$$

□

Theorem 2.3.5. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K} \subseteq \mathcal{M}$ is weak*-closed subspace such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then there exist a weak closed subspace Y of $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that:*

(i) $u_\lambda Y^* = 0$ *for all* $\lambda \in \Lambda$.

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$.

(iii) $Y = \overline{H_0^\infty Y}^{w^*}$.

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^\infty u_\lambda)$

Here $\oplus^{row}$ *is the row sum of subspaces defined in Definition 1.6.2.*

Proof. Let $\mathcal{K}_1 = [\mathcal{K} \cap L^2(\mathcal{M}, \tau)]_2$. Then $\mathcal{K}_1$ is a closed subspace of $L^2(\mathcal{M}, \tau)$ such that $H^\infty \mathcal{K}_1 \subseteq \mathcal{K}_1$. By Theorem 2.2.1, there exist a closed subspace Y_1 of $L^2(\mathcal{M}, \tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$, satisfying

(a) $u_\lambda Y_1^* = 0$ for all $\lambda \in \Lambda$.

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$.

(c) $Y_1 = [H_0^\infty Y_1]_2$, where $H_0^\infty = H^\infty \cap \ker(\Phi)$.

(d) $\mathcal{K}_1 = Y_1 \oplus \left(\oplus_{\lambda\in\Lambda} H^2 u_\lambda\right)$

Let

$$Y = \overline{Y_1 \cap \mathcal{M}}^{w^*}.$$

(i) We show that (i) is satisfied. In fact, from (a) and Lemma 1.3.21, we have

$$u_\lambda Y^* = 0 \text{ for all } \lambda \in \Lambda. \tag{2.4}$$

(ii) follows directly from (b).

(iii) We claim that

$$Y = \overline{H_0^\infty Y}^{w^*}.$$

In fact, we need only to show that $Y \subseteq \overline{H_0^\infty Y}^{w^*}$. By Proposition 2.3.1 and the definition of Y, we have

$$Y_1 = [Y_1 \cap \mathcal{M}]_2 = [\overline{Y_1 \cap \mathcal{M}}^{w*} \cap L^2(\mathcal{M}, \tau)]_2 = [Y \cap L^2(\mathcal{M}, \tau)]_2.$$

So

$$\begin{aligned} H_0^\infty Y_1 &= H_0^\infty [Y \cap L^2(\mathcal{M},\tau)]_2 \subseteq [(H_0^\infty Y) \cap L^2(\mathcal{M},\tau)]_2 \\ &\subseteq [\overline{H_0^\infty Y}^{w^*} \cap L^2(\mathcal{M},\tau)]_2. \end{aligned}$$

Thus, from (c), we have

$$Y_1 = [H_0^\infty Y_1]_2 \subseteq [\overline{H_0^\infty Y}^{w^*} \cap L^2(\mathcal{M},\tau)]_2. \tag{2.5}$$

Now, we are able to conclude that

$$\begin{aligned} Y &= \overline{Y_1 \cap \mathcal{M}}^{w^*} && \text{(by definition of } Y) \\ &\subseteq \overline{[\overline{H_0^\infty Y}^{w^*} \cap L^2(\mathcal{M},\tau)]_2 \cap \mathcal{M}}^{w^*} && \text{(by (2.5))} \\ &= \overline{H_0^\infty Y}^{w^*}. && \text{(by Proposition 2.3.2)} \end{aligned}$$

Thus

$$Y = \overline{H_0^\infty Y}^{w^*}. \tag{2.6}$$

(iv) We show that

$$\overline{Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*} = \mathcal{K}.$$

By Proposition 2.3.2, it suffices to show that

$$\overline{Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*} = \overline{[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap \mathcal{M}}^{w^*}.$$

First, we have that $Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda \subseteq \overline{[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap \mathcal{M}}^{w^*}$. In fact, $Y = \overline{Y_1 \cap \mathcal{M}}^{w^*}$ and $Y_1 \subseteq [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2$, so $Y \subseteq \overline{[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap \mathcal{M}}^{w^*}$. Moreover, for each $\lambda \in \Lambda$, by Lemma 2.3.3, we have $H^\infty u_\lambda = \overline{H^2 u_\lambda \cap \mathcal{M}}^{w^*} \subseteq \overline{[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap \mathcal{M}}^{w^*}$. So

$$Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda \subseteq \overline{[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap \mathcal{M}}^{w^*}.$$

Thus

$$\overline{Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*} \subseteq \overline{[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap \mathcal{M}}^{w^*} = \mathcal{K}. \tag{2.7}$$

Next, define $X = \overline{Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*}$. We want to show that

$$\overline{[\mathcal{K} \cap L^2(\mathcal{M}, \tau)]_2 \cap \mathcal{M}}^{w^*} \subseteq X.$$

Notice X is weak*-closed and $H^\infty X \subseteq X$. By Proposition 2.3.2,

$$X = \overline{[X \cap L^2(\mathcal{M}, \tau)]_2 \cap \mathcal{M}}^{w^*}.$$

Therefore we need only to show that $[\mathcal{K} \cap L^2(\mathcal{M}, \tau)]_2 \subseteq [X \cap L^2(\mathcal{M}, \tau)]_2$. Or, equivalently, we may show Y_1 and $\{H^2 u_\lambda\}_{\lambda\in\Lambda}$ are in $[X \cap L^2(\mathcal{M}, \tau)]_2$. By Proposition 2.3.1, we have

$$Y_1 = [Y_1 \cap \mathcal{M}]_2 = [\overline{Y_1 \cap \mathcal{M}}^{w*} \cap L^2(\mathcal{M}, \tau)]_2 = [Y \cap L^2(\mathcal{M}, \tau)]_2.$$

Thus

$$Y_1 \subseteq [X \cap L^2(\mathcal{M}, \tau)]_2. \tag{2.8}$$

By Lemma 2.3.3,

$$H^2 u_\lambda = [H^\infty u_\lambda \cap L^2(\mathcal{M}, \tau)]_2 \subseteq [X \cap L^2(\mathcal{M}, \tau)]_2 \quad \text{for each } \lambda \in \Lambda. \tag{2.9}$$

Hence, from (2.8) and (2.9), we get $[\mathcal{K} \cap L^2(\mathcal{M}, \tau)]_2 \subseteq [X \cap L^2(\mathcal{M}, \tau)]_2$ and

$$\mathcal{K} = \overline{[\mathcal{K} \cap L^2(\mathcal{M}, \tau)]_2 \cap \mathcal{M}}^{w^*} \subseteq \overline{Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*}. \tag{2.10}$$

Now, combining (2.7) and (2.10), we have

$$\mathcal{K} = \overline{Y + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\infty u_\lambda), \tag{2.11}$$

by Definition 1.6.2.

By (2.11), (2.4), (b) and (2.6), we know that Y and $\{u_\lambda\}_{\lambda\in\Lambda}$ have the desired properties. $\square$

Next, we use our result for $p = \infty$ and the density theorem to prove the case when $1 \leq p < \infty$.

Theorem 2.3.6. *Let $1 \leq p < \infty$. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then there exist a closed subspace Y of $L^p(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that:

(i) $u_\lambda Y^* = 0$ *for all* $\lambda \in \Lambda$.

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$.

(iii) $Y = [H_0^\infty Y]_p$.

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda)$

Here $\oplus^{row}$ *is the row sum of subspaces defined in Definition 1.6.2.*

Proof. Let $\mathcal{K}_1 = \overline{\mathcal{K}\cap\mathcal{M}}^{w^*}$. Then $\mathcal{K}_1$ is a weak*-closed subspace of $\mathcal{M}$ such that $H^\infty\mathcal{K}_1 \subseteq \mathcal{K}_1$. By Theorem 2.3.5, there exist a weak*-closed subspace Y_1 of $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$, satisfying

(a) $u_\lambda Y_1^* = 0$ for all $\lambda \in \Lambda$.

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$.

(c) $Y_1 = \overline{H_0^\infty Y_1}^{w^*}$.

(d) $\mathcal{K}_1 = Y_1 \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\infty u_\lambda)$

Let

$$Y = [Y_1 \cap L^p(\mathcal{M},\tau)]_p.$$

(i) From (a), the definition of Y and Lemma 2.1.2, we can conclude that

$$u_\lambda Y^* = 0 \text{ for all } \lambda \in \Lambda. \tag{2.12}$$

(ii) follows directly from (b).

(iii) We want to show that $Y = [H_0^\infty Y]_p$. In fact, we have

$$\begin{aligned}
Y &= [Y_1 \cap L^p(\mathcal{M},\tau)]_p && \text{(by definition of } Y\text{)}\\
&= [\overline{H_0^\infty Y_1}^{w^*} \cap L^p(\mathcal{M},\tau)]_p && \text{(by (c))}\\
&= [(H_0^\infty Y_1) \cap L^p(\mathcal{M},\tau)]_p && \text{(by Proposition 2.3.4)}\\
&= [\left(H_0^\infty \overline{[Y_1 \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}}^{w^*}\right) \cap L^p(\mathcal{M},\tau)]_p && \text{(by Proposition 2.3.2)}\\
&\subseteq [\overline{H_0^\infty\left([Y_1 \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}\right)}^{w^*} \cap L^p(\mathcal{M},\tau)]_p && \text{(by Lemma 1.3.21)}\\
&= [(H_0^\infty\left([Y_1 \cap L^p(\mathcal{M},\tau)]_p \cap \mathcal{M}\right)) \cap L^p(\mathcal{M},\tau)]_p && \text{(by Proposition 2.3.4)}\\
&= [(H_0^\infty\left(Y \cap \mathcal{M}\right)) \cap L^p(\mathcal{M},\tau)]_p && \text{(by definition of } Y\text{)}\\
&\subseteq [H_0^\infty Y]_p \subseteq Y, && (2.13)
\end{aligned}$$

(iv) There is only left to show that

$$\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^p u_\lambda).$$

By the definition of Y, we have

$$Y = [Y_1 \cap L^p(\mathcal{M},\tau)]_p, \tag{2.14}$$

and from Lemma 2.3.3, we have

$$H^p u_\lambda = [H^\infty u_\lambda \cap L^p(\mathcal{M},\tau)]_p, \quad \forall\ \lambda \in \Lambda. \tag{2.15}$$

Now, we have

$$
\begin{aligned}
\mathcal{K} &= [\mathcal{K}_1 \cap L^p(\mathcal{M},\tau)]_p && \text{(by Proposition 2.3.1)}\\
&= [\overline{Y_1 + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*} \cap L^p(\mathcal{M},\tau)]_p && \text{(by Definition 1.6.1)}\\
&= [(Y_1 + \sum_{\lambda\in\Lambda} H^\infty u_\lambda) \cap L^p(\mathcal{M},\tau)]_p && \text{(by Proposition 2.3.4)}\\
&= [(Y \cap L^p(\mathcal{M},\tau)) + \sum_{\lambda\in\Lambda} (H^\infty u_\lambda \cap L^p(\mathcal{M},\tau))]_p && \text{(by (a) and (b))}\\
&= [Y + \sum_{\lambda\in\Lambda} H^p u_\lambda]_p && \text{(by (2.14) and (2.15))}\\
&= Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda), && (2.16)
\end{aligned}
$$

where the last equation follows from Defintion 1.6.1.

As a summary, from (2.12), (b), (2.13), and (2.16), Y and $\{u_\lambda\}_{\lambda\in\Lambda}$ have the desired properties. This ends the proof of the theorem. □

2.4 Beurling-Blecher-Labuschagne theorem for semifinite Hardy spaces, $0 < p < 1$

Proposition 2.4.1. *Suppose $0 < p < 1$. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Assume that $\mathcal{K}$ is a closed subspace in $L^p(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then the following statements are true.*

(i) $\mathcal{K} \cap L^2(\mathcal{M},\tau) = [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap L^p(\mathcal{M},\tau)$.

(ii) $\mathcal{K} = [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_p$.

Proof. (i) We need only to show that

$$[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap L^p(\mathcal{M},\tau) \subseteq \mathcal{K} \cap L^2(\mathcal{M},\tau).$$

Let $x \in [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap L^p(\mathcal{M},\tau)$. We will show that $x \in \mathcal{K}$. By Lemma 1.5.7, there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that such that $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$ and $\lim_\lambda \|e_\lambda x - x\|_p = 0$. To show that $x \in \mathcal{K}$, it is enough to prove that $e_\lambda x \in \mathcal{K}$ for each $\lambda \in \Lambda$.

As $x \in [\mathcal{K}\cap L^2(\mathcal{M},\tau)]_2$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{K}\cap L^2(\mathcal{M},\tau)$ such that $\lim_{n\to\infty} \|x_n - x\|_2 = 0$. Thus, for each $\lambda \in \Lambda$ and some positive number q with $\frac{1}{2} + \frac{1}{q} = \frac{1}{p}$,

$$\lim_{n\to\infty} \|e_\lambda x_n - e_\lambda x\|_p = \lim_{n\to\infty} \|e_\lambda(x_n - x)\|_p \leq \lim_{n\to\infty} \|x_n - x\|_2 \|e_\lambda\|_q = 0.$$

Here, we used the fact that $\tau(e_\lambda) < \infty$, whence $\|e_\lambda\|_q < \infty$. Since $H^\infty\mathcal{K} \subseteq \mathcal{K}$ and $e_\lambda \in \mathcal{D}$, we know that $e_\lambda x_n \in \mathcal{K}$. This implies that $e_\lambda x \in \mathcal{K}$ for each $\lambda \in \Lambda$. Thus $x \in \mathcal{K}$, whence

$$[\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2 \cap L^p(\mathcal{M},\tau) \subseteq \mathcal{K} \cap L^2(\mathcal{M},\tau).$$

(ii) We need only to show that

$$\mathcal{K} \subseteq [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_p.$$

Suppose that $x \in \mathcal{K} \subseteq L^p(\mathcal{M},\tau)$. By Lemma 1.5.7, we can find a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that $\lim_\lambda \|e_\lambda x - x\|_2 = 0$ and $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$. To show that $x \in [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_p$, it suffices to prove that $e_\lambda x \in [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_p$ for each $\lambda \in \Lambda$.

Note that $x \in L^p(\mathcal{M},\tau)$ and $\tau(e_\lambda) < \infty$. By Lemma 1.5.6, there exist $h_1 \in e_\lambda H^\infty e_\lambda$ and $h_2 \in e_\lambda H^p e_\lambda$ such that (a) $h_1 h_2 = h_2 h_1 = e_\lambda$ and (b) $h_1 e_\lambda x \in \mathcal{M}$. Since $h_2 \in e_\lambda H^p e_\lambda$, there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ in $e_\lambda H^\infty e_\lambda$ such that $\lim_{n\to\infty} \|k_n - h_2\|_p = 0$. Thus

$$\begin{aligned} \lim_{n\to\infty} \|k_n h_1 e_\lambda x - e_\lambda x\|_p &= \lim_{n\to\infty} \|(k_n - h_2) h_1 e_\lambda x\|_p \\ &\leq \lim_{n\to\infty} \|(k_n - h_2)\|_p \|h_1 e_\lambda x\| = 0. \end{aligned} \tag{2.17}$$

It is not hard to check that $k_n h_1 e_\lambda x \in \mathcal{K}$. Moreover, since each $k_n \in e_\lambda H^\infty e_\lambda$, we have

$$\|k_n h_1 e_\lambda x\|_2 = \|e_\lambda k_n h_1 e_\lambda x\|_2 \leq \|e_\lambda\|_2 \|k_n\| \|h_1 e_\lambda x\| < \infty.$$

Therefore, $k_n h_1 e_\lambda x$ is also in $L^2(\mathcal{M},\tau)$. It follows that $k_n h_1 e_\lambda x \in \mathcal{K} \cap L^2(\mathcal{M},\tau)$. Combining with (2.17), we know that $e_\lambda x \in [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_p$ for each $\lambda \in \Lambda$, whence $x \in [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_p$. Thus

$$\mathcal{K} \subseteq [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_p.$$

This ends the proof of the proposition. □

Proposition 2.4.2. *Suppose* $0 < p < 1$*. Let* $\mathcal{M}$ *be a von Neumann algebra with a faithful, normal, semifinite tracial weight* τ*, and* H^∞ *be a semifinite subdiagonal subalgebra of* $\mathcal{M}$*. Assume that* S *is a subspace in* $L^2(\mathcal{M},\tau)$ *such that* $H^\infty S \subseteq S$*. Then*

$$[S \cap L^p(\mathcal{M},\tau)]_p = [[S]_2 \cap L^p(\mathcal{M},\tau)]_p.$$

Proof. We need only to show that

$$[[S]_2 \cap L^p(\mathcal{M},\tau)]_p \subseteq [S \cap L^p(\mathcal{M},\tau)]_p.$$

Or, equivalently,

$$[S]_2 \cap L^p(\mathcal{M},\tau) \subseteq [S \cap L^p(\mathcal{M},\tau)]_p.$$

Let $x \in [S]_2 \cap L^p(\mathcal{M},\tau)$. By Lemma 1.5.7, we can find a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that $\lim_\lambda \|e_\lambda x - x\|_p = 0$ and $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$. To show that $x \in [S \cap L^p(\mathcal{M},\tau)]_p$, it suffices to prove that $e_\lambda x \in [S \cap L^p(\mathcal{M},\tau)]_p$ for each $\lambda \in \Lambda$.

Note that $x \in [S]_2 \cap L^p(\mathcal{M},\tau)$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in S such that $\lim_{n\to\infty} \|x_n - x\|_2 = 0$. Therefore,

$$\|e_\lambda x_n - e_\lambda x\|_p = \|e_\lambda (x_n - x)\|_p \le \|e_\lambda\|_q \|x_n - x\|_2 \to 0, \qquad \text{as } n \to \infty, \tag{2.18}$$

where q is a positive number such that $\frac{1}{2} + \frac{1}{q} = \frac{1}{p}$. Since $H^\infty S \subseteq S$ and $e_\lambda \in \mathcal{D}$, we know that $e_\lambda x_n \in S$. Moreover, $\|e_\lambda x_n\|_p \le \|e_\lambda\|_q \|x_n\|_2 < \infty$, which implies $e_\lambda x_n \in L^p(\mathcal{M},\tau)$. This induces that $e_\lambda x_n \in S \cap L^p(\mathcal{M},\tau)$. Combining with (2.18), we have that $e_\lambda x \in [S \cap L^p(\mathcal{M},\tau)]_p$ for each $\lambda \in \Lambda$. Thus $x \in [S \cap L^p(\mathcal{M},\tau)]_p$ for each $\lambda \in \Lambda$. That is,

$$[S]_2 \cap L^p(\mathcal{M},\tau) \subseteq [S \cap L^p(\mathcal{M},\tau)]_p.$$

□

Lemma 2.4.3. *If* u *is a partial isometry in* $\mathcal{M}$ *such that* $uu^* \in \mathcal{D}$*, then* *(i)* $[(H^2u) \cap L^p(\mathcal{M}, \tau)]_p = H^p u$ *for* $0 < p < 1$;

(ii) $H^2 u = [H^p u \cap L^2(\mathcal{M}, \tau)]_2$ *for* $0 < p < 1$.

Proof. (i) Assume that $x \in H^2$ such that $xu \in (H^2u) \cap L^p(\mathcal{M}, \tau)$. Then $xuu^* \in L^p(\mathcal{M}, \tau)$, and $x(uu^*)$ is also in H^2, as $uu^* \in \mathcal{D}$. So $xuu^* \in H^2 \cap L^p(\mathcal{M}, \tau) \subseteq H^p$ by Proposition 3.2 in [2]. Note that $H^p u$ is a closed subspace in $L^p(\mathcal{M}, \tau)$. We have

$$[(H^2u) \cap L^p(\mathcal{M}, \tau)]_p \subseteq H^p u.$$

Similarly, we have

$$[(H^p u) \cap L^2(\mathcal{M}, \tau)]_2 \subseteq H^2 u. \tag{2.19}$$

Combining with Proposition 2.4.1, we have

$$H^p u = [H^p u \cap L^2(\mathcal{M}, \tau)]_p \subseteq [(H^2u) \cap L^p(\mathcal{M}, \tau)]_p.$$

Hence $[(H^2u) \cap L^p(\mathcal{M}, \tau)]_p = H^p u$, for $0 < p < 1$.

(ii) Let $x \in H^2$. By Lemma 1.5.7, we can find a net $\{e_\lambda\}_{\lambda \in \Lambda}$ of projections in $\mathcal{D}$ such that $\lim_\lambda \|e_\lambda x - x\|_2 = 0$ and $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$. From $\tau(e_\lambda) < \infty$, it is easy to verify that $e_\lambda x \in L^p(\mathcal{M}, \tau) \cap H^2$, and $L^p(\mathcal{M}, \tau) \cap H^2 \subseteq H^p$ by Proposition 3.2 in [2]. Thus $e_\lambda xu \in (H^p u) \cap L^2(\mathcal{M}, \tau)$ for each $\lambda \in \Lambda$, whence $xu \in [(H^p u) \cap L^2(\mathcal{M}, \tau)]_2$, or equivalently,

$$H^2 u \subseteq [(H^p u) \cap L^2(\mathcal{M}, \tau)]_2.$$

Combining with equation 2.19, we have

$$H^2 u = [(H^p u) \cap L^2(\mathcal{M}, \tau)]_2.$$

$\square$

Now, we can prove a Beurling-Blecher-Labuschagne Theorem for the semifinite case when $0 < p < 1$.

Theorem 2.4.4. *Let $0 < p < 1$. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$.*

Then there exist a closed subspace Y of $L^p(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$ such that:

(i) $u_\lambda Y^* = 0$ for all $\lambda \in \Lambda$.

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$.

(iii) $Y = [H_0^\infty Y]_p$.

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda)$

Here $\oplus^{row}$ is the row sum of subspaces defined in Definition 1.6.1.

Proof. Let $\mathcal{K}_1 = [\mathcal{K} \cap L^2(\mathcal{M},\tau)]_2$. Then $\mathcal{K}_1$ is a closed subspace of $L^2(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K}_1 \subseteq \mathcal{K}_1$. By Theorem 2.3.6, there exist a closed subspace Y_1 of $L^2(\mathcal{M},\tau)$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M}$, satisfying

(a) $u_\lambda Y_1^* = 0$ for all $\lambda \in \Lambda$.

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$ and $u_\lambda u_\mu =^* 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$.

(c) $Y_1 = [H_0^2 Y_1]_2$.

(d) $\mathcal{K}_1 = Y_1 \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^2 u_\lambda)$

Let

$$Y = [Y_1 \cap L^p(\mathcal{M},\tau)]_p.$$

(i) From (a), the definition of Y and Lemma 2.1.2, we can conclude that

$$u_\lambda Y^* = 0 \text{ for all } \lambda \in \Lambda. \tag{2.20}$$

(ii) follows directly from (b).

(iii) We want to show that $Y = [H_0^2 Y]_p$. First we will show that

$$[(H_0^\infty Y_1) \cap L^p(\mathcal{M}, \tau)]_p \subseteq [H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p$$

In fact, let $x \in Y_1$ and $h \in H_0^\infty$ be such that $hx \in (H_0^\infty Y_1) \cap L^p(\mathcal{M}, \tau)$. We want to show that $hx \in [H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p$. By Lemma 1.5.7, we can find a net $\{e_\lambda\}_{\lambda \in \Lambda}$ of projections in $\mathcal{D}$ such that $e_\lambda \to I$ in weak*-topology and $\tau(e_\lambda) < \infty$ for each $\lambda \in \Lambda$. By Lemma 2.1.3, we have

$$\lim_\lambda \|e_\lambda hx - hx\|_p = 0. \tag{2.21}$$

Thus, to show that $hx \in [H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p$, it suffices to prove that $e_\lambda hx \in [H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p$ for each $\lambda \in \Lambda$. Fix a $\lambda_0 \in \Lambda$. Then, for some positive number q with $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, we have

$$\lim_\lambda \|e_{\lambda_0} h e_\lambda x - e_{\lambda_0} hx\|_p \le \lim_\lambda \|e_{\lambda_0} h\|_q \|e_\lambda x - x\|_2 = 0, \tag{2.22}$$

as $x \in Y_1$. Moreover, we have $e_{\lambda_0} h \in H_0^\infty$ and $e_\lambda x \in Y_1 \cap L^p(\mathcal{M}, \tau)$, as $\|e_\lambda x\|_p \le \|e_\lambda\|_q \|x\|_2 < \infty$. Thus, $e_{\lambda_0} h e_\lambda x$ is in $H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))$ for each $\lambda \in \Lambda$. From (2.22), $e_{\lambda_0} hx$ is in $[H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p$ for each $\lambda_0 \in \Lambda$. Therefore, from (2.21), $hx \in [H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p$, or equivalently,

$$[(H_0^\infty Y_1) \cap L^p(\mathcal{M}, \tau)]_p \subseteq [H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p \tag{2.23}$$

Now, we have

$$\begin{aligned}
Y &= [Y_1 \cap L^p(\mathcal{M}, \tau)]_p && \text{(by definition of } Y\text{)} \\
&= [[H_0^2 Y_1]_2 \cap L^p(\mathcal{M}, \tau)]_p && \text{(by (c))} \\
&= [(H_0^\infty Y_1) \cap L^p(\mathcal{M}, \tau)]_p && \text{(by Proposition 2.4.2)} \\
&\subseteq [H_0^\infty (Y_1 \cap L^p(\mathcal{M}, \tau))]_p && \text{(by (2.23))} \\
&\subseteq [H_0^\infty Y]_p \subseteq Y. && \text{(by the definition of } Y\text{)}
\end{aligned}$$

Thus,

$$Y = [H_0^\infty Y]_p. \tag{2.24}$$

(iv) We have only to show that

$$\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda).$$

By the definition of Y, we have

$$Y = [Y_1 \cap L^p(\mathcal{M}, \tau)]_p. \tag{2.25}$$

And from Lemma 2.4.3, we have

$$H^p u_\lambda = [H^2 u_\lambda \cap L^p(\mathcal{M}, \tau)]_p, \quad \forall\ \lambda \in \Lambda. \tag{2.26}$$

Now, we have

$$\begin{aligned}
\mathcal{K} &= [\mathcal{K}_1 \cap L^p(\mathcal{M}, \tau)]_p && \text{(by Proposition 2.4.1)}\\
&= [[Y_1 + \sum_{\lambda\in\Lambda} H^2 u_\lambda]_2 \cap L^p(\mathcal{M}, \tau)]_p && \text{(by Definition 1.6.1)}\\
&= [(Y_1 + \sum_{\lambda\in\Lambda} H^2 u_\lambda) \cap L^p(\mathcal{M}, \tau)]_p && \text{(by Proposition 2.4.2)}\\
&= [(Y_1 \cap L^p(\mathcal{M}, \tau)) + \sum_{\lambda\in\Lambda} (H^2 u_\lambda \cap L^p(\mathcal{M}, \tau))]_p && \text{(by (a) and (b))}\\
&= [Y + \sum_{\lambda\in\Lambda} H^p u_\lambda]_p && \text{(by (2.25) and (2.26))}\\
&= Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda), && (2.27)
\end{aligned}$$

where the last equation follows from Definition 1.6.1.

As a summary, from (2.20), (b), (2.24), and (2.27), Y and $\{u_\lambda\}_{\lambda\in\Lambda}$ have desired properties. This ends the proof of the theorem. □

Corollary 2.4.5. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ.*

(i) Let $0 < p < \infty$. If $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M}, \tau)$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$, then there exists a projection $q \in \mathcal{M}$ such that $\mathcal{K} = L^p(\mathcal{M}, \tau)q$.

(ii) If $\mathcal{K}$ is a weak-closed subspace of $\mathcal{M}$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$, then there exists a projection $q \in \mathcal{M}$ such that $\mathcal{K} = \mathcal{M}q$.*

Proof. (i) Note that $\mathcal{M}$ itself is a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let $H^\infty = \mathcal{M}$. Then $\mathcal{D} = \mathcal{M}$ and Φ is the identity map from $\mathcal{M}$ to $\mathcal{M}$. Hence $H_0^\infty = \{0\}$ and $H^p = L^p(\mathcal{M}, \tau)$.

Assume that $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M}, \tau)$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$. From Theorem 2.3.6 and Theorem 2.4.4,

$$\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^p u_\lambda),$$

where Y and the $\{u_\lambda\}_{\lambda\in\Lambda}$ satisfy the conditions in Theorem 2.3.6 and Theorem 2.4.4.

From the fact that $H_0^\infty = \{0\}$, we know that $Y = \{0\}$. Since $\mathcal{D} = \mathcal{M}$, we know that

$$\begin{aligned} H^p u_\lambda &= L^p(\mathcal{M}, \tau) u_\lambda = L^p(\mathcal{M}, \tau) u_\lambda u_\lambda^* u_\lambda \\ &\subseteq L^p(\mathcal{M}, \tau) u_\lambda^* u_\lambda \subseteq L^p(\mathcal{M}, \tau) u_\lambda = H^p u_\lambda. \end{aligned}$$

So $H^p u_\lambda = L^p(\mathcal{M}, \tau) u_\lambda^* u_\lambda$ and

$$\begin{aligned} \mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^p u_\lambda) &= (\oplus_{\lambda\in\Lambda}^{row} L^p(\mathcal{M}, \tau) u_\lambda^* u_\lambda) \\ &= L^p(\mathcal{M}, \tau) \left(\sum_{\lambda\in\Lambda} u_\lambda^* u_\lambda \right) = L^p(\mathcal{M}, \tau) q, \end{aligned}$$

where $q = \sum_{\lambda\in\Lambda} u_\lambda^* u_\lambda$ is a projection in $\mathcal{M}$. This ends the proof of (i).

(ii) The proof is similar to (i). $\square$

Chapter 3

Applications for $\|\cdot\|_p$-norms

Using our results from Chapter 2, we are able to prove a Beurling-Blecher-Labuschagne-like theorem for the crossed product of a semifinite von Neumann algebra $\mathcal{M}$ by a trace-preserving action α when $0 < p < \infty$ (see the definition in Section 3.1.1). We are actually able to fully characterize the H^∞-invariant subspace of the crossed product.

Theorem 3.1.3. *Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal, tracial weight τ, and α be a trace-preserving $*$-automorphism of $\mathcal{M}$. Denote by $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ the crossed product of $\mathcal{M}$ by an action α, and still denote by τ the semifinite, faithful, normal, extended tracial weight on $\mathcal{M} \rtimes_\alpha \mathbb{Z}$.*

Let H^∞, a weak $*$-closed, nonself-adjoint subalgebra generated by $\{\Lambda(n)\Psi(x) : x \in \mathcal{M},\ n \geq 0\}$ in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$, be a semifinite subdiagonal subalgebra of $\mathcal{M} \rtimes_\alpha \mathbb{Z}$. H^∞ is a semifinite subdiagonal subalgebra of $\mathcal{M} \rtimes_\alpha \mathbb{Z}$, for which the following statements are true.

(i) Let $0 < p < \infty$. Assume that $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ satisfying

(a) $u_\lambda q = 0$ for all $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{M}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c) $\mathcal{K} = (L^p(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \tau)q) \oplus^{row} (\oplus^{row}_{\lambda \in \Lambda} H^p u_\lambda)$.

(ii) Assume that $\mathcal{K}$ is a weak $*$-closed subspace of $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ satisfying

(a) $u_\lambda q = 0$ for all $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{M}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c) $\mathcal{K} = ((\mathcal{M} \rtimes_\alpha \mathbb{Z})q) \oplus^{row} (\oplus^{row}_{\lambda \in \Lambda} H^\infty u_\lambda)$.

In [28], McAsey, Muhly and Saito prove a Beurling theorem for a crossed product. *Suppose $\mathcal{M}$ is a finite von Neumann algebra with finite trace τ and α, a trace preserving automorphism of $\mathcal{M}$, such that α fixes each element of the center $Z(\mathcal{M})$ of $\mathcal{M}$. Then let $\mathcal{A} = \mathcal{M} \rtimes_\alpha \mathbb{Z}_+$. Then every $\mathcal{A}$ and $\mathcal{A}^*$-invariant subspace $\mathcal{K}$ of $L^2(M, \tau)$ has the form $\mathcal{K} = vH^2$ for a partial isometry v in the commutant of right multiplication by $\mathcal{M}$ on $L^2(\mathcal{M}, \tau)$.* This follows from Theorem 3.1.3 when τ is finite, and $p = 2$.

McAsey, Muhly and Saito's result is a corollary of a result by Nazaki and Watatani in [30]. *Suppose $\mathcal{M}$ is a finite von Neumann algebra with trace τ, a faithful, normal, trace-preserving conditional expectation $\Phi : \mathcal{M} \to \mathcal{D}$, and $\mathcal{D} \subseteq \mathcal{M}$. We let H^∞ be a maximal subdiagonal algebra with respect to Φ, and suppose that $Z(\mathcal{D}) \subseteq Z(\mathcal{M})$. Then, if we let $\mathcal{K}$ be a H^∞-invariant subspace of $L^2(\mathcal{M}, \tau)$ such that $\mathcal{K}$ is of H^∞-type I (in the sense defined in [30]), there exists a partial isometry v in the commutant of right multiplcation by $\mathcal{M}$ such that $\mathcal{K} = vH^2$.* Again, this follows from our result in the finite case when $p = 2$.

Similarly, Saito in [39] proves another Beurling-like theorem for a finite von Neumann algebra $\mathcal{M}$. *Let a closed subspace $\mathcal{K}$ of $L^2(\mathcal{M}, \tau)$ be invariant under $\mathcal{M} \rtimes_\alpha \mathbb{Z}_+$ such that there are no subspaces of $\mathcal{K}$ with $(\mathcal{M} \rtimes_\alpha \mathbb{Z})\mathcal{K} \subseteq \mathcal{K}$ then $\mathcal{K}$ has the form $\sum_{n=0}^\infty \oplus V_n H^2$ with $\{V_n\}$ a family of partial isometries with $\{V_n V_n^*\}$ mutually orthonogal.*

We are also able to prove a Beurling-Blecher-Labuschagne-like theorem for the Schatten p-classes for $0 < p < \infty$, as described in Section 3.1.3 , using our results.

Corollary 3.1.4. *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal base $\{e_m\}_{m \in \mathbb{Z}}$. Let H^∞*

be the lower triangular subalgebra of $B(\mathcal{H})$, i.e.

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n \rangle = 0, \quad \forall n < m\}.$$

Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$ be the diagonal subalgebra of $B(\mathcal{H})$.

(i) For each $0 < p < \infty$, let $S^p(\mathcal{H})$ be the Schatten p-class. Assume that $\mathcal{K}$ is a closed subspace of $S^p(\mathcal{H})$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{D}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $B(\mathcal{H})$ satisfying

(a) $u_\lambda q = 0$ for all $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c) $\mathcal{K} = (S^p(\mathcal{H})q) \oplus^{row} (\oplus^{row}_{\lambda \in \Lambda} H^p u_\lambda)$.

(ii) Assume that $\mathcal{K}$ is a weak $*$-closed subspace of $B(\mathcal{H})$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{D}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $B(\mathcal{H})$ satisfying

(a) $u_\lambda q = 0$ for all $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c) $\mathcal{K} = (B(\mathcal{H})q) \oplus^{row} (\oplus^{row}_{\lambda \in \Lambda} H^\infty u_\lambda)$.

However, if we have that this projection q in $\mathcal{D}$ has the characteristic that $S^p(\mathcal{H})q \subseteq H^p$, then we can fully characterize, in Corollary 3.1.6, a H^∞-invariant subspace $\mathcal{K} \subseteq H^p$ when $0 < p \leq \infty$ and $\mathcal{H}$ is a separable Hilbert space with an orthonormal base.

Corollary 3.1.6. *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal base $\{e_m\}_{m \in \mathbb{Z}}$. Let H^∞ be the lower triangular subalgebra of $B(\mathcal{H})$, i.e.*

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n \rangle = 0, \quad \forall n < m\}.$$

Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$ be the diagonal subalgebra of $B(\mathcal{H})$.

(i) For each $0 < p < \infty$, if $\mathcal{K}$ is a closed subspace of H^p such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$, then there exists a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in H^∞ satisfying

(a) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(b) $\mathcal{K} = \oplus_{\lambda\in\Lambda}^{row} H^p u_\lambda$.

(ii) Assume that $\mathcal{K}$ *is a weak* $*$*-closed subspace of* H^∞ *such that* $H^\infty\mathcal{K} \subseteq \mathcal{K}$*. Then there exists a family* $\{u_\lambda\}_{\lambda\in\Lambda}$ *of partial isometries in* H^∞ *satisfying*

(a) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(b) $\mathcal{K} = \oplus_{\lambda\in\Lambda}^{row} H^\infty u_\lambda$.

Therefore, we are able to answer the question given in problem 2.0.1 and fully characterize an $\mathcal{A}$-invariant subspace of a Schatten p-class: given a subspace $\mathcal{K} \subseteq S^p(\mathcal{H})$ such that $\mathcal{A}\mathcal{K} \subseteq \mathcal{K}$, we have that $\mathcal{K} = (S^p(\mathcal{H})q) \oplus_{\lambda\in\Lambda}^{row} H^p u_\lambda$ when $0 < p < \infty$, and $\mathcal{K} = (B(\mathcal{H})q) \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^\infty u_\lambda)$ when $p = \infty$.

3.1 Invariant subspaces for analytic crossed products

3.1.1 Crossed product of a von Neumann algebra $\mathcal{M}$ by an action α

Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal tracial state τ. Let α be a trace-preserving $*$-automorphism of $\mathcal{M}$ (so $\tau(\alpha(x)) = \tau(x),\ \forall x \in \mathcal{M}^+$).

We let $l^2(\mathbb{Z})$ be the Hilbert space consisting of complex-valued functions f on $\mathbb{Z}$ such that $\sum_{m\in\mathbb{Z}} |f(m)|^2 < \infty$. We denote by $\{e_n\}_{n\in\mathbb{Z}}$ the orthonormal basis of $l^2(\mathbb{Z})$ determined by $e_n(m) = \delta(n, m)$. We also denote by $\lambda : \mathbb{Z} \to B(l^2(\mathbb{Z}))$ the left regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$, i.e. each $\lambda(n)$ is determined by $\lambda(n)(e_m) = e_{m+n}$.

Let $\mathcal{H} = L^2(\mathcal{M}, \tau) \otimes l^2(\mathbb{Z})$. Then $\mathcal{H}$ can also be written as $\oplus_{m\in\mathbb{Z}} L^2(\mathcal{M}, \tau) \otimes e_m$. Consider representations Ψ and Λ of $\mathcal{M}$ and $\mathbb{Z}$, respectively, on $\mathcal{H}$, defined by

$$\Psi(x)(\xi \otimes e_m) = (\alpha^{-m}(x)\xi) \otimes e_m, \qquad \forall\, x \in \mathcal{M},\ \forall\, \xi \in L^2(\mathcal{M}, \tau),\ \forall\, m \in \mathbb{Z}$$

$$\Lambda(n)(\xi \otimes e_m) = \xi \otimes (\lambda(n)e_m), \qquad \forall\, n, m \in \mathbb{Z}$$

It can be verified that

$$\Lambda(n)\Psi(x)\Lambda(-n) = \Psi(\alpha^n(x)), \qquad \forall\, x \in \mathcal{M},\ \forall\, n \in \mathbb{Z}.$$

Then the crossed product of $\mathcal{M}$ by an action α, denoted by $\mathcal{M} \rtimes_\alpha \mathbb{Z}$, is the von Neumann algebra generated by $\Psi(\mathcal{M})$ and $\Lambda(\mathbb{Z})$ in $B(\mathcal{H})$. If no confusion arises, we will identify $\mathcal{M}$ with its image $\Psi(\mathcal{M})$ in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$.

It is well known (for example, see Chapter 13 in [22]) that there exists a faithful, normal conditional expectation Φ from $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ onto $\mathcal{M}$ such that

$$\Phi\left(\sum_{n=-N}^{N} \Lambda(n)\Psi(x_n)\right) = x_0, \qquad \text{where } \ x_n \in \mathcal{M} \text{ for all } -N \le n \le N.$$

Moreover, there exists a semifinite, faithful, normal, extended tracial weight, still denoted by τ, on $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ satisfying

$$\tau(y) = \tau(\Phi(y)), \qquad \text{for every positive element } y \text{ in } \mathcal{M} \rtimes_\alpha \mathbb{Z}.$$

Example 3.1.1. *$\mathcal{M} = l^\infty(\mathbb{Z})$ is an abelian von Neumann algebra with a semifinite, faithful, normal tracial weight τ, determined by*

$$\tau(f) = \sum_{m \in \mathbb{Z}} f(m), \qquad \textit{for every positive element } \ f \in l^\infty(\mathbb{Z}).$$

Let α be an action on $l^\infty(\mathbb{Z})$, defined by

$$\alpha(f)(m) = f(m-1), \qquad \textit{for every element } \ f \in l^\infty(\mathbb{Z}).$$

It is not hard to verify (for example see Proposition 8.6.4 in [22]) that $l^\infty(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ is a type I_∞ factor. Thus $l^\infty(\mathbb{Z}) \rtimes_\alpha \mathbb{Z} \simeq B(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$.

3.1.2 Invariant subspace for crossed products

From the construction of crossed products, we immediately have the following result (also see Section 3 in [1]).

Lemma 3.1.2. *Let $\mathcal{M} \rtimes_\alpha \mathbb{Z}_+$ be a weak $*$-closed non-self-adjoint subalgebra generated by*

$$\{\Lambda(n)\Psi(x) : x \in \mathcal{M},\ n \geq 0\}$$

in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$. Then the following statements are true:

(i) *$\mathcal{M} \rtimes_\alpha \mathbb{Z}_+$ is a semifinite subdiagonal subalgebra with respect to $(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \Phi)$. (Such $\mathcal{M} \rtimes_\alpha \mathbb{Z}_+$ is called an analytic crossed product and will be denoted by H^∞.)*

(ii) *$H_0^\infty = \ker(\Phi) \cap H^\infty$ is a weak $*$-closed nonself-adjoint subalgebra generated by*

$$\{\Lambda(n)\Psi(x) : x \in \mathcal{M},\ n > 0\}$$

in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ satisfying

$$H_0^\infty = \Lambda(1)H^\infty.$$

(iii) *$H^\infty \cap (H^\infty)^* = \mathcal{M}$.*

Following the notation in Section 3.1.1, our next result characterizes invariant subspaces in a crossed product of a semifinite von Neumann algebra $\mathcal{M}$ by a tracing-preserving action α.

Theorem 3.1.3. *Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal tracial weight τ, and α be a trace-preserving $*$-automorphism of $\mathcal{M}$. Denote by $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ the crossed product of $\mathcal{M}$ by an action α, and still denote by τ the semifinite, faithful, normal, extended tracial weight on $\mathcal{M} \rtimes_\alpha \mathbb{Z}$.*

Let H^∞ be the weak $*$-closed non-self-adjoint subalgebra generated by $\{\Lambda(n)\Psi(x) : x \in \mathcal{M},\ n \geq 0\}$ in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$. H^∞ is a semifinite subdiagonal subalgebra of $\mathcal{M} \rtimes_\alpha \mathbb{Z}$, for which the following statements are true.

(i) *Let $0 < p < \infty$. Assume that $\mathcal{K}$ is a closed subspace of $L^p(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ satisfying*

(a) *$u_\lambda q = 0$ for all $\lambda \in \Lambda$;*

(b) $u_\lambda u_\lambda^* \in \mathcal{M}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(c) $\mathcal{K} = (L^p(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \tau)q) \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda)$.

(ii) Assume that $\mathcal{K}$ *is a weak* $*$*-closed subspace of* $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ *such that* $H^\infty \mathcal{K} \subseteq \mathcal{K}$*. Then there exist a projection* q *in* $\mathcal{M}$ *and a family* $\{u_\lambda\}_{\lambda\in\Lambda}$ *of partial isometries in* $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ *satisfying*

(a) $u_\lambda q = 0$ *for all* $\lambda \in \Lambda$*;*

(b) $u_\lambda u_\lambda^* \in \mathcal{M}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(c) $\mathcal{K} = ((\mathcal{M} \rtimes_\alpha \mathbb{Z})q) \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\infty u_\lambda)$.

Proof. (i) From Theorem 2.3.6 and Theorem 2.4.4,

$$\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda),$$

where Y is a closed subspace of $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ and $\{u_\lambda\}_{\lambda\in\Lambda}$ is a family of partial isometries in $\mathcal{M} \rtimes_\alpha \mathbb{Z}$ satisfying

(a_1) $u_\lambda Y^* = 0$ for all $\lambda \in \Lambda$;

(b_1) $u_\lambda u_\lambda^* \in \mathcal{M}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c_1) $Y = [H_0^\infty Y]_p$.

From (c_1) and Lemma 3.1.2, we have

$$Y = [H_0^\infty Y]_p = [\Lambda(1) H^\infty Y]_p \subseteq \Lambda(1)Y.$$

By induction, we have that $\Lambda(-n)Y \subseteq Y$ for every $n \in \mathbb{N}$. We already have, from the definition of H^∞, that $\Lambda(n)Y \subseteq Y$, for every $n \geq 0$, and $\psi(x)Y \subseteq Y$. So, Y is a left $\mathcal{M} \rtimes_\alpha \mathbb{Z}$-invariant subspace of $L^p(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \tau)$. From Corollary 2.4.5, there exists a projection q in $\mathcal{M}$ such that $Y = L^p(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \tau)q$. Therefore, we have

(a) $u_\lambda q = 0$ for all $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{M}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c) $\mathcal{K} = (L^p(\mathcal{M} \rtimes_\alpha \mathbb{Z}, \tau)q) \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^p u_\lambda)$.

This ends the proof of (i).

(ii) The proof is similar to (i). □

3.1.3 Invariant subspaces for Schatten p-classes

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$. Let $\tau = Tr$ be the usual trace on $B(\mathcal{H})$, i.e.

$$\tau(x) = \sum_{i\in\mathbb{Z}} \langle xe_m, e_m\rangle, \qquad \text{for all positive } x \text{ in } B(\mathcal{H}).$$

Then $B(\mathcal{H})$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. For each $0 < p < \infty$, the Schatten p-class $S^p(\mathcal{H})$ is the associated non-commuative L^p-space $L^p(B(\mathcal{H}), \tau)$.

Let

$$\mathcal{A} = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0, \quad \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$. From Example 3.1.1, $B(\mathcal{H})$ can also be realized as a crossed product $l^\infty(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ of $l^\infty(\mathbb{Z})$ by an action α, where the action α is determined by

$$\alpha(f)(m) = f(m-1), \qquad \forall\ f \in l^\infty(\mathbb{Z}).$$

Moreover, it can be verified quickly that $\mathcal{A}$, as a subalgebra of $B(\mathcal{H})$, is $l^\infty(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}_+$ (see Lemma 3.1.2) is a semifinite subdiagonal subalgebra of $l^\infty(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ (see Example 2.6 in [28]). Thus from Theorem 3.1.3, we have the following statements.

Corollary 3.1.4. *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$. Let H^∞ be the lower triangular subalgebra of $B(\mathcal{H})$, i.e.*

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0, \quad \forall n < m\}.$$

Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$ be the diagonal subalgebra of $B(\mathcal{H})$.

(i) *For each $0 < p < \infty$, let $S^p(\mathcal{H})$ be the Schatten p-class. Assume that $\mathcal{K}$ is a closed subspace of $S^p(\mathcal{H})$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{D}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $B(\mathcal{H})$ satisfying*

(a) $u_\lambda q = 0$ *for all* $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$;

(c) $\mathcal{K} = (S^p(\mathcal{H})q) \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^p u_\lambda)$.

(ii) *Assume that $\mathcal{K}$ is a weak $*$-closed subspace of $B(\mathcal{H})$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{D}$ and a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in $B(\mathcal{H})$ satisfying*

(a) $u_\lambda q = 0$ *for all* $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$;

(c) $\mathcal{K} = (B(\mathcal{H})q) \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\infty u_\lambda)$.

Remark 3.1.5. *Let $0 < p < \infty$. Assume q is a projection in $\mathcal{D}$ such that $S^p(\mathcal{H})q \subseteq H^p$. Notice that all finite rank operators are in $S^p(\mathcal{H})$. Thus $e_s \otimes e_t$, where $e_s \otimes e_t$ is a rank one operator defined for all $\xi \in H$ by $e_s \otimes e_t(\xi) = \langle \xi, e_t\rangle e_s$, is in $S^p(\mathcal{H})$ for all $s, t \in \mathbb{Z}$. Hence, for all $s, t \in \mathbb{Z}$, we have $(e_s \otimes e_t)q \in H^p$. Combining this with the fact that $q \in \mathcal{D}$ is a diagonal projection in $B(\mathcal{H})$, we may conclude that $q = 0$.*

The the next result follows directly from Corollary 3.1.4 and Remark 3.1.5.

Corollary 3.1.6. *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$.*

Let H^∞ be the lower triangular subalgebra of $B(\mathcal{H})$, i.e.

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0, \quad \forall n < m\}.$$

Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$ be the diagonal subalgebra of $B(\mathcal{H})$.

(i) *For each $0 < p < \infty$, if $\mathcal{K}$ is a closed subspace of H^p such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$, then there exists a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in H^∞ satisfying*

(a) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(b) $\mathcal{K} = \oplus_{\lambda \in \Lambda}^{row} H^p u_\lambda$.

(ii) *Assume that* $\mathcal{K}$ *is a weak* $*$*-closed subspace of* H^∞ *such that* $H^\infty \mathcal{K} \subseteq \mathcal{K}$*. Then there exists a family* $\{u_\lambda\}_{\lambda \in \Lambda}$ *of partial isometries in* H^∞ *satisfying*

(a) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(b) $\mathcal{K} = \oplus_{\lambda \in \Lambda}^{row} H^\infty u_\lambda$.

Remark 3.1.7. *Similar results hold when* H^∞ *is the upper triangular subalgebra of* $B(\mathcal{H})$*.*

Chapter 4

Invariant subspaces under unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms

Suppose that (X, Σ, ν) is a localizable measure space with the finite subset property (i.e. a measure space is *localizable* if the multiplication algebra is maximal abelian, and has the *finite subset property* if for every $A \in \Sigma$ such that $\nu(A) > 0$, there exists a $B \in \Sigma$ such that $B \subseteq A$, and $0 < \nu(B) < \infty$). We let E be a two-sided ideal of the set of complex-valued, Σ-measureable functions on X, such that all functions equal almost everywhere with respect to ν are identified. If E has a norm $\|\cdot\|_E$ such that $(E, \|\cdot\|_E)$ is a Banach lattice, then we call E a Banach function space. (See the work of de Pagter in [33]).

We let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. For every operator $x \in \mathcal{M}$, we define $d_x(\lambda) = \tau(e^{|x|}(\lambda, \infty))$ for every $\lambda \geq 0$ (where $e^{|x|}(\lambda, \infty)$ is the spectral projection of $|x|$ on the interval (λ, ∞)), and $\mu(x) = \inf\{\lambda \geq 0 : d_x(\lambda) \leq t\}$ for a given $t \geq 0$. Consider the set $\mathcal{I} = \{x \in \mathcal{M} : x \text{ is a finite rank operator in } (\mathcal{M}, \tau) \text{ and } \|\mu(x)\|_E < \infty\}$ and let $\|\cdot\|_{\mathcal{I}(\tau)} : \mathcal{I} \to [0, \infty)$ be such that $\|x\|_{\mathcal{I}(\tau)} = \|\mu(x)\|_E$ for all $x \in \mathcal{I}$. It is known that $\|\cdot\|_{\mathcal{I}(\tau)}$ defines a norm on $\mathcal{I}$ (see [33]). Denote by $\mathcal{I}(\tau)$ the closure of $\mathcal{I}$ under $\|\cdot\|_{\mathcal{I}(\tau)}$.

We briefly recall an extension of Arveson's non commutative Hardy space for a semifinite von Neumann algebra. Let H^∞ be a weak*-closed unital subalgebra of $\mathcal{M}$. Then $\mathcal{D} = H^\infty \cap (H^\infty)^*$

is a von Neumann subalgebra of $\mathcal{M}$. Assume also that there exists a faithful, normal, conditional expectation $\Phi : \mathcal{M} \to \mathcal{D}$. Then H^∞ is called a *semifinite non-commutative Hardy space* if (i) the restriction of τ on $\mathcal{D}$ is semifinite; (ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for every $x, y \in H^\infty$; (iii) $H^\infty + (H^\infty)^*$ is weak* dense in $\mathcal{M}$; and (iv)$\tau(\Phi(x)) = \tau(x)$ for every positive $x \in \mathcal{M}$.

We want to ask the following question about the space $\mathcal{I}(\tau)$:

Problem 4.0.1. *Consider a semifinite subdiagonal subalgebra H^∞ of $\mathcal{M}$ and a closed subspace $\mathcal{K}$ of $\mathcal{I}(\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. How can the subspace $\mathcal{K}$ be characterized?*

It can be shown that when $\mathcal{M}$ is diffuse, and $\|\cdot\|_{\mathcal{I}(\tau)}$ is order continuous, the norm $\|\cdot\|_{\mathcal{I}(\tau)}$ on $\mathcal{I}(\tau)$ is in the family of unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms with respect to the tracial weight τ. (See Definition 1.4.1).

Our goal for this chapter is to prove a Beurling-type theorem for a von Neumann algebra with semifinite, faithful, normal tracial weight τ, and a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ, for example, the Banach function space $\mathcal{I}(\tau)$ with the norm $\|\cdot\|_{\mathcal{I}(\tau)}$.

In 1937, J. von Neumann introduced the unitarily invariant norms on $M_n(\mathbb{C})$ as a way to metrize the matrix spaces [32]. He showed that the class of unitarily invariant norms on $M_n(\mathbb{C})$ are in correspondence with the class of symmetric gauge norms on $\mathbb{C}^n$. Specifically, he proved that for any unitarily invariant norm α, there exists a symmetric gauge norm Ψ on $\mathbb{C}^n$ such that for every finite rank operator A, $\alpha(A) = \Psi(a_1, a_2, \ldots, a_n)$, where $\{a_i\}_{1\le i\le n}$ is the spectrum of $|A|$.

Since von Neumann's result, these norms have been extended and generalized in different ways. Schatten defined unitarily invariant norms on 2-sided ideals of the continuous functions on a Hilbert space, $B(\mathcal{H})$ (for example, see [40, 41]). Chen, Hadwin and Shen defined a class of unitarily invariant, $\|\cdot\|_1$-dominating, normalized norms on a finite von Neumann algebra in [8]. Unitarily invariant norms also play an important role in the study of non-commutative Banach function spaces. For more information and history of unitarily invariant norms see Schatten [40], Hewitt and Ross [20], Goldberg and Krein [16], or Simon [43].

Typical examples of noncommutative Banach function spaces include the so called noncommutative L^p-spaces, $L^p(\mathcal{M}, \tau)$, associated with semifinite von Neumann algebras. Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. We consider $\mathcal{I}$, the set of elementary operators on $\mathcal{M}$ (when $\mathcal{M}$ is finite, $\mathcal{M} = \mathcal{I}$). We recall the construction of $L^p(\mathcal{M}, \tau)$. When $0 < p < \infty$ define a mapping $\|\cdot\|_p : \mathcal{I} \to [0, \infty)$ by $\|x\|_p = (\tau(|p|))^{1/p}$ where $|x| = \sqrt{x^* x}$ for every $x \in \mathcal{I}$. It is non-trivial to prove that $\|\cdot\|_p$ is a norm, called the *p-norm*, when $1 \leq p < \infty$. We define the space $L^p(\mathcal{M}, \tau) = \overline{\mathcal{I}}^{\|\cdot\|_p}$ for $0 < p < \infty$. When $p = \infty$, we set $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$, which acts naturally on $L^p(\mathcal{M}, \tau)$ by right or left multiplication.

In the previous chapters, we extended the work of Blecher and Labuschagne in [6] for a finite von Neumann algebra to von Neumann algebras $\mathcal{M}$ with a *semifinite*, normal, faithful tracial weight τ. *Suppose* $0 < p \leq \infty$*, and* $\mathcal{M}$ *is a von Neumann algebra with a semifinite, faithful, normal tracial weight* τ*. Let* H^∞ *be a semifinite subdiagonal subalgebra of* $\mathcal{M}$*, and* $\mathcal{D} = H^\infty \cap (H^\infty)^*$*. Suppose that* $\mathcal{K}$ *is a closed subspace of* $L^p(\mathcal{M}, \tau)$ *(if* $p = \infty$*,* $\mathcal{K}$ *is weak* closed), such that* $H^\infty \mathcal{K} \subseteq \mathcal{K}$*. Then there exists a closed subspace* $Y \subseteq L^p(\mathcal{M}, \tau)$ *and a family of partial isometries* $\{u_\lambda\} \subseteq \mathcal{M}$ *such that* $\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda \in \Lambda} H^p u_\lambda)$*, where* $Y = [H_0^\infty Y]_p$*,* $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$*, and the* u_λ *satisfy other conditions.* (See Chapter 2 for more information.)

In [8], Chen, Hadwin and Shen proved a Beurling-type theorem for unitarily invariant norms on finite von Neumann algebras. A motivation of the following chapters is to extend the result in [8] to the setting of unitarily invariant norms on *semifinite* von Neumann algebras. We define the family of unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms on the von Neumann algebra $\mathcal{M}$ with respect to the semifinite, faithful, normal tracial weight τ. Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. We let $\mathcal{I}$ be the set of finite rank operators in $(\mathcal{M}, \tau)$. A norm $\alpha : \mathcal{I} \to [0, \infty)$ is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ if α is a norm for which the following conditions hold:

(i) for any unitaries $u, v \in \mathcal{M}$ and $x \in \mathcal{I}$, $\alpha(uxv) = \alpha(x)$;

(ii) for every projection $e \in \mathcal{M}$ with $\tau(e) < \infty$ and any $x \in \mathcal{I}$, there exists $0 < c(e) < \infty$ such that $\alpha(exe) \leq c(e)\|exe\|_1$;

(iii) (a) if $\{e_\lambda\}$ is an increasing net of projections in $\mathcal{I}$ such that $\tau(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$;

(b) if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$.

Chen, Hadwin and Shen's family of norms in [8] is a subset of this family of norms. We also show that the norm $\|\cdot\|_{\mathcal{I}(\tau)}$ on a Banach function space $\mathcal{I}(\tau)$ is a unitarily invariant, $\|\cdot\|_1$-dominating, mutually continuous norm.

However, many of the methods used by Chen, Hadwin and Shen no longer apply when $\mathcal{M}$ is a semifinite von Neumann algebra. We use a similar method to extend their theorem as in Chapter 2 for $L^p(\mathcal{M},\tau)$ spaces. We therefore prove a series of density results for the $L^\alpha(\mathcal{M},\tau)$ spaces.

Lemma 4.3.2. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and that H^∞ is a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Suppose also that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then the following hold:*

1. $\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M},\tau)$
2. $\mathcal{K} = [\mathcal{K} \cap \mathcal{M}]_\alpha$

Lemma 4.3.3. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let H^∞ be a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Assume that $\mathcal{K}$ is a weak*-closed subspace of $\mathcal{M}$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then*

$$\mathcal{K} = \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}.$$

Lemma 4.3.4. *Suppose $\mathcal{M}$ is a semifinite von Neumann algebra with a faithful, normal tracial weight τ, and suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually-continuous* *norm with respect to τ. Let H^∞ be a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Assume that S is a subset of $\mathcal{M}$ such that $H^\infty S \subseteq S$. Then*

$$[S \cap L^\alpha(\mathcal{M},\tau)]_\alpha = [\overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau)]_\alpha.$$

Following these results, we are able to prove a noncommutative Beurling-Chen-Hadwin-Shen theorem for unitarily invariant, $\|\cdot\|_1$-dominating, mututally continuous norms with respect to τ on a von Neumann algebra $\mathcal{M}$ with a semifinite, faithful, normal tracial weight τ.

Theorem 4.3.1. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then, there exist a closed subspace Y of $L^\alpha(\mathcal{M},\tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that*

(i) $u_\lambda Y^* = 0$ for every $\lambda \in \Lambda$;

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$, and $u_\lambda u_\mu^* = 0$ for every $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(iii) $Y = [H_0^\infty Y]_\alpha$;

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^\alpha u_\lambda)$.

We can fully characterize $\mathcal{K}$ in the case when $K \subseteq L^\alpha(\mathcal{M},\tau)$ is $\mathcal{M}$-invariant.

Corollary 4.3.5. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let $\mathcal{K}$ be a subset of L^α such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$. Then there exists a projection q with $\mathcal{K} = \mathcal{M}q$.*

4.1 Operators affiliated with $\mathcal{M}$

Given a von Neumann algebra $\mathcal{M}$ with a semifinite, faithful, normal tracial weight τ acting on a Hilbert space $\mathcal{H}$, a *measure topology* on $\mathcal{M}$ is given by the system of neighborhoods $U_{\delta,\epsilon} = \{a \in$

$\mathcal{M} : \|ap\| \leq \epsilon$ and $\tau(p^\perp) \leq \delta$ for some projection $p \in \mathcal{M}\}$ for any $\epsilon, \delta > 0$ (for more details see [31]). We say that a_n is *Cauchy in measure* if, given ϵ and $\delta > 0$, there exists an n_0 such that if $n, m \geq n_0$, then $a_n - a_m$ is in $U_{\delta,\epsilon}$.

Definition 4.1.1. *Let $\widetilde{\mathcal{M}}$ denote the algebra of closed, densely defined (possibly unbounded) operators on $\mathcal{H}$ affiliated with $\mathcal{M}$.*

Remark 4.1.2. *$\widetilde{\mathcal{M}}$ is also the closure of $\mathcal{M}$ in the measure topology (see [31] for more information).*

4.2 Unitarily invariant norms and examples

In this section, we introduce a class of unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms on semifinite von Neumann algebras. We also introduce interesting examples from this class.

4.2.1 L^α spaces of semifinite von Neumann algebras

Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial state τ. We then let

$$\mathcal{I} = span\{xey : x, y \in \mathcal{M}, e \in \mathcal{M}, e = e^2 = e^* \text{ with } \tau(e) < \infty\}$$

be the set of elementary operators of $\mathcal{M}$ (see Remark 2.3 in [42]). Recall that for each $1 \leq p < \infty$, we define the $\|\cdot\|_p$-norm on $\mathcal{I}$ by

$$\|x\|_p = (\tau(|x|^p))^{1/p} \qquad \text{for every } x \in \mathcal{I}.$$

It is a non-trivial fact that the mapping $\|\cdot\|_p$ defines a norm on $\mathcal{I}$. We let $L^p(\mathcal{M}, \tau)$ denote the completion of $\mathcal{I}$ with respect to the $\|\cdot\|_p$-norm.

Recall from Definition 1.4.1 that we call a norm $\alpha : \mathcal{I} \to [0, \infty)$ a *unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ* on $\mathcal{I}$ if it satisfies the following characteristics:

1. α is *unitarily invariant* if for all unitaries u, v in $\mathcal{M}$ and every x in $\mathcal{I}$, $\alpha(uxv) = \alpha(x)$;

2. α is *locally $\|\cdot\|_1$-dominating* if for every projection e in $\mathcal{M}$ with $\tau(e) < \infty$, there exists $0 < c(e) < \infty$ such that $\alpha(exe) \geq c(e)\|exe\|_1$ for every $x \in \mathcal{I}$;

3. α is *mutually continuous with respect to* τ; namely

   (a) If $\{e_\lambda\}$ is an increasing net of projections in $\mathcal{I}$ such that $\tau(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$. Or, equivalently, if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $e_\lambda \to I$ in the weak* topology, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$.

   (b) If $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$.

Recall that given a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm α with respect to τ on $\mathcal{I}$, we define $L^\alpha(\mathcal{M}, \tau)$ to be the completion of $\mathcal{I}$ under α, namely,

$$L^\alpha(\mathcal{M}, \tau) = \overline{\mathcal{I}}^\alpha.$$

Notation 4.2.1. *We will denote by $[S]_\alpha$ the closure, with respect to the norm α, of a set S in $\mathcal{M}$.*

Lemma 4.2.2. *Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ, and let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Then for any $x \in L^\alpha(\mathcal{M}, \tau)$, and $a, b \in \mathcal{M}$,*

$$\alpha(axb) \leq \|a\|\alpha(x)\|b\|.$$

Proof. The proof is included here for completeness. It suffices to show that for any $x \in \mathcal{I}$, and $a, b \in \mathcal{M}$,

$$\alpha(axb) \leq \|a\|\alpha(x)\|b\|.$$

Without loss of generality, we might assume that $\|a\| < 1$. By Russo-Dye Theorem, there exist a positive integer n and unitary elements $u_1, \ldots, u_n$ in $\mathcal{M}$ such that $a = (u_1 + \cdots + u_n)/n$. Therefore,

$$\alpha(ax) = \alpha((u_1 + \cdots + u_n)x)/n \leq \alpha(x)$$

since α is unitarily invariant. So, $\alpha(ax) \leq \|a\|\alpha(x)$ for every $a \in \mathcal{M}$.

It may be proved similarly that $\alpha(xb) \leq \alpha(x)\|b\|$ for every $b \in \mathcal{M}$. $\square$

4.2.2 Examples of unitarily-invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms

Remark 4.2.3. *It is trivial to show that the $\|\cdot\|_p$-norms of $\mathcal{M}$ with $1 \leq p < \infty$ for a semifinite von Neumann algebra $\mathcal{M}$ with a faithful, normal, semifinite tracial weight τ are unitarily equivalent, $\|\cdot\|_1$-dominating, mutually continuous norms with respect to τ on $\mathcal{M}$.*

Remark 4.2.4. *It is also trivial to show that a continuous, unitarily invariant, normalized, $\|\cdot\|_1$-dominating norm on a finite von Neumann algebra $\mathcal{M}$ as given in [8] is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$.*

Proposition 4.2.5. *Suppose that $\mathcal{M}$ is a semifinite factor, and $\alpha : \mathcal{I} \to [0, \infty)$ is a unitarily invariant norm satisfying the following condition: if $\{e_\lambda\}$ is a net in $\mathcal{M}$ with $e_\lambda \to I$ in the weak* topology, then $\alpha(e_\lambda x - x) \to 0$ for each $x \in \mathcal{I}$. Then α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ.*

Proof. By assumption, α is unitarily invariant.

Let e be projection in $\mathcal{M}$ such that $\tau(e) < \infty$. Let $x = exe$ be an element in $e\mathcal{M}e$, which we denote by $\mathcal{M}_e$. As $|x| \leq \|x\|e$, we have that $\alpha(x) = \alpha(|x|) \leq \|x\|\alpha(e)$. Note $\mathcal{M}_e$ is a finite factor with a tracial state τ_e, defined by $\tau_e(y) = \tau(y)/\tau(e)$ for all $y \in \mathcal{M}_e$. By the Dixmeier Approximation Property, for every $\epsilon > 0$, there exist $c_1, c_2, \ldots, c_n$ in $[0, 1]$ with $\sum_{i=1}^n c_i = 1$, and unitaries $u_1, u_2, \ldots, u_n$ in $\mathcal{M}_e$ such that $\|\tau_e(|x|)e - \sum_{i=1}^n c_i u_i x u_i^*\| < \epsilon$. Therefore, $\alpha(\tau_e(|x|)e - \sum_{i=1}^n c_i u_i x u_i^*)) \leq \epsilon\alpha(e)$. Thus,

$$\begin{aligned}
\|x\|_1 = \tau(|x|) = \tau(e)\tau_e(|x|) &= \frac{\tau(e)}{\alpha(e)}\alpha(\tau_e(|x|)e) \\
&\leq \frac{\tau(e)}{\alpha(e)}[\alpha(\tau_e(|x|)e - \sum_{i=1}^n c_i u_i x u_i^*) + \alpha(\sum_{i=1}^n c_i u_i x u_i^*)] \\
&\leq \epsilon\tau(e) + \frac{\tau(e)}{\alpha(e)}\sum_{i=1}^n \alpha(c_i u_i x u_i^*) \\
&\leq \epsilon\tau(e) + \frac{\tau(e)}{\alpha(e)}\alpha(x).
\end{aligned}$$

Letting $\epsilon \to 0$, we find that $\tau(x) \leq \frac{\tau(e)}{\alpha(e)}\alpha(x)$ for every x in $\mathcal{M}_e$. Namely,

$$\|exe\|_1 \leq c(e)\alpha(exe) \qquad \text{for all } x \in \mathcal{I}. \tag{4.1}$$

where $c(e) = \frac{\tau(e)}{\alpha(e)}$. Thus, α is locally $\|\cdot\|_1$-dominating.

We now show that α is mutually continuous with respect to τ. Actually, we need only to show that, if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$. Assume, to the contrary, that there exist a positive number $\epsilon > 0$ and a family $\{e_n\}$ of projections in $\mathcal{I}$ such that $\alpha(e_n) < 1/n$ but $\tau(e_n) > \epsilon$ for each $n \in \mathbb{N}$. As $\mathcal{M}$ is a semifinite factor and α is unitarily invariant, we may assume further that $\{e_n\}_n$ is a decreasing sequence of projections in $\mathcal{I}$. Let $e_0 = \wedge_n e_n$. Then $\tau(e_0) \geq \epsilon$ and $\alpha(e_0) = 0$ as $e_0 \leq e_n$ implies $\alpha(e_0) \leq \alpha(e_n) < 1/n$ for each n. This is a contradiction. Therefore, if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$. $\square$

Non-commutative Banach function spaces

In this subsection, we follow the notation of de Pagter in [33]. We suppose, as before, that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial state τ. In this case, we have the ideal of the distrubtion function d_x, where x is a τ-measurable operator in $\mathcal{M}$. We define d_x by

$$d_x(\lambda) = \tau(e^{|x|}(\lambda, \infty)) \text{ for every } \lambda \geq 0,$$

where $e^{|x|}(\lambda, \infty)$ is the spectral projection of $|x|$ on (λ, ∞). It is easy to see that d_x is decreasing, right-continuous and $d_x(\lambda) \to 0$ as $\lambda \to \infty$. This allows us to define a generalized singular value function

$$\mu(x; t) = \inf\{\lambda \geq 0 : d_x(\lambda) \leq t\} \text{ for a given } t \geq 0 \text{ and for every } x \in \mathcal{M}.$$

Definition 4.2.6. *Suppose that (X, Σ, ν) is a localizable measure space with the finite subset property. Let E be a two-sided ideal of the set of all complex-valued, Σ-measurable functions on X with the identification of all functions equal a.e. with respect to ν. If E has a norm $\|\cdot\|_E$ such that $(E, \|\cdot\|_E)$ is a Banach lattice, then E is called a* Banach function space.

We assume that E is a symmetric Banach function space on $(0,\infty)$ with Lebesgue meausure (see definition 2.6 in [33]).

Following [33], we let $\mathcal{I} = \{x \in \mathcal{M} : x$ is a finite rank operator in $(\mathcal{M},\tau)$ and $\|\mu(x)\|_E < \infty\}$ and define a Banach function space $\mathcal{I}(\tau)$ equipped with a norm $\|\cdot\|_{\mathcal{I}(\tau)}$ such that

$$\|x\|_{\mathcal{I}(\tau)} = \|\mu(x)\|_E \text{ for every } x \in \mathcal{I}.$$

Denote the closure of $\mathcal{I}$ under $\|\cdot\|_{\mathcal{I}(\tau)}$by $\mathcal{I}(\tau)$ We will use the following Lemma to show that the restriction of $\|\cdot\|_{\mathcal{I}(\tau)}$ on $\mathcal{I}$ is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ.

Lemma 4.2.7. *Suppose that y_0 is an element of $\mathcal{I}$ such that $y_0 = \sum_{i=1}^n \beta_i p_i$ where $\beta_1, \beta_2, \ldots, \beta_n$ are nonnegative and $p_1, \ldots, p_n$ are projections in $\mathcal{M}$ such that $\tau(p_1) = \tau(p_2) = \cdots = \tau(p_n)$. Then*

$$\|y_0\|_{\mathcal{I}(\tau)} \geq \frac{\|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)}}{\tau(p_1 + \cdots + p_n)} \|y_0\|_1.$$

Proof. Note that y_0 is an element of $\mathcal{I}$ such that $y_0 = \sum_{i=1}^n \beta_i p_i$ where $\tau(p_1) = \tau(p_2) = \cdots = \tau(p_n)$. Now let $\beta_{n+j} = \beta_j$ for all $1 \leq j \leq n$ and $y_j = \sum_{i=1}^n \beta_{i+j} p_i$ for $1 \leq j \leq n$. Then, by definition, $\sum_{k=1}^n y_k = (\beta_1 + \cdots + \beta_n)(p_1 + \cdots + p_n)$, and also $\|y_k\|_{\mathcal{I}(\tau)} = \|y_0\|_{\mathcal{I}(\tau)}$ for all $1 \leq k \leq n$. Therefore,

$$\begin{aligned}
\|y_0\|_{\mathcal{I}(\tau)} &\geq \frac{\|\sum_{k=1}^n y_k\|_{\mathcal{I}(\tau)}}{n} \\
&\geq (\frac{\beta_1 + \cdots + \beta_n}{n}) \|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)} \\
&= \frac{\tau(y_0)}{\tau(p_1 + \cdots + p_n)} \|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)} \\
&= \|y_0\|_1 \frac{\|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)}}{\tau(p_1 + \cdots + p_n)}.
\end{aligned}$$

□

Proposition 4.2.8. *Suppose that $\mathcal{I}(\tau)$ is a Banach function space. Suppose that $\mathcal{M}$ is a diffuse von Neumann algebra with a semifinite, faithful, normal tracial state τ and with an order continuous norm $\|\cdot\|_{\mathcal{I}(\tau)}$. Then the restriction of $\|\cdot\|_{\mathcal{I}(\tau)}$ on $\mathcal{I}$ is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mututally continuous norm with respect to τ.*

Proof. Note $\|\cdot\|_{\mathcal{I}(\tau)} : \mathcal{I} \to [0, \infty)$ is a norm. Now we will verify that $\|\cdot\|_{\mathcal{I}(\tau)}$ satisfies the following conditions:

1. $\|uxv\|_{\mathcal{I}(\tau)} = \|x\|_{\mathcal{I}(\tau)}$ for all unitaries u, v in $\mathcal{M}$, and every x in $\mathcal{I}$;

2. for every projection e in $\mathcal{M}$ with $\tau(e) < \infty$, there exists $c(e) < \infty$ such that $\|exe\|_{\mathcal{I}(\tau)} \geq c(e)\|exe\|_1$ for all $x \in \mathcal{M}$;

3. a. if $\{e_\lambda\}_{\lambda\in\Lambda}$ is a net in $\mathcal{M}$ such that $e_\lambda \to I$ in the weak* topology, then $\|e_\lambda x - x\|_{\mathcal{I}(\tau)} \to 0$ for every $x \in \mathcal{I}$.

   b. if $\{e_\lambda\}_{\lambda\in\Lambda}$ is a net in $\mathcal{M}$ such that $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$, then $\tau(e_\lambda) \to 0$.

(1) We begin by showing that $\|uxv\|_{\mathcal{I}(\tau)} = \|x\|_{\mathcal{I}(\tau)}$.

Given any x and y in $\mathcal{I}$, we know that if $\tau(|x|^n) = \tau(|y|^n)$ for every $n \in \mathbb{N}$, then $\|x\|_{\mathcal{I}(\tau)} = \|y\|_{\mathcal{I}(\tau)}$ from Definition 3.4 in [33]. We have that τ is unitarily invariant by defintion, so for all unitaries u and v in $\mathcal{M}$ and x in $\mathcal{I}$,

$$\tau(|uxv|^n) = \tau(v^{-n}|x|^n v^n) = \tau(|x|^n) \text{ for every } n \in \mathbb{N}.$$

Hence $\|uxv\|_{\mathcal{I}(\tau)} = \|x\|_{\mathcal{I}(\tau)}$, and $\|\cdot\|_{\mathcal{I}(\tau)}$ is unitarily invariant.

(3) a. We show that if $\{e_\lambda\} \subseteq \mathcal{I}$ is an increasing net of projections such that $e_\lambda \to I$ in the weak* topology, then $e_\lambda x \to x$ in $\|\cdot\|_{\mathcal{I}(\tau)}$-norm for each $x \in \mathcal{I}$.

Suppose that $\{e_\lambda\} \subseteq \mathcal{I}$ is an increasing net of projections such that $e_\lambda \to I$ in the weak* topology. By definition, $\|\cdot\|_{\mathcal{I}(\tau)}$ is order continuous. So for every x in $\mathcal{I}$, $\|\sqrt{x^*(I-e_\lambda)x}\|_{\mathcal{I}(\tau)} \to 0$, and $\|(I-e_\lambda)x\|_{\mathcal{I}(\tau)} = \||(I-e_\lambda)x|\|_{\mathcal{I}(\tau)} = \|\sqrt{x^*(I-e_\lambda)x}\|_{\mathcal{I}(\tau)}$ by (1). Therefore, $\|x - e_\lambda x\|_{\mathcal{I}(\tau)} \to 0$ for every x in $\mathcal{I}$, as desired.

b. We show that if $\{e_\lambda\} \subseteq \mathcal{I}$ is a net of projections such that $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$, then $\tau(e_\lambda) \to 0$.

We suppose that $\{e_\lambda\} \subseteq \mathcal{I}$ is a net of projections such that $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$. Suppose to the contrary, that $\tau(e_\lambda) \nrightarrow 0$. There exist an $\epsilon_0 > 0$, a subsequence $\{e_{\lambda_n}\}$ of $\{e_\lambda\}_{\lambda\in\Lambda}$ such that for every $n \geq 1$, $\tau(e_{\lambda_n}) \geq \epsilon_0$. As $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$, $\|e_{\lambda_n}\|_{\mathcal{I}(\tau)} \to 0$. Recall that $\mathcal{M}$ has no minimal projection. By the properties of the norm $\|\cdot\|_{\mathcal{I}(\tau)}$, we may assume that $\{e_{\lambda_n}\}$ is a decreasing sequence of projections in $\mathcal{I}$. Thus there exist an $x = \wedge_n e_{\lambda_n}$ in $\mathcal{M}$ such that $0 \leq x \leq e_{\lambda_n}$ for every n, and $\epsilon_0 \leq \tau(x) \leq \tau(e_{\lambda_n})$. Moreover, we have that $\|e_{\lambda_n}\|_{\mathcal{I}(\tau)} \geq \|x\|_{\mathcal{I}(\tau)}$ for every n, so therefore, $\|x\|_{\mathcal{I}(\tau)} = 0$. Hence $x = 0$, which contradicts with the fact that $\epsilon_0 \leq \tau(x)$.

(2) We show that for a projection $e \in \mathcal{M}$ such that $\tau(e) < \infty$ there exists $c(e) = \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)}$ satisfying $\|exe\|_{\mathcal{I}(\tau)} \geq c(e)\|exe\|_1$ for all $x \in \mathcal{M}$.

Suppose that $e = e^2 = e^*$ is a projection in $\mathcal{M}$ such that $\tau(e) < \infty$. Let x be a positive element in $\mathcal{M}$. For any $\epsilon > 0$, there exist nonnegative numbers $\beta_1, \beta_2, \ldots, \beta_n$ and subprojections $p_1, p_2, \ldots, p_n$ of e in $\mathcal{M}$ such that $\|exe - \sum_{i=1}^n \beta_i p_i\|_{\mathcal{I}(\tau)} \leq \|e - \sum_{i=1}^n \beta_i p_i\|\|e\|_{\mathcal{I}(\tau)} < \epsilon$ and $\|exe - \sum_{i=1}^n \beta_i p_i\|_1 \leq \|e - \sum_{i=1}^n \beta_i p_i\|\|e\|_1 < \epsilon$. We call $\sum_{i=1}^n \beta_i p_i = y_0$. For each $m \in \mathbb{N}$ and $1 \leq i \leq n$, we partition $p_i = q_{i,1} + q_{i,2} + \cdots + q_{i,k_i} + q_{i,k_i+1}$ where k_i is a positive integer and $q_{i,1}, q_{i,2}, \ldots, q_{i,k_i}$ are projections in $\mathcal{M}$ such that $\tau(q_{i,1}) = \tau(q_{i,2}) = \cdots = \tau(q_{i,k_i}) = 1/m$, and $0 \leq \tau(q_{i,k_i+1}) < 1/m$. We can write

$$y_0 = \sum_{i=1}^n \beta_i (\sum_{j=1}^{k_i+1} q_{i,j}) = z_1 + z_2,$$

where $z_1 = \sum_{i=1}^n \beta_i (\sum_{j=1}^{k_i} q_{i,j})$ and $z_2 = \sum_{i=1}^n \beta_i q_{k_i+1}$.

We let $q = \sum_{i=1}^n \sum_{j=1}^{k_i} q_{i,j}$. Then, by Lemma 4.2.7,

$$\|y_0\|_{E(\tau)} \geq \|z_1\|_{\mathcal{I}(\tau)} \geq \frac{\|q\|_{\mathcal{I}(\tau)}}{\tau(q)} \|z_1\|_1.$$

Also, by the triangle inequality,

$$\|z_1\|_1 \geq \|y_0\|_1 - \|z_2\|_1 \geq \|y_0\|_1 - (\sum_{i=1}^n \beta_i)/m,$$

which approaches $\|y_0\|_1$ as $m \to \infty$. Furthermore, by (3) we have

$$\frac{\|q\|_{\mathcal{I}(\tau)}}{\tau(q)} \geq \frac{\|e\|_{\mathcal{I}(\tau)} - \sum_{i=1}^n \beta_i \|q_{i,k_i+1}\|_{\mathcal{I}(\tau)}}{\tau(e)} \to \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)} \quad \text{as } m \to \infty.$$

Therefore,

$$\|y_0\|_{\mathcal{I}(\tau)} \geq \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)} \|y_0\|_1.$$

By the choice of y_0, we conclude that

$$\|exe\|_{\mathcal{I}(\tau)} \geq \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)} \|exe\|_1,$$

for all x in $\mathcal{M}$.

□

4.2.3 Embedding from $L^\alpha(\mathcal{M}, \tau)$ into $\widetilde{M}$

We would like to show that there is a natural embedding from $L^\alpha(\mathcal{M}, \tau)$ into $\widetilde{\mathcal{M}}$.

Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ, and $\mathcal{H}$ is a Hilbert space. Recall

$$\mathcal{I} = span\{xey : x, y \in \mathcal{M}, e \in \mathcal{M}, e = e^2 = e^* \text{ with } \tau(e) < \infty\}$$

is the set of elementary operators of $\mathcal{M}$. Define $\widetilde{\mathcal{M}}$ to be the algebra of closed, densely defined operators on $\mathcal{H}$ affiliated with $\mathcal{M}$. We recall that the measure topology on $\mathcal{M}$ is given by the family of neighborhoods $U_{\delta,\epsilon} = \{a \in \mathcal{M} :$ for some projection $p \in \mathcal{M}, \|ap\| \leq \epsilon$ and $\tau(p^\perp) \leq \delta\}$ for any $\epsilon, \delta > 0$.

Suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$.

Lemma 4.2.9. *Let $\epsilon > 0$ be given. There exists $\delta_0 > 0$ such that if e is a projection in $\mathcal{I}$ with $\alpha(e) < \delta_0$, then $\tau(e) < \epsilon$.*

Proof. Suppose, to the contrary, that there exists an $\epsilon > 0$ such that for every $\delta_0 > 0$, there exists a projection e_{δ_0} in $\mathcal{I}$ such that $\alpha(e_{\delta_0}) < \delta$, and $\tau(e_{\delta_0}) \geq \epsilon$. Let $\delta_0 = 1/n$ for each $n \in \mathbb{N}$. Then there exits a sequence $\{e_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\alpha(e_n) < 1/n$, and $\tau(e_n) \geq \epsilon$. This is a contradiction, as α is mutually continuous with respect to τ (see definition 1.4.1). Therefore, the Lemma is proven. □

Lemma 4.2.10. *Suppose a sequence $\{a_n\}$ in $\mathcal{I}$ is Cauchy with respect to the norm α. Then $\{a_n\}$ is Cauchy in the measure topology.*

Proof. To prove that $\{a_n\} \subseteq \mathcal{I}$ is Cauchy in the measure topology, it suffices to show that for every $\epsilon, \delta > 0$, there exists an $N \in \mathbb{N}$ such that for $n, m > N$, there exists a projection $p_{m,n}$ satisfying $\| |a_m - a_n| p_{m,n} \| < \delta$ and $\tau((p_{m,n})^{\perp}) < \epsilon$. By Lemma 4.2.9, we know that there exists a $\delta_0 > 0$ such that

$$\text{if } e \text{ is a projection in } \mathcal{I} \text{ with } \alpha(e) < \delta_0, \text{ then } \tau(e) < \epsilon. \tag{4.2}$$

For each $m, n \in \mathbb{N}$, let $\{e_\lambda(m,n)\}$ be the spectral decomposition of $|a_m - a_n|$ in $\mathcal{M}$. By the spectral decomposition theorem, we have $|a_m - a_n| = \int_0^\infty \lambda de_\lambda(m,n)$, and $\tau(|a_m - a_n|) = \int_0^\infty \lambda d\tau(e_\lambda(m,n))$. Let $\lambda_0 = \delta_0$. Hence $\lambda_0 e_{\lambda_0}(m,n)^{\perp} \leq |a_m - a_n| e_{\lambda_0}(m,n)^{\perp}$. So

$$\alpha(\lambda_0 e_{\lambda_0}(m,n)^{\perp}) \leq \alpha(|a_m - a_n|) \text{ for all } m, n \in \mathbb{N}. \tag{4.3}$$

Recall that $\{a_n\}$ is Cauchy in α-norm. For $\epsilon_1 = \lambda_0 \delta_0 > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $\alpha(a_m - a_n) < \epsilon_1$. Combining with (4.3), we have that for every $m, n > N$, $\lambda_0 \alpha(e_{\lambda_0}(m,n)^{\perp}) < \epsilon_1$. This implies that

$$\alpha(e_{\lambda_0}(m,n)^{\perp}) < \epsilon_1 / \lambda_0 = \delta_0.$$

Because of (4.2), $\tau(e_{\lambda_0}(m,n)^{\perp}) < \epsilon$ for every $m, n > N$. Put $p_{m,n} = e_{\lambda_0}(m,n)$. Then for every $m, n > N$,

$$\| |a_m - a_n| p_{m,n} \| \leq \lambda_0 = \delta_0, \text{ and } \tau(p_{m,n}^{\perp}) < \epsilon.$$

The proof is complete. ∎

Therefore, there is a natural continuous mapping from $L^\alpha(\mathcal{M}, \tau)$ into $\widetilde{\mathcal{M}}$.

Let e be a projection in $\mathcal{M}$ such that $\tau(e) < \infty$, and let $\mathcal{M}_e = e\mathcal{M}e$. Define a faithful, normal, tracial state τ_e on $\mathcal{M}_e$ by $\tau_e(x) = \frac{1}{\tau(e)} \tau(x)$ for every x in $\mathcal{M}_e$.

It can be shown that τ_e is a finite, faithful, normal tracial state on $\mathcal{M}_e$. Suppose that α is a unitarily invariant, locally $\| \cdot \|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$. Define $\alpha_e = \alpha|_{e\mathcal{M}e}$. We define $\alpha_e' : \mathcal{M}_e \to [0, \infty]$ by $\alpha_e'(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha_e(y) \leq 1\}$ for every x in $\mathcal{M}_e$. It may be shown that α_e' is indeed a norm, and we call α_e' the *dual norm* of α_e (see [8] for more information). We define $L^{\alpha_e'}(\mathcal{M}_e, \tau) = \overline{\mathcal{M}_e}^{\alpha_e'}$.

We may also define $\overline{\alpha_e} : L^1(\mathcal{M}_e, \tau) \to [0, \infty]$ by $\overline{\alpha_e}(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha_e'(y) \leq 1\}$ for every x in $\mathcal{M}_e$, and $\overline{\alpha_e'} : L^1(\mathcal{M}_e, \tau) \to [0, \infty]$ by $\overline{\alpha_e'} = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha_e(y) \leq 1\}$ for every x in $\mathcal{M}_e$. $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$ and $L^{\overline{\alpha_e'}}(\mathcal{M}_e, \tau)$ are defined to be $\overline{\mathcal{M}_e}^{\overline{\alpha_e}}$ and $\overline{\mathcal{M}_e}^{\overline{\alpha_e}'}$ respectively.

Lemma 4.2.11. *Let α be a unitarily invariant, $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Then α_e, α_e', $\overline{\alpha_e'}$ and $\overline{\alpha_e}$ are unitarily invariant norms on $L^\alpha(\mathcal{M}, \tau)$.*

Proof. Clearly, $\alpha_e(uxv) = \alpha(uxv) = \alpha(x) = \alpha_e(x)$ for unitaries u and v and an element x in $\mathcal{M}_e \subset \mathcal{M}$. Therefore, α_e is a unitarily invariant norm.

Let u and v be unitaries, and x be an element of $L^{\alpha_e'}(\mathcal{M}_e, \tau_e)$. Then

$$\begin{aligned}
\alpha_e'(uxv) &= \sup\{|\tau(uxvy)| : y \in \mathcal{M}, \alpha_e(y) \leq 1\} \\
&= \sup\{|\tau(xuyv)| : y \in \mathcal{M}, \alpha_e(y) \leq 1\} \\
&= \sup\{|\tau(xy_0)| : y_0 \in \mathcal{M}, \alpha_e(y_0) \leq 1\} \\
&= \alpha_e'(x)
\end{aligned}$$

for every $x \in L^{\alpha_e'}(\mathcal{M}_e, \tau_e)$. Therefore, α_e' is unitarily invariant.

The proofs that $\overline{\alpha_e}$ and $\overline{\alpha_e'}$ are unitarily invariant are similar. □

Lemma 4.2.12. *Suppose α is a unitarily invariant, $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$. Then*

(i) $\|x\|_1 \leq \overline{\alpha_e}(x)$ for every $x \in L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$; and

(ii) $\|x\|_1 \leq \overline{\alpha_e}'(x)$ for every $x \in L^{\overline{\alpha_e}'}(\mathcal{M}_e, \tau)$.

Proof. (i) Suppose that x is in $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau) \subseteq L^1(\mathcal{M}_e, \tau)$. Let $x = uh$ be the polar decomposition of x in $L^1(\mathcal{M}_e, \tau)$, such that u is a unitary in $\mathcal{M}_e$, and h is positive in $L^1(\mathcal{M}_e, \tau)$. As $\overline{\alpha_e}$ is unitarily invariant (see Lemma 4.2.11),

$$\overline{\alpha_e}(x) = \overline{\alpha_e}(uh) = \overline{\alpha_e}(h). \tag{4.4}$$

By definition, $\overline{\alpha_e}(h) \geq |\tau(h)| = \|x\|_1$. Hence, combining with Equation 4.4,

$$\|x\|_1 \leq \overline{\alpha_e}(x).$$

(ii) The proof of (ii) is similar. $\square$

Lemma 4.2.13. *For every $y \in \mathcal{M}_e$ and every $z \in L^1(\mathcal{M}_e, \tau)$, $\alpha_e'(yz) \leq \|y\|\overline{\alpha_e'}(z)$.*

Proof. Suppose $y \in \mathcal{M}_e$ such that $\|y\| = 1$, and let $y = \omega|y|$ be the polar decomposition of y in $\mathcal{M}_e$, i.e. $\omega \in \mathcal{M}_e$ is unitary and $|y| \in \mathcal{M}_e$ is positive. Define $v = |y| + i\sqrt{1-|y|^2}$. Then by construction, v is unitary in $\mathcal{M}_e$, and $|y| = \frac{v+v^*}{2}$. Consider any z in $L^1(\mathcal{M}_e, \tau)$. Then we have that

$$\overline{\alpha_e'}(yz) = \overline{\alpha_e'}(\omega|y|z) = \overline{\alpha_e'}(\frac{vz+v^*z}{2}) \leq \frac{\overline{\alpha_e'}(vz) + \overline{\alpha_e'}(v^*z)}{2} \text{ for every } z \text{ in } L^1(\mathcal{M}_e, \tau),$$

and y in $\mathcal{M}_e$ such that $\|y\| = 1$. Thus $\overline{\alpha_e'}(yz) \leq \|y\|\overline{\alpha_e'}(z)$ for every z in $L^1(\mathcal{M}_e, \tau)$ and y in $\mathcal{M}_e$. $\square$

Lemma 4.2.14. *For every $x \in \mathcal{M}_e$, $\alpha_e(x) = \overline{\alpha_e}(x)$.*

Proof. First, we show that $\overline{\alpha_e}(x) \leq \alpha_e(x)$ for every x in $\mathcal{M}_e$. By definition, $|\tau(xy)| \leq \alpha_e(x)\alpha_e'(y)$ for every x and y in $\mathcal{M}_e$. Suppose $\alpha_e'(y) \leq 1$. Then $|\tau(xy)| \leq \alpha_e(x)\alpha_e'(y) < \alpha_e(x)$ for every x in $\mathcal{M}_e$, and y in $\mathcal{M}_e$ such that $\alpha_e'(y) \leq 1$. Hence

$$\overline{\alpha_e}(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}_e, \alpha_e'(y) \leq 1\} \leq \alpha_e(x) \tag{4.5}$$

by definition.

Next, we show that $\overline{\alpha_e}(x) \geq \alpha_e(x)$. Suppose x is in $\mathcal{M}_e$ with $\alpha_e(x) = 1$. Then by the Hahn-Banach Theorem, there exists a φ in $L^{\alpha_e}(\mathcal{M}_e, \tau)^{\#}$ such that $\varphi(x) = \alpha_e(x) = 1$, and $\|\varphi\| = 1$. Since φ is in $L^{\alpha_e}(\mathcal{M}_e, \tau)^{\#}$, there exists ξ in $L^{\overline{\alpha_e'}}(\mathcal{M}_e, \tau)$ such that $\varphi(x) = |\tau(x\xi)| = 1$, and $\overline{\alpha_e}'(\xi) = \|\xi\| = 1$. Let $\xi = uh$ be the polar decomposition of ξ in $L^{\overline{\alpha_e'}}(\mathcal{M}_e, \tau)$, where $u \in \mathcal{M}_e$ is unitary and $h \in L^{\overline{\alpha_e'}}(\mathcal{M}_e, \tau)$ is positive.

By Lemma 3.8 in [8], there exists a family $\{e_\lambda\}$ of projections in $\mathcal{M}_e$ such that $\|h - he_\lambda\|_1 \to 0$, and $e_\lambda h = he_\lambda \in \mathcal{M}_e$ for every $0 < \lambda < \infty$. Also, $u \in \mathcal{M}_e$, so $uhe_\lambda \in \mathcal{M}_e$. Thus $\alpha_e'(uhe_\lambda) = \overline{\alpha_e}'(uhe_\lambda) \leq \overline{\alpha_e}'(uh)\|e_\lambda\| \leq \overline{\alpha_e}'(uh) = \alpha_e'(\xi) = 1$, as $\alpha_e'(x) = \overline{\alpha_e}'(x)$ for every $x \in \mathcal{M}_e$ by Lemma 3.2 in [8]. So, $\alpha_e(x)|\tau(x\xi)| = |\tau(xuh)| = \lim_{\lambda\to\infty} |\tau(xuhe_\lambda)| \leq \sup\{|\tau(xy)| : y \in \mathcal{M}_e, \alpha_e'(y) \leq 1\} = \overline{\alpha_e}(x)$. Therefore

$$\alpha_e(x) \leq \overline{\alpha_e}(x). \tag{4.6}$$

Hence from equations 4.5 and 4.6, $\alpha_e(x) = \overline{\alpha_e}(x)$, and the Lemma is proven. $\square$

Lemma 4.2.15. $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau) = \{x \in L^1(\mathcal{M}_e) : \overline{\alpha_e}(x) < \infty\}$ *is a complete space in* α_e*-norm.*

Proof. It suffices to show that for every Cauchy sequence $\{b_n\}$ in $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$, there exists b in $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$ such that $b_n \to b$ in $\overline{\alpha_e}$-norm. Suppose that $\{b_n\}$ is a Cauchy sequence in $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$. There exists $M > 0$ such that $\overline{\alpha_e}(b_n) \leq M$ for every n.

By Lemma 4.2.12,

$$\|b_n - b_m\|_1 \leq \overline{\alpha}(b_n - b_m) \text{ for all } m, n \geq 1.$$

Therefore, $\{b_n\}$ is Cauchy in $L^1(\mathcal{M}_e, \tau)$, which is complete. So there exists a b_0 in $L^1(\mathcal{M}_e, \tau)$ such that $\|b_n - b_0\|_1 \to 0$.

First, we claim that b_0 is in $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$. Let $y \in \mathcal{M}_e$ such that $\alpha_e'(y) \leq 1$. We have that $|\tau(b_n y) - \tau(b_0 y)| = |\tau((b_n - b_0)y)| \leq \|b_n - b_0\|_1 \|y\|_\infty$ by Hölder's Inequality. However, $\|b_n - b_0\|_1 \|y\|_\infty \to 0$. Also, by the definition of $\overline{\alpha}$, we also have that $|\tau(b_0 y)| = \lim_{n\to\infty} |\tau(b_n y)| \leq \limsup_{n\to\infty} \overline{\alpha_e}(b_n)\alpha_e'(y) \leq M$. Therefore, $\overline{\alpha}(b_x) \leq M$, and $b_0 \in L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$.

Now, we show that $\overline{\alpha_e}(b_n - b_0) \to 0$. We know that $\{b_n\}$ is Cauchy in $L^{\overline{\alpha}}(\mathcal{M}_e, \tau)$, so for every $n \geq 1$,

$$\begin{aligned}
|\tau((b_n - b_0)y)| &= \lim_{m\to\infty} |\tau((b_m - b_n)y)| \\
&\leq \limsup_{m\to\infty} \overline{\alpha_e}(b_n - b_m)\alpha_e'(y) \\
&\leq \limsup_{m\to\infty} \overline{\alpha}(b_m - b_n)
\end{aligned}$$

Therefore, $\overline{\alpha_e}(b_n - b_0) \leq \limsup_{m\to\infty}(b_n - b_m)$ for every $n \geq 1$, and since $\{b_n\}$ is Cauchy in $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$,

$$\overline{\alpha_e}(b_n - b_0) \to 0 \text{ as } n \to \infty,$$

and the Lemma is proven. $\square$

Therefore $L^{\overline{\alpha_e}}(\mathcal{M}_e, \tau)$ is a Banach space with respect to $\overline{\alpha_e}$-norm.

Lemma 4.2.16. *Suppose that $e \in \mathcal{M}$ is a projection such that $\tau(e) < \infty$. Suppose $\{ea_ne\} \subseteq \mathcal{I}$ is Cauchy in α-norm, and ea_ne converges in measure to 0. Then*

(i) for every $\epsilon > 0$, there exists a $\delta > 0$ such that, if q is a projection in $\mathcal{M}$ with $\tau(q) < \delta$, $|\tau(ea_neq)| < \epsilon$ for every n;

(ii) given $\delta > 0$, $\epsilon > 0$ and $N \in \mathbb{N}$, there exists p_n, a projection in $\mathcal{M}$, such that $\|ea_nep_n\| \leq \epsilon$, and $\tau(p_n^\perp) < \delta$ for every $n \geq N$;

(iii) for every projection q in $\mathcal{I}$, $\tau(ea_neq) \to 0$ as $n \to \infty$; and

(iv) for every b in $\mathcal{M}$, $\tau(ea_neb) \to 0$ as $n \to \infty$.

Proof. (i) Suppose that, as above, $e \in \mathcal{M}$ is a projection such that $\tau(e) < \infty$ and $\{ea_ne\}$ is a Cauchy sequence in α-norm. Let $\epsilon > 0$ be given. By assumption, α is a locally $\|\cdot\|_1$-dominating norm, so there exists $c(e)$ such that $\alpha(exe) \geq c(e)\|exe\|_1$ for every $x \in \mathcal{M}$. Then, given $\frac{\epsilon}{2}c(e)$, there exists $N_0 \in \mathbb{N}$ such that for all $n, m > N_0$,

$$\alpha(ea_ne - ea_me) \leq \frac{\epsilon}{2}c(e).$$

Let $\delta = \min_{k \leq N_0}\{\frac{\epsilon}{2\|ea_ke\|_\infty}\}$. Suppose q is a projection in $\mathcal{M}$ such that $\tau(q) \leq \delta$. Then for every $k \leq N_0$, $|\tau(ea_keq)| \leq \|ea_ke\|\|q\|_1$ by Hölder's Inequality, and $\tau(q) = \|q\|_1 \leq \delta$. Hence $|\tau(ea_keq)| \leq \|ea_ke\|\delta < \epsilon/2$ for all $k \leq N_0$ by our choice of δ.

For $k > N_0$,

$$\begin{aligned}
|\tau(ea_keq)| &\leq |\tau((ea_ke - ea_{N_0}e)q)| + |\tau(ea_{N_0}eq)| \\
&\leq \|ea_ke - ea_{N_0}e\|_1\|q\| + \|ea_{N_0}e\|\|q\|_1 && \text{(by Hölder's Inequality)} \\
&\leq \frac{1}{c(e)}\alpha(ea_ke - ea_{N_0}e)\|q\| + \|ea_{N_0}e\|\delta && \text{(by Definition 1.4.1)} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Hence, (i) is proven.

(ii) Suppose that $\{ea_ne\}$ is a Cauchy sequence in α-norm and $ea_ne \to 0$ in measure. Then, by the definition of convergence in measure, for any $\epsilon > 0$, $\delta > 0$ and $N \in \mathbb{N}$, there exists p_n in $\mathcal{M}$ such that $\|ea_nep_n\| < \epsilon$ and $\tau(p_n^\perp) < \delta$ for every $n \geq N$.

(iii) Suppose that $\{ea_ne\}$ is a Cauchy sequence in α-norm such that $ea_ne \to 0$ in measure. Then by (i), given $\epsilon > 0$ and a projection q in $\mathcal{I}$, there exists a $\delta_1 > 0$ such that if $\tau(q') < \delta_1$, then $|\tau(ea_neq')| < \epsilon/2$. Let $\delta > 0$ and $\epsilon_1 = \frac{\epsilon}{2\tau(q)}$. Then by (ii), there exists $N \in \mathbb{N}$ such that $\|ea_nep_n\| < \epsilon_1$, and $\tau(p_n^\perp) < \delta$ for every $n \geq N$. Thus, for $n \geq N$ and any projection $q \in \mathcal{I}$,

$$\tau(ea_neq) = \tau(ea_ne(q - q \cap p_n)) + \tau(ea_ne(q \cap p_n)). \tag{4.7}$$

However, $\tau(q - q \cap p_n) = \tau(q \cup p_n - p_n) \leq \tau(p_n^\perp) < \delta$. Therefore,

$$|\tau(ea_ne(q - q \cap p_n))| < \epsilon/2. \tag{4.8}$$

from (i). Also,

$$\begin{aligned} |\tau(ea_ne(q \cap p_n))| &= |\tau(ea_nep_n(q \cap p_n))| \\ &\leq \|ea_nep_n\| \|q \cap p_n\|_1 \\ &\leq \epsilon_1 \tau(q \cap p_n) \\ &< \epsilon_1 \tau(q) = \epsilon/2. \end{aligned} \tag{4.9}$$

Then from equations 4.7, 4.8 and 4.9, $|\tau(ea_neq)| < \epsilon$ for any given $\epsilon > 0$. Therefore, $\tau(ea_ne) \to 0$ for every $q \in \mathcal{M}$ such that q is a projection and $\tau(q) < \infty$.

(iv) Suppose that $\{ea_ne\}$ is a Caucy sequence in α-norm. Then there exists $M > 0$ such that $\tau(ea_ne) \leq \frac{\alpha(ea_ne)}{c(e)} < \frac{M}{c(e)}$. By considering ebe instead, we may assume that $b \in \mathcal{I}$. By the spectral decomposition theorem, b can be approximated by a finite linear combination of projections q_i in

$\mathcal{M}$, i.e. there exist $q_i \in \mathcal{I}$ such that $\|b - \sum_{i=1}^n q_i\| < \epsilon \frac{c(e)}{M}$ for any given $\epsilon > 0$. Therefore,

$$\begin{aligned} |\tau(ea_neb) - \tau(ea_ne\sum_{i=1}^n q_i)| &= |\tau(ea_ne(b - \sum_{i=1}^n q_i))| \\ &\leq \|\tau(ea_ne)\|_1 \|b - \sum_{i=1}^n q_i\| \\ &\leq \frac{M}{c(e)} \epsilon \frac{c(e)}{M} < \epsilon. \end{aligned}$$

Therefore, the Lemma is proven. □

Proposition 4.2.17. *There exists a natural embedding from $L^\alpha(\mathcal{M}, \tau)$ into $\widetilde{\mathcal{M}}$.*

Proof. By Lemma 4.2.10, there exists a natural mapping from $L^\alpha(\mathcal{M}, \tau)$ to $\widetilde{\mathcal{M}}$.

It suffices to show that this mapping is an injection. Suppose that $\{a_n\} \subseteq \mathcal{I}$ is a Cauchy sequence in α-norm such that $x_n \to 0$ in measure. As $L^\alpha(\mathcal{M}, \tau)$ is complete, there exists $a \in L^\alpha(\mathcal{M}, \tau)$ such that $a_n \to a$ in α-norm. Assume that $a \neq 0$. There exists a projection e in $\mathcal{M}$ such that $\tau(e) < \infty$ and $eae \neq 0$. Thus $\{ea_ne\}$ is Cauchy in α_e-norm, $ea_ne \to 0$ in measure and $ea_ne \to eae \neq 0$ in α_e-norm. By Lemma 4.2.16, $\tau(ea_neb) \to 0$ for any $b \in \mathcal{M}$. As, $|\tau(ea_neb) - \tau(eaeb)| \leq \alpha_e(ea_ne - eae)\alpha'_e(b) \to 0$, we have

$$\tau(eaeb) = 0 \qquad \text{for all } b \in \mathcal{I}.$$

On the other hand, by Lemma 4.2.14 and definition of $\overline{\alpha_e}$, since $eae \neq 0$, there exists some $b_0 \in \mathcal{M}_e$ such that $\alpha'_e(b_0) \leq 1$ and $\tau(eaeb_0) > \frac{\alpha(eae)}{2}$. This is a contradiction. Therefore, $a = 0$, and the mapping is an embedding. □

4.3 A Beurling theorem for semifinite Hardy spaces with norm α

Theorem 4.3.1. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$.*

Then, there exist a closed subspace Y of $L^{\alpha}(\mathcal{M},\tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that

(i) $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$*;*

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$*, and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(iii) $Y = [H_0^\infty Y]_\alpha$*;*

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^\alpha u_\lambda)$*.*

First, we prove some lemmas.

Lemma 4.3.2. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and that H^∞ is a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Suppose also that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then the following hold:*

(i) $\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M},\tau)$

(ii) $\mathcal{K} = [\mathcal{K} \cap \mathcal{M}]_\alpha$

Proof. (i) It is clear that

$$\mathcal{K} \cap \mathcal{M} \subseteq \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M},\tau).$$

We will prove that

$$\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M},\tau).$$

Assume, to the contrary, that $\mathcal{K}\cap\mathcal{M} \subsetneq \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M},\tau)$. Then there exists an $x \in \overline{\mathcal{K} \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M},\tau)$, with $x \notin \mathcal{K} \cap \mathcal{M}$. By the Hahn-Banach theorem, there exists a $\varphi \in L^\alpha(\mathcal{M},\tau)^\#$ such that $\varphi(x) \neq 0$, and $\varphi(y) = 0$ for every $y \in \mathcal{K} \cap \mathcal{M}$.

Since the restriction of τ to $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is semifinite, there exists a family $\{e_\lambda\}$ of projections in $\mathcal{D}$ such that $\tau(e_\lambda) < \infty$ for every λ, and $e_\lambda \to I$ in the weak* topology. This implies that $e_\lambda x \to x$ in the weak* topology and in α-norm by condition (iiia) of definition 1.4.1.

Thus, there must exist a λ such that $e_\lambda x \notin \mathcal{K} \cap \mathcal{M}$. Also, $e_\lambda x \in e_\lambda L^\alpha(\mathcal{M}, \tau)$.

Define $\psi : \mathcal{M} \to \mathbb{C}$ by $\psi(z) = \varphi(e_\lambda z)$ for every $z \in \mathcal{M}$. Then ψ is a bounded linear functional. We will show that ψ is normal, i.e. for an increasing net f_μ of projections in $\mathcal{M}$ such that $f_\mu \to I$ in weak*-topology, then $\psi(f_\mu) \to \psi(I)$. By condition (iiia) of Defintion 1.4.1, we have that $\alpha(e_\lambda f_\mu - e_\lambda I) \to 0$, for a fixed λ. Since $\varphi \in L^\alpha(\mathcal{M}, \tau)^\#$, $\varphi(e_\lambda f_\mu) \to \varphi(e_\lambda I)$. However

$$\varphi(e_\lambda f_\mu) = \psi(f_\mu),$$

and $\varphi(e_\lambda I) = \psi(I)$. Thus, $\psi(f_\mu) \to \psi(I)$. Therefore, ψ is a normal, bounded linear functional, namely, $\psi \in L^1(\mathcal{M}, \tau)$.

There exists a $\xi \in L^1(\mathcal{M}, \tau)$ such that $\psi(z) = \tau(z\xi)$ for every $z \in \mathcal{M}$. Note that $\psi(x) = \varphi(e_\lambda x) = \tau(x\xi) \neq 0$. Thus, there exists a projection $e \in \mathcal{D}$ such that $\tau(e) < \infty$ so that $\psi(ex) = \varphi(e_\lambda ex) = \tau(ex\xi) \neq 0$, and $\psi(ey) = \varphi(e_\lambda ey) = \tau(ey\xi) = 0$ for every $y \in \mathcal{K} \cap \mathcal{M}$.

Recall that $x \in \overline{\mathcal{K} \cap \mathcal{M}}^{w^*}$. Therefore, there exists a sequence $\{y_\mu\}$ in $\mathcal{K} \cap \mathcal{M}$ such that $y_\mu \to x$ in the weak* topology. Note that $\xi e \in L^1(\mathcal{M}, \tau)$. Hence,

$$\tau(y_\mu \xi e) \to \tau(x\xi e).$$

However, $\tau(y_\mu \xi e) = 0$, so $\tau(x\xi e) = 0$, which is a contradiction. Therefore (i) is proven.

(ii) Clearly, $\mathcal{K} \cap \mathcal{M} \subseteq \mathcal{K}$, and $\mathcal{K}$ is α-norm closed, so

$$[\mathcal{K} \cap \mathcal{M}]_\alpha \subseteq \mathcal{K}.$$

We will show that

$$\mathcal{K} = [\mathcal{K} \cap \mathcal{M}]_\alpha.$$

Suppose to the contrary, that $[\mathcal{K} \cap \mathcal{M}]_\alpha \subsetneqq \mathcal{K}$. There exists an $x \in \mathcal{K}$ such that $x \notin [\mathcal{K} \cap \mathcal{M}]_\alpha$. We know that $\mathcal{D}$ is semifinite, so there exists a family of projections $\{e_\lambda\}_{\lambda \in \Lambda}$ such that $\tau(e_\lambda) < \infty$, and $e_\lambda \to I$ in the weak-* topology. By Definition 1.4.1, part (iiia), $e_\lambda x \to x$ in α-norm. So, there exists λ such that $e_\lambda x \in \mathcal{K}$, since $x \in \mathcal{K}$, and $e_\lambda x \notin [\mathcal{K} \cap \mathcal{M}]_\alpha$, as $x \notin [\mathcal{K} \cap \mathcal{M}]_\alpha$.

By Lemma 1.5.6, there exist an $h_1 \in e_\lambda H^\infty e_\lambda$ and an $h_2 \in e_\lambda H^\alpha e_\lambda$ such that $h_1 e_\lambda x \in \mathcal{M}$, and $h_1 h_2 = e_\lambda = h_2 h_1$. Thus, $e_\lambda x = h_2 h_1 e_\lambda x$, $h_1 e_\lambda x \in \mathcal{M}$, and $h_1 e_\lambda x \in \mathcal{K}$, since $H^\infty \mathcal{K} \subseteq \mathcal{K}$.

Also, $h_2 \in e_\lambda H^\alpha e_\lambda$, so there exists a sequence $\{a_n\}$ in H^∞ such that $a_n \to h_2$ in α-norm. Hence, $e_\lambda x = h_2 h_1 e x$, $a_n h_1 e_\lambda x \in K \cap \mathcal{M}$, and

$$a_n h_1 e_\lambda x \to h_2 h_1 e x$$

in α-norm.

Therefore, $e_\lambda x \in [\mathcal{K} \cap \mathcal{M}]_\alpha$, which is a contradiction. Thus, (ii) is proven. $\square$

Lemma 4.3.3. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let H^∞ be a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Assume that $\mathcal{K}$ is a weak* closed subspace of $\mathcal{M}$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then*

$$\mathcal{K} = \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*}$$

.

Proof. First we must show that

$$\mathcal{K} \subseteq \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*}.$$

Let $x \in \mathcal{K} \subseteq \mathcal{M}$. We know that τ restricted to $\mathcal{D}$ is semifinite, so there exists a net of projections $\{e_\lambda\}_{\lambda \in \Lambda}$ such that $\tau(e_\lambda) < \infty$ and $e_\lambda \to I$ in the weak* topology. Also, $e_\lambda x \to x$ in the weak* topology.

To show that

$$x \in \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*},$$

it is sufficient to show that $e_\lambda x \in [\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}$. We have that $e_\lambda x$ is in $\mathcal{K}$, as $x \in \mathcal{K}$ and $\mathcal{K}$ is H^∞-invariant. We also know $\|e_\lambda x\|_\alpha \leq \|e_\lambda\|_\alpha \|x\| < \infty$. Therefore, $e_\lambda x \in L^\alpha(\mathcal{M}, \tau)$, and $e_\lambda x \in \mathcal{K} \cap L^\alpha(\mathcal{M}, \tau) \subseteq [\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$. Thus, $x \in \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*}$.

Hence $\mathcal{K} \subseteq \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*}$.

Next, we show that

$$\overline{[\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*} \subseteq \mathcal{K}.$$

It suffices to show that $[\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M} \subseteq \mathcal{K}$ since $\mathcal{K}$ is weak*-closed.

Suppose, to the contrary, that $[\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M} \subsetneqq \mathcal{K}$. There exists an $x \in [\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M}$ such that $x \notin \mathcal{K}$. Since the restriction of τ to $\mathcal{D}$ is semifinite, there exists a net $\{e_{\lambda}\}_{\lambda \in \Lambda}$ of projections such that $\tau(e_{\lambda}) \leq \infty$ and $e_{\lambda}x \to x$ in the weak* topology.

As $x \notin \mathcal{K}$, by the Hahn-Banach theorem, there exists a $\varphi \in \mathcal{M}_{\#}$ such that $\varphi(x) \neq 0$ and $\varphi(y) = 0$ for all y in $\mathcal{K}$. As $x \in [\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M}$ and $x \notin \mathcal{K}$, there exists a λ such that $e_{\lambda}x \in [\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M}$ and $e_{\lambda}x \notin \mathcal{K}$. Since $\varphi \in \mathcal{M}_{\#}$, there exists a ξ in $L^{1}(\mathcal{M},\tau)$ such that $\varphi(z) = \tau(z\xi)$ for every $z \in \mathcal{M}$. It follows that there exists a projection $e \in \mathcal{D}$ with $\tau(e) < \infty$ so that $\tau(x\xi e) \neq 0$, and $\tau(y\xi e) = 0$ for every $y \in \mathcal{K}$.

We claim that there exists a $z = \xi e \in \mathcal{M}e$ such that $\tau(xz) \neq 0$ and $\tau(yz) = 0$ for all $y \in \mathcal{K}$.

Note that $\xi e \in L^{1}(\mathcal{M},\tau)$ since $\xi \in L^{1}(\mathcal{M},\tau)$ and $\tau(e) < \infty$. By Lemma 1.5.6, there exist $h_3 \in eH^{\infty}e$, and $h_4 \in eH^{1}e$ such that $h_3h_4 = e = h_4h_3$ and $\xi e h_3 \in \mathcal{M}$. There exists $\{k_n\}$ in H^{∞} such that $k_n \to h_4$ in $\|\cdot\|_1$-norm. So,

$$
\begin{aligned}
\lim_{n\to\infty} |\tau(ex\xi) - \tau(x\xi e h_3 k_n)| &= \lim_{n\to\infty} |\tau(x\xi e h_3 h_4) - \tau(x\xi e h_3 k_n)| \\
&\leq \lim_{n\to\infty} \|x\|\|\xi e h_3\|\|h_4 - k_n\|_1 \\
&= 0.
\end{aligned}
$$

There exists an $N \in \mathbb{N}$ such that $\tau(x\xi e h_3 k_N) \neq 0$, since $\tau(x\xi) \neq 0$. We let $z = \xi e h_3 k_N \in \mathcal{M}$. Then, $z = ze \in \mathcal{M}e$ such that $\tau(xz) = \tau(x\xi e h_3 k_N) \neq 0$, and $\tau(yz) = \tau(y\xi e h_3 k_N) = \tau((e h_3 k_N) y\xi) = 0$ for every $y \in \mathcal{K}$.

Since $x \in [\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M}$ there exists $\{x_n\}$ in $\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)$ such that $x_n \to x$ in α norm, and $ex_n \to ex$ in α-norm. Note $ey = \sqrt{eyy^*e}v = e\sqrt{eyy^*e}ev$. Therefore, $ex_n \to ex$ in $\|\cdot\|_1$-norm, as $\|ey\|_1 = \|e\sqrt{eyy^*e}e\|_1$, $\alpha(ey) = \alpha(e\sqrt{eyy^*e}e)$, and α is locally $\|\cdot\|_1$ -dominating.

We also have that $|\tau(xz - x_n z)| = |\tau((x - x_n)z)| \leq \|e(x_n - x)\|_1\|z\|$. Finally, since $\{x_n\}$ is in $\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau) \subseteq \mathcal{K}$, $\tau(x_n z) = 0$. Hence, $\tau(xz) = 0$, which is a contradiction.

Therefore, $\overline{[\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M}}^{w^*} \subseteq \mathcal{K}$.

Thus, $\mathcal{K} = \overline{[\mathcal{K} \cap L^{\alpha}(\mathcal{M},\tau)]_{\alpha} \cap \mathcal{M}}^{w^*}$. ∎

Lemma 4.3.4. *Suppose $\mathcal{M}$ is a semifinite von Neumann algebra with a faithful, normal tracial weight τ, and suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually-continuous norm with respect to τ. Let H^∞ be a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Assume that S is a subset of $\mathcal{M}$ such that $H^\infty S \subseteq S$. Then*

$$[S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha = [\overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha.$$

Proof. Clearly, $S \cap L^\alpha(\mathcal{M}, \tau) \subseteq \overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)$ so, $[S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \subseteq [\overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$.

We will show that $\overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau) \subseteq [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$. Let $x \in \overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)$. We know that there exists a net $\{e_\lambda\}$ in $\mathcal{D}$ of projections such that $\tau(e_\lambda) < \infty$, and $e_\lambda \to I$ in the weak* topology. Thus, $e_\lambda x \to x$ in the weak* topology.

We will show that $e_\lambda x \in [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$ in order to show that $x \in [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$. By Lemma 4.3.2, we have that

$$[S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M} \subseteq \overline{[S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha}^{w^*} \cap L^\alpha(\mathcal{M}, \tau).$$

Since $x \in \overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)$, there exists a net $\{x_j\}$ in S such that $x_j \to x$ in the weak*-topology. Therefore $e_\lambda x_j \to e_\lambda x$ in the weak*-topology for every $\lambda \in \Lambda$. We note that $\alpha(e_\lambda x_j) \leq \alpha(e_\lambda)\|x_j\|$, and $H^\infty S \subseteq S$. Therefore $e_\lambda x_j \in S \cap L^\alpha(\mathcal{M}, \tau)$, and $e_\lambda x_j \in \overline{[S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*}$. Thus, $e_\lambda x \in \overline{[S \cap L^\alpha[\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*}$. It is clear that $e_\lambda x \in L^\alpha(\mathcal{M}, \tau)$. By Lemma 4.3.2, $\overline{[S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M}, \tau) = [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}$. So $e_\lambda x \in [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$.

Therefore, $x \in [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$, whence $\overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau) \subseteq [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$. Hence,

$$[\overline{S}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha = [S \cap L^\alpha(\mathcal{M}, \tau)]_\alpha.$$

□

Now, we prove Theorem 4.3.1.

Proof. Let $\mathcal{K}_1 = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*}$. $\mathcal{K}_1$ is a weak* closed subspace of $\mathcal{M}$ such that $H^\infty \mathcal{K}_1 \subseteq \mathcal{K}_1$. Then by Theorem 2.3.5, there exist a weak* closed subspace $Y_1 \subseteq \mathcal{M}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M}$ such that

(a) $u_\lambda Y_1^* = 0$ for every $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$, and $u_\lambda u_\mu^* = 0$ for every $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$;

(c) $Y_1 = \overline{H_0^\infty Y_1}^{w^*}$;

(d) $\mathcal{K}_1 = Y_1 \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\infty u_\lambda)$.

Let $Y = [Y_1 \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$.

(i) We know that there exists $\{a_n\} \subseteq Y_1^*$ such that $a_n \to a$ in α-norm for some $a \in Y_1^*$. From (a), and the definition of Y_1, $a_n u_i \to a u_i$ in α-norm. Thus, we may conclude that $u_\lambda Y^* = 0$ for every $\lambda \in \Lambda$.

(ii) follows from (b).

(iii) We will show that $Y = [H_0^\infty Y]_\alpha$. We have that

$$
\begin{aligned}
Y &= [Y_1 \cap L^\alpha(\mathcal{M}, \tau)]_\alpha && \text{(by definition of } Y\text{)}\\
&= [\overline{H_0^\infty Y_1}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha && \text{(by (c))}\\
&= [H_0^\infty Y_1 \cap L^\alpha(\mathcal{M}, \tau)]_\alpha && \text{(by Lemma 4.3.4)}\\
&= [H_0^\infty(\overline{[Y_1 \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}}^{w^*}) \cap L^\alpha(\mathcal{M}, \tau)]_\alpha && \text{(by Lemma 4.3.3)}\\
&\subseteq [\overline{H_0^\infty([Y_1 \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M})}^{w^*} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha && \text{(by Theorem 1.7.8 in [35])}\\
&= [H_0^\infty([Y_1 \cap L^p(\mathcal{M}, \tau)]_\alpha \cap \mathcal{M}) \cap L^\alpha(\mathcal{M}, \tau)]_\alpha && \text{(by Lemma 4.3.4)}\\
&= [H_0^\infty(Y \cap \mathcal{M}) \cap L^\alpha(\mathcal{M}, \tau)]_\alpha && \text{(by defintion of } Y\text{)}\\
&\subseteq [H_0^\infty Y]_\alpha\\
&\subseteq Y.
\end{aligned}
$$

Hence, $Y = [H_0^\infty Y]_\alpha$ as desired.

(iv) Finally, we will show that $\mathcal{K} = Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\alpha u_\lambda)$.

Recall that $Y = [Y_1 \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$.

We claim that $[H_0^\infty Y_1 \cap L^\alpha(\mathcal{M}, \tau)]_\alpha \subseteq [H_0^\infty(Y_1 \cap L^\alpha(\mathcal{M}, \tau))]_\alpha$.

Also, by Lemma 4.3.2, $H^\alpha u_\lambda = [H^\infty u_\lambda \cap L^\alpha(\mathcal{M},\tau)]_\alpha$ for every $\lambda \in \Lambda$. Now,

$$\begin{aligned}
\mathcal{K} &= [\mathcal{K}_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha \\
&= [\overline{Y_1 + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*} \cap L^\alpha(\mathcal{M},\tau)]_\alpha && \text{(by definition of } \mathcal{K}_1\text{)} \\
&= [Y_1 + \sum_{\lambda\in\Lambda} H^\infty u_\lambda \cap L^\alpha(\mathcal{M},\tau)]_\alpha && \text{(by Lemma 4.3.4)} \\
&= [Y_1 \cap L^\alpha(\mathcal{M},\tau) + \sum_{\lambda\in\Lambda} H^\infty u_\lambda \cap L^\alpha(\mathcal{M},\tau)]_\alpha && \text{(by (a) and (b))} \\
&= [Y + \sum_{\lambda\in\Lambda} H^\alpha u_\lambda]_\alpha \\
&= Y \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\alpha u_\lambda)
\end{aligned}$$

where the last equality comes from Definition 1.6.1. □

Corollary 4.3.5. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let $\mathcal{K}$ be a subset of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$. Then there exists a projection q with $\mathcal{K} = \mathcal{M}q$.*

Proof. We note that $\mathcal{M}$ can be considered as a semifinite subdiagonal subalgebra of $\mathcal{M}$ itself. Hence, we let $\mathcal{M} = H^\infty$, and it follows that $\mathcal{D} = \mathcal{M}$ and Φ is the identity map on $\mathcal{M}$. Also, $H_0^\infty = \{0\}$ and $H^\alpha = L^\alpha(\mathcal{M},\tau)$.

Let $\mathcal{K}$ be a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$. From Theorem 4.3.1,

$$\mathcal{K} = Y \oplus_{row} (\oplus^{row}_{\lambda\in\Lambda} H^\alpha u_\lambda),$$

where $u_\lambda Y^* = 0$ for every $\lambda \in \Lambda$, $u_\lambda u_\lambda^* \in \mathcal{D}$, and $u_\lambda u_\mu^* = 0$ for every $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$, and $Y = [H_0^\infty Y]_\alpha$.

It is clear that because $H_0^\infty = \{0\}$, $Y = 0$. Also, since $\mathcal{D} = \mathcal{M}$, we have that

$$\begin{aligned}
H^\alpha u_\lambda &= L^\alpha(\mathcal{M},\tau) u_\lambda = L^\alpha(\mathcal{M},\tau) u_\lambda u_\lambda^* u_\lambda \\
&\subseteq L^\alpha(\mathcal{M},\tau) u_\lambda^* u_\lambda \subseteq L^\alpha(\mathcal{M},\tau) u_\lambda = H^\alpha u_\lambda.
\end{aligned}$$

Therefore, $H^\alpha u_\lambda = L^\alpha(\mathcal{M},\tau)u_\lambda^* u_\lambda$. Specifically, we find that

$$\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^\alpha u_\lambda) = (\oplus_{\lambda\in\Lambda}^{row} L^\alpha(\mathcal{M},\tau)u_\lambda^* u_\lambda)$$

$$L^\alpha(\mathcal{M},\tau)\big(\sum_{\lambda\in\Lambda} u_\lambda^* u_\lambda\big) = L^\alpha(\mathcal{M},\tau)q$$

where we let $\sum_{\lambda\in\Lambda} u_\lambda^* u_\lambda = q$, and q is a projection in $\mathcal{M}$. This ends the proof. $\square$

Chapter 5

Applications for unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms

In Chapter 4, we were able to prove a Beurling-Chen-Hadwin-Shen theorem for a semifinite von Neumann algebra $\mathcal{M}$. We seek to extend those results to semifinite factors, crossed products of a von Neumann algebra by an action β, and Banach function spaces, as described in Chapter 4.

When $\mathcal{M}$ is a factor, we can weaken the conditions on α.

Corollary 5.2.1. *Suppose $\mathcal{M}$ is a factor with a faithful, normal tracial weight τ. Let $\alpha : \mathcal{I} \to [0, \infty)$, where $\mathcal{I}$ is the set of elementary operators in $\mathcal{M}$, be a unitarily invariant norm such that any net $\{e_\lambda\}$ in $\mathcal{M}$ with $e_\lambda \uparrow I$ in the weak* topology implies that $\alpha((e_\lambda - I)x) \to 0$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then, there exist a closed subspace Y of $L^\alpha(\mathcal{M}, \tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that*

(i) $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$*;*

(ii) $u_\lambda u_\lambda \in \mathcal{D}$*, and* $u_\lambda u_\mu$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(iii) $Y = [H_0^\infty Y]_\alpha$*;*

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda \in \Lambda}^{row} H^\alpha u_\lambda)$*.*

Similar to our result in Chapter 3 for L^p spaces, we prove a Beurling-Chen-Hadwin-Shen theorem for the crossed product of a von Neumann algebra $\mathcal{M}$ by a trace-preserving action β with a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous with respect to the trace τ.

Corollary 5.3.2. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ, and β be a trace-preserving, *-automorphism of $\mathcal{M}$. Consider the crossed product of $\mathcal{M}$ by an action β, $\mathcal{M} \rtimes_\beta \mathbb{Z}$. Still denote the semifinite, faithful, normal, extended tracial weight on $\mathcal{M} \rtimes_\beta \mathbb{Z}$ by τ.*

*Denote by H^∞ the weak *-closed nonself-adjoint subalgebra in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which is generated by $\{\Lambda(n)\Psi(x) : x \in \mathcal{M}, n \geq 0\}$. Then H^∞ is a semifinite subdiagonal sublagebra of $\mathcal{M} \rtimes_\beta \mathbb{Z}$.*

Let $\mathcal{K}$ be a closed subspace of $L^\alpha(\mathcal{M} \rtimes_\beta \mathbb{Z}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which satisfy

(i) $u_\lambda q = 0$ for all $\lambda \in \Lambda$;

(ii) $u_\lambda u_\lambda^* \in \mathcal{M}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(iii) $\mathcal{K} = (L_\alpha(\mathcal{M} \rtimes_\beta \mathbb{Z})q) \otimes^{row} (\otimes^{row}_{\lambda \in \Lambda} H^\alpha u_\lambda)$.

As $B(\mathcal{H})$ is a factor and can be realized as the crossed product, we can also weaken the conditions on α when $\mathcal{M} = B(\mathcal{H})$. Additionally, we can fully characterize the H^∞ invariant subspace.

Corollary 5.4.2. *Suppose $\mathcal{H}$ is a separable Hilbert space with an orthonormal base $\{e_m\}_{m \in \mathbb{Z}}$, and let*

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n \rangle = 0, \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$. Then $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is the diagonal subalgebra of $B(\mathcal{H})$.

Suppose $\alpha : \mathcal{I} \to [0, \infty)$, where $\mathcal{I}$ is the set of elementary operators in $\mathcal{M}$, is a unitarily invariant norm such that any net $\{e_\lambda\}$ in $\mathcal{M}$ with $e_\lambda \uparrow I$ in the weak topology implies that $\alpha((e_\lambda - I)x) \to 0$.*

Assume that $\mathcal{K}$ is a closed subspace of H^α such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then there exists $\{u_\lambda\}_{\lambda\in\Lambda}$, a family of partial isometries in H^∞ which satisfy

(i) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \Lambda$ *such that* $\lambda \neq \mu$*;*

(ii) $\mathcal{K} = \oplus_{\lambda\in\Lambda}^{row} H^\alpha u_\lambda$.

Additionally, we prove a result for a Banach function space E with norm $\|\cdot\|_{\mathcal{I}(\tau)}$ and provide an answer for Problem 4.0.1.

Corollary 5.1.1. *Suppose that $\mathcal{I}(\tau)$ is a Banach function space on the diffuse von Neumann algebra $\mathcal{M}$ with order continuous norm $\|\cdot\|_{\mathcal{I}(\tau)}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $\mathcal{I}(\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then, there exist a closed subspace Y of $\mathcal{I}(\tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that*

(i) $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$*;*

(ii) $u_\lambda u_\lambda \in \mathcal{D}$*, and* $u_\lambda u_\mu$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(iii) $Y = [H_0^\infty Y]_{\mathcal{I}(\tau)}$*;*

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^{\mathcal{I}(\tau)} u_\lambda)$.

5.1 Invariant subspaces for non-commutative Banach function spaces

We briefly recall our discussion of a non-commutative Banach function space. Let E be a symmetric Banach function space on $(0, \infty)$ with Lebesgue measure. As before, we let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal tracial state τ and $\mathcal{I} = \{x \in \mathcal{M} : x$ is a finite rank operator in $(\mathcal{M}, \tau)$ and $\|\mu(x)\|_E < \infty\}$. We may then define a Banach function space $\mathcal{I}(\tau)$, and a norm $\|\cdot\|_{\mathcal{I}(\tau)}$ by $\|x\|_{\mathcal{I}(\tau)} = \|\mu(x)\|_E$ for every $x \in \mathcal{I}(\tau)$. We let H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$, as described in Chapter 4. The following is an easy corollary of Theorem 4.3.1 and Proposition 4.2.8.

Corollary 5.1.1. *Suppose that $\mathcal{I}(\tau)$ is a Banach function space on the diffuse von Neumann algebra $\mathcal{M}$ with order continuous norm $\|\cdot\|_{\mathcal{I}(\tau)}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $\mathcal{I}(\tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then, there exist a closed subspace Y of $\mathcal{I}(\tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that

(i) $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$*;*

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$*, and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(iii) $Y = [H_0^\infty Y]_{\mathcal{I}(\tau)}$*;*

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^{\mathcal{I}(\tau)} u_\lambda)$*.*

5.2 Invariant subspaces for factors

We also have the following corollary from Theorem 4.3.1 and Proposition 4.2.5.

Corollary 5.2.1. *Suppose $\mathcal{M}$ is a factor with a faithful, normal tracial weight τ. Let $\alpha : \mathcal{I} \to [0, \infty)$, where $\mathcal{I}$ is the set of elementary operators in $\mathcal{M}$, be a unitarily invariant norm such that any net $\{e_\lambda\}$ in $\mathcal{M}$ with $e_\lambda \uparrow I$ in the weak* topology implies that $\alpha((e_\lambda - I)x) \to 0$. Let H^∞ be a semifinite subdiagonal subalgebra of $L^\alpha(\mathcal{M}, \tau)$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then, there exist a closed subspace Y of $L^\alpha(\mathcal{M}, \tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that

(i) $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$*;*

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$*, and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(iii) $Y = [H_0^\infty Y]_\alpha$*;*

(iv) $\mathcal{K} = Y \oplus^{row} (\oplus_{\lambda\in\Lambda}^{row} H^\alpha u_\lambda)$*.*

5.3 Invariant subspaces of analytic crossed products

Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful normal tracial weight τ. We let β be a *-automorphism of $\mathcal{M}$ such that $\tau(\beta(x)) = \tau(x)$ for every $x \in \mathcal{M}^+$ (i.e. β is trace-preserving).

Let $l^2(\mathbb{Z})$ denote the Hilbert space which consists of the complex-valued functions f on $\mathbb{Z}$ which satisfy $\sum_{m\in\mathbb{Z}} |f(m)|^2 < \infty$. Let $\{e_n\}_{n\in\mathcal{Z}}$ be the orthonormal basis of $l^2(\mathbb{Z})$ such that $e_n(m) = \delta(n,m)$. We also denote the left regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ by $\lambda : \mathbb{Z} \to B(l^2(\mathbb{Z}))$, where $\lambda(n)(e_m) = e_{m+n}$.

We let $\mathcal{H} = L^2(\mathcal{M},\tau) \otimes l^2(\mathbb{Z})$, or equivalently, $H = \oplus_{m\in\mathbb{Z}} L^2(\mathcal{M},\tau) \otimes e_m$. The representations Ψ of $\mathcal{M}$ and Λ of $\mathbb{Z}$ may be defined by

$$\Psi(x)(\xi \otimes e_m) = (\beta^{-m}\xi) \otimes e_m, \qquad \text{for all } x \in \mathcal{M}, \xi \in L^2(\mathcal{M},\tau) \text{ and } m \in \mathbb{Z}$$

$$\Lambda(n)(\xi \otimes e_m) = \xi \times (\lambda(n)e_m) \qquad \text{for all } n, m \in \mathbb{Z}.$$

It is not hard to verify that

$$\Lambda(n)\Psi(x)\Lambda(-n) = \Psi(\beta^n(x)) \qquad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

We may define the crossed product of $\mathcal{M}$ by an action β, which we denote by $\mathcal{M} \rtimes_\beta \mathbb{Z}$, to be the von Neumann algebra generated by $\Psi(\mathcal{M})$ and $\Lambda(\mathbb{Z})$ in $B(\mathcal{H})$. When there is no possibility of confusion, we will identify $\mathcal{M}$ with its image $\Psi(\mathcal{M})$ under Ψ in $\mathcal{M} \rtimes_\beta \mathbb{Z}$.

In Chapter 13 of [22], amongst others, it is shown that there exists a faithful, normal conditional expectation, Φ, taking $\mathcal{M} \rtimes_\beta \mathbb{Z}$ onto $\mathcal{M}$ such that

$$\Phi\bigg(\sum_{n=-N}^{N} \Lambda(n)\Psi(x_n)\bigg) = x_0 \qquad \text{where } x_n \in \mathcal{M} \text{ for every } -N \le n \le N.$$

There also exists a semifinite, normal, extended tracial weight on $\mathcal{M} \rtimes_\beta \mathbb{Z}$, which we still denote by τ, and which satisfies

$$\tau(y) = \tau(\Phi(y)), \text{ for every postive } y \in \mathcal{M} \rtimes_\beta \mathbb{Z}.$$

Example 5.3.1. *Let $\mathcal{M} = l^\infty(\mathbb{Z})$. Then $\mathcal{M}$ is an abelian von Neumann algebra with a semifinite, faithful, normal tracial weight τ which is given by*

$$\tau(f) = \sum_{m\in\mathbb{Z}} f(m), \qquad \text{for every positive } f \in l^\infty(\mathbb{Z}).$$

We let β be an action on $l^\infty(\mathcal{Z})$, which we define by

$$\beta(f)(m) = f(m-1), \qquad \text{for every } f \in l^\infty(\mathbb{Z}) \text{ and } m \in \mathbb{Z}.$$

It is known (see, for example Proposition 8.6.4 of [22]) that $l^\infty(\mathbb{Z}) \rtimes_\beta \mathbb{Z}$ is a type I_∞ factor. Therefore, for a separable Hilbert space $\mathcal{H}$, $l^\infty(\mathbb{Z}) \rtimes_\beta \mathbb{Z} \simeq B(\mathcal{H})$.

The next result follows from our construction of crossed products. Recall the following from Lemma 3.1.2 (see also section 3 of [1]).

Take the weak *-closed, non-self-adjoint subalgebra $\mathcal{M} \rtimes_\beta \mathbb{Z}_+$ of $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which is generated by

$$\{\Lambda(n)\Psi(x) : x \in \mathcal{M}, n \geq 0\}.$$

Then the following hold:

(i) $\mathcal{M} \rtimes_\beta \mathbb{Z}_+$ is a semifinite subdiagonal subalgebra with respect to $(\mathcal{M} \rtimes_\beta \mathbb{Z}, \Phi)$. We will denote such a semifinite subdiagonal subalgebra by H^∞ and call H^∞ an analytic crossed product.

(ii) We denote by H_0^∞ the space $\ker(\Phi) \cap H^\infty$. Then H_0^∞ is a weak *-closed nonself-adjoint subalgebra which is generated in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ by

$$\{\Lambda(n)\Phi(x) : x \in \mathcal{M}, n \geq 0\}$$

and satisfies

$$H_0^\infty = \Lambda(1)H^\infty.$$

(iii) $H^\infty \cap (H^\infty)^* = \mathcal{M}$.

We are able to characterize the invariant subspaces of a crossed product of a semifinite von Neumann algebra $\mathcal{M}$ by a trace-preserving action β.

Corollary 5.3.2. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ, and β be a trace-preserving, *-automorphism of $\mathcal{M}$. Consider the crossed product of $\mathcal{M}$ by an action β, $\mathcal{M} \rtimes_\beta \mathbb{Z}$. Still denote the semifinite, faithful, normal, extended tracial weight on $\mathcal{M} \rtimes_\beta \mathbb{Z}$ by τ.*

*Denote by H^∞ the weak *-closed nonself-adjoint subalgebra in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which is generated by $\{\Lambda(n)\Psi(x) : x \in \mathcal{M}, n \geq 0\}$. Then H^∞ is a semifinite subdiagonal subalgebra of $\mathcal{M} \rtimes_\beta \mathbb{Z}$.*

Let $\mathcal{K}$ be a closed subspace of $L^\alpha(\mathcal{M} \rtimes_\beta \mathbb{Z}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{M}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which satisfy

(i) $u_\lambda q = 0$ for all $\lambda \in \Lambda$;

(ii) $u_\lambda u_\lambda^* \in \mathcal{M}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(iii) $\mathcal{K} = (L^\alpha(\mathcal{M} \rtimes_\beta \mathbb{Z})q) \otimes^{row} (\otimes^{row}_{\lambda \in \Lambda} H^\alpha u_\lambda)$.

Proof. From Theorem 4.3.1, we know that

$$K = Y \oplus^{row} (\oplus^{row}_{\lambda \in \Lambda} H^\alpha u_\lambda)$$

such that Y is a closed subspace of $\mathcal{M} \rtimes_\beta \mathbb{Z}$ and a family of partial isometries, $\{u_\lambda\}$, in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which satisfy

(a) $u_\lambda Y^* = 0$ for all $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{M}$ and $u_\lambda u_\mu^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c) $Y = [H_0^\infty Y]_\alpha$.

By Lemma 3.1.2 and (c), it is clear that

$$Y = [H_0^\infty Y]_\alpha = [\Lambda(1) H^\infty Y]_\alpha \subseteq \Lambda(1) Y.$$

We can show, by induction, that $\Lambda(-n)Y \subseteq Y$ for any n in $\mathbb{N}$. From the defintion of H^∞, we know that $\Lambda(n)Y \subset Y$ for every $n \geq 0$, and $\psi(x)Y \subseteq Y$ for every $x \in \mathcal{M}$. Therefore,

$Y \subseteq L^{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z})$ is left-$\mathcal{M} \rtimes_{\beta} \mathbb{Z}$-invariant, and from Corollary 4.3.5, there exists a projection $q \in \mathcal{M}$ with $Y = L^{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z}, \tau)q$. Therefore,

(i) $u_{\lambda} q = 0$ for all $\lambda \in \Lambda$;

(ii) $u_{\lambda} u_{\lambda}^* \in \mathcal{M}$ and $u_{\lambda} u_{\mu}^* = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(iii) $\mathcal{K} = (L_{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z})q) \otimes^{row} (\otimes^{row}_{\lambda \in \Lambda} H^{\alpha} u_{\lambda})$

hold, and the corollary is proven. □

5.4 Invariant subspaces for $B(\mathcal{H})$

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with orthonormal base $\{e_m\}_{m \in \mathbb{Z}}$. We let $\tau = Tr$ be the usual trace on $B(\mathcal{H})$, namely

$$\tau(x) = \sum_{m \in \mathcal{Z}} \langle x e_m, e_m \rangle \qquad \text{for every } x \in B(\mathcal{H}) \text{ with } x > 0.$$

With this τ, $B(\mathcal{H})$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ.

We let

$$\mathcal{A} = \{x \in B(\mathcal{H}) : \langle x e_m, e_n \rangle = 0 \quad \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$.

Recall from Example 5.3.1 that the crossed product of $l^{\infty}(\mathbb{Z})$ by an action β, denoted $l^{\infty}(\mathbb{Z}) \rtimes_{\beta} \mathbb{Z}$, where the action β is determined by

$$\beta(f)(m) = f(m-1) \qquad \text{for every } f \in l^{\infty}(\mathbb{Z}), m \in \mathbb{Z}$$

is another way to realize $B(\mathcal{H})$.

It is easy to see that $\mathcal{A}$ is $l^{\infty}(\mathbb{Z}) \rtimes_{\beta} \mathbb{Z}_{+}$, a semifinite subdiagonal subalgebra of $l^{\infty}(\mathbb{Z}) \rtimes_{\beta} \mathbb{Z}$ (see Lemma 3.1.2).

The following corollary follows from Corollary 5.3.2

Corollary 5.4.1. *Suppose $\mathcal{H}$ is a separable Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$, and let*

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0, \quad \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$. Then $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is the diagonal subalgebra of $B(\mathcal{H})$. Suppose $\alpha : \mathcal{I} \to [0,\infty)$, where $\mathcal{I}$ is the set of elementary operators in $B(\mathcal{H})$, is a unitarily invariant norm such that any net $\{e_\lambda\}$ in $B(\mathcal{H})$ with $e_\lambda \uparrow I$ in the weak topology implies that $\alpha((e_\lambda - I)x) \to 0$.*

Assume that $\mathcal{K}$ is a closed subspace of $B(\mathcal{H})$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then there exists a projection q in $\mathcal{D}$ and $\{u_\lambda\}_{\lambda\in\Lambda}$, a family of partial isometries in H^∞ which satisfy

(i) $u_\lambda q = 0$ *for every* $\lambda \in \Lambda$*;*

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$, *and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$*;*

(iii) $\mathcal{K} = (B(\mathcal{H})q) \oplus^{row} (\oplus^{row}_{\lambda\in\Lambda} H^\alpha u_\lambda)$.

The following is a corollary of Corollary 5.3.2 and Proposition 4.2.5.

Corollary 5.4.2. *Suppose $\mathcal{H}$ is a separable Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$, and let*

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0, \quad \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$. Then $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is the diagonal subalgebra of $B(\mathcal{H})$.

Suppose $\alpha : \mathcal{I} \to [0,\infty)$, where $\mathcal{I}$ is the set of elementary operators in $B(\mathcal{H})$, is an unitarily invariant norm such that any net $\{e_\lambda\}$ in $B(\mathcal{H})$ with $e_\lambda \uparrow I$ in the weak topology implies that $\alpha((e_\lambda - I)x) \to 0$.*

Assume that $\mathcal{K}$ is a closed subspace of H^α such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then there exists $\{u_\lambda\}_{\lambda\in\Lambda}$, a family of partial isometries in H^∞ which satisfy

(i) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \Lambda$ *such that* $\lambda \neq \mu$*;*

(ii) $\mathcal{K} = \oplus_{\lambda \in \Lambda}^{row} H^{\alpha} u_{\lambda}$.

Remark 5.4.3. *The result is similar when* H^{∞} *is instead the upper triangular subalgebra of* $B(\mathcal{H})$.

Remark 5.4.4. *Recall that any unitarily invariant norm* α *gives rise to a symmetric gauge norm* Ψ *on the spectrum of* $|A|$, $\{a_n\}_{1 \leq n \leq N}$, *where* A *finite rank operator. Then Corollary 5.4.2 holds for* Ψ.

Bibliography

[1] W. B. Arveson, *Analyticity in operator algebras*, Amer. J. Math. 89 (1967) 578-642.

[2] T. Bekjan, *Noncommutative Hardy space associated with semi-finite subdiagonal algebras*, J. Math. Anal. Appl. 429 (2015), no. 2, 1347-1369.

[3] T. N. Bekjan and Q. Xu, *Riesz and Szegö type factorizations for noncommutative Hardy spaces*, J. Operator Theory, 62 (2009) 215–231.

[4] T. N. Bekjan, *Noncommutative symmetric Hardy spaces*, Integr. Equ. Oper. Theory 81 (2015) 191-212.

[5] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1949) 239-255.

[6] D. Blecher and L. E. Labuschagne, *A Beurling theorem for noncommutative L^p*, J. Operator Theory, 59 (2008) 29-51.

[7] S. Bochner, *Generalized conjugate and analytic functions without expansions*, Proc. Nat. Acad. Sci. U.S.A. 45 (1959) 855-857.

[8] Y. Chen, D. Hadwin and J. Shen, *A non-commutative Beurling's theorem with respect to unitarily invariant norms*, J. Operator Theory, 75 (2016) 497-523.

[9] K. Davidson, *Nest algebras: Triangular forms for operator algebras on Hilbert space*, Longman Scientific & Technical, (1988).

[10] P. Dodds and T. Dodds, *Some properties of symmetric operator spaces*, Proc. Centre Math. Appl. Austral. Nat. Univ., 29, Austral. Nat. Univ., Canberra, 1992.

[11] P. Dodds, T. Dodds and B. Pagter, *Noncommutative Banach function spaces*, Mathematische Zeitschrift, 201 (1989) 583-597.

[12] P. Dodds, T. Dodds and B. Pagter, *Noncommutative Köthe duality*, Trans. Amer. Math. Soc. 339 (1993) 717-750.

[13] R. Exel, *Maximal subdiagonal algebras*, Amer. J. Math. 110 (1988) 775-782.

[14] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. 2 (1986) 269-300.

[15] J. Fang, D. Hadwin, E. Nordgren and J. Shen, *Tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property*, J. Funct. Anal. 255 (2008) 142-183.

[16] I. C. Goldberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Vol. 18, Translations of Mathematical Monographs, 1969.

[17] P. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. 208 (1961) 102-112.

[18] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York-London, 1964.

[19] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. 99 (1958) 165-202.

[20] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. 2, Springer-Verlag, Berlin, 1970

[21] K. Hoffman, *Analytic functions and logmodular Banach algebras*, Acta Math. 108 (1962) 271-317.

[22] R. Kadison and J. Ringrose, *Fundamentals of the thoery of operator algebras, volume II, advanced theory*, Academic Press, Inc, (1986).

[23] R. A. Kunze, *L^p-Fourier transforms on locally compact unimodular groups*, Trans. Amer. Math. Soc. 89 (1958) 519-540.

[24] G. Ji, *Maximality of semi-finite subdiagonal algebras*, J. Shaanxi Normal Univ. Sci. Ed. 28 (2000) 15-17.

[25] M. Junge and D. Sherman, *Noncommutative L^p-modules*, J. Operator Theory 53 (2005) 3-34.

[26] M. Marsalli and G. West, *Noncommutative H^p spaces* , J. Operator Theory 40 (1998) 339-355.

[27] C. A. McCarthy, C_p, Israel J. Math. 5 (1967) 249-271.

[28] M. McAsey, P. Muhly and K. Saito, *Nonselfadjoint Crossed Products (Invariant Subspaces and Maximality)*, Transactions of the American Mathematical Society, Vol. 248, No. 2 (Mar., 1979), pp. 381-409.

[29] F. Murray and J. von Neumann *On rings of operators. IV.*, Annals of Mathematics, Vol. 44, No. 2 second series, pp. 716-808.

[30] T. Nakazi and Y. Watatani, *Invariant subspace theorems for subdiagonal algebras*, J. Operator Theory 37(1997), 379-395.

[31] E. Nelson, *Notes on noncommutative integration*, J. Funct. Anal. 15 (1974) 103-116.

[32] J. von Neumann, *Some matrix-inequalities and metrization of matric-space*, Tomsk Univ. Rev. 1 (1937) 286-300.

[33] B. de Pagter, *Non-commutative Banach function spaces*, K. Boulabiar, G. Buskes, A. Triki (Eds.), Positivity, Trends in Mathematics, Birkhuser Verlag, Basel (2007), pp. 197-227

[34] G. Pisier and Q. Xu, *Noncommutative L^p-spaces*, Handbook of the geometry of Banach spaces, North-Holland, Amsterdam, 2 (2003) 1459-1517.

[35] S. Sakai, *C^*-algebras and W^*-algebras*, Springer, 1971.

[36] L. Sager, *A Beurling-Blecher-Labuschagne theorem for noncommutative Hardy spaces associated with semifinite von Neumann algebras*, Integr. Equ. Oper. Theory 86 (2016) 377-407. doi:10.1007/s00020-016-2308-z

[37] L. Sager and W. Liu, *A Beurling-Chen-Hadwin-Shen Theorem for noncommutative Hardy spaces assocaited with semifinite von Neumann algebras with unitarily invariant norms* , preprint.

[38] K. S. Saito, *A note on invariant subspaces for finite maximal subdiagonal algebras*, Proc. Amer. Math. Soc. 77 (1979) 348-352.

[39] K. S. Saito, *A simple approach to the invariant subspace structure of analytic crossed products,* J. Operator Theory 27 (1992), no. 1, 169-177.

[40] R. Schatten, *A Theory of Cross-Spaces*, Annals of Mathematics Studies, no. 26, Princeton University Press, Princeton, New Jersey, 1950.

[41] R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin, 1960.

[42] I. Segal, *A noncommutative extension of abstract integration*, Ann. Math. 57 (1952) 401-457.

[43] B. Simon, *Trace ideals and their applications*, London Mathematical Society Lecture Note Series, vol. 35, Cambridge University Press, Cambridge-New York, 1979.

[44] T. P. Srinivasan, *Simply invariant subspaces*, Bull. Amer. Math. Soc. 69 (1963) 706-709.

[45] T. Srinivasan and J. K. Wang, *Weak*-Dirichlet algebras*, Proceedings of the International Symposium on Function Algebras, Tulane University, 1965 (Chicago), Scott-Foresman, 1966, 216–249.

[46] M. Takesaki, *Theory of Operator Algebras I*, Springer, 1979.

[47] Q. Xu, *On the maximality of subdiagonal algebras*, J. Operator Theory 54 (2005), no. 1, 137-146.

[48] Q. Xu *Operator spaces and noncommutative L^p, The part on noncommutative L^p-spaces* Lectures in the Summer School on Banach spaces and Operator spaces, Nankai University China July 16 - July 20, 2007.

[49] F. Yeadon, *Noncommutative L^p-spaces*, Math. Proc. Cambridge Philos. Soc. 77 (1975) 91-102.